ON EXACT SOLUTIONS OF THE KAWAHARA, MODIFIED KAWAHARA, KK AND SK EQUATIONS USING MAPPING METHOD

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Abstract. The mapping method is used by the aid of symbolic computation system Mathematica for constructing exact solutions of the Kawahara, modified Kawahara, Kaup-Kupershmidt and Sawada-Kotra equations. The solutions obtained in this letter include Jacobi elliptic solutions, combined Jacobi elliptic solutions, rational solutions, triangular solutions, soliton solutions and combined soliton solutions.

1. Introduction

In the past decades, there has been significant progress in the development of methods for obtaining exact solutions to the nonlinear partial differential equations (NLPDEs). Many powerful methods have been presented such as Bäcklund transformation [1], Darboux transformation [2], Jacobi elliptic function method [3], tanh-function method [4], sine-cosine function method [5,6], homogenous balance method [7], modified tanh-function method [8] and mapping method [9-11]. Among these methods, some are indirect methods. The key idea of these generalized indirect methods is to use the solutions of an auxiliary ordinary differential equation (ode) to find solutions of the NLPDEs. In this letter, we will use the mapping method developed by peng [10] to obtain abundant solutions of the Kawahara, modified Kawahara, Kaup-Kupershmidt (KK) and Sawada-Kotra (SK) equations. The new auxiliary ode reads [10]

$$(\frac{d\phi}{d\xi})^2 = r + a\phi^2(\xi) + b/(2)\phi^4(\xi), \quad \xi = px + \omega t, \tag{1}$$

where r, a and b, are constants. We shall construct the exact solutions of the Kawahara, modified Kawahara, KK and SK equations by using the following solutions of Eq.(1)

$$\phi(\xi) = \begin{cases} sn\ \xi;\ cd\ \xi = cn\ \xi/dn\ \xi & : \quad r=1,\ \ a=-(m^2+1),\ \ b=2m^2; \\ cn\ \xi & : \quad r=1-m^2,\ \ a=2m^2-1,\ \ b=-2m^2; \\ dn\ \xi & : \quad r=m^2-1,\ \ a=2-m^2,\ \ b=-2; \\ ns\ \xi=1/sn\ \xi;\ dc\ \xi = dn\ \xi/cn\ \xi & : \quad r=m^2,\ \ a=-(m^2+1),\ \ b=2; \\ nc\ \xi = 1/cn\ \xi & : \quad r=-m^2,\ \ a=2m^2-1,\ \ b=2(1-m^2); \\ nd\ \xi = 1/dn\ \xi & : \quad r=-1,\ \ a=2-m^2,\ \ b=2(m^2-1); \\ cs\ \xi = cn\ \xi/sn\ \xi & : \quad r=1-m^2,\ \ a=2-m^2,\ \ b=2; \\ sc\ \xi = sn\ \xi/cn\ \xi & : \quad r=1,\ \ a=2-m^2,\ \ b=2(1-m^2); \\ sd\ \xi = sn\ \xi/dn\ \xi & : \quad r=1,\ \ a=2m^2-1,\ \ b=2m^2(m^2-1); \\ ds\ \xi = dn\ \xi/sn\ \xi & : \quad r=m^2(m^2-1),\ \ a=2m^2-1,\ \ b=2; \\ ns\ \xi \pm cs\ \xi & : \quad r=1/4,\ a=(1-2m^2)/2,\ \ b=1/2; \\ nc\ \xi \pm sc\ \xi & : \quad r=(1-m^2)/4,\ a=(1+m^2)/2,\ \ b=(1-m^2)/2; \\ ns\ \xi + ds\ \xi & : \quad r=m^4/4,\ a=(m^2-2)/2,\ \ b=1/2; \\ sn\ \xi \pm i\ cn\ \xi & : \quad r=m^2/4,\ \ a=(m^2-2)/2,\ \ b=m^2/2,\ \ i^2=-1; \\ C/\xi & : \quad r=0,\ \ a=0\ \ b=2. \end{cases} \tag{2}$$

Where C is a constant. The Jacobi elliptic functions $sn\ \xi = sn\ (\xi|m)$, $cn\ \xi = cn\ (\xi|m)$, and $dn\ \xi = dn\ (\xi|m)$, where $m(0 < m < 1)$ is the modulus of elliptic function, are double periodic and possess properties of triangular functions, namely

$$sn^2\xi + cn^2\xi = 1,\quad dn^2\,\xi + m^2 sn^2\,\xi = 1,\quad \frac{d}{d\xi} sn\,\xi = cn\,\xi dn\,\xi,$$

$$\frac{d}{d\xi} cn\,\xi = -sn\,\xi dn\,\xi,\quad \frac{d}{d\xi} dn\,\xi = -m^2 cn\,\xi sn\,\xi. \tag{3}$$

In addition, we see that other solutions are obtained from solutions (2) in case of degeneracies:

When m$\rightarrow$ 0, the Jacobi elliptic functions degenerate to the trigonometric functions solutions of the given NLPDE i.e.

$sn\,\xi \rightarrow \sin\,\xi,\;\; cn\,\xi \rightarrow \cos\,\xi,\;\; dn\,\xi \rightarrow 1.$ **(4)**

When m$\rightarrow$ 1, the Jacobi elliptic functions degenerate to the solitonic solutions of the given NLPDE i.e.

