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# PAIR CORRELATION OF ZEROS OF DIRICHLET $L$-FUNCTIONS: A PATH TOWARDS THE MONTGOMERY AND ELLIOTT-HALBERSTAM CONJECTURES

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Abstract. Assuming the Generalized Riemann Hypothesis and a pair correlation conjecture for the zeros of Dirichlet $L$-functions, we establish the truth of a conjecture of Montgomery (in its corrected form stated by Friedlander and Granville) on the magnitude of the error term in the prime number theorem in arithmetic progressions. As a consequence, we obtain that, under the same assumptions, the Elliott-Halberstam conjecture holds true.

## 1. Introduction

The study of pair correlation of the zeros of the Riemann zeta function and Dirichlet $L$-functions has its origin in the early 20th century. In the 1930s, Bohr and others (Landau, Hardy) investigated the distribution of zeros in the critical strip. In 1942, Selberg [24] proved that a positive density of the zeros of Riemann zeta function are of odd order and lie on the critical line. The density coming out of his method is around $10^{-8}$; this was dramatically increased in 1974 by Levinson [13], who improved it to more than one third. Currently it is known due to Bui, Conrey and Young [1] that more than 40% of the zeros are simple and on the critical line.

In 1973, Montgomery made a major breakthrough and conjectured that the pair correlation of zeros of the zeta function follows a distribution similar to that of the eigenvalues of random complex Hermitian or unitary matrices of large orders. That there might be such a connection, was an idea that first arose during a discussion he had with the physicist Dyson. In the intervening years, much of the focus shifted to formalizing this connection and examining its implications. A way to approach the pair correlation of Riemann zeros is to try to asymptotically evaluate sums of the form

$$\sum_{\substack{0<\gamma_j\le T,\ j=1,2\\ \zeta(1/2+i\gamma_j)=0}} f(\gamma_1-\gamma_2),$$

with $f$ taken from a class of functions as large as possible. Using Fourier analysis, see, e.g. [15], it can be shown that

$$\sum_{\substack{0<\gamma_j\le T,\ j=1,2\\ \zeta(1/2+i\gamma_j)=0}} r\left(\frac{\gamma_1-\gamma_2}{2\pi}\right) W(\gamma_1-\gamma_2) = \int_{-\infty}^{\infty} F(e^x,T)\,\hat{r}(x)dx, \tag{1}$$

where

$$W(u)=\frac{4}{4+u^2},\quad \hat{r}(x)=\int_{-\infty}^{\infty} r(u)e^{-2\pi i x u}du,$$

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is the Fourier transform of $r \in L^1(\mathbb{R})$, and

$$F(x,T) = \sum_{\substack{0<\gamma_j \le T,\ j=1,2 \\ \zeta(1/2+i\gamma_j)=0}} x^{i(\gamma_1-\gamma_2)} W(\gamma_1-\gamma_2).$$

In 1973, Montgomery [15] proved, assuming the Riemann Hypothesis (RH), that

$$F(x,T) \sim (2\pi)^{-1} T \log x \text{ uniformly for } 1 \le x \le T^{1-\varepsilon},\ T \to \infty, \tag{2}$$

a range he extended in 1987, together with Goldston [7] to $1 \le x \le T$. In [15], Montgomery also formulated the *pair correlation conjecture*, stating that

$$F(x,T) \sim (2\pi)^{-1} T \log T, \quad \text{uniformly for } T \le x \le T^A \text{ for every } A \ge 1 \tag{3}$$

(for the heuristic arguments that led him to believe in this, see [15, §3]).

Using (1), for fixed $\alpha \le \beta$, it can be seen that this conjecture is equivalent to

$$\sum_{\substack{0<\gamma,\gamma' \le T \\ \frac{2\pi\alpha}{\log T} \le \gamma-\gamma' \le \frac{2\pi\beta}{\log T}}} 1 \sim \Big( \int_\alpha^\beta \Big(1 - \Big(\frac{\sin(\pi u)}{\pi u}\Big)^2 du\Big) + \delta(\alpha,\beta) \Big) \frac{T}{2\pi} \log T, \tag{4}$$

as $T$ tends to infinity, where $\gamma, \gamma'$ are the imaginary parts of the zeros of the Riemann zeta function on the critical line and $\delta(\alpha,\beta) = 1$ if $0 \in [\alpha,\beta]$, and 0 otherwise. For an accessible introduction to the pair correlation conjecture, see, e.g., Goldston [6].

In the late 20th century, Odlyzko [19, 20] numerically studied the distribution of spacings between zeros of the Riemann zeta function. He computed 10 billion zeros near $10^{22}$-nd zero of Riemann zeta and verified the Riemann Hypothesis for those zeros. Additionally, he found that the spaces between these zeros are closely distributed according to (4). Assuming RH, he showed that the estimate (2) implies that at least $2/3$ of the zeros of Riemann zeta function are simple (it is expected that all the zeros are simple). This is a much stronger result than the unconditional results mentioned in the beginning of the introduction. In 1982, Özlük [21] in his thesis, see also [22, 23], studied the Dirichlet $L$-function analogue of Montgomery's conjecture. Consequently, under the Riemann Hypothesis for Dirichlet $L$-functions, also known as the Generalized Riemann Hypothesis (GRH), in [23] he showed that at least $11/12$ of all the zeros of such functions are simple (allowing for a certain weight function).

In 1991, Yıldırım [25] studied the pair correlation of zeros of Dirichlet $L$-functions and an analogue of the conjecture stated in equation (3). In the sequel, we will work with a redefined pair correlation function. The key difference is that our summation will not be restricted to the zeros in the upper half-plane but will also include those in the lower half-plane (note that the imaginary parts of the zeros of Dirichlet $L$-functions are typically not symmetrically distributed with respect to the real axis). In order to describe this in more detail, we need some notations.

From now on, we assume the Riemann Hypothesis for Dirichlet $L$-functions (GRH). Let $q$ be a natural number, $\chi$ a Dirichlet character mod $q$ and $L(s,\chi)$ the associated Dirichlet $L$-function. Let $\gamma_j$ be the imaginary part of the $j$th zero (ordered by height on the half line). Given two Dirichlet characters $\chi_1$ and $\chi_2$ modulo $q$, one can wonder to what extent the zeros of $L(s,\chi_1)$ are correlated with those of $L(s,\chi_2)$. In order to measure this, we define

$$G_{\chi_1,\chi_2}(x,T) = \sum_{\substack{|\gamma_j| \le T,\ j=1,2 \\ L(1/2+i\gamma_j,\chi_j)=0}} x^{i(\gamma_1-\gamma_2)} W(\gamma_1-\gamma_2).$$

Henceforth, throughout the article, whenever we write $\gamma$ and $\gamma_j$ in the summation without additional specifications, we assume, respectively, that $L(1/2+i\gamma,\chi) = 0$ and $L(1/2+i\gamma_j,\chi_j) = 0$.

