MOTIVES AND MIRROR SYMMETRY FOR CALABI–YAU ORBIFOLDS

SHABNAM KADIR AND NORIKO YUI

ABSTRACT. We consider certain families of Calabi–Yau orbifolds and their mirror partners constructed from Fermat hypersurfaces in weighted projective 4-spaces. Our focus is the topological mirror symmetry. There are at least three known ingredients to describe the topological mirror symmetry, namely, integral vertices in reflexive polytopes, monomials in graded polynomial rings (with some group actions), and periods (and Picard–Fuchs differential equations). In this paper, we will introduce Fermat motives associated to these Calabi–Yau orbifolds and then use them to give motivic interpretation of the topological mirror symmetry phenomenon between mirror pairs of Calabi–Yau orbifolds. We establish, at the Fermat (the Landau–Ginzburg) point in the moduli space, the one-to-one correspondence between the monomial classes and Fermat motives. This is done by computing the number of $\mathbb{F}_q$-rational points on our Calabi–Yau orbifolds over $\mathbb{F}_q$ in two different ways: Weil's algebraic number theoretic method involving Jacobi (Gauss) sums, and Dwork's p-adic analytic method involving Dwork characters and Gauss sums. We will discuss specific examples in detail.

Contents

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1. Introduction

This is a sequel to the article of Yui [Y05] where Calabi–Yau orbifolds of Fermat hypersurface threefolds in weighted projective 4-spaces were constructed, and their L-series (associated to the ℓ-adic Galois representations) were determined. It was often the case that the Galois representations had very high rank, which made it rather impossible to carry out the required calculations. To remedy this, we introduced Fermat motives, and then decomposed the Calabi–Yau threefolds into Fermat motives. Via cohomological realizations of these motives, we were able to calculate the motivic L-series for each motive. The global L-series was then obtained by gluing the motivic results together.

These Calabi–Yau orbifolds are all non-rigid ($h^{2,1} > 0$), and their mirror Calabi–Yau threefolds exist satisfying the following conjecture.

Conjecture 1.1 (Topological Mirror Symmetry Conjecture). Given a family of Calabi–Yau threefolds X, there is a mirror family of Calabi–Yau threefolds $\hat{X}$ such that

$$h^{2,1}(\hat{X}) = h^{1,1}(X) \quad \text{and} \quad h^{1,1}(\hat{X}) = h^{2,1}(X)$$

so that the Euler characteristics are subject to the relation

$$\chi(\hat{X}) = -\chi(X).$$

The topological mirror symmetry described above can be reformulated in the toric geometric setting à la Batyrev [Ba94]. In Batyrev's theory, mirror symmetry is described in terms of pairs of reflexive polytopes. Integral vertices of reflexive polytopes are the main ingredients in Batyrev's theory, and they correspond to monomials in graded polynomial rings. By Aspinwall, Greene and Morrison [AGM93], there is the monomial–divisor mirror map (for the corresponding cohomology groups, $H^{1,1}_{toric}$ and $H^{2,1}_{poly}$), which yields a one-to-one correspondence between toric divisors of a Calabi–Yau family and monomials in the mirror Calabi–Yau family.

Based on the theory of Dwork, Katz and Griffiths on periods (see Cox and Katz [CK99]), Candelas et al. [CORV00, CORV03] (resp. Kadir [Ka04, Ka05]) established explicitly a one-to-one correspondence between monomials and periods via Picard–Fuchs differential equations, for the quintic one-parameter (resp. the octic two-parameter) family of Calabi–Yau threefolds. In their calculations, periods decomposed into the product of subperiods. This seems to suggest that there should be motivic interpretation for such factorizations.

The concept of motives has been emerging in the physics literature, and the purpose of this paper is to give mathematically rigorous discussion on motives, restricting our attention to specific examples of Calabi–Yau threefolds. We follow the notion of motives due to Grothendieck. Starting with Fermat hypersurfaces in weighted projective 4-spaces, we define and construct explicitly the so-called Fermat motives from algebraic correspondences, as described in Shioda [S787]. Our goal is to interpret the topological mirror symmetry phenomenon for the mirror pairs of specific Calabi–Yau orbifolds in terms of Fermat motives and their cohomological realizations. As our main result, we establish a one-to-one correspondence between monomials and Fermat motives. This correspondence determines Fermat motives which are invariant under the mirror map. Since Fermat motives are defined only at the Fermat (the Landau–Ginzburg) point in the moduli space, this correspondence is established only at the Fermat point.

There are more monomials than motives, and we observe that monomials associated to conifold points seem to be associated to (mixed) Tate motives.

Incidentally, at the Fermat point, our Calabi–Yau threefolds capture the structure of CM type varieties (see Yui [Y05]), and hence our motives are also CM type motives. (G. Moore [Mo98] defined "attractive" Calabi–Yau threefolds. Among our Calabi–Yau orbifolds, there is only one such threefold, namely, $m = 6, Q = (1,1,1,1,2)$. In fact, this is the only Calabi–Yau orbifold whose weight motive is rigid, i.e., $h^{2,1}(\mathcal{M}_Q) = 0$ and $B_3(\mathcal{M}_Q) = 2$.)

Now we will describe the contents of this paper.

Section 2 is devoted to the definition of Fermat motives and their cohomological realizations. We use the definition of motives due to Grothendieck, and Manin [Ma70], which is based on algebraic correspondences and projectors. We follow the exposition of Shioda [Sh87] and Gouvêa and Yui [GY95].

In Section 3, we construct Calabi–Yau orbifolds in weighted projective 4-spaces. The starting point is the Fermat hypersurface V of degree $m \geq 5$ and dimension 3, and a finite abelian groups (which is a subgroup of the automorphism group of V). This group will determine a weight. We take the quotient of V by such a group. This gives rise to a quotient threefold with singularities. We then resolve singularities by taking the crepant resolution (which is guaranteed to exist for dimension ≤ 3). The smooth threefold thus obtained is our Calabi–Yau threefold. There are altogether 147 such Calabi–Yau orbifolds.

In Section 4, we describe the Greene–Plesser orbifolding construction of mirror partners of the Calabi–Yau orbifolds constructed in the previous section. We review the mirror construction from the paper of Greene and Plesser [GP90]. The mirror symmetry is interpreted as the duality between the two finite abelian groups associated to the mirror pair of Calabi–Yau threefolds.

In Section 5, we will construct Fermat motives for the Calabi–Yau orbifolds in Section 3, and compute their invariants (e.g., motivic Hodge numbers, motivic Betti numbers) via their cohomological realizations. For each mirror pairs of Calabi–Yau threefolds, we also determine Fermat motives which are invariant under the mirror maps. In particular, we observe that for each Calabi–Yau orbifold, the motive associated to the weight is always invariant under the mirror map. If $h^{1,1} = 1$, the weight motive is the only motive invariant under the mirror map. However, when $h^{1,1} > 1$, there are other motives apart from the weight motive that remain invariant under the mirror map.

In Section 6, we review the construction of mirror pairs of Calabi–Yau hypersurfaces in toric geometry due to Batyrev [Ba94]. We will confine ourselves to Calabi–Yau threefolds. Reflexive polytopes and their dual polytopes are the main players in Batyrev's toric mirror symmetry. A pair of reflexive polytopes (Δ, Δ^*) gives rise to a mirror pairs of Calabi–Yau hypersurfaces. It is noted that the origin is the only integral point contained both in the reflexive polytope and its dual polytope. (This fact plays a pivotal role in proving our main result.) Integral points correspond to monomials in graded polynomial rings. We will discuss, in particular, the monomial–divisor mirror map of Aspinwall, Greene and Morrison [AGM93], which gives the isomorphism between the two spaces $H^{1,1}_{toric}(X)$ and $H^{2,1}_{poly}(\hat{X})$ for a mirror pair $(X, \hat{X})$. This establishes a one-to-one correspondence between integral points in the reflexive polytope of X and monomials in the polynomial ring of $\hat{X}$.

In Section 7, we will discuss a one-to-one correspondence between monomials and periods via Picard–Fuchs differential equations. The method of Dwork–Katz–Griffiths determines the Picard–Fuchs differential equations for Calabi–Yau hypersurfaces in weighted projective spaces (of any dimension) (see Cox and Katz [CK99]). In this section, we will illustrate this correspondence focusing on the concrete calculations of periods of Candelas et al. [CORV00, CORV03] for the one-parameter deformation of the quintic Calabi–Yau threefold in the ordinary projective 4-space $\mathbb{P}^4$, and of Kadia [Ka05] for the two-parameter deformation of the octic Calabi–Yau threefold in the weighted projective space $\mathbb{P}^4(Q)$ with $Q = (1, 1, 2, 2, 2)$. Here we observe that the Picard–Fuchs differential quation decomposes into the product of smaller order Picard–Fuchs differential equations. It suggests that such a decomposition ought to have origin in the motivic decomposition of the manifold.

The Section 8 contains our main examples and the main result on the monomial–motive correspondence (Theorem 8.1). We establish a one-to-one correspondence between the class of monomials and Fermat motives at the Fermat point for the Calabi–Yau threefolds of Section 3. We prove that the motives which are invariant under the mirror maps correspond to the class of the constant monomial (and hence to the origin in the polytopes). We illustrate the monomial–motive correspondence for the quintic and the octic Calabi–Yau orbifolds.

The Section 9 contains our proof for the monomial–motive correspondence. We compute the number of $\mathbb{F}_q$–rational points (and hence congruence zeta-functions) for our Calabi–Yau threefolds over finite fields $\mathbb{F}_q$ in two different ways. On one hand, we compute them with Weil's method using Jacobi (Gauss) sums. On the other hand, we compute them with Dwork's p-adic method using Dwork's characters and Gauss sums. We then show that the two approaches reconcile at the Fermat point.

Away from the Fermat point, Calabi–Yau orbifolds with deformation parameters yield more monomials than motives. A conifold point on these Calabi–Yau orbifolds with deformation parameters is a singularity locally isomorphic to the projective quadric surface $X^2 + Y^2 + Z^2 + T^2 = 0$, and we may associate a motive (e.g., a mixed Tate motive) employing the same line of arguments as in Bloch–Esnault–Kreimer [BEK05].

Finally, Section 10 presents conclusions of this work, and further problems and future projects. The main conclusion is to bring in motives to the realm of the topological mirror symmetry for our Calabi–Yau orbifolds.

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2. Fermat motives

We will employ the definition of motives due to Grothendieck and Manin [Ma70]. The most recommended references for the generality of motives might be Deligne–Milne [DM82], and Soulé [So84]. In this paper we will be confining ourselves to the so-called "Fermat" motives arising from Fermat hypersurfaces. We will recall the construction of Fermat motives from Shioda [Sh87] and Gouvêa and Yui [GY95]. The construction works for any dimension, not only for dimension 3.

We start with the Fermat hypersurface of degree m and of dimension n in the projective space $\mathbb{P}^{n+1}$:

$$V\,:\,Z_1^m+Z_2^m+\cdots+Z_{n+2}^m=0\in\mathbb{P}^{n+1}$$

Let μ_m be the group of m-th roots of unity and let

$$\mathfrak{G}:=(\mu_m)^{n+2}/(\text{diagonal})=\{\mathbf{g}:=(g_1,g_2,\cdots,g_{n+2})\,|\,g_i\in\mu_m\quad\forall i\}/\{(g,g,\cdots,g)\}$$

be a subgroup of the automorphism group $\mathrm{Aut}(V)$ of V. The group $\mathfrak{G}$ is a finite group of order m^{n+1}, and it acts on V by component-wise multiplication of g_i for each i. The character group of $\mathfrak{G}$ is identified with the set

$$\hat{\mathfrak{G}}:=\{\mathbf{a}=(a_1,a_2,\cdots,a_{n+2})\,|\,a_i\in(\mathbb{Z}/m\mathbb{Z})\;\forall\, i,\;\sum_{i=1}^{n+2}a_i\equiv 0\pmod{m}\,\}$$

and the duality between $\mathfrak{G}$ and $\hat{\mathfrak{G}}$ is given by:

$$\hat{\mathfrak{G}}\times\mathfrak{G}\mapsto\mathbb{L}=:\mathbb{Q}(\zeta_m)\qquad(\mathbf{a},\,\mathbf{g})\mapsto\mathbf{a}(\mathbf{g}):=\prod_{i=1}^{n+2}g_i^{a_i}$$

where $\zeta_m=e^{2\pi i/m}$ is a primitive m-th root of unity. For each $\mathbf{a}\in\hat{\mathfrak{G}}$, let $A=[\mathbf{a}]$ be the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of $\mathbf{a}$. Put $\mathbb{L}_{\mathbf{a}}=\mathbb{L}_A=:\mathbb{Q}(\zeta_m^d)$ where $d=:\gcd(\mathbf{a},m)$.

Definition 2.1. For each character $\mathbf{a}\in\hat{\mathfrak{G}}$, define

$$p_{\mathbf{a}}:=\frac{1}{\#\hat{\mathfrak{G}}}\sum_{\mathbf{g}\in\mathfrak{G}}\mathbf{a}(\mathbf{g})^{-1}\mathbf{g}=\frac{1}{m^{n+1}}\sum_{\mathbf{a}\in\mathfrak{G}}\mathbf{a}(\mathbf{g})^{-1}\mathbf{g},$$

and for each $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit $A = [\mathbf{a}]$ of $\mathbf{a}$, define

$$p_A := \sum_{\mathbf{a}\in A} p_{\mathbf{a}} = \frac{1}{m^{n+1}} \sum_{\mathbf{g}\in\mathfrak{G}} (\mathrm{trace}_{\mathbb{L}_{\mathbf{a}}/\mathbb{L}}(\mathbf{a}(\mathbf{g})^{-1}))\, \mathbf{g}.$$

Then it is easy to see that $p_{\mathbf{a}}$ and p_A are elements of the group ring $\mathbb{L}[\mathfrak{G}]$ and $\mathbb{Z}[\frac{1}{m}][\mathfrak{G}]$, respectively.

Proposition 2.1. *$p_{\mathbf{a}}$ and p_A are projectors (idempotents), that is,*

$$p_{\mathbf{a}} \cdot p_{\mathbf{b}} = \begin{cases} p_{\mathbf{a}} & \textit{if } \mathbf{a} = \mathbf{b} \\ 0 & \textit{if } \mathbf{a} \neq \mathbf{b} \end{cases}$$

and

$$p_A \cdot p_B = \begin{cases} p_A & \textit{if } A = B \\ 0 & \textit{if } A \neq B \end{cases}$$

Furthermore, we have the decomposition

$$\sum_{\mathbf{a}\in\hat{\mathfrak{G}}} p_{\mathbf{a}} = 1, \quad \textit{and} \quad \sum_{A\in O(\hat{\mathfrak{G}})} p_A = 1$$

where $O(\hat{\mathfrak{G}})$ denotes the set of $(\mathbb{Z}/m\mathbb{Z})^\times$-orbits in $\hat{\mathfrak{G}}$.

Definition 2.2. Identifying each automorphism $\mathbf{g} \in \mathrm{Aut}(V)$ with its graph, the projector p_A may be regarded as an algebraic n-cycle on $V \times V$ with coefficients in $\mathbb{Z}[\frac{1}{m}]$. The pair

$$(V,\, p_A) =: \mathcal{M}_A$$

may be called the *Fermat motive* corresponding to the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit A of $\hat{\mathfrak{G}}$. Furthermore, there is the motivic decomposition of the Fermat hypersurface:

$$(V, \mathbf{1}) = \oplus_{A\in O(\hat{\mathfrak{G}})} (V, p_A) = \oplus_{A\in O(\hat{\mathfrak{G}})} \mathcal{M}_A$$

corresponding to $\sum_{A\in O(\hat{\mathfrak{G}})} p_A = 1$. (Here $\mathbf{1} := \Delta_V$.)

Note that Fermat motives are well-defined over $\mathbb{Q}$ or finite fields $\mathbb{F}_p$ for prime p such that $(m, p) = 1$.

Now we will discuss the cohomological realizations of Fermat motives.

Definition 2.3. Let $H^\bullet(V, \star)$ be any Weil cohomology group. For a Fermat motive $\mathcal{M}_A = (V, p_A)$, define

$$H^\bullet(\mathcal{M}_A, \star) := H^\bullet(V, \star)^{p_A}$$

as the image of p_A (or equivalently, the kernel of $p_A - 1$) acting on $H^\bullet(V, \star)$.

The motivic decomposition yields the decomposition of cohomology groups $H^\bullet(V, \star)$ with various coefficients $\star$, e.g., the ℓ-adic étale cohomology, the de Rham cohomology, the crystalline cohomology, etc. Here we will discuss de Rham and Betti realizations.

Proposition 2.2. *Let*

$$\mathfrak{A} := \{\mathbf{a} = (a_1, a_2, \cdots, a_{n+2}) \in \hat{\mathfrak{G}} \mid a_i \neq 0 \in (\mathbb{Z}/m\mathbb{Z}) \,\forall\, i\,\}$$

be a subset of $\hat{\mathfrak{G}}$*. For* $\mathbf{a} = (a_1, a_2, \cdots, a_{n+2}) \in \mathfrak{A}$*, let*

$$\|\mathbf{a}\| := \left(\frac{1}{m}\sum_{\mathbf{a}\in A} a_i\right) - 1.$$

(a) The $(i, n-i)$*-th motivic Hodge number is given by*

$$h^{i,n-i}(\mathcal{M}_A) = \begin{cases} \#\{\mathbf{a} \in A \mid \|\mathbf{a}\| = i & \text{for } i,\ 0 \leq i \leq n \text{ and } A \subset \mathfrak{A} \\ 1 & \text{for } i = \frac{n}{2} \text{ and } A = [(0,0,\cdots,0)] \\ 0 & \text{otherwise} \end{cases}$$

In particular, the geometric genus of $\mathcal{M}_A$ *is* $p_g(\mathcal{M}_A) := h^{0,n}(\mathcal{M}_A)$*. We have the Hodge decomposition*

$$h^{i,n-1}(V) = \sum_A h^{i,n-i}(\mathcal{M}_A)$$

and in particular,

$$p_g(V) = \sum_A p_g(\mathcal{M}_A).$$

(b) The i-th motivic Betti number is given by

$$B_i(\mathcal{M}_A) = \begin{cases} \#A & \text{for } i = n \text{ and } A \subset \mathfrak{A} \\ 1 & \text{for } i \text{ even and } A = [(0,0,\ldots,0)] \\ 0 & \text{otherwise} \end{cases}$$

For $i = n$*, we have*

$$B_n(\mathcal{M}_A) \leq \varphi(m)/gcd(\mathbf{a}, m)$$

where $\varphi(m)$ *denotes the Euler's phi-function. (We may call* $B_n(\mathcal{M}_A)$ *the dimension of the motive* $\mathcal{M}_A$*.)*

The global i-th Betti number $B_i(V)$ *is given by*

$$B_i(V) = \sum_A B_i(\mathcal{M}_A).$$

For $h^{i,n-i}(V)$ *and* $B_i(V)$*, the sum runs over* A *in the set* $O(\mathfrak{A})$ *of* $(\mathbb{Z}/m\mathbb{Z})^\times$*-orbits in* $\mathfrak{A}$ *if* n *is odd, and the set* $O(\mathfrak{A} \cup \{0\})$ *if* n *is even.*

3. Construction of Calabi–Yau orbifolds

From now on, we will confine ourselves to threefolds ($n = 3$). We will construct mirror pairs of Calabi–Yau threefolds by orbifolding method due to Greene and Plesser [GP90].

