Special values of automorphic L-functions for $\mathrm{GL}(r) \times \mathrm{GL}(r)$

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1 Introduction

The aim of this article is to compute the special values of the automorphic L-function of a product of cuspidal representations of $\mathrm{GL}(r)$ up to a number in a fixed number field (depending on the representations). The method are very similar to the ones used by H. Hida in [Hid] and the results are of course generalization of these.

More precisely, let F be a totally imaginary number field and π and π' two automorphic representations of $\mathrm{GL}(r, \mathbf{A}_F)$. According to [Clo] the representation π is defined over a finite extension $F(\pi)$ of F and we denote $F(\pi, \pi') = F(\pi)F(\pi')$ the composite field. We can define a period $C(\pi)$ for the representation π and another one $C_s(\eta)$ for a certain Eisenstein series arising from Rankin-Selberg method. We then have, Theorem 8.1,

$$\mathrm{L}_f(s, \pi \times \pi') = \frac{C(\pi)C(\pi')C_s(\eta)}{Z_\infty} c$$

with $c \in F(\pi, \pi')$ and Z_∞ an archimedean integral, for $s > 1$ integer in a certain strip (computable strip). This result is valid under some rather strong condition of compatibility of the infinite components of π and π'.

To prove this, we use the $(\mathfrak{g}, K)$-cohomology of the representations and the rationality of one-dimensional subspaces. On the infinite part, the dimension one comes from the classic theorem of [BW] on the cohomology of induced representation; on the finite part it comes from results of H. Jacquet and J. A. Shalika for the cuspidal representation and an appropriate choice for the Eisenstein series. Now the product of three elements of these one-dimensional subspaces happens to be a top-degree differential form. This differential form is defined on the manifold $G(F)\backslash G(\mathbf{A}_F)/Z(F_\infty)K_\infty K_f(\pi, \pi')$ with $K_f(\pi, \pi')$ an appropriate open compact subgroup of $G(\mathbf{A}_{F,f})$. This differential form can then be integrated over the manifold. Up to a finite number of places (including the archimedean), the L-function is represented by a Rankin–Selberg integral which is equal to the integration over the manifold. Up to the various periods $C(\pi)$, $C(\pi')$ and $C_s(\eta)$, the classes of cohomology are proportional to a rational class. The integration of the product of these rational classes leads to the rational number. We want to indicate that we where unable to prove the non vanishing of the integral Z_∞.

We would like to introduce the main notations used throughout the article. Let F be a totally imaginary number field with set of places Σ_F, infinite places S_∞ and finite places Σ_f. We let h be its class number and $d = \frac{[F:\mathbf{Q}]}{2}$. The algebraic group $G = G_r$ will be the group $\mathrm{GL}(r)$ defined over $\mathcal{O}_F$. For each $v \in \Sigma_f$, $\mathcal{O}_v$ will be the integer ring of F_v and

$K_v = G(\mathcal{O}_v)$. For $v \in S_\infty$, $K_v = \mathrm{U}_r$ and $\mathfrak{g}_v$ will be the Lie algebra of the real Lie group $G_v = G(F_v)$ (and accordingly $\mathfrak{k}_v$ for K_v). The algebraic group Z will be the center of G, $B = B_r$ will be the (standard) Borel subgroup of upper-triangular matrices and T the torus of diagonal matrices. For v an archimedean place, A_v will be the maximal split torus of B and for v finite, A_v will be the set of diagonal matrices whose elements are power of the uniformizer. When a Lie group is designed by a letter the same letter in lowercase gothic font will design its Lie algebra; a subscript $\mathbf{C}$ will indicate the complexification of a real Lie group or real Lie algebra. The set of weights for $G_r(\mathbf{C})$ and U_r will be ordered by the order associated to B. The set of positive weights will be denoted Δ^+. Geometrically,

$$G(F)\backslash G(\mathbf{A}_F)/Z(F_\infty)K_\infty K_f = \coprod_{j=1}^{h} \Gamma_j\backslash\mathcal{Z}$$

with Γ_j a congruence subgroup and $\mathcal{Z} = G^1(F_\infty)/K^1_\infty$ is the usual symmetric domain. The L-functions will be *complete* L-functions, including the infinite L-factors. The finite part will be written L_f (as above).

For each representation τ of a group H, $\tau^\vee$ will be its contragredient whenever this makes sense.

The vector spaces F^r and F_v^r will be made of row vectors and we put $\epsilon = (0, ..., 0, 1)$.

2 Critical values of motivic L-functions.

The point behind the calculus of special values of L-functions is to link these values to critical values of L-functions of motives à la Deligne. Suppose we have a motive whose L-function is equal to the automorphic L-function of $\pi \times \pi'$. Suppose at archimedean place v the representation π_v is induced from the character $\chi(z)\chi_1(a_1)...\chi_r(a_r)$ of the Borel subgroup (with $\chi_i(a_i) = \left(\frac{a_i}{|a_i|}\right)^{p_i + \frac{r-1}{2}}$, the p_i in decreasing order and $z = a_1...a_r$); similarly for π' with p'_i's. The corresponding archimedean L-factor is then equal to

$$\mathrm{L}_v(s; \pi \times \pi') = \prod_{1 \leqslant i,j \leqslant r} \Gamma(s + |p_i + p'_j + r - 1|).$$

The critical values of the global L-function are the integers s such that neither $\mathrm{L}_v(s, \pi \times \pi')$ nor $\mathrm{L}_v(1-s, \pi \times \pi')$ have poles. Given that Γ has poles for non positive integers, we must have s in $[1-m; m]$ with $m = \min_{i,j}(|p_i + p'_j + r - 1|)$. Suppose we must have the relations, consequences of proposition 6.2,

$$1 \leqslant \min_{1 \leqslant k \leqslant r} (p_k + p'_{r+1-k}) + r - 1 \tag{1}$$

$$\max_{1 \leqslant k \leqslant r-1} (p_{k+1} + p'_{r+1-k}) + r - 1 \leqslant -1 \tag{2}$$

$$
\begin{aligned}
p_k &\geqslant p_{k+1}+1 \\
p'_k &\geqslant p'_{k+1}+1
\end{aligned}
\tag{3}
$$

The p_k and p'_k are in decreasing order. So for i fixed, (1) and (2) means that the minimum of $|p_i + p'_j + r - 1|$ will be for either $j = r+1-i$ or $j = r+2-i$. So m will be the smallest of $r - 1 + \min(p_k + p'_{r+1-k})$ and $1 - r - \max(p_{k+1} + p'_{r+1-k})$. These correspond exactly to the set of values for s given by proposition 6.2. We conclude that this article computes the critical values of the given motive, provided that it exists.

3 Rankin-Selberg convolutions.

We recall here what we will need from Rankin-Selberg integrals and their link to the automorphic L-function of $\mathrm{GL}(r)\times\mathrm{GL}(r)$. We fix two irreducible cuspidal representations of G_r, π and π'. They decompose as restricted tensor products $\pi = \otimes'_v \pi_v$ and their central characters $\omega = \otimes'_v \omega_v$ and $\omega' = \otimes'_v \omega'_v$ as well. We denote $\eta = \omega\omega'$ and $\eta_v = \omega_v \omega'_v$.

3.1 Local convolutions.

Let v be a place of F. We take $\mathcal{W}(\pi_v, \psi_v)$ to be the Whittaker model of π_v with respect to the character $\theta_v(n) = \psi_v(\sum n_{i,i+1})$ of N. To unify notations, if v is finite, we take $\mathcal{W}_0(\pi_v, \psi_v)$ to be $\mathcal{W}(\pi_v, \psi_v)$ and if v is archimedean, $\mathcal{W}_0(\pi_v, \psi_v)$ is the subset of C^∞, K_v-finite vectors in $\mathcal{W}(\pi_v, \psi_v)$.

For $W \in \mathcal{W}_0(\pi_v, \psi_v)$, $W' \in \mathcal{W}_0(\pi'_v, \overline{\psi_v})$, $\Phi \in \mathcal{S}(F^r)$ and $s \in \mathbf{C}$, let

$$Z_v(s, W, W'; \Phi) = \int_{N_v\backslash G_v} W(g)W'(g)\Phi(\epsilon g)|\det g|_v^s \,\mathrm{d}g.$$

We take $H_{s,v}(p) = |\det p|_v^s |p_{rr}|_v^{-rs} \eta_v(p_{rr})^{-1}$ and we can define equivalently with $f \in J_{s,v} = \mathrm{ind}_{P_v}^{G_v} H_{s,v}$

$$Z_v(s, W, W'; f) = \int_{Z_v N_v\backslash G_v} W(g)W'(g)f(g)\,\mathrm{d}g,$$

the correspondence being

$$f(g, s; \eta, \Phi) = |\det g|_v^s \int_{F_v^\times} \Phi(a\epsilon g)|a|_v^{rs}\eta(a)\,\mathrm{d}^\times a.$$

These integrals converges for real part of s sufficiently large (actually greater than 1). Since this integral is trilinear on its last three arguments, we extend it to a function of $\mathbf{C} \times (\mathcal{W}_0(\pi_v, \psi_v) \otimes \mathcal{W}_0(\pi'_v, \overline{\psi_v}) \otimes J_{s,v})$. If v is a finite place, these integrals generate a fractional ideal of $\mathbf{C}(q^{-s})$ (with q the number of elements of the residual field of F_v). This ideal is principal and generated by $\mathrm{L}_v(\pi\times\pi', s) = \mathrm{L}(\pi_v \times \pi'_v, s) = P(q^{-s})^{-1}$ with $P(0) = 1$. If, in addition, π_v and π'_v are unramified, the ideal on which ψ_v is trivial is $\mathcal{O}_v$, W° and W'° are the essential vectors of the model (that is W° is fixed by K_v and $W^\circ(K_v) = 1$) and Φ is the characteristic function of ${\mathcal{O}_v}^r$, then $\mathrm{L}(\pi_v \times \pi'_v, s) = Z(s, W^\circ, W'^\circ; \Phi)$.

