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On Gauge Theories as Matrix Models

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The relation between the Seiberg-Witten prepotentials, Nekrasov functions and matrix models is discussed. We derive quasiclassically the matrix models of Eguchi-Yang type, describing the instantonic contribution to the deformed partition functions of supersymmetric gauge theories. The exact quasiclassical solution for the case of conformal four-dimensional theory is studied in detail, and some aspects of its relation with the recently proposed logarithmic beta-ensembles are considered. We discuss also the "quantization" of this picture in terms of two-dimensional conformal theory with extended symmetry, and stress its difference from common picture of perturbative expansion a la matrix models. Instead, the representation for Nekrasov functions in terms of conformal blocks or Whittaker vector suggests some nontrivial relation with Teichmüller spaces and quantum integrable systems.

1 Introduction

The Seiberg-Witten (SW) prepotentials play an important role in studying properties of the gauge theories at strong coupling [1, 2] (see also [3] and references therein for recent discussion of these issues). Simultaneously, they are beautiful objects of pure interest for the mathematical physics, and have been recently rederived as quasiclassical limit of the Nekrasov instanton partition functions [4, 5, 6, 7, 8, 11, 12]. The latter ones are have the form of statistical models, where summing is taken over the sets of random partitions or Young diagrams [4, 13]. Originally arosen from the integrals over the instanton moduli spaces in the double-deformed supersymmetric gauge theories, they are related directly to the correlation functions or conformal blocks in two-dimensional conformal quantum field theories [14] (see also discussion of this relation e.g. in [15]).

Quasiclassics of Nekrasov functions produce the SW geometry, so that the prepotential appears as a critical value of the free energy of the correspondent statistical model [7, 8]. This happens quite similar to the case of quasiclassics in matrix models, though some details - to be discussed below - in these two cases are still quite different. There have been already many suggestions (see recent attempts e.g. in [9, 10]) to treat the Nekrasov functions in a similar to the matrix models way beyond the quasiclassical approximation - at least in the sense of perturbative expansion, producing corrections to the prepotential. The relation seems however to be not very straightforward.

We start the demonstration of this relation in the most old and transparent example. In fact it becomes more natural, if instead of full Nekrasov function one would take only its instantonic part, leaving aside the perturbative contribution. In the simplest case, one gets in such way the Eguchi-Yang (EY) one-matrix model [16], but in the limit of *vanishing* size of the matrix [7, 17]. The SW periods variables are introduced in this context in a completely different way, not being the fractions of condensed eigenvalues, like one used to have for the common matrix models.

Basically this reflects the difference of how the parameter of genus or quasiclassical expansion is introduced. For the matrix models it is related to the rank of gauge group, with sensible expansion at its large values. In the picture of summation over partitions the string coupling is rather related to particular combination of the two deformation parameters. Differently, the same distinction comes from the completely different role of the Virasoro constraints in these two theories. For the matrix models the Virasoro constraints are equivalent to the loop equations, giving directly the spectral curve and an iterative procedure of constructing the perturbative corrections, solving the loop equations order by order (this can be thought as one of the basic features of the B-type theory). In the case of Nekrasov functions, which rather belong to the A-type string models, the Virasoro constraints determine only the gravitational dressing of Nekrasov function by the descendants of unity operator, giving no restriction to the form of this function itself.

We are going to discuss also the relation of SW geometry with the recently proposed logarithmic matrix models [18]. For this purpose we study in detail the theories with the fundamental flavors and present, in particular, its exact solution in the simplest case of Abelian theory with two flavors. This gives some hint, how the SW geometry can be rewritten in the form close to that of the matrix models, but this does not lead to the full identification of these two pictures. Instead, coming back to the pure gauge theories - where there is even no way for identification of the spectral curves - we show how reformulation of the SW geometry suggests the way of its natural quantization from the angle of view of the AGT relation, and it leads to a natural conjecture for the expression of Nekrasov function in the framework of quantized picture.

2 SW prepotentials from Nekrasov functions

The instantonic calculus [4] in $\mathcal{N} = 2$ gauge theory gives rise [7, 8] to the extended SW prepotential as a critical value of the functional $\mathcal{F} \underset{\epsilon_{1,2}\to 0}{\sim} \frac{1}{\epsilon_1\epsilon_2}\log Z$,

$$\mathcal{F} = \frac{1}{2}\int dx f''(x)F_{UV}(x) - \frac{1}{2}\int_{x_1>x_2} dx_1 dx_2 f''(x_1)f''(x_2)F(x_1-x_2) \tag{2.1}$$

giving the main contribution to the Nekrasov function $Z(\bullet|\epsilon_1,\epsilon_2)$ at vanishing deformation parameters $\epsilon_{1,2} \to 0$, when extremized w.r.t. the second derivative of the profile function $f''(x) = \frac{d^2f}{dx^2}$ of the Young diagram, with the bare UV potential

$$F_{UV}(x) = \sum_{k>0} t_k \frac{x^{k+1}}{k+1} \tag{2.2}$$

and the kernel

$$F(x) = \frac{x^2}{2}\left(\log x - \frac{3}{2}\right) \tag{2.3}$$

coming up from the (generalized) Plancherel measure in the sum over random partitions [6]. It is expected, that the main contribution is given by some large “limiting” Young diagram with the profile $f(x)$, to be found by solving the variational problem for (2.1) upon normalization conditions imposed as constraints

$$a_i = \tfrac{1}{2}\int_{\mathbf{I}_i} dx\ xf''(x), \quad i = 1,\dots,N \tag{2.4}$$

which can be in standard way taken into account by adding them with the Lagrange multipliers

$$\mathcal{F} \to \mathcal{F} + \sum_{i=1}^{N} a_i^D \left(a_i - \frac{1}{2}\int_{\mathbf{I}_i} dx\ xf''(x)\right) \tag{2.5}$$

(for the case of $U(N)$ gauge theory one has to consider solution with N cuts $\{\mathbf{I}_i\}$, $i = 1\dots,N$).

The whole setup for (2.1)-(2.5) is almost identical [19] to the standard quasiclassics of the matrix models (see e.g. [20, 21, 22, 23, 24]), but with few crucial distinctions:

- The Coulomb gas kernel in (2.1) is replaces by a multivalued kernel (2.3).
- The properties of the double derivative $f''(x)$ of the shape function for the large extremal Young diagram are essentially different from the eigenvalue density in matrix models.

