FREE LOOP SPACES, POWER MAPS AND K THEORY

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ABSTRACT. In the first part of this paper we discuss a few concepts and results about the algebraic topology in characteristic zero of the free loop space, in analogy with the topology (geometry) of algebraic varieties over a finite field. In this analogy the free loop space plays the role of the extension of the variety over the algebraic closure of the field. In the second part we use the "differential calculus " on the free loop space of a smooth manifold to provide a graded vector space valued homotopy functor SH^* whose restriction to 1-connected spaces unifies the Atiyah-Hirzebruch topological K-theory and Waldhausen algebraic K-theory.

Given a topological space X, denote by X^{S^1} the space of continuous maps (free loops) $\alpha : S^1 \to X$, equipped with the compact open topology. Here $S^1 := \{z \in C||z| = 1\}$. This space will be referred to as the free loop space of X. This space can be viewed as an extension of X by identifying X to the subset of X^{S^1} consisting of constant maps (constant loops). The group of orientation preserving isometries of S^1, which can be identified to S^1 itself, acts continuously on X^{S^1}. The action denoted by μ, $\mu : S^1 \times X^{S^1} \to X^{S^1}$, is defined by $\mu(z', \alpha)(z) = \alpha(z'z)$. The fixed point set of the action consists of the subset of constant loops, hence can be identified with X. In addition to μ we have the continuous maps

$$\varphi_k : X^{S^1} \to X^{S^1}$$

defined by $\varphi_k(\alpha)(z) = \alpha(z^k)$ and referred in the title of this paper as the *power maps*. Notice that the fixed point set of φ_k, for $k \geq 2$ consists also of the constant loops. The action μ and the maps φ_k combine into a continuous action $\overline{\mu} : \mathcal{M} \times X^{S^1} \to X^{S^1}$ of the monoid $\mathcal{M}$ whose underlying set is $N \times S^1$ and multiplication is given by

$$(n, z_1) * (m, z_2) = (nm, z_1^m z_2).$$

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Here N denotes the natural numbers.

We refer to X^{S^1} as the space of oriented parametrized curves of X. A more interesting object is the space of oriented nonparametrized curves of X; this space is not accessible to traditional technics of analysis and geometry, but from the point of view of algebraic topology, it can be satisfactory approximated by $X^{S^1}//\mu$, the homotopy quotient (,as defined in section 1,) of X^{S^1} by μ.

The map φ_k is not equivariant with respect to μ but if one considers $\mu_k : S^1 \times X^{S1} \to X^{S1}$, the S^1-action on X^{S^1} defined by $\mu_k(z, \alpha) = \mu(z^k, \alpha)$, then $\varphi_k : (X^{S^1}, \mu_k) \to (X^{S^1}, \mu)$ is S^1-equivariant and induces $\varphi'_k : (X^{S^1}//\mu_k) \to (X^{S^1}//\mu)$. The commutative diagram

$$\begin{array}{ccc} S^1 \times X^{S^1} & \xrightarrow{\omega_k \times id} & S^1 \times X^{S^1} \\ {\scriptstyle \mu_k}\downarrow & & {\scriptstyle \mu}\downarrow \\ X^{S^1} & \xrightarrow{id} & X^{S^1} \end{array}$$

with $\omega_k(z) = z^k$ induces the map $\Omega_k : (X^{S^1}//\mu_k) \to (X^{S^1}//\mu)$ which is a rational homotopy equivalence, i.e the map $(\Omega_k)_Q : (X^{S^1}//\mu_k)_Q \to (X^{S^1}//\mu)_Q$ obtained by localization at "0" in the sense of Bousfield and Kan (cf [BK]), is a homotopy equivalence. Denote by $\tilde{\varphi}_k$ the pair

$$\tilde{\varphi}_k \equiv (X^{S^1}//\mu) \overset{\Omega_k}{\leftarrow} (X^{S^1}//\mu_k) \overset{\varphi'_k}{\to} (X^{S^1}//\mu)$$

also referred to as power map in $(X^{S^1}//\mu)$. ¿From the point of view of algebraic topology in characteristic zero one can regard $\tilde{\varphi}_k$ as the homotopy class $(\tilde{\varphi}_k)_Q = (\varphi'_k)_Q \cdot ((\Omega_k)_Q)^{-1}$. It is easy to verify the equality

$$(\tilde{\varphi}_{kn})_Q = (\tilde{\varphi}_k)_Q \cdot (\tilde{\varphi}_n)_Q = (\tilde{\varphi}_n)_Q \cdot (\tilde{\varphi}_k)_Q.$$

If $X = M$ is smooth manifold modeled over a finite or infinite dimensional Hilbert space we can restrict our attention to the smooth maps $\alpha : S^1 \to M$, and equip this set with the C^∞-topology. The resulting space, denoted by $M^{S^1}_{sm}$ or (when no confusion is possible) simply by M^{S^1}, is a smooth Frechet manifold and the action μ is a smooth action. Moreover, in this case $M^{S^1}_{sm}$ is S^1-homotopy equivalent to M^{S^1}. On a Frechet manifold one can work with differential forms, vector fields etc. and deRham Theorem holds.

Given $n \geq 2$, I will view the system $\{(X, X^{S^1}, \varphi_{n^k})\}$, $k \geq 0$, and the system $\{(X, X^{S^1}//\mu, \tilde{\varphi}_{n^k}), k \geq\}$ in analogy with (the affine picture and the projective picture of) $\{V, \bar{V}, \mathbb{F}^k, k \geq 0\}$, where V is an algebraic variety over the field with q elements F_q, q a prime number, $\bar{V}$ is the variety V considered over the field $\bar{F}_q$, the algebraic closure of F_q, and $\mathbb{F} : \bar{V} \to \bar{V}$ is the Frobenius map induced by the Frobenius isomorphism $\mathbb{F} : \bar{F}_q \to \bar{F}_q$, $\mathbb{F}(x) = x^q$. Then the Weil zeta function $Z(V, z)$ is defined by

$$\log Z(V, z) = \sum_{k \geq 1} \frac{Fix(\mathbb{F}^{-k})}{k} z^k.$$

It was shown by Grothendieck and Deligne cf [FK] that when V is smooth and projective, then $Fix(\mathbb{F}^{-k})$ can be calculated as a Lefschetz number in $l-addic$ cohomology $H^q_{et}(V;Q_l)$, (with l an arbitrary prime number prime to q,) i.e.

$$Lef(\mathbb{F}^{-k}) := (-1)^q Tr(\mathbb{F}^{-k})^q | H^q_{et}(V;Q_l).$$

In this paper we will discuss two results of algebraic topology in "characteristic zero" involving X^{S^1} and $X^{S^1}//\mu$. The first result, obtained in collaboration with Z.Fiedorowicz and W.Gajda, cf [BFG],[1] is about additional structures in the cohomology with coefficients in a field of characteristic zero, for example $\mathbb{C}$, of X^{S^1} and $X^{S^1}//\mu$.

If X is 1-connected we show that there exists a "weight" decompositions[2] of the cohomology and of the equivariant cohomology of X^{S^1},

$$H^*(X^{S^1};\mathbb{C}) = \sum_{r\geq 0} H^*(X^{S^1};\mathbb{C})(r)$$

$$H^*_{S^1}(X^{S^1};\mathbb{C}) := H^*(X^{S^1}//\mu;\mathbb{C}) = \sum_{r\geq 0} H^*(X^{S^1}//\mu;\mathbb{C})(r)$$

and explain their nature in terms of power maps. These decompositions exist also for the reduced cohomologies

$$\overline{H}^*(X^{S^1};\mathbb{C}) := coker\{H^*(pt;\mathbb{C}) \to H^*(X^{S^1};\mathbb{C})\}$$

$$\overline{H}^*_{S^1}(X^{S^1};\mathbb{C}) := coker\{H^*_{S^1}(pt;\mathbb{C}) \to H^*_{S^1}(X^{S^1};\mathbb{C})\}.$$

If in addition X is a finite complex and is equipped with a "formality " structure (defined below), then this formality structure induces a refinement of the above decompositions, called the Hodge decompositions:

$$\overline{H}^*(X^{S^1};\mathbb{C})(r) = \sum_{q\geq 0} \overline{H}^{*+q,-q}(X^{S^1};\mathbb{C})(r)$$

$$\overline{H}^*_{S^1}(X^{S^1};\mathbb{C})(r) = \sum_{q\geq 0} \overline{H}^{*+q,-q}_{S^1}(X^{S^1};\mathbb{C})(r)$$

We notice that a complex analytic Kähler structure on a smooth closed manifold M specifies a formality structure on M.

We show that the numbers

$$\chi^{-q}_H(r) := \sum_{p\geq 0} (-1)^p \dim \overline{H}^{p,-q}(X^{S^1};\mathbb{C})(r)$$

[1] A number of misprints made [BFG] hard to read. This paper among others corrects these misprints and provides a better formulation of the results in [BFG]

[2] in the literature on Hochschild and cyclic homology decomposition of the same nature as the weight decompositions are improperly called Hodge decompositions

and

$$\chi_E^{-q}(r) := \sum_{p\geq 0}(-1)^p \dim \overline{H}^{p,-q}(X^{S^1}//\mu;\mathbb{C})(r)$$

are well defined when X is a finite complex, depend only on the Betti numbers $\beta^i := \dim H^i(X;\mathbb{C})$, and do not "see" the multiplicative structure of $H^*(X:\mathbb{C})$, cf Theorem 0.2. This is a rather unexpected fact since the Betti numbers of X^{S^1} and $X^{S^1}//\mu$ depend on the multiplicative structure of $H^*(X;\mathbb{C})$ and even the alternated sum $\sum(-1)^k\chi^{-k}_{H(E)}$ when makes sense, for X not formal, might depend on the multiplicative structure of the cohomology of X.

The weight and Hodge decompositions suggest a homology "zeta function" which can be defined for any 1-connected finite complex with a formality structure. This homological "zeta function" is a formal series in two variables, and presents some analogy with the Weil zeta function of smooth variety over a finite field.

The second result is the construction of a new (graded) vector space valued homotopy functor, defined in this paper only for smooth manifolds M of the homotopy type of a CW-complex of finite type[3], which I used to call to "string cohomology"[4] of M and denote by $SH^*(M)$. This functor carries "Adams operations" which are induced by the power maps defined above and "unifies" the Atiyah Hirzebruch K-theory and the Waldhausen K-theory of M, (cf. Theorem 0.3) in a way consistent with Adams operations in K-theory. This functor is defined analytically in terms of infinite sequences of smooth S^1-invariant forms on the Frechet manifold M^{S^1}. The proof of Theorem 0.3 uses the infinite dimensionality of M^{S^1} in an essential way.

If one calls a tubular neighborhood of the constant loops in M^{S^1}, *small loops*, and the complement of the constant loops in M^{S^1} *large loops*, it turns out that in the "unification" mentioned above, the small loops are responsible for Atiyah Hirzebruch K-theory, the large loops for Waldhausen K-theory and all loops, small and large together, for SH^*. The sets of small and large loops are not disjoint. Now we want to make these results more precise.