$sn\,\xi \rightarrow \tanh\,\xi,\;\; cn\,\xi \rightarrow sech\,\xi,\;\; dn\,\xi \rightarrow sech\,\xi.$ **(5)**

Detailed explanations about Jacobi elliptic functions can be found in Refs.[12,13]. The other 9 Jacobi functions are all expressible in terms of $sn\,\xi,\; cn\,\xi,\; dn\,\xi$.

2. New exact solutions of Kawahara equation

The Kawahara equation

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{6}$$

occurs in the theory of magneto-accoustic waves in plasma [14] and in the theory of shallow water waves with surface tension [15]. Its soliton solutions were given in [15,16]. Making the transformation $u(x,t) = u(\xi),\; \xi = px + \omega t$ and integrating once in term of ξ, Eq.(6) becomes the following ode.

$$\omega u + \frac{p}{2}u^2 + p^3 u'' - p^5 u^{(4)} = 0. \tag{7}$$

The solution of Eq.(6) can be expressed in the form

$$u(\xi) = \sum_{i=1}^{n} a_i \phi^i(\xi), \tag{8}$$

n is a positive integer that can be determined by balancing highest order of linear term $u^{(4)}$ with the nonlinear term u^2, thus we obtain n=4. Therefore, we may choose the following ansatz:

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + a_3\phi^3(\xi) + a_4\phi^4(\xi), \tag{9}$$

substituting (9) into (7) along with Eq.(1) and using Mathematica yields a system of equations of powers of ϕ. Setting the coefficients of ϕ^i in the system of equations to zero, we can deduce the following set of algebraic polynomials with respect to unknowns $a_0,\ a_1,\ a_2,\ a_3\ and\ a_4$ namely:

$$\begin{gathered}
-a_0a_1p - aa_1p^3 + a^2a_1p^5 - 6a_3p^3r + 6ba_1p^5r + 60aa_3p^5r - a_1\omega = 0,\\
-2a_0a_2p - a_1^2p - 8aa_2p^3 + 32a^2a_2p^5 + 72ba_2p^5r - 24p^3ra_4 + 480ap^5ra_4 - 2a_2\omega = 0,\\
3(-a_2a_1p - a_0a_3p - ba_1p^3 - 9aa_3p^3 + 10aba_1p^5 + 81a^2a_3p^5 + 126ba_3p^5r - a_3\omega) = 0,\\
2(-a_2^2p - 2a_1a_3p - 6ba_2p^3 + 120aba_2p^5 - 2a_0pa_4 - 32ap^3a_4 + 512a^2p^5a_4 + 672bp^5ra_4 - 2a_4\omega) = 0,\\
5p(-a_2a_3 - 6ba_3p^2 + 6b^2a_1p^4 + 204aba_3p^4 - a_1a_4) = 0, \qquad (\mathbf{10})\\
3p(-a_3^2 + 60b^2a_2p^4 - 2a_2a_4 - 20bp^2a_4 + 1040abp^4a_4) = 0,\\
7pa_3(90b^2p^4 - a_4) = 0,\\
4pa_4(420b^2p^4 - a_4).
\end{gathered}$$

Solving these algebraic equations with aid of Mathematica, we obtain

$$a_1 = 0,\ \ a_2 = \frac{140}{13}bp^2(52ap^2 - 1),\ \ a_3 = 0,\ \ a_4 = 420b^2p^4, \qquad (\mathbf{11}.\mathbf{a})$$

$$\omega = \frac{p}{4225(7 + 260ap^2)}[(-487 - 4225a_0 - 21840ap^2 + 473200a^2p^4)(7 + 260ap^2) - 2016], \qquad (\mathbf{11}.\mathbf{b})$$

where p satisfies the following equation

$$(3163680rb - 1406080a^3)p^6 + (85176rb - 56784a^2)p^4 + 31 = 0. \qquad (\mathbf{11}.\mathbf{c})$$

Hence, the solution of Eq.(6) reads

$$u(x,t) = a_0 + \frac{140}{13}bp^2(52ap^2 - 1)\phi^2(px + \omega t) + 420b^2p^4\phi^4(px + \omega t), \qquad (\mathbf{12})$$

Where r, a, b and a_0 are arbitrary constants, ω and p satisfy (11.b) and (11.c) respectively. Thanks to the arbitrariness of r, a and b, by choosing properly r, a, b into ode (1) such that the corresponding solution $\phi(\xi)$ of this equation is one of Jacobi elliptic functions, or combined Jacobi elliptic functions, or rational functions in relation (2), we may have multiple solutions of Eq.(6). In the following we consider several cases.

Case 1. r=1, a=-$(1 + m^2)$, $b = 2m^2$

$$u_1 = a_0 - \frac{280m^2}{13}p^2[52(1 + m^2)p^2 + 1]sn^2(px + \omega t) + 1680m^4p^4\ sn^4(px + \omega t), \qquad (\mathbf{13}.\mathbf{a})$$

$$u_2 = a_0 - \frac{280m^2}{13}p^2[52(1+m^2)p^2+1]cd^2(px+\omega t) + 1680m^4p^4\, cd^4(px+\omega t), \quad \textbf{(13.b)}$$

Where

$\omega = \frac{p}{4225[7-260(1+m^2)p^2]}[(-487-4225a_0+21840(1+m^2)p^2+473200(1+m^2)^2p^4)(7-260(1+m^2)p^2)-2016],$ **(13.c)**

$$[3163680(2m^2)+1406080(1+m^2)^3]p^6+[85176(2m^2)-56784(1+m^2)^2]p^4+31=0. \quad \textbf{(13.d)}$$