3.2 Global convolutions.

The setup is nearly the same. We take $\mathcal{W}_0(\pi, \psi)$ to be the Whittaker model of π. For $W \in \mathcal{W}_0(\pi, \psi)$, $W' \in \mathcal{W}_0(\pi', \overline{\psi})$, $\Phi \in \mathcal{S}(\mathbf{A}_F)$ and $s \in \mathbf{C}$ let

$$Z(s, W, W'; \Phi) = \int_{N(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} W(g)W'(g)\Phi(\epsilon g)|\det g|_{\mathbf{A}_F}^s \,\mathrm{d}g.$$

We take H_s to be the character of $P(\mathbf{A}_F)$ defined by $H_s(p) = |\det p|_{\mathbf{A}_F}^s |p_{rr}|_{\mathbf{A}_F}^{-rs} \eta(p_{rr})^{-1}$ and we have equivalently with $f \in J_s = \mathrm{ind}_{P(\mathbf{A}_F)}^{G(\mathbf{A}_F)} H_s$

$$Z(s, W, W'; f) = \int_{Z(\mathbf{A}_F)N(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} W(g)W'(g)f(g)\,\mathrm{d}g,$$

the correspondence being

$$f(g, s; \eta, \Phi) = |\det g|^s \int_{\mathbf{A}_F{}^\times} \Phi(a\epsilon g)|a|^{rs}\eta(a)\,\mathrm{d}^\times a.$$

Since the function Z is trilinear in its last three coefficients, we extend it as in the local case to a function of $\mathbf{C} \times (\mathcal{W}_0(\pi, \psi) \otimes \mathcal{W}_0(\pi', \overline{\psi}) \otimes J_s)$.

3.3 Product decomposition.

As representations of $G(\mathbf{A}_F)$ (by which we mean a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbf{A}_{F,f})$-module), the Whittaker models decompose as restricted tensor products; explicitly:

$$\mathcal{W}_0(\pi, \psi) \simeq \bigotimes{}' \mathcal{W}_0(\pi_v, \psi_v),$$

the product being taken all all places v of F and restricted to the essential element (at places where it is defined) at all but a finite number of places. The space of Schwartz functions decompose in the same way, the limiting element being the characteristic function of K_v.

Accordingly the zeta functions can be decomposed into products; take pure tensors in each space, that is $W = \prod W_v$, $W' = \prod W'_v$ and $\Phi = \prod \Phi_v$, then we have

$$Z(s, W, W'; \Phi) = \prod_v Z(s, W_v, W'_v; \Phi_v).$$

The product converges for real part of s large enough (greater than 1). In the product, all but a finite number of the local components are the local L-function of the product. In that way, we see that

$$Z(s, W, W'; \Phi) = \mathrm{L}(\pi \times \pi', s) \prod_{v \in S} \frac{Z(s, W_v, W'_v; \Phi_v)}{L(\pi_v \times \pi'_v, s)}$$

for a finite set of places S including the places at infinity. Note that a local L-factor can not be 0.

3.4 Other expression.

We come to another expression of the integral Z. Let Φ be a Schwartz function of ${\mathbf{A}_F}^r$. We define the Eisenstein series $E(g,s;\eta,\Phi)$ to be equal to $\sum_\gamma f(\gamma g,s;\eta,\Phi)$ where γ runs through a set of representatives of $P(F)\backslash G(F)$. This is an automorphic function of g with central character η^{-1}. This series converges for real part of s large enough (namely greater than 1).

Let φ and φ' be functions in the space of π and π' respectively and let $\Phi \in \mathcal{S}({\mathbf{A}_F}^r)$. We define

$$I(s,\varphi,\varphi';\Phi) = \int_{Z(\mathbf{A}_F)G(F)\backslash G(\mathbf{A}_F)} \varphi(g)\varphi'(g)E(g,s;\eta,\Phi)\,\mathrm{d}g.$$

Of course, as before, we can change the definition of the function E and define it as an intertwining operator between J_s and the space of automorphic forms by the same token: $E(f)(g) = \sum f(\gamma g)$ and we can extend I to be a function on $\mathbf{C} \times (\pi \otimes \pi' \otimes E(J_s))$.

To each element W in $\mathcal{W}_0(\pi,\psi)$ we can associate an element φ of the space of π by $\varphi(g) = \sum W(\xi g)$ with ξ running through $N_P(F)\backslash P(F)$. We do the same for W' and φ' we then have ([JS1, p550])

$$I(s,\varphi,\varphi';\Phi) = Z(s,W,W';\Phi) \tag{4}$$

for any Schwartz function Φ; we have equivalently

$$I(s,\varphi\otimes\varphi'\otimes E(f)) = Z(s,W\otimes W'\otimes f)$$

for any $f \in J_s$.

4 Finite dimension.

We recall or state some results about finite dimensional representations of $\mathrm{GL}_r(\mathbf{C})$ (or of U_r).

4.1 Notations, conventions, reminding.

We will here treat only the case of finite dimensional holomorphic representations of $G = \mathrm{GL}_r(\mathbf{C})$ but that can be as well applied to finite dimensional representations of U_r. The irreducible ones are parameterized by their highest weight with respect to Δ^+, whose coordinates in the basis Φ are r integers in decreasing order. We will identify a highest weight with its coordinates in Φ. The contragredient of a representation of highest weight $(\lambda_i)_{1\leqslant i\leqslant r}$ is the representation of highest weight $(-\lambda_{r+1-i})$. The representation Sym^a ($a \in \mathbf{N}$) is of highest weight $(a,0,...,0)$; if $a \leqslant 0$, Sym^a with be the contragredient of Sym^{-a} or equivalently the representation of highest weight $(0,...,0,a)$. The one-dimensional representation $\det^d$, $d \in \mathbf{Z}$ is of highest weight $(d,...,d)$; tensoring with $\det^d$ an irreducible representation of highest weight results in adding d to each coefficient. We remind for the irreducible smooth representation of G the analog of Pieri's rule:

Proposition 4.1 ([FH, Proposition 15.25, p225]) *The tensor product of the representation of highest weight* $(a_1, ..., a_r)$ *with* Sym^k *(*$k \in \mathbf{N}$*) decomposes as a direct sum of the representations of highest weight* $(b_1, ..., b_r)$ *with* $a_i \leqslant b_i \leqslant a_{i-1}$ *and* $\sum b_i = k + \sum a_i$*; each representation occurs with multiplicity one.*

4.2 One-dimensional representations in a tensor product.

We will be often confronted to tensor product of three representations. One of them, (usually the one associated to the Eisenstein series) will be (up to central character) a symmetric power. In these products we will have to find a trivial representation. This is the object of the following proposition.

Proposition 4.2 *Let* σ *and* τ *two finite dimensional irreducible representation of highest weight respectively* (λ_i) *and* (μ_i)*. Then* $\det^d$ *occurs in* $\tau \otimes \sigma \otimes \mathrm{Sym}^a$ *if and only if* $\max_{1 \leqslant i \leqslant r}(\lambda_i + \mu_{r+1-i}) \leqslant d \leqslant \min_{1 \leqslant i \leqslant r-1}(\lambda_i + \mu_{r-i})$ *and* $a = rd - \sum(\lambda_i + \mu_i) \geqslant 0$.

