CONSTANT MEAN CURVATURE TORI IN TERMS OF ELLIPTIC FUNCTIONS

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Tori with constant mean curvature have been discovered first by H.C. Wente in 1984. The construction is based on special solutions of the sinh-Gordon equation. Wente could only give an abstract existence proof for these solutions so that his description of the H-tori is not very explicit.

Based on a numerical approximation of such a solution, we could produce plots of one H-torus. In these computer generated pictures the curvature lines for the smaller principal curvature λ_1 looked almost planar. We then decided to restrict ourselves to H-tori with one family of planar curvature lines. This condition translates into a second partial differential equation which induces a separation of variables in the sinh-Gordon equation. Therefore the overdetermined system can be solved explicitly in terms of elliptic functions. We obtain a classification of all H-tori in $\mathbb{E}^3$ which have one family of planar curvature lines.

THEOREM:

There exists no H-torus such that all λ_2-curvature lines are planar. The H-tori with planar λ_1-curvature lines are naturally parametrized by those angles $\theta \in (\pi, 2\pi)$ which are rational multiples of 2π.

The symmetry group of such an H-torus contains a central reflection σ_0 *which maps any planar curvature line onto itself. Each* λ_1*-curvature line looks like a figure eight perpendicular to the fixed point plane of* σ_0 .
When the family parameter changes, they oscillate around a symmetrical figure which is a closed elastica; their two vertices move on λ_2*-curvature lines* c_+ *and* c_- *in the fixed point plane of* σ_0 *. The curves* c_+ *and* c_- *coincide precisely when the denominator* n *in* $\frac{\theta}{2\pi} = \frac{m}{n}$, $(m,n) = 1$, *is odd. (cf. Fig.1).*
θ *is the angle between the normals at two consecutive vertices of* c_+ *the planar set* $c_+ \cup c_-$ *is invariant under the dihedral group* D_n *, and the symmetry group of the immersed* H*-torus is* $D_n \times \{id, \sigma_0\}$.

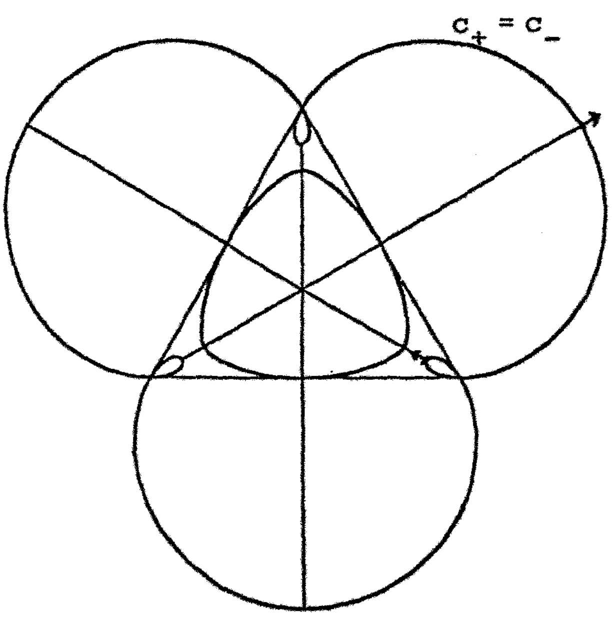


Fig.1a

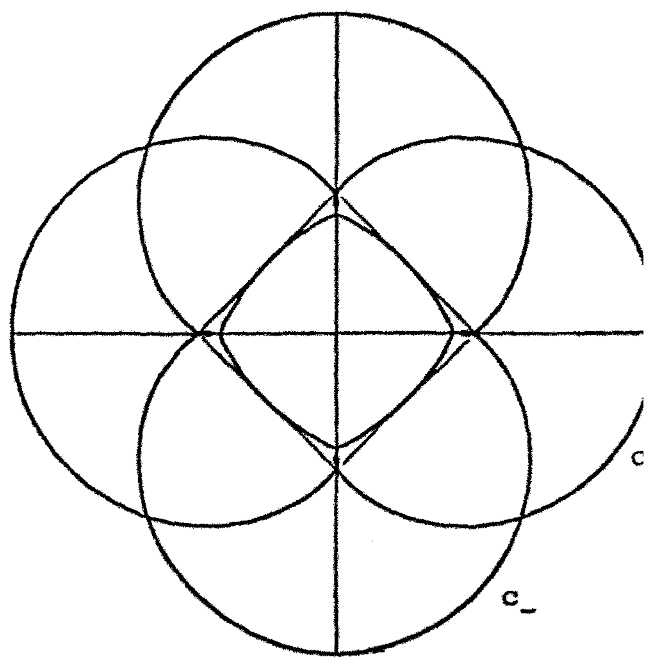


Fig.1b

Fig.1a: the set $c_+ \cup c_-$ in case $\theta = \frac{2}{3} \cdot 2\pi$: 3 symmetry axes!
Fig.1b: the set $c_+ \cup c_-$ in case $\theta = \frac{3}{4} \cdot 2\pi$: 4 symmetry axes!

We emphasize that our H-tori are known explicitly up to solving systems of ordinary differential equations, the Frenet equations for the planar curvature lines. The proof is given in section 1-4 of this paper; in section 1 the classification is reduced to a problem on the sinh-Gordon equation using standard arguments from the theory of surfaces. The actual analytical work is then carried out in section 2 and 3 which can be read independently; we calculate two two-parameter families of explicit solutions of the sinh-Gordon equation which might be of interest for other applications as well. In section 4 we finally put things together; we evaluate the closedness conditions and determine the range of the parameters. We prove the theorem and provide pictures of the H-torus with $\theta = \frac{2}{3} \cdot 2\pi$. (cf. Fig. 5).

In section 5 we digress on a problem for positive solutions of the sinh-Gordon equation on a rectangle with zero boundary values. For each rectangle which is not a priori too large we have written down in section 2 one solution with these properties. (cf. Proposition 3.4) The explicit solutions are unstable. It is still an open question whether the above Dirichlet problem for the sinh-Gordon equation has unique solutions; then we would have found all solutions. We shall not answer this question here. However, we shall show that the hypothetical bifurcations are of the simplest possible type:

THEOREM:

When the sinh-Gordon equation

$$\Delta\omega + \sinh\omega \cdot \cosh\omega = 0$$

is linearized around a positive solution on a rectangle R_{ab} *, then the linearized operator*

$$L_\omega : H_0^1(R_{ab}) \longrightarrow H^{-1}(R_{ab})$$

$$L_\omega\varphi := -\Delta\varphi - \cosh 2\omega \cdot \varphi$$

has an at most one-dimensional kernel.

1.) CLASSICAL DIFFERENTIAL GEOMETRY:

In this section we shall explain how to immerse tori $T^2 = \mathbb{R}^2/\Lambda$ with constant mean curvature into three-dimensional euclidean space $\mathbb{E}^3$. The task of finding H-tori will be reduced to an analytical problem on the sinh-Gordon equation. This translation is done by the standard Frenet theory of surfaces, and, in fact, it is quite well-known. Hence we shall be brief and omit all computations.

To begin with, we recall:

1.1. THEOREM (Hopf):

A compact immersed surface $M^2 \looparrowright \mathbb{E}^3$ *with constant mean curvature* H *is either totally umbilical, hence a standard sphere, or it has only isolated umbilics, each with strictly negative, possibly half-integral index.*

The sum of these indices is just the Euler characteristic $e(M^2)$. Therefore an H-torus cannot have any umbilic at all.

We normalize size and orientation, requiring that $H \equiv 1/2$. Moreover, we shall index the principal curvatures such that $\lambda_1 < \lambda_2$. The role of λ_1 and λ_2 is significantly different; it is only λ_1 which can - and will - change sign (cf. [HH2]). Finally we introduce a function ω by $e^{2\omega}(\lambda_2 - \lambda_1) = 1$.

1.2. PROPOSITION:

On any H-*surface* $M^2 \hookrightarrow E^3$ *(normalized to* $H \equiv 1/2$ *) a small neighborhood around any point* p *which is not an umbilic can be described by a map* $F : U \subset \mathbb{R}^2 \hookrightarrow E^3$ *such that:*

i) $F(0) = p$

ii) *the curves* $s \longmapsto F(s,t)$ *and* $t \longmapsto F(s,t)$ *are the* λ_1- *and* λ_2-*curvature lines, respectively; i.e. the Weingarten map in these coordinates diagonalizes:*

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} e^{-\omega}\sinh\omega & 0 \\ 0 & e^{-\omega}\cosh\omega \end{pmatrix} \tag{1.1}$$

iii) $(s,t) \in \mathbb{R}^2$ *are orientation preserving conformal coordinates; more precisely:*

$$F' \perp \dot{F} \quad \text{and} \quad |F'| = |\dot{F}| = e^{\omega} = (\lambda_2 - \lambda_1)^{-1/2} \quad ;$$

Here ' *and* $\dot{}$ *denote the derivatives w.r.t. the standard coordinates* s *and* t *on* $U \subset \mathbb{R}^2$, *respectively.*

In these coordinates the Gauß equation becomes:

$$\Delta\omega + \sinh\omega \cdot \cosh\omega = 0 \quad , \tag{1.2}$$

where Δ *is the coordinate Laplacian, i.e.:*
$\Delta\omega = \omega'' + \ddot{\omega}$.

<u>REMARK</u>: The transition functions between any two "adapted" coordinate systems as introduced in the proposition are just translations in $\mathbf{R}^2$ or 180°-rotations. Notice that in iii) we have also normalized the scale of the coordinate domain $\mathbf{R}^2$.

<u>Outline of PROOF</u>: It is a classical application of the Frobenius theorem to obtain coordinates s and t such that i) and ii) hold and that $F'(s,0) = e^{\omega(s,0)}$ and $\dot{F}(0,t) = e^{\omega(0,t)}$. Clearly, F' and $\dot{F}$ are perpendicular on all of U . A straightforward calculation shows that the integrability condition for having $|F'| = |\dot{F}| = e^{\omega}$ in an open neighborhood of $0 \in R^2$ are just the Codazzi equations and the condition $H \equiv 1/2$.

∎

The same computations simultaneously prove the converse.