Case 2. $r=1-m^2 \;\; a=2m^2-1 \;\; b=-2m^2$

$$u_3 = a_0 - \frac{280}{13}m^2p^2[52(2m^2-1)p^2-1]cn^2(px+\omega t)+1680m^4p^4\, cn^4(px+\omega t),$$

$\omega = \frac{p}{4225[7+260(2m^2-1)p^2]}[(-487-4225a_0-21840(2m^2-1)p^2+473200(2m^2-1)^2p^4)(7+260(2m^2-1)p^2)-2016],$ **(14)**

$$[3163680(2m^4-2m^2)-1406080(2m^2-1)^3]p^6+[85176(2m^4-2m^2)-56784(2m^2-1)^2]p^4+31=0.$$

Case 3. $r=m^2-1 \;\; a=2-m^2 \;\; b=-2$

$$u_4 = a_0 - \frac{280}{13}p^2[52(2-m^2)p^2-1]dn^2(px+\omega t)+1680p^4\, dn^4(px+\omega t),$$

$\omega = \frac{p}{4225[7+260(2-m^2)p^2]}[(-487-4225a_0-21840(2-m^2)p^2+473200(2-m^2)^2p^4)(7+260(2-m^2)p^2)-2016],$ **(15)**

$$[3163680(-2m^2+2)-1406080(2-m^2)^3]p^6+[85176(-2m^2+2)-56784(2-m^2)^2]p^4+31=0.$$

Case 4. $r=m^2 \;\; a=-m^2-1 \;\; b=2$

$$u_5 = a_0 - \frac{280}{13}p^2[52(m^2+1)p^2+1]ns^2(px+\omega t)+1680p^4\, ns^4(px+\omega t),$$

$$u_6 = a_0 - \frac{280}{13}p^2[52(m^2+1)p^2+1]dc^2(px+\omega t)+1680p^4\, dc^4(px+\omega t),$$

$\omega = \frac{p}{4225[7-260(m^2+1)p^2]}[(-487-4225a_0+21840(m^2+1)p^2+473200(m^2+1)^2p^4)(7-260(m^2+1)p^2)-2016],$ **(16)**

$$[3163680(2m^2)+1406080(m^2+1)^3]p^6+[85176(2m^2)-56784(m^2+1)^2]p^4+31=0.$$

Case 5. $r=-m^2 \;\; a=2m^2-1 \;\; b=2-2m^2$

$$u_7 = a_0 + \frac{280}{13}(1-m^2)p^2[52(2m^2-1)p^2-1]nc^2(px+\omega t)+1680(1-m^2)^2p^4\, nc^4(px+\omega t),$$

$\omega = \frac{p}{4225[7+260(2m^2-1)p^2]}[(-487-4225a_0-21840(2m^2-1)p^2+473200(2m^2-1)^2p^4)(7+260(2m^2-1)p^2)-2016],$ **(17)**

$$[3163680(2m^4-2m^2)-1406080(2m^2-1)^3]p^6+[85176(2m^4-2m^2)-56784(2m^2-1)^2]p^4+31=0.$$

Case 6. $r=-1 \;\; a=2-m^2 \;\; b=2m^2-2$

$$u_8 = a_0 + \frac{280}{13}(m^2-1)p^2[52(2-m^2)p^2-1]nd^2(px+\omega t)+1680(m^2-1)^2p^4\,nd^4(px+\omega t),$$

$$\omega = \frac{p}{4225[7+260(2-m^2)p^2]}[(-487-4225a_0-21840(2-m^2)p^2+473200(2-m^2)^2p^4)(7+260(2-m^2)p^2)-2016], \quad (\mathbf{18})$$

$$[3163680(2m^2-2)+1406080(2-m^2)^3]p^6+[85176(2m^2-2)+56784(2-m^2)^2]p^4-31=0.$$

Case 7. $r=1-m^2 \;\; a=2-m^2 \;\; b=2$

$$u_9 = a_0 + \frac{280}{13}p^2[52(2-m^2)p^2-1]cs^2(px+\omega t)+1680p^4\,cs^4(px+\omega t),$$

$$\omega = \frac{p}{4225[7+260(2-m^2)p^2]}[(-487-4225a_0-21840(2-m^2)p^2+473200(2-m^2)^2p^4)(7+260(2-m^2)p^2)-2016], \quad (\mathbf{19})$$

$$[3163680(2-2m^2)-1406080(2-m^2)^3]p^6+[85176(2-2m^2)-56784(2-m^2)^2]p^4+31=0.$$

Case 8. $r=1 \;\; a=2-m^2 \;\; b=2-2m^2$

$$u_{10} = a_0 + \frac{280}{13}(1-m^2)p^2[52(2-m^2)p^2-1]sc^2(px+\omega t)+1680(1-m^2)^2p^4\,sc^4(px+\omega t),$$

$$\omega = \frac{p}{4225[7+260(2-m^2)p^2]}[(-487-4225a_0-21840(2-m^2)p^2+473200(2-m^2)^2p^4)(7+260(2-m^2)p^2)-2016], \quad (\mathbf{20})$$

$$[3163680(2-2m^2)-1406080(2-m^2)^3]p^6+[85176(2-2m^2)-56784(2-m^2)^2]p^4+31=0.$$