Remark 4.1 *We see that it is not always possible.*

PROOF: Let the data be as in the text of the proposition. We only have to compute for which values of a the tensor product $\sigma \otimes \mathrm{Sym}^a$ contains $\tau^\vee \otimes \det^d$. To obtain the last one from σ, one needs to add $d - \mu_{r+1-i} - \lambda_i$ to the ith coefficient. To obtain that by tensoring with Sym^a, according to proposition 4.1, one has to have

$$\lambda_i \leqslant d - \mu_{r+1-i} \ (1 \leqslant i \leqslant r) \text{ et } d - \mu_{r-i} \leqslant \lambda_i \ (1 \leqslant i \leqslant r-1),$$

which leads to the conditions of the proposition. Then a must be the sum of all what have been added and that gives $a = rd - \sum(\lambda_i + \mu_{r+1-i})$. □

Corollary 4.1 *The* $\det^d$ *in the proposition above occurs with multiplicity one.*

PROOF: Suppose $\det^d$ occurs in $\sigma \otimes \tau \otimes \mathrm{Sym}^a$. According to proposition 4.1, τ occurs exactly once in $\det^d \otimes \sigma^\vee \otimes \mathrm{Sym}^a$ so that $\det^d$ has multiplicity one in the given tensor product. □

Remark 4.2 *In the particular case* $r = 2$*, the condition is*

$$\lambda_2 + \mu_2 \leqslant d \leqslant \min(\lambda_1 + \mu_2, \lambda_2 + \mu_1).$$

As $\lambda_1 \geqslant \lambda_2$ *and* $\mu_1 \geqslant \mu_2$ *this relation is satisfied for at least one value of* d *so that there exist at least one* a *such that* $\sigma \otimes \tau \otimes \mathrm{Sym}^a$ *contains a one-dimensional representation. This is of course obvious because each representation if of the type* $\det^d \mathrm{Sym}^a$.

4.3 Restriction to $G_{r-1} \times G_1$.

We inject $G_{r-1} \times G_1$ in G_r by the "standard" application $(g, x) \mapsto \begin{pmatrix} g & 0 \\ 0 & x \end{pmatrix}$. We see, as for instance in exercise 6.12, p80 of [FH]:

Proposition 4.3 *The restriction of the irreducible representation of $G_r(\mathbf{C})$ of highest weight (λ_i) to $G_{r-1}(\mathbf{C})$ decomposes in direct sum of representations of highest weight $(\mu_i)_{1 \leqslant i \leqslant r-1}$ such that for each i, $1 \leqslant i \leqslant r-1$, $\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1}$; each representation occurs with multiplicity one.*

We look for representations which have a fixed vector by $G_{r-1} \times G_1$. Their highest weight (λ_i) has to be such that

$$\begin{array}{l} \sum_{i=1}^{n} \lambda_i = 0 \text{ (central character)} \\ \lambda_1 \geqslant 0 \geqslant \lambda_2 \geqslant 0 \geqslant \ldots \geqslant \lambda_{r-1} \geqslant 0 \geqslant \lambda_r \end{array}$$

because of last proposition. It follows that we must have $\lambda_1 = -\lambda_r = k \in \mathbf{N}$ and all other λ_i are zero.

If we look for representations of $G_r(\mathbf{C})$ with a one-dimensional sub-representation of $G_{r-1}(\mathbf{C}) \times G_1(\mathbf{C})$, we have as above:

Proposition 4.4 *Let (λ_i) be the highest weight of an irreducible representation of $G_r(\mathbf{C})$. This representation has an invariant one-dimensional subspace under the action of the group $G_{r-1}(\mathbf{C}) \times G_1(\mathbf{C})$ if and only if*

$$\begin{cases} \lambda_1 = d + k & (d, k) \in \mathbf{Z} \times \mathbf{N} \\ \lambda_i = d & 2 \leqslant i \leqslant r-1 \\ \lambda_r = d - u & u \in \mathbf{N} \end{cases}$$

Then $G_{r-1}(\mathbf{C})$ act on the invariant line by determinant to the power d and $G_1(\mathbf{C})$ by the power $d + k - u$.

5 Infinite components.

In this section, v will be an infinite place, so that $F_v \simeq \mathbf{C}$. We will usually drop the subscript v or even use $P = P(F_v)$, $G = G(F_v)$ and so on. This will not create confusion because everything is purely local in this section.

5.1 Cohomology coefficients.

We will use the fundamental result of Borel and Wallach in [BW, Theorem 3.3, p93]. We rephrased it to fit our notations and our particular case.

Theorem 5.1 (Borel–Wallach) *Let $\lambda \in \mathfrak{t}^*_{\mathbf{C}}$ be a dominant weight and F_λ a simple $G_{\mathbf{C}}$-module with highest weight λ.*

(i) If $\mathrm{H}^\bullet(\mathfrak{g}_\mathbf{C}, K_\mathbf{C}; J_s \otimes F_\lambda) \neq 0$*, then there exists* $s \in W^P$ *such that*

(1) $s(\rho+\lambda)\,|_{A_0} + H_s\,|_{A_0} = 0$*;*

(2) The infinitesimal character of $H_s\,\big|_{A_0\backslash B}$ *is equal to the one parameterized by* $-s(\rho+\lambda)$ *(restricted to* $A_0\backslash B$*).*

Such an s *is unique.*

(ii) If $s \in W^P$ *satisfies conditions (1) and (2) then, for every* $q \in \mathbf{N}$*, we have*

$$\mathrm{H}^{q+\ell(s)}(\mathfrak{g}_\mathbf{C}, K_\mathbf{C}; J_s \otimes F_\lambda) = (\mathrm{H}^\bullet({}^0\mathfrak{m}, K_P; H_\sigma \otimes E_{s(\rho+\lambda)-\rho})) \otimes {\bigwedge}^\bullet \mathfrak{a}_\mathbf{C}^*)^q$$

and all other cohomology groups are 0.

The algebra $\mathfrak{g}_\mathbf{C}$ is $\mathcal{M}_r(\mathbf{C} \otimes_\mathbf{R} \mathbf{C})$. We will fix some notations. We denote one of the elements in $\mathbf{C}$ whose square is -1 by J. Thus $\mathbf{C} \otimes_\mathbf{R} \mathbf{C} = \mathbf{C} \oplus J\mathbf{C}$ with J not part of the $\mathbf{C}$'s on the right hand side of the equality. Then $\mathfrak{g}_\mathbf{C} = \mathcal{M}_r(\mathbf{C}) \oplus J\mathcal{M}_r(\mathbf{C})$. For the diagonal torus, the weights are $(x_k - x_l) \pm i(y_k - y_l)$ where $x_k + Jy_k$ is the k-th coordinate on the diagonal (and x_k and y_k are $\mathbf{C}$-valued); the corresponding weight vectors are $\frac{1 \mp J}{2} E_{kl}$. The weights will be denoted by a pair of r-tuples of integers $((\lambda_k), (\lambdabar_k))$; the Weil group is the group $\mathfrak{S}_r \times \mathfrak{S}_r$ (each $\mathfrak{S}_r$ acting on each r-tuple) so that the dominant weights will be chosen to be the ones with decreasing values in each r-tuple.

A little bit of combinatorics gives the following result.

Proposition 5.1 *Let* η *be a character of* $\mathbf{C}^\times$ *of the form* $\eta(z) = \left(\frac{z}{\bar z}\right)^{k_\eta}$ *with* k_η *half integer and for* $s \in \mathbf{C}$*, let* H_s *the character of* $P(\mathbf{C})$ *defined by* $H_s(p) = |\det p|^s |p_{rr}|^{-rs} \eta(p_{rr})^{-1}$*. Let* V *be a finite dimensional irreducible representation of* $\mathrm{GL}(r, \mathbf{C})$ *tensor product of the representation of highest weight* (λ_j) *by the complex conjugate of the one of highest weight* $(\lambdabar_j)$*. The cohomology* $\mathrm{H}^\bullet(\mathfrak{g}_\mathbf{C}, K_\mathbf{C}; J_s \otimes V)$ *is non trivial if and only if* k_η *and* s *are integers and one of the following conditions are satisfied:*

(0) $s \geqslant 1 + \frac{|k_\eta|}{r}$,
$\lambda_1 = 1 - r + k_\eta + (r-1)s$,
$\lambdabar_1 = 1 - r - k_\eta + (r-1)s$ *et*
$\lambda_j = \lambdabar_j = 1 - s$ $(2 \leqslant j \leqslant r)$*. We then take* $\ell(s) = 2(r-1)$*.*

(1) $1 + \frac{k_\eta}{r} \leqslant s \leqslant -\frac{k_\eta}{r}$,
$\lambda_j = -s$ $(1 \leqslant j \leqslant r-1)$,
$\lambda_r = k_\eta + (r-1)s$,
$\lambdabar_1 = 1 - r - k_\eta + (r-1)s$ *et*
$\lambdabar_j = 1 - s$ $(2 \leqslant j \leqslant r)$*. We then take* $\ell(s) = r - 1$*.*

(2) $1 - \frac{k_\eta}{r} \leqslant s \leqslant \frac{k_\eta}{r}$,
$\lambda_1 = 1 - r + k_\eta + (r-1)s$,
$\lambda_j = 1 - s$ $(2 \leqslant j \leqslant r)$,
$\lambdabar_j = -s$ $(1 \leqslant j \leqslant r-1)$ *et*
$\lambdabar_r = -k_\eta + (r-1)s$*. We then take* $\ell(s) = r - 1$*.*

(3) $s \leqslant -\frac{|k_\eta|}{r}$,
$\lambda_j = \not\lambda_j = -s$ $(1 \leqslant j \leqslant r-1)$,
$\lambda_r = k_\eta + (r-1)s$ *et*
$\not\lambda_r = -k_\eta + (r-1)s$. *We then take* $\ell(s) = 0$.

