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FOLDING LINES IN OUTER SPACE DETERMINE THE DUAL LAMINATION OF THEIR LIMIT $\mathbb{R}$-TREE

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Abstract. The purpose of this note is to draw a connection between folding lines and dual laminations for $\mathbb{R}$-trees, two subjects within the domain of Outer space and free group automorphisms which have received recently much attention (e.g. see [1], [2], [4], [11], [16], [17]). We prove that for any folding line $(\widetilde{\Gamma}_t)_{t \in I}$ in Outer space which converges towards an $\mathbb{R}$-tree T with dense orbits, the intersection of the *totally illegal* laminations of all $\widetilde{\Gamma}_t$ is equal to the dual lamination $L^2(T)$ of T.

1. Introduction

Throughout this paper F_N denotes the non-abelian free group of rank $N \geq 2$. In this introduction we assume familiarity of the reader with the basic terminology around automorphisms of free groups and Outer space; some details are given in §1, and more of them as well as some back ground can be found for example in [15] or [18].

For any train track representative $f : \tau \to \tau$ of an iwip automorphism φ of F_N the dual lamination $L^2(T_+)$ of the forward limit tree $T_+ = T_+(\varphi)$ is given by biinfinite paths γ in τ which are *totally illegal* in that they have the property that any legal subpath γ' of γ must have length $L(\gamma') \leq C$, where C is a constant (called the *backtracking constant*) that only depends on f. Indeed (see [12]), $L^2(T_+)$ is equal to the set of all such paths γ, if one imposes the additional condition that $f^t(\gamma)$ is also totally illegal, for any exponent $t \geq 0$.

In order to see whether any such statement could be true for more general trees T in place of T_+, we first observe that, rather than changing γ by applying powers of the train track map f (which wouldn't necessarily be available for a more general such T), one can stick to γ and change τ, by composing the marking isomorphism $\theta : F_N \to \pi_1\tau$ from the right with powers of φ. By passing to the universal coverings, this gives a sequence of metric simplicial trees $\widetilde{\tau}_t$ with free action of F_N as deck transformations, via the marking isomorphisms $\theta_t = \theta \circ \varphi^t : F_N \to \pi_1\tau$. They define a sequence of points $[\widetilde{\tau}_t]$ in Culler-Vogtmann's Outer space CV_N which converge to $[T_+] \in \overline{\mathrm{CV}}_N$ (= the Thurston compactification of Outer space). Equivalently, we can pass to the non-projectivied versions of Outer space and its

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closure, denoted cv_N and $\overline{\mathrm{cv}}_N$ respectively, where the $\widetilde{\tau}_t$ converge to T_+, (after proper rescaling). We can now reformulate the above description of the dual lamination of T_+ as follows:

The algebraic lamination $L^2(T_+)$ is equal to the set of all pairs $(X, Y) \in \partial F_N \times \partial F_N$ which have the property that on any of the $\widetilde{\tau}_t$ the biinfinite geodesic $\gamma_t = \gamma_t(X, Y)$ which represents (X, Y) is totally illegal.

Thus a possible generalization of this statement could be given by considering sequences of free simplicial $\mathbb{R}$-trees $\widetilde{\Gamma}_t \in \mathrm{cv}_N$ which converge to $T \in \overline{\mathrm{cv}}_N$, if one generalizes the notion of legal paths properly. This has been done in [15] by considering *edge-isometric* F_N-equivariant maps $i : \widetilde{\Gamma} \to T$ and the gate structure $\mathfrak{G}(i)$ defined on $\widetilde{\Gamma}$ by such a map i; the details of these definitions are reviewed here in §2. In particular, any such gate structure $\mathfrak{G}(i)$ defines an algebraic lamination $L^2(\mathfrak{G}(i))$, called the *totally illegal* lamination with respect to $\mathfrak{G}(i)$, which is defined as the set of all pairs $(X, Y) \in \partial^2 F_N := \partial F_N \times \partial F_N \smallsetminus \{(X, X) \mid X \in \partial F_N\}$ which have the property that the biinfinite geodesic $\gamma = \gamma(X, Y)$ in $\widetilde{\Gamma}$ that represents (X, Y), is totally illegal with respect to $\mathfrak{G}(i)$.

