Lower deformations of isolated hypersurface singularities

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Abstract

Given a smooth real or complex algebraic variety V and a hypersurface $W \subset V$ with isolated singular points, we study deformations of these singular points which can be realized when varying W in the linear system $|W|$. For isolated singularities of hypersurfaces we define *lower deformations* which have an explicit topological-analytic description, and we show that such deformations can be independently realized under some conditions even if the linear system $|W|$ is not a joint versal deformation of all singular point of W. We give few applications of lower deformations, among them construction of divisors with prescribed isolated singularities and construction of real algebraic curves with prescribed topology.

1 Introduction

Let V be a smooth real or complex algebraic variety of dimension $n \geq 2$, and W be a divisor on V with isolated singular points. Assuming that $\dim |W| > 0$, we ask two natural questions:

- *What deformations of the singular points of W are possible when varying W in $|W|$?*
- *Given a collection of isolated hypersurface singularities, does there exist a divisor $W' \in |W|$ with the given singularities ?*

The questions are related in the following way: knowing deformations of a divisor with complicated singularities, one can construct divisors with prescribed simpler singularities. The classical example is Severi's theorem [22] claiming that singular

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points of a plane nodal algebraic curve can be independently smoothed out or preserved when varying the curve in the space of curves of the same degree, and this is the basic construction of plane curves of a given degree with a given number of nodes.

If the linear system $|W|$ is a joint versal deformation of the multisingularity of W then all possible local deformations of singular points of W are independently realizable when varying W in $|W|$. However, often we do not know what are "possible local deformations" of singular points. On the other hand, the above versality condition for $|W|$ is quite restrictive. It can be expressed in the following form [1, 2, 10, 11]

$$h^1(V, \mathcal{J}_{X^{ea}/V} \otimes \mathcal{O}_V(W)) = 0 \ , \tag{1}$$

where $\mathcal{J}_{X^{ea}/V}$ is the ideal sheaf of the zero-dimensional scheme X^{ea}, concentrated at the singular locus $\mathrm{Sing}(W)$ of W and defined at each point $z \in \mathrm{Sing}(W)$ by the ideal $I^{ea}_{z_i} =< f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} >$ in the local ring $\mathcal{O}_{V,z}$, with $f(x_1, ..., x_n) = 0$ being an equation of the germ (W, z) in some coordinates $x_1, ..., x_n$ in a neighborhood of z on V. If (1) fails, one looses control on the simultaneous deformation of singular points of W.

The aim of this paper is to introduce a class of deformations of singular points (called below "lower"), which have an explicit topological analytic description and are independently realizable in variation of W in $|W|$ under a condition weaker than (1). In section 2 we give a geometric description of lower deformations of semiquasihomogeneous and Newton nondegenerate singular points, in section 3 we establish a general sufficient cohomological criterion for the independence of lower deformations (Theorem 1), in sections 4, 5 we express the above cohomological independence criterion through numerical invariants of the divisor W and its singularities (Lemmas 2, 3, Corollary 1 and Lemmas 6, 7, 8, Corollary 2). As application, we give a criterion for the existence of hypersurfaces with prescribed isolated singularities and consider some special deformations of real plane algebraic curves (Lemma 9, Corollary 3 in section 6, Lemma 12 in section 7).

We were motivated by the following:

- The Viro method of constructing real nonsingular hypersurfaces with prescribed topology [30, 31, 32] (see also [20]) appeared to be very efficient in the topology of real algebraic varieties. One procedure in this method, the gluing of polynomials whose Newton polyhedra form a subdivision of a convex polyhedron, has been extended on polynomials with given singular and critical points [25, 26] (for other generalizations see [17]). In the present paper we give similar generalizations for another procedure in the Viro method, a nonsingular deformation of Newton nondegenerate singularities of a given hypersurface which are located at the fundamental points of the coordinate system. Namely, we allow any number on singular points in an arbitrary position and do not require the Newton nondegeneracy. Moreover, we allow also singular deformation and describe both the real and complex parts of the deformation.

- The existence problem for divisors with prescribed isolated singularities has an asymptotically proper solution in the case of plane curves [13]. In the present paper we develop the techniques of [13] to be applicable to any algebraic varieties of any dimension.

- There are few isotopy types of real curves of degrees 6 in $\mathbb{R}^2$ and of degree 8 in $\mathbb{R}P^2$ which cannot be realized by the original Viro method [6, 19, 23], but require different statements on deformation of singular curves. Our generalization includes all these statements as very special cases.

- Biran [5] proposed a construction of ample divisors on algebraic varieties which seems to be an analogue of the Viro construction for the problem considered. One of our goals has been to generalize the Viro construction to divisors on any smooth algebraic variety.

- In the above notation, the germ at W of the family of hypersurfaces $F \in |W|$ with singular points of the same analytic type as for W, is smooth and has "expected" dimension if and only if $|W|$ is a joint versal deformation of all singular points of W, i.e. arbitrary local deformations of all singular points of W can be independently realized by variation of W in $|W|$. The meaning of our result is that if the germ at W of the family of hypersurfaces $F \in |W|$ with singular points of the same topological type is smooth and has an expected dimension then the lower deformations of all singular points of W can be independently realized by variation of W in $|W|$.

Our approach to the problem studied in based on the Viro method of gluing polynomials [30, 31, 32] and its modifications given in [23, 26, 13].

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2 Singular points and their deformations

For the sake of notation simplicity we shall denote $\mathbf{x} = (x_1, ..., x_n)$ a point in $\mathbb{C}^n$, $\mathbf{i} = (i_1, ..., i_n)$ a point in $\mathbb{Z}^n_+$, $\mathbb{Z}_+ = \{n \in \mathbb{Z} \ : \ n \geq 0\}$, $|\mathbf{i}| = i_1 + ... + i_n$, and $\mathbf{x}^\mathbf{i} = x_1^{i_1} ... x_n^{i_n}$ a monomial.

2.1 Equivalence of singular points and related zero-dimensional schemes

Let V be a smooth algebraic projective variety of dimension $n \geq 2$. Isolated singular points z_1, z_2 of hypersurfaces $W_1, W_2 \subset V$, respectively, are called analytically equivalent if the complete local rings $\hat{\mathcal{O}}_{W_1,z_1}$ and $\hat{\mathcal{O}}_{W_2,z_2}$ are isomorphic. This means that there exist neighborhoods U_1, U_2 of z_1, z_2 in V (in metric topology) and an analytic diffeomorphism of U_1 onto U_2 which takes $W_1 \cap U_1$ to $W_2 \cap U_2$. In the above notation, the points $z_1 \in W_1$, $z_2 \in W_2$ are called topologically equivalent if there exists a homeomorphism of U_1 onto U_2 which takes $W_1 \cap U_1$ to $W_2 \cap U_2$. If V, W_1, W_2, z_1, z_2 are real, i.e. invariant with respect to an antiholomorphic involution (complex conjugation), then we suppose the above isomorphism, diffeomorphism, or homeomorphism to be equivariant.

Let z be an isolated singular point of a hypersurface $W \subset V$ defined in a neighborhood of z by an equation $F(\mathbf{x}) = 0$ in some local coordinates on V. For any sufficiently large integer k the germ at F of the stratum in $\mathcal{O}_{V,z}/\mathfrak{m}_z^k$, where $\mathfrak{m}_z \subset \mathcal{O}_{V,z}$ is the maximal ideal, which consists of elements $G \in \mathcal{O}_{V,z}/\mathfrak{m}_z^k$ defining hypersurface germ on V with a singular point analytically equivalent to $z \in W$, is smooth and its tangent space can be identified with $I_z^{ea}/\mathfrak{m}_z^k$, where $I_z^{ea} \subset \mathcal{O}_{V,z}$ is the ideal generated by $F, \frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n}$.

If $n = 2$, then similarly for any sufficiently large integer k the germ at F of the stratum in $\mathcal{O}_{V,z}/\mathfrak{m}_z^k$ which consists of elements $G \in \mathcal{O}_{V,z}/\mathfrak{m}_z^k$ defining a curve germ with a singular point topologically equivalent to $z \in W$, is smooth and its tangent space can be identified with $I_z^{es}/\mathfrak{m}_z^k$, where $I_z^{es} \subset \mathcal{O}_{V,z}$ is the so-called equisingular ideal [8, 33].

In the sequel we will be given an analytic or topological (for $n = 2$) equivalence of isolated singular points, which will be denoted by $\mathcal{S}$. This data, for a singular point $z \in W \subset V$, defines an ideal $I_z^{\mathcal{S}} \subset \mathcal{O}_{V,z}$ equal to I_z^{ea} or I_z^{es}, and thereby a zero-dimensional scheme $X_z^{\mathcal{S}}$ concentrated at z.

2.2 Newton nondegenerate and semiquasihomogeneous singular points

Let a holomorphic function germ

$$F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \quad F(\mathbf{x}) = \sum_{\mathbf{i}} A_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \tag{2}$$

define an isolated singularity at the origin. Let $\Delta(F)$ be the convex hull of all points $\mathbf{i} + \mathbf{j}$ such that $A_{\mathbf{i}} \neq 0$ and $\mathbf{j} = (j_1, ..., j_n) \in \mathbb{Z}_+^n$. For a face σ of $\Delta(F)$ denote by $K(\sigma)$ the convex hull of $\sigma \cup \{0\}$. By the Newton diagram $\Gamma(F)$ of F (at the origin) we mean the union of those faces σ of $\Delta(F)$ which satisfy

$$\dim K(\sigma) = \dim \sigma + 1, \quad K(\sigma) \cap \Delta(F) = \sigma \ .$$

Denote by $K(\Gamma(F))$ the union of the closures of $K(\sigma)$ taken over all faces $\sigma \subset \Gamma(F)$.

Define the *truncation F^σ of F on the face $\sigma \subset \Gamma(F)$* by $F^\sigma(\mathbf{x}) = \sum_{\mathbf{i} \in \sigma} A_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$. Any truncation F^σ is a quasihomogeneous polynomial or power series. A polynomial truncation F^σ is called *nondegenerate* if the hypersurface $F^\sigma = 0$ has no singular points in $(\mathbb{C}^*)^n$.

A singular point at the origin defined by $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is called *Newton nondegenerate* if $\Gamma(F)$ is compact and intersect all the coordinate axes, and all truncations F^σ, $\sigma \subset \Gamma(F)$, are nondegenerate.

Let z be an isolated singular point of a hypersurface W in a smooth complex projective algebraic variety V of dimension $n \geq 2$. We say that z is *generalized Newton nondegenerate* (briefly, *GNND*) if there exists a holomorphic germ $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ defining a Newton nondegenerate singularity at the origin such that the compete local rings $\hat{\mathcal{O}}_{V,z}$ and $\hat{\mathcal{O}}_{\mathbb{C}^n,0}/(F)$ are isomorphic (or, equivalently, there is an isomorphism $\varphi : \mathcal{O}_{V,z} \to \mathcal{O}_{\mathbb{C}^n,0}$ such that $F = 0$ is an equation of the germ (W, z) in the coordinates induced by φ. We call F a *representative of the GNND singular point* $z \in W$.

We say that a Newton nondegenerate singular point defined by a germ $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is *semiquasihomogeneous* if there is a face $\sigma \subset \Gamma(F)$ of dimension $n-1$ (called *main face*) such that F^σ has no critical points in $\mathbb{C}^n \backslash \{0\}$. Correspondingly, we call a GNND point z of a hypersurface $W \subset V$ *generalized semiquasihomogeneous* (briefly, *GSQH*), if it has a semiquasihomogeneous representative. For a given GSQH point, the main face does not depend on its representative [21]. We call it the main face of a GSQH singular point. If the main face σ meets all the coordinate axes at the points with coordinate m, then such a semiquasihomogeneous singular point is called an *ordinary singular point of multiplicity* m.

2.3 Zero-dimensional schemes associated with GSQH and GNND singular points

Let $z \in W \subset V$ be a GSQH singular point with a representative $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $F(\mathbf{x}) = \sum A_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$. Let $\sigma \subset \Gamma(F)$ be the main face which lies on the hyperplane given by $l(\mathbf{i}) = 0$ with $l(0) > 0$, then we define two ideals $I(F)$ and $\widetilde{I}(F)$ in $\mathcal{O}_{\mathbb{C}^n,0}$:

$$I(F) = \{G \in \mathcal{O}_{\mathbb{C}^n,0} \ : \ G = \sum_{l(\mathbf{i})<0} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}\} \ ,$$

$$\widetilde{I}(F) = \{G \in \mathcal{O}_{\mathbb{C}^n,0} \ : \ G = \sum_{l(\mathbf{i})\leq 0} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}\} \ .$$

Here

$$\dim \mathcal{O}_{\mathbb{C}^n,0}/I(F) = \#\{\mathbf{i} : \ l(\mathbf{i}) \geq 0\}, \quad \dim \mathcal{O}_{\mathbb{C}^n,0}/\widetilde{I}(F) = \#\{\mathbf{i} \ : \ l(\mathbf{i}) > 0\}.$$

If $\varphi : \mathcal{O}_{V,z} \to \mathcal{O}_{\mathbb{C}^n,0}$ is an isomorphism such that $F = 0$ is an equation of the germ (W, z) on V in the coordinates induced by φ, then we obtain ideals

$$I(W, z) = \varphi^*(I(F)) + I_z^{ea}, \quad \widetilde{I}(W, z) = \varphi^*(\widetilde{I}(F)) + I_z^{ea} \ .$$

They define in V zero-dimensional subschemes X_z^{sqh}, $\widetilde{X}_z^{sqh}$, respectively, concentrated at the point z.