The cohomology is then equal to $\mathbf{C}$ *in degree* $\ell(s)$, $\mathbf{C}^2$ *in degree* $\ell(s)+1$, $\mathbf{C}$ *in degree* $\ell(s)+2$ *and 0 everywhere else.*

Remark 5.1 *We are interested in the cases* (1) *and* (2)*; these are possible only if* $1 - \frac{|k_\eta|}{r} \leqslant \frac{|k_\eta|}{r}$, *which leads to* $|k_\eta| \geqslant \frac{r}{2}$.

The situation for the cuspidal representations is already known. The π_v part of the cuspidal representation is unitarily induced from the Borel subgroup of the character $\chi(b) = \chi(z)\prod \chi_j(b_{jj})$ with $\chi_j(z) = \left(\frac{z}{\bar{z}}\right)^{p_j + \frac{r-1}{2}}$ and $z = \prod b_{jj}$. The p_j's are supposed to be in strictly decreasing order. The finite dimensional irreducible representation τ such that $\mathrm{H}^\bullet(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}}; \pi_v \otimes \tau) \neq 0$ is the tensor product of the representation of highest weight $\lambda_\tau = (-p_{r+1-j} + j - r)$, the conjugate of the one of highest weight $(p_j + j - 1)$ and the inverse of $\chi \circ \det$.

Definition 5.1 *We define* τ, τ' *and* τ'' *be the finite dimensional representations in which* π, π' *and* J_s *respectively have cohomology (provided that this is possible for* J_s*). We let* k_η *be* $\sum(p_j + p'_j) + r(r-1)$.

5.2 Minimal K-types.

The representation π_v (or π'_v) is unitarizable thus the K-type supporting cohomology in degree $\frac{r(r-1)}{2}$ is the minimal K-type. We want to compute it in terms of the parameters of the character of B from which the representation is induced. The representations π_v and τ are unitary representations so that the differentiation is trivial and the cohomology is equal to the complex ([BW, Proposition 3.1, p52]).

Proposition 5.2 *The minimal* K*-type of* π_v *is of highest weight* $(2p_j + r - 1)$.

Proof: Let $\mathfrak{p}$ be the Lie algebra of matrices $M \in \mathfrak{g}_{\mathbf{C}}$ such that ${}^t\overline{M} = M$ (the conjugacy is taken with respect to the complex structure of $\mathfrak{g}$, not the one of $\mathfrak{g}_{\mathbf{C}}$). We let $\tilde{\mathfrak{p}}$ be the sub-algebra of $\mathfrak{p}$ with zero trace. The complex computing the cohomology of π_v is:

$$C^i(\tilde{\mathfrak{g}}_{\mathbf{C}}, \tilde{K}_{\mathbf{C}}; \pi_v \otimes \tau) = (\bigwedge^i \tilde{\mathfrak{p}}^* \otimes \pi_v \otimes \tau)^{K_{\mathbf{C}}}.$$

The first non zero group is the one for $i = \frac{r(r-1)}{2}$ in which it is a one-dimensional $\mathbf{C}$-vector space.

As a $K_{\mathbf{C}}$ representation, $\tilde{\mathfrak{p}}$ is of highest weight $(1, 0, ..., 0, -1)$ (and thus is auto-dual). If V is the standard representation of $\mathfrak{k}_{\mathbf{C}}$, we note that $V \otimes V^{\vee} \simeq \mathbf{C} \oplus \tilde{\mathfrak{p}}$. We then have two different expressions for exterior powers of $V \otimes V^{\vee}$: the easiest one is

$$\bigwedge^{i}(V \otimes V^{\vee}) = \bigwedge^{i-1} \tilde{\mathfrak{p}} \oplus \bigwedge^{i} \tilde{\mathfrak{p}}$$

and the other one given by [FH, p80]:

$$\bigwedge^{i}(V \otimes V^{\vee}) = \bigoplus_{\lambda} \mathbf{S}_{\lambda}(V) \otimes \mathbf{S}_{\lambda'}(V^{\vee})$$

with λ running through partitions of i with at most r lines and at most r columns, λ' is the conjugate partition to λ and $\mathbf{S}_{\lambda}$ is the Schur functor associated to λ. In our case, $\mathbf{S}_{\lambda}(V)$ is the representation of highest weight λ and $\mathbf{S}_{\lambda'}(V^{\vee})$ is the contragredient of the representation of highest weight λ'.

We are interested in the case $i = \frac{r(r-1)}{2}$. We take for λ the partition $(r-j)_{1 \leqslant j \leqslant r}$. Then $\lambda' = \lambda$ and $\mathbf{S}_{\lambda}(V) \otimes \mathbf{S}_{\lambda'}(V^{\vee})$ contains the representation τ' of highest weight $\lambda_{\tau'} = (r+1-2j)$ because it is the sum of the highest weights of the given representations. As a representation of $K_{\mathbf{C}}$, τ is the tensor square of the representation of highest weight λ_{τ}. We thus see that $\tau \otimes \tau'$ contains the irreducible representation of highest weight $(-2p_{r+1-j} - r + 1)$ as it is the sum of the highest weight of the representations of $K_{\mathbf{C}}$. We see now that the representation of highest weight $(2p_j + r - 1)$ occurs is π_v, because of Frobenius reciprocity law and because ρ is trivial on $B \cap K$. Thus there is a non trivial vector in $(\bigwedge^{i}(V \otimes V^{\vee}) \otimes \pi_v \otimes \tau)^{K_{\mathbf{C}}}$. It can come either from $C^{i-1}(\tilde{\mathfrak{g}}_{\mathbf{C}}, \tilde{K}_{\mathbf{C}}; \pi_v \otimes \tau)$ or $C^{i}(\tilde{\mathfrak{g}}_{\mathbf{C}}, \tilde{K}_{\mathbf{C}}; \pi_v \otimes \tau)$. Since the H^{i-1} is zero and is equal to C^{i-1}, this means that this vector is part of $C^{i} = \mathrm{H}^{i}$ and the proposition is proved. $\square$

5.3 K-type supporting Eisenstein cohomology.

We begin by the list of the K-types contained in an induced representation from P to G.

Proposition 5.3 *Let σ be the (unitary) character of $\mathbf{C}$ such that $\sigma(z) = \left(\frac{z}{\bar{z}}\right)^{-k}$, $k \in \mathbf{Z}$ and let H_s be the character of P defined by $H_s(p) = |\det p|_{\mathbf{C}}^{s} |p_{rr}|^{-rs} \sigma(p_{rr})$. All the K-types contained in $J_s = \mathrm{ind}_P^G H_s$ are all the representations of K with highest weight $(\lambda - 2k, 0, ..., 0, -\lambda)$ for $\lambda \geqslant \max(0, 2k)$; each of these K-type has multiplicity one. A vector of highest weight for the K-type $(\lambda - 2k, 0, ..., 0, -\lambda)$ is the function from G_v to $\mathbf{C}$ given by*

$$f_{\lambda}(g) = \frac{|\det g|_v^s {g_{r1}}^{\lambda-2k} \overline{g_{rr}}^{\lambda}}{(\sum_i |g_{ri}|_v)^{rs-k+\lambda}}.$$

Proof: As K representations, the values of s are irrelevant, so that we take s to be equal to 0 and put $H = H_0$. As a representation of K, $\mathrm{ind}_P^G H$ is naturally isomorphic to $\mathrm{ind}_{P \cap K}^K (H|_{P \cap K})$. We then use Frobenius reciprocity law which states that for any K representation V

$$\mathrm{Hom}_K(\mathrm{ind}_{P \cap K}^K (H|_{P \cap K}), V) \simeq \mathrm{Hom}_{P \cap K}(H|_{P \cap K}, V|_{P \cap K}).$$

The group $P\cap K$ is isomorphic to $U_{r-1}\times U_1$ and through that isomorphism the restriction of H to $P\cap K$ is the character $((0),-2k)$ where (0) is the null weight of U_{r-1}. If V is the representation of highest weight $(\lambda_i)_{1\leqslant i\leqslant r}$ its restriction to $P\cap K$ decomposes in direct sum of representations of highest weight $((\mu_i)_{1\leqslant i\leqslant r-1},\mu_r)$ with:

- $\forall i, 1\leqslant i\leqslant r-1,\ \lambda_i\geqslant\mu_i\geqslant\lambda_{i+1}$.
- $\sum_{i=1}^r \lambda_i=\sum_{i=1}^r \mu_i$ (equality of central character).

This is the content of proposition 4.3. In that case all μ_i's are zero except possibly μ_1 and μ_r which gives the first result of the proposition.

We suppose $k\geqslant 0$, the situation is exactly the same when $k<0$. Under the action of K, the vector f_λ generates a vector space contained in the space W of functions

$$g\longmapsto \frac{P((g_{rj})_j)\overline{Q}((g_{rj})_j)}{(\sum|g_{ri}|_{\mathbf{C}})^{\lambda-k}}$$

with $(P,Q)\in \mathrm{Sym}^{\lambda-2k}\times\mathrm{Sym}^{\lambda}$. The space W is finite dimensional and f_λ is a highest weight vector of W. $\square$

Proposition 5.4 *The K-type of J_s supporting cohomology in degree $r-1$ is the minimal K-type of J_s, that is the one of highest weight* $(\max(0,-2k),0,...,0,\min(0,-2k))$.