The power maps $\varphi_k : X^{S^1} \to X^{S^1}$ and $(\tilde{\varphi}_k)_Q : (X^{S^1}//\mu)_Q \to (X^{S^1}//\mu)_Q$ induce endomorphisms

$$\Psi_k^* : H^*(X^{S^1};\mathbb{C}) \to H^*(X^{S^1};\mathbb{C}) \text{ and } \tilde{\Psi}_k^* : H^*_{S^1}(X^{S^1};\mathbb{C}) \to H^*_{S^1}(X^{S^1};\mathbb{C})$$

which are automorphisms when X is simply connected.

Proposition 0.1. *([BFG]) If X is 1-connected and of the homotopy type of a CW-complex of finite type then:*

1) The eigenvalues of Ψ_k^* and of $\tilde{\Psi}_k^*$ are among $k^0 = 1, k^1, k^2, \ldots$

[3] finite type= finitely many cells in each dimension

[4] I understand that M.Konsevici has already used the term "string cohomology" for another functor so the reader can read SH as "special cohomology"

2) The eigenspace corresponding to k^r is independent of k and therefore denoted by $H^*(X^{S^1};\mathbb{C})(r)$ resp. $H^*(X^{S^1}//\mu,\mathbb{C})(r)$.

3) $H^*(X^{S^1};\mathbb{C})(0) = H^*(X,\mathbb{C})$, $\quad H^{*-1}(X^{S^1}//\mu;\mathbb{C})(0) = H^*(X,\mathbb{C})$, and $H^r(X^{S^1}//\mu,\mathbb{C})(p) = H^r(X^{S^1};\mathbb{C})(p) = 0$, if $p \geq r+1$.

Proposition 0.1 will be proven for $X = M$ a smooth 1-connected smooth (Hilbert) manifold[5] by constructing the isomorphisms

$$\theta_H^* : \overline{H}^*(M^{S^1};\mathbb{C}) \to \overline{H}H_{-*}(\Omega^*(M), d_M^*)$$

$$\theta_E^* : \overline{H}^*(M^{S^1}//\mu;\mathbb{C}) \to \overline{H}C_{-*+1}(\Omega^*(M), d_M^*)$$

between the reduced cohomology resp. reduced equivariant cohomology of M^{S^1} and Hochschild resp. cyclic homology of $(\Omega^*(M), d_M^*)$, the deRham algebra of M. This is a differential graded algebra which is commutative in the graded sense. These isomorphisms intertwine the endomorphisms Ψ_k^* and $\tilde{\Psi}_k^*$ with the Adams operations in Hochschild and cyclic homology of $(\Omega^*(M), d_M)$ (cf [BFG]). The operations Ψ_k^* and $\tilde{\Psi}_k^*$ are always defined for commutative algebras.

By a "formality" structure on a smooth manifold M we mean a triple consisting of a $\mathbb{C}$-commutative differential graded algebra $(\mathcal{A}^*, d_{\mathcal{A}}^*)$ and two morphisms $\alpha_1 : (\mathcal{A}^*, d_{\mathcal{A}}^*) \to (\Omega^*(M), d_M^*)$ and $\alpha_1 : (\mathcal{A}^*, d_{\mathcal{A}}^*) \to (H^*(M;\mathbb{C}), d^* = 0)$ which induce isomorphisms in cohomology. A complex analytic Kähler structure on the closed manifold M, provides a canonical "formality" structure on M cf [DGMS]. The existence of a formality structure is an homotopy invariant property.

The Hochschild or cyclic homology of a commutative graded algebra when viewed as a degree $+1$ commutative differential graded algebra $\mathcal{B}^*$ with differential equal to zero, as well as their reduced versions have natural decompositions

$$HH^*(\mathcal{B}^*) = \sum_{q\geq 0} HH^{*+q,-q}(\mathcal{B}^*), \quad HC^*(\mathcal{B}^*) = \sum_{q\geq 0} HC^{*+q,-q}(\mathcal{B}^*)$$

which diagonalize the Adams operations. Using the formality structure one can identify the reduced Hochschild and cyclic homologies of $(\Omega^*(M), d_M^*)$ and of $(H^*(X;\mathbb{C}), 0)$ and using the isomorphisms $\theta_{H,(E)}^*$ one obtains decompositions of $\overline{H}^*(X^{S^1};\mathbb{C})$ and $\overline{H}_{S^1}^*(X^{S^1};\mathbb{C})$ which diagonalize the endomorphisms Ψ_k^* and $\tilde{\Psi}_k^*$,

$$\overline{H}^*(X^{S^1};\mathbb{C})(r) = \sum_{q\geq 0} \overline{H}^{*+q,-q}(X^{S^1};\mathbb{C})(r)$$

$$\overline{H}_{S^1}^*(X^{S^1};\mathbb{C})(r) = \sum_{q\geq 0} \overline{H}_{S^1}^{*+q,-q}(X^{S^1};\mathbb{C})(r).$$

Denote by

$$\beta^i := \dim \overline{H}^i(X;\mathbb{C})$$

[5] any X satisfying the hypotheses of Proposition 0.1 is homotopy equivalent with such manifold

$$b^i_H := \dim \overline{H}^i(X^{S^1};\mathbb{C}),\quad b^i_E := \dim \overline{H}^i_{S^1}(X^{S^1};\mathbb{C})$$

$$b^i_H(r) := \dim \overline{H}^i(X^{S^1};\mathbb{C})(r),\quad b^i_E(r) := \dim \overline{H}^i_{S^1}(X^{S^1};\mathbb{C})(r)$$

and by

$$b^{p,-q}_H(r) := \dim \overline{H}^{p,-q}(X^{S^1};\mathbb{C})(r),\quad b^{p,-q}_E(r) := \dim \overline{H}^{p,-q}_{S^1}(X^{S^1};\mathbb{C})(r).$$

While the Hodge decompositions depend on the formality structure, the numbers $b^{p,-q}_{H(E)}(r)$ do not and are the same for different formality structures.

Let

$$\mathcal{P}_H(z,\lambda,u) := \sum_{q,r,p\geq 0} b^{p,-q}_H(r) z^p \lambda^r u^q$$

and

$$\mathcal{P}_E(z,\lambda,u) := \sum_{q,r,p\geq 0} b^{p,-q}_E(r) z^p \lambda^r u^q.$$

Following P. Hanlon [H], denote by[6] $\bar{P}_X(z) = \sum_{i\geq 0} \beta_i(X) z^i$ and form the power series

$$Q_H(z,\lambda) = (1+\bar{P}_X(z))\prod_l (1+\bar{P}_X(z^l))^{-\epsilon(l)}$$

$$Q_E(z,\lambda) = \frac{\bar{P}_X(z)}{(1-\lambda)}(\prod_l (1+\bar{P}_X(z^l))^{-\epsilon(l)} - \lambda)$$

where

$$\epsilon(l) = \frac{1}{l}\sum_{d|l} \mu(d)\lambda^{l/d}.$$

Here $\mu(d)$ denotes the Möbius function $\mu : N \to \{+1,0,-1\}$ defined by $\mu(1)=1$, $\mu(1)=0$ if the prime factors decomposition $d = p_1p_2\dots p_s$ has at least two primes equal and $\mu(d) = (-1)^s$ otherwise.

Theorem 0.2. *(cf* [BFG]*) It M is* 1*-connected smooth (Hilbert) manifold of the homotopy type of a CW-complex of finite type with a formality structure, then:*
1) $\mathcal{P}_{H(E)}(z,\lambda,u) \in \mathbb{Z}[z][[\lambda,u]]$ *hence* $\mathcal{P}_H(-1,\lambda,u)$ *and* $\mathcal{P}_E(-1,\lambda,u)$ *are defined*
2) $\mathcal{P}_H(-1,\lambda,u) = Q_H(u,\lambda)$ *and* $\mathcal{P}_E(-1,\lambda,u) = Q_E(u,\lambda)$.

Theorem 0.2 implies that Euler Poincaré characteristics $\chi^{-q}_H(r)$ and $\chi^{-q}_E(r)$ are completely determined by the Betti numbers of X.

In the analogy mentioned above $\overline{H}^*_{S^1}(X^{S^1};\mathbb{C})$ is the analogue of the reduced algebraic K-theory of the variety V. There are reasons to view the vector spaces

$$\sum_{q,r\geq 0} \overline{H}^{p,-q}(X^{S^1};\mathbb{C})(r) \text{ and } \sum_{q,r\geq 0} \overline{H}^{p,-q}_{S^1}(X^{S^1};\mathbb{C})(r)$$

[6] Notice that $\beta_0(X) = 0$

as the analogue of the reduced $l-adic$ cohomology of V in degree p and since these vector spaces might be of infinite dimension it is convenient to write them as formal power series

$$\sum_{q,r\geq 0} \overline{H}^{p,-q}(X^{S^1};\mathbb{C})(r)\lambda^r u^q, \text{ and } \sum_{q,r\geq 0} \overline{H}^{p,-q}_{S^1}(X^{S^1};\mathbb{C})(r)\lambda^r u^q.$$

Define

$$\overline{Lef}^H(\varphi_n^{-k}) = \sum_{r,q,\geq 0,\ i\geq 0}\sum (-1)^i Tr(\Psi_n)^{-k}|_{\overline{H}^{i,-q}(X^{S^1})(r)}\lambda^r u^q =$$
$$\sum_{r,q,\geq 0,\ i\geq 0}\sum (-1)^i n^{(-kr)} b_H^{i,-q}(r)\lambda^r u^q$$
$$\overline{Lef}^E(\varphi_n^{-k}) = \sum_{r,q,\geq 0,\ i\geq 0}\sum (-1)^i Tr(\Psi_n)^{-k}|_{\overline{H}^{i,-q}_{S^1}(X^{S^1}(r)}\lambda^r u^q =$$
$$\sum_{r,q,\geq 0,\ i\geq 0}\sum (-1)^i n^{(-kr)} b_E^{i,-q}(r)\lambda^r u^q$$

and in analogy with Weil zeta function of a smooth projective variety over finite field introduce the formal power series

$$\log Z^{H(E)}(n,X,\lambda,u,z) = \sum_{r\geq 1}\frac{(Lef^{H(E)}(\varphi_n^{-1})^k)(\lambda,u)}{k}z^k = \sum \chi^{-q}_{H(E)}(r)\frac{\lambda^r z^k u^q}{n^{rk}k}$$

It can be shown that because the cohomology of X is finite dimensional in each dimension, given any q, $dimH^q(X^{S^1};\mathbb{C})$ and $dimH^q(X^{S^1}//\mu;\mathbb{C})$ are finite and therefore for all but finitely many pairs (p,r) the numbers $b_H^{p,-q}(r)$ and $b_E^{p,-q}(r)$ are zero. This implies that $\log Z^{H(E)}(n,X,\lambda,u,z)$ is a formal power series in u,z with coefficients polynomials in λ and therefore one can evaluate at $\lambda=1$ and obtain the formal power series in u,z,

$$\log Z^H(n,X,u,z) := \log Z^H(n,X,1,u,z),$$

$$\log Z^E(n,X,u,z) := \log Z^E(n,X,1,u,z).$$

These series are called the" homological Zeta power series" of the formal space X, and are completely determined by the Betti numbers of X. So far we know very little about the analyticity of these formal power series as functions of two variables and about their geometric relevance, but we expect that they have partition functions interpretations.