The extremal equation for the (2.1) gives the system of N integral equations

$$\sum_{k>0} t_k z^k - \int dx f''(x)(z-x)\left(\log|z-x|-1\right) = a_i^D, \qquad z \in \mathbf{I}_i, \quad i=1,\dots,N \tag{2.6}$$

on each segment of the support. Generally the solution can be expressed in terms of the Abelian integrals on the double cover

$$y^2 = \prod_{i=1}^{N}(z-x_i^+)(z-x_i^-) \tag{2.7}$$

which is a hyperelliptic curve of genus $g = N-1$. Define

$$\begin{gathered} S(z) = F'_{UV}(z) - \int dx f''(x)(z-x)(\log(z-x)-1) - a^D = \\ \underset{z\to\infty}{=} \sum_{k>0} t_k z^k - 2N\cdot z(\log z - 1) + 2\sum_{i=1}^{N} a_i \cdot \log z + \dots \end{gathered} \tag{2.8}$$

where the integral is taken over the whole support $\mathbf{I} = \cup_{i=1}^N \mathbf{I}_i$, $a^D = \frac{1}{N}\sum_{j=1}^N a_j^D$, and consider its differential, or

$$\Phi(z) = \frac{dS}{dz} = \sum_{k>0} k t_k z^{k-1} - \int dx f''(x)\log(z-x) \tag{2.9}$$

satisfying

$$\Phi(x+i0) + \Phi(x-i0) = 0, \qquad x \in \mathbf{I}_i, \quad i=1,\dots,N \tag{2.10}$$

on each cut, and normalized to

$$\begin{gathered} \Phi(x_N^+) = 0, \\ \Phi(x_j^- \pm i0) = \Phi(x_{j-1}^+ \pm i0) = \pm 2\pi i(N-j+1), \qquad j=2,\dots,N \\ \Phi(x_1^-) = \pm 2\pi i N \end{gathered} \tag{2.11}$$

If all $t_k = 0$ for $k \neq 1$ and $e^{t_1} = \Lambda^{2N}$, the derivative $\Phi = \frac{dS}{dz}$ is an Abelian integral on the curve (2.7) with the asymptotic

$$\Phi \underset{P\to P_\pm}{=} \mp 2N\log z \pm 2N\log\Lambda + O(z^{-1}) \tag{2.12}$$

whose jumps are integer-valued due to (2.11), or $\oint d\Phi \sim 4\pi i\mathbb{Z}$. It means that the hyperelliptic curve (2.7) can be seen also as an algebraic Riemann surface for the function $w = \exp(-\Phi/2)$, satisfying quadratic equation

$$\Lambda^N\left(w + \frac{1}{w}\right) = P_N(z) = \prod_{i=1}^{N}(z-v_i) \tag{2.13}$$

or

$$y^2 = P_N(z)^2 - 4\Lambda^{2N} \tag{2.14}$$

i.e. $\{x_i^\pm\}$ are roots of $P_N(z) \mp 2\Lambda^N = 0$, and

$$y = \Lambda^N \left(w - \frac{1}{w} \right) \tag{2.15}$$

Note, that in the simplest $N = 1$ case the form (2.13), i.e.

$$z - v = \Lambda \left(w + \frac{1}{w} \right) \tag{2.16}$$

can be always achieved by change of the variables, i.e. it can be used as well for the switched on higher flows [7, 8]. However generally, for nonvanishing higher couplings in the UV potential (2.2) the profile function cannot be obtained as a jump of any algebraic function on the curve (2.7), unlike the case of resolvent and density in matrix model.

The generating differential (2.9) is now

$$dS = -2 \log w dz = -d(2z \log w) + 2z \frac{dw}{w} \tag{2.17}$$

just the Legendre transform of the SW differential $dS_{SW} \sim z\frac{dw}{w}$ on the curve (2.13), (2.14). It periods

$$a_i = \frac{1}{2\pi i} \oint_{A_i} z \frac{dw}{w} \tag{2.18}$$

coincide with the SW integrals and the only nontrivial residues at infinity give

$$\begin{gathered} \mathrm{res}_{P_+} \left(z^{-1} dS \right) = -\mathrm{res}_{P_-} \left(z^{-1} dS \right) = \log \Lambda^{2N} \\ \mathrm{res}_{P_+} \left(dS \right) = -\mathrm{res}_{P_-} \left(dS \right) = 2 \sum_{j=1}^N v_j \end{gathered} \tag{2.19}$$

The differential (2.17) satisfies the condition

$$\delta dS \sim \frac{\delta w}{w} dz = \frac{\delta P(z)}{y} dz \underset{\sum_{j=1}^N a_j = 0}{=} \text{ holomorphic} \tag{2.20}$$

where the variation is taken at constant co-ordinate z and constant scale factor Λ. This provides the integrability of the gradient formulas

$$\frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B_i} z \frac{dw}{w} \tag{2.21}$$

reducing it consistency to the symmetricity of the period matrix - a particular case of the Riemann bilinear relations.

3 Eguchi-Yang matrix model

Let us now substitute into (2.1) the shift of the variable [7, 17]

$$\begin{gathered} f(x) = |x-a| + g(x) \\ f'(x) = 2\ \mathrm{sign}(x-a) + g'(x) \equiv 2\ \mathrm{sign}(x-a) + \rho(x) \\ f''(x) = 2\delta(x-a) + g''(x) \end{gathered} \tag{3.1}$$

where the function $g(x)$ and its derivative $g'(x) = \rho(x)$ vanish at the ends of the cut $\mathbf{I} = (x_-, x_+)$:

$$g(x_\pm) = 0, \quad g'(x_\pm) = \rho(x_\pm) = 0 \tag{3.2}$$

One obviously gets

$$\begin{gathered} \mathcal{F} = F_{UV}(a) + \frac{1}{2}\int dx g''(x) F_{UV}(x) - \int dx g''(x) F(x-a) - \\ -\frac{1}{2}\int_{x_1 > x_2} dx_1 dx_2 g''(x_1) g''(x_2) F(x_1 - x_2) \equiv F_{UV}(a) - \mathcal{W} \end{gathered} \tag{3.3}$$

where

$$\begin{gathered} \mathcal{W} = \frac{1}{2}\int dx \rho(x) W(x) - \frac{1}{2}\int_{x_1 > x_2} dx_1 dx_2 \rho(x_1)\rho(x_2)\log(x_1 - x_2) \\ W(x) = F'_{UV}(x) - 2(x-a)\left(\log(x-a) - 1\right) \end{gathered} \tag{3.4}$$

The variational problem for (3.4) is solved under constraint, following from (2.4)

$$-\int dx\ x g''(x) = \int dx \rho(x) = 0 \tag{3.5}$$

The dependence of $\mathcal{W}$ upon the variable a can be easily obtained by redefinition of the x-variable $x \to x + a$ and renaming $\rho(x+a) \to \rho(x)$

$$\begin{gathered} \mathcal{W} = \frac{1}{2}\int dx \rho(x)\left(F'_{UV}(x+a) - 2x(\log x - 1)\right) - \\ -\frac{1}{2}\int_{x_1 > x_2} dx_1 dx_2 \rho(x_1)\rho(x_2)\log(x_1 - x_2) \end{gathered} \tag{3.6}$$

so that

$$\frac{\partial \mathcal{W}}{\partial a} = \tfrac{1}{2}\int dx \rho(x) F''_{UV}(x+a) = \sum_{k>1} k t_k \frac{\partial \mathcal{W}}{\partial t_{k-1}} \tag{3.7}$$

and it means that

$$\mathcal{W}(a; t_1, t_2, \ldots) = \mathcal{W}_0(t_1, t_2, \ldots)|_{t_1 \to F''_{UV}(a)} \tag{3.8}$$

and

$$
\begin{aligned}
\mathcal{W}_0 \equiv \mathcal{W}\big|_{a=0} = \frac{1}{2}\int dx\rho(x)\left(F'_{UV}(x) - 2x(\log x - 1)\right) - \\
-\frac{1}{2}\int_{x_1>x_2} dx_1 dx_2 \rho(x_1)\rho(x_2)\log(x_1 - x_2)
\end{aligned}
\tag{3.9}
$$

is just the effective potential of the EY matrix model [16], constrained by vanishing of the total density of the eigenvalues (3.5).