Question 1.1. Let $T \in \overline{\mathrm{cv}}_N$ be an $\mathbb{R}$-tree (say with dense orbits), and let $\widetilde{\Gamma}_t \in \mathrm{cv}_N$ be any sequence of free simplicial $\mathbb{R}$-trees, provided with edge-isometric maps $i_t : \widetilde{\Gamma}_t \to T$. Assume that the $\widetilde{\Gamma}_t$ converge in $\overline{\mathrm{cv}}_N$ to T. Under which conditions can one conclude the equality

$$L^2(T) = \cap L^2(\mathfrak{G}(i_t))\,?$$

This question is phrased too general to be true as stated without imposing further conditions on the maps i_t: for example the condition, that $\mathfrak{G}(i_t)$ has at least two gates at any vertex, seems necessary.

In this paper we restrict the set of trees $\widetilde{\Gamma}_t$ further, still in the spirit of generalizing the sequence of $\widetilde{\tau}_t$ as above: The latter constitute not just *any* sequence of points in Outer space converging to T_+, but rather each $\widetilde{\tau}_{t+1}$ is obtained from *folding at $\widetilde{\tau}_t$ in the direction of T_+*. In §3 these terms will be defined precisely; they go back to one of the early papers on folding in Outer space, see [14]. In particular, we will define in §3 the term *folding line* $(\widetilde{\Gamma}_t)_{t\in I}$ *directed towards* T, where the direction of the folding line is given by a "continuously moving" family of F_N-equivariant edge-isometric maps $i_t : \widetilde{\Gamma}_t \to T$.

Theorem 1.2. *Let $T \in \overline{cv}_N$ be an $\mathbb{R}$-tree with dense orbits, and let $(\widetilde{\Gamma}_t)_{t\in I}$ be a folding line which is directed towards T via F_N-equivariant edge-isometric maps $i_t : \widetilde{\Gamma} \to T$. Assume that the limit volume $vol_{\lim}(\widetilde{\Gamma}_t)$ of the Γ_t is equal to 0. Then the dual algebraic lamination $L^2(T)$ is equal to the set of pairs $(X, Y) \in \partial^2 F_N$ which are represented in any $\widetilde{\Gamma}_t$ by a biinfinite path $\gamma_t(X, Y)$ that is totally illegal with respect to $\mathfrak{G}(i_t)$:*

$$L^2(T) = \bigcap_{t\in I} L^2(\mathfrak{G}(i_t))$$

Remark 1.3. For an $\mathbb{R}$-tree T with dense F_N-orbits of points, as in the above Theorem 1.2, the condition that $\mathrm{vol}_{\lim}(\widetilde{\Gamma}_t) = 0$ is equivalent to the assertion that the sequence of $\widetilde{\Gamma}_t$ converges in $\overline{\mathrm{cv}}_N$ for increasing t to the tree T. Moreover, for any folding path $\mathcal{F} : I \to \mathrm{CV}_N, t \mapsto [\widetilde{\Gamma}_t]$ with forward limit point $[T]$ there are canonical F_N-equivariant edge-injective maps $i_t : \widetilde{\Gamma}_t \to T$, and we can rescale the edge lengths of the Γ_t to make the maps i_t edge-isometric: this would transform the original folding path $\mathcal{F}$ into a directed folding line as in Theorem 1.2. But the laminations $L^2(\mathfrak{G}(i_t))$ do not depend on the edge lengths of the Γ_t. This shows that the equality of Theorem 1.2 is actually true for any folding path $\mathcal{F} : I \to \mathrm{CV}_N, t \mapsto [\widetilde{\Gamma}_t]$ with forward limit point $[T]$ with dense orbits.

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2. BASICS

We denote by cv_N the unprojectivized Outer space, i.e. the set of simplicial $\mathbb{R}$-trees $\widetilde{\Gamma}$, provided with a minimal free F_N-action by isometries. Alternatively, a point in cv_N is given by a metric graph Γ which is connected and has no valence 1 vertices, and which is provided with a *marking isomorphism* $\theta : F_N \to \pi_1\Gamma$. Then $\widetilde{\Gamma}$ can be viewed as universal covering of Γ with lifted edge lengths.

We denote by $\overline{\mathrm{cv}}_N$ the bordification of cv_N given by the set of all $\mathbb{R}$-trees T provided with a minimal very small action of F_N by isometries.

For any $\widetilde{\Gamma} \in \mathrm{cv}_N$ and any $T \in \overline{\mathrm{cv}}_N$ an F_N-equivariant map $i : \widetilde{\Gamma} \to T$ is called *edge-isometric* if i restricts on any edge e of $\widetilde{\Gamma}_t$ to an isometry onto $i(e)$. The *bounded back tracking constant* $\mathrm{BBT}(i)$ is defined to be the infimum of all $c \geq 0$ such that for any geodesic segment $[x, y] \subset \widetilde{\Gamma}$ the image $i([x, y])$ is contained in the c-neighborhood $\mathcal{N}_c([i(x), i(y)]) \subset T$ of the geodesic segment $[i(x), i(y)]$ in T. It has been shown in [9] that for any edge-isometric map $i : \widetilde{\Gamma} \to T$ one has

$$\mathrm{BBT}(i) \leq \mathrm{vol}(\Gamma) =: \sum_{e \in Edges(\Gamma)} L(e)\,,$$

where $L(e)$ denotes the length of the edge e.