Suppose M is a smooth Hilbert manifold. As we have already indicated M^{S^1} is a Frechet smooth manifold of infinite dimension. Denote by L the smooth vector field provided by the "tangents to the orbits" of the smooth action μ, and by $i_L : \Omega^*(M^{S^1}) \to \Omega^{*-1}(M^{S^1})$ the contraction with respect to L. Let $(\Omega^*_{inv}(M^{S^1}), d^*_{M^{S^1}})$ be the differential graded algebra of S^1-invariant differential forms. We will always consider complex valued differential forms. The contraction i_L leaves the subalgebra

$\Omega^*_{inv}(M^{S^1})$ invariant and restricts to a derivation of degree -1 in this subalgebra. Let us form the cochain complex

$$\mathbf{C}^r_- = \prod_{k\geq 0} \Omega^{r+2k}_{inv}(M^{S^1}),\ D^r_- : C^r_- \to C^{r+1}_-$$

with $D_- = \bar{d} + \bar{i}_L$, $\bar{i}_L(\omega_r, \omega_{r+2}, \dots) = (i_L\omega_{r+2}, i_L\omega_{r+4} \dots)$, and $\bar{d}(\omega_r, \omega_{r+2} \dots) = (d\omega_r, d\omega_{r+2}, \dots)$. Denote by $SH^*(M)$ the cohomology of $(\mathbf{C}^*_-, D^*_-)$. Let $\mathbf{K}^r(X)$ denote the Atiyah Hirzebruch complex K-theory and $\tilde{A}_r(X)$ denotes the reduced Waldhausen K-theory (cf [B2]) of X in degree r.

Theorem 0.3. *1) The assignment $M \rightsquigarrow SH^*(M)$ is a Z_+-graded vector space valued homotopy functor defined on the category of smooth manifolds of the homotopy type of CW complexes of finite type and of smooth maps.*

2)There exist natural transformations

$$ch_r : \mathbf{K}^r(X) \to SH^r(X)$$

and, between the functors SH^r and $\tilde{A}_r$ when restricted to the full subcategory of 1-connected manifolds, the natural transformations

$$C_r : SH^r(X) \to Hom(\tilde{A}_{r-1}(X), \mathbb{C})$$

so that

$$0 \to \mathbf{K}^r(X) \otimes \mathbb{C} \to SH^r(X) \to Hom(\tilde{A}_r(X), \mathbb{C}) \to 0$$

is a short exact sequence for any r.

3) The power maps ϕ_n induce endomorphisms $S\Phi^r_n$ in $SH^r(X)$ so that ch_r intertwines the Adams operations in topological K-theory with $S\Phi^r_n$.[7]

The proof of these results are based on the following facts:
Given a minimal model of M (in the sense of rational homotopy theory, cf [L] or [DGMS]), one can construct explicit minimal models for M^{S^1} and $M^{S^1}//\mu$ as well as models for power maps cf [VS], [VB] and [BFG]. Using these models one can show that the Hochschild and cyclic homology of $(\Omega^*(M), d^*_M)$ are isomorphic to the cohomology of M^{S^1} and of $M^{S^1}//\mu$ by isomorphisms which intertwine the Adams operations in Hochschild and cyclic homology with the automorphisms induced by the power maps φ_k and $\tilde{\varphi}_k$. This fact permits the verification of Propositions 0.1, and is used in the proof of Theorem 0.3. Theorem 0.2 requires in addition results about partial Euler Poincaré characteristics in Hochschild and cyclic homology of commutative graded algebras due to Ph. Hanlon. The proof of Theorem 0.3 also needs the identifications of the reduced equivariant cohomology of X^{S^1} with $Hom(\tilde{A}_r(X), \mathbb{C})$ established in [B2].

In section 1 we introduce the necessary concepts and outline the main steps in the proof of Propositions 0.1 and Theorem 0.2 and in section 2 we outline the proof of Theorem 0.3.

[7]it is possible to define Adams operations in the rational Waldhausen algebraic K-theory and then C_r intertwines $S\Phi^r_n$ with the dual of the Adams operations in rational Waldhausen K-theory.

Section 1

1: An $\mathcal{M}$-space (smooth manifold) consists of a space (smooth manifold) Y together with a continuous (smooth) action $\overline{\mu} : \mathcal{M} \times Y \to Y$. To give an $\mathcal{M}-$ action $\overline{\mu}$ is equivalent to give an S^1- action $\mu : S^1 \times Y \to Y$ and the continuous (smooth) maps $\varphi_k : Y \to Y, k \geq 1$, which satisfy

$$\varphi_1 = id, \;\; \varphi_{kr} = \varphi_k \cdot \varphi_r \tag{1.1}$$

$$\varphi_k(\mu(e^{ik\theta}, x)) = \mu(e^{i\theta}, \varphi_k(x)). \tag{1.2}$$

The relationship between $\overline{\mu}$, μ and φ_k is provided by the equality

$$\overline{\mu}((k, e^{i\theta}), x) = \varphi_k(\mu(e^{i\theta}, x)) \tag{1.3}$$

The action $\mu : S^1 \times Y \to Y$ induces the fibration (smooth bundle)

$$Y//\mu = ES^1 \times_{S^1} Y \xrightarrow{\pi} BS^1 \tag{1.4}$$

with fiber Y, where ES^1 is the unit sphere $S^\infty = \{v \in l_2 | \sum_{i \geq 1} |z_i|^2 = 1\}$ in the infinite dimensional complex Hilbert space $l_2 \equiv \{v = (z_1, \cdots, z_n, \cdots) | \sum_{i\geq 1} |z_i|^2 < \infty\}$ and BS^1 is the quotient space $S^\infty//\mu_0$ where $\mu_0 : S^1 \times S^\infty \to S^\infty$ is given by $\mu_0(e^{i\theta}, v) = e^{i\theta}v$. Notice that ES^1 and BS^1 are smooth Hilbert manifolds and $p : ES^1 \to BS^1$, the canonical projection, is a smooth submersion. The action μ provides a canonical vector field L on Y, the "tangents to the orbits", which at any $x \in Y$ associates the tangent vector to the orbit through x. Denote by $i_L : \Omega^*(Y) \to \Omega^{*-1}(Y)$ the contraction with respect to L. The following two observations are important:

Observation 1.1: There exists a smooth 1-form $\gamma \in \Omega^1(S^\infty)$ so that:
a) γ is S^1- invariant
b) if L_0 denotes the canonical vector field for μ_0, then $i_{L_0}\gamma = 0$
c)$d\gamma = \tilde{u}$ is the pull back of a closed 2$-$form u on $S^\infty//\mu_0$ representing the Euler class of the S^1-principal bundle $S^\infty \to S^\infty//\mu_0$.

Any connection in the principal bundle $S^\infty \to S^\infty//\mu_0$ provides such γ and $\tilde{u}$.

Observation 1.2: If Y is a smooth manifold equipped with the smooth action $\mu : S^1 \times Y \to Y$ and L denotes the canonical vector field, then the smooth forms $\Omega^*(Y//\mu)$ identifies to the smooth forms on $S^\infty \times Y$ which are invariant with respect to the diagonal action on $S^\infty \times Y$ and satisfy $i_{L_0+L}(\omega) = 0$. Note that $L_0 + L$ is the canonical vector field associated to the diagonal action of μ_0 and μ.

Let $\omega_k : S^1 \to S^1$ denote the group homomorphism induced by $\omega_k(e^{i\theta}) = (e^{ik\theta})$ and denote by $E\omega_k : S^\infty \to S^\infty$ the smooth map defined by

$$E\omega_k(z_1, \cdots, z_n \cdots) = \left(\frac{z_1^k}{(\sum_i |z_i|^{2k})^{1/2}}, \cdots, \frac{z_n^k}{(\sum_i |z_i|^{2k})^{1/2}}, \cdots \right).$$

Since $E\omega_k(\mu_0(e^{i\theta}, v)) = \mu_0(\omega_k(e^{i\theta}), E\omega_k(v))$, the smooth map $E\omega_k$ induces the smooth map $B\omega_k : BS^1 \to BS^1$.

The commutative diagram

$$\begin{array}{ccccc} S^1 \times Y & \overset{\omega_n \times id}{\leftarrow} & S^1 \times Y & \overset{id \times \varphi_k}{\rightarrow} & S^1 \times Y \\ \downarrow \mu_k & & \downarrow \mu_{nk} & & \downarrow \mu_k \\ Y & \overset{id}{\leftarrow} & Y & \overset{\varphi_k}{\rightarrow} & Y. \end{array}$$

induces the commutative diagram

$$(1.5) \qquad \begin{array}{ccccc} Y//\mu_n & \overset{\Omega^n_{nk}}{\leftarrow} & Y//\mu_{nk} & \overset{\varphi'^n_{nk}}{\rightarrow} & Y//\mu_n \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ BS^1 & \overset{B\omega_n}{\leftarrow} & BS^1 & \overset{id}{\rightarrow} & BS^1. \end{array}$$

We put $\Omega^1_n := \Omega_n$ and $\varphi'^1_n = \varphi'_n$ and one can verify

$$(1.6) \qquad \Omega_n \cdot \Omega^n_{nk} = \Omega_{nk}, \quad \varphi'_n \cdot \varphi'^n_{nk} = \varphi'_{nk}, \quad \varphi'_n \cdot \Omega^n_{nk} = \Omega_k \cdot \varphi'^k_n k.$$

We denote by $\tilde{\varphi}_k$ the pair (Ω_k, φ'_k)

$$\tilde{\varphi}_k \equiv \{(Y//\mu) \overset{\Omega_k}{\leftarrow} (Y//\mu_n) \overset{\varphi'_k}{\rightarrow} (Y//\mu)\}.$$

In the rational category (of spaces localized at "0" in the sense of Bousfield Kan, cf [BK]), we regard $\tilde{\varphi}_k$ as the homotopy class $(\tilde{\varphi}_k)_Q = (\varphi'_k)_Q \cdot ((\Omega_k)_Q)^{-1}$. Using (1.6) one can verify that

$$(\tilde{\varphi}_{kn})_Q = (\tilde{\varphi}_k)_Q \cdot (\tilde{\varphi}_n)_Q = (\tilde{\varphi}_n)_Q \cdot (\tilde{\varphi}_k)_Q.$$

The fibration (1.4) induces the Gysin sequence

$$(1.7) \quad \to H^*(Y;\mathbb{C}) \xrightarrow{\delta^*} H^{*-1}(Y//\mu;\mathbb{C}) \xrightarrow{S^{*-1}} H^{*+1}(Y//\mu;\mathbb{C}) \xrightarrow{J^{*+1}} H^{*+1}(Y;\mathbb{C}) \to,$$

and the maps φ_k and $\tilde{\varphi}_k$ induce the endomorphisms $\Psi_k : H^*(Y;\mathbb{C}) \to H^*(Y;\mathbb{C})$ and $\tilde{\Psi}_k : H^*(Y//\mu;\mathbb{C}) \to H^*(Y//\mu;\mathbb{C})$. It is not hard to check that δ^* intertwines Ψ^*_k with $\tilde{\Psi}^{*-1}_k$, J^* intertwines $\tilde{\Psi}^*_k$ with Ψ^*_k and S^* intertwines $\tilde{\Psi}^*_k$ with $k\tilde{\Psi}^{*+2}_k$.