Let $z \in W \subset V$ be a GNND singular point with a representative $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Let $\Gamma(F)$ be the Newton diagram of F, and $\Delta(F)$, $K(\Gamma(F))$ be the unbounded and bounded closed domains in the nonnegative orthant of $\mathbb{R}^n$ bounded by $\Gamma(F)$ and the coordinate hyperplanes. Define two ideals $I(F)$ and $\widetilde{I}(F)$ in $\mathcal{O}_{\mathbb{C}^n,0}$:

$$I(F) = \{G \in \mathcal{O}_{\mathbb{C}^n,0} \ : \ G = \sum_{\mathbf{i}\in\Delta(F)} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}\} \ ,$$

$$\widetilde{I}(F)=\{G\in\mathcal{O}_{\mathbb{C}^n,0}\ :\ G=\sum_{\mathbf{i}\in\Delta(F)\setminus\Gamma(F)}\alpha_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\}\ .$$

Here

$$\dim\mathcal{O}_{\mathbb{C}^n,0}/I(F)=\#(\mathbb{Z}^n\cap(K(\Gamma(F))\setminus\Gamma(F))),\quad \dim\mathcal{O}_{\mathbb{C}^n,0}/\widetilde{I}(F)=\#(\mathbb{Z}^n\cap K(\Gamma(F)))\ .$$

If $\varphi:\mathcal{O}_{V,z}\to\mathcal{O}_{\mathbb{C}^n,0}$ is an isomorphism such that $F=0$ is an equation of the germ (W,z) on V in the coordinates induced by φ, then we obtain ideals

$$I(W,z)=\varphi^*(I(F))+I^{ea}_z,\quad \widetilde{I}(W,z)=\varphi^*(\widetilde{I}(F))+I^{ea}_z\ .$$

They define in V zero-dimensional subschemes X^{nnd}_z, $\widetilde{X}^{nnd}_z$, respectively, concentrated at the point z.

2.4 Deformations of singular points

In the above notation, let $B\subset V$ be a sufficiently small compact neighborhood of z in V, diffeomorphic to a closed $2n$-dimensional ball, and such that $W_0=W\cap B$ is a compact variety with an isolated singular point z and a boundary $\partial W_0\subset\partial B$ which is transversal to ∂B along ∂W_0 and is homeomorphic to a cone over ∂W_0.

Under a (one-parametric) deformation of a singular point $z\in W\subset V$ we mean an analytic hypersurface $\mathcal{W}\subset B\times D_\varepsilon$, $D_\varepsilon=\{t\in\mathbb{C}\ :\ |t|<\varepsilon\}$, such that

- $(\pi\big|_{\mathcal{W}})^{-1}(0)=W_0$, where $\pi:B\times D_\varepsilon\to D_\varepsilon$ is the projection,
- for $0<|t|<\varepsilon$, $(\pi\big|_{\mathcal{W}})^{-1}(t):=W_t$ is a compact variety in B with isolated singularities in $\mathrm{Int}(B)$ and boundary $\partial W_t\subset\partial B$, so that W_t is transversal to ∂B along ∂W_t.

Given an equivalence $\mathcal{S}$ of hypersurface isolated singular points, we say that a deformation W_t, $t\in(\mathbb{C},0)$ (or $t\in[0,\varepsilon)$ in the real case), of (W,z) is *$\mathcal{S}$-compatible*, if for any $t_1,t_2\in(\mathbb{C},0)$, $t_1t_2\ne0$, there exists a bijection between $\mathrm{Sing}(W_{t_1})$ and $\mathrm{Sing}(W_{t_2})$ such that the corresponding points in $\mathrm{Sing}(W_{t_1})$, $\mathrm{Sing}(W_{t_2})$ are $\mathcal{S}$-equivalent.

If z, W, V, $\mathcal{W}$ are real, i.e. invariant with respect to the complex conjugation Conj, then we take only real nonnegative values of the parameter t, and consider $X\subset B\times[0,\varepsilon)$.

2.5 Models for deformations of GSQH singular points

Let $z\in W\subset V$ be a GSQH singular point. Accepting the notation of section 2.2, introduce the space of polynomials

$$\mathcal{P}(\sigma)=\left\{\sum_{l(\mathbf{i})\ge0}\alpha_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\right\}$$

and an affine subspace

$$\mathcal{P}(\sigma, F) = \{P \in \mathcal{P}(\sigma) \ : \ P^\sigma = F^\sigma\} \ ,$$

where F^σ, P^σ are the truncations of $F(\mathbf{x})$, $P(\mathbf{x})$, respectively, on σ. Let a polynomial $P \in \mathcal{P}(\sigma, F)$ define a hypersurface $\Phi = \{P = 0\} \subset \mathbb{C}^n$ having only isolated singular points. There is $R_0 > 0$ such that any $(2n-1)$-sphere S_R of radius $R \le R_0$ centered at the origin intersects Φ transversally. Denote by $B^{2n} \subset \mathbb{C}^n$ the $2n$-dimensional closed ball of radius R_0 centered at the origin. The pair $(B^{2n}, \Phi \cap B^{2n})$ is said to be a *model for an $\mathcal{S}$-compatible deformation W_t, $t \in D_\varepsilon$, of the GSQH singular point $z \in W$* if, for any $t \in D_\varepsilon \backslash \{0\}$, there exists a homeomorphism $(B, W_t) \to (B^{2n}, Y)$ which takes $\mathrm{Sing}(W_t)$ to $\mathrm{Sing}(Y)$ so that the corresponding singular points are $\mathcal{S}$-equivalent (here B is a small (closed) neighborhood of z in V).

Given an integer $l_0 > \max\{|\mathbf{i}| \ : \ l(\mathbf{i}) \ge 0\}$, we consider P, $\mathcal{P}(\sigma)$, $\mathcal{P}(\sigma, F)$ to be included in the space $\Sigma(l_0)$ of polynomials of degree $\le l_0$. For any singular point w of $\Phi \cap B^{2n}$, introduce the germ $M_d(w, P)$ at P of the set of polynomials in $\Sigma(d)$ close to P and having a singular point $\mathcal{S}$-equivalent to (Φ, w) in a neighborhood of w. We say that a model $(B^{2n}, \Phi \cap B^{2n})$ for deformations of the GSQH singular point $z \in W$ satisfies the $\mathcal{S}$-transversality (resp., strong $\mathcal{S}$-transversality) condition if there exists sufficiently large integer d such that

- the intersection $M_d(P)$ of the germs $M_d(w, P)$, $w \in \mathrm{Sing}(\Phi \cap B^{2n})$, is smooth, and
- $M_d(P)$ intersects with $\mathcal{P}(\sigma)$ (resp., $\mathcal{P}(\sigma, F)$) in $\Sigma(d)$ transversally.

Remark 1 The $\mathcal{S}$-transversality is defined in accordance with the similar notion in [26], section 3.

2.6 Models for deformations of GNND singular points

Let $z \in W \subset V$ be a GSQH singular point, and $F = \sum_{\mathbf{i} \in \Delta(F)} A_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ be its representative. In the notation of subsection 2.2, let us be given

- a subdivision of $K(\Gamma(F))$ into convex lattice polyhedra $\Delta_1, ..., \Delta_N$ and a convex piece-wise linear real-valued function ν on the nonnegative orthant of $\mathbb{R}^n$ such that $\Delta_1, ..., \Delta_N, \Delta(F)$ are its linearity domains, $\nu\big|_{\Delta(F)} = 0$,
- a set of numbers $\{A_{\mathbf{i}} \ : \ \mathbf{i} \in (K(\Gamma(F)) \backslash \Gamma(F)) \cap \mathbb{Z}^n_+\}$ (real in the real case, complex in the general case) such that, for any Δ_k and any proper face $\sigma \subset \Delta_k$, the polynomial $F^\sigma(\mathbf{x}) = \sum_{\mathbf{i} \in \sigma} A_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ has no singular points in $(\mathbb{C}^*)^n$.

Note that the coefficients $A_{\mathbf{i}}$, $\mathbf{i} \in \Gamma(F)$, are taken from (2), and that all the polynomials $F_k(\mathbf{x}) = \sum_{\mathbf{i} \in \Delta_k} A_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ define hypersurfaces in $\mathbb{C}^n$ with isolated singular points only.

The following construction is basically introduced in [18]. For any $k = 1, ..., N$ we have the momentum map of the positive orthant $\mathbb{R}^n_+$ into the polyhedron Δ_k

$$\mu_k(\mathbf{x}) = \frac{\sum_{\mathbf{i}\in\Delta_k} \mathbf{i}\cdot\mathbf{x}^{\mathbf{i}}}{\sum_{\mathbf{i}\in\Delta_k} \mathbf{x}^{\mathbf{i}}} \stackrel{\text{def}}{=} (\mu_k^{(1)}(\mathbf{x}), ..., \mu_k^{(n)}(\mathbf{x})) \ ,$$

which is a diffeomorphism of $\mathbb{R}^n_+$ onto $\mathrm{Int}(\Delta_k)$ (see [3, 4]). We introduce the complexification of the polyhedron Δ_k

$$\mathbb{C}\Delta_k = \{(w_1, ..., w_n) \in \mathbb{C}^n \ : \ (|w_1|, ..., |w_n|) \in \Delta_k\},$$

and the complexification of the momentum map

$$\mathbb{C}\mu_k : (\mathbb{C}^*)^n \to \mathbb{C}\Delta_k, \quad \mathbb{C}\mu_k(w_1, ..., w_n) = (\mu_k^{(1)}(\mathbf{x})v_1, \ ..., \ \mu_k^{(n)}(\mathbf{x})v_n),$$

$$\mathbf{x} = (|w_1|, ..., |w_n|), \quad v_1 = \frac{w_1}{|w_1|}, \ ..., \ v_n = \frac{w_n}{|w_n|} \ ,$$

which is a diffeomorphism of $(\mathbb{C}^*)^n$ onto $\mathrm{Int}(\mathbb{C}\Delta_k) \cap (\mathbb{C}^*)^n$. By the (complex) chart $\mathrm{Ch}(F_k)$ of the polynomial F_k we call the closure of the set $\mathbb{C}\mu_k(\{F_k = 0\} \cap (\mathbb{C}^*)^n)$.

Proposition 1 *(1) The set* $\mathbb{C}K(\Gamma(F)) = \mathbb{C}\Delta_1 \cup ... \cup \mathbb{C}\Delta_N$ *is homeomorphic to a closed ball* B^{2n}*. The set* $\Phi = \mathrm{Ch}(F_1) \cup ... \cup \mathrm{Ch}(F_N)$ *is a (topological)* $(2n-2)$*-manifold with the boundary* $\Phi \cap \partial(\mathbb{C}K(\Gamma(F)))$ *and the finite singular locus* $\mathrm{Sing}(\Phi) = \mathbb{C}\mu_1(\mathrm{Sing}(F_1)) \cup ... \cup \mathbb{C}\mu_N(\mathrm{Sing}(F_N))$.

(2) Let $\{F = 0\}$ *intersect the boundary of a closed ball* B^{2n} *centered at the origin transversally, and* $\{F = 0\} \cap B^{2n}$ *be a cone over* $\{F = 0\} \cap \partial(B^{2n})$*. Let functions* $\widetilde{A}_{\mathbf{i}}(t)$*,* $t \ge 0$*, with* $\widetilde{A}_{\mathbf{i}}(0) = A_{\mathbf{i}}$*,* $\mathbf{i} \in \mathbb{Z}^n_+$*, be such that, for a sufficiently small* $t > 0$*, there is a bijection between the singular point set of the hypersurface* $\widetilde{F}(\mathbf{x}) = \sum \widetilde{A}_{\mathbf{i}}(t)\mathbf{x}^{\mathbf{i}}t^{\nu(\mathbf{i})} = 0$ *in* $\mathrm{Int}(B^{2n})$ *and the disjoint union* $\mathrm{Sing}(F_1, ..., F_N)$ *of the sets* $\mathrm{Sing}(F_1) \cap (\mathbb{C}^*)^n$*, ...,* $\mathrm{Sing}(F_N) \cap (\mathbb{C}^*)^n$*, so that the corresponding singular points are* $\mathcal{S}$*-equivalent. Then there exists a homeomorphism of* $(B^{2n}, \{\widetilde{F} = 0\} \cap B^{2n})$ *onto* $(\mathbb{C}K(\Gamma(F)), \Phi)$ *which extends the above bijection* $\mathrm{Sing}(\widetilde{F}) \cap B^{2n} \leftrightarrow \mathrm{Sing}(F_1, ..., F_N)$.

If all the data is defined over the reals and $\mathrm{Sing}(F_1, ..., F_N) = \emptyset$, the statement of Proposition 1 for the real parts of $\mathbb{C}K(\Gamma(F))$, Φ, B^{2n} and $\{\widetilde{F} = 0\}$ is a particular case of Viro's theorem [30]. In fact, the proof of Viro's theorem [30] provides the same claim with $\mathrm{Sing}(F_1, ..., F_N) \ne \emptyset$. In full generality Proposition 1 is proven in [18] in a similar way. Here we omit the proof.