<u>1.3. PROPOSITION:</u>

Any solution ω *to equation 1.2. defined on a connected open neighborhood* $U \subset \mathbf{R}^2$ *of* 0 *uniquely determines an* $H \equiv 1/2$ *immersion* $F : U \to \mathbf{E}^3$ *by means of the Frenet data 1.1, once the following inital data are specified:*

i) *the point* $p = F(0)$

and ii) *the oriented, orthonormal frame*

$$e^{-\omega} \cdot F'(0),\ e^{-\omega} \cdot \dot{F}(0) .$$

Since this proposition will be basic for the construction of the H-tori, we shall give the Frenet equations which determine F in terms of ω explicitly in the (s,t)-coordinates:

$$
(1.3)\qquad
\begin{aligned}
F'' &= \omega' F' - \dot{\omega}\dot{F} + e^{\omega}\sinh\omega \cdot N\\
\dot{F}' &= \dot{\omega}F' + \omega'\dot{F}\\
\ddot{F} &= -\omega' F' + \dot{\omega}\dot{F} + e^{\omega}\cosh\omega \cdot N\\
N' &= -e^{-\omega}\sinh\omega \cdot F'\\
\dot{N} &= -e^{-\omega}\cosh\omega \cdot \dot{F}
\end{aligned}
$$

Here N denotes the unit normal field associated with the Weingarten map A . Later on we shall also use the curvature κ_i and the torsion τ_i of the λ_i-curvature lines viewed as space curves in $E^3 (i=1,2)$:

$$
(1.4)\qquad
\begin{aligned}
\kappa_1^2 &= e^{-2\omega} \cdot (\dot{\omega}^2 + \sinh^2\omega) \quad ; \quad \kappa_1^{geod} = -e^{-\omega} \cdot \dot{\omega} \quad ,\\
\kappa_2^2 &= e^{-2\omega} \cdot (\omega'^2 + \cosh^2\omega) \quad ; \quad \kappa_2^{geod} = e^{-\omega} \cdot \omega' \quad ,\\
\kappa_1^2 \cdot \tau_1 &= e^{-3\omega} \cdot (\sinh\omega \cdot \dot{\omega}' - \cosh\omega \cdot \dot{\omega}\omega') \quad ,\\
\kappa_2^2 \cdot \tau_2 &= e^{-3\omega} \cdot (\cosh\omega \cdot \dot{\omega}' - \sinh\omega \cdot \dot{\omega}\omega') \quad .
\end{aligned}
$$

Theorem 1.4 turns the local description into a global result.

1.4. THEOREM:

Proposition 1.2 and 1.3 determine natural injections:

$$\{H \equiv 1/2\text{-tori } T^2 \hookrightarrow E^3 \text{ with base point } p_0\}/\{\text{motions in } E^3\}$$

$$\xrightarrow{i_1} \mathbb{M} = \{\omega : \mathbb{R}^2 \to \mathbb{R} \mid \omega \text{ solves 1.2 and is invariant w.r.t. some lattice } \Lambda \subset \mathbb{R}^2\}$$

$$\xrightarrow{i_2} \{H \equiv 1/2\text{-immersions } F : \mathbb{R}^2 \hookrightarrow E^3 \mid F(0) = p_0, e^{-\omega}\cdot F'(0) = b_1, \text{ and } e^{-\omega}\cdot \dot{F}(0) = b_2\},$$

where b_1, b_2, b_3 *stand for the canonical basis of* E^3. *Moreover, whenever* ω *lies in the image of* i_1, *then the group* $\Lambda_\omega := \{\lambda \in \mathbb{R}^2 \mid \omega(\lambda(x)) = \omega(\lambda + x) = \omega(x),\ \forall x \in \mathbb{R}^2\}$ *of translational symmetries contains all decktransformations of the H-torus. Conversely, the map* i_2 *determines a homomorphism* $\chi_\omega : \Lambda_\omega \longrightarrow \{\text{motions in } E^3\}$, *and the immersion* $F = i_2(\omega)$ *is equivariant under* Λ_ω *w.r.t.* χ_ω, *i.e.:*

$$\chi_\omega(\lambda) \circ F = F \circ \lambda^{-1}, \quad \forall \lambda \in \Lambda_\omega .$$

For generic $\omega \in \mathbb{M}$ the group of translational symmetries is just a lattice in $\mathbb{R}^2$. However this group can be non-discrete in special cases:

i) $\Lambda_\omega = \mathbb{R}^2$: then ω must be a constant and hence vanish identically. The corresponding H-surface is a cylinder with radius 1.

ii) $\mathbf{R}\cdot\lambda_0 \subset \Lambda_\omega \neq \mathbf{R}^2$ for some $\lambda_0 \neq 0$:
Here ω is determined by an ordinary differential equation along the axis $\lambda_0^\perp \subset \mathbf{R}^2$. When $\lambda_0 = (1,0)$ or $\lambda_0 = (0,1)$, then one gets surfaces of revolution with constant mean curvature. These are classically known and called Delaunay surfaces. (cf.[DEL]). For each choice of $\lambda_0^\perp$ there exists a one-parameter family of such surfaces; for one of them the meridean curves have no self-intersections, whereas for the other family they do have.

Clearly i_2 is a kind of left-inverse of i_1 . This helps in characterizing the subset $\mathfrak{M}^* := \{\omega \in \mathfrak{M} \mid F = i_2(\omega)$ closes up to a torus $\}$.

<u>1.5. COROLLARY</u>:

$$\mathfrak{M}^* = \mathrm{im}(i_1) = \{\omega \in \mathfrak{M} \mid \chi_\omega \ \underline{\text{has compact image}}\ \} \quad .$$

<u>PROOF</u>: Observe that by definition Λ_ω is cocompact for all $\omega \in \mathfrak{M}$!

∎

Clearly the right-hand side in this corollary is a condition on $[\chi_\omega]$, the equivalence class of χ_ω under conjugation in the group of motions. This just restates the fact that the <u>closedness condition</u> depends only on ω and not on the choice of the initial conditions one can make when applying Proposition 1.3.

We shall translate this group-theoretic closedness condition and express it in terms of functionals on $\omega \in \mathfrak{M}$ only for special solutions ω.

1.6. DEFINITION:

Given $a,b > 0$, we say that a solution $\omega : \mathbf{R}^2 \longrightarrow \mathbf{R}$ of the sinh-Gordon equation 1.2 lies in:

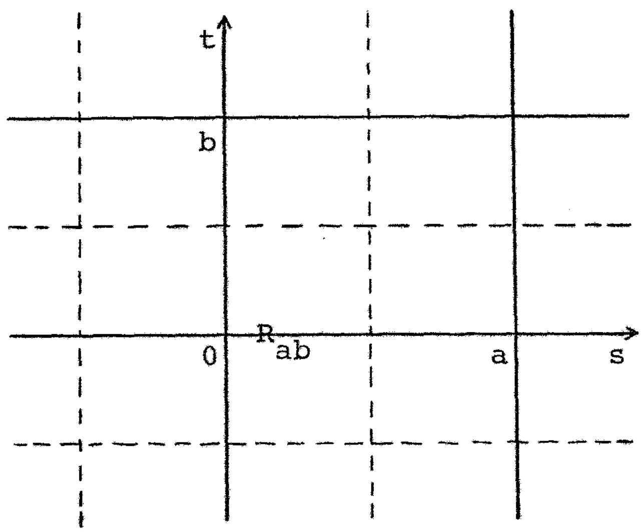


Fig. 2

i) $\mathfrak{M}^{\Gamma}_{ab}$, iff it is invariant under the action of a group Γ_{ab}, generated by the reflections at the four lines $s = 0$, $s = a$, $t = 0$, and $t = b$.

ii) $\mathfrak{M}^{D}_{ab}$, iff the rectangle $R_{ab} = \left(-\frac{a}{2}, \frac{a}{2}\right) \times \left(-\frac{b}{2}, \frac{b}{2}\right)$ is a connected component of $\{(s,t) \in \mathbf{R}^2 \mid \omega(s,t) > 0\}$.

REMARKS:

i) Solutions $\omega \in \mathfrak{M}^D_{ab}$ vanish on the boundary of the rectangle R_{ab} and can be recovered from their restriction $\omega|R_{ab}$ when extending it in the obvious way as an odd function w.r.t. all boundary edges. In fact, this procedure has originally been used by Wente in order to obtain an abstract existence result for equation 1.2.

ii) Notice that $\mathfrak{M}^D_{ab} \subset \mathfrak{M}^\Gamma_{ab}$ for all positive a and b. The additional symmetries w.r.t. the axes of the rectangle R_{ab} follow directly from a theorem by Gidas-Ni-Nirenberg (cf. [GNN]).

Let us now assume that $\omega \in \mathfrak{M}^\Gamma_{ab}$ for some $a,b > 0$. The fixed point set of a reflection $\tau \in \Gamma_{ab}$ is a horizontal or a vertical line in $\mathbb{R}^2$, and therefore it is not only a geodesic but also a curvature line. Under F it is therefore mapped onto a planar curve c_τ in $\mathbb{E}^3$. Because of the fundamental existence and uniqueness theorem the immersed surface is symmetric w.r.t. the plane which contains c_τ and is perpendicular to the surface. Hence we obtain a homomorphism $\Psi_{ab} : \Gamma_{ab} \longrightarrow \mathrm{Isom}(\mathbb{E}^3)$ which coincides with χ_ω on the finite index subgroup $\Gamma_{ab} \cap \Lambda_\omega$, and in fact the immersion F is equivariant under Γ_{ab} with respect to Ψ_{ab}.

Since Γ_{ab} preserves curvature lines, we have proven:

1.7. PROPOSTION:

$\omega \in \mathfrak{M}_{ab}$ describes a torus, i.e. lies in $\mathfrak{M}^*$, iff all λ_1-λ_2-curvature lines close up, or equivalently iff the planar λ_1-curvature line $c_1 : s \longmapsto F(s,0)$ and the planar λ_2-curvature line $c_2 : t \longmapsto F(0,t)$ both close up.