Fix a Fermat hypersurface threefold $V = V(m, 3, \mathbf{1})$ of degree m and coefficients $\mathbf{1}$ defined over $\mathbb{Q}$:

$$V : Z_1^m + Z_2^m + Z_3^m + Z_4^m + Z_5^m = 0 \subset \mathbb{P}^4_{\mathbb{Q}}.$$

Let $Q = (q_1, q_2, q_3, q_4, q_5) \in \mathbb{N}^5$ be a vector consisting of 5-tuples of positive rational integers. We call Q a *weight* if it satisfies the following two conditions:

(1) $\gcd(q_1, \cdots, \widehat{q_j}, \cdots, q_5) = 1$ for every j, $1 \leq j \leq 5$, and
(2) $q_j \,|\, m = \deg(\mathrm{V})$ for every j, $1 \leq j \leq 5$.
(Here $\hat{}$ means the exclusion of that component.)

For each j, $1 \leq j \leq 5$, let μ_{q_j} denote the finite group scheme of q_j-th roots of unity over $\mathbb{Q}$, namely,

$$\mu_{q_j} := \mathrm{Spec}\,(\mathbb{Q}[T]/(T^{q_j} - 1)).$$

Taking the direct product of μ_{q_j}'s and define

$$G = G_Q := \mu_{q_1} \times \cdots \times \mu_{q_5}.$$

Then $G = G_Q$ is a finite automorphism group scheme of $V_{\mathbb{Q}}$, whose action over the closure $\bar{\mathbb{Q}}$ is described as follows:

$$(Z_1, \cdots, Z_5) \mapsto (\zeta_{q_1}^{e_1} Z_1, \cdots, \zeta_{q_5}^{e_5} Z_5), \quad e_j \in \mathbb{Z}/q_j\mathbb{Z} \text{ for each } j$$

where ζ_{q_j} denotes a primitive q_j–th root of unity in $\bar{\mathbb{Q}}$ for every j, $1 \leq j \leq 5$.

Take the quotient, $Y := V/G_Q$. Then Y is, in general, a singular variety defined over $\mathbb{Q}$ in the weighed projective 4-space $\mathbb{P}^4(Q)$. Let $\Sigma(Y)$ denote the singular loci of Y. Then $\Sigma(Y)$ is finite, and all singularities are at most abelian (indeed, cyclic) quotient singularities.

Next we wish to construct a smooth Calabi–Yau threefold from Y. For this we first look for the sufficient condition for which this quotient variety Y is a singular Calabi–Yau threefold, that is, the canonical sheaf ω_Y to be trivial. The so-called Calabi–Yau condition is imposed on the weight $Q = (q_1, q_2, \cdots, q_5)$ by

(3) $q_1 + q_2 + q_3 + q_4 + q_5 = m$ or equivalently $\sum_{i=1}^5 \frac{q_i}{m} = 1$.

Indeed, in the weighted projective 4-space $\mathbb{P}^4(Q)$, the canonical sheaf is given by (see Dolgachev [Dol82])

$$\omega_Y \simeq \mathcal{O}(m - q_1 - q_2 - q_3 - q_4 - q_5).$$

We require the triviality of ω_Y, which gives rise to the sufficient condition (3). With the condition (3), Y has trivial canonical bundle away from its singularities.

Definition 3.1. For a fixed m, we call $Q = (q_1, q_2, q_3, q_4, q_5)$ an *admissible* weight if it satisfies the conditions (1), (2) and (3), and we will call $< m, Q >$ an *admissible* pair.

Now we look for a smooth resolution of singularities without disturbing the triviality of the canonical bundle. Such a resolution is called a *crepant* resolution. The existence of the crepant resolution for our singular Calabi–Yau threefold Y is proved by Greene, Roan and Yau [GRY91]. We summarize their results reformulated by Yui [Y05] suitable for the arithmetic discussion and applications.

Theorem 3.1. *Let $V = V(m, 3, \mathbf{1})$ be the Fermat hypersurface threefold of degree $m \geq 5$ with coefficients $\mathbf{1}$ defined over $\mathbb{Q}$. Let $Q = (q_1, q_2, q_3, q_4, q_5)$ be a weight.*

(a) There are 147 admissible pairs $< m, Q >$ giving rise to singular Calabi–Yau threefolds Y. The smallest is $< 5, (1, 1, 1, 1, 1) >$ and the largest is $< 1807, (1, 42, 258, 602, 903) >$.

In the weighted projective 4-space $\mathbb{P}^4_{\mathbb{Q}}(Q)$, Y is defined by the equation

$$Y_1^{m/q_1} + Y_2^{m/q_2} + Y_3^{m/q_3} + Y_4^{m/q_4} + Y_5^{m/q_5} = 0 \subset \mathbb{P}^4_{\mathbb{Q}}(Q)$$

with $\deg(Y_j) = q_j$ for each j, $1 \leq j \leq 5$.

(b) Y has at most abelian (cyclic) quotient singularities. Let $\Sigma(Y)$ be the singular loci of Y. Let ℓ be an integer ranging over the set $\{0, 1, \cdots, m-1\}$. For each ℓ, we define the two sets of indices:

$$S_\ell := \{\, j \mid 1 \leq j \leq 5, \ell/m_j \in \mathbb{Z}\,\} \quad \text{and} \quad \mathrm{I}_\ell = \{1, 2, 3, 4, 5\} \setminus \mathrm{S}_\ell.$$

Further put $c_{I_\ell} := gcd\{q_j \mid j \notin I_\ell\} = gcd\{q_j \mid j \in S_\ell\}$. Let

$$Y_{I_\ell} := Y \cap \{Y_j = 0,\ j \in I_\ell\}.$$

Then $Y_{I_\ell} \in \Sigma(Y)$, and it is defined over $\mathbb{Q}$.

More concretely, the following assertions hold:

(b1) $\#S_\ell \neq 4$ for any ℓ, and $\#S_\ell = 5$ for $\ell = 0$.

(b2) If $\#S_\ell = 3$ for some ℓ, then there is a set I_ℓ with $\#I_\ell = 2$, $c_{I_\ell} \geq 2$. In this case $\Sigma(Y)$ contains a dimension 1 singularity.

Suppose that C is an irreducible component of $Y_{I_\ell} = Y \cap \{Y_j = 0 \mid j \in I_\ell\}$, then C is a smooth weighted diagonal curve defined over $\mathbb{Q}$ of degree $m' := m/c_{I_\ell}$ and reduced weight $Q' := (q'_i, q'_j, q'_k)$ where $i, j, k \notin I_\ell$ and $q'_\bullet = q_\bullet/c_{I_\ell}$. The multiplicity of C in Y_{I_ℓ} is $m_C := c_{I_\ell} - 1$. The genus $g(C)$ of C is given by the coefficient of $t^{m'-(q'_i+q'_j+q'_k)}$ in the formal power series

$$\frac{1 - t^{m'}}{(1 - t^{q'_i})(1 - t^{q'_j})(1 - t^{q'_k})}.$$

(b3) If $\#S_\ell = 2$ for some ℓ, then there are dimension 0 singularities. Let $P \in \Sigma(Y)$ be a singular point. Then P is a cyclic quotient singularity, and $\pi^{-1}(P)$ is a smooth rational surface birationally equivalent to $\mathbb{P}^2$. It is defined over a finite Galois extension of $\mathbb{Q}$; its Galois orbit is defined over $\mathbb{Q}$.

(b4) If $\#S_\ell = 1$, Y has no singularity.

(c) There exists the crepant resolution $\pi : X \to Y$, and X is a smooth Calabi–Yau threefold defined over $\mathbb{Q}$. The third Betti number of X can be computed by

$$B_3(X) = B_3(Y) + \sum_C m_C\, B_1(C)$$

where C runs over all the smooth irreducible curves in $\Sigma(Y)$ with multiplicity $m_C = c_{I_\ell} - 1$. ($\pi^{-1}(C)$ is a smooth ruled surface birationally equivalent to $C \times \mathbb{P}^1$ and it is defined over $\mathbb{Q}$.)

Furthermore, $B_3(Y)$ can be computed by Vafa's formula (see Roan [Ro90]) as follows. For each integer $\ell \in \{0, 1, 2, \cdots, m-1\}$, let $S_\ell = \{\, j\,|1 \le j \le 5, \ell\frac{q_i}{m} \in \mathbb{Z}\}$ be the set of indices defined in Proposition 3.1, and define the integer β_ℓ by

$$\beta_\ell := \frac{1}{m}\sum_{r=0}^{m-1} \prod_{\ell\frac{q_i}{m}\in\mathbb{Z}, r\frac{q_i}{m}\in\mathbb{Z}} \left(1 - \frac{m}{q_i}\right).$$

Here we employ the convention that the product $\prod_{\ell\frac{q_i}{m}, r\frac{q_i}{m}}(1-\frac{m}{q_i}) = 1$ if there is no q_i with $\ell\frac{q_i}{m} \in \mathbb{Z}, r\frac{q_i}{m} \in \mathbb{Z}$. Then

$$B_3(Y) = -\beta_0, \quad \text{and} \quad B_3(X) = B_3(Y) - \sum_{\#s_\ell = 3} \beta_\ell,$$

where the sum runs over the set s_ℓ consisting of three elements. The Euler characteristic of X is given by

$$\chi(X) = \sum_{\ell=0}^{m-1} \beta_\ell.$$

Consequently,

$$B_2(X) = h^{1,1}(X) = -1 + \frac{1}{2}\sum_{\ell=0}^{m-1}\sum_{\#s_\ell<3} \beta_\ell.$$

In particular, all cycles generating $H^{1,1}(X)$ arise from singular points on Y.

Remark 3.1. Here we used Vafa's method developed for computing orbifold Euler characteristics. (At glance, it is not transparent why β_ℓ is an integer. Hendrik Lenstra gave an elementary proof to establish the integrality of β_ℓ's.) It should be noted that Vafa's method is a special case of the calculation of *stringy* Euler characteristics by toric method developed by Batyrev [Ba94]. In Section 5 we will give another approach for the computation of Betti numbers and Euler characteristics of the 147 Calabi–Yau orbifolds using Fermat motives.

4. Construction of mirror Calabi–Yau orbifolds

The Calabi–Yau orbifolds constructed in Theorem 3.1 are not rigid, i.e., $h^{1,2} \neq 0$. Therefore, their mirror Calabi–Yau threefolds exist in the sense of Topological Mirror Symmetry Conjecture 1.1. In this section, we will construct mirror partners of these Calabi–Yau orbifolds by applying again orbifolding construction on perturbed Calabi–Yau hypersurfaces by certain (finite abelian) groups of discrete symmetries. This construction is due to Greene and Plesser [GP90] (see also Greene–Roan and Yau [GRY91], Roan [Ro90], [Ro91], [Ro94]). Now we will describe the mirror construction.

Theorem 4.1. *Let X be a Calabi–Yau orbifold corresponding to $< m, Q >$ with $Q = (q_1, q_2, q_3, q_4, q_5)$ in Theorem 3.1 obtained from the weighted Fermat hypersurface:*

$$Y_1^{m_1} + Y_2^{m_2} + Y_3^{m_3} + Y_4^{m_4} + Y_5^{m_5} = 0 \subset \mathbb{P}^4(Q)$$

where we put $m_j = m/q_j$ for $1 \leq j \leq 5$. Let $\mathcal{W}_Q$ be the generic weighted Fermat hypersurface with deformation parameters, and let $\{\mathcal{W}_Q = 0\}$ be its zero locus in $\mathbb{P}^4(Q)$. Then $\{\mathcal{W}_Q = 0\}$ is a Calabi–Yau threefold. Let Ω denote the unique (up to a scalar multiplication) holomorphic 3-form on $\mathcal{W}_Q$. Then

$$\Omega = Res_{\mathcal{W}_Q=0} d\frac{d\mu}{\mathcal{W}_Q}$$

where

$$d\mu = \sum_{j=1}^{5} (-1)^j q_j Y_j dY_1 \cdots \wedge \widehat{dY_j} \wedge \cdots \wedge dY_5.$$

For each j, $1 \leq j \leq 5$, let g_{m_j} be a primitive m_j-th root of unity and let μ_{m_j} be the cyclic group generated by $g_j = e^{2\pi i/m_j}$. Define the group $\hat{G}$ by

$$\hat{G} = \{\, \mathbf{g} = (g_1, g_2, g_3, g_4, g_5) \,|\, g_i^{m_i} = 1,\ \prod_{i=1}^{5} g_i = 1 \,\} / \{(g, g, \cdots, g)\}.$$

Then $\hat{G}$ is a finite abelian group (scheme) of order $\prod_{j=1}^{5} m_j/m^2 = m^3/\prod_{i=1}^{5} q_i$, and it leaves Ω invariant.

This gives rise to a $\hat{G}$-invariant hypersurface

$$\mathcal{M}_Q := \sum_{I=(i_1,\cdots,i_5)} c_I Y_1^{i_1} \cdots Y_5^{i_5}$$

in $\mathbb{P}^4(Q)$ quotiented out by $\hat{G}$, where the sum runs over

$$I = (i_1, \cdots, i_5) \in (\mathbb{Z}/m_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_5\mathbb{Z})$$

subject to the relation $q_1 i_1 + \cdots + q_5 i_5 = m$. The zero locus $\{\mathcal{M}_Q = 0\}$ defines a Calabi–Yau threefold.

Then there exists a resolution $\hat{X} := \{\mathcal{M}_Q = 0\} \subset \mathbb{P}^4(Q)/\hat{G}$ which is a mirror partner of the family of Calabi–Yau orbifold $X := \{\mathcal{W}_Q = 0\}$ satisfying Topological Mirror Symmetry Conjecture 1.1:

$$h^{2,1}(\hat{X}) = h^{1,1}(X),\ h^{1,1}(\hat{X}) = h^{2,1}(X),\ \chi(\hat{X}) = -\chi(X).$$

The orbifolding construction was described by Greene and Plesser [GP90]. The calculations of Hodge numbers and Euler characteristics were carried out in Candelas-Lynker-Schimmrigk [CLS90]. Roan [R091] gave a mathematical proof of these results.

m	Q	G_Q	$\hat{G}$	generators
5	$(1,1,1,1,1)$	$\{1\}$	$(\mathbb{Z}/5\mathbb{Z})^3$	$(1,0,0,4,0),(1,0,4,0,0),(1,4,0,0,0)$
6	$(1,1,1,1,2)$	$(\mathbb{Z}/2\mathbb{Z})$	$(\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/6\mathbb{Z})^2$	$(0,2,2,2,0),(5,0,0,1,0),(0,5,0,1,0)$
8	$(1,1,1,1,4)$	$(\mathbb{Z}/4\mathbb{Z})$	$(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/8\mathbb{Z})^2$	$(7,0,0,1,0),(0,7,0,1,0),(0,0,7,1,0)$
10	$(1,1,1,2,5)$	$(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/5\mathbb{Z})$	$(\mathbb{Z}/10\mathbb{Z})^2$	$(9,0,1,0,0,),(0,9,1,0,0)$

TABLE 1. Mirror Calabi–Yau orbifolds I

Remark 4.1. In Theorem 4.1, defining equations for mirror Calabi–Yau threefolds are given by deformations of weighted Fermat hypersurfaces threefolds. Generically, these defining equations may have as many independent deformation parameters as $h^{2,1}_{poly} < h^{2,1}$. (The Hodge number $h^{2,1}_{poly}$ is defined in terms of toric geometry. For the definition, see Sections 6 and 7 below.) The Hodge numbers are invariant under Kähler deformations, i.e, when deformations preserve Kähler structure of the manifold in question (see Nakamura [Na75]), so $h^{1,1}$ remains unchanged under deformations. In particular, by taking a special point of the generic deformation, we can have a defining equation for our Calabi–Yau family with a small number of deformation parameters.

Example 4.1. We consider one-parameter deformations of Calabi–Yau threefolds obtained from weighted Fermat hypersurfaces corresponding to a pair $< m, Q >$ in Theorem 4.1:

$$Y_1^{m/q_1} + Y_2^{m/q_2} + Y_3^{m/q_3} + Y_4^{m/q_4} + Y_5^{m/q_5} - m\,\psi Y_1Y_2Y_3Y_4Y_5 = 0 \subset \mathbb{P}^4(Q)$$

where λ is a parameter (subject to the relation $\psi^m = 1$). Then its zero locus $\mathcal{W}_Q = 0$ gives rise to a family of Calabi–Yau threefolds.

TABLE 1 lists Calabi–Yau threefolds $\{\mathcal{W}_Q = 0\}$ corresponding to the pairs $< m, Q >$ with $h^{1,1}(X) = 1$, and their mirror families of Calabi–Yau threefolds $\mathcal{M}_Q$. Here a generator $\mathbf{g} = (g_1, \cdots, g_5)$ for $\hat{G}$ stands for $(e^{2\pi i g_1/m}, \cdots, e^{2\pi i g_5/m})$. The action of $\hat{G}$ on X is

$$(Y_1, Y_2, Y_3, Y_4, Y_5) \mapsto (g_1Y_1, g_2Y_2, g_3Y_3, g_4Y_4, g_5Y_5).$$

For instance, for $m = 5$ and the generator $(1, 0, 0, 4, 0)$, the action is read as

$$(Y_1, Y_2, Y_3, Y_4, Y_5) \mapsto (e^{2\pi i/5}Y_1, Y_2, Y_3, e^{2\pi i4/5}Y_4, Y_5).$$

The above construction of mirror pairs of Calabi–Yau orbifolds may be characterized in terms of the "duality" between the two finite abelian groups. See Greene–Plesser [GP90], and for a nice exposition, Morrison [Mor97].

Theorem 4.2. *For the above mirror pairs of Calabi–Yau orbifolds $(X, \hat{X})$, the mirror symmetry can be described as the "duality" between the two finite abelian groups, that is, there is the group $G = G_Q$ associated to the original Calabi–Yau orbifold X:*

$$G = G_Q = \prod_{j=1}^{5} \mu_{q_j} \quad \textit{of order} \quad \textstyle\prod_{j=1}^5 q_j.$$

m	Q	$\hat{G}$	*generator*
8	$(1,1,2,2,2)$	$(\mathbb{Z}/8\mathbb{Z})^2$	$(1,7,0,0,0),(7,1,0,0,0)$
12	$(1,1,2,2,6)$	$(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/6\mathbb{Z})^2$	$(6,6,0,0,0),(4,0,2,2,0),(0,4,2,2,0)$
12	$(1,2,2,3,4)$	$(\mathbb{Z}/6\mathbb{Z})^2$	$(4,2,2,0,0),(0,2,2,0,4)$
14	$(1,2,2,2,7)$	$(\mathbb{Z}/7\mathbb{Z})^2$	$(2,2,2,2,4),(4,4,4,4,2)$
18	$(1,1,1,6,9)$	$(\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/6\mathbb{Z})^2$	$(6,6,6,0,0),(3,3,3,0,9),(3,3,3,3,6)$

TABLE 2. Mirror Calabi–Yau threefolds II

On the mirror side, there is the group $\hat{G}$ associated to the mirror Calabi–Yau orbifold $\hat{X}$:

$$\hat{G}=\{\mathbf{g}=(g_1,\cdots,g_5)\,|\,g_j^{m_j}=1,\ \prod_{j=1}^{5}g_j=1\}/\mu_m \quad \textit{of order} \quad m^3/\prod_{j=1}^5 q_j.$$

These two abelian groups are subject to the "duality" relation:

$$\#G_Q\times\#\hat{G}=m^3.$$

Example 4.2. Now we consider two-parameter deformations of Calabi–Yau orbifolds with $h^{1,1}(X)=2$. Note that for these examples listed in TABLE 2, the groups G_Q are more complicated than the cases $h^{1,1}(X)=1$ discussed above, but in any event, we have the 'duality" relation: $\#G_Q\times\#\hat{G}=m^3$.