Recall that the cohomology of the homotopy quotient $Y//\mu$ is by definition the S^1-equivariant cohomology of Y with respect to the action μ and it is usually denoted by $H^*_{S^1}(Y,\mu;\mathbb{C})$. Denote by $\overline{H}^*(Y;\mathbb{C})$ and $\overline{H}^*_{S^1}(Y,\mu;\mathbb{C})$ the reduced cohomologies

$$\overline{H}^*(Y;\mathbb{C}) = coker(H(p) : H^*(pt;\mathbb{C}) \to H^*(Y;\mathbb{C}))$$

$$\overline{H}^*_{S^1}(Y,\mu;\mathbb{C}) = coker(H_{S^1}(p) : H^*_{S^1}(pt;\mathbb{C}) \to H^*_{S^1}(Y,\mu;\mathbb{C})).$$

where $p : Y \to pt$ is the map from Y to the space pt consisting of one point, and observe that the power maps φ_k and $(\tilde{\varphi}_k)_Q$ induce endomorphisms Ψ^*_k and $\tilde{\Psi}^*_k$ in reduced cohomology as well as reduced equivariant cohomology of Y. It is easy to check that if the $\mathcal{M}$-space Y has fixed points, then the Gysin sequence for reduced cohomologies remains exact.

2: Consider the category of commutative differential graded algebras with differential of degree +1 over the field of characteristic zero $\mathbb{C}$, abbreviated CDGA. The objects in this category are pairs $(\mathcal{A}^*, d^*_{\mathcal{A}})$, $\mathcal{A}^*$ a unital augmentable commutative differential graded algebra with differential $d^*_{\mathcal{A}}$ of degree +1, i.e

$$(1.8) \qquad d(a_1 \cdot a_2) = d(a_1) \cdot a_2 + (-1)^{dega_1} a_1 \cdot d(a_2), \quad d^{*+1}_{\mathcal{A}} d^*_{\mathcal{A}} = 0,$$

and the morphisms $f = f^* : (\mathcal{A}^*, d^*_{\mathcal{A}}) \to (\mathcal{B}^*, d^*_{\mathcal{B}})$ are degree zero linear maps which are compatible with the products, preserve the unit and intertwine the differentials $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$. (Recall that "augmentable" means that there exists morphisms $\epsilon : (\mathcal{A}^*, d^*_{\mathcal{A}}) \to (\mathcal{K}^*, 0)$ where $\mathcal{K}^*$ denotes the unital graded algebra whose components $\mathcal{K}^i = 0$ if $i \neq 0$ and $\mathcal{K}^0 = \mathbb{C}$. In order to lighten the notation we will sometimes write $(\mathcal{A}, d_{\mathcal{A}})$ instead of $(\mathcal{A}^*, d^*_{\mathcal{A}})$. The category CDGA is the algebraic analogue of the category of topological spaces and continuous maps and we can provide algebraic analogues for all previous concepts and constructions.

For a commutative differential graded algebra $(\mathcal{A}^*, d^*_{\mathcal{A}})$, the graded vector space $H^*(\mathcal{A}^*, d^*) = Ker(d^*)/Im(d^{*-1})$ is a commutative graded algebra whose multiplication is induced by the multiplication in $\mathcal{A}^*$. A morphism $f = \{f^*\} : (\mathcal{A}^*, d^*_{\mathcal{A}}) \to (\mathcal{B}^*, d^*_{\mathcal{B}})$ induces a degree zero linear map which is an algebra homomorphism $H^*(f) : H^*(\mathcal{A}^*, d^*_{\mathcal{A}}) \to H^*(\mathcal{B}^*, d^*_{\mathcal{B}})$. In analogy with topological spaces one has a concept of homotopy between morphisms of CDGA (cf [L] and [Ha]) and it can be shown that two homotopic morphisms induce the same linear maps in cohomology. A morphism f, so that $H^k(f)$'s are isomorphisms for all k, is called a quasi isomorphism.

An S^1-commutative differential graded algebra consists of a commutative differential graded algebra $(\mathcal{A}^*, d^*_{\mathcal{A}})$ as above together with a degree −1 differential $i^*_{\mathcal{A}} : \mathcal{A}^* \to \mathcal{A}^{*-1}$ which anticommutes with $d^*_{\mathcal{A}}$. This means that:

$$(1.9) \qquad i_{\mathcal{A}}(\omega_1 \cdot \omega_2) = i_{\mathcal{A}}(\omega_1) \cdot \omega_2 + (-1)^{deg\omega_1} \omega_1 \cdot i_{\mathcal{A}}(\omega_2), \quad i^{*+1}_{\mathcal{A}} i^*_{\mathcal{A}} = 0,$$

$$(1.10) \qquad i^{*+1}_{\mathcal{A}} d^*_{\mathcal{A}} + d^{*-1}_{\mathcal{A}} i^*_{\mathcal{A}} = 0.$$

An $\mathcal{M}$-commutative differential graded algebra consists of a commutative S^1- differential graded algebra
$(\mathcal{A}^*, d^*_{\mathcal{A}}, i^*_{\mathcal{A}})$ together with the morphisms $\phi_k = \{\phi^*_k\} : (\mathcal{A}^*, d^*) \to (\mathcal{A}^*, d^*_{\mathcal{A}})$, $k \geq 1$ which satisfy

$$(1.11) \qquad \phi_1 = id, \ \phi_{kr} = \phi_k \cdot \phi_r$$

$$(1.12) \qquad \phi_k \cdot i_{\mathcal{A}} = k i_{\mathcal{A}} \cdot \phi_k.$$

Denote by S^1-CDGA resp. $\mathcal{M}$-CDGA the categories of S^1 resp. $\mathcal{M}$-commutative differential graded algebras with the obvious morphisms. Observe that if $(\mathcal{A}^*, d^*_{\mathcal{A}}, i^*_{\mathcal{A}})$ resp. $(\mathcal{A}^*, d^*_{\mathcal{A}}, i^*_{\mathcal{A}}, \phi_k)$ is in S^1- CDGA resp. $\mathcal{M}-$ CDGA, then for any integer n, $(\mathcal{A}^*, d^*_{\mathcal{A}}, ni^*_{\mathcal{A}})$ resp. $(\mathcal{A}^*, d^*_{\mathcal{A}}, ni^*_{\mathcal{A}}, \phi_k)$, is in S^1- CDGA resp. $\mathcal{M}-$ CDGA. The CDGA $\mathcal{K}^*$ can be viewed as an $\mathcal{M}$-CDGA with $i^*_{\mathcal{K}} = 0$ and $\phi_k = id$, and then an S^1-CDGA as well. We will sometimes write $(\mathcal{A}, d_{\mathcal{A}}, i_{\mathcal{A}})$ instead of $(\mathcal{A}^*, d^*_{\mathcal{A}}, i^*_{\mathcal{A}})$.

A first example of $\mathcal{M}$- CDGA is given by an $\mathcal{M}$-smooth manifold consisting of a smooth manifold Y, a smooth action $\mu : S^1 \times Y \to Y$, and smooth maps $\varphi_k : Y \to Y$. The associated $\mathcal{M}$-DCGA consists of the differential graded algebra $(\Omega^*_{inv}(Y), d^*_Y)$, the degree -1 differential i_L given by contraction with respect to the canonical vector field L, and the morphisms ϕ_k's induced by the smooth maps φ_k. Observe also that while $(\Omega^*_{inv}(Y), d^*_Y, i^*_L)$ is the S^1-CDGA associated to the the S^1-space (Y, μ), $(\Omega^*_{inv}(Y), d^*_Y, ni^*_L)$ is the S^1-CDGA associated to (Y, μ_n). We will pay particular attention to the case $Y = M^{S^1}_{sm}$ where M is a smooth manifold.

A second example is provided by the following construction. Consider a free CDGA of the form $(\Lambda[V], d)$ where $V = \oplus_{i \geq 1} V_i$ is a graded vector space with $V_0 = 0$, and denote by $\overline{V} = \oplus_{i \geq 0} \overline{V}_i$ the graded vector space with $\overline{V}_i = V_{i+1}$. Equip the free commutative graded algebra $\Lambda[V \oplus \overline{V}]$ with the unique degree -1 derivation $\iota_V : \Lambda[V \oplus \overline{V}] \to \Lambda[V \oplus \overline{V}]$ defined by

$$\iota_V(v) = \overline{v}, \text{and } \iota_V(\overline{v}) = 0, \tag{1.13}$$

and extend d to the unique degree $+1$ derivation $\delta_n : \Lambda[V \oplus \overline{V}] \to \Lambda[V \oplus \overline{V}]$ defined by

$$\delta_n(v) = d(v), \text{and } \delta_n(\overline{v}) = -n\iota_V(d(v)). \tag{1.14}$$

Define $\phi_k : (\Lambda[V \oplus \overline{V}], \delta_n) \to (\Lambda[V \oplus \overline{V}], \delta_n)$ by the formulas

$$\phi_k(v) = v, \quad \phi_k(\underline{v}) = k\underline{v}. \tag{1.15}$$

It is easy to check that for any n, $(\Lambda[V \oplus \overline{V}], \delta_n, \iota_n = n\iota_V, \phi_k, k \geq 1)$ is a $\mathcal{M}$- CDGA. Notice that if $f : (\Lambda[V], d) \to (\mathcal{A}, d_{\mathcal{A}})$ is a morphism in CDGA and $(\mathcal{A}, d_{\mathcal{A}}, \iota_{\mathcal{A}})$ is in S^1- CDGA then f extends uniquely to the morphism $F_n : (\Lambda[V \oplus \overline{V}], \delta_n, \iota_n) \to (\mathcal{A}, d_{\mathcal{A}}, n\iota_{\mathcal{A}})$ in the category S^1- CDGA, defined by

$$F_n(v) = f(v), \; F_n(\overline{v}) = n\iota_{\mathcal{A}}(f(v)). \tag{1.16}$$

We will put

$$\delta := \delta_1, \; \iota := \iota_1, \text{and}, \; F := F_1.$$

To any $(\mathcal{A}^*, d^*_{\mathcal{A}}, i^*_{\mathcal{A}},)$ in S^1- CDGA one can associate the extension

$$(\Lambda[u], d = 0) \xrightarrow{i} (\mathcal{A}^*[u], d^*_{\mathcal{A}}[u]) \xrightarrow{e} (\mathcal{A}^*, d^*_{\mathcal{A}}) \tag{1.17,}$$

where $\Lambda[u]$ denotes the free commutative graded algebra generated by u of degree 2, $d^*_{\mathcal{A}}[u]$ is defined by

(1.18')
$$d^*_{\mathcal{A}}[u](a\otimes u^r) = d^*_{\mathcal{A}}(a)\otimes u^r + i_L(a)\otimes u^{r+1}$$

and e^* is defined by

(1.18)
$$e(a\otimes u^r) = 0, \quad r>0$$
$$e(a\otimes 1) = a.$$

If $(\mathcal{A}^*, d^*_{\mathcal{A}}, i^*_{\mathcal{A}}, \phi_k)$ is in $\mathcal{M}-$ CDGA then ϕ_k induces the morphism $\tilde{\phi}_k : (\mathcal{A}^*[u], d^*_{\mathcal{A}}[u]) \to (\mathcal{A}^*[u], d^*_{\mathcal{A}}[u])$ defined by

(1.18")
$$\tilde{\phi}_k(a\otimes u^r) = \frac{1}{k^r}\phi_k(a)\otimes u^r.$$

Clearly e intertwines $\tilde{\phi}_k$ with ϕ_k.