This is easily made explicit in terms of ω using a standard fact on Coxeter groups:

the planar curve c_j closes up, iff either $\theta_j \in (\mathbb{Q}\setminus\mathbb{Z}) \cdot \pi$ or

(*) $\quad \theta_j \in \mathbb{Z} \cdot \pi \quad$ and $\quad \rho_j = 0 \quad ; \quad j = 1,2$.

Here we have made use of the following quantities:

i) the angle θ_j between two consequtive symmetry planes perpendicular to c_j, i.e. the angles:

$$\theta_1 = \sphericalangle (F'(0,0), F'(a,0)) = \int_0^a \sinh \omega(s,0)\, ds \quad ,$$

(1.5)

$$\theta_2 = \sphericalangle (\dot{F}(0,0), \dot{F}(0,b)) = \int_0^b \cosh \omega(0,t)\, dt \quad , \quad \text{and}$$

ii) the oriented distance function ρ_j between such a pair of planes, which is defined in case the planes are parallel. It is given by:

(1.6)

$$\rho_1 = e^{-\omega(c,0)} \cdot \langle F'(0,0), F(a,0) - F(0,0) \rangle$$

$$\rho_2 = e^{-\omega(0,0)} \cdot \langle \dot{F}(0,0), F(0,b) - F(0,0) \rangle$$

REMARKS:

i) The fact that we are working with a reflection group Γ_{ab} and a homomorphism Ψ_{ab} into $\mathrm{Isom}(E^3)$ implies directly that $\theta_1 \in \mathbb{Z} \cdot \pi$ or $\theta_2 \in \mathbb{Z} \cdot \pi$, i.e. that at least one pair of symmetry planes is parallel.

ii) In case $\omega \in \mathbb{M}^D_{ab} \subset \mathbb{M}^\Gamma_{ab}$, it is clear from the above formula that $\theta_1 = 0$.

iii) Wente has given a different closedness argument. His reasoning is based on the fact that for $\omega \in \mathbb{M}^D_{ab}$ the curves $s \longmapsto F\left(s,\left(n+\frac{1}{2}\right)b\right)$, $n \in \mathbb{Z}$, lie in planes tangential to the immersed surface. Along the same lines one can prove that for these more special solutions ω the curves $t \longmapsto F\left(\left(n+\frac{1}{2}\right)a,t\right)$; $n \in \mathbb{Z}$, lie on unit spheres which are tangential to the immersed surface. Moreover, if the surface closes up, all these spheres must coincide.

2. EXPLICIT SOLUTIONS OF THE SINH-GORDON EQUATION:

In this section we shall classify all real-analytic solutions $\omega : R^2 \longrightarrow R$ of the following overdetermined system:

$$(2.1)\qquad \begin{aligned} \Delta\omega + \sinh\omega \cdot \cosh\omega &= 0 \\ \sinh\omega \cdot \dot{\omega}' - \cosh\omega \cdot \dot{\omega}\omega' &= 0 \end{aligned} \quad .$$

Actually by elliptic regularity ω is automatically real-analytic, once it is assumed to be locally bounded and measurable.

We were lead to the second equation basically for geometric reasons; it is precisely the condition that all λ_1-curvature lines of the corresponding H-surface are planar curves (cf. formulae 1.4). It will lead to explicit solutions of the sinh-Gordon equation; it induces a separation of variables and reduces system 2.1 to solving ordinary differential equations. Likewise we could have asked for planar λ_2-curvature lines and supplement the sinh-Gordon equation with

$$(2.1.ii)' \qquad \cosh\omega \cdot \dot{\omega}' - \sinh\omega \cdot \dot{\omega}\omega' = 0$$

instead of equation 2.1.ii. We shall give the corresponding results in section 3 ; however, in section 4 it will turn out that none of the corresponding H-surfaces is compact.

To analyse system 2.1, we first consider the function $W = \cosh\omega$ and look for real-analytic solutions of the following system:

$$
(2.2)\qquad
\begin{aligned}
&(W^2-1)\cdot\Delta W - W\cdot|\nabla W|^2 + W\cdot(W^2-1)^2 = 0\\
&(W^2-1)\cdot\dot{W}' - 2W\cdot\dot{W}\cdot W' = 0 \quad .
\end{aligned}
$$

2.1. THEOREM:

The real-analytic solutions $W:\mathbf{R}^2\to\mathbf{R}$ *of system 2.2 are precisely the functions given by*

$$
(2.3)\qquad W = (1+f^2+g^2)^{-1}\cdot(f'+\dot{g}) \quad ,
$$

where $s\longmapsto f(s)$ *and* $t\longmapsto g(t)$ *are meromorphic functions of one variable which solve*

$$
(2.4)\qquad
\begin{aligned}
f'^2 &= f^4 + (1+c-d)f^2 + c\\
f'' &= 2f^3 + (1+c-d)f\\
\dot{g}^2 &= g^4 + (1-c+d)g^2 + d\\
\ddot{g} &= 2g^3 + (1-c+d)g
\end{aligned}
$$

for some constants $c,d\in\mathbf{R}$.

Moreover, f *and* g *can be recovered from* W *by*

$$(2.5) \qquad \begin{aligned} W' &= -f(s)\cdot(W^2-1) \\ \dot{W} &= -g(t)\cdot(W^2-1) \end{aligned} \qquad ,$$

except when $W^2 \equiv 1$.

The proof of this theorem and all the following results will be deferred to the end of the section.

REMARKS:

i) When $W^2 \equiv 1$, then clearly $(1+c-d)^2 - 4c = (1-c+d)^2 - 4d = 0$.

ii) The second order equations for f and g in 2.4 cannot be dropped, since they exclude certain enveloping solutions of the first order equations.

iii) We point out that $W = \pm 1$ at the poles of f and g . The theorem will be proved by purely local calculations, and we see that in fact any germ of a real-analytic solution W of system 2.2. extends uniquely to a globally bounded solution defined on all of $\mathbf{R}^2$.

The functions $s \longmapsto W(s,t)$ can be viewed as a family of elliptic functions parametrized by the elliptic curve determined by g .

2.2. PROPOSITION:

Let f,g *and* W *be as in Theorem 2.1. Then* W *is a solution of the following first order differential equations:*

$$W'^2 = \left(W^2-1\right) \cdot \left(\frac{c}{1+g^2} - (1+g^2)\left(W - \frac{\dot{g}}{1+g^2}\right)^2\right) \quad ,$$

(2.6)

$$\dot{W}^2 = \left(W^2-1\right) \cdot \left(\frac{d}{1+f^2} - (1+f^2)\left(W - \frac{f'}{1+f^2}\right)^2\right) \quad .$$

It remains to characterize the solutions W which arise as $\cosh\omega$ for some real-anlytic solution ω of system 2.1. Clearly $W \geq 1$, and hence $c = \alpha^2 \geq 0$ and $d = \beta^2 \geq 0$ (c.f. 2.6).

2.3. THEOREM:

i) *There exists a two-parameter family of real analytic solutions* ω *of system 2.1 which is defined on* $\mathfrak{P} = \{(\alpha,\beta) \in \mathbf{R}^2 \mid \alpha,\beta \geq 0$ *and* $\alpha + \beta \geq 1\}$ *by means of the equations:*

(2.7) $\cosh\omega = W = (1+f^2+g^2)^{-1} \cdot (f'+\dot{g}) \quad , \quad \omega(0,0) \geq 0$

(2.8)
$$\omega' = -f(s) \cdot \sinh\omega$$
$$\dot{\omega} = -g(t) \cdot \sinh\omega$$

where $f(s)$ *and* $g(t)$ *are the elliptic functions determined by:*

(2.9)
$$f'^2 = f^4 + (1+\alpha^2-\beta^2)f^2 + \alpha^2 \quad , \quad f(0)=0, \; f'(0)=\alpha$$
$$\dot{g}^2 = g^4 + (1-\alpha^2+\beta^2)g^2 + \beta^2 \quad , \quad g(0)=0, \; \dot{g}(0) = \beta \quad .$$

ii) *Conversely, if* ω *is any real-analytic solution of system 2.1, then* ω *is globally bounded,* $|\omega|$ *achieves its maximum, and up to a translation* ω *or* $-\omega$ *is contained in the above family.*

Observe that the ambiguity of arccosh in a neighborhood of a zero of ω is fixed by equations 2.8.

We shall briefly discuss the special solutions on the boundary ∂P of the parameter space. It is easily checked that the trivial solution $\omega \equiv 0$ corresponds to the whole segment $\alpha + \beta = 1$, $\alpha,\beta \geqq 0$ in ∂P. The rays $\alpha = 0$, $\beta \geqq 1$ and $\beta = 0$, $\alpha \geqq 1$ parametrize solutions ω which are one-dimensional in the sense that they either depend only on t or only on s. They describe the two types of Delaunay surfaces mentioned in section 1.

For the sake of simplicity let us now assume that $\alpha,\beta > 0$ and $\alpha + \beta > 1$. The qualitative behaviour of f depends in a crucial way on the question whether the quartic $f^4 + (1+\alpha^2-\beta^2)f^2 + \alpha^2$ has real zeroes or not. In fact, on the real axis f qualitatively resembles the functions tan, tanh, or sin, when $\beta > \alpha + 1$, $\beta = \alpha + 1$, or $\beta < \alpha + 1$, respectively. Another way to express this difference is the following: Since $\alpha \neq 0$, we can speak of the smallest positive zero $a = a(\alpha,\beta)$ of f. When $\beta > \alpha + 1$ $(\beta < \alpha + 1)$, then $a(\alpha,\beta)$ is a half period (resp. a full period) of f. On the borderline $\beta = \alpha + 1$ the function f has no positive zero and $a = +\infty$.

A similar statement holds for the smallest positive zero $b = b(\alpha,\beta)$ of g.