Moreover, we remark that rather than $0 < \gamma \le T$, we require that $|\gamma| \le T$, thus taking into account that the imaginary parts of the zeros of Dirichlet $L$-functions are (mostly) not symmetrical with respect to the real axis.

Note that $G_{\chi_1,\chi_2}(x,T)$ is an analogue of $F(x,T)$. We make a more global object out of $G_{\chi_1,\chi_2}(x,T)$ by considering

$$F_q(x,T) = \sum_{\chi_1,\chi_2 \pmod q} \overline{\chi_1}(a)\chi_2(a) G_{\chi_1,\chi_2}(x,T).$$

Observe that $F_q(x,T)$ depends on $a$. However, in our estimates the actual value of $a$ will not play a role and so, in accordance with the existent literature, we suppress the $a$ dependence. Now, using [2, Ch. 16, eqn 1] and [2, Ch. 16, Lemma], we have

$$|G_{\chi_1,\chi_2}(x,T)| \le \sum_{|\gamma_j| \le T,\ j=1,2} \frac{1}{4+(\gamma_1-\gamma_2)^2} \ll T(\log(qT))^2,$$

uniformly in $x$ as $T \to \infty$. Therefore, we trivially have

$$F_q(x,T) \ll T(\varphi(q)\log(qT))^2, \tag{5}$$

uniformly in $x$ as $T \to \infty$, where $\varphi$ denotes the Euler's totient function. Yıldırım [25] defined the $G_{\chi_1,\chi_2}(x,T)$ analogue

$$G^+_{\chi_1,\chi_2}(x,T) = \sum_{0<\gamma_j \le T} x^{i(\gamma_1-\gamma_2)} W(\gamma_1-\gamma_2),$$

and proved the following result.

**Theorem 1 (Yıldırım [25]).** *Under GRH, as $x \to \infty$, we have*

$$\sum_{\chi_1,\chi_2 \pmod q} \overline{\chi_1}(a)\chi_2(a) G^+_{\chi_1,\chi_2}(x,T) \sim \frac{\varphi(q)}{2\pi} T \log x \tag{6}$$

*uniformly for*

$$1 \le q \le \sqrt{x}\log^{-3} x \quad \textit{and} \quad \frac{x}{q}\log x \le T \le \exp(\sqrt[4]{x}).$$

For smaller $T$ (with respect to $x$), he conjectured:

**Conjecture 1 (Pair correlation for Dirichlet $L$-functions [25]).** *Let $q = 1$ or $q$ be a prime and $0 < \eta \le 1$ be a fixed real number. Then, under GRH, as $x \to \infty$,*

$$\sum_{\chi_1,\chi_2 \pmod q} \overline{\chi_1}(a)\chi_2(a) G^+_{\chi_1,\chi_2}(x,T) \sim \frac{\varphi(q)}{2\pi} T \log(qT), \tag{7}$$

*uniformly for*

$$q \le \min(\sqrt{x}\log^{-3} x, x^{1-\eta}\log x) \quad \textit{and} \quad x^\eta \le T < \frac{x}{q}\log x.$$

We remark that in Conjecture 1 and in his results about the mean square of primes in arithmetic progressions, Yıldırım assumed that $q$ is prime in order to avoid the contribution of imprimitive characters; we do not have such a limitation in our application.

We prove the following result, which is inspired by [11, eq. 3].

**Theorem 2.** *Let $\varepsilon > 0$. Under GRH, as $x \to \infty$, we have*

$$F_q(x,T) \ll \varphi(q) T \log x,$$

*uniformly for*

$$1 \le q \le x^{1-\varepsilon} \quad \textit{and} \quad x \le T \le \exp(\sqrt{x}).$$

**Remark 1.** In fact, by using the Brun-Titchmarsh estimate in its whole range of validity, it is easy to see that the $q$-uniformity range in Theorem 2 can be slightly extended to $1 \le q \le x \log^{-(1+\varepsilon)} x$, at the cost of having the slightly worse estimate $F_q(x,T) \ll \varphi(q) T \log^2 x / \log\log x$.

The different uniformity ranges for $q, T$ in our result compared with the ones in Yıldırım's are due to the fact we wish to work with $q$ larger than $\sqrt{x}$, so outside the range in which the Bombieri-Vinogradov theorem works. As a consequence, having $q$ so large forces us to have $T$ larger than $x$ (instead of $x/q \log x$ as in Yıldırım's result) and to obtain an upper bound for $F_q(x,T)$ instead than an asymptotic formula.

Motivated by conjectures made by Montgomery, see (3), and Yıldırım, see (7), for the remaining values of $T$ with respect to $x$, we believe:

**Conjecture 2.** *Let $\varepsilon > 0$ fixed. Under GRH, as $x \to \infty$, we have*

$$F_q(x,T) \ll \varphi(q) T x^{\varepsilon}, \tag{8}$$

*uniformly for*

$$1 \le q \le x^{1-\varepsilon} \quad \textit{and} \quad x^{\varepsilon} \le T < x.$$

In Theorem 3 we refine Theorem 1. In it the allowed $T$-range is extended, as we show that the logarithmic term from the lower bound for $T$ can be removed by utilizing the Goldston-Montgomery version of Montgomery-Vaughan's mean value theorem for Dirichlet series (see Lemma 3), instead of the original version given in [17]. Furthermore, correcting a typo in [25] as mentioned in Footnote 3 we are able to replace $\exp(x^{1/4})$ by $\exp(x^{3/4})$ in the upper bound for $T$.

**Theorem 3.** *Let $\varepsilon > 0$. Under GRH, as $x \to \infty$, we have*

$$F_q(x,T) \sim \frac{\varphi(q)}{\pi} T \log x \tag{9}$$

*uniformly for*

$$1 \le q \le \sqrt{x}\ \log^{-(2+\varepsilon)} x, \quad \textit{and} \quad \frac{x}{\varphi(q)} \le T \le \exp(x^{3/4}).$$

Comparing (6) with (9) one can wonder about a missing factor 2 at the denominator but this is justified by the fact that we are summing the imaginary parts of the zeros over a range which is twice the one used in (6).

Note that the asymptotic estimate proved in Theorem 3 is in the $q$-aspect the square root of the trivial estimate (5). In achieving this the orthogonality of the Dirichlet characters plays a key role.