The pair $(\mathbb{C}K(\Gamma(F)), \Phi)$ as above is said to be a *model for an* $\mathcal{S}$*-compatible deformation* W_t*,* $t \in D_\varepsilon$*, of the GNND singular point* $z \in W$ if, for any $t \in D_\varepsilon \backslash \{0\}$, there exists a homeomorphism $(B, W_t) \to (\mathbb{C}K(\Gamma(F)), \Phi)$ which takes $\mathrm{Sing}(W_t)$ to $\mathrm{Sing}(F_1, ..., F_N)$ so that the corresponding singular points are $\mathcal{S}$-equivalent (here B is a small (closed) neighborhood of z in V).

We shall define the $\mathcal{S}$-transversality for the above model $(\mathbb{C}K(\Gamma(F)), \Phi)$ (cf. [26], section 3).

Let Gr be the dual graph of the subdivision $\mathbb{R}^n_+ = \Delta(F) \cup \Delta_1 \cup ... \cup \Delta_N$: namely, its vertices are incident to $\Delta(F), \Delta_1, ..., \Delta_N$, and its arcs are incident to common facets of these polyhedra. Define $\mathcal{G}$ to be the set of oriented graphs $\widetilde{G}$ with support G and without oriented cycles. It is clear that $\mathcal{G} \neq \emptyset$. For any $G \in \mathcal{G}$ we denote by $\Delta_{i,+}(G)$ the union of facets of Δ_i, which correspond to the arcs of $\widetilde{G}$ coming in Δ_i, $i = 1, ..., N$.

For a polynomial $F_i(\mathbf{x})$ and an $d \geq \deg F_i$ denote by $M_d(F_i)$ the germ at F_i of the set of polynomials of degree $\leq d$ which, in a neighborhood of $\mathrm{Sing}(F_i) \cap (\mathbb{C}^*)^n$, define singular points $\mathcal{S}$-equivalent to the corresponding singular points of $\{F_i = 0\}$. The triple $(\Delta_i, \Delta_{i,+}(G), F_i)$ is called $\mathcal{S}$-transversal if

- $M_d(F_i)$ is smooth for sufficiently large d,
- if $n \geq 3$ then $M_d(F_i)$ $(d >> 0)$ intersects transversally with the set of polynomials with Newton polyhedron Δ_i having the same coefficients at $\mathbf{i} \in \Delta_{i,+}(G)$ as F_i; if $n = 2$ then $M_d(F_i)$ $(d >> 0)$ intersects transversally with the set of polynomials with Newton polyhedron Δ_i, whose truncations on any connected component of $\Delta_{i,+}(G)$ is proportional to the corresponding truncation of F_i.

Remark 2 The $\mathcal{S}$-transversality condition for $n \geq 3$ is stronger than that for $n = 2$, if $\Delta_{k,+}(G)$ is not connected. In [26], the latter one is called weak $\mathcal{S}$-transversality.

The model $(\mathbb{C}K(\Gamma(F)), \Phi)$ is called $\mathcal{S}$-transversal if there exists a graph $G \in \mathcal{G}$ such that no arc of G goes out of $\Delta(F)$ and all the triples $(\Delta_i, \Delta_{i,+}(G), F_i)$, $i = 1, ..., N$, are $\mathcal{S}$-transversal. The model $(\mathbb{C}K(\Gamma(F)), \Phi)$ is called strongly $\mathcal{S}$-transversal if there exists a graph $G \in \mathcal{G}$ such that no arc of G goes into $\Delta(F)$ and all the triples $(\Delta_i, \Delta_{i,+}(G), F_i)$, $i = 1, ..., N$, are $\mathcal{S}$-transversal.

Remark 3 The case of GSQH singular point with all notions introduced (models, $\mathcal{S}$-transversality) is, in fact, a particular case of a GNND singular point. Nevertheless, we treat it separately, because it is important by itself, all definitions look much simpler that the general ones. Finally, if the main face σ of a GSQH singular point intersects the coordinate axes not at integral points, then the treatment of this situation as a general GNND singular point would be unnecessarily complicated.

3 Independence of lower deformations of GSQH and GNND singular points

Theorem 1 *Let V be a smooth algebraic projective variety of dimension $n \geq 2$ and $W \subset V$ be a hypersurface with a finite singular set* $\mathrm{Sing}(W)$*. Let* $\mathrm{Sing}(W)$ *be divided into disjoint subsets $S, S_1, S_2, S_3, S_4, S_5$ (each of them may be empty), such that the singular points $z \in S_2 \cup S_3$ are GSQH, the singular points in $S_4 \cup S_5$ are GNND. Let us be given, for any point $z \in S_2 \cup S_3$, a deformation model (B^{2n}, Φ_z), for any point*

$z \in S_4 \cup S_5$, a deformation model $(\mathbb{C}K(\Gamma(F_z)), \Phi_z)$, which is strongly $\mathcal{S}$-transversal if $z \in S_2 \cup S_4$ and $\mathcal{S}$-transversal if $z \in S_3 \cup S_5$. Define a zero-dimensional scheme

$$X = \bigcup_{z \in S} X_z^{\mathcal{S}} \cup \bigcup_{z \in S_1} X_z^{ea} \cup \bigcup_{z \in S_2} X_z^{sqh} \cup \bigcup_{z \in S_3} \widetilde{X}_z^{sqh} \cup \bigcup_{z \in S_4} X_z^{nnd} \cup \bigcup_{z \in S_5} \widetilde{X}_z^{nnd}$$

and its ideal sheaf $\mathcal{J}_{X/V}$ on V. If

$$h^1(V, \mathcal{J}_{X/V} \otimes \mathcal{O}_V(W)) = 0 \tag{3}$$

then there exists a family $W_t \in |W|$, $t \in \mathbb{C}$, $|t| < \varepsilon$, such that $W_0 = W$ and, the family W_t realizes

- *for any $z \in S$, an $\mathcal{S}$-equisingular deformation of $z \in W$,*
- *for any $z \in S_1$, an arbitrary deformation prescribed up to local diffeomorphism of a neighborhood of z in V,*
- *for any $z \in S_2 \cup S_3$, an $\mathcal{S}$-compatible deformation of $z \in W$ with the model (B^{2n}, Φ_z),*
- *for any $z \in S_4 \cup S_5$, an $\mathcal{S}$-compatible deformation of $z \in W$ with the model $(\mathbb{C}K(\Gamma(F_z)), \Phi_z)$.*

In addition, for $t \neq 0$,

$$h^1(V, \mathcal{J}_{X_t/V} \otimes \mathcal{O}_V(W_t)) = 0 \ , \tag{4}$$

where $X_t = \cup_z X_z^{\mathcal{S}}$ with z running over all singular points of W_t (this means, in particular, that the germ at W_t of the $\mathcal{S}$-equisingular stratum in $|W|$ is smooth and has an expected dimension).

If all the data are invariant with respect to Conj*, then there exists an equivariant family W_t, $t \in [0, \varepsilon)$, with the above properties.*

Remark 4 (1) Note that all simple singularities are GSQH and for them $X_z^{\mathcal{S}} = X_z^{ea}$. Hence they should be included into the sets S and S_1, when applying Theorem 1.

(2) Comparing with Lemma [23], Theorem 1 not only extends the application range of the statement on independence of deformations, but is *stronger* even in the situation of [23]. Indeed, the condition of Lemma [23] is equivalent to the h^1-vanishing for the ideal sheaf of a zero-dimensional scheme *bigger* than that in Theorem 1, for instance, such a scheme for an ordinary singular point is defined by the ideal $I(F)$ in [23] (cf. [24]), and is defined by the ideal $I(F) + I^{ea} \neq I(F)$ in Theorem 1.

Proof. For the reader's convenience we present our argument in few steps.

Step 1. Assume that $S_2 = \{z\}$, $S_1 = S_3 = S_4 = S_5 = \emptyset$.

1.1. From the exact sequence

$$0 \to \mathcal{J}_{X/V} \otimes \mathcal{O}_V(W) \to \mathcal{O}_V(W) \to \mathcal{O}_X \to 0 \tag{5}$$

and (3) we obtain

$$H^0(V, \mathcal{O}_V(W)) = H^0(V, \mathcal{J}_{X/V} \otimes \mathcal{O}_V(W)) \oplus H^0(X, \mathcal{O}_X)$$

$$= H^0(V, \mathcal{J}_{X/V} \otimes \mathcal{O}_V(W)) \oplus \mathcal{O}_{V,z}/I_z^{sqh} \ . \tag{6}$$

For the point z we have an isomorphism $\varphi : \mathcal{O}_{V,z} \to \mathcal{O}_{\mathbb{C}^n,0}$ and a semiquasihomogeneous representative $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that $F(\mathbf{x}) = 0$ is an equation of the germ (W, z) in the coordinates defined in a neighborhood of z in V by φ. We can write

$$F(\mathbf{x}) = \sum_{l(\mathbf{i}) \le 0} A_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \ , \tag{7}$$

where $l(\mathbf{i}) = 0$ is an equation of the hyperplane containing the main face σ of the Newton diagram of F, $l(0) > 0$ and l is integral-valued at integral points. On the other hand the model Φ_z of deformation for z can be described in B^{2n} by an equation

$$\Phi_z(\mathbf{x}) = \sum_{l(\mathbf{i}) \ge 0} A_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \ .$$

1.2. Let $\Lambda, \Lambda_1, ..., \Lambda_n \subset \mathbb{Z}_+^n$ be (finite) sets such that $\{\mathbf{x}^{\mathbf{i}} \ : \ \mathbf{i} \in \Lambda\}$ is projected to a basis of $\mathcal{O}_{\mathbb{C}^n,0}/(I(F) + < \partial F/\partial x_1, ..., \partial F/\partial x_n >)$, and

$$\{\mathbf{x}^{\mathbf{i}} \ : \ \mathbf{i} \in \Lambda\} \cup \bigcup_{j=1}^{n} \{\mathbf{x}^{\mathbf{j}} \frac{\partial F}{\partial x_j} \ : \ \mathbf{j} \in \Lambda_j\}$$

is projected to a basis of $\mathcal{O}_{\mathbb{C}^n,0}/I(F)$. Denote by $[P(\mathbf{x})]$ the sum of those monomials of a polynomial $P(\mathbf{x})$ which do not belong to $I(F)$.

Relation (6) and the isomorphism φ imply that there exist a basis of $H^0(V, \mathcal{O}_V(W))$, consisting of elements represented in $\mathcal{O}_{V,z} = \mathcal{O}_{\mathbb{C}^n,0}$ as

$$F_{\mathbf{i}}(\mathbf{x}) = \mathbf{x}^{\mathbf{i}} + \sum_{\substack{1 \le j \le n \\ \mathbf{j} \in \Lambda_j}} A_{\mathbf{i},j\mathbf{j}} \left[\mathbf{x}^{\mathbf{j}} \frac{\partial F}{\partial x_j} \right] + \sum_{l(\mathbf{m}) \le 0} A_{\mathbf{im}} \mathbf{x}^{\mathbf{m}}, \ \mathbf{i} \in \Lambda,$$

$$F_i(\mathbf{x}) = \sum_{\substack{1 \le j \le n \\ \mathbf{j} \in \Lambda_j}} A_{i,j\mathbf{j}} \left[\mathbf{x}^{\mathbf{j}} \frac{\partial F}{\partial x_j} \right] + \sum_{l(\mathbf{m}) \le 0} A_{i\mathbf{m}} \mathbf{x}^{\mathbf{m}}, \ i = 0, ..., s.$$

Denote by $B_{\mathbf{i}}$, $\mathbf{i} \in \Lambda$, B_i, $i = 0, ..., s$, the coordinates in $H^0(V, \mathcal{O}_V(W))$ with respect to this basis. Without loss of generality, assume that F is encoded by $B_{\mathbf{i}} = 0$, $\mathbf{i} \in \Lambda$, $B_0 = 1$, $B_i = 0$, $i = 1, ..., s$.

1.3. Consider a deformation of F in $\varphi_z(H^0(V, \mathcal{O}_V(W))) \subset \mathcal{O}_{\mathbb{C}^n,0}$ defined by

$$F_t(\mathbf{x}) = F(\mathbf{x}) + \sum_{\mathbf{i} \in \Lambda} a_{\mathbf{i}} F_{\mathbf{i}}(\mathbf{x}) + \sum_{i=1}^{s} a_i F_i(\mathbf{x}), \tag{8}$$

and depending on small parameters $a_{\mathbf{i}}$, $\mathbf{i}\in\Lambda$, a_i, $i=1,...,s$, which later will be specified as smooth functions of a small parameter t.

The requirement of Theorem 1 means that the elements of this deformation belong to the germ at W of the $\mathcal{S}$-equisingular stratum Σ in $H^0(V,\mathcal{O}_V(W))$ corresponding to the points $w\in S$, and, on the other hand, they realize the given model (B^{2n},Φ_z) in a neighborhood of z.