PROOF: The induced representation J_s is not unitarizable which makes things slightly harder. We will use Zuckerman-Jantzen translation functor. For more information see [Vog] and of course [Zuc]. We will use the translation functor which brings J_s to an induced representation with coefficientless cohomology, and we will get in the process an unitarizable representation.

We will proceed the other way round, it's actually easier. We suppose that the characters χ and χ' used in the parameterization of π_v and π'_v are trivial. As a representation of $\mathfrak{g}_{\mathbf{C}}$, the representation τ'' is irreducible and has highest weight

$$\big((r-1)(s-1)+k_\eta,1-s,...,1-s),(-s,...,-s,(r-1)s-k_\eta)\big)$$

so that there is up to a scalar a unique vector of weight

$$\big((1-s,...,1-s,(r-1)(s-1)+k_\eta),(-s,...,-s,(r-1)s-k_\eta)\big)$$

because this weight is extremal. Call one of these vectors v_s. Under the action of $P(F_v)$ the line $\mathbf{C}v_s$ is a one-dimensional representation. We call H' the character of $P(F_v)$ defined by $H'(p)={p_{rr}}^r$ and J' the induced representation $J'=\mathrm{ind}_{P(F_v)}^{G(F_v)}H'$. As characters of $P(F_v)$, $H_s\otimes\mathbf{C}v_s$ and H' are isomorphic. Moreover, $\mathrm{ind}_P^G(H_s\otimes\mathbf{C}v_s)\subset\mathrm{ind}_P^G(H_s\otimes\tau'')$ and as stated by [Vog, Lemma 4.5.2(a), p206], $\mathrm{ind}_P^G(H_s\otimes\tau'')\simeq\mathrm{ind}_P^G(H_s)\otimes\tau''=J_s\otimes\tau''$. This last isomorphism is reciprocal to the one defined by:

$$\begin{aligned} T'_s: J_s\otimes\tau'' &\longrightarrow \mathrm{ind}_P^G(H_s\otimes\tau'')\\ f\otimes v &\longmapsto (g\mapsto f(g)\otimes(gv))\end{aligned}$$

By composing these various morphisms, we obtain an injective morphism from J' to $J_s \otimes \tau''$:

$$J' = \operatorname{ind}_P^G H' \simeq \operatorname{ind}_P^G (H_s \otimes \mathbf{C} v_s) \longrightarrow \operatorname{ind}_P^G (H_s \otimes \tau'') \simeq J_s \otimes \tau''.$$

Using theorem 5.1, we see that J' is indeed a cohomological representation with trivial coefficients and that the only non trivial cohomology groups are those of degree $r-1$, r and $r+1$ (of dimension respectively 1, 2 and 1). Let's compare the cohomologies of J_s and J'. The complexes are respectively

$$\mathrm{C}^{\bullet}(J_s) = \left(\bigwedge^{\bullet} (\mathfrak{g}/\mathfrak{k})^* \otimes J_s \otimes \tau'' \right)^K$$

and

$$\mathrm{C}^{\bullet}(J') = \left(\bigwedge^{\bullet} (\mathfrak{g}/\mathfrak{k})^* \otimes J' \right)^K .$$

The action of T_s induces and injection $T_s^{\bullet}$ from $\mathrm{C}^{\bullet}(J')$ to $\mathrm{C}^{\bullet}(J_s)$. But now J' is unitarizable because it is unitarily induced from the character $H'_{\circ} = |\delta_P|^{-1/2} H'$ which is given by $H'_{\circ}(p) = \left(\frac{\det p}{\overline{\det p}}\right)^{\frac{1}{2}} \left(\frac{p_{rr}}{\overline{p_{rr}}}\right)^{-\frac{r}{2}}$ so that it is unitary. That way, the differentiation of the complex $\mathrm{C}^{\bullet}(J')$ is trivial and the cohomology is isomorphic to the complex. Thus $\mathrm{C}^{r-2}(J')$ is zero and $\mathrm{C}^{r-1}(J')$ is one-dimensional. As in proposition 5.3 the K-types of J' are of the form $(1+\lambda, 1, ..., 1, 1-r-\lambda)$ for $\lambda \in \mathbf{N}$ and each K-type has multiplicity one. As J' is unitary, the cohomology is supported in the $(r-1)$th group by the minimal K-type, that is $(1, ..., 1, 1-r)$.

The part of $J_s \otimes \tau''$ which has same infinitesimal character as J' is exactly the image of T_s because of [Vog, Proposition 7.4.1, p488]. Now the differentiation operator of the complex $\mathrm{C}^{\bullet}(J_s)$ is defined by the action of the Lie algebra $\mathfrak{g}$ so that it commutes with the action of center of the universal enveloping algebra. Thus if a given K type is in the image of the differentiation, its preimage has the same infinitesimal character. The image of $\mathrm{C}^{r-1}(J')$ in $\mathrm{C}^{r-1}(J_s)$ is a one-dimensional vector space L. Its image by the differentiation d has same infinitesimal character hence it is the image of T_s^r. So now $(T_s^r)^{-1}(\mathrm{d}L)$ is in the image of the differentiation of $\mathrm{C}^{r-1}(J')$ which is trivial. As $T_s^{\bullet}$ is injective, that means that $\mathrm{d}L = 0$. Now this L cannot be in the image of d because its preimage would be in the image of $T_s^{r-1} \circ \mathrm{d}$ which is zero. Thus $T_s^{r-1}(\mathrm{C}^{r-1}(J')) = \mathrm{H}^{r-1}(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}}; J_s \otimes \tau'')$.

On the level of K-types, T_s is simply translation by the character of $\mathbf{C}v_s$ because it is simply tensoring with this vector. Thus the K-type of J_s supporting cohomology is the minimal K-type because it is the image of the minimal K-type of J' by T_s.

If then χ and χ' are non trivial, we tensorize J_s by $\chi\chi' \circ \det$ and τ'' by the inverse, but the result remains the same. □

6 Cohomology.

6.1 Commutative diagram.

We fix the following notations. The sub-algebra $\mathfrak{p} \subset \mathfrak{g}_{\mathbf{C}}$ is the sub-algebra of $\mathrm{Lie}(B)_{\mathbf{C}}$ such that diagonal elements are "real". Whenever there is a tilde over a letter, this means that the center is mod-ed out. We denote $p_c = \frac{r(r-1)}{2}d$ and $p_E = (r-1)d$ (recall that $[F:\mathbf{Q}] = 2d$). We use the natural basis of weight vectors on $\tilde{\mathfrak{p}}$. We define a morphism from $\bigwedge^{p_c}\tilde{\mathfrak{p}}^* \otimes \bigwedge^{p_c}\tilde{\mathfrak{p}}^* \otimes \bigwedge^{p_E}\tilde{\mathfrak{p}}^*$ to $\mathbf{C}$ by using the natural contraction to $\left(\bigwedge^{r^2-1}\tilde{\mathfrak{p}}^*\right)^{\otimes d}$ and dividing by the wedge product of the elements of the basis of $\tilde{\mathfrak{p}}^*$ we chose.

We consider the following diagram:

$$\begin{tikzcd}
\bigwedge^{p_c}\tilde{\mathfrak{p}}^* \otimes \bigwedge^{p_c}\tilde{\mathfrak{p}}^* \otimes \bigwedge^{p_E}\tilde{\mathfrak{p}}^* \bigotimes \pi \otimes \pi' \otimes E(J_s) \bigotimes \tau \otimes \tau' \otimes \tau'' \arrow[r, "m\otimes Z(s)\otimes \mathrm{id}"] & \mathbf{C} \bigotimes \mathbf{C} \bigotimes \tau \otimes \tau' \otimes \tau'' \arrow[d, "\times\otimes\mathrm{id}"] \\
\mathrm{H}^{p_c}(\widetilde{\mathfrak{g}_{\mathbf{C}}}, \widetilde{K_{\mathbf{C}}}; \pi\otimes\tau) \otimes \mathrm{H}^{p_c}(\widetilde{\mathfrak{g}_{\mathbf{C}}}, \widetilde{K_{\mathbf{C}}}; \pi'\otimes\tau') \otimes \mathrm{H}^{p_E}(\widetilde{\mathfrak{g}_{\mathbf{C}}}, \widetilde{K_{\mathbf{C}}}; E(J_s)\otimes\tau'') \arrow[u] \arrow[r, "\vee"'] & \tau\otimes\tau'\otimes\tau''
\end{tikzcd} \tag{5}$$

with the following arrows:

- The ascending arrow is the inclusion of the K_∞-invariants in the total space of vectors. Note that differentiations are trivial so that the cohomologies are equal to the complexes. Note also that the tensor elements have been reorganized.
- The descending arrow is only the product of the scalars with the vector.
- The bottom arrow is the cup product of the differential forms on the manifold.