The extension (1.17) induces the Gysin sequence
(1.19)
$$\to H^*(\mathcal{A}, d_{\mathcal{A}}) \xrightarrow{\delta^*} H^{*-1}(\mathcal{A}[u], d_{\mathcal{A}}[u]) \xrightarrow{S^{*-1}} H^{*+1}(\mathcal{A}[u], d_{\mathcal{A}}[u]) \xrightarrow{E^{*+1}} H^{*+1}(\mathcal{A}, d_{\mathcal{A}}) \to,$$

where δ^* is induced by $a \to i_{\mathcal{A}}(a)$, S^* by the multiplication by u and E^* by e. The morphisms ϕ_k and $\tilde{\phi}_k$ induce the endomorphisms $\Phi_k : H^*(\mathcal{A}, d_{\mathcal{A}}) \to H^*(\mathcal{A}, d_{\mathcal{A}})$ resp. $\tilde{\Phi}_k : H^*(\mathcal{A}[u], d_{\mathcal{A}}[u]) \to H^*(\mathcal{A}[u], d_{\mathcal{A}}[u])$. Observe that δ^* intertwines Φ^*_k with $\tilde{\Phi}_k^{*-1}$, J^* intertwines $\tilde{\Phi}_k^{*}$ with Φ^*_k and S^* intertwines $\tilde{\Phi}_k^{*}$ with $k\tilde{\Phi}_k^{*+2}$.

If $f : (\mathcal{A}, d_{\mathcal{A}}, i_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}}, i_{\mathcal{B}})$ is a morphism in S^1- CDGA's then it induces the commutative diagram (1.20) whose horizontal lines are the extensions (1.17) for $(\mathcal{A}, d_{\mathcal{A}}, i_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}}, i_{\mathcal{B}})$.

(1.20)
$$\begin{array}{ccccc} (\Lambda[u], d=0) & \to & (\mathcal{A}[u], d_{\mathcal{A}}[u]) & \to & (\mathcal{A}, d_{\mathcal{A}}) \\ \downarrow Id & & \downarrow f[u] & & \downarrow f \\ (\Lambda[u], d=0) & \to & (\mathcal{B}[u], d_{\mathcal{B}}[u]) & \to & (\mathcal{B}, d_{\mathcal{B}}) \end{array}$$

The diagram (1.20) induces the commutative diagram (1.21) whose horizontal lines are the Gysin sequences associated to the horizontal lines of (1.20).
(1.21)
$$\begin{array}{cccc} \to H^*(\mathcal{A}, d_{\mathcal{A}}) \overset{\delta^*_{\mathcal{A}}}{\to} & H^{*-1}(\mathcal{A}[u], d_{\mathcal{A}}[u]) & \overset{S^{*-1}}{\to} H^{*+1}(\mathcal{A}[u], d_{\mathcal{A}}[u]) & \overset{J^{*+1}}{\to} H^{*+1}(\mathcal{A}, d_{\mathcal{A}}) \\ \downarrow H^*(f) & \downarrow H^{*-1}(f[u]) & \downarrow H^{*+1}(f[u]) & \downarrow H^{*+1}(f) \\ \to H^*(\mathcal{B}, d_{\mathcal{B}}) \overset{\delta^*_{\mathcal{B}}}{\to} & H^{*-1}(\mathcal{B}[u], d_{\mathcal{B}}[u]) & \overset{S^{*-1}}{\to} H^{*+1}(\mathcal{B}[u], d_{\mathcal{B}}[u]) & \overset{J^{*+1}}{\to} H^{*+1}(\mathcal{B}, d_{\mathcal{B}}) \end{array}$$

Let $(\mathcal{A}, d_{\mathcal{A}}, i_{\mathcal{A}}, \phi_{\mathcal{A},k})$ and $(\mathcal{B}, d_{\mathcal{B}}, i_{\mathcal{B}}, \phi_{\mathcal{B},k})$ be two $\mathcal{M}-$ CDGA's. The morphism $f : (\mathcal{A}, d_{\mathcal{A}}, i_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}}, i_{\mathcal{B}})$ is called "$\mathcal{M}-$ homotopic" if f and $f[u]$ intertwine $\phi_{\mathcal{A},k}$ with $\phi_{\mathcal{B},k}$ and $\phi_{\mathcal{A},k}[u]$ with $\phi_{\mathcal{B},k}[u]$ up to homotopy. This implies that the vertical arrows in the diagram (1.21) intertwine $\Phi_{\mathcal{A},k}$'s and $\tilde{\Phi}_{\mathcal{A},k}$'s with $\Phi_{\mathcal{B},k}$'s and with $\tilde{\Phi}_{\mathcal{B},k}$'s.

Suppose $\mu : S^1 \times Y \to Y$, is a the smooth S^1- manifold and $(\Omega^*_{inv}(Y), d^*_Y, i^*_L)$, is the S^1- CDGA of invariant differential forms. Suppose $f : (\mathcal{A}, d_{\mathcal{A}}, \iota_{\mathcal{A}}) \to (\Omega^*_{inv}(Y), d^*_Y, i^*_L)$ is a morphism in S^1- CDGA's. Then f induces a morphism $f(u) : (\mathcal{A}^*[u], d_{\mathcal{A}}[u]) \to (\Omega^*(Y//\mu), d^*_{Y//\mu})$ constructed in the following way: denote by $\pi_1 : S^\infty \times Y \to S^\infty$ and $\pi_2 : S^\infty \times Y \to Y$the canonical projections and define $f[u]$ by the formula

$$f(u)(\sum a_r \otimes u^r) = \sum I(a_r) \wedge (\pi_1^*(\tilde{u}))^r$$

where $\tilde{u}$ is the 2-form in Observation 1.1 and

$$I(a) = f(a) + (-1)^{dega-1}\pi_2^*(f(i_{\mathcal{A}}(a)) \wedge \pi_1^*(\gamma)).$$

It is not hard to see that $f[u]$ is a morphism of CDGA's and that its image consists only of smooth forms which, by Observation 1.2, identify to pull backs of smooth forms on $Y//\mu$.

Proposition 1.3. *If $f : (\mathcal{A}, d_{\mathcal{A}}) \to (\Omega^*_{inv}(Y), d^*_Y)$ is a quasi isomorphism then so is $f[u]$.*

To check Proposition 1.3 we first verify the statement for $f = Id$. In this case the the proof is the same as the proof that the equivariant cohomology defined using invariant forms and the contraction i_L is the same as the deRham cohomology of the smooth manifold $Y//\mu$, cf [AB]. The general case follows from the commutative diagram (1.21) and the case $f = Id$.

It is convenient to write $H^*_{S^1}(\mathcal{A}, d_{\mathcal{A}}, \iota_{\mathcal{A}}) := H^*(\mathcal{A}[u], d[u])$ and to consider the reduced versions of these cohomologies, $\overline{H}^*(\mathcal{A}, d_{\mathcal{A}})$ and $\overline{H}^*_{S^1}(\mathcal{A}, d_{\mathcal{A}}, \iota_{\mathcal{A}})$,

$$\overline{H}^*(\mathcal{A}, d_{\mathcal{A}}) = coker(H(i) : H^*(\mathcal{K}, 0, 0) \to H^*(\mathcal{A}, d_{\mathcal{A}}))$$

$$\overline{H}^*_{S^1}(\mathcal{A}, d_{\mathcal{A}}, \iota_{\mathcal{A}}) = coker(H_{S^1}(i) : H^*_{S^1}(\mathcal{K}, 0, 0) \to H^*_{S^1}(\mathcal{A}, d_{\mathcal{A}}, \iota)),$$

where $i : (\mathcal{K}, 0, 0) \to (\mathcal{A}, d_{\mathcal{A}}, \iota)$ is the morphism in S^1-CDGA's defined by the unit of $\mathcal{A}$. The power maps φ_k and $\tilde{\varphi}_k$ induce endomorphisms $\Phi_k{}^*$ and $\tilde{\Phi}_k{}^*$ in cohomology and equivariant cohomology ordinary and reduced. Notice that if $(\mathcal{A}, d_{\mathcal{A}}, \iota_{\mathcal{A}})$ is augmentable (i.e there exists the morphism $\epsilon : (\mathcal{A}, d_{\mathcal{A}}, \iota) \to (\mathcal{K}, 0, 0)$ so that $\epsilon i = id$, then the Gysin sequence in reduced cohomologies remains exact.

3. Recall that given a connected smooth manifold M, a model for M is a pair $((\Lambda[V], d_V), \theta)$, with $(\Lambda[V], d_V)$ a free connected[8] commutative differential graded algebra and $\theta : (\Lambda[V], d_V) \to (\Omega^*(M), d_M)$ a quasi isomorphism. The commutative differential graded algebra $(\Lambda[V], d_V)$ is called minimal if d_V restricts to zero on V_1 and d_V is decomposable, i.e $d_V(V) \in (\Lambda^+[V])^2$; here $\Lambda^+[V]$ denotes the ideal of the elements of positive degree.

The following results, due to D. Sullivan (cf [L],[DGMS]), constitute the so called Sullivan's minimal model theory.

[8]i.e $V_0 = 0$

Theorem 1.4. *(1) If $f : (\Lambda[V], d_V) \to (\Lambda[W], d_W)$ is a quasi isomorphism between two minimal CDGA's then f is an isomorphism.*

(2) There exists minimal models for any 1-connected smooth manifold M which has the homotopy type of a CW-complex of finite type.