Case 9. $r=1 \;\; a=2m^2-1 \;\; b=2m^4-2m^2$

$$u_{11} = a_0+\frac{280}{13}(m^4-m^2)p^2[52(2m^2-1)p^2-1]sd^2(px+\omega t)+1680(m^4-m^2)^2p^4\,sd^4(px+\omega t),$$

$$\omega = \frac{p}{4225[7+260(2m^2-1)p^2]}[(-487-4225a_0-21840(2m^2-1)p^2+473200(2m^2-1)^2p^4)(7+260(2m^2-1)p^2)-2016], \quad (\mathbf{21})$$

$$[3163680(2m^4-2m^2)-1406080(2m^2-1)^3]p^6+[85176(2m^4-2m^2)-56784(2m^2-1)^2]p^4+31=0.$$

Case 10. $r=m^4-m^2 \;\; a=2m^2-1 \;\; b=2$

$$u_{12} = a_0 + \frac{280}{13}p^2[52(2m^2-1)p^2-1]ds^2(px+\omega t)+1680p^4\,ds^4(px+\omega t),$$

$$\omega = \frac{p}{4225[7+260(2m^2-1)p^2]}[(-487-4225a_0-21840(2m^2-1)p^2+473200(2m^2-1)^2p^4)(7+260(2m^2-1)p^2)-2016], \quad (\mathbf{22})$$

$$[3163680(2m^4-2m^2)-1406080(2m^2-1)^3]p^6+[85176(2m^4-2m^2)-56784(2m^2-1)^2]p^4+31=0.$$

Case 11. $r=\frac{1}{4} \;\; a=\frac{1-2m^2}{2} \;\; b=\frac{1}{2}$

$$u_{13} = a_0+\frac{70}{13}p^2[26(1-2m^2)p^2-1][ns(px+\omega t)\pm cs(px+\omega t)]^2+105p^4[ns(px+\omega t)\pm cs(px+\omega t)]^4,$$

$\omega = \frac{p}{4225[7+130(1-2m^2)p^2]}[(-487-4225a_0-21840(\frac{1-2m^2}{2})p^2+473200(\frac{1-2m^2}{2})^2p^4)(7+130(1-2m^2)p^2)-2016]$, **(23)**

$$[3163680(\frac{1}{8})-1406080(\frac{1-2m^2}{2})^3]p^6+[85176(\frac{1}{8})-56784(\frac{1-2m^2}{2})^2]p^4+31=0.$$

Case 12. $r=\frac{1-m^2}{4}$ $a=\frac{1+m^2}{2}$ $b=\frac{1-m^2}{2}$

$u_{14}=a_0+\frac{70}{13}(1-m^2)p^2[26(1+m^2)p^2-1][nc(px+\omega t)\pm sc(px+\omega t)]^2+105(1-m^2)^2p^4[nc(px+\omega t)\pm sc(px+\omega t)]^4$,

$\omega = \frac{p}{4225[7+130(1+m^2)p^2]}[(-487-4225a_0-21840(\frac{1+m^2}{2})p^2+473200(\frac{1+m^2}{2})^2p^4)(7+130(1+m^2)p^2)-2016]$, **(24)**

$$[3163680(\frac{(1-m^2)^2}{8})-1406080(\frac{1+m^2}{2})^3]p^6+[85176(\frac{(1-m^2)^2}{8})-56784(\frac{1+m^2}{2})^2]p^4+31=0.$$

Case 13. $r=\frac{m^4}{4}$ $a=\frac{m^2-2}{2}$ $b=\frac{1}{2}$

$$u_{15}=a_0+\frac{70}{13}p^2[26(m^2-2)p^2-1][ns(px+\omega t)+cs(px+\omega t)]^2+105p^4[ns(px+\omega t)+cs(px+\omega t)]^4,$$

$\omega = \frac{p}{4225[7+130(m^2-2)p^2]}[(-487-4225a_0-21840(\frac{m^2-2}{2})p^2+473200(\frac{m^2-2}{2})^2p^4)(7+130(m^2-2)p^2)-2016]$, **(25)**

$$[3163680\frac{m^4}{8}-1406080(\frac{m^2-2}{2})^3]p^6+[85176\frac{m^4}{8}-56784(\frac{m^2-2}{2})^2]p^4+31=0.$$

Case 14 $r=\frac{m^2}{4}$ $a=\frac{m^2-2}{2}$ $b=\frac{m^2}{2}$

$u_{16}=a_0+\frac{70}{13}m^2p^2[26(m^2-2)p^2-1][sn(px+\omega t)\pm i\ cn(px+\omega t)]^2+105m^4p^4[sn(px+\omega t)\pm i\ cn(px+\omega t)]^4$,

$\omega = \frac{p}{4225[7+130(m^2-2)p^2]}[(-487-4225a_0-21840(\frac{m^2-2}{2})p^2+473200(\frac{m^2-2}{2})^2p^4)(7+130(m^2-2)p^2)-2016]$, **(26)**

$$[3163680(\frac{m^4}{8})-1406080(\frac{m^2-2}{2})^3]p^6+[85176(\frac{m^4}{8})-56784(\frac{m^2-2}{2})^2]p^4+31=0.$$