Finally we should point out that on $\mathfrak{P}$ the lattices of f and g viewed as elliptic functions defined on $\mathbb{C}$ are completely unrelated. In fact, one choice for the cross-ratios for f and g is:

$$\frac{1}{4\alpha}\left(\beta^2-(\alpha-1)^2\right) \quad \text{and} \quad \frac{1}{4\beta}\left(\alpha^2-(\beta-1)^2\right) .$$

2.4. PROPOSITION: (see Fig.3)

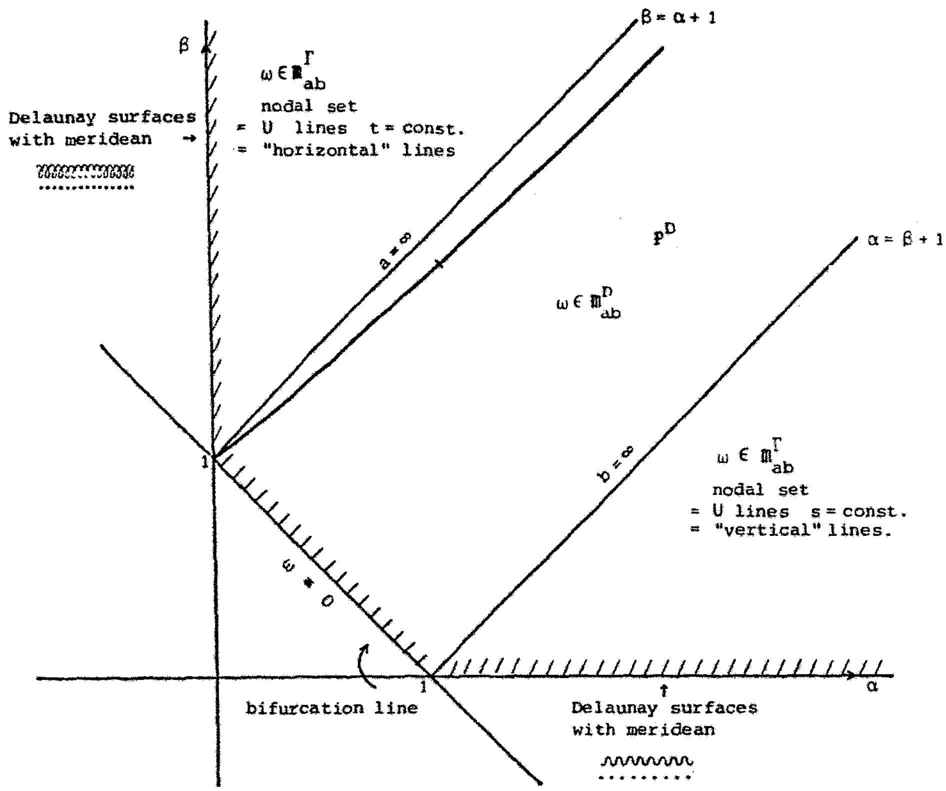
Suppose that $\alpha,\beta > 0$ *and* $\alpha+\beta > 1$.

i) *Except when* $|\alpha-\beta| = 1$, *the solution* ω *is invariant under the reflection group* Γ_{ab} *with* a *and* b *as above. (notation as introduced in section 1).*

ii) *If* $|\alpha-\beta| < 1$, *then the connected components of* $\{(s,t) \in \mathbb{R}^2 \mid \omega(s,t) \neq 0\}$ *are rectangles; more precisely* $\omega \in \mathfrak{M}^D_{ab}$ *with* a *and* b *as above . Otherwise* $\beta \geq \alpha+1$ *or* $\alpha \geq \beta+1$, *and the complement of the nodal set of* ω *consists of horizontal or vertical strips, respectively.*

iii) *The strip* $\mathfrak{P}^D = \{(\alpha,\beta) \in \mathfrak{P} \mid |\alpha-\beta| < 1\}$ *around the diagonal in the parameter space is mapped diffeomorphically onto the set* $\mathfrak{R}^D = \{(a_0,b_0) \in \mathbb{R}^2_+ \mid a_0^{-2} + b_0^{-2} \geq \pi^{-2}\}$. *This set contains the edgelengths of all rectangles* $\mathfrak{R}_{a_0 b_0}$ *which* *can arise as a connected component of the complement of the nodal set for a solution ω of the sinh-Gordon equation 1.2.*

Part iii) of this proposition asserts that we have not lost any rectangle, when imposing the additional condition 2.1.ii in the beginning of this section. In fact, the subfamily parametrized by $\mathcal{P}^D$ is just the family of "Dirichlet solutions" which bifurcates away from 0 along the curve of rectangles with $a_0^{-2}+b_0^{-2}=\pi^{-2}$. This gives a natural interpretation to the loss of uniqueness which we encountered in our parametrization for $\alpha+\beta=1$.



<u>Fig.3</u>

<u>FIGURE 3</u>: the different pieces of the parameter space $\mathcal{P}$.

PROOF of Theorem 2.1:

Step 1: a real analytic function $W : \mathbf{R}^2 \longrightarrow \mathbf{R}$ solves system 2.2, if and only if there exist meromorphic functions f and g of one variable only such that equations 2.3 and 2.5 hold.

Let us first consider the case when $W^2 \not\equiv 1$. Then equations 2.5 are just the integrated version of

$$0 = ((W^2-1)^{-1} \cdot \dot{W})' = ((W^2-1)^{-1} \cdot W')^{\cdot} \quad ,$$

i.e. of equation 2.2 ii . Equation 2.3 merely restates 2.2i in terms of f and g using equations 2.5.

In case $W^2 \equiv 1$, the only condition which is not a priori empty is equation 2.3. It is then a condition on f and g and leads to ordinary differential equations $f' = W \cdot (\alpha + f^2)$ and $\dot{g} = W \cdot (\beta + g^2)$ with $\alpha + \beta = 1$. These are clearly solvable. Hence we have not excluded the case $W^2 \equiv 1$ when passing from system 2.2 to the equations 2.3 and 2.5.

Step 2: equations 2.3 and 2.5 imply that with suitable constants $\bar{c}$ and $\bar{d}$, we have:

$$\text{(2.4)}' \qquad \begin{aligned} f'' &= 2f^3 + \bar{c} \cdot f \\ \ddot{g} &= 2g^3 + \bar{d} \cdot g \end{aligned} \quad .$$

Hence f and g are elliptic functions.

Substituting W into equation 2.5i yields:

(2.10) $$\frac{f''}{1+f^2+g^2} - f \cdot \frac{f'^2 - \dot{g}^2}{(1+f^2+g^2)^2} = f \quad .$$

Multiplying with $2f'$ and integrating w.r.t. s, we obtain with a suitable constant $k(t)$

$$\frac{f'^2 - \dot{g}^2}{1+f^2+g^2} = f^2 + k(t) \quad .$$

We multiply the above equations with $1+f^2+g^2$ and f, respectively, and add them up:

$$f'' = 2f^3 + (1+g^2(t) + k(t)) \cdot f \quad .$$

For fixed t this is just the claimed differential equation for f. Since f does not depend on t, we can in fact pick any of the values of $1 + g^2(t) + k(t)$ for $\bar{c}$. The differential equation for g follows similarly.

<u>Step 3</u>: The equation (2.4)' have first integrals:

(2.4)" $$\begin{aligned} f'^2 &= f^4 + \bar{c} \cdot f^2 + c \\ \dot{g}^2 &= g^4 + \bar{d} \cdot g^2 + d \quad . \end{aligned}$$

Moreover any solution to (2.4') and (2.4)" defines a real-analytic function $W : R^2 \longrightarrow R$ by means of formula 2.3.

<u>Claim</u>: These functions f,g and W obey equations 2.5, if f and g solve equations 2.4.

We insert formulae (2.4') and 2.4") into equation (2.10). It follows that equation 2.5i is equivalent to:

$$(2.11i) \qquad 0 = (\bar{c} + \bar{d} - 2) \cdot f \cdot g^2 + (d - c + \bar{c} - 1) \cdot f \quad .$$

Similarly equation 2.5ii is seen to be equivalent to

$$(2.11ii) \qquad 0 = (\bar{c} + \bar{d} - 2) \cdot f^2 \cdot g + (c - d + \bar{d} - 1) \cdot g \quad .$$

This proves the if-direction in the claim. In order to obtain the only-if part as well, let us first assume that $W \not\equiv 0$. Then we may further on assume w.r.g. that g is not a constant. If moreover f does not vanish identically, then it follows from equation 2.11i that $\bar{c} = 1 + c = d$ and $\bar{d} = 1 - c + d$. If $f \equiv 0$, then equation 2.1ii yields $\bar{d} = 1 - c + d$; the claim follows, since $\bar{c}$ can be modified arbitrarily. We point out that in this case c must be zero. The remaining case $W \equiv 0$ is even easier: equations 2.5 directly imply that $f \equiv g \equiv 0$, hence $c = d = 0$, and both $\bar{c}$ and $\bar{d}$ can be chosen arbitrarily.

∎

PROOF of Proposition 2.2:

It is sufficient to prove formula 2.6i. Because of formula 2.5i we must compute that

$$f^2 \cdot (W^2 - 1) = \frac{c}{1+g^2} - (1+g^2)(W - \frac{\dot{g}}{1+g^2})^2 \quad . \tag{2.12}$$

For this purpose we multiply equation 2.3 with

$$(1 + f^2 + g^2)W - 2\dot{g} = f' - \dot{g} \quad ,$$

and obtain:

$$f^2 \cdot W^2 + W \cdot ((1+g^2)W - 2\dot{g}) = \frac{f'^2 - \dot{g}^2}{1+f^2+g^2} = f^2 - g^2 + c - d \quad .$$

Equation 2.12 follows when collecting terms, using 2.4 once more.

∎

PROOF of Theorem 2.3:

i) We recall that system 2.2 follows from system 2.1 when substituting $W = \cosh \omega$. Conversely a real-analytic solution W of system 2.2 which is ≥ 1 defines a solution ω of the original system. A little care is necessary at the places where $W = 1$. Since the functions f and g in theorem 2.2 have only simple poles with principal parts $(s_\infty - s)^{-1}$ and $(t_\infty - t)^{-1}$, it follows from equations 2.5 that the solution W can take the value $+1$ only with multiplicity 2 , except when $W \equiv 1$. Hence a solution W which is > 1 at some point is ≥ 1 everwhere, and moreover there is no problem with the function arccosh being well-defined. We see that ω can have only simple zeroes, provided $\omega \not\equiv 0$, and equations 2.8 merely restate formulae 2.5 in the variable. The claim follows, since $W(0,0) = \alpha + \beta \geq 1$ and $W \equiv 1$ for $\alpha + \beta = 1$.

ii) It is sufficient to show that for any real-analytic solution $W \geq 1$ which is not identically 1 the functions f and g must have zeroes s_0 and t_0 such that $\alpha = f'(s_0) \geq 0$ and $\beta = \dot{g}(t_0) \geq 0$.