Proposition 6.1 *Up to a multiplication of m by a constant, the diagram is commutative.*

PROOF: this is clear, this is only the contents of the equality of the Z and the I function as seen in (4). What remains to prove is the following.

> **Lemma 6.1** *The operator m is non trivial on the product of K_∞-types of the $\bigwedge^{p_\nabla}\tilde{\mathfrak{p}}^*$ occurring in the cohomologies.*
>
> PROOF: In the case of the infinite part of the cuspidal representations, the K_∞-type of $\bigwedge^{p_c}\tilde{\mathfrak{p}}^*$ occurring is the one with highest weight $k_c = (r+1-2j)$ while for the Eisenstein part it is the one of highest weight $k_E = (r-1, -1, ..., -1)$. The representation of highest weight k_c is its own contragredient. When it is tensored with the one of highest weight k_E (which is $\mathrm{Sym}^r \otimes \det^{-1}$) we see by using Pieri's rule (Proposition 4.1) that one can obtain only once the representation of highest weight k_c (graphically, one has to add each square of the Young diagram of Sym^r to one line of the Young diagram for k_c). This means that the trivial representations occurs with multiplicity one in the tensor product of the 3 K_∞-types. Now, if m is non vanishing on one vector of the tensor product it will be the intertwining operator. between the tensor product and the trivial representation.
>
> In the k_c representations, the vectors $v_h = \bigwedge_{1\leqslant j<k\leqslant r} e_{jk}$ and $v_l = \bigwedge_{1\leqslant k<j\leqslant r} e_{jk}$ are respectively highest and lowest weight vectors. We define the sequence of vectors $\mathbf{v}_k$

by

$$\begin{cases} \mathbf{v}_0 &= \bigwedge_{2\leqslant l\leqslant r} e_{1l} \\ \mathbf{v}_{k+1} &= (E_{1,r-k} - E_{r-k,1}) \cdot \mathbf{v}_k \end{cases}$$

We can then check inductively that $\mathbf{v}_k$ has a non zero (integer) coefficient against $\bigwedge_{2\leqslant l\leqslant r-k} e_{1l} \wedge \bigwedge_{r-k\leqslant l<r} (e_{ll} - e_{rr})$ and all other components are of degree smaller than k in the $e_{ll} - e_{rr}$. This proves that there is a vector v_m with a non zero coefficient in $\bigwedge_{1\leqslant l<r} (e_{ll} - e_{rr})$ in the representation of highest weight k_E and $m(v_h \otimes v_b \otimes v_m)$ is a non zero integer. □

We even have proved that the constant is rational. □

6.2 Tensor product of minimal K_∞-types.

We have to ponder when the diagram (5) gives a non zero multiple of the Z function in the bottom-right corner. The first point is that there must be a trivial representation in the tensor product $\tau \otimes \tau' \otimes \tau''$. This is easily accomplished using proposition 4.2:

Proposition 6.2 *The tensor product $\tau \otimes \tau' \otimes \tau''$ contains the trivial representation if and only if s is in the following bounds:*

- *if $k_\eta \geqslant \frac{r}{2}$*

$$\begin{cases} r + \max\limits_{1\leqslant j\leqslant r-1}(p_{j+1} + p'_{r+1-j}) \leqslant s \leqslant r - 1 + \min\limits_{1\leqslant i\leqslant r}(p_j + p'_{r+1-j}) \\ 2 - r - \min\limits_{1\leqslant i\leqslant r}(p_j + p'_{r+1-j}) \leqslant s \leqslant 1 - r - \max\limits_{1\leqslant i\leqslant r-1}(p_{j+1} + p'_{r+1-j}) \end{cases}$$

- *if $k_\eta \leqslant -\frac{r}{2}$*

$$\begin{cases} -r + \max\limits_{1\leqslant j\leqslant r}(p_j + p'_{r+1-j}) \leqslant s \leqslant -r - 1 + \min\limits_{1\leqslant j\leqslant r-1}(p_j + p'_{r-j}) \\ 2 + r - \min\limits_{1\leqslant j\leqslant r-1}(p_j + p'_{r-j}) \leqslant s \leqslant 1 + r - \max\limits_{1\leqslant j\leqslant r}(p_j + p'_{r+1-j}) \end{cases}$$

In that case, the trivial representation has multiplicity one in the tensor product.

This is obviously not always possible. In the case $k_\eta \geqslant \frac{r}{2}$, the parameters p_j and p'_j must satisfy the conditions $\max(p_{j+1}+p'_{r+1-j}) \leqslant -1+\min(p_j+p'_{r+1-j})$, $\max(p_{j+1}+p'_{r+1-j})+r \leqslant 1-r-\max(p_{j+1}+p'_{r+1-j})$ and $-\min(p_j+p'_{r+1-j})+2-r \leqslant -1+r-\min(p_j+p'_{r+1-j})$. They lead respectively to $p_j \geqslant 1+p_{j+1}$, $\max(p_{j+1}+p'_{r+1-j})+r \leqslant \frac{1}{2}$ and $\min(p_j+p'_{r+1-j})+r-1 \geqslant \frac{1}{2}$. Since $k_\eta = \sum(p_j + p'_j) + r(r-1)$ we see that the last inequality implies $k_\eta \geqslant r$. Then we see that $k_\eta \geqslant r - 1 + \min(p_j + p'_{r+1-j})$ so that the s verifying the proposition 6.2 falls in case *(2)* of 5.1.

We define the *computable strip* of $\pi \times \pi'$.

Definition 6.1 *The set of values for s defined by proposition 6.2 is called the* computable strip *of the pair of representations $\pi \times \pi'$.*

The last point is that the Z integral must be non-vanishing on the minimal K_∞-types of the representations.

Proposition 6.3 *Whenever s is in the computable strip of $\pi \times \pi'$, the tensor product of the K_∞-types of π_∞, π'_∞ and $E(J_s)_\infty$ contains the trivial representation with multiplicity one.*

PROOF: this is only proposition 4.2. $\square$
Unfortunately we did not manage to prove that Z is an intertwining operator with the trivial representation so that we have to suppose it:

Conjecture 6.1 *The integral Z is non trivial on the tensor product of the minimal K_∞-types of π, π' and $E(J_s)$.*

7 Rationality.

We want to prove that the cohomology spaces we are interested in are rational subspaces of the de Rham cohomology. In the case of the cuspidal cohomology, this is a result of L. Clozel ([Clo, Théorème 3.19, p129]).

7.1 Eisenstein series.

This section recalls only what is written in [Har1, p102ff].

Let φ be a character the Borel subgroup $B(\mathbf{A}_F)$ of $G(\mathbf{A}_F)$. We will denote I_φ the unitarily induced representation of φ from B to G:

$$I_\varphi = \mathrm{Ind}_{B(\mathbf{A}_F)}^{G(\mathbf{A}_F)} \varphi.$$

If α is a root, we define N_α to be the one parameter subgroup corresponding to α. Explicitly, if $\alpha = e_j - e_k$, $j \neq k$, $N_\alpha(A) = I_r + AE_{jk}$ where A is any $\mathcal{O}_F$-algebra. If w is an element of W, we define

$$N^{(w)} = \prod_{\substack{a>0 \\ w\alpha<0}} N_\alpha.$$

Definition 7.1 *Let $T(w, \varphi)$ be the non-normalized intertwining operator from I_φ to $I_{w^{-1}\varphi}$ defined by:*

$$\forall f \in I_\varphi, \forall g \in G(\mathbf{A}_F), T(w,\varphi)(f)(g) = \int_{N(\mathbf{A}_F)} f(wng)\,\mathrm{d}n.$$

The operators $T(w,\varphi)$ are multiplicative in the following sense: for all w, w' in W, $T(ww',\varphi) = T(w', w^{-1}\varphi) \circ T(w,\varphi)$. This operator is meromorphic in φ but not holomorphic: it may have (and has) poles. We decompose $T(w,\varphi)$ into the product of a scalar function $c(w,\varphi)$ and a holomorphic operator $T^{\mathrm{loc}}(w,\varphi)$. To define the scalar function (and thus determine the operator), let us define for a root $\alpha = e_j - e_k$, χ_α to be the group morphism:

$$\begin{array}{rl} \chi_\alpha : \mathbf{A}_F{}^\times & \to B(\mathbf{A}_F) \\ t & \mapsto I_r + (t-1)E_{jj} + (t-1)E_{kk} \end{array}$$

If w is in W, and k is the number of positive roots that become negative under the action of w^{-1} let

$$c(w,\varphi) = |d_F|^{-\frac{k}{2}} \prod_{\substack{a>0 \\ w^{-1}\alpha<0}} \frac{\mathrm{L}(\varphi\circ\chi_\alpha)}{\mathrm{L}(\varphi\circ\chi_\alpha\cdot|\,|)}$$

The corresponding operator $T^{\mathrm{loc}}(w,\varphi)$ is holomorphic and as a consequence of what is written on the bottom of page 120 of [Har1], it is non zero and even surjective on Ind_P^G.