Remark 1

When m→ 0, these solutions (13)-(26) degenerate as triangular function-like solutions of Eq.(6), namely

$$u_{17}=a_0-\frac{280}{13}p^2(52p^2+1)csc^2(px+\omega t)+1680p^4\,csc^4(px+\omega t),$$

$$u_{18}=a_0-\frac{280}{13}p^2(52p^2+1)sec^2(px+\omega t)+1680p^4\,sec^4(px+\omega t),$$

$$\omega=\frac{p}{4225(7-260p^2)}[(-487-4225a_0+21840p^2+473200p^4)(7-260p^2)-2016] \quad \mathbf{(27)}$$

$$1406080p^6-56784p^4+31=0.$$

$$u_{19} = a_0 + \frac{280}{13}p^2(104p^2-1)\cot^2(px+\omega t) + 1680p^4\cot^4(px+\omega t),$$

$$u_{20} = a_0 + \frac{280}{13}p^2[104p^2-1]\tan^2(px+\omega t) + 1680p^4\tan^4(px+\omega t),$$

$$\omega = \frac{p}{4225(7+520p^2)}[(-487-4225a_0-21840(2)p^2+473200(2)^2p^4)(7+520p^2)-2016], \quad \textbf{(28)}$$

$$[3163680(2)-1406080(2)^3]p^6+[85176(2)-56784(2)^2]p^4+31=0.$$

$$u_{21} = a_0+\frac{70}{13}p^2(26p^2-1)[csc(px+\omega t)\pm\cot(px+\omega t)]^2+105p^4[csc(px+\omega t)\pm\cot(px+\omega t)]^4,$$

$$u_{22} = a_0+\tfrac{70}{13}p^2(26p^2-1)[sec(px+\omega t)\pm\tan(px+\omega t)]^2+105p^4[sec(px+\omega t)\pm\tan(px+\omega t)]^4,$$

$$\omega = \frac{p}{4225(7+130p^2)}[(-487-4225a_0-21840(\frac{1}{2})p^2+473200(\frac{1}{2})^2p^4)(7+260(\frac{1}{2})p^2)-2016], \quad \textbf{(29)}$$

$$[3163680(\frac{1}{8})-1406080(\frac{1}{2})^3]p^6+[85176(\frac{1}{8})-56784(\frac{1}{2})^2]p^4+31=0.$$

Remark 2

When m$\rightarrow$ 1, these solutions (13)-(26) degenerate as soliton-like solutions of Eq.(6), namely

$$u_{23} = a_0 - \frac{280}{13}p^2[104p^2+1]\tanh^2(px+\omega t) + 1680p^4\tanh^4(px+\omega t),$$

$$u_{24} = a_0 - \frac{280}{13}p^2[104p^2+1]\coth^2(px+\omega t) + 1680p^4\coth^4(px+\omega t), \quad \textbf{(30)}$$

$$\omega = \frac{p}{4225[7-520p^2]}[(-487-4225a_0+21840(2)p^2+473200(2)^2p^4)(7-520p^2)-2016],$$

$$[3163680(2)+1406080(2)^3]p^6+[85176(2)-56784(2)^2]p^4+31=0.$$

$$u_{25} = a_0 - \frac{280}{13}p^2(52p^2-1)sech^2(px+\omega t) + 1680p^4\,sech^4(px+\omega t),$$

$$u_{26} = a_0 + \frac{280}{13}p^2(52p^2-1)csch^2(px+\omega t) + 1680p^4\,csch^4(px+\omega t), \quad \textbf{(31)}$$

$$\omega = \frac{p}{4225[7+260p^2]}[(-487-4225a_0-21840p^2+473200p^4)(7+260p^2)-2016],$$

$$-1406080p^6-56784p^4+31=0.$$

$u_{27} = a_0 - \frac{70}{13}p^2(26p^2+1)[\coth(px+\omega t) \pm\ csch(px+\omega t)]^2 + 105p^4[\coth(px+\omega t) \pm csch(px+\omega t)]^4$,

$u_{28} = a_0 - \frac{70}{13}p^2(26p^2-1)[\tanh(px+\omega t) \pm i\ sech(px+\omega t)]^2 + 105p^4[\tanh(px+\omega t) \pm i\ sech(px+\omega t)]^4$,

$$\omega = \frac{p}{4225[7-130p^2]}[(-487-4225a_0+21840(\frac{1}{2})p^2+473200(\frac{1}{2})^2p^4)(7-130p^2)-2016], \quad \textbf{(32)}$$

$$[3163680(\frac{1}{8})+1406080(\frac{1}{2})^3]p^6+[85176(\frac{1}{8})-56784(\frac{1}{2})^2]p^4+31=0.$$

All the above solutions are quite new and different from the ones reported in [16]

3. Exact solutions of modified Kawahara, SK and KK equations

3.1. **modified Kawahara equation**

. The modified Kawahara equation equation reads

$$u_t + u_x + u^2u_x + lu_{xxx} + su_{xxxxx} = 0, \qquad \mathbf{(33)}$$

where r, s are nonzero real constants. This equation arises also in the theory of shallow water waves [17] and its exact solution was obtained in [16,18]. After making the transformation u(x,t)=$u(\xi)$, $\xi = px + \omega t$, Eq.(33) becomes

$$(p+\omega)u + \frac{p}{3}u^3 + lp^3u'' + sp^5u^{(4)} = 0, \qquad \mathbf{(34)}$$