Let us first show that f must have a zero s_0 . We view equation 2.5i as a Riccati equation for the bounded functions $s \longmapsto W(s,t)$. It follows that f cannot be bounded away from zero uniformly on $\mathbf{R}$. If f had no zero at all, we could hence conclude from equation 2.4i that $c = 0$ and $d \leq 1$. Passing to the limit $f \longrightarrow 0$ the condition $W \geq 1$ would give $\dot{g} \geq 1 + g^2$ so that $d \geq 1$ because of 2.4iii. Combining these inequalities, we would get $f' = f^2$ and $\dot{g} = 1 + g^2$, hence $W \equiv 1$, the desired contradiction.

Let us now suppose that f had only zeroes with f' negative. Since all poles of f have principal parts $(s_\infty - s)^{-1}$, we conclude that f is monotone decaying with one zero only. But then W must become singular because of the Riccati equation 2.5i, again a contradiction.

The same arguments apply to g , hence the proof.

∎

PROOF of Proposition 2.4:

i) & ii) One can easily see from formula 2.8 that ω is even at all zeroes of f and g and odd at all poles of these functions, hence both the claims.

iii) First let us analyse the map $\mathcal{P}^D \longrightarrow \mathcal{R}^D \subset \mathbf{R}_+^2$, $(\alpha,\beta) \longmapsto (a(\alpha,\beta), b(\alpha,\beta))$. It will be convenient to parametrize $\mathcal{P}^D$ as follows:

$$\alpha = m+\delta \ , \ \beta = m-\delta \ , \quad (m,\delta) \in \left[\frac{1}{2}, \infty\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \quad . \tag{2.13}$$

The mapping is then given by:

$$\begin{aligned} a(m+\delta, m-\delta) &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(m+\delta)(1+x^4)+(1+4m\delta)x^2}} \\ b(m+\delta, m-\delta) &= \int_{-\infty}^{\infty} \frac{dy}{\sqrt{(m-\delta)(1+y^4)+(1-4m\delta)y^2}} \end{aligned} \quad . \tag{2.14}$$

It follows that $\frac{\partial a}{\partial m}$, $\frac{\partial a}{\partial \delta}$, and $\frac{\partial b}{\partial m}$ are all less than zero, whereas $\frac{\partial b}{\partial \delta} > 0$. Hence the map (a,b) is injective on $\mathcal{P}^D$ and has maximal rank. To prove that its image is all of $\mathcal{R}^D$, it is sufficient to know the following limits and boundary values:

$$
(2.15)\quad
\begin{array}{llll}
a = \pi/\sqrt{\frac{1}{2}+\delta} & , \ b = \pi/\sqrt{\frac{1}{2}-\delta} & \text{for} & m = \frac{1}{2} \ , \\
a \to 0 & , \ b \to 0 & \text{for} & m \to \infty \ , \\
a \to +\infty & , \ b \to \pi/\sqrt{\frac{1}{2}+m} & \text{for} & \delta \to -\frac{1}{2} \ , \\
\text{and} \quad a \to \pi/\sqrt{\frac{1}{2}+m} & , \ b \to +\infty & \text{for} & \delta \to \frac{1}{2} \ .
\end{array}
$$

It remains to show that on rectangles greater than those parametrized by $\mathbb{R}^D$ there exists no positive solution of the sinh-Gordon equation 1.2 with zero boundary values. Note that ω can be viewed as the first Dirichlet-eigenfunction of the linear operator $-\Delta\Psi - \frac{\sinh\omega}{\omega}\cdot\Psi$. This operator is bounded from above by $-\Delta\Psi - \Psi$; the claim follows since λ_1 for this operator is known on rectangles.

■

3.) THE SECOND FAMILY OF EXPLICIT SOLUTIONS

This time the starting point is the system

$$(3.1)\qquad \begin{aligned} \Delta\omega + \sinh\omega \cdot \cosh\omega &= 0 \\ \cosh\omega \cdot \dot{\omega}' - \sinh\omega \cdot \dot{\omega}\omega' &= 0 \quad . \end{aligned}$$

The discussion begins completely analogous to what has been done in section 2; only sinh and cosh are interchanged, and hence a couple of signs will be different.

The substitution $W = \sinh\omega$ yields

$$(3.2)\qquad \begin{aligned} (W^2+1)\Delta W - W \cdot |\nabla W|^2 + W \cdot (W^2+1)^2 &= 0 \\ (W^2+1)\dot{W}' - 2 \cdot W \cdot \dot{W}W' &= 0 \quad . \end{aligned}$$

3.1. THEOREM:

The real solutions $W : \mathbf{R}^2 \to \mathbf{R}$ *of system 3.2 are precisely the functions given by:*

$$(3.3)\qquad W = (1 + f^2 + g^2)^{-1} \cdot (f' + \dot{g})$$

where $s \longmapsto f(s)$ *and* $t \longmapsto g(t)$ *are real-valued functions of one variable which solve*

$$(3.4)\qquad \begin{aligned} - f'^2 &= f^4 + (1+c-d)f^2 + c \\ - f'' &= 2f^3 + (1+c-d)f \\ - \dot{g}^2 &= g^4 + (1-c+d)g^2 + d \\ - \ddot{g} &= 2g^3 + (1-c+d)g \end{aligned}$$

for some constants $c, d \leq 0$.

Moreover f *and* g *can be recovered from* W *by*

$$(3.5)\qquad \begin{aligned} W' &= -f(s)\cdot(W^2+1) \\ \dot{W} &= -g(t)\cdot(W^2+1) \end{aligned} \quad .$$

The proof is almost the same as for theorem 2.1, except that the special case $W^2 \equiv 1$ has no counterpart. On the other hand we have encountered the conditions $c, d \leq 0$ when requiring that the equations 3.4 have real-valued solutions f and g .

3.2. PROPOSITION:

If f,g, *and* W *are as in Theorem 3.1, then:*

$$(3.6)\qquad \begin{aligned} W'^2 &= (W^2+1)\cdot\left(\frac{-c}{1+g^2} - (1+g^2)\left(W - \frac{\dot{g}}{1+g^2}\right)^2\right) \\ \dot{W}^2 &= (W^2+1)\cdot\left(\frac{-d}{1+f^2} - (1+f^2)\left(W - \frac{f'}{1+f^2}\right)^2\right) \end{aligned} \quad .$$

Since $\sinh : \mathbf{R} \to \mathbf{R}$ is a diffeomorphism there is no problem in translating back to the original variable ω . Equations (3.5) become:

$$(3.7)\qquad \begin{aligned} \omega' &= -f(s)\cdot\cosh\omega \\ \dot{\omega} &= -g(t)\cdot\cosh\omega \end{aligned} \quad .$$

The normalization to $f(0) = g(0) = 0$, $f'(0) = \alpha \geq 0$, $g'(0) = \beta \geq 0$, i.e. $c = -\alpha^2$ and $d = -\beta^2$ is also easy. It follows directly from the fact that both f and g oscillate around 0 .

The analoque of Proposition 2.4 becomes just as easy as that of Theorem 2.3: for all $\alpha, \beta > 0$ the solution $\omega = \operatorname{arcsinh} W$ lies in $\mathfrak{M}^{\Gamma}_{ab} \setminus \mathfrak{M}^{D}_{ab}$ where $a = a(\alpha,\beta)$ and $b = b(\alpha,\beta)$ again denote the smallest positive zero of f and g , respectively.

Note that the axis $\alpha = 0$ or $\beta = 0$ again parametrize "one-dimensional" solutions, the Delaunay surfaces.

∎

4.) GEOMETRIC INTERPRETATION OF THE SOLUTIONS ω :

In this section we shall concentrate on the family of solutions to the Sinh-Gordon equation which has been constructed in Theorem 2.3. We shall refer to it as family $\mathcal{A}$ in the sequel. Some of our results will have no analogue for the H-surfaces obtained from family $\mathcal{B}$, the explicit solutions given in section 3. For instance, the latter family contains no closed H-tori, whereas family $\mathcal{A}$ does. In fact, the evaluation of the closedness criterion is a major goal in this section .

To begin with, we carry out one more integration in the Frenet equations for the families of planar curvature lines. In case of family $\mathcal{A}$ these are the λ_1 - curvature lines, and we consider the functions

$$(4.1)\qquad \begin{aligned} \varphi(s,t) &= \measuredangle\,(F'(s,t),\ F'(0,t)) \qquad \text{and} \\ U(s,t) &= \cos\varphi(s,t) = e^{-\omega(s,t)-\omega(0,t)} \cdot \langle F'(s,t), F'(0,t)\rangle \ . \end{aligned}$$

4.1. LEMMA:

$$(4.2)\qquad \alpha \cdot U = (1+g^2)\cdot W - \dot{g} = f' \cdot \frac{1+g^2}{1+f^2+g^2} - f^2 \cdot \frac{\dot{g}}{1+f^2+g^2} \quad .$$

$$(4.3)\qquad \begin{aligned} U'^2 &= (1+g^2)^{-1}\cdot(1-U^2)\cdot((\alpha U+\dot{g})^2 - (1+g^2)^2) \\ \dot{U}^2 &= (1+f^2)^{-1}\cdot(1-U^2)\cdot((\beta U+f')^2 - (1+f^2)^2) \quad . \end{aligned}$$

PROOF: Clearly $\varphi' = |F'| \cdot \kappa_1 = \sqrt{1+g^2} \cdot \sinh \omega = \sqrt{1+g^2} \cdot \sqrt{W^2-1}$.

Hence we obtain, using formula 2.6:

$$\frac{d\varphi}{dW} = \frac{\varphi'}{W'} = -\left(\frac{\alpha^2}{(1+g^2)^2} - \left(W - \frac{\dot{g}}{1+g^2}\right)^2\right)^{-\frac{1}{2}} .$$

Formula 4.2 follows by integration, checking that $W(0,t) = \frac{\dot{g}+\alpha}{1+g^2}$ and $U(0,t) = 1$ by definition.