For any of these Calabi–Yau orbifolds X, its mirror Calabi–Yau threefold is defined by a hypersurface $\mathcal{W}_Q=0$ in the weighted projective space $\mathbb{P}^4(Q)/G$ where $\mathcal{W}_Q$ is of the form:

$$\mathcal{W}_Q=Y_1^{m_1}+\cdots+Y_5^{m_5}-m\psi Y_1Y_2\cdots Y_5-k\phi\begin{cases}Y_1^4Y_2^4 & \text{for } m=8\\ Y_1^6Y_2^6 & \text{for } m=12\\ Y_1^6Y_4^2 & \text{for } m=12\\ Y_1^7Y_5 & \text{for } m=14\\ Y_1^6Y_2^6Y_3^6 & \text{for } m=18\end{cases}$$

where ψ and ϕ are parameters and k is a positive integer such that $k|m$. The mirror of the Calabi–Yau threefolds with $m=8$ and $m=12$, $Q=(1,1,2,2,6)$ with two parameters was studied in Candelas et al. [COFKM93].

Remark 4.2. Theorem 4.2 is valid even when one passes to a pair of finite groups $(H,\hat{H})$ where H is a subgroup of $(\mathbb{Z}/m\mathbb{Z})^3$ and $\hat{H}$ is a dual group of H in the sense that $\#H\times\#\hat{H}=m^3$. See Klemm and Theisen [KT93] for examples of one-modulus Calabi–Yau threefolds of this type. See Problem 10.3.

Remark 4.3. One-(or multi)-parameter deformations of Calabi–Yau orbifolds constructed in Theorem 3.1 are no longer dominated by product varieties. (Confer Schoen [Sch96].) Consequently, these deformations are no longer of CM type varieties.

5. Fermat motives and mirror maps

In this section, we first construct Fermat motives arising from the Calabi–Yau threefolds constructed in Theorem 3.1 at special points of deformations, i.e., at the Fermat points (where all the deformation parameters are set to zero). We will interpret the mirror symmetry phenomenon for our Calabi–Yau orbifolds and their mirror partners in terms of Fermat motives. In particular, we will determine Fermat motives which are invariant under the mirror map for each mirror pair of Calabi–Yau orbifolds.

Now we will construct Fermat motives for the Calabi–Yau orbifolds X constructed from Fermat hypersurface of degree m and weight $Q=(q_1,q_2,q_3,q_4,q_5)$. Recall from Section 2 that associated to the Fermat hypersurface threefold

$$V\,:\,Z_1^m+Z_2^m+Z_3^m+Z_4^m+Z_5^m=0\subset\mathbb{P}^4,$$

we have a group

$$\mathfrak{G}:=\{\mathbf{g}=(g_1,g_2,g_3,g_4,g_5)\,|\,g_i\in\mu_m\}/\{(g,g,\cdots,g)\}\subset\mathrm{Aut}(V)$$

and its dual group

$$\hat{\mathfrak{G}}=\{\mathbf{a}=(a_1,a_2,a_3,a_4,a_5)\,|\,a_i\in(\mathbb{Z}/m\mathbb{Z}),\ \sum_{i=1}^5 a_i\equiv 0\pmod{m}\}.$$

We consider its subset

$$\mathfrak{A}=\{\mathbf{a}\in\hat{\mathfrak{G}}\,|\,a_i\neq 0\in(\mathbb{Z}/m\mathbb{Z})\,\forall\, i\}.$$

There is a pairing between $\mathfrak{G}$ and $\mathfrak{A}$ given by

$$(\mathbf{g},\mathbf{a})\mapsto\prod_{i=1}^5 g_i^{a_i}\in\mathbb{L}=\mathbb{Q}(\zeta_m).$$

Now we pass onto weighted Fermat Calabi–Yau threefolds. We ought to bring in weights to our discussion.

Theorem 5.1. (Yui [Y05]) *Let Y be the singular Calabi–Yau orbifold corresponding to an admissible pair $<m,Q>$ with $=(q_1,q_2,q_3,q_4,q_5)$ in Theorem 3.1. Define*

$$\mathfrak{A}(Q):=\{\mathbf{a}=(a_1,a_2,a_3,a_4,a_5)\in\mathfrak{A}\,|\,a_i\in(q_i\mathbb{Z}/m\mathbb{Z})\,\forall\, i\}.$$

For each character $\mathbf{a}\in\mathfrak{A}(Q)$, let $A=[\mathbf{a}]$ be the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of $\mathbf{a}$, i.e., $A=\{t\mathbf{a}\,|\,t\in(\mathbb{Z}/m\mathbb{Z})^\times\}$. Let $p_{\mathbf{a}}$ and p_A be the projectors defined as in Definition 2.1. Then $(Y,p_A)=\mathcal{M}_A$ is the Fermat motive corresponding to A. Furthermore, we have the motivic decomposition

$$(Y,\mathbf{1})=\oplus_{A\in O(\mathfrak{A}(Q))}\mathcal{M}_A$$

where A runs over the set of $(\mathbb{Z}/m\mathbb{Z})^\times$-orbits in $\mathfrak{A}(Q)$.

Then the following assertions hold.

(a) We have

$$h^{3,0}(Y)=1=h^{3,0}(\mathcal{M}_Q)$$

A	dim	mult	$h^{3,0}$	$h^{2,1}$	$\sum B_3(\mathcal{M}_A)$
$[1,1,1,1,1]$	4	1	1	1	4
$[1,1,1,3,4]$	4	20	0	2	80
$[1,1,2,2,4]$	4	30	0	2	120

TABLE 3. Fermat motives for the quintic Calabi–Yau threefold

$$h^{2,1}(Y) = \sum_{A \in O(\mathfrak{A}(Q))} h^{2,1}(\mathcal{M}_A)$$

and

$$B_3(Y) = \sum_{A \in O(\mathfrak{A}(Q))} B_3(\mathcal{M}_A).$$

(b) Let $\pi : X \to Y$ be the crepant resolution of Y, and let $\hat{\Sigma}(Y) := \pi^{-1}(\Sigma(Y))$ be the pull-back of the singular locus $\Sigma(Y)$. Let $C \in \Sigma(Y)$ be an irreducible curve of genus $g(C)$. Then

$$h^{2,1}(\hat{\Sigma}(Y)) = \sum_C m_C g(C)$$

where the sum runs over all distinct irreducible curves in $\Sigma(Y)$ with multiplicity m_C.

(c) For the crepant resolution X of Y, we have

$$h^{3,0}(X) = 1 = h^{3,0}(\mathcal{M}_A)$$

$$h^{2,1}(X) = h^{2,1}(Y) + h^{2,1}(\hat{\Sigma}(Y)$$

and

$$B_3(X) = B_3(Y) + 2\sum_C m_C g(C).$$

Furthermore,

$$B_2(X) = h^{1,1}(X) = 1 + \sum_P m_P$$

where P is a singular point in $\Sigma(Y)$ with multiplicity m_P.

Example 5.1. Let $m = 5$ and $Q = (1,1,1,1,1)$. This is the quintic Calabi–Yau threefold X, which is smooth. The Fermat motives are tabulated in TABLE 3.

Here each motive has dimension $\varphi(5) = 4$ and multiplicity is given by the number of permutations of the components. Furthermore, Y is smooth, so $Y = X$. Summing over the motivic Hodge and Betti numbers, we get the global Hodge and Betti numbers:

$$h^{3,0}(X) = 1,\ h^{2,1}(X) = 1 + 2 \times 20 + 2 \times 30 = 101,\ h^{1,1}(X) = 1.$$

The third Betti number and the Euler characteristic are given by

$$B_3(X) = 4 + 80 + 120 = 204,\ E(X) = 2(h^{1,1}(X) - h^{2,1}(X)) = 2(1 - 101) = -200.$$

A	dim	mult	$h^{3,0}$	$h^{2,1}$	$\sum B_3(\mathcal{M}_A)$
$[1,1,2,2,2]$	4	1	1	1	4
$[7,3,2,2,2]$	4	1	0	2	4
$[6,4,2,2,2]$	2	2	0	2	4
$[7,1,2,2,4]$	4	6	0	12	24
$[6,2,2,2,4]$	2	6	0	6	12
$[4,4,2,2,4]$	2	3	0	3	6
$[5,1,2,2,6]$	4	3	0	6	12
$[4,2,2,2,6]$	2	6	0	6	12
$[3,3,2,2,6]$	4	3	0	6	12
$[5,1,2,4,4]$	4	3	0	6	12
$[4,2,2,4,4]$	2	6	0	6	12
$[3,3,2,4,4]$	4	3	0	6	12
$[3,1,2,4,6]$	4	6	0	12	24
$[2,2,2,4,6]$	2	6	0	6	12
$[3,1,4,4,4]$	4	1	0	2	4
$[2,2,4,4,4]$	2	1	0	1	2

TABLE 4. Fermat motives for the octic Calabi–Yau threefold

Example 5.2. Let $m = 8$ and $Q = (1, 1, 2, 2, 2)$. This is an octic Calabi–Yau threefold X. The Fermat motives for the singular Calabi–Yau threefold Y are tabulated in TABLE 4.

Note that the dimension of each motive is either 4 or 2 and the latter occurs when $\gcd(\mathbf{a}, 8) \neq 1$.

Summing up the motivic Hodge and Betti numbers, we obtain

$$h^{3,0}(Y) = 1,\ h^{2,1}(Y) = 83 \quad \text{and} \quad B_3(Y) = \sum_A B_3(\mathcal{M}_A) = 168.$$

There is one curve singularity $C : Y_3^4 + Y_4^4 + Y_5^4 = 0$ which has genus $g(C) = 3$ with multiplicity $m_C = 1$. Passing onto the crepant resolution X of Y, we then get

$$h^{3,0}(X) = 1, \quad h^{2,1}(X) = 3 + 83 = 86, \quad h^{1,1}(X) = 2$$

and

$$B_3(X) = 168 + 2 \times 3 = 174.$$

Finally the Euler characteristic is

$$E(X) = 2(2 - 86) = 2(-84) = -168.$$

Remark 5.1. This gives an alternative method for computing the global Betti numbers and Euler characteristic for the Calabi–Yau orbifolds X corresponding to admissible pairs $< m, Q >$ from the formula of Vafa described in Theorem 3.1 (c).

Now we pass onto mirror partners of Calabi–Yau orbifolds. We need to bring in weights into the duality.

Definition 5.1. Let X be a Calabi–Yau orbifold of degree m and weight $Q=(q_1,q_2,q_3,q_4,q_5)$. Let ζ_m denote a primitive m-th root of unity. Let $m_i=m/q_i$ and let μ_{m_i} denote the group (scheme) of m_i-th roots of unity, for each i, $0\le i\le 5$. Let

$$\mathfrak{A}(Q)=\{\,\mathbf{a}=(a_1,a_2,a_3,a_4,a_5)\in\mathfrak{A}\mid a_i\in(q_i\mathbb{Z}/m\mathbb{Z})\,\}.$$

Define

$$\hat{G}=\{\,\mathbf{g}=(g_1,g_2,g_3,g_4,g_5)\mid g_i\in\mu_{m_i},\ \prod_{i=1}^{5}g_i=1\,\}/\{(g,g,\cdots,g)\,\}.$$

Then $\hat{G}$ is the group of discrete symmetry on X leaving the holomorphic 3-form Ω invariant, and $\hat{G}$ acts on the set $\mathfrak{A}(Q)$ by

$$\hat{G}\times\mathfrak{A}(Q)\to\mathbb{Q}(\zeta_m)\ :\ (\mathbf{g},\mathbf{a})\mapsto\prod_{i=1}^{5}g_i^{a_i}=g_1^{a_1}g_2^{a_2}\cdots g_5^{a_5}.$$

(1) Let $\mathbf{a}\in\mathfrak{A}(Q)$, and $\mathbf{g}\in\hat{G}$ be a generator of $\hat{G}$. We say that $\mathbf{a}$ is *invariant under* $\mathbf{g}$ if

$$\prod_{i=1}^{5}g_i^{a_i}=g_1^{a_1}g_2^{a_2}g_3^{a_3}g_4^{a_4}g_5^{a_5}=1.$$

(2) Let $A=[\mathbf{a}]$ be the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of $\mathbf{a}$ and let $\mathcal{M}_A$ be the corresponding Fermat motive. We say that $\mathcal{M}_A$ is *invariant under* $\hat{G}$ if any $\mathbf{a}\in A$ is invariant under every generator $\mathbf{g}$ of $\hat{G}$.

Example 5.3. We consider Calabi–Yau orbifolds X and their mirror partners with Kähler modulus 1, i.e., $h^{1,1}(X)=1$. There are altogether four such Calabi–Yau orbifolds among the 147 cases. All these four Calabi–Yau orbifolds, no singularities of dimension 1.

(a) Let $m=5$ and $Q=(1,1,1,1,1)$. We know that $h^{2,1}=101$, $B_3=204$, and that $\hat{G}=(\mathbb{Z}/5\mathbb{Z})^3$ generated by

$$\mathbf{g}=(1,0,0,4,0),(1,0,4,0,0),\quad\text{and}\quad(1,4,0,0,0).$$

Let $\mathbf{a}=(1,1,1,1,1)$. Then for $\mathbf{g}=(1,0,0,4,0)$, we have

$$\prod_{i=1}^{5}g_i^{a_i}=(\zeta_5)^1\cdot 1^1\cdot 1^1\cdot(\zeta_5^4)^1\cdot 1^1=\zeta_5^5=1.$$

and similarly for the other two generators. Now consider the Fermat motive corresponding to

$$[Q]=[(1,1,1,1,1)]=\{(1,1,1,1,1),(2,2,2,2,2),(4,4,4,4,4),(3,3,3,3,3)\}.$$

We can compute that all elements in $[Q]$ are invariant under $\hat{G}$. Hence the motive $\mathcal{M}_Q$ is invariant under $\hat{G}$.

On the other hand, let $\mathbf{a} = (1,1,1,3,4)$. Then for $\mathbf{g} = (1,0,0,4,0)$, we have

$$\prod_{i=1}^{5} g_i^{a_i} = (\zeta_5)^1 \cdot 1^1 \cdot 1^1 \cdot (\zeta_5^4)^3 \cdot 1^4 = (\zeta_5)^{13} \neq 1,$$

and similarly for the other two generators. Also for $\mathbf{a} = (1,1,2,2,4)$ and $\mathbf{g} = (1,0,0,4,0)$, we have

$$\prod_{i=5}^{5} g_i^{a_i} = (\zeta_5)^1 \cdot 1^1 \cdot 1^2 \cdot (\zeta_5^4)^2 \cdot 1^4 = (\zeta_5)^9 \neq 1,$$

and similarly for the other two generators. Consequently, the Fermat motives $\mathcal{M}_A$ for $A = [(1,1,1,3,4)], [(1,1,2,2,4)]$ are not invariant under $\hat{G}$.

(b) Let $m = 6$ and $Q = (1,1,1,1,2)$. We know that $h^{2,1} = 103$, $B_3 = 208$ and that $\hat{G} = (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})^2$ with generators

$$\mathbf{g} = (0,2,2,2,0), (5,0,0,1,0), (0,5,0,1,0).$$

The Fermat motives for this Calabi–Yau orbifold are:

$$(1,1,1,1,2), (1,1,1,5,4), (1,1,2,4,4), (1,1,3,3,4), (1,1,3,5,2),$$
$$(1,1,4,4,2), (1,2,2,3,4), (1,2,2,5,2), (1,2,3,4,2), (1,3,3,3,2),$$
$$(2,2,2,2,4), (2,2,2,4,2), (2,2,3,3,2)$$

and they are all of dimension 2.

Among these, the Fermat motives corresponding to $(1,1,1,1,2)$ and $(2,2,2,2,4)$ are invariant under $\hat{G}$. For instance, for $\mathbf{a} = (2,2,2,4,2)$ and $\mathbf{g} = (5,0,0,1,0)$, we have

$$\prod_{i=1}^{5} g_i^{a_i} = (\zeta_6^5)^2 \cdot 1^2 \cdot 1^2 \cdot (\zeta_6)^2 \cdot 1^2 = (\zeta_6)^{12} = 1,$$

and similarly for the other generators. On the other hand, for $\mathbf{a} = (1,1,3,3,4)$ and $\mathbf{g} = (5,0,0,1,0)$, we compute

$$\prod_{i=1}^{5} g_i^{a_i} = (\zeta_6)^8 \neq 1,$$

and also for the other two generators. Therefore, the two Fermat motives, $\mathcal{M}_A$ with $A = [Q] = [(1,1,1,1,2)], [(2,2,2,2,4)]$ are invariant under $\hat{G}$.

(c) Let $m = 8$ and $Q = (1,1,1,1,4)$. Then $h^{2,2,1} = 149$, $B_3 = 300$, and that $\hat{G} = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^2$ with generators $\mathbf{g} = (7,0,0,1,0), (0,7,0,1,0), (0,0,7,1,0)$. After calculating the action of $\hat{G}$ on each Fermat motive, we see that only the motive corresponding to weight Q of dimension 4 is invariant.

(d) Let $m = 10$ and $Q = (1,1,1,2,5)$. We know that $h^{1,1} = 1 = B_2$, $h^{2,1} = 145$ and $B_3 = 284$, and that $\hat{G} = (\mathbb{Z}/10\mathbb{Z})^2$ with generators $\mathbf{g} = (9,0,1,0,0), (0,9,1,0,0)$. We compute the products $\prod_{i=1}^{5} g_i^{a_i}$ for all motives and generators and the outcome is that only the motive $\mathcal{M}_Q$ corresponding to the weight Q of dimension 4 is invariant under $\hat{G}$.

In each case of (a),(c) and (d), the Fermat motive $\mathcal{M}_Q$ corresponding to the weight Q is the only motive left invariant under the mirror map. It is of dimension 4, and we have $B_3(\hat{X}) = 4 = 2(1 + h^{1,1}(X))$, while, in the case (b), there are two Fermat motives $\mathcal{M}_Q$ and $\mathcal{M}_{[2,2,2,2,4]}$ which are invariant under the mirror map, and we have $B_3(\hat{X}) = 2 + 2 = 2(1 + h^{1,1}(X))$.

Example 5.4. We consider examples of Calabi–Yau orbifolds X and their mirror partners with Käher modulus two, i.e., $h^{1,1}(X) = 2$. There are six such examples among our 147 Calabi–Yau orbifolds. Some of these six Calabi–Yau orbifolds have curve singularities.