(3) Let $h : M_1 \to M_2$ be a smooth map between two smooth manifolds as in (2). If $\Omega(h) : (\Omega^*(M_2), d^*_{M_2}) \to (\Omega^*(M_1), d^*_{M_1})$ denotes the morphism between the corresponding deRham algebras induced by h and $\theta_i : (\Lambda[V_i], d_{V_i}) \to (\Omega^*(M_i), d_{M_i})$, $i = 1, 2$ are minimal models, then there exists $f : (\Lambda[V_2], d_{V_2}) \to (\Lambda[V_1], d_{V_1})$ which makes the following diagram homotopy commutative

$$\begin{array}{ccc} (\Lambda[V_2], d_{V_2}) & \xrightarrow{\theta_2} & (\Omega^*(M_2), d^*_{M_2}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Omega(h)} \\ (\Lambda[V_1], d_{V_1}) & \xrightarrow{\theta_1} & (\Omega^*(M_1), d^*_{M_1}) \end{array} \quad . \tag{1.22}$$

(4) Given θ_i, the assignment $h \rightsquigarrow f$ provides a well defined map from the set of homotopy classes of smooth maps $h : M_1 \to M_2$ to the set of homotopy classes of morphisms $f : (\Lambda[V_2], d_{V_2}) \to (\Lambda[V_1], d_{V_1})$.

The main tool used in the proof of Proposition 0.1 and of the Theorems 0.2 and 0.3 is provided by the following theorem:

Theorem 1.5. *(1) Suppose $\theta : (\Lambda[V], d_V) \to (\Omega^*(M), d^*_M)$ is a minimal model for the 1-connected manifold M. Then there exists the morphism*

$$\theta_n : (\Lambda[V \oplus \overline{V}], \delta_n, \iota_n) \to (\Omega^*_{inv}(M^{S^1}_{sm}), d^*_{M^{S^1}}, ni^*_L)$$

in S^1-CDGA so that $\theta_n : (\Lambda[V \oplus \overline{V}], \delta_n) \to (\Omega^*_{inv}(M^{S^1}_{sm}), d^*_{M^{S^1}})$ is a quasi isomorphism.

(2) With respect to the $\mathcal{M}$-structure on $(\Lambda[V \oplus \overline{V}], \delta, \iota_V)$ provided by the power maps defined in (1.15) and the $\mathcal{M}$-structure on $\Omega^*_{inv}(M^{S^1}_{sm})$ induced by the maps φ_k, θ is an $\mathcal{M}$-homotopic morphism.

About the proof of (1): For $n = 1$ this statement was first proven in [VS] and [VB], cf Theorem 3.2 in [BFG]. For n arbitrary this is Proposition 3.3 in[BFG].

Observe that Theorem 1.5 (1) implies that

$$\theta_n : (\Lambda[V \oplus \overline{V}], \delta_n) \to (\Omega^*_{inv}(M^{S^1}), d^*_{M^{S^1}}) \subset (\Omega^*(M^{S^1}), d^*_{M^{S^1}}) \tag{1.23}$$

and

(1.24)

$$\tilde{\theta}_n : (\Lambda[V \oplus \overline{V} \oplus [u]], \delta_n) \to (\Omega^*_{inv}(M^{S^1})[u], d^*_{M^{S^1}}[u]_n) \to (\Omega^*(M^{S^1}//\mu_n), d^*_{M^{S^1}//\mu_n})$$

are minimal models. Here $[u]$ denotes the one dimensional graded vector space concentrated in the degree 2, generated by the symbol u, the index "n" for $d_{M^{S^1}}[u]_n$

indicates that this differential is constructed as in (1.17), (1.18), from $d_{M^{S^1}}$ and ni_L.

About the proof of (2): In view of Theorem 1.4 (4) this statement is equivalent to the fact that the morphisms

$$\phi_k : (\Lambda(V \oplus \overline{V}), \delta) \to (\Lambda[V \oplus \overline{V}], \delta) \tag{1.25}$$

and

$$\phi_k[u] : (\Lambda(V \oplus \overline{V})[u], \delta[u]) \to (\Lambda(V \oplus \overline{V})[u], \delta[u]) \tag{1.26}$$

represent φ_k and $(\tilde{\varphi}_k)_Q$, with respect to the minimal models $\theta = \theta_1$and $\theta[u] = \theta_1[u]$. To establish (1.26) it suffices to show that the map $\varphi'_k : M^{S^1}//\mu_k \to M^{S^1}//\mu$ induced by $\varphi_k : M^{S^1} \to M^{S^1}$ is represented with respect to the minimal models $\tilde{\theta}_n[u]$ and $\theta[u]$ by

$$\phi_k[u] : (\Lambda[V \oplus \overline{V}][u], \delta_n[u]) \to (\Lambda[V \oplus \overline{V}][u], \delta[u]) \tag{1.27}$$

Indeed, it is not hard to see that the map Ω_n is represented with respect to the same minimal models by $\Omega^*_n : (\Lambda[V \oplus \overline{V}][u], \delta[u]) \to (\Lambda[V \oplus \overline{V}][u], \delta_n[u])$, defined by

$$\Omega^*_n(v) = v,\ \Omega_n(\overline{v}) = \overline{v},\ \Omega_n(u) = nu. \tag{1.8}$$

The statement follows from the equality $(\tilde{\varphi}_k)_Q = (\varphi'_k)_Q \cdot ((\Omega_k)_Q)^{-1}$. The verification of (1.25) and (1.27) is done in Proposition 3.4 in [BFG].
Few misprints in [BFG] have remained uncorrected in the Erratum. On page 279, the vertical arrows in the first diagram and the horizontal arrow φ_k in the second diagram should have the directions changed.

The above theorem implies that the Gysin sequence (1.7), for $Y = M^{S^1}_{sm}$, identifies to the Gysin sequence (1.19) for $(\Lambda[V \oplus \overline{V}], \delta, \iota)$ defined in (1.13)-(1.15) and the identification is compatible with the endomorphisms Φ_k's and $\tilde{\Phi}_k$'s.

4. Recall that for the category of associative unital differential graded algebras over a field of characteristic zero one has two Z_+-graded vector spaces valued functors, the Hochschild and cyclic homology. For any such algebra A, $HH_*(A)$, the Hochschild homology, and $HC_*(A)$, the cyclic homology, are related by a long exact sequence (Connes exact sequence).

$$\cdots \to HH_*(A) \xrightarrow{J^*} HC_*(A) \xrightarrow{S_*} HC_{*-2}(A) \xrightarrow{B_{*-2}} HH_{*-1}(A) \to \cdots, \tag{1.29}$$

If $f : A \to B$ is a morphism of unital algebras one has the commutative diagram

$$\begin{array}{cccccccc} \to HH_*(A) & \overset{J^A}{\to} & HC_*(A) & \overset{S^A}{\to} & HC_{*-2}(A) & \overset{b^A_{*-2}}{\to} & HH_{*-1}(A) \\ \downarrow HH(f)_* & & \downarrow HC(f)_* & & \downarrow HC(f)_{*-2} & & \downarrow HH(f)_{*-1} \\ \to HH_*(B) & \overset{J^B}{\to} & HC_*(B) & \overset{S^B}{\to} & HC_{*-2}(B) & \overset{b^B_{*-2}}{\to} & HH_{*-1}(B) \end{array} \tag{1.30}$$

One can define a reduced version of the Hochschild resp. cyclic homologies

$$\overline{HH}_*(A) := coker(HH_*(i) : HH_*(\mathbb{C}) \to HH_*(A)$$

$$\overline{HC}_*(A) := coker(HC_*(i) : HC_*(\mathbb{C}) \to HC_*(A)$$

where $i : \mathbb{C} \to A$ denotes the morphism induced by the unit of A. The sequence (1.29) induces a similar sequence for the reduced cohomologies which, when A is augmentable, remains exact. If in addition A is commutative both Hochschild and cyclic homologies (reduced homologies) carry natural Adams operations
$Ad_k : HH_*(A)(\overline{H}H_*(A)) \to HH_*(A)(\overline{H}H_*(A))$ and
$\tilde{A}d_k : HC_*(A)(\overline{H}C_*(A)) \to HC_*(A)(\overline{H}C_*(A))$. These operations satisfy $Ad_1 = id$, $Ad_{kr} = Ad_k \cdot Ad_r$ and have the following intertwining properties

$$J_k \cdot Ad_k = \tilde{A}d_k \cdot J_k, \quad S_k \cdot \tilde{A}d_k = k\tilde{A}d_{k-2} \cdot S_k, \quad b_k \cdot \tilde{A}d_k = Ad_k \cdot b_k \tag{1.32}$$

It was observed ([B2],[BFG]) that Hochschild and cyclic homology as well as the exact sequence (1.29) can be extended[9]as Z-graded vector spaces valued functors to unital differential graded algebras with differential of degree +1. Similarly Adams operations in Hochschild and cyclic homology can be extended to commutative differential graded algebras. Here is a summary of these extensions; for details we reffer to [BFG].

To a CDGA, $(\mathcal{A}, d_{\mathcal{A}})$, we associate the bicomplex

$$(T(\mathcal{A})_{p,-q}, D^I_{p,-q}, D^E_{p,-q})p, q \geq 1$$

with $D^I_{p,-q} : T(\mathcal{A})_{p,-q} \to T(\mathcal{A})_{p,-q-1}$ and $D^E_{p,-q} : T(\mathcal{A})_{p,-q} \to T(\mathcal{A})_{p-1,-q}$ defined as follows:

$$T(\mathcal{A})_{p,-q} = \bigoplus_{i_0+\cdots+i_p=q} \mathcal{A}_{i_0} \otimes \mathcal{A}_{i_1} \otimes \cdots \mathcal{A}_{i_p}, \tag{1.33}$$

$$\text{(1.34')} \quad D^I_{p,-q}(a_{i_0}\otimes\cdots a_{i_p}) = da_{i_0}\otimes\cdots a_{i_p} + \sum_{l=1}^{p}(-1)^{i_0+\cdots+i_{l-1}} a_{i_0}\otimes\cdots da_{i_l}\otimes\cdots a_{i_p},$$

$$D^E_{p,-q}(a_{i_0} \otimes \cdots a_{i_p}) = \sum_{l=0}^{p-1}(-1)^l a_{i_0} \otimes \cdots a_{i_l}a_{i_{l+1}} \otimes \cdots a_{i_p} +$$

$$\text{(1.34'')} \quad +(-1)^{p+i_p(i_0+\cdots+i_{p-1})}a_{i_p}a_{i_0} \otimes a_{i_1} \otimes \cdots a_{i_{p-1}}.$$