Formulae (4.3) are just restatements of equation 2.6.

■

4.2. DESCRIPTION OF THE H-SURFACES WITH PLANAR λ_1-CURVATURE LINES:

i) *the angle between the normal vector* N *on the surface and any plane containing a* λ_1*-curvature line is constant along this curve: The tangent of this angle is the function* g *which has been introduced in section 2 for purely analytical reasons.*

ii) *When* $\beta > \alpha + 1$, *then* $\theta_1 = \pi$ *and* $\rho_1 < 0$ *(see (1.6)) except when* $\alpha = 0$. *The* λ_1*-curvature lines look like a sequence of the letter* ℓ ; *one should imagine that the circles of the Delaunay surface burst like this, when one tries to bend the surface.*

iii) *When* $\beta < \alpha + 1$ *, then* $\theta_1 = 0$ *and the tangent vector of a* λ_1*-curvature line oscillates around the normal on a symmetry plane, intersected perpendicularly by the curvature line. The amplitude in angle is less than* π *. The only closed planar curves have the shape of a figure 8.*

iv) *When* $\beta < \alpha - 1$ *, then the tangent vector of the* λ_1*-curvature lines* $s \longmapsto F(s,(n+\frac{1}{2})\cdot b)$ *,* $n \in \mathbb{Z}$ *, remains in a fixed open halfspace; its projection onto* $F'(0,(n+\frac{1}{2})\cdot b)$ *is positive everywhere. Therefore none of the planar* λ_1*-curvature lines can close up.*

v) *In order to get an immersion with closed planar* λ_1*-curvature lines, it is necessary that either* $\alpha = 0$ *(→ Delaunay surfaces) or that* (α,β) *lies in the open diagonal strip* P^D *.*

PROOF: i) this follows directly from formulae 1.4 and 2.8.

ii,iii) We use the lemma. By a homotopy argument it is sufficient to consider only those λ_1-curvature lines $s \longmapsto f(s,t)$ for which $g(t) = 0$ and $\dot{g}(t) = \beta$. Then the quartic on the right hand side of equation 4.3i has zeroes:

$$-1 \ , \ 1 \ , \ -\frac{\beta-1}{\alpha} \ , \ -\frac{1+\beta}{\alpha} \ .$$

It follows that for $\beta < \alpha + 1$, the function $U = \cos\varphi$ oscillates between $\frac{1-\beta}{\alpha}$ and 1 , hence claim iii). In case

$\beta > \alpha + 1$ we see that $\theta_1 = \pi$ and U oscillates between -1 and 1. In order to see that $\rho_1 < 0$, we compute:

$$\left|\frac{dF}{d\varphi}\right| = \left|\frac{F'}{\varphi'}\right| = \frac{1}{\kappa_1} .$$

Now it is sufficient to observe that $\kappa_1 = \sqrt{1+g^2} \cdot \frac{1}{2}(1-e^{-2\omega})$ is positive and monotone decreasing in s on $[0,a]$.

iv) The hypothesis $\beta < \alpha - 1$ implies that the quadric $G^2 + (1-\alpha^2+\beta^2)G + \beta^2$ has a smallest positive root G in the interval $(0,\alpha-1)$. Hence the function g must oscillate between $-\sqrt{G}$ and $+\sqrt{G}$, taking its extremal values for $t = (n+\frac{1}{2}) \cdot b$, $n \in \mathbf{Z}$. Inserting this information into equation 4.3i, we see that U' vanishes if and only if $U = \pm 1$ or $U = \pm \frac{1}{\alpha} \cdot (G+1)$. Since $U(0) = 1$, it is now clear that U oscillates in the interval $\left[\frac{1}{\alpha}(G+1), 1\right]$. In particular, U remains positive everywhere. The claim now follows from the definition of U and from Proposition 1.7.

v) This statement essentially summarizes information contained in the previous parts of the lemma. Only the boundary of $\mathcal{P}^D$ in the parameter space $\mathcal{P}$ has to be considered in addition. When $\beta = \alpha + 1$, the curvature function of the λ_1-curvature lines is not periodic (since $f(s) = \sqrt{\alpha} \cdot \tanh \sqrt{\alpha}\, s$), and the case $\beta = \alpha - 1$ can be included as a limit of the argument given in iv).

■

For $(\alpha,\beta) \in \mathfrak{P}^D$, the function g has poles, and hence there are special λ_1-curvature lines which lie in planes tangential to the surface. It turns out that closedness of the λ_1-curvature lines, i.e. the condition $\rho_1 = 0$, can be easily checked for these "singular" λ_1-curvature lines. It is a straightforward computation, using formula 2.9, to pass to the limit $g \to \infty$ in equations (4.2) and (4.3i):

$$\begin{aligned} U_\infty &= f' - f^2 \\ U_\infty'^2 &= 2\alpha \cdot (U_\infty + q)(1 - U_\infty^2) \end{aligned} \qquad (4.4)$$

where $1 + q = \frac{1}{2\alpha}(\beta^2 - (\alpha-1)^2)$, which is twice a cross ratio of f . Recall that $|F'| = 1$ along these special curvature lines. So they are up to congruence uniquely determined by the function $U_\infty = \cos\varphi_\infty$. Observe that $q \in (-1,1)$ since we have restricted ourselves to $(\alpha,\beta) \in \mathfrak{P}^D$.

4.3. LEMMA:

Closedness of the λ_1-curvature lines is a condition on q, i.e. on the conformal type of the lattice of the elliptic function f :

$$0 = \int_{-q}^{1} \frac{U\,dU}{\sqrt{(1-U^2)(U+q)}} \qquad (4.5)$$

This equation determines a unique $q \in (0,1)$ *, which is approximately* 0.652229... .

The above reasoning showes that the right-hand side of formula 4.5 is just $\frac{1}{2}\sqrt{2\alpha}$ times the quantity ρ_1 introduced in fomula 1.6. This explains the lemma.

In fact, the special λ_1-curvature lines enjoy a nice geometric property:

4.4. PROPOSITION:

All λ_1*-curvature lines (of family* $\mathbb{A}$ *) which lie in a plane tangential to the H-surface are elasticae, i.e. critical points of the absolute squared curvature functional*

$$F(c) = \int_c |\kappa(s)|^2 \, ds \quad .$$

PROOF: We shall compute that

$$(4.6) \qquad \kappa_1'^2 = -\frac{1}{4}\left(\kappa_1^2 - 2\alpha(1+q)\right)\left(\kappa_1^2 + 2\alpha(1-q)\right) \quad .$$

Since on these curves s is indeed arclength, we have thus verified the Euler equations of F in the plane (c.f. [LS]). Here α is a scaling parameter whereas q determines the shape of the curve.

Clearly we have $\kappa_1 = \varphi'$. Therefore:

$$\kappa_1^2 = (1 - U_\infty^2)^{-1} \cdot U_\infty'^2 = 2\alpha(U_\infty + q) \quad ,$$

and formula 4.6 follows directly from equation 4.4.

∎

It remains to investigate, when the λ_2-curvature lines close up as well. Using Proposition 2.2 and Theorem 2.3, we see that the functional θ_2 , introduced in formula 1.5, becomes:

$$\theta_2 = \int_0^b W(0,t)\,dt = \int_0^b \frac{\alpha+\dot{g}}{1+g^2}\,dt$$

i.e.

$$(4.7)\qquad \theta_2 = \begin{cases} \pi + \displaystyle\int_{-\infty}^{\infty} \frac{\alpha \cdot dg}{(1+g^2)\sqrt{g^4+(1-\alpha^2+\beta^2)g^2+\beta^2}} & \text{for } \alpha<\beta+1 \\[2ex] \displaystyle\int_{\alpha-\beta}^{\alpha+\beta} \frac{W\,dW}{\sqrt{(W^2-1)(\alpha+\beta-W)(\beta-\alpha+W)}} & \text{for } \alpha>\beta+1 \end{cases} .$$

We recall that in the interior of the parameter space $\mathcal{P}$ the set $\rho_1 = 0$ is given by the hyperbola:

$$(4.8)\qquad \beta^2 = (\alpha+q)^2 + 1 - q^2 ,$$

where q is as in Lemma 4.3. This curve clearly lies in the strip $\alpha < \beta < \alpha + 1$; it begins at the point $(\alpha,\beta) = (0,1)$.

LEMMA 4.5:

i) θ_2 *is monotone increasing along the curve* $\rho_1 = 0$.

ii) $\theta_2(0,1) = \pi$

iii) $\lim_{\alpha\to\infty} \theta_2(\alpha,\alpha+\varepsilon) = 2\pi$, *uniformly for* $\varepsilon \in [0,1]$.