(a) Let $m = 8$ and $Q = (1, 1, 2, 2, 2)$. We know that $h^{2,1} = 86$ and $B_3 = 168 + (6)$ where 6 comes from the singular locus, and that $G_Q = (\mathbb{Z}/2\mathbb{Z})^3$ and $\hat{G} = (\mathbb{Z}/8\mathbb{Z})^2$ with generators $\mathbf{g} = (1, 7, 0, 0, 0), (7, 1, 0, 0, 0)$. The Fermat motives constituting singular part of this Calabi–Yau orbifold correspond to:

$$(1,1,2,2,2),(1,1,2,6,6),(1,1,4,4,6),(1,3,2,4,6),(1,3,2,2,6),(1,3,4,4,4),$$
$$(1,5,2,2,6),(1,5,2,4,4),(1,5,6,6,6),(1,7,2,2,4),$$

of dimension 4, and

$$(2,2,2,4,6),(2,2,4,4,4),(2,4,2,2,6),(2,4,2,4,4),(2,6,2,2,4),(4,4,2,2,4)$$

of dimension 2.

We compute the product $\prod_{i=1}^{5} g_i^{a_i}$ for all these motives and all generators $\mathbf{g}$. For the weight $Q = (1, 1, 2, 2, 2)$ and $\mathbf{g} = (1, 7, 0, 0, 0)$, we have

$$\prod_{i=1}^{5} g_i^{a_i} = (\zeta_8)^1 \cdot (\zeta_8^7)^1 \cdot 1^2 \cdot 1^2 \cdot 1^2 = 1$$

and similarly for $\mathbf{g} = (7, 1, 0, 0, 0)$. Therefore the motive $\mathcal{M}_Q$ is invariant under $\hat{G}$. The motive $\mathcal{M}_Q$ has dimension 4. Also for $(2, 2, 4, 4, 4)$ and $\mathbf{g} = (1, 7, 0, 0, 0)$, we have

$$\prod_{i=1}^{5} g_i^{a_i} = (\zeta_8)^2 \cdot (\zeta_8^7)^2 \cdot 1^4 \cdot 1^4 \cdot 1^4 = 1$$

and similarly for $\mathbf{g} = (7, 1, 0, 0, 0)$. The motive $\mathcal{M}_{[2,2,4,4,4]}$ has dimension 2. All the remaining Fermat motives are not invariant under $\hat{G}$.

(b) For $m = 12$ and $Q = (1, 1, 2, 2, 6)$, we know that $h^{2,1} = 128$ and $B_3 = 258$. The weight motive $\mathcal{M}_{[1,1,2,2,6]}$ has dimension 4. There is another motive of dimension 2 which is invariant under the mirror operation.

(c) For $m = 12$, $Q = (1, 2, 2, 3, 4)$, we know that $h^{2,1} = 74$, $B_3 = 150$. The weight motive $\mathcal{M}_{[1,2,2,3,4]}$ has dimension 4 and it is the only motive which is invariant under the mirror operation.

(d) Let $m = 14$ and $Q = (1, 2, 2, 2, 7)$. Then $h^{2,1} = 122$, $B_3 = 216 + (30)$ where 30 comes from the singularity, and $\hat{G} = (\mathbb{Z}/7\mathbb{Z})^2$ with generators $\mathbf{g} = (2, 2, 2, 2, 4)$ and $(4, 4, 4, 4, 2)$.

We calculate that only the motive $\mathcal{M}_Q$ corresponding to weight Q is invariant under $\hat{G}$, and it has dimension 6.

(e) Let $m = 18$ and $Q = (1, 1, 1, 6, 9)$. Then $h^{2,1} = 272$ and $B_3 = 546$, and $\hat{G} = (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})^2$ with generators $\mathbf{g} = (6, 6, 6, 0, 0), (3, 3, 3, 0, 9), (3, 3, 3, 3, 6)$. We calculate that there is only one motive that is left invariant under $\hat{G}$, namely, the Fermat motive $\mathcal{M}_Q$ corresponding to the weight Q, and its dimension is 6.

Theorem 5.2. *Let $(X, \hat{X})$ be a mirror pair of Calabi–Yau orbifold corresponding to an admissible pair $< m, Q >$ with $Q = (q_1, q_2, q_3, q_4, q_5)$. Let $\mathbf{g} = (g_1, g_2, g_3, g_4, g_5)$ be any generator of $\hat{G}$. Let $\mathcal{M}_A = [a_1, a_2, a_3, a_4, a_5]$ be a Fermat motive.*

Suppose that

$$\prod_{i=1}^{5} g_i^{a_i} = 1 \ \forall \ \mathbf{a} \in A$$

then $\mathcal{M}_A$ is invariant under the mirror map.

In particular, if $h^{1,1}(X) = 1$, then the Fermat motive $\mathcal{M}_Q$ corresponding to the weight Q is the only Fermat motive left invariant under $\hat{G}$.

If $h^{1,1}(X) > 1$, there is an algorithm to determine the other Fermat motives which are invariant under the mirror map.

This theorem will be proved in Section 9 below. We ought to understand possible relations between motives and monomials and integral points in reflexive polytopes in toric geometric setting à la Batyrev [Ba94].

Remark 5.2. When $h^{1,1}(X) = 1$, Goto [G06] has recently showed that the formal group arising from X is invariant under the mirror map. In particular, the height h of the formal group at bounded above by 2.

6. Batyrev's mirror symmetry

Batyrev [Ba94] gives a combinatorial construction of mirror pairs of Calabi–Yau hypersurfaces in the toric geometric setting. Here is a brief summary of Batyrev's construction and the main ingredients in his theory. A nice reference on this topic might be Cox and Katz [CK99].

Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional polytope, and let $\Delta^* = \mathrm{Hom}(\Delta, \mathbb{Z})$ be the dual polytope. Denote by $< *, * >$ the nondegenerate pairing between the n-dimensional $\mathbb{R}$-vector spaces $\Delta_{\mathbb{R}}$ and $\Delta^*_{\mathbb{R}}$. Then

$$\Delta^* := \{\mathbf{y} = (y_1, y_2, \cdots, y_n) \mid < \mathbf{y}, \mathbf{x} >:= \sum_{i=1}^{n} y_i x_i \geq -1 \, \forall \, \mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Delta \, \}.$$

The polytope Δ is said to be *reflexive* if it has the following properties:

(1) Δ is convex integral polytope, i.e., all vertices of Δ are integral,
(2) Δ contains the origin $v_0 = (0, 0, \cdots, 0)$ as an interior point, and

(3) each codimension one face is of the form

$$\{\,\mathbf{x}\in\Delta_{\mathbb{R}}\,|\,<\mathbf{y},\mathbf{x}>=-1 \text{ for some } \mathbf{y}\in\Delta^*_{\mathbb{R}}\,\}.$$

If Δ is reflexive, then its dual Δ^* is again reflexive, and $(\Delta^*)^*=\Delta$. (Indeed, this fact is the basis for Batyrev's mirror symmetry.)

To a reflexive polytope Δ, one associates a complete rational fan $\Sigma(\Delta)$ as follows: For every l-dimensional face $\Theta_l\subset\Delta$, define an n-dimensional cone $\sigma(\Theta_l)$ by

$$\sigma(\Theta_l):=\{\lambda(p'-p)\,|\,\lambda\in\mathbb{R}_{\geq 0}, p\in\Delta,\, p'\in\Theta_\ell\,\}.$$

That is, $\sigma(\Theta_l)$ consists of all vectors $\lambda(p-p')$ where $\lambda\in\mathbb{R}_{\geq 0}$, $p\in\Delta$, $p'\in\Theta_l$. Then the fan $\Sigma(\Delta)$ is given as the collection of all $(n-l)$-dimensional dual cones $\sigma^*(\Theta_l)$ for $l=0,1,\cdots,n$ for all faces of Δ, and the complete fan defines the toric variety $\mathbb{P}_\Delta$. (That is, $\mathbb{P}_{\Sigma(\Delta)}$ is defined by a compactification of the algebraic torus $(\mathbb{C}^*)^n$ using combinatorial data encoded in the fan $\Sigma(\Delta)$.)

Lemma 6.1. (Batyrev [Ba94]) *Denote by v_i $(i=0,1,\cdots,s)$ the integral points in Δ. Consider the affine space $\mathbb{C}^{s+1}$ with smooth coordinates $\mathbf{c}=(c_0,c_1,\cdots,c_s)$. Let Z_f denote the zero locus of the Laurent polynomial*

$$f_\Delta(\mathbf{c},\mathbf{X})=\sum_{i=0}^{s}c_i\mathbf{X}^\mu\in\mathbb{C}[X_1^{\pm1},X_2^{\pm1},\cdots,X_n^{\pm1}]$$

where $\mathbf{X}^\mu=X_1^{\mu_1}X_2^{\mu_2}\cdots X_n^{\mu_n}$ in the algebraic torus $(\mathbb{C}^*)^n\subset\mathbb{P}_\Delta$. Let $\bar{Z}_f$ be the closure of Z_f in $\mathbb{P}_\Delta$. The Δ-regularity conditions for hypersurfaces imply that the singularities of hypersurfaces are induced only by singularities of the ambient toric variety $\mathbb{P}_\Delta$. As a consequence, one obtains a simultaneous resolution of all members of the family of Calabi–Yau hypersurfaces, and hence one has a different family of smooth Calabi–Yau hypersurfaces X_Δ associated to Δ.

Similarly, starting with the dual reflexive polytope Δ^* and carrying out the same construction as above, one obtains a family of smooth Calabi–Yau hypersurfaces X_{Δ^*} associated to Δ^*.

We thus obtain a pair of different families of smooth Calabi–Yau hypersurfaces. The remarkable theorem of Batyrev is formulated as follows.

Theorem 6.2. (Batyrev) *If $n\leq 4$, there is a crepant resolution X_Δ (resp. X_{Δ^*}) of singularities for the hypersurface $\bar{Z}_{f_\Delta}$ (resp. $\bar{Z}_{f_{\Delta^*}}$).*

When $n=4$, the resolutions X_Δ and X_{Δ^*} are mirror symmetric in the sense of Topological Mirror Symmetry Conjecture 1.1, that is,

$$h^{1,1}(X_\Delta)=h^{2,1}(X_{\Delta^*}),\quad h^{2,1}(X_\Delta)=h^{1,1}(X_{\Delta^*}).$$

The Hodge numbers are given combinatorially by

$$h^{1,1}(X_\Delta)=h^{2,1}(X_{\Delta^*})=l(\Delta^*)-(4+1)-\sum_{codim\Theta^*=1}l'(\Theta^*)+\sum_{codim\Theta^*=2}l'(\Theta^*)l'(\Theta),$$

$$h^{2,1}(X_\Delta) = h^{1,1}(X_{\Delta^*}) = l(\Delta) - (4+1) - \sum_{codim\Theta=1} l'(\Theta) + \sum_{codim\Theta=2} l'(\Theta)l'(\Theta^*).$$

Here Θ, Θ^* are faces of Δ and Δ^*, and (Θ, Θ^*) is a dual pair. $l(\Delta)$ denotes the number of integral points in Δ, and $l(\Theta)$ (resp. $l'(\Theta)$) the number of integral points (resp. the number of integral points in the interior) of the face Θ.

Remark 6.1. The formulae for Hodge numbers give yet another method of computing Hodge numbers (and hence Betti numbers) and the Euler characteristic of our Calabi–Yau orbifolds. Via Batyrev's combinational approach, there are altogether 473 800 776 families of Calabi–Yau threefolds corresponding to 4-dimensional reflexive polytopes. See Kreuzer and Skarke [KS].

Our mirror pairs of Calabi–Yau hypersurfaces obtained in Section 4 by orbifolding construction can be reformulated in the framework of Batyrev's mirror construction. Confer Batyrev [Ba94].

Proposition 6.3. *Let $< m, Q >$ be an admissible pair where $Q = (q_1, q_2, q_3, q_4, q_5)$ satisfies the Calabi–Yau condition: $m = q +_1 +q_2 + \cdots + q_5$, or equivalently $\sum_{i=1}^5 \frac{q_i}{m} = 1$ given in Theorem 3.1. Let*

$$\Delta(Q) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{Q}^5 \mid \sum_{i=1}^{5} q_i x_i = 0,\ x_i \geq -1\ (1 \leq i \leq 5)\,\}.$$

Then $\Delta(Q)$ is a reflexive polytope and the toric variety $\mathbb{P}_{\Delta^*(Q)}$ is isomorphic to the weighted projective 4-space $\mathbb{P}^4(Q)$, and the Calabi–Yau threefold $X_{\Delta(Q)}$ is isomorphic to some Calabi–Yau orbifolds in Theorem 4.1.

More precisely, we have the following assertions.

(a) Put $m_i = m/q_i$ for each i, $1 \leq i \leq 5$. Then the quasi-homogeneous equation

$$Y_1^{m_1} + Y_2^{m_2} + Y_3^{m_3} + Y_4^{m_4} + Y_5^{m_5} = 0$$

defines a $\Delta(Q)$-regular Calabi–Yau hypersurface of Fermat type in the weighted projective 4-space $\mathbb{P}^4(Q)$. Then the family of Calabi–Yau hypersurfaces consists of quotients of deformations of this hypersurface by the fundamental group $\pi_1(\Delta(Q))$. Here $\pi_1(\Delta(Q))$ is isomorphic to the kernel of the surjective homomorphism

$$(\mu_{m_1} \times \mu_{m_2} \times \mu_{m_3} \times \mu_{m_4} \times \mu_{m_5})/\mu_n \to \mu_m$$

and it has order $m^3/q_1q_2\cdots q_5$.

(b) For the dual reflexive polytope $\Delta^*(Q)$, the fundamental group $\pi_1(\Delta^*(Q))$ is isomorphic to the kernel of the surjective homomorphism

$$(\mu_{m_1} \times \mu_{m_2} \times \cdots \times \mu_{m_5})/\mu_m \to \mu_m \quad : \quad \bar{\gamma}_Q(g_1^{a_1} \cdots g_5^{a_5}) = g^{q_1a_1 + \cdots q_5a_5}$$

and it has order $q_1q_2q_3q_4q_5 = \#G_Q$.

(c) In the toric geometric setting, mirror pairs of Calabi–Yau hypersurfaces is described by a pair of reflexive polytopes, that is, $\pi_1(\Delta(Q))$ and $\pi_1(\Delta^*(Q))$ are dual finite abelian

m	Q	$\Delta^*(Q)$
8	$(1,1,2,2,2)$	$(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),$ $(0,0,0,1),(-1,-2,-2,-2),(-1,-1,-1,0)$
12	$(1,1,2,2,6)$	$(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),$ $(0,0,0,1),(-1,-2,-2,-6),(-3,-1,-1,0)$
12	$(1,2,2,3,4)$	$(0,0,0,1),(1,0,0,0),(0,1,0,0),(0,0,1,0),$ $(0,0,0,1),(-2,-2,-3,-4),(-1,-1,-1,-2)$
14	$(1,2,2,27)$	$(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),$ $(0,0,0,1),(-2,-2,-2,-7),(-1,-1,-1,-3)$
18	$(1,1,1,6,9)$	$(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),$ $(0,0,0,1),(-1,-1,-6,-9),(0,0,-2,-3)$

TABLE 5. Vertices for Dual Polytopes

groups in the sense that

$$\#\pi_1(\Delta(Q)) \times \#\pi_1(\Delta^*(Q)) = m^3$$

coinciding with the mirror duality in the orbifolding construction of Theorem 4.1.

(d) The mirror map for the families of Calabi–Yau hypersurfaces is the following Galois correspondence: If $\mathcal{F}(\Delta(Q))$ and $\mathcal{F}(\Delta^*(Q))$ are quotients respectively by $\pi_1(\Delta(Q))$ and $\pi_1(\Delta^*(Q))$ of some subfamilies of deformations of Calabi–Yau hypersurfaces obtained by the construction in Lemma 6.1.

Reflexive polytopes for Calabi–Yau hypersurfaces of Fermat type are discussed in a number of literatures listing integral vertices, e.g., Batyrev [Ba94], Hosono-Lian and Yui [HLY95]. Here are some examples.

Example 6.1. In our examples of one-parameter Calabi–Yau orbifolds with $h^{1,1} = 1$ in Example 4.1, we have $q_1 = 1$ and $q_2 \leq q_3 \leq q_4 \leq q_5$. The convex polytope $\Delta(Q)$ is the convex hull of the integral vertices given as follows:

$$(0,0,0,0), (-1,-1,-1,-1), ((\frac{m}{q_2}) - 1, -1,-1,-1), (-1, (\frac{m}{q_3} - 1), -1, -1)$$
$$(-1,-1,(\frac{m}{q_4} - 1), -1), (-1,-1,-1,(\frac{m}{q_5} - 1)).$$

The dual polytope $\Delta^*(Q)$ consists of the following vectors:

$$(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(-q_2,-q_3,-q_4,-q_5).$$

Example 6.2. For the two-parameter families of Calabi–Yau orbifolds with $h^{1,1} = 2$ in Example 4.2, we again have $q_1 = 1$ and $q_2 \leq q_3 \leq q_4 \leq q_5$. We list the dual polytopes for these examples in TABLE 5.

Proposition 6.4. *For our mirror paris of Calabi–Yau hypersufaces constructed in Theorem 4.1 and Proposition 6.3, the origin is the only integral point present in both the reflexive polytope $\Delta(Q)$ and its dual reflexive polytope $\Delta^*(Q)$.*

In Batyrev's mirror symmetry, the essential information is encoded in the *Newton polyhedron* of the defining equation of the Calabi–Yau hypersurface, which is the convex hull of the monomials appearing in the hypersurface. Suppose that a family of Calabi–Yau hypersurfaces in $\mathbb{P}^4(Q)$ is the zero locus of the generic equation

$$f(\mathbf{c}, \mathbf{X}) = 0 \subset \mathbb{P}^4(Q)$$

of degree m and weight $Q = (q_1, q_2, q_3, q_4, q_5)$ with deformation parameters. The Newton polyhedron of f always contains the monomial $x_1 \cdots x_5$. Corresponding to this monomial, we have the integral point $(1, 1, \cdots, 1)$. Translate it to the origin. Then the Newton polyhedron of f is given by

$$\Delta(Q) = \text{convex hull of} \{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{Q}^5 \mid \sum_{i=1}^{5} q_i x_i = 0,\ x_i \geq -1\, (1 \leq i \leq 5) \}$$

The shifted by $(1, 1, \cdots, 1)$ convex polytope is the intersection of $\mathbb{Q}^5$ with the affine hyperplane

$$q_1 x_1 + q_2 x_2 + \cdots + q_5 x_5 = q_1 + q_2 + \cdots + q_5 = m.$$

Since $X_{\Delta(Q)}$ is isomorphic to some of our Calabi–Yau orbifolds in Theorem 4.1, we obtain the following result.