Besides the consequences on the simplicity of the zeros mentioned at the beginning of this introduction, it is not surprising that these results and conjectures can give information on the distribution of primes too. In 1978, Gallagher and Mueller [5] were the first to establish a result connecting the pair correlation of zeros of $\zeta(s)$ and prime number distribution. They proved that assuming RH, and pair correlation conjecture for zeros of Riemann zeta function,

$$\psi(x) = x + o(\sqrt{x} \log^2 x) \text{ as } x \to \infty.^{1} \tag{10}$$

This improves on the classic result that, assuming RH, the above estimate holds with error term $O(\sqrt{x}\log^2 x)$. However, In 1982, Heath-Brown [9] was able to obtain the estimate in (10) assuming the weaker hypothesis $F(x,T) = o(T \log^2 T)$, as $T \to \infty$, uniformly for $T \le x \le T^A$ for

[1] We recall some standard prime number theory notations in Section 2.

every constant $A \geq 1$. In the same year, Goldston and Montgomery [7, Thm 2] established an equivalence between an asymptotic result for the distribution of primes and the pair correlation conjecture (3). Assuming Conjecture 1, Yıldırım [25] was able to establish, following the work of Goldston-Montgomery [7], an asymptotic formula for the mean square of primes in arithmetic progressions.

We recall that in 1837, Dirichlet (see for example [18, Chp. 2]) proved that there are infinitely many primes congruent to $a \pmod q$ for $(a,q)=1$. The method used by Hadamard to prove the Prime Number Theorem allowed him to show also, cf. [18, p. 206], that

$$\psi(x;q,a) - \frac{x}{\varphi(q)} = o\left(\frac{x}{q}\right), \tag{11}$$

as $x \to \infty$. In 1936 Walfisz proved, for any fixed $A > 0$, that (11) holds uniformly for $q \leq \log^A x$. Under GRH, for any $(a,q)=1$ it is known, uniformly for $1 \leq q < x$, that

$$\psi(x;q,a) - \frac{x}{\varphi(q)} = O\big(\sqrt{x}\,\log^2 x\big) \tag{12}$$

as $x \to \infty$. See for instance, [2, p. 125]. Moreover, estimate (12) implies that under GRH, for any $\varepsilon > 0$, $(a,q)=1$, the estimate (11) holds uniformly for $1 \leq q \leq \sqrt{x}\log^{-(2+\varepsilon)} x$, as $x \to \infty$.

Friedlander and Granville [4, Cor. 2] proved that the estimate (11) can not hold uniformly for the range $1 \leq q < x/\log^B x$, where $B > 0$ is any arbitrary fixed number. In the same paper they also formulated the following corrected form of another conjecture of Montgomery, see [4, p. 366].

**Conjecture 3 (Montgomery).** *For any $\varepsilon > 0$, $(a,q)=1$, uniformly for $1 \leq q < x$ we have*

$$\psi(x;q,a) - \frac{x}{\varphi(q)} \ll \sqrt{\frac{x}{q}}\, x^{\varepsilon}. \tag{13}$$

For $1 \leq q \leq x^{\varepsilon}$ it is clear that equation (13) follows immediately from (12). Hence it suffices to restrict our analysis to the range $x^{\varepsilon} < q < x$.

Our main result states that if both GRH and Conjecture 2 hold true, then the estimate (13) holds in almost the whole $q$-range given in Montgomery's Conjecture 3.

**Theorem 4.** *Let $\varepsilon > 0$ and $(a,q)=1$. Assume that both GRH and Conjecture 2 hold true. Then, (13) holds for $x$ sufficiently large and uniformly for $1 \leq q \leq x^{1-\varepsilon}$.*

Another very important conjecture on the distribution of primes is:

**Conjecture 4 (Elliott-Halberstam [3]).** *For every $\varepsilon > 0$ and $A > 0$,*

$$\sum_{q \leq x^{1-\varepsilon}} \max_{(a,q)=1} \left| \psi(x;q,a) - \frac{x}{\varphi(q)} \right| \ll_{A,\varepsilon} \frac{x}{(\log x)^A}.$$

If true, this is close to best possible as Friedlander and Granville [4], using the Maier matrix method [14], showed the conjecture to be false when $x^{1-\epsilon}$ is replaced by $x\log^{-B} x$, with $B > 0$ arbitrary. Indeed, Maier's matrix method can be used to establish limitations to a very successful probabilistic prime number distribution model of Cramèr, cf. Granville [8].

So far, an estimate like the one in Conjecture 4 is known only in the interval $q \leq \sqrt{x}\log^{-B} x$, with $B > 0$ thanks to the Bombieri-Vinogradov result.

Theorem 4 has the following straightforward elegant corollary supporting the Elliott-Halberstam conjecture.

**Corollary 5.** *Let $\varepsilon > 0$ and $(a, q) = 1$. Assume that both GRH and Conjecture 2 hold true. Then, for $x$ sufficiently large, we have*

$$\sum_{q \le x^{1-4\epsilon}} \max_{(a,q)=1} \left| \psi(x; q, a) - \frac{x}{\varphi(q)} \right| \ll x^{1-\varepsilon}.$$

We remark that the $x^{\varepsilon}$-factor present on the right hand side of (13) implies that the $q$-range of the sum in Corollary 5 cannot be further enlarged. In fact, assuming the truth of Conjecture 2 in a $q$-uniformity range larger than the one stated would have no consequence on Corollary 5. We hence decided to assume the the truth of Conjecture 2 only in the uniformity ranges strictly needed to obtain Corollary 5.

Moreover, it is also possible to show that a weaker form of (13) follows by assuming a weaker form of the pair correlation conjecture for the Dirichlet $L$-functions. This weaker form involves an additional factor satisfying $1 \le g(q) \le \varphi(q)$ in the upper bound, the original Conjecture 2 arising on putting $g(q) = 1$ and, essentially, the trivial estimate for $F_q(x, T)$, equation (5), on choosing $g(q) = \varphi(q)$.