The two differentials satisfy:

$$(D^I)^2 = 0, (D^E)^2 = 0 \text{ and } D^I D^E + D^E D^I = 0. \tag{1.35}$$

[9] a unital algebra can be regarded as DGA concentrated in the degree 0

One defines the action of the symmetric group S_m on $\mathcal{A}^{\otimes m}$, the m-fold tensor product (over $\mathbb{C}$) of the graded algebra $\mathcal{A}$ with itself, by the formula

$$\rho_m(\sigma, a_{i_1} \otimes \cdots a_{i_m}) = (-1)^{\epsilon(\sigma; i_1, \cdots, i_m)} a_{i_{\sigma(1)}} \otimes \cdots \otimes a_{i_{\sigma(m)}}$$

with $\epsilon(\sigma; i_1, \cdots, i_m)$ given as in [BFG] pp 274. Let $\tau : T(\mathcal{A})_{p,-q} \to T(\mathcal{A})_{p,-q}$ be given by

$$(1.36) \qquad \tau(a_{i_o} \otimes \cdots a_{i_m}) = (-1)^m \rho_{m+1}(\omega_{m+1}, a_{i_0} \otimes \cdots a_{i_m}) =$$

$$= (-1)^{m + i_m(i_0 + \cdots + i_{m-1})} a_{i_m} \otimes a_{i_0} \otimes \cdots a_{i_{m-1}},$$

with ω_{m+1} the cyclic permutation of $\{0, 1, 2, \cdots, m\}$. The total complex $(T_*(\mathcal{A}), D_*)$ is defined by

$$(1.37) \qquad T_r(\mathcal{A}) = \oplus_{p-q=r} T(\mathcal{A})_{p,-q}, \; D_r = \sum_{p-q=r} D^I_{p,-q} + D^E_{p,-q},$$

and its homology is called the Hochschild homology of $(\mathcal{A}, d_{\mathcal{A}})$. Put $CoinvT_*(\mathcal{A}) = T_*(\mathcal{A})/Im(1-\tau_*)$. Since $D_*(Im(1-\tau_*) \subset Im(1-\tau_*)$, $(CoinvT_*(\mathcal{A}), D_*)$ is again a chain complex and the canonical projection $\pi : (T_*(\mathcal{A}), D) \to (CoinvT_*(\mathcal{A}), D)$ is a morphism of cochain complexes. The homology of $(CoinvT_*(\mathcal{A}), D)$ is called the cyclic homology of $(\mathcal{A}, d_{\mathcal{A}})$ and π induces the morphism $J^* : HH_*(\mathcal{A}, d_{\mathcal{A}}) \to HC_*(\mathcal{A}, d_{\mathcal{A}})$. Since our algebras are unital, one defines the reduced Hochschild and cyclic homology $\overline{HH}_*(\mathcal{A}, d_{\mathcal{A}})$ and $\overline{HC}_*(\mathcal{A}, d_{\mathcal{A}})$ as the homologies of the reduced complexes $(T_*(\mathcal{A})/T_*(\mathbb{C}), D)$ and $(CoinvT_*(\mathcal{A})/CoinvT_*(\mathbb{C}), D)$. It is clear from definitions that the (reduced) Hochschild and cyclic homologies are functors from the category CDGA's to the category of $Z-$ graded vector spaces. It is easy to verify (cf [B], [BV1]) that:

Proposition 1.8. *If $f : (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ is a quasi isomorphism then $HH_*(f)$, $HC_*(f)$, $\overline{HH}_*(f)$, and $\overline{HC}_*(f)$ are isomorphisms.*

To define Adams operations one construct first the elements $\psi^k_n = \sum \alpha^k_\sigma \sigma$, $\sigma \in S_n$, $\alpha^k_\sigma \in Z$, in $Z(S_n)$, the group ring of the symmetric group S_n, as in [L] p.13 (cf also [BFG]).Then one defines the linear maps $Ad_k : (T_*(\mathcal{A}), D) \to (T_*(\mathcal{A}), D)$ and $\tilde{A}d_k : (CoinvT_*(\mathcal{A}), D) \to (CoinvT_*(\mathcal{A}), D)$ by the formula:

$$Ad_k(a_{i_1} \otimes \cdots a_{i_n}) = \sum_{\sigma \in S_n, k} \alpha^k_\sigma \rho(\sigma, a_{i_1} \otimes \cdots a_{i_n})$$

Ad_k induce in homology the endomorphisms (Adams operations)

$$Ad_k : HH_*(\mathcal{A}, d_{\mathcal{A}}) \to HH_*(\mathcal{A}, d_{\mathcal{A}}) \text{ and } Ad_k : HC_*(\mathcal{A}, d_{\mathcal{A}}) \to HC_*(\mathcal{A}, d_{\mathcal{A}}).$$

which are natural transformations of functors.

It is shown in [BV][10] that $(T_*(\mathcal{A}), D)$ resp. $(CoinvT_*(\mathcal{A}), D)$ decomposes canonically as a sum of subcomplexes

$$(T_*(\mathcal{A}), D) = \bigoplus_{i\geq 0}(T_*(\mathcal{A})(i), D_*(i))$$

resp.

$$(CoinvT_*(\mathcal{A}), D) = \bigoplus_{i\geq 0}(CoinvT_*(\mathcal{A})(i), D_*(i)),$$

and therefore $\overline{HH}_*(\mathcal{A}, d_{\mathcal{A}})$ and $\overline{HC}_*(\mathcal{A}, d_{\mathcal{A}})$ decompose canonically as

$$\overline{HH}_*(\mathcal{A}, d_{\mathcal{A}}) = \sum_{i\geq 0}\overline{HH}_*(\mathcal{A}, d_{\mathcal{A}})(i) \tag{1.38}$$

and

$$\overline{HC}_*(\mathcal{A}, d_{\mathcal{A}}) = \sum_{i\geq 0}\overline{HC}_*(\mathcal{A}, d_{\mathcal{A}})(i). \tag{1.39}$$

Moreover

$$\begin{aligned} \overline{HH}_*(\mathcal{A}, d_{\mathcal{A}})(0) &= \overline{HC}_{*-1}(\mathcal{A}, d_{\mathcal{A}})(0) = \overline{H}^*(\mathcal{A}, d_{\mathcal{A}}) \\ \overline{HH}_n(\mathcal{A}, d_{\mathcal{A}})(r) &= \overline{HC}_{n-1}(\mathcal{A}, d_{\mathcal{A}})(r) = 0 \text{ if } r > n, \end{aligned} \tag{1.40}$$

with $\overline{HH}_*(\mathcal{A}, d_{\mathcal{A}})(i)$, resp. $\overline{HC}_*(\mathcal{A}, d_{\mathcal{A}})(i+1)$ eigenspaces of eigenvalues k^i for the linear maps Ad_k. These decompositions are referred to as the WEIGHT decompositions.

If in addition $d_{\mathcal{A}} = 0$, we have the additional decompositions
(1.41)

$$(T_*(\mathcal{A}), D_*) = \sum_{p\geq 0}(T(\mathcal{A})_{p,-*}, D^E_{p,-*}),\ (CoinvT_*(\mathcal{A}), D_*) = \sum_{p\geq 0}(T(\mathcal{A})_{p,-*}, D^E_{p,-*})$$

compatible with Adams operations and therefore the additional decompositions:

$$\overline{HH}_{-n}(\mathcal{A}, d_{\mathcal{A}})(i) = \bigoplus_{-n=p-r} \overline{HH}_{p,-r}(\mathcal{A}, d_{\mathcal{A}})(i), \tag{1.42'}$$

$$\overline{HC}_{-n}(\mathcal{A}, d_{\mathcal{A}})(i) = \bigoplus_{-n=p-r} \overline{HC}_{p,-r}(\mathcal{A}, d_{\mathcal{A}})(i) \tag{1.42"}$$

The decompositions (1.42') and (1.42") will be called HODGE decompositions.

Let $(\mathcal{A}, d_{\mathcal{A}} = 0)$ be a CDGA with $dim\mathcal{A}_r = \beta_r < \infty$. Introduce

$$\overline{P}_H(z) := \sum_{r>0}\beta_r z^r, \Pi(z, \lambda) = \sum_{i,q}\left(\sum_p(-1)^p dim(\tilde{HH}_{p,-q}(\mathcal{A}, 0)(i)\right)\lambda^i z^q$$

$$C\Pi(z, \lambda) = \sum_{i,q}\left(\sum_p(-1)^p dim(\tilde{HC}_{p,-q}(\mathcal{A}, 0)(i)\right)\lambda^i z^q.$$

In [H] Ph.Hanlon has proven the following result:

[10] for commutative differential graded algebras with differential of degree -1, but the same arguments hold for CDGA's as above

Theorem 1.9. [H]. *If* $(\mathcal{A}, d_{\mathcal{A}} = 0)$ *is a CDGA with* $\mathcal{A}_0 = \mathbb{C}$ *and* $dim\mathcal{A}_r = \beta_r < \infty$, *then*

$$\Pi(z,\lambda) = P(z) \prod_{l \in N} (1 + P(z^l))^{-1/l \sum_{d|l} \mu(d) z^{l/d}},$$

$$C\Pi(z,\lambda) = P(z)/(1-\lambda)\{\prod_{l \in N} (1 + P(z^l))^{-1/l \sum_{d|l} \mu(d) z^{l/d}} - \lambda\}.$$

The following result is a graded version of Hochschild- Konstant-Rosenberg theorem and was proven in [BV] for commutative differential graded algebras with differential of degree -1, but the same arguments hold in the case the differential is of degree $+1$.

Theorem 1.10. [BFG]. *If* $(\Lambda[V], d_V)$ *is a connected free CDGA, then there exists the natural isomorphisms* $h_* : \overline{HH}_{-*}(\Lambda[V], d_V) \to \overline{H}^*(\Lambda[V \oplus \overline{V}], \delta)$ *and* $c_* : \overline{HC}_{-*}(\Lambda[V], d_V) \to \overline{H}^{*-1}_{S^1}(\Lambda[V \oplus \overline{V}], \delta)$ *which intertwine the reduced Adams operations with the reduced power maps and identify the reduced Connes exact sequence of* $(\Lambda[V], d_V)$ *with reduced Gysin sequence of* $(\Lambda[V \oplus \overline{V}], \delta_V, \iota_V)$.

The following Corollary is useful in the calculation of the weight decompositions.

Corollary 1.11. *Using the isomorphisms provided by Theorem 1.10 the weight decomposition described above identifies to the decompositions (1.45) and (1.46).*

$$(1.45) \qquad (\Lambda[V \oplus \overline{V}], \delta) = \sum_{i \geq 0} (\Lambda[V] \otimes \overline{V}^{\otimes i}, \delta)$$

$$(1.46) \qquad (\Lambda[V \oplus \overline{V}][u], \delta) = \sum_{i \geq 0} (\Lambda[V][u] \otimes \overline{V}^{\otimes i}, \delta[u])$$

Proof of Proposition 0.1: It suffices to prove Proposition 0.1 for reduced cohomology resp. reduced equivariant cohomology of M^{S^1}. These cohomologies and the action of the power maps on these cohomologies can be calculated with the help of the explicit minimal models for X^{S^1} and $X^{S^1}//\mu$ and of the representation of ϕ_k and $(\tilde{\phi}_k)_Q$ with respect to these minimal models given in Theorem 1.5 . Once this noticed the verification of 1),2),3) can be done easily with the help of Corollary 1.11.