1.4. By (3) Σ is smooth and transversal to the space $H^0(V,\mathcal{J}_{X_z^{sqh}}(W))$. Hence there is a smooth germ $\Sigma'\subset\Sigma$ defined by

$$B_0=1,\quad B_i=A_i+\psi_i(\{B_{\mathbf{i}}\ :\ \mathbf{i}\in\Lambda\}),\quad i=1,...,s,$$

where ψ_i, $i=1,...,s$, are smooth functions vanishing at zero. Therefore the elements of (8) belong to Σ' if

$$a_i=\psi_i(\{a_{\mathbf{i}}\ :\ \mathbf{i}\in\Lambda\}),\quad i=1,...,s. \tag{9}$$

1.5. Consider a local coordinate change $\mathbf{x}\mapsto T(\mathbf{x})$ such that

$$T_j(\mathbf{x})=x_j+\sum_{\mathbf{j}\in\Lambda_j}a_{j,\mathbf{j}}\mathbf{x}^{\mathbf{j}},\quad j=1,...,n, \tag{10}$$

where $a_{j,\mathbf{j}}$, $\mathbf{j}\in\Lambda_j$, $j=1,...,n$, are small parameters. Then one derives

$$\widetilde{F}_t(\mathbf{x})=F_t(T(x))=F_t(\mathbf{x})+\sum_{j=1}^n\sum_{\mathbf{j}\in\Lambda_j}a_{j,\mathbf{j}}\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}+O(\{a_{\mathbf{i}}^2\},\{a_{j,\mathbf{j}}^2\})$$

$$=\sum_{\mathbf{k}\in\Lambda}c_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}+\sum_{l(\mathbf{k})\le 0}c_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}+\sum_{j=1}^n\sum_{\mathbf{j}\in\Lambda_j}c_{j,\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right],$$

where

$$c_{\mathbf{k}}=a_{\mathbf{k}}+O(\{a_{\mathbf{i}}^2\},\{a_{j,\mathbf{j}}^2\}),\quad \mathbf{k}\in\Lambda, \tag{11}$$

$$c_{j,\mathbf{j}}=\textstyle\sum_{\mathbf{i}\in\Lambda}a_{\mathbf{i}}A_{\mathbf{i},j\mathbf{j}}+\sum_{i=1}^s a_iA_{i,j\mathbf{j}}+a_{j,\mathbf{j}}+O(\{a_{\mathbf{i}}^2\},\{a_{j,\mathbf{j}}^2\}),\quad \mathbf{j}\in\Lambda_j,\ j=1,...,n, \tag{12}$$

$$c_{\mathbf{k}}=A_{\mathbf{k}}+\psi_{\mathbf{k}}^{(1)}(\{a_{\mathbf{i}}^2\},\{a_{j,\mathbf{j}}^2\}),\quad l(\mathbf{k})\le 0, \tag{13}$$

with $\psi_{\mathbf{k}}^{(1)}$ being certain smooth functions vanishing at zero. By the implicit function theorem, equations (9), (11), (12) are solvable with respect to $a_{\mathbf{i}}$, $\mathbf{i}\in\Lambda$, a_i, $i=1,...,s$, $a_{j,\mathbf{j}}$, $\mathbf{j}\in\Lambda_j$, $j=1,...,n$, as functions of

$$c_{\mathbf{k}},\ \mathbf{k}\in\Lambda,\quad c_{j,\mathbf{j}},\ \mathbf{j}\in\Lambda_j,\ j=1,...,n. \tag{14}$$

Hence the family $\widetilde{F}_t$ can be reparameterized by the free parameters (14):

$$\widetilde{F}_t(\mathbf{x})=\sum_{\mathbf{k}\in\Lambda}c_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}+\sum_{j=1}^n\sum_{\mathbf{j}\in\Lambda_j}c_{j,\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]+\sum_{l(\mathbf{k})\le 0}(A_{\mathbf{k}}+\psi_{\mathbf{k}}^{(2)})\mathbf{x}^{\mathbf{k}}$$

with $\psi^{(2)}_{\mathbf{k}}$ being smooth functions of (14) vanishing at zero.

Now we impose the following condition on the parameters (14):

$$\sum_{\mathbf{k}\in\Lambda} c_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} + \sum_{j=1}^{n}\sum_{\mathbf{j}\in\Lambda_j} c_{j,\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right] = \sum_{l(\mathbf{k})>0} d_{\mathbf{k}} t^{l(\mathbf{k})}\mathbf{x}^{\mathbf{k}},$$

where $d_{\mathbf{k}}$ is close to $A_{\mathbf{k}}$ for any $\mathbf{k}\in\mathbb{Z}^n_+$ with $l(\mathbf{k})>0$, and t is a small parameter. The latter relation has a solution with respect to (14) as functions of $\{d_{\mathbf{k}}t^{l(\mathbf{k})} \ : \ l(\mathbf{k})>0\}$. Let $\gamma_1 = -\frac{\partial l}{\partial i_1}$, ..., $\gamma_n = -\frac{\partial l}{\partial i_n}$. Passing to the parameters $\{d_{\mathbf{k}}\}$ and t and rescaling as follows, we obtain the family

$$\hat{F}_t(\mathbf{x}) = \widetilde{F}_t(x_1t^{\gamma_1}, ..., x_nt^{\gamma_n})t^{-l(0)} = \sum_{l(\mathbf{k})>0} d_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} + \sum_{l(\mathbf{k})\le 0} t^{-l(\mathbf{k})}(A_{\mathbf{k}} + t\psi^{(3)}_{\mathbf{k}})\mathbf{x}^{\mathbf{k}} \ ,$$

where $\phi^{(3)}_{\mathbf{k}}$ are smooth functions of $\{d_{\mathbf{k}}\}$ and t.

1.6. The family $\hat{F}_t(\mathbf{x})$ is included in the space $\mathcal{F}$ of functions in $\mathbf{x}\in\mathbb{C}^n$ analytic in the ball B^{2n}. The subfamily $\hat{F}_0(\mathbf{x})$ (depending on parameters $d_{\mathbf{k}}$, $l(\mathbf{k})>0$) is a germ at Φ_z of the space of polynomials $\mathcal{P}(\sigma)$ introduced in subsection 2.5. Due to the strong $\mathcal{S}$-transversality, $\hat{F}_0(\mathbf{x})$ transversally intersects the germ $M\subset\mathcal{F}$ at Φ_z of functions which are $\mathcal{S}$-equisingular to Φ_z in B^{2n}. One can choose $\Lambda'\subset\{\mathbf{i}\in\mathbb{Z}^n_+ \ : \ l(\mathbf{i})>0\}$ so that the family $\hat{F}'_0(\mathbf{x})$ with fixed parameters $d_{\mathbf{k}} = A_{\mathbf{k}}$, $\mathbf{k}\notin\Lambda'$, and free parameters $d_{\mathbf{k}}$, $\mathbf{k}\in\Lambda'$, intersects M transversally and only at Φ_z. Assume that for any small fixed t, fixed parameters $d_{\mathbf{k}} = A_{\mathbf{k}}$, $\mathbf{k}\notin\Lambda'$, and free parameters $d_{\mathbf{k}}$, $\mathbf{k}\in\Lambda'$, the family $\hat{F}'_t(\mathbf{x})$ meets M at one point as well, and this point corresponds to the parameters $d_{\mathbf{k}}(t)$, $\mathbf{k}\in\Lambda'$. These values of $d_{\mathbf{k}}$ define the initial parameters $a_{\mathbf{i}}$, $\mathbf{i}\in\Lambda$, as functions of t so that the deformation (8) is $\mathcal{S}$-equisingular outside a neighborhood of z and realizes the given model Φ_z in a neighborhood of z.

1.7. It remains to justify the latter argument, since the implicit function theorem in general is not valid in infinite-dimensional spaces. Let μ be the maximal Milnor number of the singular points $\mathbf{x}_1, ..., \mathbf{x}_q$ of Φ_z. Due to Tougeron's theorem [28], the condition for an element $\Psi\in\mathcal{F}$, close to Φ_z, to be $\mathcal{S}$-equisingular to Φ_z at $\mathbf{x}_i$ is expressed by a system of analytic equations

$$\psi^{(4)}_{i,1}(\mathrm{jet}_{\mathbf{x}_i,\mu+1}(\Psi)) = ... = \psi^{(4)}_{i,s_i}(\mathrm{jet}_{\mathbf{x}_i,\mu+1}(\Psi)) = 0 \ ,$$

whose variables are the coefficients of the $(\mu+1)$-jet $\mathrm{jet}_{\mathbf{x}_i,\mu+1}(\Psi)$ of Ψ at $\mathbf{x}_i$, and whose Jacobian with respect to these variables has rank s_i at $\Psi = \Phi_z$. Introducing new qn variables being the coordinates of points $\overline{\mathbf{x}}_1, ..., \overline{\mathbf{x}}_q\in\mathbb{C}^n$, close to $\mathbf{x}_1, ..., \mathbf{x}_q$, respectively, one can express the relation $\hat{F}'_t(\mathbf{x})\in M$ as

$$\psi^{(4)}_{i,j}(\mathrm{jet}_{\overline{\mathbf{x}}_i,\mu+1}(\hat{F}'_t)) = 0, \quad j = 1, ..., s_i, \quad i = 1, ..., q. \tag{15}$$

Since the coefficients of the jets $\mathrm{jet}_{\overline{\mathbf{x}}_i,\mu+1}(\hat{F}'_t)$ are analytic functions in

$$\overline{\mathbf{x}}_1, ..., \overline{\mathbf{x}}_q, \quad d_{\mathbf{k}}, \ \mathbf{k}\in\Lambda', \quad t, \tag{16}$$

(15) turns into a system of $s_1+...+s_q$ equations in the $s_1+...+s_q+1$ variables (16). The transversality of the intersection of the family $\hat{F}'_0$ with M means that, for $t=0$, the determinant of the Jacobian of functions (15) with respect to variables (16) is not zero. Hence the system (15) has a solution $d_{\mathbf{k}}=d_{\mathbf{k}}(t)$, $\mathbf{k}\in\Lambda'$ (and $\overline{\mathbf{x}}_i=\overline{\mathbf{x}}_i(t)$, $i=1,...,q$), which completes the proof of Theorem 1 in case $S_2=\{z\}$, $S_3=\emptyset$.

1.8. At last, note that the transversality of the intersection of the family $\hat{F}'_t$ with M, established in 1.6, 1.7, means that the germ at W_t ($t\ne 0$) of the $\mathcal{S}$-equisingular stratum in $|W|$, corresponding to the singular points of W_t in a neighborhood of z, is smooth, has an expected dimension and intersects transversally with Σ', hence so does with $\Sigma\supset\Sigma'$, the germ at W of the $\mathcal{S}$-equisingular stratum corresponding to points in S. In the language of zero-dimensional schemes, this is equivalent to (4).

Step 2. Assume that $S_1=S_2=S_4=S_5=\emptyset$, $S_3=\{z\}$.

In this case the proof goes word-for-word as the proof in Step 1 with the following changes:

- substitute $\widetilde{I}^{sqh}_z$ for I^{sqh}_z, $\widetilde{I}(F)$ for $I(F)$,
- substitute in 1.2-1.6 (in summations etc.) the set $\{\mathbf{i}\in\mathbb{Z}^n_+\ :\ l(\mathbf{i})\ge 0\}$ for $\{\mathbf{i}\in\mathbb{Z}^n_+\ :\ l(\mathbf{i})>0\}$, the set $\{\mathbf{i}\in\mathbb{Z}^n_+\ :\ l(\mathbf{i})<0\}$ for $\{\mathbf{i}\in\mathbb{Z}^n_+\ :\ l(\mathbf{i})\le 0\}$.

Step 3. Assume that $S_1=S_2=S_3=S_4=\emptyset$, $S_5=\{z\}$, and the Newton diagram of a representative of the germ (W,z) has at least two facets (otherwise we get the case in Step 2). Here we combine the argument from Step 1 and from the proof of Theorems 3.1, 3.2 [26].

3.1. As in Step 1, from the exact sequence (5) and (3) we obtain

$$H^0(V,\mathcal{O}_V(W))=H^0(V,\mathcal{J}_{X/V}\otimes\mathcal{O}_V(W))\oplus\mathcal{O}_{V,z}/\widetilde{I}^{nnd}_z\ . \tag{17}$$

For the point z we have an isomorphism $\varphi:\mathcal{O}_{V,z}\to\mathcal{O}_{\mathbb{C}^n,0}$ and a Newton nondegenerate representative $F_z=F:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ such that $F(\mathbf{x})=0$ is an equation of the germ (W,z) in the coordinates defined in a neighborhood of z in V by φ. We can write

$$F(\mathbf{x})=\sum_{\mathbf{i}\in\Delta(F)}A_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\ . \tag{18}$$

On the other hand the model Φ_z of deformation for z is given by the data $\Delta_1,...,\Delta_N$, $\nu:\mathbb{R}^n_+\to\mathbb{R}$, $\{A_{\mathbf{i}}\ :\ \mathbf{i}\in K(\Gamma(F))\backslash\Gamma(F)\}$, as described in section 2.6.