**Conjecture 5 (Weak pair correlation for Dirichlet $L$-functions).** *Let $\varepsilon > 0$ fixed and $g(q)$ be an arithmetic function satisfying $1 \le g(q) \le \varphi(q)$. Under GRH, as $x \to \infty$, we have*

$$F_q(x, T) \ll \varphi(q) g(q) T x^{\varepsilon}$$

*uniformly for*

$$1 \le q \le x^{1-\varepsilon} \quad \textit{and} \quad x^{\varepsilon} \le T < x.$$

**Theorem 6.** *Let $\varepsilon > 0$, $(a, q) = 1$ and $g(q)$ be an arithmetic function satisfying $1 \le g(q) \le \varphi(q)$. Assume that both GRH and Conjecture 5 hold true. Then, uniformly for $1 \le q \le x^{1-\varepsilon}$ we have, for sufficiently large $x$, that*

$$\psi(x; q, a) - \frac{x}{\varphi(q)} \ll \sqrt{\frac{x g(q)}{q}}\, x^{\varepsilon}.$$

For instance, if $g(q) = \varphi(q)^{\alpha}$, $0 \le \alpha \le 1$, then assuming GRH and Conjecture 5, Theorem 6 implies that

$$\sum_{q \le Q} \max_{(a,q)=1} \left| \psi(x; q, a) - \frac{x}{\varphi(q)} \right| \ll x^{1-\varepsilon}$$

for $Q := Q(\alpha, x) \le x^{1/(1+\alpha)-4\varepsilon}$. Note that $x^{1/2-4\varepsilon} \le Q \le x^{1-4\varepsilon}$ for such $\alpha$'s.

The paper is organised as follows. In §2, we recall results we need to prove our theorems. In §3, we prove Theorems 2–3 and the §4 is devoted to proofs of the main results of this paper, i.e, Theorems 4 and 6. The crux of the proof of Theorem 4 lies in the following observations. As noted earlier, the conjectural estimate represents a substantial improvement compared to the trivial bound in both the $q$ and $T$ aspects when discussing the pair correlation of zeros of Dirichlet $L$-functions (see (5), (7) & (8)). On the other hand, it is relatively less significant in the context of the correlation of zeros of the Riemann zeta. This leads to the following key point: while proving Theorem 4, the analysis proceeds smoothly in case of "not too small" zeros, aligning with the case of modulus 1 (see [9, Thm 1]). However, when accounting for the contributions of low-lying zeros, the trivial estimate becomes inadequate. To address this difficulty, we dissect the problem into dyadic intervals and subsequently transition from local to global intervals. For more details, please refer to §4.

## 2. Preliminaries

2.1. **Prime number distribution.** In this section, we recall the material we need on the distribution of prime numbers, using the notations

$$\pi(t) = \sum_{p \le t} 1, \qquad \pi(t; q, a) = \sum_{\substack{p \le t \\ p \equiv a \pmod q}} 1,$$

and

$$\psi(t) = \sum_{n \le t} \Lambda(n), \qquad \psi(t; q, a) = \sum_{\substack{n \le t \\ n \equiv a \pmod q}} \Lambda(n), \qquad \psi(t, \chi) = \sum_{n \le t} \Lambda(n)\chi(n),$$

where $\Lambda$ denotes the von Mangoldt function and $\chi$ a Dirichlet character $\pmod q$. For fixed coprime integers $a$ and $q$, we have asymptotic equidistribution:

$$\pi(t; q, a) \sim \frac{\pi(t)}{\varphi(q)}, \quad \text{and} \quad \psi(t; q, a) \sim \frac{\psi(t)}{\varphi(q)} \quad (t \to \infty),$$

with $\varphi$ Euler's totient.

An important tool we will use is the following theorem (for a proof, see, e.g., Montgomery-Vaughan [16, Theorem 2]).

**Classical Theorem 1 (Brun-Titchmarsh theorem).** *Let $x, y > 0$ and $a, q$ be positive integers such that $(a, q) = 1$. Then, uniformly for all $y > q$, we have*

$$\pi(x + y; q, a) - \pi(x; q, a) < \frac{2y}{\varphi(q) \log(y/q)}.$$

Starting point for our deliberations is an explicit truncated form of the von Mangoldt explicit formula, which we state in the classical, respectively Dirichlet $L$-function case.

**Lemma 1.** [2, Ch. 17] *Let $2 \le Z \le x$. Assuming RH, we have, as $x \to \infty$,*

$$\psi(x) = x - \sum_{\substack{|\gamma| \le Z \\ \zeta(1/2 + i\gamma) = 0}} \frac{x^{1/2 + i\gamma}}{1/2 + i\gamma} + O\Big(\frac{x \log^2(xZ)}{Z}\Big).$$

**Lemma 2.** [2, Ch. 19] *If $\chi$ is a non principal character modulo $q$ and $2 \le Z \le x$, then assuming GRH, we have, as $x \to \infty$,*

$$\psi(x, \chi) = -\sum_{|\gamma| \le Z} \frac{x^{1/2 + i\gamma}}{1/2 + i\gamma} + O\Big(\frac{x \log^2(qx)}{Z}\Big).$$

We will apply the Cauchy-Schwarz inequality ($L^2$-norm form) multiple times in our proofs so we record it here: For any square-integrable complex valued functions $f$ and $g$, we have

$$\Big|\int_a^b f(t)g(t)dt\Big| \le \Big(\int_a^b |f(t)|^2 dt\Big)^{\frac{1}{2}} \Big(\int_a^b |g(t)|^2 dt\Big)^{\frac{1}{2}}.$$

Now we record the following crucial lemma, an analogue of Montgomery-Vaughan's mean value theorem for Dirichlet series [17].

**Lemma 3.** [7, 12] *Let $\mathcal{S}(t) = \sum_{\mu \in \mathcal{M}} c(\mu) e^{2\pi i \mu t}$ be a Fourier series with $\mathcal{M}$ be a countable set of real numbers and with $c(\mu)$ real Fourier coefficients. If $\sum_{\mu \in \mathcal{M}} |c(\mu)| < \infty$, then uniformly for*

*$T \geq 1$ and $1/2T \leq \delta \leq 1/2$, we have*

$$\int_{-T}^{T} |\mathcal{S}(t)|^2 dt = \big(2T + O(\delta^{-1})\big) \sum_{\mu\in\mathcal{M}} |c(\mu)|^2 + O\big(T \sum_{\substack{\mu,\nu\in\mathcal{M}\\ 0<|\mu-\nu|<\delta}} |c(\mu)c(\nu)|\big).$$

## 3. Proofs of Theorems 2–3

The idea of the proof of Theorem 2 is to integrate both sides of the identity (14) from $-T$ to $T$. The left hand side then gives $2\pi F_q(x,T)$ plus an error term. For the right hand side an upper bound is derived. This is done by finding upper bounds for $\int_{-T}^{T} |R_j(x,t)|^2 dt$ for $j = 1,\ldots,4$ and then applying the Cauchy-Schwarz inequality.