An $\mathbb{R}$-tree $T \in \overline{\mathrm{cv}}_N$ is said to have *dense orbits* if the orbit $F_N \cdot x$ is dense in T, for some (and hence any) point $x \in T$. In [13] for any $T \in \overline{\mathrm{cv}}_N$ with dense orbits, a map $\mathcal{Q} : \partial F_N \to \overline{T} \cup \partial T$ (where $\overline{T}$ denotes the metric completion of T and ∂T its Gromov boundary) has been defined such that for any $(X, Y) \in \partial^2 F_N$ and any biinfinite geodesic $\gamma = \gamma(X, Y)$ in $\widetilde{\Gamma}$ which *realizes* (X, Y) (i.e. γ connects the end of $\widetilde{\Gamma}$ defined by X to that defined by Y) one has:

$$i(\gamma) \subset \mathcal{N}_{BBT(i)}([\mathcal{Q}(X), \mathcal{Q}(Y)])$$

The *algebraic lamination dual to* T has been defined in [6] as

$$L^2(T) = \{(X, Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}.$$

This gives immediately:

Lemma 2.1. *For any* $(X, Y) \in L^2(T)$ *and any edge isometric map* $i : \widetilde{\Gamma} \to T$ *the geodesic path* $\gamma(X, Y)$ *in* $\widetilde{\Gamma}$ *which realizes* (X, Y) *is mapped by* i *into the* $BBT(i)$*-neighborhood of* $\mathcal{Q}(X) = \mathcal{Q}(Y)$. □

Any edge-isometric map $i : \widetilde{\Gamma} \to T$ as before defines a *gate structure* $\mathfrak{G}(i)$ on $\widetilde{\Gamma}$: Two edges e, e' with common initial vertex v belong to the same *gate* $\mathfrak{g}$ at v if i maps non-degenerate initial segments of e and e' to the same segment in T. A *turn* (e, e') in Γ, i.e. e and e' are edges with common initial vertex, is *legal* if and only if e and e' do not belong to the same gate. A path γ in $\widetilde{\Gamma}$ is *legal* if it only runs over legal turns: if γ contains the subpath $e\, e'$, then $(\overline{e}, e')$ must be a legal turn. Here and elsewhere we denote by $\overline{e}$ the edge e with reversed orientation. A turn or path which is not legal is called *illegal*. We observe directly from these definitions:

Lemma 2.2. *A path* γ *in* $\widetilde{\Gamma}$ *is legal if and only if* γ *is mapped by* i *isometrically onto the image path* $i(\gamma) \subset T$. □

Definition 2.3. We define the *totally illegal* lamination $L^2(\mathfrak{G}(i))$ to be the set of all pairs $(X, Y) \in \partial^2 F_N$ such that any legal subpath γ of the geodesic realization $\gamma(X, Y)$ in $\widetilde{\Gamma}$ has length $L(\gamma) \leq 2\mathrm{BBT}(i)$.

Proposition 2.4. *For every* $T \in \overline{cv}_N, \widetilde{\Gamma} \in cv_N$ *and any edge-isometric map* $i : \widetilde{\Gamma} \to T$ *one has:*

$$L^2(T) \subset L^2(\mathfrak{G}(i))$$

Proof. This is a direct consequence of the Lemmata 2.1 and 2.2. □

3. Folding lines

A *folding line* in cv_N is a path $I \to \mathrm{cv}_N$, for some (not necessarily compact) connected subset $I \subset \mathbb{R}$, which satisfies certain rules, which can vary slightly according to the version one preferes to work with. In our context, a folding line $(\Gamma_t)_{t \in I}$ is always *directed towards* some $T \in \overline{\mathrm{cv}}_N$, i.e. for any $t \in I$ the metric tree $\widetilde{\Gamma}_t$ is equipped with an edge-isometric F_N-equivariant map $i : \widetilde{\Gamma}_t \to T$, and thus with an induced gate structure $\mathfrak{G}(i_t)$ as defined in the last section.