Theorem 1.10 combined with Proposition 1.8 provide a new proof of a result of J.D.Jones's (cf [J]), concerning the isomorphism between the cohomologies $\overline{H}^*(M^{S^1}; \mathbb{C})$ resp. $\overline{H}^*_{S^1}(M^{S^1}; \mathbb{C})$ and the reduced Hochschild resp. cyclic cohomologies of the de Rham algebra $(\Omega(M), d)$. This new proof permits also to verify that the isomorphisms h_* and c_* intertwine the Adams operations with the power maps.

A formality structure on M,

$$(H^*(M;\mathbb{C}), d = 0) \overset{\alpha_2}{\leftarrow} (\mathcal{A}, d_{\mathcal{A}}) \overset{\alpha_1}{\rightarrow} (\Omega(M), d)$$

identifies, using Proposition 2.8, the Hochschild resp. cyclic homology of $(\Omega(M)^*, d^*_M)$ with the Hochschild resp. cyclic homology of $(H^*(M;\mathbb{C}), d^* = 0)$. This identification transports the Hodge decomposition (1.42), on $\overline{H}^*(M^{S^1};\mathbb{C})(i)$ and on $\overline{H}^*_{S^1}(M^{S^1};\mathbb{C})(i)$. Theorem 0.2 follows then from, Theorem 1.11.

Using the model $(\Lambda[V \oplus \overline{V}], \delta)$ of the smooth action $\mu : S^1 \times M^{S^1} \to M^{S^1}$ (cf Theorem 1.5) one can derive (cf [VB] Corollary 4) the following result first proven by Goodwillie [G] by a different method:

Theorem 1.12. *If M is a smooth 1-connected (Hilbert) manifold of the homotopy type of a CW complex of finite type, then*

$$\overrightarrow{\lim}\{\cdots \overline{H}^{*+2k}_{S^1}(M^{S^1};\mathbb{C}) \xrightarrow{S^{*+2k}} \overline{H}^{*+2k+2}_{S^1}(M^{S^1};\mathbb{C}) \to \cdots\} = 0$$

(For the nonsimply connected manifolds the above limit depends only on the fundamental group cf [B3].)

SECTION 2

In Introduction, for a smooth (Hilbert) manifold M, we have introduced the functor $SH^*(M)$ using $(\Omega^*_{inv}(M^{S^1}_{sm}), d^* = d^*_{M^{S^1}}, \iota^*_L)$. We have introduced the cochain complex $(\mathbf{C}^*_-, D^*_- : \mathbf{C}^*_- \to \mathbf{C}^{*+1}_-)$ with $\mathbf{C}^r_- := \prod_{k\geq 0}\Omega^{r+2k}_{inv}(M^{S^1})$ and

$$D^r_-(\omega_r, \omega_{r+2}, \omega_{r+2}, \cdots) = (d(\omega_r) + \iota_L(\omega_{r+2}), d(\omega_{r+2}) + \iota_L(\omega_{r+4}), \cdots) \tag{2.1}$$

and then we defined $SH^*(M)$ as the cohomology of $(\mathbf{C}^*_-, D^*_-)$.

The power maps ϕ_k define the endomorphisms $\Phi_k : (C^*_-, D^*_-) \to (C^*_-, D^*_-)$ by the formula

$$\Phi_k(\omega_r, \omega_{r+2}, \omega_{r+2}, \cdots) = (\phi_k(\omega_r), 1/k\phi_k(\omega_{r+2}), 1/k^2\phi_k(\omega_{r+2}), \cdots) \tag{2.2}$$

and these endomorphismes induce $S\Phi^*_k : SH^*(M) \to SH^*(M)$. $SH^*(M)$ is a functor from the category of smooth Hilbert manifolds and smooth maps to the category of Z_+-graded vector spaces and Φ^*_k are natural transformations.

In order to prove Theorem 0.3 we have to introduce a few additional complexes: $(P\mathbf{C}^*, D^*)$, $(\overset{\circ}{P}\mathbf{C}^*_-, D^*_-)$ and $(\mathbf{C}^*_+, D^*_+)$, with the last two subcomplexes of the first. They are defined as follows:

$$\mathbf{C}^r_+ := \prod_{k\geq 0}\Omega^{r-2k}_{inv}(M^{S^1}), \tag{2.3}$$

$$P\mathbf{C}^{k=even} := \prod_{i\geq 0}\Omega^{2i}_{inv}(M^{S^1}),\ P\mathbf{C}^{k=odd} := \prod_{i\geq 0}\Omega^{2i+1}_{inv}(M^{S^1}) \tag{2.3'}$$

$$(2.3") \qquad \overset{\circ}{P}C^{k=even} := \sum_{i\geq 0} \Omega^{2i}_{inv}(M^{S^1}),\ PC^{k=odd} := \sum_{i\geq 0} \Omega^{2i+1}_{inv}(M^{S^1})$$

and $D = \bar{d}_{M^{S^1}} + \bar{\iota}_L$ where

$$(2.4') \qquad \bar{d}_{M^{S^1}}(\omega_0, \omega_2, \cdots) = (d\omega_0, d\omega_2, \cdots),\quad d = d_{M^{S^1}},$$

$$\bar{d}_{M^{S^1}}(\omega_1, \omega_3, \cdots) = (0, d\omega_1, d\omega_3, \cdots),\quad d = d_{M^{S^1}},$$

$$(2.5) \qquad \bar{\iota}_L(\omega_0, \omega_2, \omega_4, \cdots) = (\iota\omega_2, \iota\omega_4, \cdots),\quad \iota = \iota_L,$$

$$\bar{\iota}_L(\omega_1, \omega_3, \cdots) = (\iota\omega_1, \iota\omega_3, \cdots),\quad \iota = \iota_L.$$

Define also $\Phi_k : (\mathbf{C}^*_+, D^*_+) \to (\mathbf{C}^*_+, D^*_+)$ by the formula

$$(2.6) \qquad \Phi^r_k(\omega_r, \omega_{r-2}, \omega_{r-4}, \cdots) = (\phi_k(\omega_r), k\phi_k(\omega_{r-2}), k^2\phi_k(\omega_{r-2}), \cdots)$$

and $\Phi^*_k : (P\mathbf{C}^*, D^*) \to (P\mathbf{C}^*, D^*)$, with $\Phi^n_k : P\mathbf{C}^n \to P\mathbf{C}^n$ by the formula

$$(2.7) \qquad \Phi^n_k(\prod_{i\geq 0} \omega_{2i+\epsilon}) = \prod_{i\geq 0} k^{i-n}\phi_k(\omega_{2i+\epsilon})$$

where $\epsilon = 0$ if $n = even$ and $\epsilon = 1$ if $n = odd$.
Observe that the cohomology of $(\mathbf{C}^*_+, D^*_+)$ is exactly the equivariant cohomology $H^*_{S^1}(M^{S^1}; \mathbb{C})$ and the cohomology of $(\overset{\circ}{P}\mathbf{C}^*_-, D^*_-)$ is

$$\lim_{\rightarrow}\{\cdots \overline{H}^{*+2k}_{S^1}(M^{S^1}) \overset{S^{*+2k}}{\longrightarrow} \overline{H}^{*+2k+2}_{S^1}(M^{S^1}) \to \cdots\}.$$

Denote by $\mathbb{P}H_{S^1}(M^{S^1})$ the cohomology of $(P\mathbf{C}^*, D^*)$. In order to understand the relationship between all these cohomologies observe first that there exists the commutative diagram of cochain complexes where the horizontal lines are short exact sequences and the vertical arrows are induced by inclusions.

$$(2.8) \qquad \begin{array}{ccccccc} 0 \to & (\mathbf{C}^{*-2}_+, D^{*-2}_+) & \overset{S^{*-2}}{\to} & (\mathbf{C}^*_+, D^*_+) & \overset{J^*}{\to} & (\Omega^*(M^{S^1}), d_{M^{S^1}}) & \overset{\delta_*}{\to} 0 \\ & \downarrow Id^{*-2} & & \downarrow In^* & & \downarrow In^* & \\ 0 \to & (\mathbf{C}^{*-2}_+, D^{*-2}_+) & \overset{s^*}{\to} & (P\mathbf{C}^*, D^*) & \overset{l^*}{\to} & (\mathbf{C}^*_-, D^*_-) & \overset{h^*}{\to} 0 \end{array}$$

This diagram induces the commutative diagram (2.9), whose horizontal sequences are long exact sequences
(2.9)

$$\begin{array}{ccccccccc} \to & H^{*-2}_{S^1}(M^{S^1}) & \overset{S^{*-2}}{\to} & H^*_{S^1}(M^{S^1}) & \overset{J^*}{\to} & H^*(M^{S^1}) & \overset{\delta_*}{\to} & H^{*-1}_{S^1}(M^{S^1}) & \to \\ & \downarrow Id^* & & \downarrow & & \downarrow & & \downarrow Id* & \\ \to & H^{*-2}_{S^1}(M^{S^1}) & \overset{s^*}{\to} & \mathbb{P}H^*(M^{S^1}) & \overset{l^*}{\to} & SH^*(M) & \overset{h^*}{\to} & H^{*-1}_{S^1}(M^{S^1}) & \to \end{array}$$

Proof of Theorem 0.3: It is shown in[JP], see also [B1] section 5, that $\mathbb{P}H^*_{S^1}(M^{S^1})$ is isomorphic to $\prod_{i\geq 0} H^{2i}(M; \mathbb{C})$ if r is even and to $\prod_{i\geq 0} H^{2i+1}(M; \mathbb{C})$ if r is odd, hence it is isomorphic to $\mathbb{K}^r(M) \otimes \mathbb{C}$ where $\mathbb{K}^r(M)$ denotes Atiyah-Hirzebruch K-theory. Moreover by this identification the operations $\Phi_k : \mathbb{P}H^*_{S^1}(M^{S^1}) \to \mathbb{P}H^*_{S^1}(M^{S^1})$ induced by the map of complexes Φ_k are intertwined with the Adams's operations in K-theory. The naturality of the diagram (2.9) and the homotopy invariance of $\mathbb{K}^r(M) \otimes \mathbb{C}$ and of $H^*_{S^1}(M^{S^1})$ imply the fact that the functor $SH^*(M)$ is a homotopy functor. It is implicit in [B2] (see also [B1] sections 2 and 4) that if M is 1-connected then the reduced equivariant cohomology $\overline{H}^*_{S^1}(M^{S^1})$ identifies to $Hom(\tilde{A}_{*-1}(M), \mathbb{C})$. Proposition 1.12 implies that for M 1-connected, s^* factors through $\mathbb{P}H^*_{S^1}(pt^{S^1})$. The a long exact sequence provided by second line in the diagram 2.9, combined with these observations lead to the proof of Theorem 0.3.

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