It means, that the dependence on zero Toda time is introduced here in quite uncommon for the matrix models way (3.7), (3.8). On the level of the Toda chain hierarchy, which governs the dynamics over the parameters of the potential [28], one deals here with two completely different solutions to the Toda equation

$$
\frac{\partial^2 \mathcal{F}}{\partial t_1^2} = \exp\left(\frac{\partial^2 \mathcal{F}}{\partial t_0^2}\right)
\tag{3.10}
$$

namely

$$
\mathcal{F}_{\mathrm{mamo}} = \tfrac{1}{2}t_0^2\left(\log t_0 - \frac{3}{2}\right) + \tfrac{1}{2}t_0 t_1^2 + \ldots, \qquad t_0 = \hbar N
\tag{3.11}
$$

for the standard matrix model (see e.g. [24]), but

$$
\mathcal{F}_{\mathrm{EY}} = \tfrac{1}{2}t_0 t_1^2 + e^{t_1} + \ldots, \qquad t_0 = a
\tag{3.12}
$$

The essential difference comes in the zero-time dependence: for the standard matrix model it contains the function (2.3), giving rise to the logarithm into the second derivative, "canceled" by exponentiating in (3.10), while in the EY case the absence of logarithms of $t_0 = a$ requires necessarily the exponential dependence on t_1. Moreover, the W-boson masses, which are associated to the periods of dual to (2.8) SW differential, are *not* related with the filling fractions of the matrix model with the EY potential (3.4).

Note also, since

$$
W(x) = F'_{UV}(x) + 2a(\log a - 1) + \sum_{n>0}\frac{x^{n+1}}{n(n+1)a^n}
\tag{3.13}
$$

the effective potential of the matrix model satisfies the property

$$
\begin{aligned}
\mathcal{W} = \mathcal{W}(t_1; \hat{t}_2, \hat{t}_3, \ldots) \\
\hat{t}_k = t_k + \frac{1}{k(k-1)a^{k-1}}, \quad k \geq 2
\end{aligned}
\tag{3.14}
$$

and has a natural expansion at large values of a.

For the non-Abelian case instead of (3.1) one has to make a substitution

$$
\begin{aligned}
f(x) = L(x;\mathbf{a}) + g(x) \equiv \sum_{j=1}^{N} |x-a_j| + g(x) \\
f'(x) = 2\sum_{j=1}^{N} \ \mathrm{sign}(x-a_j) + g'(x) \equiv 2\sum_{j=1}^{N} \ \mathrm{sign}(x-a_j) + \rho(x) \\
f''(x) = 2\sum_{j=1}^{N} \delta(x-a_j) + g''(x)
\end{aligned}
\tag{3.15}
$$

after which again the function $g(x)$ and its derivative $g'(x) = \rho(x)$ vanish at the ends of all cuts $\mathbf{I}_j = (x_j^-, x_j^+)$, $\mathbf{I} = \cup_{j=1}^N \mathbf{I}_j$:

$$
g(x_j^{\pm}) = 0, \quad g'(x_j^{\pm}) = \rho(x_j^{\pm}) = 0, \quad j = 1,\ldots,N \tag{3.16}
$$

Now, instead of (3.3) one gets

$$
\begin{aligned}
\mathcal{F} = \sum_{j=1}^{N} F_{UV}(a_j) - \sum_{i\neq j} F(a_i - a_j) + \\
+\frac{1}{2}\int dx g''(x) F_{UV}(x) - \int dx g''(x) \sum_{j=1}^{N} F(x-a_j) - \\
-\frac{1}{2}\int_{x_1>x_2} dx_1 dx_2 g''(x_1) g''(x_2) F(x_1-x_2) \equiv \\
= \sum_{j=1}^{N} F_{UV}(a_j) - \sum_{i\neq j} F(a_i-a_j) - \mathcal{W} \equiv \mathcal{F}_0 - \mathcal{W}
\end{aligned}
\tag{3.17}
$$

where $\mathcal{F}_0$ is just the sum of the classical and perturbative contributions to the prepotential (see Appendix A), while the instantonic part is again given by effective potential of a "matrix model"

$$
\begin{aligned}
\mathcal{W} = \frac{1}{2}\int dx \rho(x) W(x) - \frac{1}{2}\int_{x_1>x_2} dx_1 dx_2 \rho(x_1)\rho(x_2)\log(x_1-x_2) \\
W(x) = F'_{UV}(x) - 2\sum_{j=1}^{N} (x-a_j)\left(\log(x-a_j) - 1\right)
\end{aligned}
\tag{3.18}
$$

The variational problem for (3.18) is again solved under constraint (3.5) at each component of the cut $\mathbf{I}$, since

$$
\sum_{j=1}^{N} a_j^D \left(a_j - \tfrac{1}{2}\int_{\mathbf{I}_j} dx x f''(x) \right) = -\tfrac{1}{2}\sum_{j=1}^{N} a_j^D \int_{\mathbf{I}_j} dx x g''(x) = \tfrac{1}{2}\sum_{j=1}^{N} a_j^D \int_{\mathbf{I}_j} dx \rho(x) \tag{3.19}
$$

We have found therefore, that in the non-Abelian case the instantonic partition function is described by the EY type matrix model with vanishing filling fractions (3.19), where the role of the SW periods is played by the impurities in the potential.

4 Supersymmetric QCD and matrix models

In order to analyze possible outcome of this similarity, let us now turn to the case of supersymmetric QCD, or the $\mathcal{N} = 2$ Yang-Mills theory with extra fundamental flavors of matter [25, 26]. We shall concentrate mostly on four-dimensional conformal theory with vanishing beta-function, i.e. with the number of flavors $N_f = 2N$ and vanishing beta-function $\beta = 2N - N_f = 0$, and then come back to the pure gauge theories, taking the limit of infinite masses.

4.1 Abelian theory with two flavors: exact solution

In the conformal case of Abelian theory with two flavors $f = 1, 2$ the functional (2.1) is changed just by substitution $F_{UV}(x) \to F_{UV}(x) + \sum_f F(x - m_f)$, (see Appendix A for a simple argument from the point of view of perturbative prepotentials)

$$\begin{gathered}\mathcal{F} = \frac{1}{2}\int dx f''(x)(F_{UV}(x) + \sum_f F(x - m_f)) - \\ -\frac{1}{2}\int_{x_1>x_2} dx_1 dx_2 f''(x_1) f''(x_2) F(x_1 - x_2) + a^D \left(a - \tfrac{1}{2}\int dx\ x f''(x)\right)\end{gathered} \tag{4.1}$$

so that the function

$$\begin{gathered}S(z) = 2\frac{d}{dz}\frac{\delta\mathcal{F}}{\delta f''(z)} = F'_{UV}(z) + \sum_f F'(z - m_f) - \int F'(z - x) f''(x) dx - a^D = \\ \underset{z\to\infty}{=} \sum_{k>0} t_k z^k + \left(2a - \sum_f m_f\right)\log z - \left(a^D + \sum_f m_f\right) + O\left(\frac{1}{z}\right) \\ S(z) \underset{z\to m_f}{=} (z - m_f)\log(z - m_f) + \ldots, \quad f = 1, 2\end{gathered} \tag{4.2}$$

acquires extra singularities at $z \to m_f$, $f = 1, 2$, while the EY $z\log z$-singularity at $z \to \infty$ is canceled due to vanishing beta-function (for two flavors).