Theorem 6.5. *There is a one-to-one correspondence between integral points in $\Delta(Q)$ and monomials of degree m in the graded polynomial ring $\mathcal{R} = \mathbb{C}[X_1, X_2, X_3, X_4, X_5]$ of a Calabi–Yau orbifold X associated to an admissible pair $< m, Q >$.*

Remark 6.2. In fact, this is a concrete realization of the more general high-power monomial–divisor mirror map constructed by Batyrev [Ba94] and Aspinwall–Greene and Morrison [AGM93] giving the isomorphism between the subspaces $H^{1,1}_{toric}(X)$ and $H^{2,1}_{poly}(\hat{X})$. The space $H^{2,1}_{poly}(\hat{X}) \subset H^{2,1}(\hat{X})$ is isomorphic to the first-order polynomial deformations, and can be generated by monomials. Its dimension $h^{2,1}_{poly}(\hat{X})$ is equal to the number of generators of the Mori cone (which is dual to the Kähler cone of the mirror X). The generators of the Mori cone correspond to divisor classes which generate the subspace $H^{1,1}_{toric}(X)$. This gives rise to a natural isomorphism

$$H^{1,1}_{toric}(X) \simeq H^{2,1}_{poly}(\hat{X})$$

and such an isomorphism can be interpreted as the differential of the expected mirror map between the moduli spaces.

7. Monomials and periods

We will briefly describe the Dwork–Katz–Griffiths reduction method [CK99] for determining the Picard–Fuchs equations for hypersurfaces in a weighted projective space $\mathbb{P}^4(Q)$ with weight $Q = (q_1, q_2, q_3, q_4, q_5)$. The Picard–Fuchs differential equation is the device which establish the one-to-one correspondence between monomials and periods.

Suppose that a hypersurface $P(\mathbf{X}) := P(X_1, \cdots, X_5) = 0$ defines a Calabi–Yau threefold X of degree m and weight $Q = (q_1, q_2, \cdots, q_5)$, and let $P_{\mathbf{c}}(\mathbf{X})$ be a Calabi–Yau threefold $X_{\mathbf{c}}$ with a deformation parameter $\mathbf{c}$ (in practice, these parameters are complex structure moduli.) Let $\Omega_{\mathbf{c}}$ denote the unique (up to a scalar multiplication) holomorphic 3-form on $X_{\mathbf{c}}$. Then the period $\Pi(\mathbf{c})$ can be written as

$$\Pi(\mathbf{c}) = \int_{\Gamma_i} \Omega_{\mathbf{c}} = \int_{\gamma}\int_{\Gamma} \frac{\omega}{P_{\mathbf{c}}} \tag{1}$$

where

$$\omega = \sum_{i=1}^{5} (-1)^i q_i X_i dX_1 \wedge \ldots \wedge d\hat{X}_i \wedge \ldots \wedge dX_5$$

is the unique (up to scalar) holomorphic 3 form on $X_{\mathbf{c}}$. Here $\Gamma_i \in H_3(X_{\mathbf{c}}, \mathbb{Z})$ is a topological 3-cycle, and γ is a small curve around the hypersurface $P = 0$ in the 4-dimensional embedding space. Since $H_3(X, \mathbb{Z})$ has rank $B_3(X)$, there are in total $B_3(X)$ periods.

Definition 7.1. Let $\mathbb{C}[X_1, X_2, X_3, X_4, X_5]$ be the weighted polynomial ring with weight $Q = (q_1, q_2, q_3, q_4, q_5)$. Let $\mathbf{X}^{\mathbf{v}} = X_1^{v_1} X_2^{v_2} \cdots X_5^{v_5}$ be a monomial in $\mathbb{C}[X_1, X_2, \cdots, X_5]$. The *degree* of $\mathbf{X}^{\mathbf{v}}$, denoted by $w(\mathbf{v})$ is defined by by

$$w(\mathbf{v}) = \sum_{i=1}^{5} q_i v_i = q_1 v_1 + q_2 v_2 + \cdots + q_5 v_5.$$

We see that there are in total $B_3(X)$ monomials, which are divided into monomials of degree $w(\mathbf{v}) = 0, m, 2m, 3m$ with cardinalities 1, $h^{2,1}$, $h^{1,2}$, 1, respectively.

Proposition 7.1. *Let $\mathbf{X}^{\mathbf{v}} = X_1^{v_1} X_2^{v_2} \cdots X_5^{v_5}$ be a monomial of degree $w(\mathbf{v}) = \sum_{i=1}^{5} q_i v_i$ in the weighted polynomial ring $\mathbb{C}[X_1, X_2, X_3, X_4, X_5]$ with weight $(q_1, q_2, \cdots, q_5)$. Then there is a one-to-one correspondence between monomials and periods. That is, to every monomial $\mathbf{X}^{\mathbf{v}}$, there corresponds a period*

$$\Pi_{\mathbf{v}} = \int_{\Gamma_{\mathbf{v}}} \frac{\mathbf{X}^{\mathbf{v}}}{P_{\mathbf{c}}^{w(\mathbf{v})+1}}$$

where $\Gamma_{\mathbf{v}}$ is a topological cycle corresponding to $\mathbf{v}$ in $H_3(\mathbf{X}_{\mathbf{c}}, \mathbb{Z})$.

Our examples of mirror pairs of Calabi–Yau threefolds in weighted projective 4-spaces are equipped with large automorphism groups. If $(X, \hat{X})$ is a mirror pair of families of Calabi–Yau threefolds corresponding to $< m, Q >$, then the automorphism group consists of $\hat{G}$ and permutations of the weighted projective coordinates. We may choose a set of basis elements for the set of monomials under this action.

The action of the automorphisms on monomials can be transfered by Proposition 7.1 to the actions of partial differential operators on periods, and under these actions modulo the exact ones, periods decompose into the disjoin union of orbits.

Observe that $\frac{\partial}{\partial X_i}\left(\frac{g(\mathbf{X})}{P_{\mathbf{c}}^r}\right)\omega$ is exact if $g(\mathbf{X})$ is homogeneous with degree such that the whole expression has degree zero. This leads to the partial integration rule

$$\frac{h\, \partial_i P}{P_{\mathbf{c}}^r} = \frac{1}{r-1} \frac{\partial_i h}{P_{\mathbf{c}}^{r-1}}$$

with $\partial_i = \frac{\partial}{\partial X_i}$. One then takes derivatives of the expressions

$$\Pi_{\mathbf{v}} = \int_\Gamma \frac{\mathbf{X}^{\mathbf{v}}}{P_{\mathbf{c}}^{w(\mathbf{v})+1}}$$

with respect to the moduli $\mathbf{c}$ of the defining equation.

If one produces an expression such that the numerator in the integrand is not one of the basis elements, one relates it, using the equations $\partial_i P = \ldots$, to the basis and uses the relation given above. This leads to a system of first order differential equations (known as 'Gauss–Manin equations') for the $\Pi_{\mathbf{v}}$ which can be rewritten as a system of partial differential equations for the period, which is the Picard–Fuchs equations:

$$\partial_{c_k}\Pi(\mathbf{c}) = M^{(k)}(a)\Pi(\mathbf{c}),\ k = 1, \ldots, h^{2,1}.$$

Furthermore, the Picard–Fuchs differential equations (or equivalently, the periods) are equipped with the following operators:

$$\mathcal{D}_i\left(\frac{\mathbf{X}^{\mathbf{v}}}{P_{\mathbf{c}}^{w(\mathbf{v})+1}}\right) := \partial_i\left(X_i\,\frac{\mathbf{X}^{\mathbf{v}}}{P_{\mathbf{c}}^{w(\mathbf{v})+1}}\right)$$

for $i = 1, 2, \cdots, 5$. Since

$$\int_\Gamma d^5\mathbf{X}\mathcal{D}_i\left(\frac{\mathbf{X}^{\mathbf{v}}}{P_{\mathbf{c}}^{w(\mathbf{v})+1}}\right) = 0,$$

this gives rise to relations (identities) between the differential form associated to the monimial $\mathbf{X}^{\mathbf{v}}$ and those associated to $\mathbf{X}^{\mathbf{v}}$ multiplied by $\hat{G}$-invariant monomials. Hence under the action of $\mathcal{D}_i$ $(i = 1, 2, \cdots, 5)$, the Picard–Fuchs differential equation (and hence the period) decomposes into the product of differential equations.

Summing up, the Picard–Fuchs differential equation is determined by those complex structure parameters for which there is a monomial perturbation in the defining equation, and there are $h^{2,1}_{poly}$ for these. The number of deformation parameters is equal to the number of generators of the Mori cone (which is dual to the Kähler cone) in the mirror. The generators of the Mori cone come from relations between the divisors, in this way we get the secondary fan which describes the complex structure moduli space in terms of "large structure coordinates".

Based on the theory of Dwork–Katz–Griffiths, these operators are explicitly defined for the one-parameter family of the quintic Calabi–Yau threefolds by Candelas et al. [CORV00, CORV03], and for the two-parameter octic family of Calabi–Yau threefolds by Kadir [Ka04, Ka05]. Also they have the so-called Picard–Fuchs diagrams to illustrate the decomposition of the Picard–Fuchs differential equation and the periods. Clearly, such decompositions ought to have origin in the motivic decomposition of the manifolds. We obtain the following result:

Theorem 7.2. *(a) Periods are divided into disjoint set of equivalent classes under the action of the partial differential operators $\mathcal{D}_i$ $(i = 1, 2, \cdots, 5)$.*

(b) Equivalently, the Picard–Fuchs differential equation decomposes to the product of the Picard–Fuchs differential equations corresponding to equivalent classes.

#	Monomial $\mathbf{v}$	deg $w(\mathbf{v})$	mult	length of orbit
1	$(0,0,0,0,0)$	0	1	5
2	$(4,1,0,0,0)$	5	20	5
3	$(3,2,0,0,0)$	5	20	5
4	$(3,1,1,0,0)$	5	30	5
5	$(2,2,1,0,0)$	5	30	5
6	$(4,0,3,2,1)$	10	24	1

Table 6. Monomials for the quintic Calabi–Yau threefold

(c) There is a one-to-one correspondence between the set of monomial classes of under the action of $\hat{G}$-invariant monomials and the equivalence classes of periods under the action of $\mathcal{D}_i$ operators.

8. The monomial–motive correspondence: Examples

In this section, we will formulate our main findings, namely, a one-to-one correspondence between motives and monomials for a mirror pairs of Calabi–Yau hypersurfaces in weighted projective 4-spaces. This correspondence is established at the Fermat points in the moduli spaces.

First we illustrate the correspondence by two examples. A monomial $X_1^{v_1}X_2^{v_2}X_3^{v_3}X_4^{v_4}X_5^{v_5}$ is represented by the exponent $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5)$ and $(0,0,0,0,0)$ represents the constant.

Example 8.1. Consider the quintic hypersurfaces in the projective $\mathbb{P}^4$.

$$X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 - 5\psi X_1X_2X_3X_4X_5 = 0.$$

So it has the weight $Q = (1,1,1,1,1)$. There are six classes of monomials (cf. Candelas et al. [CORV00, CORV03]). Here multiplicity is the number of permutations.

In the set of monomials, the $\hat{G}$-invariant monomials are

$$X_i^5\,(1 \le i \le 5) \quad \text{and} \quad X_1X_2X_3X_4X_5.$$

The (repeated) multiplication by the $\hat{G}$-invariant monomial $X_1X_2\cdots X_5 = (1,1,1,1,1)$ to each monomial and reducing each component modulo 5 establishes the monomial–motive correspondence. The correspondences are:

$$[\mathbf{a}] = [a_1, a_2, a_3, a_4, a_5] \mapsto \text{class of} \quad X_1^{a_1/5}X_2^{a_2/5}X_3^{a_3/5}X_4^{a_4/5}X_5^{a_5/5}$$

and

$$\mathbf{v} = (v_1, v_2, v_3, v_4, v_5) \mapsto [\mathbf{v} + t(1,1,1,1,1)] \in \mathfrak{A}(Q) \quad \text{for } t \in (\mathbb{Z}/5\mathbb{Z})^\times.$$

For instance, we have

$$(0,0,0,0,0) \mapsto (1,1,1,1,1) \leftrightarrow [1,1,1,1,1]$$
$$(4,1,0,0,0) \mapsto (1,3,2,2,2) \mapsto (3,4,1,1,1) \leftrightarrow [1,1,1,3,4]$$

#	monomial $\mathbf{v}$	deg $R([\mathbf{v}],t)$	$\lambda_{\mathbf{v}}$	motive	dim	mult
1	(0,0,0,0,0)	4	1	[1,1,1,1,1]	4	1
2 3	(4,1,0,0,0) (3,2,0,0,0)	4	20	[1,1,1,3,4]	4	20
4 5	(3,1,1,0,0) (2,2,1,0,0)	4	30	[1,1,2,2,4]	4	30
6	(4,0,3,2,1)	0	24	?	?	

TABLE 7. Monomial–motive correspondence for the quintic Calabi–Yau threefold

$$(3,2,0,0,0) \mapsto (4,3,1,1,1) \leftrightarrow [1,1,1,3,4]$$
$$(3,1,1,0,0) \mapsto (4,2,2,1,1) \leftrightarrow [1,1,2,2,4]$$
$$(2,2,1,0,0) \mapsto (3,3,2,1,1) \mapsto (1,1,4,2,2) \leftrightarrow [1,1,2,2,4]$$

The length of orbit of each monomials of these actions is listed in the last column of the table.

The monomial $(4,0,3,2,1)$ only contributes at a conifold point, and hence it is not realizable at the Fermat point.

The new notation introduced in TABLE 7 and TABLE 8 ought to be explained: For a monomial $\mathbf{v}$, $R([\mathbf{v}],t)$ denotes the polynomial factor corresponding to the orbit $[\mathbf{v}]$ and $\lambda_{\mathbf{v}}$ its degree. (For details, see below.)

In Section 4, we observed that the motive corresponding to the weight $[1,1,1,1,1]$ is the only motive that is invariant under the mirror symmetry operation by the group $\hat{G}$. Also in Section 6, we showed in the toric geometric setting, that the only the origin is left invariant under the mirror map. The monomial–motive correspondence is consistent with these two facts.

Example 8.2. Consider the octic Calabi–Yau hypersurfaces in the weighted projective 4-space $\mathbb{P}^4(Q)$ with $Q=(1,1,2,2,2)$ with two deformation parameters:

$$X_1^8+X_2^8+X_3^4+X_4^4+X_5^5-2\phi X_1^4X_2^4-8\psi X_1X_2X_3X_4X_5=0$$

in $\mathbb{P}^4(Q)$. In this case, there are 15 classes of monomials (see Kadir [Ka04, Ka05]).

Here again the monomial $(4,0,3,2,1)$ only contributes at a conifold point, and hence is not realizable at the Fermat point.

In this case, the $\hat{G}$-invariant monomials are

$$X_1^8, X_2^8, X_3^4, X_4^4, X_5^4$$

and

$$X_1^4X_2^4 \quad \text{and} \quad X_1X_2X_3X_4X_5.$$

Given a monomial $\mathbf{v}=(v_1,v_2,v_3,v_4,v_5)$, adding $(1,1,1,1,1)$ or $(4,4,0,0,0)$ repeatedly and then reducing the first two components modulo 8 and the last three components by

#	monomial $\mathbf{v}$	deg $w(\mathbf{v})$	mult	#	monomial	deg	mult
1	(0,0,0,0,0)	0	1	9	(0,4,0,3,3)	16	3
2	(0,2,1,1,1)	8	2	10	(4,0,1,1,0)	8	3
3	(6,2,0,0,0)	8	1	11	(2,0,3,0,0)	8	6
4	(0,0,0,2,2)	8	3	12	(6,0,1,0,0)	8	6
5	(2,0,1,3,3)	16	6	13	(0,0,3,1,0)	8	6
6	(4,0,2,0,0)	8	3	14	(2,0,2,1,0)	8	6
7	(0,0,2,1,1)	8	3	15	(4,0,3,2,1)	16	6
8	(2,2,1,1,0)	8	3				

TABLE 8. Monomials for the octic Calabi–Yau threefold

modulo 4 yield the monomial–motive correspondence. The correspondences are given as follows with $Q = (q_1, q_2, q_3, q_4, q_5) = (1, 1, 2, 2, 2)$.

$$[\mathbf{a}] = [a_1, a_2, a_3, a_4, a_5] \mapsto \text{class of } X_1^{a_1/q_1} X_2^{a_2/q_2} X_3^{a_3/q_3} X_4^{a_4/q_4} X_5^{a_5/q_5}$$

and

$$\mathbf{v} = (v_1, v_2, v_3, v_4, v_5) \mapsto [\mathbf{v} + \prod (1,1,2,2,2)^t (4,4,0,0,0)^s] \in \mathfrak{A}(Q)$$

for some $t, s \in \mathbb{N}$. Here are some examples. First we describe motives to monomials correspondence.

$$[1,1,2,2,2] \mapsto X_1 X_2 X_3^{2/2} X_4^{2/2} X_5^{2/2} = \prod_{i=1}^{5} X_i \mapsto (0,0,0,0,0)$$

$$[2,2,4,4,4] \mapsto X_1^2 X_2^2 X_3^{4/2} X_4^{4/2} X_5^{4/2} = (\prod_{i=1}^{5} X_i)^2 \mapsto (0,0,0,0,0)$$

$$[5,1,2,4,4] \mapsto X_1^5 X_2 X_3^{2/2} X_4^{4/2} X_5^{4/2} = \prod_{i=1}^{5} X_i (X_1^4 X_4 X_5) \mapsto (4,0,0,1,1)$$

$$[3,1,2,4,6] \mapsto X_1^3 X_2 X_3 X_4^2 X_5^3 = (\prod_{i=1}^{5} X_i)(X_1^2 X_4 X_5^2) \mapsto (2,0,0,1,2)$$

Conversely, here are examples of monomials to motives correspondence.

$$(6,2,0,0,0) \mapsto [7,3,2,2,2]$$

$$(3,5,2,1,1) \mapsto (7,9,2,1,1) \mapsto (7,1,2,1,1) \mapsto [7,1,4,2,2]$$

#	Monomial $\mathbf{v}$	deg $R([\mathbf{v}], t)$	$\lambda_{\mathbf{v}}$	motive	dim	mult
1	$(0,0,0,0,0)$	6	1	$[1,1,2,2,2]$	4	1
				$[2,2,4,4,4]$	2	1
2	$(0,2,1,1,1)$	4	2	$[3,1,4,4,4]$	4	1
				$[6,4,2,2,2]$	2	2
3	$(6,2,0,0,0)$	4	1	$[7,3,2,2,2]$	4	1
4	$(0,0,0,2,2)$	4	3	$[3,3,2,2,6]$	4	3
5	$(2,0,1,3,3)$	2	6	$[4,2,2,2,6]$	2	6
6	$(4,0,2,0,0)$	4	3	$[5,1,2,2,6]$	4	3
7	$(0,0,2,1,1)$	3	3	$[4,4,2,2,4]$	2	3
8	$(2,2,1,1,0)$	3	3	$[3,3,2,2,4]$	4	3
9	$(0,4,0,3,3)$	4	3	$[6,2,2,2,4]$	2	6
10	$(4,0,1,1,0)$	4	3	$[5,1,2,2,4]$	4	3
11	$(2,0,3,0,0)$	3	6	$[4,2,2,4,4]$	2	6
12	$(6,0,1,0,0)$	3	6	$[7,1,2,2,4]$	4	6
13	$(0,0,3,1,0)$	2	6	$[2,2,2,4,6]$	2	6
14	$(2,0,2,1,0)$	4	6	$[3,1,2,4,6]$	2	12
15	$(4,0,3,2,1)$	0	6	?	?	?

TABLE 9. Monomial–motive correspondence for the octic Calabi–Yau threefold

In the above examples, the monomial–motive correspondences are obtained by combinatorial matching of monomials and motives. In the rest of this section we will establish this correspondence mathematically.