3.2. We choose (finite) sets $\Lambda,\Lambda_1,...,\Lambda_n\subset\mathbb{Z}^n_+$ as in 1.2 and take a basis of $H^0(V,\mathcal{O}_V(W))$ consisting of elements represented in $\mathcal{O}_{V,z}=\mathcal{O}_{\mathbb{C}^n,0}$ by

$$F_{\mathbf{i}}(\mathbf{x})=\mathbf{x}^{\mathbf{i}}+\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j}}A_{\mathbf{i},j\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]+A_{\mathbf{i},0}[F]+\sum_{\mathbf{m}\in\Delta(F)}A_{\mathbf{im}}\mathbf{x}^{\mathbf{m}},\ \mathbf{i}\in\Lambda,$$

$$F_i(\mathbf{x})=\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j}}A_{i,j\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]+A_{i,0}[F]\sum_{\mathbf{m}\in\Delta(F)}A_{i\mathbf{m}}\mathbf{x}^{\mathbf{m}},\ i=0,...,s,$$

(here F is nonzero in the ideal $\widetilde{I}(F)+ < \frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n} >$ if $\Gamma(F)$ contains at least two facets). Denote by $B_{\mathbf{i}}$, $\mathbf{i} \in \Lambda$, B_i, $i = 0, ..., s$, the coordinates in $H^0(V, \mathcal{O}_V(W))$ with respect to this basis. Without loss of generality, assume that F is encoded by $B_{\mathbf{i}} = 0$, $\mathbf{i} \in \Lambda$, $B_0 = 1$, $B_i = 0$, $i = 1, ..., s$.

3.3. Consider a deformation of F in $\varphi_z(H^0(V, \mathcal{O}_V(W))) \subset \mathcal{O}_{\mathbb{C}^n,0}$ defined by (8) and depending on small parameters $a_{\mathbf{i}}$, $\mathbf{i} \in \Lambda$, a_i, $i = 1, ..., s$.

As in 1.4, the elements of (8) belong to a smooth germ $\Sigma' \subset \Sigma$ if the parameters $a_{\mathbf{i}}$, $\mathbf{i} \in \Lambda$, a_i, $i = 1, ..., s$, satisfy (9).

3.4. We introduce the local coordinate change (10), but transform the family F_t in a slightly different way:

$$\widetilde{F}_t(\mathbf{x}) = (1 + a_0)F_t(T(\mathbf{x})) = F_t(\mathbf{x}) + a_0 F(\mathbf{x}) \sum_{j=1}^{n} \sum_{\mathbf{j} \in \Lambda_j} a_{j,\mathbf{j}} \mathbf{x}^{\mathbf{j}} \frac{\partial F}{\partial x_j} + O(a_0^2, \{a_{\mathbf{i}}^2\}, \{a_{j,\mathbf{j}}^2\}) \ . \tag{19}$$

As in 1.5, one can introduce new small parameters (14) and c_0, and reparameterize F_t in the form

$$\widetilde{F}_t(\mathbf{x}) = \sum_{\mathbf{k} \in \Lambda} c_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} + c_0[F] + \sum_{j=1}^{n} \sum_{\mathbf{j} \in \Lambda_j} c_{j,\mathbf{j}} \left[\mathbf{x}^{\mathbf{j}} \frac{\partial F}{\partial x_j} \right] + \sum_{\mathbf{k} \in \Delta(F) \backslash \Gamma(F)} (A_{\mathbf{k}} + \psi_{\mathbf{k}}^{(2)}) \mathbf{x}^{\mathbf{k}}$$

with $\psi_{\mathbf{k}}^{(2)}$ being smooth functions of (14) and c_0 vanishing at zero. Then we impose the following condition on the parameters (14) and c_0:

$$\sum_{\mathbf{k} \in \Lambda} c_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} + c_0[F] + \sum_{j=1}^{n} \sum_{\mathbf{j} \in \Lambda_j} c_{j,\mathbf{j}} \left[\mathbf{x}^{\mathbf{j}} \frac{\partial F}{\partial x_j} \right] = \sum_{\mathbf{k} \in K(\Gamma(F))} d_{\mathbf{k}} t^{\nu(\mathbf{k})} \mathbf{x}^{\mathbf{k}},$$

where $d_{\mathbf{k}}$ is close to $A_{\mathbf{k}}$ for any $\mathbf{k} \in K(\Gamma(F))$, and t is a small parameter. The latter relation has a solution with respect to (14) as functions of $\{d_{\mathbf{k}} t^{\nu(\mathbf{k})} \ : \ \mathbf{k} \in K(\Gamma(F)) \backslash \Gamma(F)\}$, and leads us to the family

$$\hat{F}_t(\mathbf{x}) = \sum_{\mathbf{k} \in K(\Gamma(F))} d_{\mathbf{k}} t^{\nu(\mathbf{k})} \mathbf{x}^{\mathbf{k}} + \sum_{\mathbf{k} \in \Delta(F) \backslash \Gamma(F)} (A_{\mathbf{k}} + t\psi_{\mathbf{k}}^{(3)}) \mathbf{x}^{\mathbf{k}} \ , \tag{20}$$

where $\phi_{\mathbf{k}}^{(3)}$ are smooth functions of $\{d_{\mathbf{k}}\}$ and t.

To complete the proof in the case considered, we should show that the parameters $d_{\mathbf{k}}$, $\mathbf{k} \in K(\Gamma(F))$, can be defined as functions of t equal to $A_{\mathbf{k}}$ as $t = 0$ and such that the singularities of $\hat{F}_t = 0$ in a neighborhood of the origin are $\mathrm{Sing}(F_1, ..., F_N)$ (up to $\mathcal{S}$-equivalence). Then by construction and Proposition 1 one would obtain that the deformation (8) satisfies (9) and realizes the given model $(\mathbb{C}K(\Gamma(F)), \Phi_z)$ in a neighborhood of z.

3.5. The proof of the latter claim essentially coincides with the proof of Theorems 3.1, 3.2, 3.6 in [26], section 3.3: it is indeed the same if F is a polynomial, and it should be completed with the argument from 1.7 if F is a power series.

Remark 5 If $V=\mathbb{P}^n$ then the situation considered in Step 3 is a particular case of Theorems 3.1, 3.6 [26] with the following difference. Taking the Viro-type polynomial in Theorems 3.1, 3.6 [26] as a deformation of a polynomial F with a Newton nondegenerate singular point, one would have all the coefficients at $\mathbf{i}\in K(\Gamma(F))$ as independent free parameters, whereas in our situation, in the beginning, one has the free coefficients $a_{\mathbf{i}}$, $\mathbf{i}\in\Lambda$, only. The rest of necessary free parameters (as in formula (20)) one obtains from the transformation (19).

Step 4. Assume that $S_1=S_2=S_3=S_5=\emptyset$, $S_4=\{z\}$. Then the proof goes as in Step 3 with the following changes (cf. Step 2):

- one should get rid of $[F]$, a_0, c_0 in all the formulae,
- the ideals $\widetilde{I}_z^{nnd}$, $\widetilde{I}(F)$ should be replaced by the ideals I_z^{nnd}, $I(F)$, respectively,
- the sets $\Delta(F)\backslash\Gamma(F)$, $K(\Gamma(F))$ (in summations etc.) should be replaced by the sets $\Delta(F)$, $K(\Gamma(F))\backslash\Gamma(F)$, respectively.

Step 5. Assume that $S_1=\emptyset$, but S_2,S_3,S_4,S_5 may be nonempty.

The exact sequence (5) implies that $H^0(V,\mathcal{O}_V(W))$ splits into the direct sum

$$H^0(V,\mathcal{J}_{X/V}\otimes\mathcal{O}_V(W))\oplus\bigoplus_{z\in S_2}\frac{\mathcal{O}_{V,z}}{I_x^{sqh}}\oplus\bigoplus_{z\in S_3}\frac{\mathcal{O}_{V,z}}{\widetilde{I}_x^{sqh}}\oplus\bigoplus_{z\in S_4}\frac{\mathcal{O}_{V,z}}{I_x^{nnd}}\oplus\bigoplus_{z\in S_5}\frac{\mathcal{O}_{V,z}}{\widetilde{I}_x^{nnd}}.$$

This allows us to introduce the following base in $H^0(V,\mathcal{O}_V(W))$:

$$F_{\mathbf{i}}^{(z)}(\mathbf{x})=\mathbf{x}^{\mathbf{i}}+\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j^{(z)}}}A_{\mathbf{i},j\mathbf{j}}^{(z)}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]^{(z)}+\sum_{l^{(z)}(\mathbf{m})\le 0}A_{\mathbf{im}}^{(z)}\mathbf{x}^{\mathbf{m}},\ \mathbf{i}\in\Lambda^{(z)},\quad z\in S_2,$$

$$F_{\mathbf{i}}^{(z)}(\mathbf{x})=\mathbf{x}^{\mathbf{i}}+\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j^{(z)}}}A_{\mathbf{i},j\mathbf{j}}^{(z)}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]^{(z)}+\sum_{l^{(z)}(\mathbf{m})<0}A_{\mathbf{im}}^{(z)}\mathbf{x}^{\mathbf{m}},\ \mathbf{i}\in\Lambda^{(z)},\quad z\in S_3,$$

$$F_{\mathbf{i}}^{(z)}(\mathbf{x})=\mathbf{x}^{\mathbf{i}}+\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j}}A_{\mathbf{i},j\mathbf{j}}^{(z)}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]^{(z)}+\sum_{\mathbf{m}\in\Delta^{(z)}(F)}A_{\mathbf{im}}^{(z)}\mathbf{x}^{\mathbf{m}},\ \mathbf{i}\in\Lambda^{(z)},\quad z\in S_4,$$

$$F_{\mathbf{i}}^{(z)}(\mathbf{x})=\mathbf{x}^{\mathbf{i}}+\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j}}A_{\mathbf{i},j\mathbf{j}}^{(z)}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]^{(z)}+A_{\mathbf{i},0}^{(z)}[F]^{(z)}+\sum_{\mathbf{m}\in\Delta^{(z)}(F)\backslash K(\Gamma^{(z)}(F))}A_{\mathbf{im}}^{(z)}\mathbf{x}^{\mathbf{m}},$$

$$\mathbf{i}\in\Lambda^{(z)},\quad z\in S_5,$$

and F_i, $i=0,...,s$, where (i) each $F_{\mathbf{i}}^{(z)}$ is represented by its image in $\mathcal{O}_{V,z}=\mathcal{O}_{\mathbb{C}^n,0}$, (ii) the sets $\Lambda^{(z)},\Lambda_1^{(z)},...,\Lambda_n^{(z)}$ and the symbols $[*]^{(z)}$ are defined for any point $z\in$

$S_2\cup...\cup S_5$ as described above, (iii) for any $i=1,...,s$ and any $z\in S_2\cup...\cup S_5$, the element $F_i\in H^0(V,\mathcal{O}_V(W))$ has an image

$$\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j}} A^{(z)}_{i,j\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]^{(z)}+\sum_{l(\mathbf{m})\le 0}A^{(z)}_{i\mathbf{m}}\mathbf{x}^{\mathbf{m}},\quad\text{or}$$

$$\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j}} A^{(z)}_{i,j\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]^{(z)}+\sum_{l(\mathbf{m})<0}A^{(z)}_{i\mathbf{m}}\mathbf{x}^{\mathbf{m}},\quad\text{or}$$

$$\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j}} A^{(z)}_{i,j\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]^{(z)}+\sum_{\mathbf{m}\in\Delta^{(z)}(F)}A^{(z)}_{i\mathbf{m}}\mathbf{x}^{\mathbf{m}},\quad\text{or}$$

$$\sum_{\substack{1\le j\le n\\ \mathbf{j}\in\Lambda_j}} A^{(z)}_{i,j\mathbf{j}}\left[\mathbf{x}^{\mathbf{j}}\frac{\partial F}{\partial x_j}\right]^{(z)}+A^{(z)}_{i,0}[F]^{(z)}+\sum_{\mathbf{m}\in\Delta^{(z)}(F)\backslash K(\Gamma^{(z)}(f))}A^{(z)}_{i\mathbf{m}}\mathbf{x}^{\mathbf{m}},$$

according as $z\in S_2$, or S_3, or S_4, or S_5. Finally, we assume that $F_0=F$.

The existence of such a basis in $H^0(V,\mathcal{O}_V(W))$ allows us to perform the above construction of the required deformation for all singular points $z\in S_2\cup...\cup S_5$ independently, and leads to the desired result.

Step 6. Assume that $S_1\ne\emptyset$.

Considering the set S_1 as a part of S and taking $X^{\mathcal{S}}_zX^{ea}_z$, $z\in S_1$, we construct the required deformation of all the points $z\in S_2\cup...\cup S_5$, while the points $z\in S\cup S_1$ keep their $\mathcal{S}$-type (may be, slightly moving in V). For an element W_t, $t\ne 0$, of the family constructed, we have (4). In this equality $X^{\mathcal{S}}_z=X^{ea}_z$ for all $z\in S'_1$ (the singular points coming from $S_1\subset\mathrm{Sing}(W)$); hence the germ Σ'' at W_t of the $\mathcal{S}$-equisingular stratum in $|W|$, corresponding to the singular points in $\mathrm{Sing}(W_t)\backslash S'_1$, is a joint versal deformation of all the points $z\in S'_1$, which are analytically equivalent to the corresponding singular points of W and have the same deformations. Thus, one can deform W_t inside Σ'' realizing prescribed deformations of the singular points $z\in S'_1$ simultaneously.

It remains to prove (4) for the hypersurfaces in the latter deformation. This follows immediately from the fact that all equisingular (in the analytic sense) strata in a versal deformation base of an isolated hypersurface singularity are smooth and have an expected dimension. □

4 Numerical criteria for the h^1-vanishing in Theorem 1

Here we restrict ourselves to the case $V=\mathbb{C}P^n$ or $\mathbb{R}P^n$ and $W\subset V$ is a hypersurface of degree d.