Balancing the highest derivative term $u^{(4)}$ with the highest power nonlinear term u^3 gives n=2. Then we assume the following form of solutions

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi), \qquad \mathbf{(35)}$$

Where ϕ satisfy ode (1). Substituting Eq.(35) into (34) and using Mathematica gives the following system

$$\begin{gathered}
a_1(p + a_0^2p + alp^3 + a^2p^5s + 6bp^5rs + \omega) = 0,\\
2(a_2p + a_0^2a_2p + a_0a_1^2p + 4aa_2lp^3 + 16a^2a_2p^5s + 36ba_2p^5rs + a_2\omega) = 0,\\
a_1p(6a_0a_2 + a_1^2 + 3blp^2 + 30abp^4s) = 0,\\
4a_2p(a_0a_2 + a_1^2 + 3blp^2 + 60abp^4s) = 0,\\
5a_1p(a_2^2 + 6b^2p^4s) = 0,\\
2a_2p(a_2^2 + 90b^2p^4s) = 0.
\end{gathered} \qquad \mathbf{(36)}$$

It is easy to find a solution from above system, namely

$$a_1 = 0, \;\; a_0 = \mp\frac{1}{\sqrt{-10s}}(l + 20asp^2), \;\; a_2 = \pm 3bp^2\sqrt{-10s}, \quad \mathbf{(37)}$$

$$\omega = -p[1 - \frac{1}{10s}(l + 20asp^2)^2 + 4alp^2 + 16a^2p^4s + 36bp^4rs].$$

Therefore the solution of Eq.(33) is given by

$$u = a_0 + a_2\phi^2(\xi), \qquad \mathbf{(38.1)}$$

$$\xi = px + \omega t, \qquad \mathbf{(38.2)}$$

$$a_0 = \mp\frac{1}{\sqrt{-10s}}(l + 20asp^2), \;\; a_2 = \pm 3bp^2\sqrt{-10s}, \quad \mathbf{(39.1)}$$

$$\omega = -p[1 - \frac{1}{10s}(l + 20asp^2)^2 + 4alp^2 + 16a^2p^4s + 36bp^4rs].(\mathbf{39.2})$$

p, r, a and b are arbitrary functions. Since we have r, a and b arbitrary we may choose properly the relations betweem them, so that one of the solutions in relation (2) is recovered. This will follow later.

3.2. **KK and SK equations**

. We consider the Kaup-Kupershmidt type equation [19]

$$u_t + 45u^2u_x - 15\gamma u_x u_{xx} - 15uu_{xxx} + u_{xxxxx} = 0. \qquad (\mathbf{40})$$

This equation is known to be integrable for two values of γ, $\gamma = 5/2$ and $\gamma = 1$. This corresponds to the Kaup-Kupershmidt (KK) and Sawada-Kotra (SK) respectively. Although they are quite similar, KK and SK equations are fundamentally different since there is no scaling that transforms one into the other [19]. The SK and KK equations have been widely studied in the past two decades [20-22] here, with the aid of mapping method we will proposed new solutions in unified way. Using the transformation $\xi = px + \omega t$ we change the NLPDE (40) in the following ode

$$\omega u' + 45pu^2u' - 15\gamma p^3u'u'' - 15p^3uu''' + p^5u^{(5)} = 0, \qquad (\mathbf{41})$$

The balance procedure gives n=2, therefore the solution can be expanded in the following

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi), \qquad (\mathbf{42})$$

Where ϕ satisfies ode (1). Substituting Eq.(42) into (41) and using Mathematica gives the following system

$$a_1(45a_0^2p - 15aa_0p^3 - 30a_2p^3r\gamma + a^2p^5 + 6bp^5r + \omega) = 0,$$

$$90a_0^2a_2p + 90a_0a_1^2p - 15aa_1^2p^3 + 32a^2a_2p^5 + 72ba_2p^5r - 15aa_1^2p^3\gamma - 60a_2^2p^3r\gamma - 120a_2p^3aa_0 + 2g\omega = 0,$$

$$15a_1p(18a_0a_2 + 3a_1^2 - 3ba_0p^2 - 9aa_2p^2 + 2abp^4 - 6aa_2p^2\gamma) = 0, \qquad (\mathbf{43})$$

$$15p(12a_0a_2^2 + 12a_2a_1^2 - 12ba_0a_2p^2 - 8aa_2^2p^2 - 3ba_1^2p^2 + 16aba_2p^4 - 8aa_2^2p^2\gamma - ba_1^2p^2\gamma) = 0,$$

$$15pa_1(15a_2^2 - 15ba_2p^2 + 2b^2p^4 - 5ba_2p^2\gamma) = 0,$$

$$90a_2p(a_2^2 - 2ba_2p^2 + 2b^2p^4 - ba_2p^2\gamma) = 0.$$

Solving these algebraic equations with the aid of Mathematica, and looking for solutions which have maximum of parameters r, a and b arbitrary we obtain the following results