First we re-access the situation. Let $\mathcal{R} = \mathbb{C}[X_1, X_2, X_3, X_4, X_5]$ be the weighted polynomial ring of the family of Calabi–Yau hypersurfaces in weighted projective 4-space. Let $\mathcal{S}$ denote the set of all monomials $\mathbf{X}^{\mathbf{v}} = X_1^{v_1} X_2^{v_2} \cdots X_5^{v_5}$. Then $\mathcal{S}$ has the group of automorphisms consisting of $\hat{G}$ and permutations of weighted coordinates. Under the action of automorphisms, $\mathcal{S}$ can be decomposed into the disjoint sets of orbits of monomials, which we call a monomial class. A monomial class consists of monomials $\mathbf{X}^{\mathbf{v}}$ and its multiples by $\hat{G}$-invariant monomials. We may employ the notation $([v_1], [v_2], [v_3], [v_4], [v_5])$for the monomial class.

Since periods are in one-to-one correspondence with monomials, the decomposition of monomials to the set of orbits under the action of automorphisms is carried over to periods. On the set of periods, there are the differential operators $\mathcal{D}_i$ described in Section 7. Under these differential operators, periods decompose into the product of subperiods, which are in one-to-one correspondence with the disjoint sets of orbits of monomials.

Theorem 8.1. *For a Calabi–Yau threefold X associated to an admissible pair $< m, Q >$ with $Q = (q_1, q_2, q_3, q_4, q_5)$, there is a one-to-one correspondence at the Fermat point between monomial classes and Fermat motives. The correspondences are given as follows:*

A motive $A = [\mathbf{a}] = [a_1, a_2, a_3, a_4, a_5]$ *with* $\mathbf{a} \in \mathfrak{A}(Q)$ *corresponds to the monomial class* $X_1^{a_1/q_1} X_2^{a_2/q_2} X_3^{a_3/q_3} X_4^{a_4/q_4} X_5^{a_5/q_5}$.

Conversely, a monomial class $X_1^{v_1} X_2^{v_2} X_3^{v_3} X_4^{v_4} X_5^{v_5}$ *with* $([v_1], [v_2], [v_3], [v_4], [v_5])$ *representing the class of monomials, corresponds to the motive* $A = [[v_1]q_1, [v_2]q_2, [v_3]q_3, [v_4]q_4, [v_5]q_5] \in \mathfrak{A}(Q)$.

(Note that the constant monomial class contain the monomial $(1, 1, 1, 1, 1)$ *which corresponds to the weight motive* $\mathcal{M}_Q = [q_1, q_2, q_3, q_4, q_5]$.*)*

9. Proof of the monomial–motive correspondence

We will prove Theorem 8.1 by comparing the two expressions for the congruence zeta-function of X over $\mathbb{F}_q$ obtained in two different ways. One is based on Weil's method and involves Fermat motives. The other is based on Dwork's method and involves monomials. We will show that the two approaches reconcile at the Fermat point giving the same congruence zeta-function.

Let X be a Calabi–Yau threefold defined over $\mathbb{Q}$. To compute the number of $\mathbb{F}_q$-rational points on the reduction $X_{\mathbb{F}_q}$ of X to the finite field $\mathbb{F}_q$ of characteristic p. (We need to choose a "good prime" p, for our examples, good primes are those primes $p \nmid m$.) Denote by $\#X(\mathbb{F}_q)$ the number of $\mathbb{F}_q$-rational points on $X_{\mathbb{F}_q}$. The congruence zeta function of $X_{\mathbb{F}_q}$ is defined by concocting the numbers $\#X_{\mathbb{F}_{q^r}}$ for all integers $r \geq 1$:

$$Z(X_{\mathbb{F}_q}, t) = \exp\left(\sum_{r=1}^{\infty} \#X_{\mathbb{F}_{q^r}} \frac{t^r}{r}\right).$$

Let $\bar{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$ and write $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Choose a prime $\ell \neq p$. Then by the work developed by Grothendieck and reached the summit by Deligne [D74], $Z(X_{\mathbb{F}_q}, t)$ has the interpretation in terms of ℓ-adic etale cohomology groups $H^i_{et}(\bar{X}, \mathbb{Q}_\ell)$ as

$$Z(X_{\mathbb{F}_q}, t) = \prod_{i=0}^{6} P_i(X_{\mathbb{F}_q}, t)^{(-1)^{i-1}}$$

where

$$P_i(X_{\mathbb{F}_q}, t) := det(1 - t\, Frob_q \mid H^i(\bar{X},\, Q_\ell)) \in 1 + \mathbb{Z}[t]$$

is an integral polynomial of degree equal to the i-th Betti number $B_i(X)$, and it satisfies the Riemann Hypothesis, namely, its reciprocal roots are algebraic integers with absolute value $q^{i/2}$.

For diagonal hypersurfaces, the congruence zeta-functions were explicitly described in Weil [We49]. The number $\#X(\mathbb{F}_q)$ were computed using Jacobi sums and Gauss sums. For our Calabi–Yau orbifolds associated to admissible pairs $< m, Q >$, the congruence zeta-functions are of the form

$$Z(X_{\mathbb{F}_q}, t) = \frac{P_3(X_{\mathbb{F}_q}, t)}{(1-t)(1-qt)^{h^{1,1}(X)}(1-q^2t)^{h^{2,2}(X)}(1-q^3t)}$$

only the middle cohomology group H^3 (and hence P_3) contains essential information.

Proposition 9.1. *(*Yui [Y05]*) Let X be a Calabi–Yau orbifold corresponding to an admissible pair $< m, Q >$ with $Q = (q_1, q_2, q_3, q_4, q_5)$. Let q be a power of prime p such that $q \equiv 1 \pmod{m}$. Then*

$$P_3(X_{\mathbb{F}_q}, t) = \prod_{\mathbf{a}\in\mathfrak{A}(Q)} (1 - J(\mathbf{a})\, t)$$

with

$$\mathfrak{A}(Q) = \{\mathbf{a} \in (\mathbb{Z}/m\mathbb{Z})^5 \mid a_i \neq 0, \sum_{i=1}^{5} a_i \equiv 0 \pmod{m} \}.$$

Here $J(\mathbf{a})$ is the Jacobi sum defined as

$$J(\mathbf{a}) = (-1)^3 \sum \chi(x_1)^{a_1}\chi(x_2)^{a_2}\chi(x_3)^{a_3}\chi(x_4)^{a_4}\chi(x_5)^{a_5}$$

where $\chi : \mathbb{F}_q \to \mu_m$ is a multiplicative character of order m, and the sum runs over $(x_1, x_2, \cdots, x_5) \in (\mathbb{F}_q^*)^5$ such that $\sum_{i=1}^5 x_i = 0 \in \mathbb{F}_q$.

Furthermore, the Jacobi sum is expressed as the product of Gauss sums as follows:

$$J(\mathbf{a}) = (-1)^3 \frac{1}{q} G(\chi^{a_1})G(\chi^{a_2}) \cdots G(\chi^{a_5})$$

where $G(\chi)$ is the Gauss sum defined by

$$G(\chi) = G(\chi, \psi) := \sum_{x\in\mathbb{F}_q^*} \chi(x)\psi(x)$$

with ψ an additive character of $\mathbb{F}_q$.

(If we use a different normalization by putting minus sign in front of the sum in the definition of Gauss sums, then $G(\chi_1, \phi) = 1$ for the trivial character χ_1. In this case, the minus sign in front of $J(\mathbf{a})$ can be dropped. There are two types of normalizations in literature.)

Furthermore, $P_3(X, t)$ factors into the product:

$$P_3(X_{\mathbb{F}_q}, t) = \prod_{A\in O(\mathfrak{A}(Q))} P_3(\mathcal{M}_A, t)$$

where

$$P_3(\mathcal{M}_A, t) = \prod_{\mathbf{a}\in A} (1 - J(\mathbf{a})\, t) \in 1 + \mathbb{Z}[t] \quad \text{with deg } P_3(\mathcal{M}_A, t) = B_3(\mathcal{M}_A).$$

By Weil [We49], we know that Gauss sums are algebraic integers with absolute value $q^{1/2}$ in the field $\mathbb{Q}(\zeta_m, \zeta_q)$ over $\mathbb{Q}$ where ζ_m (resp. ζ_q) is a primitive m-th (resp. q-th) root of unity. Therefore, Jacobi sums $J(\mathbf{a})$ are algebraic integers in $\mathbb{L} = \mathbb{Q}(\zeta_m)$ with absolute value $q^{3/2}$. Weil's formula for $|X(\mathbb{F}_q)|$ in terms of Gauss and Jacobi sums.

Lemma 9.2. (Weil's formula) *Let X be a Calabi–Yau threefold corresponding to an admissible pair $< m, Q >$. Then the number of $\mathbb{F}_q$-rational point on X is given by the* *formula:*

$$\#X(\mathbb{F}_q) = 1 + q + q^2 + q^3 - \sum_{\mathbf{a}\in\mathfrak{A}(Q)} J(\mathbf{a}) = 1 + q + q^2 + q^3 + \frac{1}{q} \sum_{\mathbf{a}\in\mathfrak{A}(Q)} \prod_{i=1}^{5} G(\chi^{a_i}).$$

We may further factor the last term into sums involving Fermat motives as

$$N_{motive} := \sum_{\mathbf{a}\in\mathfrak{A}(Q)} J(\mathbf{a}) = \sum_{A\in O(\mathfrak{A}(Q))} \sum_{\mathbf{a}\in A} J(\mathbf{a}).$$

Now we consider a mirror partner $\hat{X}$ of a Calabi–Yau orbifold X corresponding to $< m, Q >$, and its congruence zeta-function at the Fermat point.

Theorem 9.3. *Let $(X, \hat{X})$ be a mirror pair of Calabi–Yau orbifolds over $\mathbb{Q}$ corresponding to an admissible pair $< m, Q >$ constructed in Section 4. Let $X_{\mathbb{F}_q}$ (resp. $\hat{X}_{\mathbb{F}_q}$) be the reduction of X (resp. $\hat{X}$) to the finite fields $\mathbb{F}_q$ of characteristic p (where p is a good prime). Then the congruence zeta-function of $\hat{X}$ at the Fermat point is given as follows:*

$$Z(\hat{X}_{\mathbb{F}_q}, t) = \frac{P_3(\hat{X}_{\mathbb{F}_q}, t)}{(1-t)(1-qt)^{h^{2,1}}(1-q^2t)^{h^{1,2}}(1-q^3t)}$$

with

$$P_3(\hat{X}_{\mathbb{F}_q}, t) = \prod_A P_3(\mathcal{M}_A, t)$$

where the product runs over all A such that corresponding Fermat motives $\mathcal{M}_A$ are invariant under the mirror map (i.e., the action of $\hat{G}$).

There is another method of calculating the number of $\mathbb{F}_q$-rational points on the family of Calabi–Yau hypersurfaces. This method was developed by Dwork [D60]. The number of $\mathbb{F}_q$-rational point is computed using p-adic analytic method, namely, p-adic Gamma function, the Dwork character, and the Gross–Koblitz formula. This method works for more general hypersurfaces, e.g., deformations of diagonal hypersurfaces.

For this, we review the Dwork character, Gauss sums, the Gross–Koblitz formula from Candelas et al. [CORV00] (tailored especially for physicists).

Let $\mathbb{F}_p$ be the prime field of p elements and let $\mathbb{Z}_p^* \simeq \mu_{p-1}$ be the group of $(p-1)$-th roots of unity. The Teichmüller character $\omega_p : \mathbb{F}_p^* \to \mathbb{Z}_p^*$ is a multiplicative character of $\mathbb{F}_p^*$ of order $p-1$, so that $\omega_p(x) \equiv x \pmod{p}$. Let π be an element in the algebraic closure $\bar{\mathbb{Q}}_p$ such that $\pi^{p-1} = -p$, and let F be the function defined by $F(x) = exp(X + X^p/p)$. The Dwork character Θ is an additive character of $\mathbb{F}_p$ defined by

$$\Theta(x) = \Theta_{p^0}(x) = F(\pi\omega_p(x)).$$

Then we can define a Gauss sum

$$G_n = G(\Theta, \omega_p^n) = \sum_{x\in\mathbb{F}_p^*} \Theta(x)\omega_p^n(x).$$

We have the inversion formula

$$\Theta(x)=\frac{1}{p-1}\sum_{k=0}^{p-2}G_{-k}\omega_p^k(x).$$

For a finite extension $\mathbb{F}_q$ ($q=p^r$)) of $\mathbb{F}_p$, the Teichmüller character ω_q is defined as a multiplicative character of $\mathbb{F}_q^*\to\mu_{q_1}$ of the maximal order $q-1$, and the Dwork character Θ_r is an additive character of $\mathbb{F}_q$ defined by composing Θ_{p^0} with the trace $\mathrm{tr}:\mathbb{F}_q\to\mathbb{F}_p,\ \mathrm{tr}(x)=x+x^p+x^{p^2}+\cdots+x^{p^{r-1}}$. That is,

$$\Theta_r(x)=\prod_{i=0}^{r-1}\Theta_{p^i}(x).$$

A Gauss sum is defined by

$$G_{r,n}=G(\Theta_r,\omega_q^n)=\sum_{x\in\mathbb{F}_q^*}\Theta_r(x)\omega_q^n(x).$$

Lemma 9.4. *(The Gross–Koblitz formula) Let ω_p be the Teichmüller character of $\mathbb{F}_p^*$. Let Γ_p be the p-adic Gamma function defined by*

$$\Gamma_p(n):=(-1)^n\prod_{1\le i<n,p\nmid i}i$$

taking values in the p-adic integer ring $\mathbb{Z}_p$. Then the Gauss sum $G(\Theta,\omega_p)$ is expressed by the Gross–Koblitz formula for the case $p-1\nmid n$:

$$G_n=p(p)^{-\langle\frac{n}{p-i}\rangle}\Gamma_p\left(1-\left\langle\frac{n}{p-1}\right\rangle\right).$$

Here $\langle x\rangle$ denotes the fractional part for $x\in\mathbb{R}$.

For a finite extension $\mathbb{F}_q$ ($q=p^r$), the Gross–Koblitz formula for the case $q-1\nmid n$ is given by

$$G_{r,n}=G(\Theta_r,\omega_q^n)=(-1)^{r+1}q\pi^{-S(n)}\prod_{i=0}^{r-1}\Gamma_p\left(1-\left\langle\frac{p^in}{q-1}\right\rangle\right)$$

in $\mathbb{Q}_p(\pi)$ where $S(n)$ denotes the sum of the p-adic digits of n.

Lemma 9.5. *(Dwork's formula) Let X be a variety defined as the zero set of a polynomial $P(\mathbf{X})\in\mathbb{F}_q[X_1,X_2,\cdots,X_5]$. Then*

$$\sum_{y\in\mathbb{F}_q}\Theta(yP(X))=\begin{cases}0 & \text{if } P(X)\neq 0\\ q & \text{if } P(X)=0\end{cases}$$

Hence we have

$$q\#X(\mathbb{F}_q)=\sum_{X_i\in\mathbb{F}_q}\sum_{y\in\mathbb{F}_q}\Theta(yP(\mathbf{X})).$$

The congruence zeta-function $Z(X_{\mathbb{F}_q}, t)$ of $X_{\mathbb{F}_q}$ is defined same as above by counting the number $\#X_{\mathbb{F}_q}$ for each $r \geq 1$. The Dwork theory has established the rationality of $Z(X_{\mathbb{F}_q}, t)$ ([Dw60].) We can compute the congruence zeta-function for our mirror pairs of Calabi–Yau threefolds.

Let $< m, Q >$ be an admissible pair, and let $\Delta(Q)$ be a reflexive polytope. Let $f(\mathbf{c}, \mathbf{X}) = 0 \in \mathbb{P}^4(Q)$ be a defining equation of degree m over $\mathbb{Q}$ with deformation parameter $\mathbf{c}$. Let $X_{\Delta(Q)}$ denote a family of Calabi–Yau threefolds. Let $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5)$ denote a monomial $X_1^{v_1} X_2^{v_2} \cdots X_5^{v_5}$ in the weighted polynomial coordinate ring $\mathbb{C}[X_1, X_2, \cdots, X_5]$ of $X_{\Delta(Q)}$, and let $\lambda_{\mathbf{v}}$ stands for its multiplicity, i.e., the number of weighted permutations. Let $\mathcal{S}$ be the set of all monomials. Then we know that $\mathcal{S}$ decomposes into the disjoint sets $[\mathbf{v}]$ of orbits of monomials $\mathbf{v}$ under the action of the automorphism group. Denote by $O(\mathcal{S})$ the set of orbits.

Theorem 9.6. *Let $< m, Q >$ be an admissible pair, and let X_Δ be a family of Calabi–Yau threefolds. For a good prime p, let $X_{\Delta/\mathbb{F}_q}$ denote the reduction of X_Δ over the finite field $\mathbb{F}_q$. Then the congruence zeta-function of $X_{\Delta/\mathbb{F}_q}$ is a rational function over $\mathbb{Z}$ and is given by*

$$Z(X_{\Delta/\mathbb{F}_q}, t) = \frac{R(X_{\Delta/\mathbb{F}_q}, t)}{(1-t)(1-qt)^{h^{1,1}(X_\Delta)}(1-q^2t)^{h^{2,2}(X_\Delta)}(1-q^3t)}.$$

Here $R(X_{\Delta/\mathbb{F}_q}, t)$ is an integral polynomial of degree $B_3(X_\Delta)$ and has the form:

$$R(X_{\Delta(Q)/\mathbb{F}_q}, t) = \prod_{[\mathbf{v}] \in O(\mathcal{S})} R([\mathbf{v}], t) = R(\mathbf{0}, t) \prod_{[\mathbf{v}] \neq [\mathbf{0}]} R([\mathbf{v}], t).$$

Furthermore, $R([\mathbf{v}], t)$ is an integral polynomial (not necessarily irreducible), and

$$\sum_{[\mathbf{v}] \in O(\mathcal{S})} \lambda_{\mathbf{v}} deg(R) = B_3(X_{\Delta(Q)})$$

where $\lambda_{\mathbf{v}}$ denotes the multiplicity of $\mathbf{v}$.

Now consider a mirror partner $X_{\Delta^*(Q)}$ and its congruence zeta-function.

Theorem 9.7. *Let $(X_{\Delta(Q)}, X_{\Delta^*(Q)})$ be a mirror pair of Calabi–Yau threefolds defined over $\mathbb{Q}$ corresponding to a pair of reflexive polytopes $(\Delta(Q), \Delta^*(Q))$ constructed in Section 6. Let $X_{\Delta(Q)/\mathbb{F}_q}$ (resp. $X_{\Delta^*(Q)/\mathbb{F}_q}$) be the reduction of $X_{\Delta(Q)}$ (resp. $X_{\Delta^*(Q)}$) to the finite field $\mathbb{F}_q$ of characteristic p (where p is a good prime). Then the congruence zeta-function of $X_{\Delta^*(Q)/\mathbb{F}_q}$ at the Fermat point is given as follows:*

$$Z(X_{\Delta^*(Q)/\mathbb{F}_q}, t) = \frac{R([\mathbf{0}], t)}{(1-t)(1-qt)^{h^{2,1}(X_{\Delta(Q)})}(1-q^2t)^{h^{1,2}(X_{\Delta(Q)})}(1-q^3t)}$$

where $R([\mathbf{0}], t)$ is defined as in Theorem 9.6.