More transparently it is seen for

$$
\begin{aligned}
\Phi(z) = \frac{dS}{dz} &= F''_{UV}(z) + \sum_f \log(z - m_f) - \int \log(z-x) f''(x) dx = \\
&= \sum_{k>0} k t_k z^{k-1} + \sum_f \log\left(1 - \frac{m_f}{z}\right) - \int \log\left(1 - \frac{x}{z}\right) f''(x) dx = \\
&\underset{z\to\infty}{=} \sum_{k>0} k t_k z^{k-1} + \frac{2a - \sum_f m_f}{z} + \sum_{k>0} \frac{1}{z^{k+1}} \left(2\frac{\partial \mathcal{F}}{\partial t_k} - \frac{1}{k} \sum_f m_f^k \right) \\
\Phi(z) &\underset{z\to m_f}{=} \log(z - m_f) + O(1), \quad f = 1,2
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
\frac{d\Phi}{dz} &= F'''_{UV}(z) + \sum_f \frac{1}{z - m_f} - \int \frac{f''(x) dx}{z - x} = \\
&\underset{z\to\infty}{=} \sum_{k>1} k(k-1) t_k z^{k-2} - \frac{2a - \sum_f m_f}{z^2} + O\left(\frac{1}{z^3}\right) \\
d\Phi(z) &\underset{z\to m_f}{=} \frac{dz}{z - m_f} + \ldots, \quad f = 1,2
\end{aligned}
\tag{4.4}
$$

The solution can be still constructed as an odd under $y \leftrightarrow -y$ differential on a cylinder

$$
y^2 = (z - x_+)(z - x_-)
\tag{4.5}
$$

i.e.

$$
d\Phi = \frac{\psi(z) dz}{(z - m_1)(z - m_2) y}
\tag{4.6}
$$

On small phase space, where only $t_1 \neq 0$, the asymptotic (4.4) requires numerator of (4.6) to be a linear function

$$
\psi(z)\big|_{t_k = t_1 \delta_{k,1}} = \psi_1 z + \psi_0
\tag{4.7}
$$

where

$$
\begin{aligned}
\psi_1 &= y_1 + y_2 = \sum_f m_f - 2a \\
\psi_0 &= -m_1 y_2 - m_2 y_1
\end{aligned}
\tag{4.8}
$$

and

$$
y_f = y(m_f), \quad f = 1,2
\tag{4.9}
$$

The solution to (4.6) can be conveniently described by the following anzatz

$$
d\Phi = -\frac{1}{\sqrt{1 - 4\zeta^2}} \frac{dz}{y} \frac{\left(2v - \sum_f m_f\right) z + 2m_1 m_2 - v \sum_f m_f}{(z - m_1)(z - m_2)} \equiv -2\frac{dW}{W}
\tag{4.10}
$$

where we have introduced

$$
\begin{aligned}
&W+\frac{1}{W}=\frac{z-v}{\sqrt{Q(z)}}, \qquad Y^2=(1-4\zeta^2)y^2=(z-v)^2-4Q(z)\\
&Q(z)=\zeta^2(z-m_1)(z-m_2), \qquad y_f=\frac{m_f-v}{\sqrt{1-4\zeta^2}}, \quad f=1,2
\end{aligned}
\tag{4.11}
$$

with[1]

$$
\begin{aligned}
&\zeta=\frac{1}{2\cosh\frac{t_1}{2}}, \qquad \sqrt{1-4\zeta^2}=-\tanh\frac{t_1}{2}\\
&\frac{2v-\sum_f m_f}{\sqrt{1-4\zeta^2}}=-\left(2v-\sum_f m_f\right)\coth\frac{t_1}{2}=2a-\sum_f m_f
\end{aligned}
\tag{4.12}
$$

At $e^{t_1}\to 0$ the last relation turns into $v=a$, and in this limit at large masses one can introduce finite scale $\zeta^2 m_1 m_2=\Lambda^2$.

In order to write the generation function (4.2) one needs to introduce first the (similar to (2.16)) uniformization of (4.5) via

$$
\begin{aligned}
&\frac{L}{2}\left(\varpi+\frac{1}{\varpi}\right)=z-V, \qquad \frac{L}{2}\left(\varpi-\frac{1}{\varpi}\right)=y, \qquad \frac{d\varpi}{\varpi}=\frac{dz}{y}\\
&V=\frac{x_++x_-}{2}=\frac{v-2\zeta^2(m_1+m_2)}{1-4\zeta^2}\\
&L=\frac{x_+-x_-}{2}=\frac{2\zeta}{1-4\zeta^2}\sqrt{\zeta^2(m_1-m_2)^2+(v-m_1)(v-m_2)}
\end{aligned}
\tag{4.13}
$$

and define the auxiliary functions

$$
\begin{aligned}
&\chi_f=\frac{\varpi-\varpi_f}{\varpi\varpi_f-1}, \qquad f=1,2\\
&\chi_f^2-\sigma_f\chi_f+1=0, \quad \sigma_f=\frac{2}{L}\frac{(m_f-V)(z-V)-L^2}{z-m_f}\\
&\chi_f \underset{z\to\infty}{=} \varpi_f=\frac{1}{L}(m_f-V+y_f), \quad f=1,2
\end{aligned}
\tag{4.14}
$$

The generation function (4.2) now reads

$$
\begin{aligned}
&S=\sum_{k>0}t_k\Omega_k+(2a-m_1-m_2)\log\varpi-\sum_f(z-m_f)\log\chi_f-y\log\Xi^2\\
&\Omega_k=z^k(\varpi)_+-z^k(\varpi)_- \ , \qquad k>0\\
&\zeta=\frac{1}{\Xi+\frac{1}{\Xi}}, \qquad \log(\varpi_1\varpi_2)=\frac{1}{\Xi^2}
\end{aligned}
\tag{4.15}
$$

[1]The sign of the root is chosen to fit with the weak-coupling regime, when $e^{t_1}\to 0$ and $\tanh\frac{t_1}{2}<0$.

where we have just used the basis of odd functions on (4.5) with the only singularities at two infinities, e.g.

$$\Omega_0 = \log \varpi, \quad \Omega_1 = y, \quad \dots \tag{4.16}$$

If only $t_1 \neq 0$, $\Xi^2 = e^{t_1}$, and one gets from (4.15)

$$S(z)\big|_{t_k = t_1\delta_{k,1}} = -2z \log W + (2a - m_1 - m_2) \log \varpi + \sum_{f=1,2} m_f \log \chi_f \tag{4.17}$$

where $W^2 = \chi_1\chi_2$ is defined in (4.11).

Using these formulas it is easy to compute the resulting prepotential for (4.10), (4.12), which reads (for the only nonvanishing t_1, and up to the linear terms $\sim (m_1 + m_2)a$, which do not influence onto the second derivatives - coupling constants, and can be eliminated by adding the linear terms $\sim (m_1 + m_2)x$ to the potential in (4.1))

$$\mathcal{F} = \tfrac{1}{2}a^2 t_1 - (a - m_1)(a - m_2) \log\left(1 - e^{t_1}\right) + \mathcal{F}_{\text{pert}}(a; \mathbf{m}) \tag{4.18}$$

and contains

$$\mathcal{F}_{\text{pert}}(a; \mathbf{m}) = \sum_f F(a - m_f) \tag{4.19}$$

so that

$$\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \log \frac{e^{t_1}\prod_f (a - m_f)}{\left(1 - e^{t_1}\right)^2} = \log a^{N_f} + \tau_{\text{conf}} \tag{4.20}$$

where

$$\tau_{\text{conf}} = \log \frac{e^{t_1}\prod_f \left(1 - \frac{m_f}{a}\right)}{\left(1 - e^{t_1}\right)^2} \underset{m_f \to 0}{=} \log \frac{e^{t_1}}{\left(1 - e^{t_1}\right)^2} \tag{4.21}$$

does not depend at vanishing masses on the condensate and gives rise to the non-perturbative renormalization of the coupling

$$\begin{gathered} e^{\tau/2} = \frac{1}{e^{-t_1/2} - e^{t_1/2}}, \quad \text{or} \\ \tau = t_1 - 2\log\left(1 - e^{t_1}\right) = t_1 + 2\sum_{k>0} \frac{e^{kt_1}}{k} \end{gathered} \tag{4.22}$$

which is a toy-model analog of the instanton renormalization given by the Zamolodchikov asymptotic formula [27].