Before defining our version of folding line formally, we want to give the reader an intuitive idea:

A *generic* vertex v of $\widetilde{\Gamma}$ has precisely two gates, one consisting of only one edge e_0, and the other of two edges e_1, e_2, such that all e_i have v as initial vertex. (To be precise, it is quite possible that e_0 is equal to the edge $\overline{e}_i$ obtained by reversing the orientation of e_i, for $i = 1$ or $i = 2$). At any such generic vertex the folding line is locally defined by moving $i(v)$ on T continuously into the initial segment of $i(e_1) = i(e_2)$, with "speed" depending on the chosen folding line, and by pulling back the resulting edge lengths from T to $\widetilde{\Gamma}_t$ to obtain edge-isometric maps $i_t : \widetilde{\Gamma}_t \to T$ at all times.

For non-generic vertices the folding line will also fold initial segments of edges which lie in the same gate, but the situation can be more complicated in that (i) local folding may takes place simultaneously in more than one gate (so that the vertex itself "multiplies" into several vertices, or (ii) a gate which contains 3 or more edges can split into several gates, if the folding between the various edge pairs takes place with distinct speed (again "multiplying" the original vertex v).

In order to give a formal definition of a folding line $(\Gamma_t)_{t\in I}$, one has to define first a local version of it: A local folding line starting at any $\widetilde{\Gamma}_t$, provided as above with an F_N-equivariant edge-isometric map $i_t : \widetilde{\Gamma}_t \to T$, is given by assigning to any pair of edges e, e' of $\widetilde{\Gamma}$ in the same gate (with respect to i_t) a monotonically (not necessarily strictly) increasing *identification function* $id_{e,e'} = id_{e',e} : [t, t'] \to [0, \frac{L(\widetilde{\Gamma}_t)}{3}]$ with $id_{e,e'}(t) = 0$, where $t' \in \mathbb{R}$ is any value with $t' > t$, and $L(\widetilde{\Gamma}_t) = \min\limits_{e \in Edges(\Gamma_t)} L(e)$. We also require that for any triple of edges e_1, e_2, e_3 in the same gate one has $id_{e_i,e_j}(s) \geq \min(id_{e_i,e_k}(s), id_{e_k,e_j}(s))$, for any choice of parameters satisfying $\{i, j, k\} = \{1, 2, 3\}$ and $t \leq s \leq t'$. Then the family of those identification functions defines, for any parameter value $s \in [t, t']$ an F_N-equivariant quotient tree $\widetilde{\Gamma}_s$ obtained by identifying, for any pair of edges e, e' in the same gate, initial segments of length $id_{e,e'}(s)$. The map i_t factors through the quotient map $i_{s,t} : \widetilde{\Gamma}_t \to \widetilde{\Gamma}_s$ and hence induces an F_N-equivariant edge-isometric map $i_s : \widetilde{\Gamma}_s \to T$. The map $[t, t'] \to \mathrm{cv}_N$ which associates to each $s \in [t, t']$ the tree $\widetilde{\Gamma}_s$ equipped with the map i_s is called a *local folding line directed towards T which starts at $\widetilde{\Gamma}_t$.*

Definition 3.1. For any interval $I \subset \mathbb{R}$ and any $\mathbb{R}$-tree $T \in \overline{\mathrm{cv}}_N$, a family $(\widetilde{\Gamma}_t)_{t\in I}$, consisting of metric simplicial trees $\widetilde{\Gamma}_t \in \mathrm{cv}_N$ equipped with an F_N-equivariant edge-isometric map $i_t : \widetilde{\Gamma}_t \to T$, is called a *folding line directed towards* T, if for all $t \in I$ the restriction $(\widetilde{\Gamma}_t)_{t\in[t,t']}$ for some suitable parameter value $t' > t$ is a local folding line directed towards T which starts at $\widetilde{\Gamma}_t$.

A particularly interesting folding parameter has been exhibited in [14] by exhibiting "natural" identification functions at the illegal turns for any of the $\widetilde{\Gamma}_t$, which lead to the definition of *canonical folding lines.* However, these will not be used here.

Lemma 3.2. *Let* $(\Gamma_t)_{t\in I}$ *be a folding line in* cv_N*, directed towards some* $T \in \overline{cv}_N$*. Then for any parameter values* $t \leq t'$ *in* I *there exists a map* $i_{t',t} : \widetilde{\Gamma}_t \to \widetilde{\Gamma}_{t'}$ *which has the following properies:*

(1) *For all* $t \leq t'$ *in* I *one has*

$$i_t = i_{t'} \circ i_{t',t}$$

(2) *For all* $t \leq t' \leq t''$ *in* I *one has*

$$i_{t'',t} = i_{t,t'} \circ i_{t',t}$$

(3) *For all* $t \leq t'$ *in* I *the maps* $i_{t',t}$ *is surjective and edge-isometric.*