*Proof of Theorem 2.* Under GRH, following the argument in Landau [10, p. 353] we obtain for $q \geq 1$, $(a,q) = 1$, $x > 1$, and for primitive characters $\chi$ (when $q > 1$) that

$$\sum_{n\leq x}{}' \frac{\Lambda(n)\chi(n)}{n^s} = \delta_q \frac{x^{1-s}}{1-s} + \sum_{\ell=0}^{\infty} \frac{x^{-2\ell-s-\mathfrak{a}}}{2\ell+s+\mathfrak{a}} - \sum_{\rho} \frac{x^{\rho-s}}{\rho-s} - \frac{L'}{L}(s,\chi),$$

where $'$ over the summation means only half of the term with $n = x$ is included in the sum, the sum over the non-trivial zeroes is interpreted in the symmetrical sense as $\lim_{Z\to\infty}\sum_{|\gamma|<Z}$, $s \in \mathbb{C}$, $s \neq 1$, $s \neq \rho$, $s \neq -(2\ell+\mathfrak{a})$, $\mathfrak{a} = 0$ if $\chi(-1) = 1$, $\mathfrak{a} = 1$ if $\chi(-1) = -1$, and $\delta_q = 1$ if $q = 1$ and $\delta_q = 0$ otherwise.[2] Letting $s = 3/2 + it$, following the proof of Montgomery [15, p. 185–186], and summing over the Dirichlet characters, we can write

$$\Big| \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi}(a) \sum_{\gamma} \frac{2x^{i\gamma}}{1+(t-\gamma)^2}\Big|^2 = \Big|\sum_{j=1}^{4} R_j(x,t)\Big|^2, \tag{14}$$

where

$$R_1(x,t) = -\frac{1}{\sqrt{x}} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi}(a)\Big(\sum_{n\leq x} \Lambda(n)\chi(n) \left(\frac{x}{n}\right)^{-\frac{1}{2}+it} + \sum_{n>x} \Lambda(n)\chi(n) \left(\frac{x}{n}\right)^{\frac{3}{2}+it}\Big),$$

$$R_2(x,t) = x^{-1+it} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi}(a)(\log(q^*\tau) + O(1)),$$

$$R_3(x,t) = O(x^{-\frac{1}{2}}\tau^{-1}\varphi(q) + \delta_q x^{\frac{1}{2}}\tau^{-2}),$$

$$R_4(x,t) = \frac{1}{\sqrt{x}} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi}(a)\Big( \sum_{\substack{n\leq x\\ \chi(n)\neq\chi^*(n)}} \Lambda(n)\chi^*(n) \left(\frac{x}{n}\right)^{-\frac{1}{2}+it} + \sum_{\substack{n>x\\ \chi(n)\neq\chi^*(n)}} \Lambda(n)\chi^*(n) \left(\frac{x}{n}\right)^{\frac{3}{2}+it}\Big).$$

The precise definition of $R_2$ and $R_3$ is not relevant for our purposes, the estimates above will suffice. Here $\tau = |t| + 2$, and $\chi$ (mod $q$) is a character induced by the primitive character $\chi^*$ (mod $q^*$). The last term in the parentheses is a correction term for non primitive $\chi$[3]. Using the orthogonality of Dirichlet characters, we see that

$$R_1(x,t) = -\frac{\varphi(q)}{\sqrt{x}}\Big( \sum_{\substack{n\leq x\\ n\equiv a \ (\mathrm{mod}\ q)}} \Lambda(n) \left(\frac{x}{n}\right)^{-\frac{1}{2}+it} + \sum_{\substack{n>x\\ n\equiv a \ (\mathrm{mod}\ q)}} \Lambda(n) \left(\frac{x}{n}\right)^{\frac{3}{2}+it}\Big), \tag{15}$$

[2]The term with $\delta_q$ gives the contribution of the pole at 1 of the Riemann zeta function, see the proof of eq. (18) in [15], and it should be inserted into [25, formula (6)] too.

[3]Remark that in the analogous formula in [25, eq. (8)] in the contribution for primitive characters for $x^{1/2}$ one should read $x^{-1/2}$.

and

$$
\begin{aligned}
&R_1(x,t)+R_4(x,t)\\
&=-\frac{1}{\sqrt{x}}\sum_{\chi \pmod q}\overline{\chi}(a)\Big(\sum_{\substack{n\le x\\ \chi(n)=\chi'(n)}}\Lambda(n)\chi(n)\left(\frac{x}{n}\right)^{-\frac12+it}+\sum_{\substack{n>x\\ \chi(n)=\chi'(n)}}\Lambda(n)\chi(n)\left(\frac{x}{n}\right)^{\frac32+it}\Big)\\
&=-\frac{1}{\sqrt{x}}\sum_{\chi \pmod q}\overline{\chi}(a)\Big(\sum_{\substack{n\le x\\ (n,q)=1}}\Lambda(n)\chi(n)\left(\frac{x}{n}\right)^{-\frac12+it}+\sum_{\substack{n>x\\ (n,q)=1}}\Lambda(n)\chi(n)\left(\frac{x}{n}\right)^{\frac32+it}\Big)\\
&=-\frac{\varphi(q)}{\sqrt{x}}\Big(\sum_{\substack{n\le x\\ n\equiv a \pmod q}}\Lambda(n)\left(\frac{x}{n}\right)^{-\frac12+it}+\sum_{\substack{n>x\\ n\equiv a \pmod q}}\Lambda(n)\left(\frac{x}{n}\right)^{\frac32+it}\Big)=R_1(x,t),
\end{aligned}
$$

so $R_4(x,t)=0$.

We integrate both sides of equation (14) from $t=-T$ to $t=T$, where $T$ will be specified later. The left hand side of equation (14) can be written as

$$\sum_{\chi_1,\chi_2}\overline{\chi_1}(a)\chi_2(a)\sum_{\gamma_j,\ j=1,2}\frac{4x^{i(\gamma_1-\gamma_2)}}{(1+(t-\gamma_1)^2)(1+(t-\gamma_2)^2)}.$$

Now we integrate this expression from $t=-T$ to $t=T$. We claim that

$$
\begin{aligned}
&\int_{-T}^{T}\sum_{\chi_1,\chi_2}\overline{\chi_1}(a)\chi_2(a)\sum_{\gamma_j,\ j=1,2}\frac{4x^{i(\gamma_1-\gamma_2)}}{(1+(t-\gamma_1)^2)(1+(t-\gamma_2)^2)}dt\\
&=\int_{-\infty}^{\infty}\sum_{\chi_1,\chi_2}\overline{\chi_1}(a)\chi_2(a)\sum_{\substack{|\gamma_j|\le T\\ j=1,2}}\frac{4x^{i(\gamma_1-\gamma_2)}}{(1+(t-\gamma_1)^2)(1+(t-\gamma_2)^2)}dt+O(\varphi(q)^2\log T\log^2(qT))\\
&=2\pi F_q(x,T)+O(\varphi(q)^2\log T\log^2(qT)). \qquad (16)
\end{aligned}
$$