The formulas for the EY matrix model (3.4) remain almost intact, except for the potential

$$\begin{gathered} W(x) \to W(x; \mathbf{m}) = F'_{UV}(x) + \sum_f F'(x - m_f) - 2F'(x - a) = \\ = \sum_{k>0} t_k x^k + \sum_f (x - m_f)\left(\log(x - m_f) - 1\right) - 2(x - a)\left(\log(x - a) - 1\right) \end{gathered} \tag{4.23}$$

For $a = 0$ and $m_f = 0$, $f = 1, 2$ the potential (4.23) turns back into the potential a standard 1-matrix model.

4.2 Different parameterizations of the curves

We have already seen that to construct the exact solution explicitly one rather needs the parameterization (4.13), than the common for supersymmetric QCD form of the curve (4.11). Let us point out now, that the curve (4.11), can be also presented in the form

$$\begin{gathered} q_1(z)w^2 - (z-v)w + q_2(z) = 0 \\ q_f(z) = \zeta(z-m_f), \quad f=1,2 \\ w^2 = \frac{q_2}{q_1}W^2 = \frac{q_2}{q_1}\chi_1\chi_2 \end{gathered} \tag{4.24}$$

Introducing another new variable

$$x = \frac{z}{w} = \frac{z}{W}\sqrt{\frac{q_1(z)}{q_2(z)}} \tag{4.25}$$

one can rewrite (4.24) as

$$\begin{aligned} x = \frac{\zeta m_1 w^2 - vw + \zeta m_2}{\zeta w\left(w^2 - \frac{w}{\zeta} + 1\right)} \underset{\zeta=\frac{1}{\Xi+1/\Xi}}{=} \frac{m_1 w^2 - v\left(\Xi + \frac{1}{\Xi}\right)w + m_2}{w(w-\Xi)\left(w - \frac{1}{\Xi}\right)} = \\ = \frac{m_2}{w} + \frac{a-m_2}{w-\Xi} - \frac{a-m_1}{w-\frac{1}{\Xi}} \end{aligned} \tag{4.26}$$

where we have used (4.12) and also remind that on small phase space $\Xi^2 = e^{t_1}$ is just the UV bare coupling. The SW differential becomes

$$\begin{aligned} dS_{SW} = z\frac{dW}{W} = z\frac{dw}{w} + \frac{z}{2}\left(\frac{dq_1}{q_1} - \frac{dq_2}{q_2}\right) = xdw + \frac{m_1}{2}\frac{dz}{z-m_1} - \frac{m_2}{2}\frac{dz}{z-m_2} \equiv \\ \equiv dS_G + \frac{m_1}{2}\frac{dz}{z-m_1} - \frac{m_2}{2}\frac{dz}{z-m_2} \end{aligned} \tag{4.27}$$

The first term in the r.h.s. $dS_G = xdw$, appeared naturally in the context of [30] from D-brane considerations, it is normalized due to (4.26) as

$$\begin{gathered} \mathrm{res}_{w=0}xdw = m_2, \quad \mathrm{res}_{w=\infty}xdw = -m_1 \\ \mathrm{res}_{w=\Xi}xdw = a-m_2, \quad \mathrm{res}_{w=\frac{1}{\Xi}}xdw = -(a-m_1) \end{gathered} \tag{4.28}$$

This representation is especially natural from the point of view of comparison with the conformal representation [5]

$$\begin{aligned} Z(x;a,m_1,m_2) = Z_{\text{pert}} \cdot \langle e^{im_1\phi(\infty)} e^{i(a-m_1)\phi(1)} x^{L_0} e^{-i(a-m_2)\phi(1)} e^{-im_2\phi(0)} \rangle = \\ = Z_{\text{pert}} \cdot x^{\frac{m_2^2}{2} + \frac{(a-m_2)^2}{2}} \langle e^{im_1\phi(\infty)} e^{i(a-m_1)\phi(1)} e^{-i(a-m_2)\phi(x)} e^{-im_2\phi(0)} \rangle = \\ = Z_{\text{pert}} \cdot x^{a^2/2}(1-x)^{-(a-m_1)(a-m_2)} \end{aligned} \tag{4.29}$$

of the $U(1)$ partition function with $N_f = 2$ in terms of a theory of two-dimensional free scalar field. One can absorb the perturbative contribution inside the correlator by redefinition of the scalar product in two-dimensional theory, see the details below.

From (4.26) one also easily finds, that

$$\begin{gathered}\frac{\partial}{\partial a} dS_G = \frac{\partial x}{\partial a} dw = \frac{\left(\Xi - \frac{1}{\Xi}\right) dw}{(w - \Xi)\left(w - \frac{1}{\Xi}\right)} = d\omega \\ \mathrm{res}_{w=\Xi}\ d\omega = 1, \quad \mathrm{res}_{w=\frac{1}{\Xi}}\ d\omega = -1\end{gathered} \tag{4.30}$$

is just a degenerate analog of the holomorphic differential.

4.3 Supersymmetric QCD, conformal theory and logarithmic potentials

In the case of nonabelian supersymmetric QCD (the N-cut solution) the formula for the EY matrix model potential (with fundamental matter) obviously changes for

$$W(x) = F'_{UV}(x) + \sum_{f=1}^{N_f} (x - m_f) \left(\log(x - m_f) - 1\right) - 2 \sum_{j=1}^{N} (x - a_j) \left(\log(x - a_j) - 1\right) \tag{4.31}$$

It means, that in conformal case $N_f = 2N$ the derivative of this potential does not have any longer a logarithmic singularity at infinity

$$\begin{aligned} W'(x) &= F''_{UV}(x) + \sum_f \log(x - m_f) - 2 \sum_j \log(x - a_j) = \\ &= F''_{UV}(x) + \sum_f \log\left(1 - \frac{m_f}{x}\right) - 2 \sum_j \log\left(1 - \frac{a_j}{x}\right)\end{aligned} \tag{4.32}$$

but still has logarithmic singularities at $x = a_j$ and $x = m_f$.