Lemma 2 *In the notation of Theorem 1, (3) holds if*

$$\deg X < 4(d-1) \ .$$

Proof. Since $X \subset \bigcup_{z \in \mathrm{Sing}(W)} X_z^{ea}$, the required statement can be derived in the same way as Theorems 1, 2 [27] (see also [9]). $\square$

In the case $n = 2$ there is a number of h^1-vanishing criteria for zero-dimensional schemes associated with singular plane algebraic curves, some of them we present below.

First, we recall some singular point invariants.

Definition 1 (see [26], section 4.1, and [24]) Let z be an isolated singular point of a plane curve C, T be its embedded resolution tree, $C_{(q)}$ be the strict transform of C corresponding to an infinitely near point $q \in T$, and $E_{(q)}$ be the reduced exceptional divisor. We define topological invariants associated with z:

- Let z be of type A_{2k-1}, $k \geq 1$ (equivalent to $y^2 + x^{2k}$). Then C has at z two non-singular branches P_1, P_2, and we put $b(C, P_1) = b(C, P_2) = k - 1$.
- Let z be of type A_{2k}, $k \geq 1$ (equivalent to $y^2 + x^{2k+1}$). Then C has at z one branch P, and we put $b(P, C) = 2k - 1$.
- Let z be not of type A_k. A point $q \in T$ is called terminal if it is non-singular both for $C_{(q)}$ and $E_{(q)}$. We remove from T all the points following terminal points. For a local branch Q of C centered at z, define $b'(C, Q)$ to be the sum of the multiplicities of the strict transforms $Q_{(q)}$ of Q at non-terminal points $q \in T$, and define $b''(C, Q)$ to be $(Q_{(q)} \cdot E_{(q)}) - 1$ at the (unique) terminal point q on Q, where $(* \cdot *)$ is the intersection number. Put $b(C, Q) = b'(C, Q) + b''(C, Q)$.

Let $z \in C$ be a GSQH point with a representative F_z and the main edge σ of the Newton diagram $\Gamma(F_z)$. We assume that if the straight line through σ meets a coordinate axis at an integral point than this point belongs to σ (clearly, such a representative F_z exists). Put $\widetilde{b}(C, Q) = b(C, Q)$ for branches Q which are not determined by σ, and $\widetilde{b}(C, Q) = p + q$ for branches Q which are determined by σ, where $F_z^\sigma(x, y) = x^\alpha y^\beta \prod_{k=1}^m (y^q + \lambda_k x^p)$, $(p, q) = 1$.

Let $z \in C$ be a GNND point with a representative F_z and the Newton diagram $\Gamma(F_z)$. Put $\widetilde{b}(C, Q) = p + q$ for a branch Q determined by an edge $\sigma \subset \Gamma(F)$, where $F_z^\sigma(x, y) = x^\alpha y^\beta \prod_{k=1}^m (y^q + \lambda_k x^p)$, $(p, q) = 1$.

At last, put $b(C, z)$ (resp. $\widetilde{b}(C, z)$) be the sum of $b(C, Q)$ (resp. $\widetilde{b}(C, Q)$) over all branches centered at z.

Note that $b(C, z)$ is equal to $\mu(C, z) - \bmod (C, z) - isod_z(\mathcal{N}^{es}_{C/\mathbb{C}P^2}, \mathcal{O}_C)$ with μ being the Milnor number, $\bmod$ the modality, $isod_z(\mathcal{N}^{es}_{C/\mathbb{C}P^2}, \mathcal{O}_C)$ the isomorphism defect defined in [11], sections 3, 4.

Definition 2 (see [14], section 2.1) Let z be an isolated singular point of a plane curve C, and $Y \subset X_z^{ea}$ be a zero-dimensional scheme. Then we define

$$\gamma(C,Y) = \max_D \frac{(\deg(D \cap Y) + \min\{(D \cdot C)_z - \deg(D \cap Y),\ \deg(D \cap Y)\})^2}{\min\{(D \cdot C)_z - \deg(D \cap Y),\ \deg(D \cap Y)\}}\ ,$$

where D runs over all curves through z which have no component in common with C.

According to [14], Lemma 2.2, $\gamma(C,Y) \le (\deg Y + 1)^2$.

Lemma 3 *In the notation of Theorem 1, let W be a reduced plane curve of degree d. Then (3) holds if one of the following conditions is satisfied:*

(1) $\sum_{z \in \mathrm{Sing}(W)} (\deg X_z - isod_z(\mathcal{J}_{X/\mathbb{C}P^2}, \mathcal{O}_W)) < 4(d-1)$, where $isod_z(\mathcal{J}_{X/\mathbb{C}P^2}, \mathcal{O}_{W'})$, defined as in [11], section 3, always is a nonnegative integer, or

(2) W is irreducible and

$$\sum_{z \in \mathrm{Sing}(W)} \gamma(W, X_z) \le (d+3)^2\ ,$$

where X_z is a component of the scheme X concentrated at z, or

(3) W is irreducible and

$$\sum_{z \in S^a \cup S_1} (\tau(W,z) - isod_z(\mathcal{N}^{ea}_{W/\mathbb{C}P^2}, \mathcal{O}_W)) + \sum_{z \in S^t \cup S_2 \cup S_4} b(W,z) + \sum_{z \in S_3 \cup S_5} \widetilde{b}(W,z) < 3d\ ,$$

where $isod_z(\mathcal{N}^{ea}_{W/\mathbb{C}P^2}, \mathcal{O}_W)$, defined as in [11], section 3, always is a positive integer, and S^a, S^t are the parts of S, where the singular points are considered up to analytical or topological equivalence, respectively, or

(4) W splits into irreducible components $W_1, ..., W_r$, the sets S^a, S_1 are empty, and

$$\sum_{z \in S \cup S_2 \cup S_4} \sum_{Q_z^{(i)}} b(W, Q_z^{(i)}) + \sum_{z \in S_3 \cup S_5} \sum_{Q_z^{(i)}} \widetilde{b}(W, Q_z^{(i)}) < 3 \cdot \deg W_i, \quad i = 1, ..., r\ . \tag{21}$$

This follows from [14], Proposition 2.1, [11], Corollary 3.9, [26], Theorem 4.1.

Corollary 1 *Let $W \subset \mathbb{C}P^2$ be an irreducible curve of degree d, and* $\mathrm{Sing}(W) = S_2$ *consist of ordinary singular points of multiplicities $m_1, ..., m_s$. Then (3) holds if*

$$4 \cdot \sum_{m_i = 2} 1 + 18 \cdot \sum_{m_i = 3} 1 + \frac{16}{7} \cdot \sum_{m_i > 3} m_i^2 \le (d+3)^2\ ,$$

or

$$\sum_{i=1}^{s} m_i < 3d\ .$$

This follows from Lemma 3 and the upper bound for $\gamma(W, X_z^{sqh})$ in [14], Corollary 2.5.

5 Criteria for the $\mathcal{S}$-transversality of deformation models for GSQH and GNND singular points

Let, in the notation of section 2.5, $(B^{2n},\Phi\cap B^{2n})$ be a model for a deformation of a GSQH singular point $z\in W\subset V$, defined by a polynomial $P\in\mathcal{P}(\sigma,F)$. Denote $d=\deg P$. The homogeneous ideals generated by the monomials $x_0^{d-i_1-...-i_n}x_1^{i_1}...x_n^{i_n}$, $l(i_1,...,i_n)\ge 0$, or by the monomials $x_0^{d-i_1-...-i_n}x_1^{i_1}...x_n^{i_n}$, $l(i_1,...,i_n)>0$, define in $\mathbb{C}P^n$ subschemes Y and $\widetilde{Y}$, respectively. Put

$$Z=Y\cup X(P),\quad \widetilde{Z}=\widetilde{Y}\cup X(P),\quad X(P)=\bigcup_{w\in\mathrm{Sing}(P)\backslash Y}X_w^{\mathcal{S}}\ .$$

Proposition 4 *The model $(B^{2n},\Phi\cap B^{2n})$ is $\mathcal{S}$-transversal if and only if $h^1(\mathbb{C}P^n,\mathcal{J}_{Z/\mathbb{C}P^n}(d))=0$, and it is strongly $\mathcal{S}$-transversal if and only if $h^1(\mathbb{C}P^n,\mathcal{J}_{\widetilde{Z}/\mathbb{C}P^n}(d))=0$.*

Proof. Note that

$$H^0(\mathbb{C}P^n,\mathcal{J}_{Y/\mathbb{C}P^n}(d))=\left\{\sum_{l(i_1,...,i_n)\ge 0}a_{i_1...i_n}x_0^{d-i_1-...-i_n}x_1^{i_1}...x_n^{i_n}\right\},$$

$$H^0(\mathbb{C}P^n,\mathcal{J}_{\widetilde{Y}/\mathbb{C}P^n}(d))=\left\{\sum_{l(i_1,...,i_n)>0}a_{i_1...i_n}x_0^{d-i_1-...-i_n}x_1^{i_1}...x_n^{i_n}\right\}.$$

Hence the exact sequences

$$\begin{aligned}0\to H^0(\mathbb{C}P^n,\mathcal{J}_{Z/\mathbb{C}P^n}(d)) &\to H^0(\mathbb{C}P^n,\mathcal{J}_{Y/\mathbb{C}P^n}(d))\to H^0(X(P),\mathcal{O}_{X(P)})\\ &\to H^1(\mathbb{C}P^n,\mathcal{J}_{Z/\mathbb{C}P^n}(d))\to H^1(\mathbb{C}P^n,\mathcal{J}_{Y/\mathbb{C}P^n}(d))=0,\end{aligned}$$

$$\begin{aligned}0\to H^0(\mathbb{C}P^n,\mathcal{J}_{\widetilde{Z}/\mathbb{C}P^n}(d)) &\to H^0(\mathbb{C}P^n,\mathcal{J}_{\widetilde{Y}/\mathbb{C}P^n}(d))\to H^0(X(P),\mathcal{O}_{X(P)})\\ &\to H^1(\mathbb{C}P^n,\mathcal{J}_{\widetilde{Z}/\mathbb{C}P^n}(d))\to H^1(\mathbb{C}P^n,\mathcal{J}_{\widetilde{Y}/\mathbb{C}P^n}(d))=0,\end{aligned}$$

prove the required statement. The h^1-vanishing for the ideal sheaves of Y and $\widetilde{Y}$ pointed out in the latter expressions, can easily be proven by induction on n and $\deg Y$, $\deg\widetilde{Y}$: using the fact that Y and $\widetilde{Y}$ are contained in a d-multiple hyperplane $\{x_0=0\}$, one subsequently divides the schemes Y, $\widetilde{Y}$ by $\{x_0=0\}$ and applies the "Horace's method" (see detail in [15, 16], or in section 6.2 below). □

In the notation of section 2.6, let $(\mathbb{C}K(\Gamma(F)),\Phi)$ be a model of deformation for a GNND singular point $z\in W\subset V$, defined by a subdivision $K(\Gamma(F))=\Delta_1\cup...\cup\Delta_N$ and polynomials $F_1,...,F_N$ with Newton polyhedra $\Delta_1,...,\Delta_N$, respectively. Given an oriented graph $G\in\mathcal{G}$ and a triple $(\Delta_k,\Delta_{k,+}(G),F_k)$, $1\le k\le N$, assume that F_k is divided by maximal possible powers of $x_1,...,x_n$ and Δ_k is shifted in $\mathbb{R}^n_+$ correspondingly, and introduce the scheme $Y_k\subset\mathbb{C}P^n$ defined by the homogeneous ideal, whose generators are

- for $n \geq 3$, the monomials $x_0^{d_k-i_1-...-i_n}x_1^{i_1}...x_n^{i_n}$, $(i_1,...,i_n) \in \Delta_k \backslash \Delta_{k,+}(G)$, where $d_k = \deg F_k$;
- for $n = 2$, the monomials $x_0^{d_k-i_1-i_2}x_1^{i_1}x_2^{i_2}$, $(i_1, i_2) \in \Delta_k \backslash \Delta_{k,+}(G)$, and the truncations F_k^σ, where σ runs over all connected components of $\Delta_{k,+}(G)$.

Put

$$Z_k = Y_k \cup \bigcup_{w \in \mathrm{Sing}(F_k) \cap (\mathbb{C}^*)^n} X_w^{\mathcal{S}} .$$

Proposition 5 *The triple* $(\Delta_k, \Delta_{k,+}(G), F_k)$ *is* $\mathcal{S}$*-transversal if and only if* $h^1(\mathbb{C}P^n, \mathcal{J}_{Z_k/\mathbb{C}P^n}(d_k)) = 0$.

Proof basically coincides with that of Proposition 4 with the following remark: the vanishing of $H^1(\mathbb{C}P^n, \mathcal{J}_{Y_k/\mathbb{C}P^n}(d_k))$ is derived again by means of Horace's method from the fact that Y_k is contained in the union of multiple coordinate hyperplanes with the total degree $\leq d_k$. $\Box$

Below we present few numerical sufficient criteria for the $\mathcal{S}$-transversality of deformation models.