For $\gamma = 5/2$

Case 1

$$b = 1, \;\; a_0 = \frac{8ap^2}{3}, \;\; a_2 = 4p^2, \;\; a_1 = 0, \;\; \omega = -176a^2p^5 + 264p^5r. \qquad (\mathbf{44})$$

where p, a and r are arbitrary constants. From the relation (2), we obtain for this case the following solutions

subcase 1: $b = 1, \;\; m = \frac{1}{\sqrt{2}}, \;\; a = -\frac{3}{2}, \;\; r = 1,$

$$u = -4p^2[1 - sn^2(\xi|1/\sqrt{2})], \qquad (\mathbf{45.1})$$

$$u = -4p^2[1 - cd^2(\xi|1/\sqrt{2})],$$

$$\xi = px - 132p^5t,$$

subcase 2: $b = 1, \;\; m = \frac{1}{\sqrt{2}}, \;\; a = 0, \;\; r = -1,$

$$u = 4p^2 \; nc^2(\xi|1/\sqrt{2}), \qquad (\mathbf{45.2})$$

$$\xi = px - 132p^5t,$$

subcase 3: $b = 1, \;\; m = \frac{1}{\sqrt{2}}, \;\; a = \frac{3}{2}, \;\; r = 1,$

$$u = 4p^2[1 + sc^2(\xi|1/\sqrt{2})], \qquad (\mathbf{45.3})$$

$$\xi = px - 132p^5t$$

Case 2

$$a_0 = \frac{ap^2}{3}, \;\; a_1 = 0, \;\; a_2 = \frac{bp^2}{2}, \;\; \omega = -a^2p^5 + \frac{3}{2}bp^5. \qquad (\mathbf{46})$$

Therefore the solutions for this case read

$$u = a_0 + a_2\phi^2(\xi), \qquad (\mathbf{47.1})$$

$$\xi = px + \omega t, \qquad (\mathbf{47.2})$$

Where

$$a_0 = \frac{ap^2}{3}, \quad a_2 = \frac{bp^2}{2}, \;\; \omega = -a^2p^5 + \frac{3}{2}bp^5. \qquad (\mathbf{48})$$

With p, r, a and b being arbitrary constants. The explicit solution $\phi(\xi)$ will be given later.

For γ = 1

We obtain three subcases solutions namely

$$a_0 = \frac{4ap^2}{3},\;\; a_1 = 0,\;\; a_2 = bp^2,\;\; \omega = 16a^2p^5 + 6bp^5r, \tag{49.1}$$

$$a_0 = \frac{4ap^2}{3},\;\; a_1 = 0,\;\; a_2 = 2bp^2,\;\; \omega = 16a^2p^5 - 24bp^5r, \tag{49.2}$$

$$a_0 \neq \frac{4ap^2}{3},\;\; a_1 = 0,\;\; a_2 = bp^2,\;\; \omega = -45pa_0^2 + 60aa_0p^3 + 16a^2p^5 + 6bp^5r. \tag{49.3}$$

In all these subcases p, r, a and b are arbitrary constants. Therefore the solution is given by

$$u = a_0 + a_2\phi^2(\xi), \tag{50.1}$$

$$\xi = px + \omega t, \tag{50.2}$$

Where

$(a_0, a_2, \omega) = (\frac{4ap^2}{3}, bp^2, 16a^2p^5 + 6bp^5r) \vee (\frac{4ap^2}{3}, 2bp^2, 16a^2p^5 - 24bp^5r) \vee (\neq \frac{4ap^2}{3}, bp^2, -45pa_0^2 + 60aa_0p^3 + 16a^2p^5 + 6bp^5r)$ **(51)**

Since for the cases (38),(47) and (50) the r, a and b are arbitrary constants, we may choose appropriate values for these parameters in order to obtain Jacobi elliptic function, combined Jacobi elliptic function and rational solutions from relation (2). The solution will be given in term of values of the three parameters as $u^{(l)}_{(r,a,b)}$, (l) is the number of solution.

$$\begin{aligned}
u^{(1)}_{(1,-m^2-1,2m^2)} &= a_0 + a_2\; sn^2\xi,\\
u^{(2)}_{(1,-m^2-1,2m^2)} &= a_0 + a_2\; cd^2\xi,\\
u^{(3)}_{(1-m^2,2m^2-1,-2m^2)} &= a_0 + a_2\; cn^2\xi,\\
u^{(4)}_{(m^2-1,2-m^2,-2)} &= a_0 + a_2\; dn^2\xi,\\
u^{(5)}_{(m^2,-m^2-1,2)} &= a_0 + a_2\; ns^2\xi,\\
u^{(6)}_{(m^2,-m^2-1,2)} &= a_0 + a_2\; dc^2\xi,\\
u^{(7)}_{(-m^2,2m^2-1,2-2m^2)} &= a_0 + a_2\; nc^2\xi,\\
u^{(8)}_{(-1,2-m^2,2m^2-2)} &= a_0 + a_2\; nd^2\xi,\\
u^{(9)}_{(1-m^2,2-m^2,2)} &= a_0 + a_2\; cs^2\xi,\\
u^{(10)}_{(1,2-m^2,2-2m^2)} &= a_0 + a_2\; sc^2\xi,\\
u^{(11)}_{(1,2m^2-1,2m^4-2m^2)} &= a_0 + a_2\; sd^2\xi,\\
u^{(12)}_{(m^4-m^2,2m^2-1,2)} &= a_0 + a_2\; ds^2\xi,
\end{aligned} \tag{52}$$

$$u^{(13)}_{(1/4,1/2-m^2,1/2)} = a_0 + a_2[ns\ \xi \pm cs\ \xi]^2,$$

$$u^{(14)}_{(1/4-m^2/4,1/2+m^2/2,1/2-m^2/2)} = a_0 + a_2[nc\ \xi \pm sc\ \xi]^2,$$