Generally, for the $U(N)$ theory with $N_f \le 2N$ flavors on small phase space one usually starts with the analog [25, 26] of the representation (4.11)

$$\begin{gathered} W + \frac{1}{W} = \frac{P(z)}{\sqrt{Q(z)}}, \qquad Y^2 = P(z)^2 - 4Q(z) \\ P(z) = z^N - \sum_{k=0}^{N-2} u_k z^k, \qquad Q(z) = \Lambda^{2N - N_f} \prod_{f=1}^{N_f} (z - m_f)\end{gathered} \tag{4.33}$$

with the generating differential

$$dS_{SW} = \frac{z dz}{Y} \left(P'(z) - \frac{P(z) Q'(z)}{2Q(z)}\right) \tag{4.34}$$

In the conformal case $N_f = 2N$ the constant Λ becomes dimensionless, and (4.33) is changed for

$$W + \frac{1}{W} = \frac{P(z)}{\sqrt{Q(z)}}, \qquad Y^2 = (1-4\zeta^2)y^2 = P(z)^2 - 4Q(z)$$
$$Q(z) = \zeta^2 \prod_{f=1}^{2N}(z-m_f) \tag{4.35}$$

For the vanishing masses the differential (4.34) becomes holomorphic

$$dS_{SW} \underset{Q(z)=\zeta z^{N_f}}{=} \frac{dz}{Y}\left(zP'(z) - \frac{N_f}{2}P(z)\right) =$$
$$= \frac{dz}{Y}\left(\left(N - \frac{N_f}{2}\right)z^N + \sum_{k=0}^{N-2}\left(\frac{N_f}{2} - k\right)u_k z^k\right) = \tag{4.36}$$
$$\underset{N_f=2N}{=} \frac{dz}{Y}\sum_{k=0}^{N-2}(N-k)\,u_k z^k$$

For example, in the $N = 2$, $N_f = 4$ case the differential (4.36) is proportional to the only holomorphic differential on torus (see e.g. [27])

$$dS_{SW} \sim u\frac{dz}{Y} = \frac{u}{\sqrt{1-4\zeta^2}}\frac{dz}{y} \sim \frac{d\xi}{\eta} \tag{4.37}$$

with

$$\eta^2 = \prod_{f=0,1,\lambda}(\xi - q_f) = \xi(\xi-1)(\xi-\lambda)$$
$$\zeta^2 = \frac{\lambda}{(1+\lambda)^2} = \frac{1}{\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^2} \underset{\lambda=\frac{1}{\sqrt{\Xi}}}{=} \frac{1}{\left(\Xi + \frac{1}{\Xi}\right)^2} \tag{4.38}$$

and since $\Xi = e^{\tau_0/2}$ this is identical to the formula (4.12) from the $N = 1$ case.

However, for the nonvanishing masses it is more convenient to use the analog of the representation (4.24)

$$q(z)w^2 - P(z)w + \tilde{q}(z) = 0 \tag{4.39}$$

where $P(z)$ is the same polynomial of power N as in (4.33), while for $N_f = 2N$ one can introduce

$$q(z) = \zeta\prod_{f=1}^{N}(z-m_f), \qquad \tilde{q}(z) = \zeta\prod_{f=1}^{N}(z-\tilde{m}_f) \tag{4.40}$$

for some splitting of the matter multiplets into N "fundamental" and N "anti-fundamental" - with the masses m and $\tilde{m}$ correspondingly. Again, like in (4.24), we have introduced here

$$w^2 = \frac{\tilde{q}(z)}{q(z)}W^2 \tag{4.41}$$

For $N = 2$ in (4.40), equation (4.39) becomes as well a quadratic equation in z-variable, and can be therefore written [31] as

$$C_2(w)z^2 - C_1(w)z + C_0(w) = 0 \tag{4.42}$$

with, as in (4.26)

$$C_2(w) = \zeta w^2 - w + \zeta \underset{\zeta=\Xi+\frac{1}{\Xi}}{=} \left(\Xi + \frac{1}{\Xi}\right)(w - \Xi)\left(w - \frac{1}{\Xi}\right) \tag{4.43}$$

Equation (4.42) can be further re-written as

$$\tilde{z}^2 \equiv \left(z - \frac{C_1}{2C_2}\right)^2 = \frac{C_1^2}{4C_2^2} - \frac{C_0}{C_2} \equiv x^2 w^2 \tag{4.44}$$

or

$$x^2 = \frac{C_1(w)^2}{4w^2 C_2(w)^2} - \frac{C_0(w)}{w^2 C_2(w)} \tag{4.45}$$

At vanishing masses the last equation turns into

$$x^2 = -\frac{C_0(w)}{w^2 C_2(w)} = -\frac{u}{\zeta w(w - \Xi)\left(w - \frac{1}{\Xi}\right)} = -\frac{u}{\zeta \Xi^3}\frac{1}{\xi(\xi - 1)(\xi - \lambda)} \tag{4.46}$$

in terms of the elliptic curve (4.38).

Generally, instead of the holomorphic differential (4.37) one should write

$$dS_{SW} = \frac{\sqrt{p(\xi)}d\xi}{\xi(\xi - 1)(\xi - \lambda)} \tag{4.47}$$

for some polynomial

$$p(\xi) = \sum_{j=0}^{4} p_j \xi^j \underset{m_f \to 0}{\to} C \cdot \xi(\xi - 1)(\xi - \lambda) \tag{4.48}$$

reproducing the holomorphic differential (4.37) in the limit of vanishing masses. Due to relation between the coefficients of $p(\xi)$ and the residues

$$\begin{aligned} \mathrm{res}_{\xi=0}\ dS_{SW} &= \sqrt{p(0)} = \sqrt{p_0} = \lambda m_0 \\ \mathrm{res}_{\xi=\infty}\ dS_{SW} &= \sqrt{p_4} = -m_\infty \\ \mathrm{res}_{\xi=1}\ dS_{SW} &= \sqrt{p(1)} = (1 - \lambda)m_1 \\ \mathrm{res}_{\xi=\lambda}\ dS_{SW} &= \sqrt{p(\lambda)} = \lambda(\lambda - 1)m_\lambda \end{aligned} \tag{4.49}$$

one gets

$$\delta_{\mathrm{moduli}} p(\xi) = \delta_{\mathrm{moduli}} C \cdot \xi(\xi - 1)(\xi - \lambda) \tag{4.50}$$

and it means that for the variation of the SW differential one can write

$$\delta_{\rm moduli} dS_{SW} \sim \frac{d\xi}{\sqrt{p(\xi)}}\frac{\delta_{\rm moduli}p(\xi)}{\xi(\xi-1)(\xi-\lambda)} = \delta_{\rm moduli}C\frac{d\xi}{\sqrt{p(\xi)}} \tag{4.51}$$

so that the r.h.s. is obviously holomorphic.

One can also present the 2-differential $\mathcal{T}(d\xi)^2 = (dS_{SW})^2$ as

$$\begin{gathered}\mathcal{T} = \left(\frac{dS_{SW}}{d\xi}\right)^2 = \frac{p(\xi)}{\xi^2(\xi-1)^2(\xi-\lambda)^2} = \\ = \sum_{f=0,1,\lambda}\left(\frac{m_f^2}{(\xi-q_f)^2}+\frac{C_f}{\xi-q_f}\right) = \left(\sum_{f=0,1,\lambda}\frac{m_f}{\xi-q_f}\right)^2 + \frac{\mathcal{C}(\xi)}{\xi(\xi-1)(\xi-\lambda)}\end{gathered} \tag{4.52}$$

where

$$\sum_{f=0,1,\lambda} C_f = 0, \qquad C_1+\lambda C_\lambda = -\sum_{A=0,1,\lambda\infty} m_A^2 \tag{4.53}$$

and

$$\mathcal{C}(\xi) = -\xi\left[\left(\sum_{f=0,1,\lambda} m_f\right)^2 + m_\infty^2\right] + \left[C_0 - 2m_0m_1\lambda - 2m_0m_\lambda\right] \tag{4.54}$$

is a linear function with $\delta_{\rm moduli}\mathcal{C}(\xi) = \lambda\delta_{\rm moduli}C_0$. Formula (4.52) may be assigned with two possible interpretations:

- It has a sense of average of the stress-energy tensor in some two-dimensional conformal field theory

$$\mathcal{T}(\xi) = \frac{\langle T(\xi)\prod_{A=0,1,\lambda,\infty}V_A(q_A)\rangle}{\langle\prod_{A=0,1,\lambda,\infty}V_A(q_A)\rangle} \tag{4.55}$$

  with four primary operators of dimensions m_A^2, $A=0,1,\lambda\infty$ (m_∞ can be determined from the second equation in (4.53)) and C_f being the corresponding accessor parameters (see e.g. [29] and references therein);

- The r.h.s. (4.52) has an obvious form of the matrix model curve, if one takes formally the logarithmic potential

$$\mathcal{V}(\xi) = \sum_{f=0,1,\lambda} m_f\log(\xi-q_f) \tag{4.56}$$

  and the SW periods can be then identified with the filling fractions [18].