Lemma 6 *A nonsingular deformation model of a GSQH or GNND singular point is strongly* $\mathcal{S}$*-transversal.*

Proof. Straightforward from Propositions 4, 5 and their proofs. $\Box$

Lemma 7 *In the above notation, let* $z \in W$ *be an ordinary singular point of multiplicity* d*. Then a deformation model given by a polynomial* P *(of degree* d*) is* $\mathcal{S}$*-transversal if* $\deg X(P) < 4(d-1)$*, and is strongly* $\mathcal{S}$*-transversal if* $\deg X(P) < 3(d-1)$.

Proof. Both the statements can be proven as Theorems 1, 2 [27] (see also [9]). $\Box$

In the case $n = 2$ (planar curve singularities) we can say more.

If F is a polynomial in two variables with Newton polygon Δ such that the truncation F^σ, for any edge $\sigma \subset \partial\Delta$, has no singular points in $(\mathbb{C}^*)^2$, then each edge σ determines (up to topological equivalence) certain local branches of the projective closure of the curve $F(x, y) = 0$, centered on the lines $x_0 = 0, x_1 = 0, x_2 = 0$. The number of these branches is equal to the number of the minimal segments on σ with integral endpoints. If G is a component of F then $\rho(\sigma, G)$ is the number of the local branches of G determined by the edge σ.

Lemma 8 *In the notation of section 2.6, let* $n = 2$*. The triple* $(\Delta_k, \Delta_{k,+}(G), F_k$ *is transversal with respect to the topological equivalence of singular points if*

- *the curve* $F_k = 0$ *is irreducible in* $(\mathbb{C}^*)^2$ *and*

$$\sum_{z \in \mathrm{Sing}(F_k) \cap (\mathbb{C}^*)^2} b(F_k, z) < \sum_{\substack{\sigma \subset \partial\Delta \\ \sigma \not\subset \Delta_{k,+}(G)}} \rho(\sigma, F_k) \ ,$$

- *the curve* $\{F_k = 0\}$ *splits into components* $C_1, ..., C_s$, *and*

$$\sum_{Q_i} b(F_k, Q_i) < \sum_{\substack{\sigma \subset \partial\Delta \\ \sigma \not\subset \Delta_{k,+}(G)}} \rho(\sigma, C_i), \quad i = 1, ..., s,$$

  where Q_i *runs over all local branches of* C_i *centered at* $\mathrm{Sing}(F_k) \cap (\mathbb{C}^*)^2$.

The triple $(\Delta_k, \Delta_{k,+}(G), F_k$ *is transversal with respect to the analytic equivalence of singular points if*

- *the curve* $\{F_k = 0\} \cap (\mathbb{C}^*)^2$ *is irreducible and*

$$\sum_{w} (\tau(F_k, w) - 1) < \sum_{\substack{\sigma \subset \partial\Delta \\ \sigma \not\subset \Delta_{k,+}(G)}} \rho(\sigma, F_k) \ ,$$

  where w *runs through all singular points of* $F_k = 0$ *in* $(\mathbb{C}^*)^2$,

- *the curve* $\{F_k = 0\} \cap (\mathbb{C}^*)^2$ *splits into components* $C_1, ..., C_s$ *and*

$$\sum_{w} (\tau(C_i, w) - 1 + (C_i \cdot \hat{C}_i)_w) < \sum_{\substack{\sigma \subset \partial\Delta \\ \sigma \not\subset \Delta_{k,+}(G)}} \rho(\sigma, C_i), \quad i = 1, ..., s,$$

  where w *runs through all singular points of* $F_k = 0$ *in* $(\mathbb{C}^*)^2$, *and* $(C_i \cdot \hat{C}_i)_w$ *is the intersection number at* w *of* C_i *with* $\hat{C}_i$, *the union of all components of* $\{F_k = 0\} \cap (\mathbb{C}^*)^2$ *but* C_i.

This is a reformulation of Theorem 4.1 [26].

Corollary 2 *If* $n = 2$ *and a deformation model of a GSQH or GNND singular point has nodes as its only singularities, then it is strongly* $\mathcal{S}$*-transversal.*

This is a reformulation of Theorem 4.2 [26].

6 Existence of hypersurfaces with prescribed isolated singularities

6.1 Main construction

Given a linear system $|W|$ on a smooth algebraic variety V of dimension $n \geq 2$ and a classification $\mathcal{S}$ of isolated hypersurface singular points, we give a sufficient condition for the existence of an element $W' \in |W|$ with a given (up to $\mathcal{S}$-equivalence) collection of isolated singularities.

Definition 3 For a type $s \in \mathcal{S}$ define $e(s)$ to be the minimal degree of a polynomial $F_s(\mathbf{x})$ such that

- the leading $e(s)$-form of F has no critical points in $\mathbb{C}^n \backslash \{0\}$,
- $\mathrm{Sing}(F_s)$ consists of one point z, and this singular point is of type s,
- the germ at F_s of the family of polynomials of degree $e(s)$ with a singular point of type s in a neighborhood of z is smooth and has an expected dimension (i.e. $h^1(\mathbb{C}P^n, \mathcal{J}_{X^S_z/\mathbb{C}P^n}(e(s))) = 0$).

Lemma 9 *Given singularity types $s_1, ..., s_r \in \mathcal{S}$ and a hypersurface $W' \in |W|$ having r ordinary singular points of multiplicities $e(s_1)+1$, ..., $e(s_r)+1$, respectively, as its only singularities and satisfying*

$$h^1(V, \mathcal{J}_{X(W')/V} \otimes \mathcal{O}_V(W)) = 0 \ , \tag{22}$$

where $X(W') = \bigcup_{z \in \mathrm{Sing}(W')} X_z^{sqh}$, then there exists $W'' \in |W|$ having r singular points of types $s_1, ..., s_r$, respectively, as its only singularities.

Proof. As we saw in the proof of Theorem 1, condition (22) means that, given coordinates in neighborhoods of the points $\mathrm{Sing}(W') = \{z_1, ..., z_r\}$, the jets $\mathrm{jet}_{e(s_i)+1}(W'_i)$ (where W'_i is the image of W' in $\mathcal{O}_{V,z_i}$), are homogeneous forms of degrees $e(s_i)+1$, $i = 1, ..., r$, and, in addition, any variations of the jets $\mathrm{jet}_{e(s_i)}(W'_i)$, $i = 1, ..., r$, are simultaneously realizable, when varying W' in $|W|$. So, first, we deform W' into $\widetilde{W'} \in |W|$ such that $\mathrm{jet}_{e(s_i)}(\widetilde{W'_i})$ is the leading form of the polynomial $F_{s_i}(\mathbf{x})$ (which can be assumed nondegenerate without loss of generality), $i = 1, ..., r$. Then we apply Theorem 1, since X in (22) is $\bigcup_z \widetilde{X}_z^{sqh}$ for $\widetilde{W'}$, and the deformation models F_{s_i}, $i = 1, ..., r$, are $\mathcal{S}$-transversal. $\square$

6.2 Multidimensional generalization of a theorem of Hirschowitz and projective hypersurfaces with prescribed isolated singularities

Let X be the zero-dimensional subscheme of $\mathbb{C}P^n$, $n \geq 1$, concentrated at distinct points $z_1, ..., z_r \in \mathbb{C}P^n$ and defined at z_i by the power $\mathfrak{m}_i^{m_i}$ of the maximal ideal $\mathfrak{m}_i \subset \mathcal{O}_{\mathbb{C}P^n, z_i}$, $i = 1, ..., r$. We call X *the scheme of fat points $z_1, ..., z_r$ with multiplicities $m_1, ..., m_r$, respectively.* If $r = 1$, or $n = 1$ then the condition $m_1 + ..., m_r \leq d+1$ is necessary and sufficient for

$$h^1(\mathbb{C}P^n, \mathcal{J}_X(d)) = 0 \ . \tag{23}$$

We shall give a sufficient condition for (23) for $r \geq 2$, $n \geq 2$. Given $k \geq 1$, put

$$M(n, 2k) = 2 \cdot \binom{k+n}{n} \ , \quad M(n, 2k-1) = \binom{k+n}{n} + \binom{k+n-1}{n} \ .$$

Proposition 10 *If $z_1, ..., z_r$ are generic points in $\mathbb{C}P^n$ and*

$$\sum_{i=1}^{r} \binom{m_i + n - 1}{n} < M(n, d) \tag{24}$$

then (23) holds true.

Remark 6 (1) For $n = 2$ Proposition 10 coincides with Hirschowitz's theorem [16].

(2) Estimate (24) is sharp for $r \geq 2$ in the following sense. Equality in (24) can be realized by the scheme of two points z_1, z_2 of multiplicities $m_1 = m_2 = k + 1$ if $d = 2k$, or $m_1 = k + 1$, $m_2 = k$ if $d = 2k - 1$. In both the cases $m_1 + m_2 = d + 2$ which implies $h^1(\mathbb{C}P^n, \mathcal{J}_X(d)) > 0$, what one gets restricting the sheaf $\mathcal{J}_X(d)$ on the straight line through z_1, z_2.

Proposition 11 *Under the condition of Proposition 10, there exists a hypersurface of degree d in $\mathbb{C}P^n$ which has ordinary singularities of multiplicities $m_1, ..., m_r$ at $z_1, ..., z_r$, respectively, and no singular points in $\mathbb{C}P^n \backslash \{z_1, ..., z_r\}$.*

Corollary 3 *Given singularity types $s_1, ..., s_r \in \mathcal{S}$, for any integer d satisfying*

$$\sum_{i=1}^{r} \binom{e(s_i) + n}{n} < M(n, d) \ ,$$

there exists a hypersurface of degree d in $\mathbb{C}P^n$ which has r singular point of types $s_1, ..., s_r$, respectively, as its only singularities. Such a hypersurface can be chosen irreducible if $n \geq 3$, or

$$n = 2, \quad \sum_{i=1}^{r} \binom{e(s_i) + 2}{2} < \frac{d^2 + 6d - 1}{4} - \left[\frac{d}{2}\right] \ .$$

Straightforward from Lemma 9, Proposition 11, and [12], Lemma 5.7 (irreducibility condition for $n = 2$).

6.3 Proof of Proposition 10

Due to the semicontinuity of h^1, the statement of Proposition 10 follows from the next claim: given nonnegative $r' \leq r$ such that $z_1, ..., z_{r'}$ are generic points in the hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$, $z_{r'+1}, ..., z_r$ are generic points in $\mathbb{C}P^n \backslash \mathbb{C}P^{n-1}$, and $m_1, ..., m_r$ satisfy (24) and

$$\sum_{i=1}^{r'} \binom{m_i + n - 2}{n - 1} < M(n - 1, d) \ , \tag{25}$$

then (23) holds true.

We prove the latter claim by the double induction on d and n. For $n \leq 2$ it is proven in [16]. For $d \leq 1$ it is trivial. So, assume that $n \geq 3$, $d \geq 2$.

Step 1. First, we consider the case $r' = r$, where (25) turns into

$$\sum_{i=1}^{r} \binom{m_i + n - 2}{n - 1} < M(n-1, d) \ . \tag{26}$$

Note that this particular claim implies Proposition 10 as well.

We have the following exact sequence of sheaves

$$0 \to \mathcal{J}_Y(d-1) \to \mathcal{J}_X(d) \to \mathcal{J}_Z(d) \to 0$$

which implies the exact sequence

$$H^1(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) \to H^1(\mathbb{C}P^n, \mathcal{J}_X(d)) \to H^1(\mathbb{C}P^{n-1}, \mathcal{J}_Z(d)) \ , \tag{27}$$

where $Y = X : \mathbb{C}P^{n-1}$ is the scheme of fat points $z_1, ..., z_r$ in $\mathbb{C}P^n$ of multiplicities $m_1 - 1, ..., m_r - 1$, respectively, $Z = X \cap \mathbb{C}P^{n-1}$ is the scheme of fat points $z_1, ..., z_r$ in $\mathbb{C}P^{n-1}$ of multiplicities $m_1, ..., m_r$, respectively. By the induction assumption and (26), $h^1(\mathbb{C}P^{n-1}, \mathcal{J}_Z(d)) = 0$. Hence, to show (23), one needs $h^1(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) = 0$. The latter relation follows from the induction assumption as well, provided

$$\sum_{i=1}^{r} \binom{m_i + n - 2}{n} < M(n, d-1) \ , \quad \sum_{i=1}^{r} \binom{m_i + n - 3}{n - 1} < M(n-1, d-1) \ . \tag{28}$$

Indeed,

$$\binom{m_i + n - 2}{n} = \binom{m_i + n - 1}{n} \cdot \frac{m_i - 1}{m_i + n - 1} \le \binom{m_i + n - 1}{n} \cdot \frac{d - 1}{d + n - 1} \ .$$

We show that $M(n, d-1)/M(n, d) \ge (d-1)/(d+n-1)$: if $d = 2k$, $k \ge 1$, then

$$\frac{M(n, d-1)}{M(n, d)} \ge \frac{2k - 1}{2k + n - 1} \quad \iff$$

$$(2k + n - 1) \cdot \left(\binom{k + n}{n} + \binom{k + n - 1}{n} \right) \ge 2(2k - 1) \cdot \binom{k + n}{n} \quad \iff$$

$$(2k + n - 1) \cdot \binom{k + n - 1}{n} \ge (2k - n - 1) \cdot \binom{k + n}{n} \quad \iff$$

$$(2k + n - 1) \ge (2k - n - 1) \frac{k + n}{k} \quad \iff \quad n^2 + n \ge 0 \ ,$$

which is true, and if $d = 2k + 1$, $k \ge 1$, then

$$\frac{M(n, d-1)}{M(n, d)} \ge \frac{2k - 1}{2k + n - 1} \quad \iff$$

$$(2k + n) \cdot \binom{k + n}{n} \ge 2k \cdot \binom{k + n + 1}{n} \quad \iff$$

$$(2k + 2n) \ge 2k \cdot \frac{k + 1 + n}{k + 1} \quad \iff \quad n \ge 0,$$

which is true. Similarly one derives the second inequality in (28).