$$u^{(15)}_{(m^4/4,m^2/2-1,1/2)} = a_0 + a_2[ns\ \xi + ds\ \xi]^2,$$

$$u^{(16)}_{(m^2/4,m^2/2-1,m^2/2)} = a_0 + a_2[sn\ \xi \pm i\ cn\ \xi]^2,$$

$$u^{(17)}_{(0,0,2)} = a_0 + a_2\frac{C^2}{\xi^2}. \qquad \mathbf{(53)}$$

Remark 3

When m→ 1, the solutions (52) degenerate as soliton, combined soliton solutions , namely

$$u^{(18)}_{(1,-2,2)} = a_0 + a_2 \tanh^2 \xi,$$

$$u^{(19)}_{(1,-2,2)} = a_0 + a_2 \coth^2 \xi,$$

$$u^{(20)}_{(0,1,-2)} = a_0 + a_2\ sech^2\xi, \qquad \mathbf{(54)}$$

$$u^{(21)}_{(0,1,2)} = a_0 + a_2\ csch^2\xi,$$

$$u^{(22)}_{(1/4,-1/2,1/2)} = a_0 + a_2[\coth\ \xi \pm csch\ \xi]^2,$$

$$u^{(23)}_{(1/4,-1/2,1/2)} = a_0 + a_2[\tanh\ \xi \pm i\ sech\ \xi]^2,$$

Remark 4

When m→ 0, the solutions (52) degenerate as triangular function namely

$$u^{(24)}_{(0,-1,2)} = a_0 + a_2\ sec^2\xi,$$

$$u^{(25)}_{(0,-1,2)} = a_0 + a_2\ csc^2\xi,$$

$$u^{(26)}_{(1,2,2)} = a_0 + a_2 \cot^2 \xi, \qquad \mathbf{(55)}$$

$$u^{(27)}_{(1,2,2)} = a_0 + a_2 \tan^2 \xi,$$

$$u^{(28)}_{(1/4,1/2,1/2)} = a_0 + a_2[csc\ \xi \pm \cot\ \xi]^2,$$

$$u^{(29)}_{(1/4,1/2,1/2)} = a_0 + a_2[sec\ \xi \pm \tan\ \xi]^2,$$

The solutions (52)-(55) are exact solutions, of the modified Kawahara equation, when $a_0,\ a_2\ and\ \omega$ are given by Eq.(39), of the KK equation (case 2) when $a_0,\ a_2\ and\ \omega$ are satisfied by Eq.(46) and of the SK equation when $a_0,\ a_2\ and\ \omega$ are given by Eq.(51). While $\xi = px + \omega t$.

4. CONCLUSION

Based on the mapping method and the use of symbolic computation system Mathematica, we construct explicit, rich and new solutions such as Jacobi elliptic function, combined Jacobi elliptic function, rational, triangular function, soliton and combined soliton solutions for the Kawahara, modified Kawahara, Kaup-Kupershmidt and Sawada-Kotra equations.

Acknowledgement.
This research was carried out during the author's stay at the Max–Planck Institut für Mathematik in Bonn. I thank this institution for the generous hospitality and support. I thank Matilde Marcolli for useful discussions.

References

[1] M. Wadati, H. Sanuki, K. Konno, Prog. Theor. Phys. 53 (1975) 419.
[2] V.A. Matveev, M.A. Salle, Darboux transformations and solitons. Berlin, Heidelberg: Springer-Verlag; (1991).
[3] Z. Fu, S.K. Liu, S.D. Liu, Q. Zhao, Phys. Lett. A 290 (2001) 72.
[4] E.J. Parkes, B.R.Duffy, Comp.Phys.Commun. 98, (1996) 288.
[5] C.T. Yan, Phys. Lett. A 224 (1996) 77.
[6] Z.Y. Yan, H.Q. Zhang, Phys. Lett.A 252 (1999) 291.
[7] M.L. Wang, Phys. Lett. A 216 (1996) 67.
[8] S.A. Elwakil, S.K. El-labany, M.A. Zahhran, R. Sabry, Phys. Lett. A 299 (2002) 179.
[9] Y.Z. Peng, Chin.J. Phys.41 (2003) 103.
[10] Y.Z. Peng, Phys. Lett. A 314 (2003) 401.
[11] Y. Zhou, M. Wang, Y. Wang, Phys. Lett. A 308 (2003) 31.
[12] D.V. Patrick, Elliptic function and elliptic curves. Cambridge: cambidge University Press, 1973.
[13] K. Chamdrasekharan, Elliptic functions. Berlin: Springer-Verlag, 1985.
[14] T. Kawahara, J. Phys. Soc. Jpn 33 (1972) 260.
[15] J.K. Hunter, J. Scheule, Physica D 32 (1988) 253.
[16] Sirendaoreji, will be published in Chaos, Solitons and Fractals 19 (2004) 147.
[1] 7] 17 J.K. Hunter et al. , Physica D 32 (1988) 253.
[18] Z.B. Li, Z.S. Qing, Math. Appl. Scientia 17 (1997) 81.
[19] A. Parker, Inverse Probl. 17 (2001) 885.
[20] E. Fan, Chaos Solitons and Fractals 16 (2003) 819.
[21] A. Parker, Physica D 137 (2000) 25.
[22] R.A. Zait, Chaos, Soliton and Fractals 15 (2003) 673.

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