We do not find the last observation to be very useful, and in the rest of the paper we would discuss rather the first one, digressing back to the case of the pure gauge theory.

5 Pure gauge theories and Whittaker vectors

If one takes in (4.24) the decoupling matter limit $\zeta \to 0$ and $m_{1,2} \to \infty$ so that $\Lambda^2 = \zeta^2 m_1 m_2 =$ fixed [2], one gets instead of (4.26) the equation

$$x = \frac{\Lambda}{w^2} + \frac{v}{w} + \Lambda = \frac{\langle\Psi|J(w)|\Psi\rangle}{\langle\Psi|\Psi\rangle} \equiv \langle J(w)\rangle \tag{5.1}$$

which is an average of the $U(1)$-current $J(w) = \sum_{n\in\mathbb{Z}} \frac{J_n}{w^{n+1}}$ over the state, satisfying

$$\begin{gathered} J_n|\Psi\rangle = 0, \quad n > 1 \\ J_1|\Psi\rangle = \Lambda|\Psi\rangle, \quad J_0|\Psi\rangle = a|\Psi\rangle \end{gathered} \tag{5.2}$$

Since $[J_n, J_m] = n\delta_{n+m,0}$, equations (5.3) can be immediately solved explicitly by

$$\begin{gathered} |\Psi\rangle = \Lambda^{L_0} e^{J_{-1}}|a\rangle \\ J_n|a\rangle = 0,\ n > 0, \qquad J_0|a\rangle = a|a\rangle \end{gathered} \tag{5.3}$$

in terms of the coherent state in the (charged) Fock module, so that

$$\langle\Psi|\Psi\rangle = \langle a|e^{J_1}\Lambda^{2L_0}e^{J_{-1}}|a\rangle = \Lambda^{a^2}e^{\Lambda^2} \underset{\Lambda^2=e^{t_1}}{=} \exp\left(\tfrac{1}{2}a^2 t_1 + e^{t_1}\right) \tag{5.4}$$

The generating differential can be treated as just as an average of the $\widehat{U(1)}$-current

$$\langle J(w)\rangle = dS_{SW} = x dw \tag{5.5}$$

and its more conventional form (2.16) is restored by substitution $x = \frac{z}{w}$.

For the pure $U(2)$ gauge theory instead of (5.1) one gets an equation [30]

$$x^2 = \frac{\Lambda^2}{w^3} + \frac{u}{w^2} + \frac{\Lambda^2}{w} = \frac{\langle\Psi|T(w)|\Psi\rangle}{\langle\Psi|\Psi\rangle} \equiv \langle T(w)\rangle \tag{5.6}$$

which has a sense of averaging of the stress-tensor $T(w) = \sum_{n\in\mathbb{Z}} \frac{L_n}{w^{n+2}}$ over the Whittaker [12, 32] state $|\Psi\rangle \in \mathcal{H}_{\Delta,c}$, satisfying $L_n|\Psi\rangle = 0$ if $n > 1$ and

$$L_1|\Psi\rangle = \Lambda^2|\Psi\rangle \tag{5.7}$$

An important consequence of (5.6), proven in [33], is that the Nekrasov function of the corresponding theory is given by the scalar product, or the matrix element

$$Z(a;\epsilon_1,\epsilon_2) = \langle\Psi|\Psi\rangle = \langle\Psi_1|\Lambda^{4L_0}|\Psi_1\rangle \tag{5.8}$$

[2] Literally, one needs to perform this limit in symmetric way, putting $\Lambda = -\zeta m_1 = -\zeta m_2$.

after identification

$$\Delta = -\frac{a^2}{\epsilon_1\epsilon_2} + \frac{\epsilon^2}{4\epsilon_1\epsilon_2}, \qquad c = 1 + \frac{6\epsilon^2}{\epsilon_1\epsilon_2}, \quad \epsilon = \epsilon_1 + \epsilon_2 \tag{5.9}$$

where

$$\begin{gathered} |\Psi\rangle|_{\Lambda=0} = |\Omega\rangle \in \mathcal{H}_{\Delta,c} \\ L_n|\Omega\rangle = 0, n > 0, \qquad L_0|\Omega\rangle = \Delta|\Omega\rangle \end{gathered} \tag{5.10}$$

i.e. $|\Omega\rangle \sim |\Delta\rangle$ is proportional to the highest-weight vector, but with a non-standard normalization

$$\langle\Omega|\Omega\rangle \sim \Gamma_2(a|\epsilon_1,\epsilon_2)\Gamma_2(-a|\epsilon_1,\epsilon_2) \tag{5.11}$$

to the product of the inverse Barnes double-Gamma functions, and

$$|\Psi_1\rangle = |\Psi\rangle|_{\Lambda=1} \tag{5.12}$$

The nontrivial part of (5.6) contains the only equality

$$\frac{\langle\Psi|L_0|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{1}{4}\frac{\partial}{\partial\log\Lambda}\langle\Psi|\Psi\rangle = u \underset{\epsilon_{1,2}\to 0}{=} \frac{1}{4}\frac{\partial}{\partial\log\Lambda}\mathcal{F}_{SW} \tag{5.13}$$

being a sort of renormalization group equation. Again, by $x = \frac{z}{w}$ the curve (5.6) turns into that with $N = 2$ from the family (2.13), and the generating differential is now [30]

$$\sqrt{\langle T(w)\rangle} = dS_{SW} = xdw = z\frac{dw}{w} \tag{5.14}$$

Generally for the pure $U(N)$ gauge theories one should write the set of "formal differential operators"

$$\begin{gathered} \langle\Psi|\mathcal{D}_N|\Psi\rangle = 0 \\ \mathcal{D}_N \equiv D^N - T(w)D^{N-2} - \mathcal{W}^{(3)}(w)D^{N-3} - \ldots - \mathcal{W}^{(N)}(w) \end{gathered} \tag{5.15}$$

with the coefficients acting in the highest-weight module $\mathcal{H}_{\mathbf{a},c}$ of the W_N-algebra, and $|\Psi\rangle$ is now the Whittaker vector

$$\begin{gathered} \mathcal{W}_1^{(N)}|\Psi\rangle = \Lambda^N|\Psi\rangle \\ \mathcal{W}_n^{(N)}|\Psi\rangle = 0,\ n > 1, \qquad \mathcal{W}_n^{(K)}|\Psi\rangle = 0,\ n > 0,\ K < N \end{gathered} \tag{5.16}$$

and

$$\mathcal{W}_0^{(K)}|\Psi\rangle = s_K(\mathbf{a};\epsilon_1,\epsilon_2)|\Psi\rangle = \sum_{j=1}^N \left(\frac{a_j^K}{\epsilon_1\epsilon_2} + \ldots\right)|\Psi\rangle, \quad K = 1,\ldots,N \tag{5.17}$$

where also everywhere

$$D|\Psi\rangle = x|\Psi\rangle \tag{5.18}$$

This set of equations (again by substitution $x = \frac{z}{w}$) gives rise to the SW curves (2.13) endowed with generating differential $dS_{SW} = xdw = z\frac{dw}{w}$, with a natural conjecture

$$Z = \langle\Psi|\Lambda^{2NL_0}|\Psi\rangle \tag{5.19}$$

provided by normalization

$$\langle\Psi|\Psi\rangle|_{\Lambda=0} \sim \prod_{i,j} \Gamma_2(a_i - a_j|\epsilon_1, \epsilon_2) \sim Z_{\text{pert}} \tag{5.20}$$

written again in terms of the Barnes double-gamma function.