We compare the expression for the congruence zeta-function for X over $\mathbb{F}_q$ obtained from Weil's method and that from Dwork method at the Fermat point in the moduli space (i.e., at the point where all parameters are set to zero).

Theorem 9.8. *Let $< m, Q >$ be an admissible pair. Let $(X, \hat{X})$ be a mirror pair of Calabi–Yau threefolds. The for a good prime p, the two expressions for the denominator of the congruence zeta-function for X coincide at the Fermat point. That is,*

$$P_3(X_{\mathbb{F}_q}, t) = \prod_{A \in O(\mathfrak{A}(Q))} P_3(\mathcal{M}_A, t)^{dim(A)} = R(X_{\Delta(Q)/\mathbb{F}_q}, t) = \prod_{[\mathbf{v}] \in O(\mathcal{S})} R(\mathbf{v}, t)^{\lambda(\mathbf{v})}.$$

In particular, the factor $R([\mathbf{0}], t)$ coincides with the factor $\prod_A P_3(\mathcal{M}_A, t)$ where A runs over the $\hat{G}$-invariant motives. Moreover, this is the only factor present in the denominator of the congruence zeta function of X and $\hat{X}$.

For the calculation of the polynomials $P_3(\mathcal{M}_A, t)$, we have the following results for the cases when the polynomials become powers of linear polynomials over $\mathbb{Q}$, and also when the polynomials are irreducible over $\mathbb{Q}$.

Proposition 9.9. (Yui[05]) *Fix an admissible pair $< m, Q >$, and let $\mathcal{M}_A$ be a Fermat motive of $\mathbf{a} \in \mathfrak{A}(Q)$. Let p be a good prime and let f be the order of p mod m, i.e., f is the smallest positive integer such that $p^f \equiv 1 \pmod{m}$.*

(a) If f is even and $p^{f/2} + 1 \equiv 0 \pmod{m}$, then putting $q = p^f$, we have

$$P_3(\mathcal{M}_A, t) = (1 - q^{3/2} t)^{\varphi(m)}.$$

(b) Let $\mathcal{M}_Q$ be the motive corresponding to the weight $Q = (q_1, q_2, q_3, q_4, q_5)$. Then for $p \equiv 1 \pmod{m}$, $P_{\mathcal{M}_Q}(t)$ is always irreducible over $\mathbb{Q}$.

Example 9.1. For $m = 5$ and $Q = (1, 1, 1, 1, 1)$, the polynomials $P_3(\mathcal{M}_A, t)$ are given as follows. By the (Newton) slopes, we mean the (normalized) p-adic order of the reciprocal roots of $P_3(\mathcal{M}_A, t)$, so that they take rational values in $[0, 3]$ with multiplicity $(*)$.

(a) When $p \equiv 1 \pmod 5$ (e.g., $p = 11$), we have

$$P_3([1, 1, 1, 1, 1], t) = 1 + 89t + 3^3 \cdot 11 \cdot 13t^2 + 11^3 \cdot 89t^3 + 11^6 t^4 \quad \text{with slopes } 0, 1, 2, 3$$

$$P_3([1, 1, 1, 3, 4], t) = 1 - 11t - 3^2 11^2 t^2 - 11^4 t^3 + 11^6 t^4 = P_3([1, 1, 2, 2, 4], t) \quad \text{with slopes } 1(2), 2(2).$$

Both polynomials are irreducible over $\mathbb{Q}$.

(b) When $p \not\equiv 1 \pmod 5$, then $p \equiv 2, 3$ or $4 \pmod 5$. If $p \equiv 2, 3 \pmod 5$, then $p^4 \equiv 1 \pmod 5$ and $p^2 \equiv -1 \pmod 5$. In this case, for every motive $\mathcal{M}_A$, we have

$$P_3(\mathcal{M}_A, t) = (1 - p^{4 \cdot \frac{3}{2}} t)^4 \quad \text{with slopes } 3/2(4).$$

If $p \equiv 4 \pmod 5$, then $p^2 \equiv 1 \pmod 5$ and $p \equiv -1 \pmod 5$. In this case, for every motive $\mathcal{M}_A$, we have

$$P_3(\mathcal{M}_A, t) = (1 - p^{2 \cdot \frac{3}{2}} t)^4 \quad \text{with slopes } 3/2(4).$$

On the other hand, using Dwork's method, the polynomials $R([\mathbf{0}], t)$, $R([4, 1, 0, 0, 0], t)$ and $R([3, 2, 0, 0, 0], t)$ are computed by Candelas et al. [CORV00, CORV03] as follows. For $p = 11$,

$$R([\mathbf{0}], t) = 1 + 89t + 3^3 \cdot 13 \cdot 11t^2 + 89 \cdot 11^3 t^3 + 11^6 t^4 \quad \text{with slopes } 0, 1, 2, 3$$

and

$$R([4,1,0,0,0],t) = R([3,2,0,0,0],t) = 1 - 11t - 3^2 11^2 t^2 - 11^4 t^3 + 11^6 t^4$$

with slopes $1(2), 2(2)$.

Example 9.2. *For $m = 8$ and $Q = (1,1,2,2,2)$, the polynomials $P_3(\mathcal{M}_A, t)$ are given as follows.*

(a) When $p \equiv 1 \pmod 8$ (e.g., $p = 17$), we have

$$P_3([1,1,2,2,2],t) = 1 - 2^2 3^2 \cdot 5t + 2 \cdot 17 \cdot 467t^2 - 2^2 3^2 \cdot 5 \cdot 17^3 t^3 + 17^6 t^4 \quad \textit{with slopes } 0,1,2,3$$

$$P_3([1,1,2,6,6],t) = (1 + 2 \cdot 3 \cdot 17t + 17^3 t^2)^2 = P_3([1,1,4,4,6],t)$$

$$P_3([1,3,2,4,6],t) = (1 - 2 \cdot 3 \cdot 17t + 17^3 t^2)^2 = P_3([1,3,4,4,4],t)$$

$$P_3([1,5,2,2,6],t) = (1 - 2 \cdot 17t + 17^3 t^2)^2 = P_3([1,5,2,4,4],t) = P_3([1,5,6,6,6],t)$$

all with slopes $1(2), 2(2)$. For all the remaining motives (see TABLE 4),

$$P_3(\mathcal{M}_A, t) = (1 + 2 \cdot 17t + 17^3 t^2)^2 \quad \textit{with slopes } 1(2), 2(2).$$

In particular,

$$P_3([2,2,4,4,4],t) = (1 + 2 \cdot 17t + 17^3 t^2)^2.$$

(b) When $p \not\equiv 1 \pmod 8$, $p \equiv 3, 5$ or $7 \pmod 8$, and in all cases $p^2 \equiv 1 \pmod 8$.

If $p \equiv 7 \pmod 8$, we have, for very motive $\mathcal{M}_A$,

$$P_3(\mathcal{M}_A, t) = (1 - p^3 t)^4 \quad \textit{with slopes } 3/2(4).$$

If $p \equiv 3 \pmod 8$ (e.g., for $p = 11$), we have

$$P_3([1,1,2,2,2],t) = (1 - 2 \cdot 7 \cdot 11^2 t + 11^6 t^2)^2 = P_3([1,1,2,6,6],t) = P_3([1,1,4,4,6],t)$$

$$P_3([1,3,2,4,6],t) = (1 + 2 \cdot 7 \cdot 11^2 t + 11^6 t^2)^2 = P_3([1,3,4,4,4],t)$$

all with slopes $1(2), 2(2)$. For the remaining motives $\mathcal{M}_A$,

$$P_3(\mathcal{M}_A, t) = (1 \pm 11^3 t)^4 \quad \textit{with slopes } 3/2(4).$$

In particular,

$$P_3([2,2,4,4,4],t) = (1 - 11^3 t^2)^4.$$

If $p \equiv 5 \pmod 8$ (e.g., for $p = 13$), we have

$$P_3([1,1,2,2,2],t) = (1 + 2 \cdot 7 \cdot 13 \cdot 17t + 13^6 t^2)^2 \quad \textit{with slopes } 1(2), 2(2)$$

$$P_3([1,1,2,6,6],t) = (1 - 13^3 t)^4 = P_3([1,1,4,4,6],t) = P_3([1,3,2,4,6],t) = P_3([1,3,4,4,4],t)$$

with slopes $3/2(4)$. For the remaining motives $\mathcal{M}_A$,

$$P_3(\mathcal{M}_A, t) = (1 \pm 2 \cdot 5 \cdot 13^2 + 13^6 t^2)^2 \quad \textit{with slopes } 1(2), 2(2).$$

In particular,

$$P_3([2,2,4,4,4],t) = (1 + 2 \cdot 5 \cdot 13^2 t + 13^6 t^2)^2.$$

Kadir [Ka04] computed the polynomials $R([\mathbf{v}], t)$ using Dwork's method, the results are shown in Table 10. The following notation is used:

Monomial $\mathbf{v}$	$p=11$	$p=13$	$p=17$
$(0,0,0,0,0)$	$(0)_2(0,2.7.11^2)_4$	$(6.13)_2(0,-238.13)_4$	$(-2.17)_2(180,934.17)_4$
$(0,2,1,1,1)$	$(0)_2[(-6.11)_2(6.11)_2]^{1/2}$	$(0)_2(-6.13)_2$	$(-2.17)_2(6.17)_2$
$(6,2,0,0,0)$	$(0)_2^2$	$(-4.13)_2(4.13)_2$	$(2.17)_2^2$
$(0,0,0,2,2)$	$(0,2.7.11^2)_4$	$(0)_2$	$(-6.17)_2^2$
$(2,0,1,3,3)$	$(0)_2$	$(-6.13)_2$	$(-2.17)_2$
$(4,0,2,0,0)$	$(0)_2^2$	$(-4.13)_2(4.13)_2$	$(2.17)_2^2$
$(0,0,2,1,1)$	$(0)_2^{1/2}(0,2.7.11^2)_4^{1/2}$	$(0)_2(6.13)_2$	$(-6.17)_2^2(-2.17)_2$
$(2,2,1,1,0)$	$(0)_2^{1/2}(0,2.7.11^2)_4^{1/2}$		
$(6,0,1,0,0)$	$(0)_2^{1/2}(11\sqrt{11})_1(-11\sqrt{11})_1$	$(-4.13)_2(4.13)_2(6.13)_2$	$(-2.17)_2^3$
$(2,0,3,0,0)$	$(0)_2^{1/2}(11\sqrt{11})_1(-11\sqrt{11})_1$		
$(0,4,0,3,3)$	$(0)_2^2$	$(-4.13)_2(4.13)_2\times$	$(-2.17)_2^2(2.17)_2^2$
$(4,0,1,1,0)$	$(0)_2^2$	$(-6.13)_2(6.13)_2$	
$(0,0,3,1,0)$	$(0)_2$	$(-6.13)_2$	$(-2.17)_2$
$(2,0,2,1,0)$	$[(-6.11)_2(6.11)_2]^{1/2}$	$(0)_2$	$(6.17)_2$
$(4,0,2,3,1)$	1	1	1

TABLE 10. Polynomials for $p=11,13,17$

Notation	*Polynomial, for prime* p
$(a)_1$	$(1+at)$
$(a)_2$	$(1+at+p^3t^2)$
$(a,b)_4$	$(1+at+bt^2+ap^3t^3+p^6t^4)$

Alternatively, since the basic constituents of the congruence zeta-function are the number of $\mathbb{F}_{q^r}$-rational points on X, we compare the expression for the number of $\mathbb{F}_q$-rational points, N_{motive} by Weil's method and the number of $\mathbb{F}_q$-rational pints, N_{mon} by Dwork's method.

Proof of Theorem 9.8. We now prove Theorem 9.8 by showing that $N_{motive}=N_{mon}$. we now count the number of points on the variety excluding the contribution coming from any exceptional divisor required to smooth the ambient weighted projective space. The piece coming from the exceptional divisor needs to be added by hand using Weil's method, but emerges naturally from toric geometry using Dwork's method (namely toric geometry forbids the vanishing of coordinates associated to points lying on the same cone; see [CORV00, CORV03] and [Ka04, Ka05] for examples). Using Dwork's method, as outlined above, we obtain the following expression for the case when we have an admissible pair $<m,Q>$ with $Q=(q_1,q_2,q_3,q_4,q_5)$:

$$q\nu=\sum_{y\in\mathbb{F}_p}\sum_{\mathbf{x}\in(\mathbb{F}_p)^5}\Theta\left(y\left(\sum_{i=1}^{5}x_i^{\frac{m}{q_i}}\right)\right)=q^5+\sum_{y\in\mathbb{F}_p^*}\sum_{\mathbf{x}\in(\mathbb{F}_p)^5}\Theta\left(y\left(\sum_{i=1}^{5}x_i^{\frac{m}{q_i}}\right)\right)+\nu_{exceptional}.$$

Dividing both sides by q and bringing in Gauss sums, we get:

$$\begin{aligned}
\nu &= q^4 + \frac{1}{q(q-1)^5} \sum_{y\in\mathbb{F}_p^*} \sum_{\mathbf{x}\in(\mathbb{F}_p^*)^5} \prod_{i=1}^{5} G_{-s_i}\omega_q^{\frac{ms_i}{q_i}}(x_i)\omega_q^{\sum s_i}(y) \\
&-G_0\frac{1}{q(q-1)^4} \sum_{j=1}^{5} \sum_{y\in\mathbb{F}_p^*} \sum_{\substack{x_i\in\mathbb{F}_p^* \\ i\neq j}} \prod_{i\neq j} G_{-s_i}\omega_q^{\frac{ms_i}{q_i}}(x_i)\omega_q^{\sum s_i}(y) \\
&+G_0^2\frac{1}{q(q-1)^3} \sum_{j,k} \sum_{y\in\mathbb{F}_p^*} \sum_{\substack{x_i\in\mathbb{F}_p^* \\ i\neq j}} \prod_{i\notin\{j,k\}} G_{-s_i}\omega_q^{\frac{ms_i}{q_i}}(x_i)\omega_q^{\sum s_i}(y) \\
&-G_0^3\frac{1}{q(q-1)^2} \sum_{j,k,l} \sum_{y\in\mathbb{F}_p^*} \sum_{\substack{x_i\in\mathbb{F}_p^* \\ i\neq j}} \prod_{i\notin\{j,k,l\}} G_{-s_i}\omega_q^{\frac{ms_i}{q_i}}(x_i)\omega_q^{\sum s_i}(y) \\
&+G_0^4\frac{1}{q(q-1)} \sum_{j,k,l,n} \sum_{y\in\mathbb{F}_p^*} \sum_{\substack{x_i\in\mathbb{F}_p^* \\ i\neq j}} \prod_{i\notin\{j,k,l,n\}} G_{-s_i}\omega_q^{\frac{ms_i}{q_i}}(x_i)\omega_q^{\sum s_i}(y) \\
&-G_0^5\frac{q-1}{q} + \nu_{exceptional}.
\end{aligned}$$

Further, we pass onto the following expression for ν:

$$\begin{aligned}
\nu &= q^4 + \frac{q-1}{q} \sum_{s_i=1}^{q-2} \prod_{i=1}^{5} G_{-s_i} - G_0\frac{q-1}{q} \sum_{j=1}^{5} \sum_{\substack{s_i=1 \\ i\neq j}}^{q-2} \prod_{i\neq j} G_{-s_i} \\
&+G_0^2\frac{q-1}{q} \sum_{j,k} \sum_{\substack{s_i=1 \\ i\notin\{j,k\}}}^{q-2} \prod_{i\notin\{j,k\}} G_{-s_i} - G_0^3\frac{q-1}{q} \sum_{j,k,l} \sum_{\substack{s_i=1 \\ i\notin\{j,k,l\}}}^{q-2} \prod_{i\notin\{j,k,l\}} G_{-s_i} \\
&+G_0^4\frac{q-1}{q} \sum_{j,k,l,n} \sum_{\substack{s_i=1 \\ i\notin\{j,k,l,n\}}}^{q-2} \prod_{i\notin\{j,k,l,n\}} G_{-s_i} - G_0^5\frac{q-1}{q} + \nu_{exceptional},
\end{aligned}$$

where passing from the former to the latter expression we had to impose the following conditions:

$$\begin{array}{lll}
q-1| & \frac{ms_i}{q_i} & \forall i=1,\ldots,5 \\
q-1| & \sum_{i=1}^{5} s_i &
\end{array}$$

Hence introduce a vector of integers, $\mathbf{v^t} = (v_1, v_2, v_3, v_4, v_5)$, such that:

$$\begin{array}{rcll}
v_i(q-1) &=& \frac{s_i m}{q_i} & \forall i=1,\ldots,5 \\
v(q-1) &=& \sum_{i=1}^{5} s_i &
\end{array}$$

and of course $0 \le s_i \le q-2, \quad i=1,\dots,5$. It is easy to see that we need to consider vectors $\mathbf{v}$ such that:

$$\sum_{i=1}^{5} q_i v_i = mv, \quad 0 \le q_i v_i \le \left\lfloor \frac{m(q-2)}{q-1} \right\rfloor \quad (\forall i = 1,\dots,5)$$

$$0 \le v \le \left\lfloor \frac{5(q-2)}{q-1} \right\rfloor \quad (= 4 \quad \text{for } q \ge 7)$$

Hence,

$$\nu \;=\; q^4 + \frac{q-1}{q} \sum_{\substack{\mathbf{v} \\ \prod v_i \ne 0}} \prod_{i=1}^{5} G_{-\left(\frac{q-1}{m} v_i q_i\right)} + \nu_{exceptional}.$$

The reason all monomials $\mathbf{v}$ with zero entries are excluded is as follows: Consider a monomial $\mathbf{v}$ with exactly r zero entries. In the second equation for ν, it will appear exactly $\binom{r}{0}$ times in the first sum, and $(-1)^u\binom{r}{u}$ times in the $(u+1)$-th sum. Hence in total it appears $\sum_{u=0}^{r}(-1)^u\binom{r}{u} = 0$ times. Clearly this argument works in any dimension.

Finally we obtain:

$$N_{mon} = \frac{1}{q} \sum_{\substack{\mathbf{v} \\ \prod v_i \ne 0}} \prod_{i=1}^{5} G_{-\left(\frac{q-1}{m} v_i q_i\right)} = \frac{1}{q} \sum_{\substack{\mathbf{v} \\ \prod v_i \ne 0}} \prod_{i=1}^{5} G_{\left(\frac{q-1}{m}\right)(m - v_i q_i)},$$

where the last equality comes from the fact that $G_{r,n} = G_{r,q-1+n}$. If we define the following:

$$q_i v_i = a_i', \quad a_i = m - a_i', \quad \forall i = 1, \dots 5$$

we see that

$$vm = \sum_{i=1}^{5} a_i = \sum_{i=1}^{5} (m - a_i') = m(5 - v').$$

It is clear that the vectors with components a_i are precisely those in $\mathfrak{A}(Q)$; hence

$$N_{mon} \;=\; \frac{1}{q} \sum_{\substack{\mathbf{v} \\ \prod v_i \ne 0}} \prod_{i=1}^{5} G_{\left(\frac{q-1}{m} a_i\right)}$$

On the other hand, we know that the expression for the sum $\sum_{\mathbf{a}\in\mathfrak{A}(Q)} J(\mathbf{a})$ in terms of Weil's method is

$$\frac{1}{q} \sum_{\mathbf{a}\in\mathfrak{A}(Q)} \prod_{i=1}^{5} G(\chi^{a_i}).$$

Therefore, we obtain:

$$N_{motive} = N_{mon}.$$

This completes the proof of Theorem 9.8.