PROOF:

i) From formula 4.7 we calculate that for $\beta > \alpha$

$$\frac{\partial}{\partial\alpha}\,\theta_2 = \int_{-\infty}^{\infty} \frac{(g^2+\beta^2)\,dg}{\sqrt{g^4+(1-\alpha^2+\beta^2)g^2+\beta^2}^{\,3}} > 0$$

and

$$\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\,\theta_2 = \int_{-\infty}^{\infty} \frac{g^2+\beta\cdot(\beta-\alpha)}{\sqrt{g^4+(1-\alpha^2+\beta^2)g^2+\beta^2}^{\,3}} > 0 \quad .$$

The tangent vector of the curve $\rho_1 = 0$ is a convex combination of $\frac{\partial}{\partial\alpha}$ and $\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}$.

iii) Since the factor $\alpha/\sqrt{g^4+(1+2\alpha\varepsilon+\varepsilon^2)g^2+(\alpha+\varepsilon)^2}$ in formula 4.7 converges monotonically to 1 for $\alpha \longrightarrow \infty$, it is clear that

$$\lim_{\alpha\to\infty} \theta_2(\alpha,\alpha+\varepsilon) = \int_{-\infty}^{\infty} \frac{dg}{1+g^2} \quad .$$

Uniformity in ε follows from the inequality:

$$\theta_2(\alpha,\alpha+1) \leq \theta_2(\alpha,\alpha+\varepsilon) \leq \theta_2(\alpha,\alpha) \quad .$$

∎

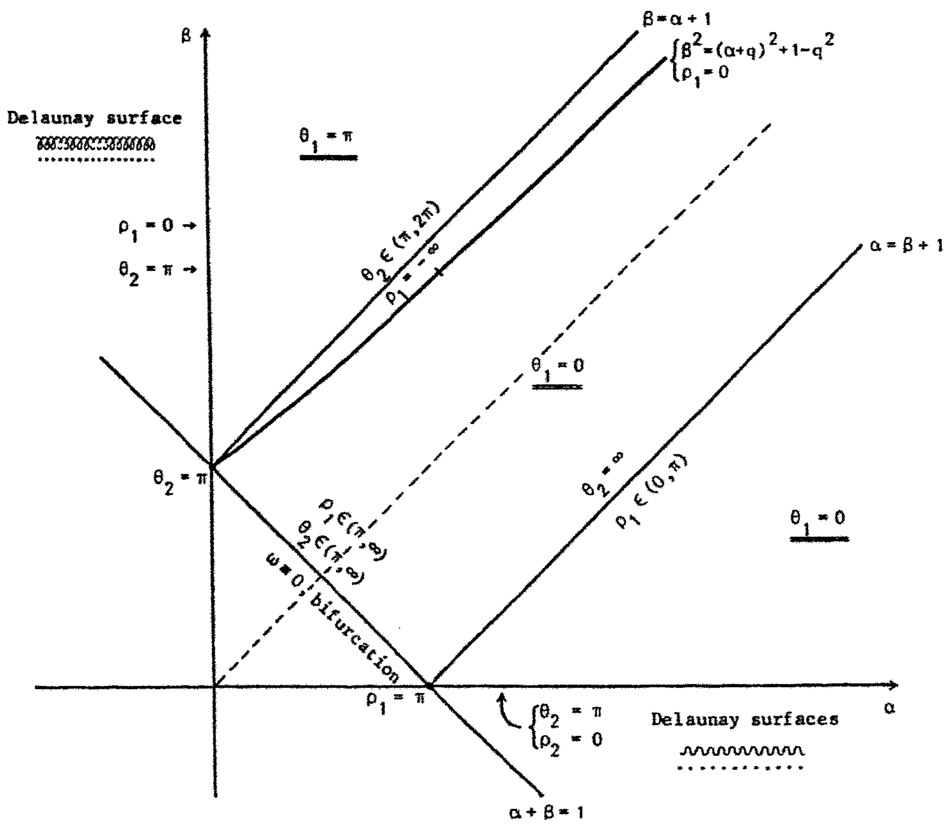


Fig.4

FIGURE 4: the parameter space $\mathbb{P}$ and the closedness discussion in terms of $\theta_1, \rho_1, \theta_2$, and ρ_2.

We summarize the above discussion (recall that $\lambda_1 < \lambda_2$ by convention):

4.6. THEOREM:

For any $\theta \in (\pi, 2\pi)$ *such that* $\frac{1}{2\pi} \cdot \theta$ *is rational there exists precisely one torus with* $H \equiv 1/2$ *, planar* λ_1*-curvature lines, and* $\theta_2 = \theta$ *. These are all immersed tori in* $\mathbf{E}^3$ *with* $H \equiv 1/2$ *and planar* λ_1*-curvature lines.*

It follows that the index of the deck transformation group in Λ_ω , the invariance group of the curvature functions, is always ≥ 3 . In this sense the simplest possible H-torus in $\mathbf{E}^3$ has $\theta_2 = \frac{2}{3} \cdot 2\pi$. We shall depict this torus in Fig. 5.

Not only the pictures look pretty non-standard. It is a matter of fact that this immersion of a torus into $\mathbf{E}^3$ is not regularly homotopic to a standard immersion. This follows directly when counting the number of twists for strips around curves representing a homology basis. (cf. [PIN]).

Finally we point out that Lemma 4.1 and Proposition 4.2 carry over to the second family of solutions with planar λ_2-curvature lines. It will turn out that $\theta_2 = \pi$ and the λ_2-curvature lines look like a sequence of the letter ℓ - i.e. $\rho_2 \neq 0$ - except when they are the circles of a Delaunay surface.

4.7. COROLLARY:

In Theorem 4.6 we have in fact classified all H-tori with one family of planar curvature lines.

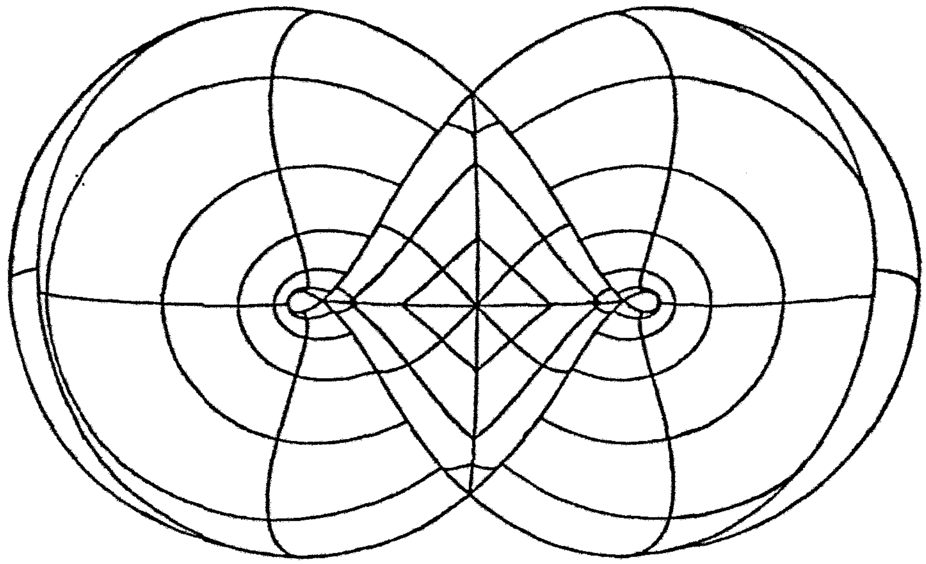


Fig.5a

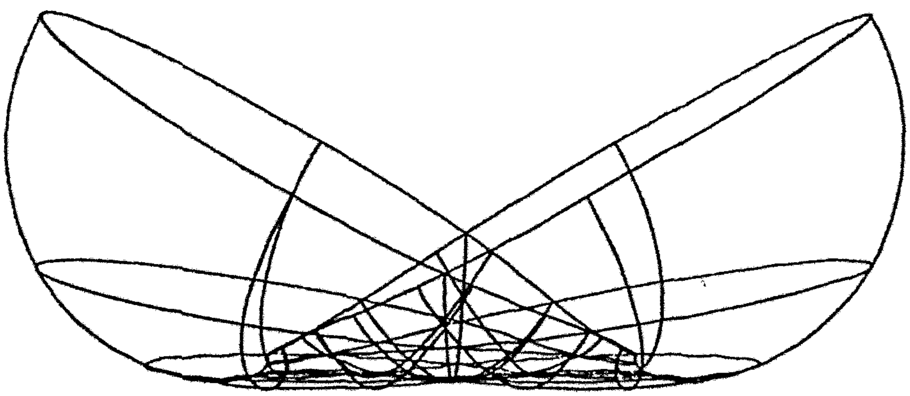


Fig.5b

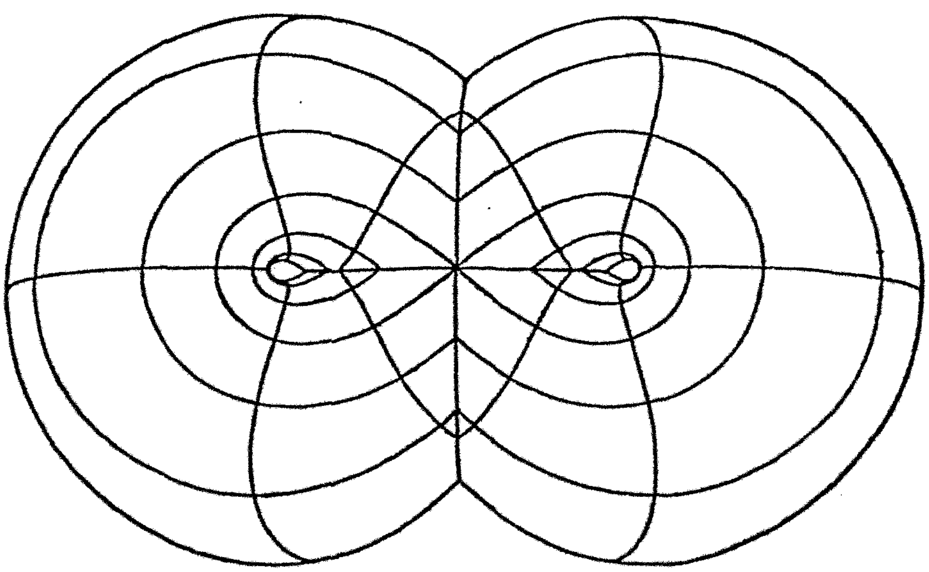


Fig.5c

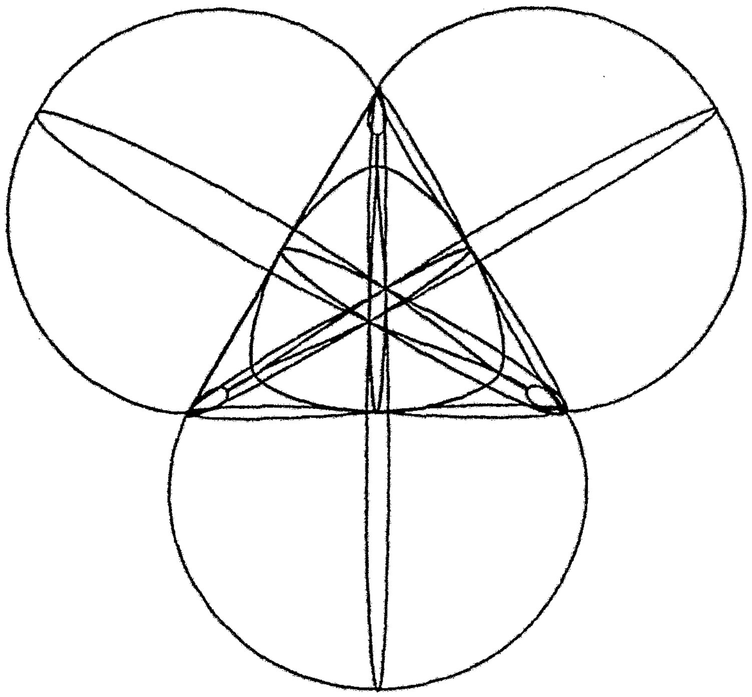


Fig.5d

FIGURE 5: curvature lines on the constant mean curvature torus with $\theta = \frac{2}{3} \cdot 2\pi$.

a-c) top, side, and bottom view of the cylindrical piece parametrized by an appropriate fundamental domain of the corresponding solution ω . It is a cylinder over a planar figure 8, which changes its shape.

d) skew parallel projection (like in Fig.5b) of the full torus which is glued from 3 congruent cylindrical pieces. All non-planar λ_2-curvature lines have been suppressed; whereas all self-intersection lines have been included.
For a more schematic view compare Fig.1a.