To prove equation (16), we first recall that for $\chi$ mod $q$, there are $\ll \log(qT)$ zeros such that $L(1/2+i\gamma,\chi)=0$ and $T\le\gamma\le T+1$, $T\ge 2$. This implies for $|t|\le T$ that

$$\sum_{|\gamma|>T}\frac{1}{1+(t-\gamma)^2}\ll\frac{\log(qT)}{T-t+1}. \qquad (17)$$

We now recall [2, Ch. 16, Lemma], i.e.,

$$\sum_{\gamma}\frac{1}{1+(t-\gamma)^2}\ll\log(q\tau). \qquad (18)$$

Using (17)-(18) we obtain that

$$\int_{-T}^{T}\sum_{\chi_1,\chi_2}\overline{\chi_1}(a)\chi_2(a)\sum_{\substack{\gamma_1,\gamma_2\\ |\gamma_2|>T}}\frac{4x^{i(\gamma_1-\gamma_2)}}{(1+(t-\gamma_1)^2)(1+(t-\gamma_2)^2)}dt\ll\varphi(q)^2(\log T)\log^2(qT), \qquad (19)$$

For $|t|>T$, we have

$$\sum_{|\gamma|\le T}\frac{1}{1+(t-\gamma)^2}\ll\frac{\log(q|t|)}{|t|-T+1},$$

so that

$$\int_{|t|>T} \sum_{\substack{|\gamma_j|\le T\\ j=1,2}} \frac{dt}{(1+(t-\gamma_1)^2)(1+(t-\gamma_2)^2)} \ll \int_{|t|>T} \frac{\log^2(q|t|)}{(|t|-T+1)^2} dt \ll \log^2(qT).$$

This implies that

$$\int_{|t|>T} \sum_{\chi_1,\chi_2} \overline{\chi_1}(a)\chi_2(a) \sum_{\substack{|\gamma_j|\le T\\ j=1,2}} \frac{4x^{i(\gamma_1-\gamma_2)}}{(1+(t-\gamma_1)^2)(1+(t-\gamma_2)^2)} dt \ll \varphi(q)^2 \log^2(qT). \tag{20}$$

Now (16) follows on combining equations (19)–(20) and applying Cauchy's residue theorem to evaluate the second integral in (16).

Integrating the right hand side of equation (14) from $t=-T$ to $t=T$, we obtain[4]

$$\int_{-T}^{T} |R_2(x,t)|^2 dt \ll \frac{\varphi(q)^2 T \log^2(qT)}{x^2}, \tag{21}$$

and

$$\int_{-T}^{T} |R_3(x,t)|^2 dt \ll \frac{\varphi(q)^2}{x} + x \ll x \tag{22}$$

for $1 \le q \le x$.

The mean square of $R_1(x,t)$ is evaluated in the following lemma.

**Lemma 4.** *As $x \to \infty$ and uniformly for $q \le x^{1-\varepsilon}$ and $x/\varphi(q) \le T$ we have that*

$$\frac{1}{\varphi(q)^2} \int_{-T}^{T} |R_1(x,t)|^2 dt = 2TS(x) + O\Big(\frac{\sqrt{TxS(x)}}{\varphi(q)}\Big), \tag{23}$$

*where*

$$S(x) = \frac{1}{x^2} \sum_{\substack{n\le x\\ n\equiv a \ (\mathrm{mod}\ q)}} n\Lambda(n)^2 + x^2 \sum_{\substack{n>x\\ n\equiv a \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)^2}{n^3}.$$

*Proof.* Recalling the definition of $R_1(x,t)$ given in (15), we follow the argument given in [12]. Using Lemma 3, we have

$$\frac{1}{\varphi(q)^2} \int_{-T}^{T} |R_1(x,t)|^2 dt = 2TS(x) + O(\delta^{-1}S(x) + E_1 + E_2), \tag{24}$$

where

$$E_1 := T \sum_{\substack{n<x\\ n\equiv a \ (\mathrm{mod}\ q)}} \Big( \sum_{\substack{m\le x\\ m\equiv a \ (\mathrm{mod}\ q)\\ 0<|\log(n/m)|<2\pi\delta}} \Lambda(n)\Lambda(m) \frac{(nm)^{1/2}}{x^2} + \sum_{\substack{m>x\\ m\equiv a \ (\mathrm{mod}\ q)\\ 0<|\log(n/m)|<2\pi\delta}} \Lambda(n)\Lambda(m) \frac{n^{1/2}}{m^{3/2}} \Big),$$

[4]Equation (21) is the analogue of the first of the two integral estimates appearing in the middle of [25, p. 330], where for $x$ in both argument and bound one should read $x^2$. Taking this into account we arrive at a better upper bound for $T$ in Theorem 3.

and

$$E_2 := T\sum_{r=0}^{\infty}\sum_{\substack{n=2^r x\\ n\equiv a \ (\mathrm{mod}\ q)}}^{2^{r+1}x}\Big(\sum_{\substack{m\le x\\ m\equiv a \ (\mathrm{mod}\ q)\\ 0<|\log(n/m)|<2\pi\delta}}\Lambda(n)\Lambda(m)\frac{m^{1/2}}{n^{3/2}}+\sum_{\substack{m>x\\ m\equiv a \ (\mathrm{mod}\ q)\\ 0<|\log(n/m)|<2\pi\delta}}\Lambda(n)\Lambda(m)\frac{x^2}{(nm)^{3/2}}\Big).$$

Using the Brun-Titchmarsh Classical Theorem 1 we obtain

$$S(x)\ll\frac{\log x}{\varphi(q)}, \tag{25}$$

as $x\to\infty$, uniformly in $q\le x^{1-\varepsilon}$. Choosing

$$\delta=\frac{\varphi(q)}{2}\sqrt{\frac{S(x)}{Tx}}, \tag{26}$$

for sufficiently large $x$ we have that $1/T\le 2\delta\le 1$, when $x/\varphi(q)\le T$ and $q\le x^{1-\varepsilon}$.