With these definitions, we can express the constant term of the Eisenstein series. If $f \in I_\varphi$ and $\mathrm{Eis}(g,f) = \sum_{\gamma\in B\backslash G(F)} f(\gamma g)$, we have

$$\int_{N(F)\backslash N(\mathbf{A}_F)} \mathrm{Eis}(ng,f)\,\mathrm{d}n = \sum_{w\in W} T(w,\varphi)(f)(g).$$

7.2 Rationality of cohomology.

Proposition 7.1 *Let V be a finite dimensional representation of $G(F_\infty)$. The space $\mathrm{H}^{(r-1)d}(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}}; E(J_s)\otimes V)$ is a rational subspace of $\mathrm{H}^{(r-1)d}(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}}; \mathcal{A}\otimes V)$.*

Remark 7.1 *The cohomology space is zero most of the time, except when $V = \tau''$ is the right coefficient space. We suppose it that way.*

PROOF: We will use the general methods of Günter Harder as in [Har1] and [Har2] and in particular [Har1, 4.3, p85]. The principle is the following: let r_B be the restriction to the Borel stratum of the Borel-Serre compactification; if V is a representation such that $\mathrm{Hom}(V_f, \ker r_B) = 0$ then for any intertwining operator $T : V \to \mathcal{A}$, $T(V)$ maps to a rational subspace of the cohomology if $r_B \circ T^\bullet$ is non trivial ($T^\bullet$ is the mapping induced on cohomology by T).

Here we take $V = J_s$ and $T = E$. We already know that $\mathrm{Hom}(J_{s,f}, \ker r_B) = 0$ ([Har2, 2.2.4, p125]). All what remains to prove is that $r_B \circ E^\bullet$ is non zero. Harder has already proved that $r_B \circ \mathrm{Eis}^\bullet_{\max}$ is non zero, so that it is enough to prove that $\mathrm{Eis}_{\max}$ is isomorphic to $E(J_s)$. Now this is clear because we just have to compare the constant terms; they are both linear combinations of the operators T^{loc}, they coincide (up to a factor) and these T^{loc} are onto J_s. $\square$

7.3 Rational vectors.

7.3.1 Essential vector

Let v be a finite place of F. Suppose the maximal ideal on which ψ_v is trivial is $\mathcal{O}_v$. In theorem (4.1), page 208 of [JPSS1], it is shown that there exists an unique function W° called essential vector of the representation such that

- $\forall g \in G_v$, $\forall k \in G_{r-1}(\mathcal{O}_v)$, $W^\circ\left(g\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right) = W^\circ(g)$
- a certain integral is covariant.

If the representation is non ramified, this is the only vector fixed by K_v such that its value at I_r is 1.

Now let ψ_1 be a character of $F_v^\times$, and let a_1 be an element of F_v such that for all $x \in F_v$, $\psi_1(x) = \psi_v(a_1 x)$. We define an isomorphism between $\mathcal{W}_0(\pi_v, \psi_v)$ and $\mathcal{W}_0(\pi_v, \psi_1)$ sending a vector W of the first space to the function

$$g \longmapsto W(\operatorname*{diag}_{1\leqslant j\leqslant r}(a_1^{r-j})g).$$

This isomorphism sends the essential vector to the essential vector if it is already defined. If not, we define the essential vector W° of $\mathcal{W}_0(\pi_v, \psi_1)$ by requiring that it is mapped to an essential vector by this isomorphism.

7.3.2 Cuspidal representations

Let's see first the cuspidal representation. We decompose it as $\pi = \left(\bigoplus_{v\in\Sigma_f} \pi_v\right) \otimes \pi_f$. According to [Clo, Théorème 3.19, p129], the cohomology $\mathrm{H}^\bullet(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}}; \pi_\infty \otimes \tau)$ is defined over some finite extension $F(\pi)$ of F. We take E to be equal to $F(\pi, \pi') = F(\pi)F(\pi')$.

As stated in [JPSS1, Theorem (5.1), p211], there exist an ideal $n(\pi)$ of $\mathcal{O}_F$ such that the compact open subgroup of $G(\mathbf{A}_{F,f})$ defined by

$$K_f(\pi) = \left\{ A \in G\left(\prod_{v\in\Sigma_f} \mathcal{O}_v\right) \Big/ A \equiv \begin{pmatrix} k_1 & * \\ 0 & 1 \end{pmatrix} \pmod{n(\pi)}, k_1 \in G_{r-1}\left(\prod \mathcal{O}_v\right) \right\}$$

has a one-dimensional space of fixed vector in π_f. The essential vector W_v°, [JPSS1, definition (4.4), p211], of $\mathcal{W}_0(\pi_v, \psi_v)$ has the property that $W_f^\circ = \bigotimes W_v^\circ$ is in $\pi_f^{K_f(\pi)}$. We denote $K(\pi) = Z_G(F_\infty)K_\infty K_f(\pi)$. The space

$$\mathrm{H}^{\frac{r(r-1)}{2}d}(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}}; \pi \otimes \tau)^{K_f(\pi)} = \left(\bigwedge^{\frac{r(r-1)}{2}d} (\mathfrak{g}/\mathfrak{k})^* \otimes \pi \otimes \tau\right)^{K(\pi)}$$

is thus one-dimensional.

On each irreducible finite dimensional representation of G_v, there is a natural $\mathbf{Q}$-rational structure associated to each highest weight vector. It comes from the natural $\mathbf{Q}$-rational structure of the exterior powers of the standard representation and tensors thereof. By picking a highest weight vector in any finite-dimensional G_v, we fix an embedding into a tensor product of exterior powers of G_v, giving it a $\mathbf{Q}$-rational structure. This of course applies as well to (irreducible finite-dimensional) representations of K_v.

We choose a highest weight vector in τ and give to τ the corresponding $\mathbf{Q}$-rational structure. On the minimal K_v type of $\mathcal{W}_0(\pi_v, \psi_v)$ we fix a $\mathbf{Q}$-rational structure by picking a non-zero highest weight vector. For $\mathfrak{p}^*$ we take the usual basis as a $\mathbf{Q}$-rational basis. We have now fixed a $\mathbf{Q}$-rational structure on the abovementioned one-dimensional space. But it still has the $F(\pi)$-rational structure coming from cohomology. We denote $C(\pi)$ the quotient of the two rational structures (that is: a rational vector coming from the cohomology, is $C(\pi)$ times a rational vector coming from the Whittaker models). This number $C(\pi)$ is well defined up to an element of $F(\pi)$. This number does not depend on the rational structure given to the coefficient system because it is a ratio.

7.3.3 Eisenstein series representations

For the Eisenstein series, the idea is the same. On the infinite part, the only thing to do is to choose the "good" multiple. We need to choose a finite part. We decompose the ideal $n(\pi)$ as $n(\pi) = \prod_{v\in\Sigma_f} n_v(\pi)$ with $n_v(\pi)$ a power of the prime ideal corresponding to v. With that, we take

$$K_v(\pi) = \left\{ A \in G(\mathcal{O}_v) \Big/ A \equiv \begin{pmatrix} k_1 & * \\ 0 & 1 \end{pmatrix} \pmod{n_v(\pi)}, k_1 \in G_{r-1}(\mathcal{O}_v) \right\}$$

for each finite place v and we do the same for π'. We choose for Φ_v the characteristic function of $\epsilon K_v(\pi, \pi')$ with $K_v(\pi, \pi') = K_v(\pi) \cap K_v(\pi')$. Gluing together all these local components, we obtain a Schwartz function for ${\mathbf{A}_F}^r$ and the associated Eisenstein series. We use it, as for the cuspidal representations as a rational vector and we define $C_s(\eta)$ to be the corresponding period. We can compute this $C_s(\eta)$. Let Δ[1] be the positive real number such that $\frac{\mathrm{d}x}{\Delta}$ is an F-rational differential form on $\mathbf{C}$. Let's start from $\omega = \sum_{i=1}^{\dim \tau''} E(f_i) v_i \,\mathrm{d}g_i$. The v_i's are the elements of the rational basis of τ'' we chose. We take v_1 to be a rational vector of highest weight,

$$f_1(g) = \frac{|\det g|_{\mathbf{A}_F}^s |g_{rr}|_{\mathbf{A}_F}^{k_\eta} \eta(g_{rr})^{-1}}{\left(\sum_{j=1}^r |g_{jr}|_{\mathbf{A}_F}\right)^{k_\eta + rs}}$$

and $\mathrm{d}g_1 = \mathrm{d}g_{1,r} \wedge \ldots \wedge \mathrm{d}g_{r-1,r}$. To evaluate the rationality of this differential form, we first restrict it to the Borel stratum of the Borel-Serre compactification (operation rational

[1] The definition is a bit long and can be found p493 of [Hid]

since $E(J_s)$ is embedded in $\mathrm{Eis}_{\max}$), then take the cup-product with the rational form $\frac{1}{\Delta}^{r-1}\,\mathrm{d}\overline{g_{1,r}}\wedge\ldots\wedge\mathrm{d}\overline{g_{r-1,r}}$ and integrate over the domain whose representatives are the matrices

$$\begin{pmatrix} 1 & & & \\ \mathbf{0} & & I_{r-2} & \\ x_1 & \ldots & x_{r-1} & 1 \end{pmatrix}$$

The result is the constant term of $E(f_1)$ along the corresponding parabolic (times $\frac{v_1}{\Delta}^{r-1}$) evaluated at I_r; this is just $f_1(I_r) = 1$. ¿From that we see that $\Delta^{r-1}.\omega$ is a rational form, so that $C_s(\eta) = \Delta^{r-1}$. We will need an additional notation: we call $D(s,\eta)$ the E-vector space generated by the rational vectors of $E(J_s)$.