$\frac{m}{n} = \frac{4}{7}$

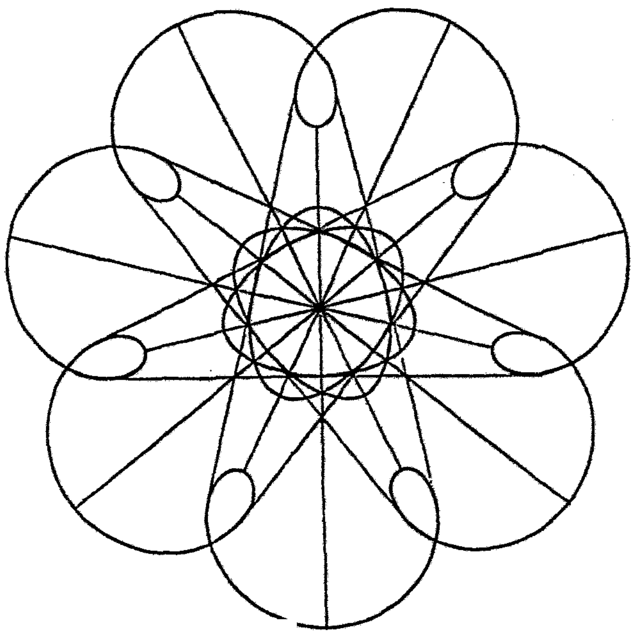


Fig.6a

$\frac{m}{n} = \frac{5}{7}$

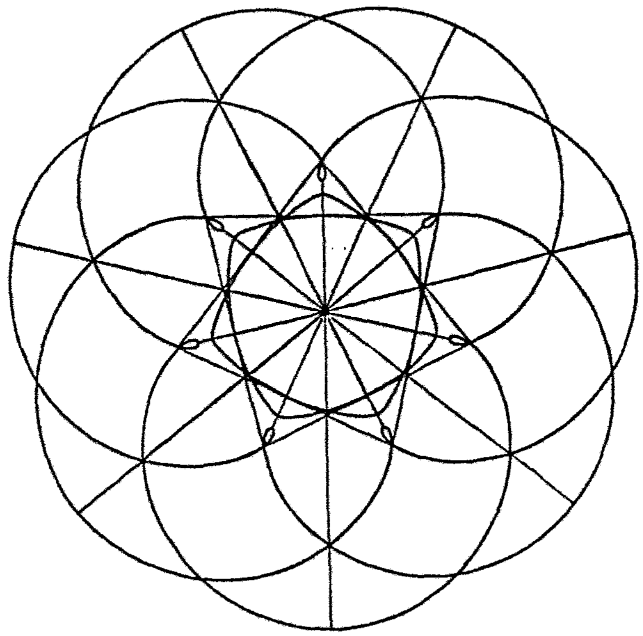


Fig.6b

FIGURE 6: two more examples show how the planar λ_2-curvature line in the central symmetry plane and the self-intersection line in this plane can look. When compared with Fig.1 and Fig.5 these examples illustrate how the global picture of the H-torus jumps when changing the angle parameter θ ; the fundamental cylindrical pieces (cf. Fig.5a-c)) nevertheless vary smoothly.

5. ON THE SPECTRUM OF THE LINEARIZED SINH-GORDON EQUATION:

In this section we consider for an arbitrary function $\omega \in \mathfrak{m}^D_{ab}$ the linearized operator:

$$L_\omega(\varphi) := -\Delta\varphi - \cosh(2\omega)\cdot\varphi \quad .$$

We are interested in the Dirichlet spectrum of L_ω on the rectangle $R_{ab} = -\frac{a}{2}, \frac{a}{2} \times -\frac{b}{2}, \frac{b}{2}$. Our basic tool will be the domain dependance of the first Dirichlet eigenvalue $\lambda_1(L_\omega, \Omega)$. We shall consider the half-rectangles

$$R^{11}_{ab} = \left(-\frac{a}{2}, 0\right) \times \left(-\frac{b}{2}, \frac{b}{2}\right) \quad , \quad R^{12}_{ab} = \left(0, \frac{a}{2}\right) \times \left(-\frac{b}{2}, \frac{b}{2}\right) \quad ,$$

$$R^{21}_{ab} = \left(-\frac{a}{2}, \frac{a}{2}\right) \times \left(-\frac{b}{2}, 0\right) \quad , \quad R^{22}_{ab} = \left(-\frac{a}{2}, \frac{a}{2}\right) \times \left(-0, \frac{b}{2}\right) \quad ,$$

and put

$$\lambda_{hor} = \lambda_1\left(L_\omega, R^{11}_{ab}\right) = \lambda_1\left(L_\omega, R^{12}_{ab}\right)$$

$$\lambda_{vert} = \lambda_1\left(L_\omega, R^{21}_{ab}\right) = \lambda_1\left(L_\omega, R^{22}_{ab}\right)$$

Then

$$\lambda_{odd} := \min\{\lambda_{hor}, \lambda_{vert}\}$$

is the smallest Dirichlet eigenvalue such that one of the corresponding eigenfunctions is odd w.r.t. at least one axis of the rectangle R_{ab} .

5.1. PROPOSITION:

i) $\lambda_{odd} > 0$

ii) $\lambda_1(L_\omega, R_{ab}) < 0$, *i.e.* ω *is an unstable solution of the sinh-Gordon equation.*

PROOF:

i) Since $-\Delta\omega - \frac{1}{2}\sinh 2\omega = 0$, it is clear that $L_\omega(\omega') = 0$. Considering the nodal set of ω' , we see that $\lambda_{hor} > \lambda_1\left(L_\omega, (-a,0) \times \left(-\frac{b}{2}, \frac{b}{2}\right)\right) = 0$. Similarly we prove that $\lambda_{vert} > 0$.

ii) This follows from the inequality $L_\omega < \mathcal{P}_\omega$, where

$$\mathcal{P}_\omega(\varphi) = -\Delta\varphi - \frac{\sinh 2\omega}{2\omega} \cdot \varphi \quad .$$

Notice that $\lambda_1(\mathcal{P}_\omega, R_{ab}) = 0$!

∎

5.2. THEOREM:

If $\lambda < \lambda_{odd}$ *is a Dirichlet eigenvalue of* L_ω *on* R_{ab} *, then the corresponding eigenspace* $E(\lambda)$ *is 1-dimensional. Especially,* $\dim \ker L_\omega \leq 1$.

We shall establish two lemmas first.

5.3. LEMMA:

Let φ *be an eigenfunction of* L_ω *with eigenvalue* λ . *Suppose that one component* C *of* $\{(s,t) \in R_{ab} \mid \varphi(s,t) \neq 0\}$ *is contained in one of the four half-rectangles* R_{ab}^{11}, R_{ab}^{12}, R_{ab}^{21}, *or* R_{ab}^{22} .

Then $\lambda \geq \lambda_{odd} > 0$.

This follows directly from Proposition 5.1i and the domain monotonicity of Dirichlet eigenvalues. It is an easy corollary that all Dirichlet eigenfunctions with eigenvalue $\lambda < \lambda_{odd}$ must be even w.r.t. both axes of the rectangle R_{ab} .

5.4. LEMMA:

Suppose that φ *is a Dirichlet eigenfunction of* L_ω *with eigenvalue* $\lambda < \lambda_{odd}$. *Then* φ *does not change sign in a sufficiently small neighborhood* U *of* ∂R_{ab} .

PROOF:

Let us assume conversely that there are components C_+ and C_- of the sets $\{(s,t) \in R_{ab} \mid \varphi(s,t) < 0\}$ resp. $\{(s,t) \mid \varphi(s,t) > 0\}$ which touch the boundary. By symmetry we may assume moreover that both C_+ and C_- intersect the right upper quadrant $\left[0, \frac{a}{2}\right) \times \left[0, \frac{b}{2}\right)$. It follows from the Jordan curve theorem that one of the components C_+ or C_- must lie entirely either in the half-rectanle R_{ab}^{12} or in R_{ab}^{22}. Lemma 5.3 therefore yields the contradiction $\lambda \geq \lambda_{odd}$.

∎

PROOF of the theorem:

Suppose that there are two linearly independant eigenfunctions $\varphi, \tilde{\varphi} \in E(\lambda)$. A well-known argument based on the maximum principle shows that the normal derivative $\partial_\nu \varphi$ vanishes nowhere on the boundary ∂R_{ab} except at the four corners (cf. [GNN]). Since by Lemma 5.4 none of the linear combinations $\tilde{\varphi} - c \cdot \varphi$, $c \in \mathbf{R}$, changes sign in a small neighborhood of the boundary ∂R_{ab}, we conclude that $\partial_\nu \tilde{\varphi} = c_0 \cdot \partial_\nu \varphi$ for a suitable constant $c_0 \in \mathbf{R}$. Using the maximum principle as above, we see that $\tilde{\varphi} - c_0 \cdot \varphi = 0$, a contradiction.

∎

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