Uniformly for $q\le x^{1-\varepsilon}$, we have, for sufficiently large $x$, that

$$\begin{aligned}E_1&\ll\frac{T}{x}\sum_{k\le 10^4\delta x}\sum_{\substack{n\le 10^4 x\\ n\equiv a \ (\mathrm{mod}\ q)}}\sum_{\substack{m\le 10^4 x\\ m\equiv a \ (\mathrm{mod}\ q)\\ |n-m|=k}}\Lambda(n)\Lambda(m)\ll\frac{T}{x}\sum_{\substack{n\le 10^4 x\\ n\equiv a \ (\mathrm{mod}\ q)}}\Lambda(n)\sum_{\substack{n<m\le n+10^4\delta x\\ m\equiv a \ (\mathrm{mod}\ q)}}\Lambda(m)\\ &\ll\frac{T}{x}\sum_{\substack{n\le 10^4 x\\ n\equiv a \ (\mathrm{mod}\ q)}}\Lambda(n)(\psi(n+10^4\delta x;q,a)-\psi(n;q,a))\ll\frac{T\delta x}{\varphi(q)^2},\end{aligned} \tag{27}$$

where we have used the Brun-Titchmarsh Classical Theorem 1 in the final step. Similarly, we also obtain

$$E_2\ll\frac{T}{x}\sum_{r=0}^{\infty}\frac{1}{2^{3r}}\sum_{k\le 10^4 2^{r+1}\delta x}\sum_{\substack{n\le 10^4 2^{r+1}x\\ n\equiv a \ (\mathrm{mod}\ q)}}\sum_{\substack{m\le 10^4 2^{r+1}x\\ m\equiv a \ (\mathrm{mod}\ q)\\ |n-m|=k}}\Lambda(n)\Lambda(m)\ll\frac{T\delta x}{\varphi(q)^2}, \tag{28}$$

as $x\to\infty$, uniformly in $q\le x^{1-\varepsilon}$. Combining (24) and (27)–(28), and substituting the value of $\delta$ as in (26), we have the proof. □

From Lemma 4, see (23), and (25) it follows that

$$\int_{-T}^{T}|R_1(x,t)|^2dt\ll\varphi(q)T\log x+(\varphi(q)Tx\log x)^{1/2}, \tag{29}$$

as $x\to\infty$, uniformly in the range $q\le x^{1-\varepsilon}$.

Using the Cauchy-Schwarz inequality, we can easily obtain that

$$\begin{aligned}\int_{-T}^{T}\Big|\sum_{j=1}^{3}R_j(x,t)\Big|^2dt&=\sum_{j=1}^{3}\int_{-T}^{T}\big|R_j(x,t)\big|^2dt\\ &\quad+O\Big(\sum_{j=1}^{3}\sum_{\substack{k=1\\ k\ne j}}^{3}\Big(\int_{-T}^{T}\big|R_j(x,t)\big|^2dt\Big)^{1/2}\Big(\int_{-T}^{T}\big|R_k(x,t)\big|^2dt\Big)^{1/2}\Big).\end{aligned} \tag{30}$$

Combining (21)–(22) and (29)–(30) it follows that

$$\int_{-T}^{T} \Big|\sum_{j=1}^{3} R_j(x,t)\Big|^2 dt \ll \varphi(q)T\log x + (\varphi(q)Tx\log x)^{1/2} + x + \frac{\varphi(q)^2 T\log^2(qT)}{x^2}$$
$$\ll \varphi(q)T\log x + (\varphi(q)Tx\log x)^{1/2}, \tag{31}$$

uniformly in the range $q \le x^{1-\varepsilon}$ and $x/\varphi(q) \le T \le \exp(\sqrt{x})$ as $x \to \infty$. Recalling (16) we also need that $\varphi(q)\log T\log^2(qT) = o(T\log x)$, as $x \to \infty$, and this implies $T \ge x$. Summarising, (16) and (31) imply that

$$F_q(x,T) \ll \varphi(q)T\log x,$$

uniformly in the range $q \le x^{1-\varepsilon}$ and $x \le T \le \exp(\sqrt{x})$ as $x \to \infty$. □

*Proof of Theorem 3.* Using the Prime Number Theorem in arithmetic progressions under GRH in $S(x)$ (defined in Lemma 4), we have

$$S(x) = \frac{\log x}{\varphi(q)} + O\Big(\frac{\log^3 x}{\sqrt{x}}\Big), \tag{32}$$

as $x \to \infty$, uniformly in $q \le \sqrt{x}\log^{-(2+\varepsilon)} x$. From Lemma 4, see (23), and (32) it follows that

$$\int_{-T}^{T} |R_1(x,t)|^2 dt = 2\varphi(q)T\log x + O\Big(\frac{\varphi(q)^2 T\log^3 x}{x^{1/2}}\Big) + O\Big(\big(\varphi(q)Tx\log x(1+\frac{\varphi(q)\log^2 x}{x^{1/2}})\big)^{1/2}\Big)$$
$$= 2\varphi(q)T\log x(1+o(1)) + O\big((\varphi(q)Tx\log x)^{1/2}\big), \tag{33}$$

as $x \to \infty$, uniformly in $q \le \sqrt{x}\log^{-(2+\varepsilon)} x$.

From (21)–(22) and (30)–(33) we obtain

$$\int_{-T}^{T} \Big|\sum_{j=1}^{3} R_j(x,t)\Big|^2 dt = 2\varphi(q)T\log x(1+o(1)) + O\Big((\varphi(q)Tx\log x)^{1/2} + x + \frac{\varphi(q)^2 T\log^2(qT)}{x^2}\Big)$$
$$= 2\varphi(q)T\log x(1+o(1)), \tag{34}$$

as $x \to \infty$, uniformly in $q \le \sqrt{x}\log^{-(2+\varepsilon)} x$ when $x/\varphi(q) \le T \le \exp(x^{3/4})$. In this range of $q$ and $T$, we also have $\varphi(q)\log T\log^2(qT) = o(T\log x)$, as $x \to \infty$. Equations (16) and (34) imply that

$$F_q(x,T) \sim \frac{\varphi(q)}{\pi} T\log x,$$

as $x \to \infty$, uniformly in $q \le \sqrt{x}\log^{-(2+\varepsilon)} x$ when $x/\varphi(q) \le T \le \exp(x^{3/4})$. □

## 4. Proof of Theorems 4 and 6

Let $(a,q) = 1$. It is a simple consequence of the orthogonality of Dirichlet characters that

$$\psi(x;q,a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)\psi(x,\chi)$$

Lemmas 1 and 2 imply that under GRH, and for $Z \le x$ that

$$\psi(x;q,a) = \frac{\psi(x,\chi_0)}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0 \pmod q} \overline{\chi}(a)\psi(x,\chi)$$
$$= \frac{\psi(x) + O(\log q\,\log x)}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0 \pmod q} \overline{\chi}(a)\psi(x,\chi)$$

$$= \frac{1}{\varphi(q)}\Big(x - \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{|\gamma|\le Z} \frac{x^{1/2+i\gamma}}{1/2+i\gamma}\Big) + O\Big(\frac{x\log^2(qx)}{Z}\Big), \tag{35}$$

as $x \to \infty$. Note that if $Z = x$ and $q \le x^{1-\varepsilon}$, then the error term in the previous estimate becomes $O(\log^2 x)$.