8 Special value.

Theorem 8.1 *Let π and π' be two irreducible automorphic cuspidal cohomological representations of G_r. Let η be the product of their central characters. Let $s_0 > 1$ be an integer greater than 1 in the computable strip of $\pi\times\pi'$. Let $\sum\varphi\otimes\varphi'\otimes\Phi$ be an element of the trivial K_∞-type of $\mathcal{W}_0(\pi_\infty,\psi_\infty)\otimes\mathcal{W}_0(\pi'_\infty,\overline{\psi_\infty})\otimes\mathcal{S}(\mathbf{A}_F{}^r)$ such that each φ, φ' and $f(s,g;\eta,\Phi)$ are rational elements of each minimal K_∞-type. Then:*

$$\mathrm{L}_f(s_0,\pi\times\pi') = \frac{\Delta^{r-1}C(\pi)C(\pi')}{\sum Z_\infty(s_0,\varphi\otimes\varphi'\otimes\Phi)}c$$

where $c\in F(\pi,\pi')$.

Proof: it is now straightforward. We take two irreducible cuspidal cohomological representations π and π' and $s_0 > 1$ an integer in the computable strip of $\pi\times\pi'$. We begin with pure tensors. Let φ and φ' be non zero rational elements of respectively $\mathcal{W}_0(\pi_\infty,\psi_\infty)$ and $\mathcal{W}_0(\pi'_\infty,\overline{\psi_\infty})$. We take then $W = \varphi\otimes\bigotimes_{v\in\Sigma_f} W^\circ(\pi_v,\psi_v)$ and W' accordingly (changing ψ to $\overline{\psi}$). The isomorphism between π and $\mathcal{W}(\pi,\psi)$ is canonical because it is fixed by Fourier decomposition thus to W and W' correspond $\varphi_t\in\pi$ and $\varphi'_t\in\pi'$.

Now take Φ to be a non zero element of $D(s_0,\eta)$. We need to find local components for finite places. Let $v\in\Sigma_f$. We will show that for a good choice of $\Phi_v\in\mathcal{S}(F_v^r)$, $Z_v = Z(s_0,W_v^\circ,W_v'^\circ;\Phi_v)$ is a non zero element of $F(\pi,\pi')(q_v^{-s_0})$ (where $q_v = \mathrm{N}\,v$ is the number of element of the residue field of F_v). Suppose first that the ideal on which ψ_v is trivial is $\mathcal{O}_v$. We then have $W_v^\circ(I_r) = W_v'^\circ(I_r) = 1$. We put $K_v(\pi,\pi') = K_v(\pi)\cap K_v(\pi')$ and take Φ_v to be the characteristic function of $\epsilon K_v(\pi,\pi')$ in F_v^r. As $K_v(\pi)$ and $K_v(\pi')$ are open compact subgroup of $G_r(F_v)$, $K_v(\pi,\pi')$ is an open compact subgroup of $G_r(F_v)$ and $\Phi_v\in\mathcal{S}(F_v^r)$. If G_v is decomposed as $G_v = P_vK_v(\pi,\pi')$.

$$Z_v = \int_{K_v(\pi,\pi')}\int_{N\backslash P/P\cap K_v(\pi,\pi')} W_v^\circ(pk)W_v'^\circ(pk)\Phi_v(\epsilon pk)|\det p|^{-s_0}\,\mathrm{d}p\mathrm{d}k.$$

The integral on $N_v \backslash P_v$ is the integral on $\big(N_{r-1}(F_v)\backslash G_{r-1}(F_v)\big) \times G_1(F_v)$ injected in G_v as in paragraph 4.3. We look for the (p,k) such that the integrand in non vanishing. We have $\Phi_v(\epsilon pk) = \Phi_v(\epsilon p_{rr}k)$. As Φ_v is the characteristic function of $\epsilon K_v(\pi,\pi')$, we must have $p_{rr} \equiv 1 \pmod{n_v(\pi,\pi')}$ with $n_v(\pi,\pi') = n_v(\pi) \cap n_v(\pi')$. It follows that $k \in (P \cap K)K_v(\pi,\pi')$. But all functions are invariant under the action of $K_{r-1,v} = \mathrm{GL}_{r-1}(\mathcal{O}_v)$, thus it comes:

$$\begin{aligned} Z_v &= c \int_{A_v/A_v \cap K_v} W_v^\circ(a) W_v'^\circ(a) |\det a|^{-s_0}\, \mathrm{d}a \\ &= c \sum_{\substack{n_1,\dots,n_r \\ n_r \geqslant 0}} W_v^\circ(d) W_v'^\circ(d) q^{-(n_1+\dots+n_r)s_0} \\ &= c \sum_{n_1 \geqslant \dots \geqslant n_r \geqslant 0} W_v^\circ(d) W_v'^\circ(d) q^{-(n_1+\dots+n_r)s_0} \end{aligned}$$

where, with ϖ an uniformizer of F_v, $d = \mathrm{diag}(\varpi^{n_1}, \dots, \varpi^{n_r})$ and $c > 0$ is the measure of $(P \cap K)K_v(\pi,\pi')$. The last equality comes from the explicit calculus of W_v° in [JPSS1, p212] and [Shi]. Measures are normalized by requiring that all $\mathrm{GL}_\alpha(\mathcal{O}_v)$, $1 \leqslant \alpha \leqslant r$ have measure 1. Then Z_v is non zero because the constant coefficient is equal to c, that is the inverse of the number of elements of $K_v/(P \cap K)K_v(\pi,\pi')$. Now Z_v is an entire series in q^{-s_0}. The coefficient of q^{-ls_0} is equal to $\sum_d W_v^\circ(d) W_v'^\circ(d)$ where d runs through the set of matrices of determinant q^{-l}. In [JPSS1, p212], the value of $W_v^\circ(d)$ is computed using the value of non-ramified Whittaker models. In [Shi] it is proven that this value is a polynomial with integral coefficients in the eigenvalues of the Hecke operators thus are elements of $F(\pi)$. It follows that $Z_v \in F(\pi,\pi')[[q^{-s_0}]]$. In [JPSS2, p392], it is proven that Z_v is a rational fraction with denominator $\prod(1 - \alpha_i q^{-s_0})$ where the α_i's depend only from π and π'; with this setup, $\mathrm{L}_v(s_0; \pi \times \pi') = \prod(1 - \alpha_i q^{-s_0})^{-1}$. But now, from [JPSS2] and in particular proposition 8.1, theorem 8.2 and proposition 9.4 with the following remarks, each $\alpha_i \in F(\pi,\pi')$. So that summing up, $\frac{Z_v}{\mathrm{L}_v(s_0;\pi\times\pi')} \in F(\pi,\pi')[q^{-s_0}]$. Note that if all representations are unramified, this polynomial is 1. But $q^{-s_0} \in \mathbf{Q}$ thus $\frac{Z_v}{\mathrm{L}_v(s_0;\pi\times\pi')} \in F(\pi,\pi')$. We remove the restriction on ψ_v as we did in subsection 7.3.1.

We now build the global Schwartz function by multiplying all these Φ_v's together: $\Phi_t = \Phi.\prod_{v \in \Sigma_f} \Phi_v$.

Let S be the set of finite places where (at least) one of the representations π, π' or ψ ramifies. As a local L-factor is never zero, we have

$$Z(s_0, W \otimes W' \otimes \Phi_t) = Z_\infty(s_0, \varphi \otimes \varphi' \otimes \Phi)\mathrm{L}_f(s_0; \pi \times \pi') \prod_{v \in S} \frac{Z(s_0, W_v^\circ \otimes W_v'^\circ \otimes \Phi_v)}{\mathrm{L}_v(s_0, \pi \times \pi')}.$$

We now come back to our trivial K_∞-type. The finite part of all vectors are equal so that they factor out of the sum. As for the infinite part, they add up correctly to an element of the trivial K_∞-type. Since diagram (5) is commutative and from the definition of $C(\pi)$, $C(\pi')$ and $C_{s_0}(\eta)$, the integral

$$\sum Z\left(s_0, \frac{\varphi_t}{C(\pi)} \otimes \frac{\varphi_t'}{C(\pi')} \otimes \frac{\Phi_t}{C_{s_0}(\eta)}\right)$$

is an element of $F(\pi, \pi')$. $\square$

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