Proof. For any parameter values t and t' sufficiently close, the map $i_{t',t}$ is simply given by the isometric identification of initial edge segments of edges in the same gate, and it clearly has the stated properties (1) - (3). Furthermore, if $i_{t_k,t}$ is defined for any sequence of t_k with $\lim t_k = t'$, then $i_{t',t}$ is obtained by passing to the limit of the quotient maps $i_{t_k,t}$. The properties (1) - (3) pass over to the limit. It follows now directly from a maximum argument on $t' - t$ that by proper concatenation of $i_{t',t}$ for close parameter values we can define those maps for any $t, t' \in I$, and that the properties (1) - (3) always hold. □

A crucial property in order to prove our main result is given by the following:

Proposition 3.3. *Any of the folding maps* $i_{t',t} : \widetilde{\Gamma}_t \to \widetilde{\Gamma}_{t'}$ *maps legal paths to legal paths.*

Proof. This follows directly from Lemma 3.2 (1), and from Lemma 2.2, applied to i_t and to $i_{t'}$. □

Remark 3.4. For any folding line $(\widetilde{\Gamma}_t)_{t\in I}$ as above, directed towards a tree $T \in \overline{\text{cv}}_N$, the co-volume $\text{vol}(\Gamma_t)$ of the trees $\widetilde{\Gamma}_t$ decreases monotonically during the folding process, so that there is a well defined *limit volume* $\text{vol}_{\lim}(\widetilde{\Gamma}_t) := \inf\{\text{vol}(\widetilde{\Gamma}_t) \mid t \in I\}$. It is not hard to show that $\text{vol}_{\lim}(\widetilde{\Gamma}_t) = 0$ implies that the $\widetilde{\Gamma}_t$ converge in $\overline{\text{cv}}_N$ for increasing t to the point T, and that for T with dense orbits the converse implication also holds. Note that the inequality at the beginning of §2 implies that in this case one has also that the backtracking constant $\text{BBT}(i_t)$ converges to 0.

In Proposition 2.4 it has been shown already that $L^2(T)$ is contained in any of the lamination $L^2(\mathfrak{G}(i_t))$. The following proposition gives a converse statement, thus proving Theorem 1.2.

Proposition 3.5. *Let* $(\widetilde{\Gamma}_t)_{t\in I}$ *be a folding line in* cv_N *directed towards* $T \in \overline{cv}_N$*, and assume that the limit volume satisfies* $vol_{\lim}(\widetilde{\Gamma}_t) = 0$*. Then the intersection of all* $L^2(\mathfrak{G}(i_t))$ *is contained in* $L^2(T)$*.*

Proof. It suffices to show that for any $(X, Y) \in \partial^2 F_N \smallsetminus L^2(T)$ there is some $t \in I$ such that (X, Y) is not contained in $L^2(\mathfrak{G}(i_t))$. For any such X and Y we know by definition of $L^2(T)$ that $\mathcal{Q}(X) \neq \mathcal{Q}(Y) \in T$. We now consider a parameter value $t \in I$ with $\mathrm{BBT}(i_t) \leq \frac{d(\mathcal{Q}(X), \mathcal{Q}(Y))}{5}$ and consider points x, y on $[\mathcal{Q}(X), \mathcal{Q}(Y)] \in T$ with $d(x, y) > 2\mathrm{BBT}(i_t)$ such that $[x, y]$ is disjoint from $\mathcal{N}_{BBT(i_t)}(\mathcal{Q}(X))$ and from $\mathcal{N}_{BBT(i_t)}(\mathcal{Q}(Y))$. Let γ be the shortest subpath of $\gamma(X, Y)$ such that $i(\gamma)$ contains x, y.

By Lemma 2.2 the path γ can not be legal. We now pass to a larger parameter value $t' > t$ and observe that the path $i_{t',t}(\gamma)$ reduces in $\widetilde{\Gamma}'$ to a geodesic subpath γ' of the path $\gamma'_t(X, Y)$ which also has the property that $i(\gamma')$ contains $[x, y]$ (by property (1) of Lemma 3.2). But by Proposition 3.3 the number of illegal turns does not increase when passing from γ to γ'. It follows directly that for sufficiently small $\mathrm{BBT}(i_{t'})$ the path γ' must contain a legal subpath γ'' of length $L(\gamma'') \geq 2\mathrm{BBT}(i_{t'})$. Thus, again by Lemma 2.2, (X, Y) does not belong to $L^2(\mathfrak{G}(i_{t'}))$, which shows our claim. □

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