As a consequence of the proof of Theorem 9.8, we can now establish the motive–monomial correspondence at the Fermat point.

For a character $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5) \in \mathfrak{A}(Q)$, there corresponds the monomial

$$X_1^{a_1/q_1} X_2^{a_2/q_2} X_3^{a_3/q_3} X_4^{a_4/q_4} X_5^{a_5/q_5}.$$

Now the $(\mathbb{Z}/m/Z)^{\times}$-orbit of $\mathbf{a}$ gives rise to the Fermat motive $\mathcal{M}_A$, and on the other hand, multiplications by $\hat{G}$-invariant monomials yields the corresponding monomial class. These two actions are compatible. Conversely, starting with a monomial $X_1^{v_1} X_2^{v_2} X_3^{v_3} X_4^{v_4} X_5^{v_5}$, let $([v_1], [v_2], [v_3], [v_4], [v_5])$ be the equivalence class of monomials under multiplication of the $\hat{G}$-invariant monomials. Then there corresponds the motive

$$[[v_1]q_1, [v_2]q_2, [v_3]q_3, [v_4]q_4, [v_5]q_5] \in \mathfrak{A}(Q).$$

Note that the constant monomial class corresponds to the weight motive $\mathcal{M}_Q = [q_1, q_2, q_3, q_4, q_5]$.

This establishes the motive-monomial correspondence at the Fermat point.

Example 9.3. For instance for the quintic threefolds the monomial classes $(4, 1, 0, 0, 0)$ and $(3, 2, 0, 0, 0)$ combine to give only one motive $[1, 1, 1, 3, 4]$:

Monomials	Motives	Monomials	Motives
$(4,1,0,0,0)$		$(2,3,0,0,0)$	
$(0,2,1,1,1)$		$(3,4,1,1,1)$	$\in [1,1,1,3,4]$
$(1,3,2,2,2)$	$\in [1,1,1,3,4]$	$(4,0,2,2,2)$	
$(2,4,3,3,3)$	$\in [1,1,1,3,4]$	$(0,1,3,3,3)$	
$(3,0,4,4,4)$		$(1,2,4,4,4)$	$\in [1,1,1,3,4]$

Example 9.4. For the octic threefolds the monomial classes $(2, 0, 3, 0, 0)$ and $(6, 0, 1, 0, 0)$ together correspond to two different motives, $[4, 2, 2, 4, 4]$ and $[7, 1, 2, 2, 4]$:

Monomials	Motives	Monomials	Motives
$(2,0,3,0,0)$		$(6,0,1,0,0)$	
$(3,1,0,1,1)$		$(7,1,2,1,1)$	$\in [7,1,4,2,2]$
$(4,2,1,2,2)$	$\in [4,2,2,2,4]$	$(0,2,3,2,2)$	
$(5,3,2,3,3)$	$\in [7,1,2,2,4]$	$(1,3,0,3,3)$	
$(6,4,3,0,0)$		$(2,4,1,0,0)$	
$(7,5,0,1,1)$		$(3,5,2,1,1)$	$\in [7,1,4,2,2]$
$(0,6,1,2,2)$		$(4,6,3,2,2)$	$\in [4,2,2,2,4]$
$(1,7,2,3,3)$	$\in [7,1,4,2,2]$	$(5,7,0,3,3)$	

Example 9.5. It should be noted that whenever a monomial class only makes a contributions at special points in the moduli space, such as conifold points (i.e. not at the Fermat point), none of the monomials in the class have no non-zero entries, and hence, as expected, no correspondence with Fermat motives can be made.

$Q = (1, 1, 2, 2, 2)$
$(4, 0, 3, 2, 1)$
$(5, 1, 0, 3, 2)$
$(6, 2, 1, 0, 3)$
$(7, 3, 2, 1, 0)$
$(0, 4, 3, 2, 1)$
$(1, 5, 0, 3, 2)$
$(2, 6, 1, 0, 3)$
$(3, 7, 2, 1, 0)$

$Q = (1, 1, 1, 1, 1)$
$(4, 0, 3, 2, 1)$
$(0, 1, 4, 3, 2)$
$(1, 2, 0, 4, 3)$
$(2, 3, 1, 0, 4)$
$(3, 4, 2, 1, 0)$

The above correspondence can be easily seen in the existing expressions derived for the number of points for the following cases: the one-parameter family of the quintic Calabi–Yau threefolds by Candelas et al. [CORV00, CORV03] and for the two-parameter family of the octic Calabi–Yau threefolds by Kadir [Ka04, Ka05].

Example 9.6. Let $m = 5$ and $Q = (1, 1, 1, 1, 1)$. Then the N_{mon} computed in terms of Gauss sums formed from the Dwork character as follows.

(a) When $5|q - 1$

$$N_{mon} = \frac{1}{q}\sum_{a=0}^{3}\sum_{\substack{\mathbf{v}\\ v_i+a\neq 0}} \lambda_{\mathbf{v}} \prod_{i=1}^{5} G_{-(v_i+a)k}$$

with $k = \frac{q-1}{5}$

(b) When $5 \nmid q - 1$

$$N_{mon} = \frac{1}{q} G_0^5$$

The corresponding expression in terms of Jacobi sums from Weil's method is as follows:

$$\sum_{\mathbf{a}\in\mathfrak{A}(Q)} J(\mathbf{a}) = \frac{1}{q}\sum_{\mathbf{a}\in\mathfrak{A}(Q)} \prod_{i=1}^{5} G(\chi^{a_i}).$$

Therefore,

$$N_{motive} = N_{mon}.$$

Example 9.7. For the two-parameter model in with $Q = (1, 1, 2, 2, 2)$ the expression for N_{mon} is more complicated. At the Fermat point, they are given as follows.

(a) When $8|q - 1$

$$N_{mon} = \frac{1}{q}\sum_{b=0}^{1}\sum_{a=0}^{3}\sum_{\mathbf{v}} \lambda_{\mathbf{v}} \prod_{i=1,2} G_{-(v_i+a+4b)k} \prod_{i=3,4,5} G_{-2(v_i+a)k}$$

There is a condition on the sum over the monomial that $v_i + a + 4b \neq 0$ for $i = 1, 2$, and $v_i + a \neq 0$ for $i = 3, 4, 5$.

(b) When $4|q-1$, $8 \nmid q-1$

$$N_{mon} = \frac{1}{q} \sum_{a=0}^{3} \sum_{\mathbf{v}} \lambda_{\mathbf{v}} \prod_{i=1,2} G_{-\frac{(v_i+2a)}{2}l} \prod_{i=3,4,5} G_{-(v_i+2a)l}.$$

There is a condition on the sum over the monomials that $v_i + 2a \neq 0$ for $i = 1, \ldots 5$.

(c) When $4 \nmid q-1$

$$N_{mon} = \frac{1}{q} \sum_{b=0}^{1} \sum_{\substack{\mathbf{v} \\ v_i+4b\neq 0}} \lambda_{\mathbf{v}} \prod_{i=1,2} G_{-\frac{(v_i+4b)}{4}m} \prod_{i=3,4,5} G_{-\frac{v_i}{2}m}.$$

There is a condition on the sum over the monomial that $v_i + 4b \neq 0$ for $i = 1, 2$, and $v_i \neq 0$ for $i = 3, 4, 5$.

10. Conclusions and further problems

In the above we had four objects to describe the topological mirror symmetry for Calabi–Yau hypersurfaces of dimension 3. They are:

(a) Reflexive polytopes and their integral vertices (Toric geometry);

(b) Monomials in the graded polynomial ring, and the action of the group of automorphisms of the coordinate ring, which gives rise to monomial classes (Toric geometry via coordinate rings);

(c) Picard–Fuchs differential equations (and hence periods), and the action of the differential operators $\mathcal{D}_i$, which gives rise to Picard–Fuchs differential systems (Differential equations and Periods), and

(d) Fermat motives (i.e., characters in $\mathfrak{A}(Q)$ and the action of the Galois group $(\mathbb{Z}/m\mathbb{Z})^\times$ of the m-th cyclotomic field L over $\mathbb{Q}$) (Algebraic number theory).

The bijection between (a) and (b) was established by Batyrev [Ba94], and Aspinwall–Greene and Morrison [AGM93], the bijection between (b) and (c) follows from the work of Dwork, Katz, Griffiths (see Cox and Katz [CK99]); the explicit bijections were constructed for the quintic Calabi–Yau threefolds by Candelas et al. [CORV00, CORV03], and for the octic Calabi–Yau threefolds with two deformation parameters by Kadir [Ka04, Ka05].

The point of this article is to establish a one-to-one correspondence between Fermat motives and any of the above three objects in (a), (b) and (c) in the equivariant fashion compatible with the various actions. This correspondence was established at the Fermat (the Landau–ginzburg) point in the moduli space of the family of Calabi–Yau threefolds.

Though Fermat motives are defined only at the Fermat point (and also motives are algebraic in nature), through their correspondences to monomials, they appear to contain some information about Calabi–Yau orbifolds with deformation parameters.

This raises the following questions.

Problem 10.1. We now introduce parameters into the defining hypersurface equations of our Calabi–Yau orbifolds. Batyrev's method produces monomials for these Calabi–Yau hypersurfaces. Our motives are defined at the Fermat point putting deformation parameters equal to zero. When deformation parameters are non-zero, we still have monomials which correspond to Fermat motives, but acquire more monomials. Some monomials do arise from conifold singularities. How can one interpret these extra monomials from the motivic point of view?

Naïvely, one way to start would be to group the monomials multiplied by weight under their transformation properties under $(\mathbb{Z}/m\mathbb{Z})^\times$. For instance considering Example 9.4 once more and multiplying the monomials by the weight $(1, 1, 2, 2, 2)$ we get:

Weighted Monomials	Motives
$(2, 0, 6, 0, 0)$	$[2, 0, 6, 0, 0]$
$(6, 0, 2, 0, 0)$	
$(3, 1, 0, 2, 2)$	$[3, 1, 0, 2, 2]$
$(1, 3, 0, 6, 6)$	
$(7, 5, 0, 2, 2)$	
$(5, 7, 0, 6, 6)$	
$(4, 2, 2, 4, 4)$	$[4, 2, 2, 2, 4]$
$(4, 6, 6, 4, 4)$	

Weighted Monomials	Motives
$(5, 3, 4, 6, 6)$	$[7, 1, 2, 2, 4]$
$(3, 5, 4, 2, 2)$	
$(7, 1, 4, 2, 2)$	
$(1, 7, 4, 6, 6)$	
$(6, 4, 6, 0, 0)$	$[6, 4, 6, 0, 0]$
$(2, 4, 2, 0, 0)$	
$(0, 6, 2, 4, 4)$	$[0, 6, 2, 4, 4]$
$(0, 2, 6, 4, 4)$	

Hence we obtain 4 new "fictitious motives" containing zero entries, $[2, 0, 6, 0, 0]$, $[3, 1, 0, 2, 2]$, $[6, 4, 6, 0, 0]$ and $[0, 6, 2, 4, 4]$, in addition to the two found at the Fermat point, $[4, 2, 2, 2, 4]$ and $[7, 1, 2, 2, 4]$.

The "fictious motives" containing zeroes may be related to lower dimensional genuine motives of Fermat varieties of the same degree (there may be a twisting by a character). For instance, we drop the component 0 in both $[3, 1, 0, 2, 2]$ and $[0, 6, 2, 4, 4]$ we obtain $[3, 1, 2, 2]$ and $[6, 2, 4, 4]$ which may be considered as motives arising from the Fermat surface $Z_1^8 + Z_2^8 + Z_3^8 + Z_4^8 = 0 \in \mathbb{P}^4$ of geometric genus $p_g = 35$. The motive $[3, 1, 2, 2]$ has dimension 4 and multiplicity 6. The Hodge numbers are $h^{2,0} = 2, h^{1,1} = 4, h^{2,0} = 2$ so that the 2nd Betti number is $B_2 = 8$. The motive $[6, 2, 4, 4]$ has dimension 2 and multiplicity 1. The Hodge numbers are $h^{2,0} = 0, h^{1,1} = 2, h^{0,2} = 0$. However, it is not clear how these Fermat motives of lower dimensions come into the picture of the octic Calabi–Yau threefold.

When there is a conifold singularity, it is locally isomorphic to the quadric $X^2 + Y^2 + Z^2 + T^2 = 0$, and there should correspond a twisted Tate motive (an extension of a Tate motive), which in turn should come from some monimial class.

Exploring the role played by the deformation parameters will be a project in the future. It is well known that Fermat hypersurfaces are dominated by the product of Fermat hypersurfaces of lower dimensions of the same degree (see, for instance Hunt and Schimmrigk [HS99]). However, the situation is totally different if one passes onto deformations of Fermat hypersurfaces. Chad Schoen [Sch96] showed that the one-parameter deformation of the quintic Calabi–Yau threefold

$$Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 + Z_5^5 - 5\psi Z_1 Z_2 Z_3 Z_4 Z_5 = 0 \in \mathbb{P}^4 \times \mathbb{P}^1$$

is not dominated by product varieties. This would imply that extra monomials $(4, 0, 3, 2, 1)$ of degree 10 and multiplicity 24, which corresponds to the conifold singular point $\psi = 1$, are not arising from lower dimensional Fermat surfaces of degree 5. At $\psi = 1$, they acquire 125 nodes, and resolving them with $\mathbb{P}^1$ yields the rigid Calabi–Yau threefold considered by Chad Schoen [Sch86]. Candelas et al. [CORV03] explained the role of these extra monomials in the mirror construction. From their calculations of the local zeta-functions, clearly we see the appearance of the Tate motives corresponding to these extra monomials.

Problem 10.2. The study of the zeta-functions and L-series of Calabi–Yau orbifolds with deformation parameters is an ongoing projct with Y. Goto and R. Kloosterman ([GKY]), where we use a rigid cohomology theory, e.g., Monsky–Washnitzer p-adic cohomology theory. It is our hope to describe the correspondence between monomials and motives equivariantly in terms of this p-adic cohomology theory and the other cohomology theories (e.g., étale, Betti).

Problem 10.3. The duality described between the two finite abelian groups $G = G_Q$ and $\hat{G}$ in Section 4 may be extended further to pairs of quotients $(X/H, \hat{X}/\hat{H})$ where $H \subset \hat{G}$ and $\hat{H}$ is the complement of H in $\hat{G}$ such that $\#H \times \#\hat{H} = m^3$. H and $\hat{H}$ act on X and $\hat{X}$ respectively. Indeed, mirror symmetry can be extended to the mirror pairs of Calabi–Yau orbifolds corresponding to $(H, \hat{H})$ (Klemm and Theisen [KT93]), where H is normalized in some cases.

Establish motive-monomial correspondence for these Calabi–Yau threefolds. For this, one should calculate which monomials (or equivalently, motives) are preserved under the orbifolding operation by various different groups.

Example 10.3.1: Consider the case

$$m = 5,\ Q = (1, 1, 1, 1, 1) \,:\, Y_1^5 + Y_2^5 + Y_3^5 + Y_4^5 + Y_5^5 = 0$$

with the groups $(G_Q, \hat{G}) = (\{1\}, (\mathbb{Z}/5\mathbb{Z})^3)$, and the relevant dual pairs of subgroups. They are tabulated below where the number $\hat{\#}$ indicates a mirror partner of $\#$.

Example 10.3.2: Next, we consider Calabi–Yau orbifolds corresponding to

$$m = 10,\ Q = (1, 1, 1, 2, 5) \,:\, Y_1^{10} + Y_2^{10} + Y_3^{10} + Y_4^5 + Y_5^2 = 0$$

with the groups $(G_Q, \hat{G}) = ((\mathbb{Z}/10\mathbb{Z}), (\mathbb{Z}/10\mathbb{Z})^2)$. The relevant subgroups H are normalized by factoring out the group $G_Q = (\mathbb{Z}/10\mathbb{Z})$, that is, $(\mathbb{Z}/2\mathbb{Z})$ actually means $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/10\mathbb{Z})$.

	H	generators	$h^{1,2}$	$h^{1,1}$	χ
1	$\{1\}$		101	1	-200
2	$(\mathbb{Z}/5\mathbb{Z})$	$(1,0,0,4,0)$	49	5	-88
3	$(\mathbb{Z}/5\mathbb{Z})$	$(1,2,3,4,0)$	21	1	-40
4	$(\mathbb{Z}/5\mathbb{Z})^2$	$(1,0,0,4,0),(1,2,3,4,0)$	21	17	-8
$\hat{4}$	$(\mathbb{Z}/5\mathbb{Z})$	$(1,2,2,0,0)$	17	21	8
$\hat{3}$	$(\mathbb{Z}/5\mathbb{Z})^2$	$(1,2,3,4,0),(1,0,2,2,0)$	1	21	40
$\hat{2}$	$(\mathbb{Z}/5\mathbb{Z})^2$	$(1,0,0,4,0),(1,0,4,0,0)$	5	49	88
$\hat{1}$	$(\mathbb{Z}/5\mathbb{Z})^3$	$(1,0,0,4,0),(1,0,4,0,0),(1,4,0,0,0)$	1	101	200

Table 11. TABLE

	H	generators	$h^{2,1}$	$h^{1,1}$	χ
1	$\{1\}$		145	1	-288
2	$(\mathbb{Z}/2\mathbb{Z})$	$(0,5,5,0,0)$	99	3	-192
3	$(\mathbb{Z}/2\mathbb{Z})^2$	$(0,5,5,0,0),(5,5,0,0,0)$	67	7	-120
4	$(\mathbb{Z}/5\mathbb{Z})$	$(0,4,4,1,0)$	47	11	-72
5	$(\mathbb{Z}/5\mathbb{Z})$	$(0,8,2,0,0)$	37	13	-48
6	$(\mathbb{Z}/10\mathbb{Z})$	$(9,0,1,0,0)$	39	15	-48
7	$(\mathbb{Z}/10\mathbb{Z})$	$(0,5,3,1,0)$	29	17	-24
$\hat{7}$	$(\mathbb{Z}/10\mathbb{Z})$	$(0,7,1,1,0)$	17	29	24
$\hat{6}$	$(\mathbb{Z}/10\mathbb{Z})$	$(0,3,3,2,0)$	15	39	48
$\hat{5}$	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/10\mathbb{Z})$	$(5,5,0,0,0),(0,5,3,1,0)$	13	37	48
$\hat{4}$	$(\mathbb{Z}/10\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$	$(9,0,1,0,0),(0,5,5,0,0)$	11	47	72
$\hat{3}$	$(\mathbb{Z}/\mathbb{Z})^2$	$(0,8,2,0,0),(8,0,2,0,0)$	7	67	120
$\hat{2}$	$(\mathbb{Z}/10\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$	$(9,0,1,0,0),(0,8,2,0,0)$	3	99	192
$\hat{1}$	$(\mathbb{Z}/10\mathbb{Z})^2$	$(9,0,1,0,0),(0,9,1,0,0)$	1	145	288

Table 12. TABLE

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