In the $N = 1$ and $N = 2$ cases equations (5.16) (i.e. $\langle\Psi|D - J(w)|\Psi\rangle = 0$ for $W_1 = \widehat{U(1)}$ and $\langle\Psi|D^2 - T(w)|\Psi\rangle = 0$ for $W_2 = Vir$ can be explicitly solved w.r.t. $x = \langle D\rangle$ variable, and we come back to (5.1) and (5.6) correspondingly. To switch on higher times one just has to generalize the Hamiltonian in (5.19)

$$Z(t_1, \ldots, t_N) \sim \langle\Psi| \exp\left(\sum_{K=1}^{N} t_K \mathcal{W}_0^{(K)}\right) |\Psi\rangle \tag{5.21}$$

by introducing higher W-flows. Arising of the Gelfand-Dikij-type operator (5.15) in this context together with the W-flows in (5.21) suggests a nontrivial relation with so-called W-gravity [35] and geometry of generalized Teichmüller spaces.

6 Conclusion

We have discussed in these notes some relations between the formulation of supersymmetric gauge theories, coming from the instanton partition functions, and the matrix models. As we pointed out - there are several parallels of this kind, but neither of them seems to be very essential by itself.

Let us finish instead with the following important remark. The quasiclassical picture of the matrix model is indeed very similar to the SW theory, and they are both described in terms of a quasiclassical integrable system, so that the prepotential is in fact a restricted Krichever tau-function [36]. The simplest part of integrable dynamics in this case is parameterized by the parameters of potential (2.2), and reproduces in the simplest case the dispersionless well-known hierarchy of the Toda equation (3.10) (for the "higher-genera" analogs of dispersionless equations see [37]). The dependence on smooth periods (2.18) is far more transcendental and, on quasiclassical level, is basically exhausted by the gradient formulas (2.21) and its consequences, like residue formulas, the WDVV equations, etc.

The formulas of sect. 5 demonstrate, that the nontrivial dynamics over quasiclassical period variables has even far less trivial dispersive analogs. While the naive "quantization" of the dynamics over the times (2.2) leads (in absence of the "smooth variables" (2.18)) just to the string solution of the full hierarchy of the Toda or KP type with the tau-functions presented by the matrix elements in the two-dimensional theory of free fermions, the dispersive analogs of the full Krichever tau-function are matrix elements in non-trivial two-dimensional conformal theory, generally with extended symmetry. They also seem to be directly related with the quantization of Teichmüller spaces [38] (higher Teichmüller spaces for the case of W-gravity), Liouville theory [29, 39], and quantum-mechanical integrable dynamics in the systems of Toda type (see e.g. [40, 41, 42]). All these relations clearly deserve further investigation, and we are going to return to them elsewhere.

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A Perturbative prepotentials

The instantonic expansion $\mathcal{F} = \sum_{k\geq 0} \mathcal{F}_k$ in the non-Abelian theory starts with the perturbative prepotential

$$\mathcal{F}_0 = \sum_{j=1}^{N} F_{UV}(a_j) - \sum_{i\neq j} F(a_i - a_j) \tag{A.1}$$

defined entirely in terms of the ultraviolet or classical (2.2) and perturbative (2.3) prepotentials. It is totally characterized by degenerate differential of (2.9)

$$d\Phi_0 = F'''_{UV}(z)dz - 2\frac{dP_N(z)}{P_N(z)} = F'''_{UV}(z)dz - 2\sum_{j=1}^{N} \frac{dz}{z - v_j} \tag{A.2}$$

and the coefficients of the polynomial $P_N(z)$ (A.2) (as in (2.13)) coincide with the perturbative values of the SW periods

$$a_i = -\tfrac{1}{2}\mathrm{res}_{v_i} z d\Phi_0 = v_i \tag{A.3}$$

The perturbative generating differential is $dS_0 = \Phi_0 dz$, with

$$\Phi_0 = F''_{UV}(z) - 2\sum_{j=1}^{N} \log(z - v_j) \tag{A.4}$$

and satisfies

$$\begin{aligned} \frac{\partial dS_0}{\partial a_j} &= 2\frac{dz}{z - v_j}, \qquad j = 1, \dots, N \\ \frac{\partial dS_0}{\partial t_k} &= kz^{k-1}dz, \qquad k > 0 \end{aligned} \tag{A.5}$$

what gives rise to

$$S_0(z) = F'_{UV}(z) - 2\sum_{j=1}^{N}(z - v_j)\left(\log(z - v_j) - 1\right) \tag{A.6}$$

Equations (2.21)

$$a_j^D = \frac{\partial \mathcal{F}_0}{\partial a_j} = S_0(a_j) \tag{A.7}$$

completely determine (A.1), since on this stage one makes no difference between v_j and a_j.

For the theory with N_f fundamental multiplets instead of the formula (A.1) one has

$$\mathcal{F}_0 = \mathcal{F}_{\rm cl} + \mathcal{F}_{\rm pert} = \sum_{j=1}^{N} F_{UV}(a_j) - \sum_{i \neq j}^{N} F(a_i - a_j) + \sum_{f=1}^{N_f}\sum_{j=1}^{N} F(a_j + m_f) \tag{A.8}$$

which can be obtained from (A.1) just by formal modification of the UV prepotential via

$$F_{UV}(x) \to F_{UV}(x) + \sum_{f=1}^{N_f} F(x + m_f) \tag{A.9}$$

which can be further used, after its substitution to the functional (2.1) to compute the full partition function for the theory with matter.

The "quantum" - or, better, double-deformed - version of the perturbative prepotentials (A.1), (A.8) can be written in terms of the Barnes double-gamma functions

$$\Gamma_2(x|\epsilon_1, \epsilon_2) \sim \prod_{n,m \geq 0} \frac{1}{x + n\epsilon_1 + m\epsilon_2} \tag{A.10}$$

where the infinite product can be understand, say, via the zeta-regularization (see e.g. [34])

$$\log \Gamma_2(x|\epsilon_1, \epsilon_2) = \frac{d}{ds} \sum_{n,m \geq 0} (x + n\epsilon_1 + m\epsilon_2)^{-s} \Bigg|_{s=0} \tag{A.11}$$

analytically continued to $s = 0$. The relation to (A.1), (A.8) is established via the asymptotic

$$\begin{aligned} \log \Gamma_2(x|\epsilon_1, \epsilon_2) \underset{\epsilon_{1,2}\to 0}{=} -\frac{1}{\epsilon_1\epsilon_2} F(x) + \text{less singular} = \\ = -\frac{1}{\epsilon_1\epsilon_2}\frac{x^2}{2}\left(\log x - \frac{3}{4}\right) + \ldots \end{aligned} \tag{A.12}$$

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