Step 2. Assume that $r' < r$, and

$$\sum_{i=1}^{r'} \binom{m_i+n-2}{n-1} \ge M(n,d) - M(n,d-1). \tag{29}$$

Once again we have the exact sequence (27) with $Y \subset \mathbb{C}P^n$ being the scheme of fat points $z_1, ..., z_r$ of multiplicities $m_1 - 1, ..., m_{r'} - 1, m_{r'+1}, ..., m_r$, respectively, and $Z \subset \mathbb{C}P^{n-1}$ being the scheme of fat points $z_1, ..., z_{r'}$ of multiplicities $m_1, ..., m_{r'}$, respectively. By (25) and the induction assumption, $h^1(\mathbb{C}P^{n-1}, \mathcal{J}_Z(d)) = 0$. Since by (29)

$$\sum_{i=1}^{r'} \binom{m_i+n-2}{n} + \sum_{i=r'+1}^{r} \binom{m_i+n-1}{n} = \sum_{i=1}^{r} \binom{m_i+n-1}{n} - \sum_{i=1}^{r'} \binom{m_i+n-2}{n-1}$$

$$< M(n,d) - (M(n,d) - M(n,d-1)) = M(n,d-1) \ ,$$

and by (25)

$$\sum_{i=1}^{r'} \binom{m_i+n-3}{n-1} \le M(n-1,d-1) \ ,$$

which one derives as the second inequality in (28), the induction assumption for the scheme Y and degree $d-1$ implies $h^1(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) = 0$, that together with (27) completes the proof of (23).

Step 3. Assume that $r' < r$, $d = 2k$, $k \ge 1$, and

$$\sum_{i=1}^{r'} \binom{m_i+n-2}{n-1} < M(n,d) - M(n,d-1) = \binom{k+n-1}{n-1} \ . \tag{30}$$

If one of $m_{r'+1}, ..., m_r$, say $m_{r'+1}$, does not exceed $k+1$, then one can specialize the point $z_{r'+1}$ into a generic point in $\mathbb{C}P^{n-1} \backslash \{z_1, ..., z_{r'}\}$. Condition (25) for $z_1, ..., z_{r'+1}$ still holds:

$$\sum_{i=1}^{r'+1} \binom{m_i+n-2}{n-1} < \binom{k+n-1}{n-1} + \binom{k+n-1}{n-1} = M(n-1,d) \ ,$$

hence one arrives either to the situations of Step 1 or 2, or in the beginning of Step 3.

If one of $m_{r'+1}, ..., m_r$, say m_r, exceeds $k+2$, then one has

$$\binom{m_r+n-2}{n-1} \ge \binom{k+n}{n-1} > \binom{k+n-1}{n-1} = M(n,d) - M(n,d-1) \ . \tag{31}$$

Let $L \subset \mathbb{C}P^n$ be a generic hyperplane through z_r. The one has the exact sequence

$$H^1(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) \to H^1(\mathbb{C}P^n, \mathcal{J}_X(d)) \to H^1(L, \mathcal{J}_Z(d)) \ , \tag{32}$$

where $Y = X : L \subset \mathbb{C}P^n$ is the scheme of fat points $z_1, ..., z_r$ of multiplicities $m_1, ..., m_{r-1}, m_r - 1$, respectively, $Z = X \cap L \subset L$ is the scheme of the fat point z_r of multiplicity m_r. Since $m_r \le d$, we have $h^1(\mathbb{C}P^{n-1}, \mathcal{J}_Z(d)) = 0$. Since by (31)

$$\sum_{i=1}^{r-1} \binom{m_i + n - 1}{n} + \binom{m_r + n - 2}{n} = \sum_{i=1}^{r} \binom{m_i + n - 1}{n} - \binom{m_r + n - 2}{n-1}$$

$$< M(n,d) - (M(n,d) - M(n,d-1)) = M(n,d-1) \ ,$$

and by (30)

$$\sum_{i=1}^{r'} \binom{m_i + n - 2}{n-1} < \binom{k+n-1}{n-1} < \binom{k+n-1}{n-1} + \binom{k+n-2}{n-1} = M(n-1,d-1) \ ,$$

by the induction assumption $h^1(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) = 0$, that together with (32) completes the proof of (23).

Step 4. Assume that $r' < r$, $d = 2k+1$, $k \ge 1$, and

$$\sum_{i=1}^{r'} \binom{m_i + n - 2}{n-1} < M(n,d) - M(n,d-1) = \binom{k+n}{n-1} \ . \tag{33}$$

If, for some $j = r'+1, ..., r$,

$$\sum_{i=1}^{r'} \binom{m_i + n - 2}{n-1} + \binom{m_j + n - 2}{n-1} < M(n-1,d) \ ,$$

then one specializes the point z_j on $\mathbb{C}P^{n-1}$ and comes to Steps 1,2, or 3, or in the beginning of Step 4.

Assume that for any $j = r'+1, ..., r$

$$\sum_{i=1}^{r'} \binom{m_i + n - 2}{n-1} + \binom{m_j + n - 2}{n-1} \ge M(n-1,d) \ . \tag{34}$$

By (33) this means that

$$\binom{m_j + n - 2}{n-1} > M(n-1,d) - \binom{k+n}{n-1} = \binom{k+n+1}{n-1} \ , \tag{35}$$

hence $m_j \ge k+3$. The latter inequality and (25) imply that

$$\binom{k+n+2}{n} < M(n,d) = \binom{k+n+1}{n} + \binom{k+n}{n}$$

$$\Longleftrightarrow \quad n(n+k+1) < (k+2)(k+1) \quad \Longrightarrow \quad n \le k \ . \tag{36}$$

Take a generic hyperplane $L \subset \mathbb{C}P^n$ through z_r and consider the exact sequence (32). As in Step 3, $h^1(\mathbb{C}P^{n-1}, \mathcal{J}_Z(d)) = 0$. By (35),

$$\sum_{i=1}^{r'} \binom{m_i+n-1}{n} + \binom{m_r+n-2}{n} = \sum_{i=1}^{r'} \binom{m_i+n-1}{n} + \binom{m_r+n-2}{n}$$

$$-\binom{m_r+n-2}{n-1} < M(n,d) - \binom{k+n+1}{n-1} < M(n,d) - \binom{k+n}{n-1} = M(n,d-1) \ .$$

In addition, by (33)

$$\sum_{i=1}^{r'} \binom{m_i+n-2}{n-1} < \binom{k+n}{n-1} \le M(n-1,d-1) = 2 \cdot \binom{k+n-1}{n-1} \ .$$

The latter inequality follows from

$$\binom{k+n}{n-1} \le 2 \cdot \binom{k+n-1}{n-1} \quad \Longleftrightarrow \quad \binom{k+n-1}{n-2} \le \binom{k+n-1}{n-1} \ ,$$

which is true, because $n \le k$ in (36). Hence by the induction assumption for the scheme Y and degree $d-1$, $h^1(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) = 0$, that together with (32) completes the proof of (23).

6.4 Proof of Proposition 11

The required statement follows from the next one: given nonnegative $r' \le r$ such that $z_1, ..., z_{r'}$ are generic points in the hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$, $z_{r'+1}, ..., z_r$ are generic points in $\mathbb{C}P^n \backslash \mathbb{C}P^{n-1}$, and $m_1, ..., m_r$ satisfy (24) and (25), then there exists $F \in H^0(\mathbb{C}P^n, \mathcal{J}_X(d)) \backslash \{0\}$ such that the hypersurface $\{F = 0\} \subset \mathbb{C}P^n$ has ordinary singularities of multiplicities $m_1, ..., m_r$ at $z_1, ..., z_r$, respectively, and no singular points in $\mathbb{C}P^n \backslash \{z_1, ..., z_r\}$.

Indeed, given a family of r-tuples $z_1(t), ..., z_r(t) \in \mathbb{C}P^n$, $t \in (\mathbb{C}, 0)$, such that, for $t \ne 0$, $z_1(t), ..., z_r(t)$ are generic points in $\mathbb{C}P^n$, and, for $t = 0$, $z_1(0), ..., z_{r'}(0)$ are generic points in $\mathbb{C}P^{n-1}$, $z_{r'+1}(0), ..., z_r(0)$ are generic points in $\mathbb{C}P^n \backslash \mathbb{P}^{n-1}$, by Proposition 10 $h^0(\mathbb{C}P^n, \mathcal{J}_{X(t)}(d)) = \text{const} > 0$, $t \in (\mathbb{C}, 0)$. Hence any element $F_0 \in H^0(\mathbb{C}P^n, \mathcal{J}_{X(0)}(d))$ can be included in a family $F_t \in H^0(\mathbb{C}P^n, \mathcal{J}_{X(t)}(d))$, $t \in (\mathbb{C}, 0)$. Hence F_t possesses the required properties for almost all t if F_0 does.

Then one proves the claim stated above along the same plan as in the proof of Proposition 10. In all the cases we have $h^1(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) = 0$, hence the exact sequence

$$0 \to H^0(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) \to H^0(\mathbb{C}P^n, \mathcal{J}_X(d)) \to H^0(L, \mathcal{J}_Z(d)) \to 0,$$

where L stands for $\mathbb{C}P^{n-1}$, or a generic hyperplane through z_r, and Y, Z have the corresponding meaning. So, we have the canonical decomposition

$$H^0(\mathbb{C}P^n, \mathcal{J}_X(d)) = L \cdot H^0(\mathbb{C}P^n, \mathcal{J}_Y(d-1)) \oplus H^0(L, \mathcal{J}_Z(d)) \ .$$

The induction step is as follows: if $F_1 \in H^0(\mathbb{C}P^n, \mathcal{J}_Y(d-1))$ defines a hypersurface with ordinary singularities at $z_1, ..., z_r$ of multiplicities $m_1 - 1, ..., m_{r'} - 1, m_{r'+1}, ..., m_r$, respectively, and without singularities in $\mathbb{C}P^n \backslash \{z_1, ..., z_r\}$, and $F_2 \in H^0(L, \mathcal{J}_Z(d))$ defines a hypersurface in L with ordinary singularities at $z_1, ..., z_{r'}$ of multiplicities $m_1, ..., m_{r'}$, respectively, and without singularities in $L \backslash \{z_1, ..., z_{r'}\}$, then, for a generic $\lambda, \mu \in \mathbb{C}$, $F = \lambda F_1 L + \mu F_2 \in H^0(\mathbb{C}P^n, \mathcal{J}_X(d))$ defines in $\mathbb{C}P^n$ a hypersurface having the required properties.

7 Deformation of plane real algebraic curves

Nonsingular deformations (smoothings) of real plane singular curves are an essential source of isotopy types of nonsingular plane curves [30, 31, 32]. Original Viro's method allows simultaneous lower nonsingular deformations of at most three Newton nondegenerate singular points, which should be in a regular position with respect to the projective coordinate axes. Few known constructions which cannot be done by the standard Viro method appeared to be particular cases of the generalization of Viro's method presented here.

Example 1 For the first time, a non-standard Viro method was used in the construction of a curve of degree 8 of type $< 4 + 3 < 5 >>$ [23]. This construction is based on Lemma [23] which a particular case of Lemma 3(3): the restrictions to singular point invariants are the same, whereas the statement concerns only nonsingular deformations.

Example 2 Two other applications of a non-standard Viro method were proposed in [6, 19]. In particular, Chevallier [6] constructed nonsingular curves of degree 8 with 22 ovals of new isotopy types: $< 16+1 < 2+1 < 2 >>$, $13+1 < 2+1 < 5 >>>$, $2 + 1 < 2 + 1 < 16 >>>$, $< 5 + 1 < 2 + 1 < 13 >>>$. The basic step in these construction is a deformation of a singular point of the union of three or four conics which have one common point, where they intersect with multiplicity 4. Note that such a singular point cannot be semiquasihomogeneous for a curve of degree 6 or 8 (but is topologically equivalent to $y^3 + x^{12}$ or $y^4 + x^{16}$, hence is GSQH). We shall show that this is a very special case covered by Lemma 3(4):

Lemma 12 *Let real conics $C_1, ..., C_r$ have one common point z and $(C_i \cdot C_j)_z = 4$, $1 \le i < j \le r$. Then any nonsingular lower deformation of the GSQH singular point $z \in C_1 ... C_r$ can be realized when varying $C_1 ... C_r$ in the space of curves of degree $2r$.*

Proof. By Lemma 6 a nonsingular lower deformation is transversal. In the notation of Lemma 3, $S = S_1 = S_3 = S_4 = S_5 = \emptyset$, $S_2 = \{z\}$, C_i has only one local branch Q_i centered at z, and $b(W, Q_i) = 4$. Hence inequality (21) holds for any $i = 1, ..., r$:

$$b(W, Q_i) = 4 < 6 = 3 \cdot \deg W_i \quad \square$$

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