We use now the analogue of several ideas contained in a paper of Heath-Brown [9]. Letting

$$\Sigma(x,T,v) = \Sigma(x,T,v;q,a) = \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{|\gamma|\le T} x^{i\gamma} e^{iv\gamma}, \tag{36}$$

remarking $W(u) = \int_{-\infty}^{+\infty} e^{ivu} e^{-2|v|} dv$ and arguing as in [11], it is easy to obtain that

$$F_q(x,T) = \int_{-\infty}^{+\infty} |\Sigma(x,T,v)|^2 \, e^{-2|v|} dv. \tag{37}$$

**Lemma 5.** *For $x \ge 2$ and $T > U \ge 0$, we have*

$$|\Sigma(x,T,0) - \Sigma(x,U,0)| \ll \big(T \max_{U\le t\le T} F_q(x,t)\big)^{1/2}.$$

*Proof.* Define

$$g(x,T,U,v) = \Sigma(x,T,v) - \Sigma(x,U,v) = \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{U<|\gamma|\le T} x^{i\gamma} e^{i\gamma v}. \tag{38}$$

Then

$$\begin{aligned}
\int_{-\infty}^{\infty} |g(x,T,U,v)|^2 e^{-2|v|} dv &= \sum_{\chi_1,\chi_2 \pmod q} \overline{\chi}_1(a)\chi_2(a) \sum_{\substack{U<|\gamma_j|\le T\\ j=1,2}} x^{i(\gamma_1-\gamma_2)} \int_{-\infty}^{\infty} e^{i(\gamma_1-\gamma_2)v} e^{-2|v|} dv \\
&= \sum_{\chi_1,\chi_2 \pmod q} \overline{\chi}_1(a)\chi_2(a) \sum_{\substack{U<|\gamma_j|\le T\\ j=1,2}} x^{i(\gamma_1-\gamma_2)} W(\gamma_1-\gamma_2) \\
&= \sum_{\chi_1,\chi_2 \pmod q} \overline{\chi}_1(a)\chi_2(a) \, \big(G_{\chi_1,\chi_2}(x,T) - G_{\chi_1,\chi_2}(x,U)\big) \\
&= F_q(x,T) - F_q(x,U).
\end{aligned}$$

Using the Sobolev-Gallagher inequality stated as in [11, Lemma 2], we have

$$|g(x,T,U,0)|^2 \ll T \max_{U\le t\le T} F_q(x,t).$$

The lemma now immediately follows from (38). ◻

**Lemma 6.** *For $x \ge 2$ and $U \ge 0$, we have*

$$\int_x^{2x} |\Sigma(t,U,0)|^2 dt \ll x F_q(x,U).$$

*Proof.* Recalling (36) and substituting $t = xe^v$, we have

$$\begin{aligned}
\int_x^{2x} |\Sigma(t,U,0)|^2 dt &= x \int_0^{\log 2} |\Sigma(x,U,v)|^2 e^v dv \\
&\ll x \int_{-\infty}^{+\infty} |\Sigma(x,U,v)|^2 e^{-2|v|} dv = x F_q(x,U),
\end{aligned}$$

where the final estimate is a consequence of (37). ◻

**Lemma 7.** *Let $\varepsilon > 0$, $(a,q) = 1$ and*
$$I(x,q,T) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{|\gamma|\le T} \frac{x^{1/2+i\gamma}}{1/2+i\gamma}.$$
*Let $q \ge x^{\varepsilon}$, $x > q^{1+2\varepsilon}$ and $J$ be the maximal integer such that $(x/2^J)^{1-\varepsilon} \ge q$. Assuming GRH and Conjecture 2 for $j = 0,\dots,J$ we have*
$$\Big|I\big(\frac{x}{2^j},q,T\big) - I\big(\frac{x}{2^{j+1}},q,T\big)\Big| \ll \sqrt{\frac{x}{2^j q}}\, x^{\varepsilon},$$
*uniformly for $q \le (x/2^j)^{1-\varepsilon}$ and $x^{\varepsilon} \le T \le x/2^j$.*

*Proof.* Let $U = x^{\varepsilon}$. Let $y = x/2^{j+1}$, $j = 0,\dots,J$, $q \le y^{1-\varepsilon}$. Using partial summation, Conjecture 2 and Lemma 5, for $c = 1,2$ we obtain
$$\begin{aligned}\Big|\sum_{\chi \pmod q} \overline{\chi}(a) \sum_{U<|\gamma|\le T} \frac{(cy)^{i\gamma}}{1/2+i\gamma}\Big| &\ll \frac{|\Sigma(cy,T,0) - \Sigma(cy,U,0)|}{T} + \int_U^T |\Sigma(cy,w,0) - \Sigma(cy,U,0)| \frac{dw}{w^2} \\ &\ll \frac{1}{\sqrt{T}}\Big(\max_{U<t\le T} F_q(cy,t)\Big)^{1/2} + \int_U^T w^{-3/2}\Big(\max_{U<t\le w} F_q(cy,t)\Big)^{1/2} dw \\ &\ll \sqrt{q}\, y^{\varepsilon} \ll \sqrt{q}\, x^{\varepsilon},\end{aligned}$$
as $x \to \infty$, uniformly for $q \le y^{1-\varepsilon}$. This implies that for $U \le T \le y$, we have
$$\frac{1}{\varphi(q)}\Big| \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{U<|\gamma|\le T} \frac{(2y)^{1/2+i\gamma} - y^{1/2+i\gamma}}{1/2+i\gamma}\Big| \ll \sqrt{\frac{y}{q}} x^{\varepsilon}, \tag{39}$$
as $x \to \infty$, uniformly in $q \le y^{1-\varepsilon}$.

Recall that $y = x/2^{j+1}$, $j = 0,\dots,J$. Using Conjecture 2 and Lemma 6, we have
$$\begin{aligned}\varphi(q)\big|I(2y,q,U) - I(y,q,U)\big| = \Big|\int_y^{2y} w^{-1/2}\Sigma(w,U,0)dw\Big| &\ll \Big(\int_y^{2y} |\Sigma(w,U,0)|^2 dw\Big)^{1/2} \\ &\ll (yF_q(y,U))^{1/2} \ll (qUy^{1+\varepsilon})^{1/2} \ll \sqrt{q}\, y^{1/2} x^{\varepsilon},\end{aligned} \tag{40}$$
as $x \to \infty$, where we have used the Cauchy-Schwarz inequality in the previous estimate. The lemma follows on combining (39)–(40) and recalling that $q/\varphi(q) \ll \log\log q$. □

*Proof of Theorem 4.* Let $x > q^{1+2\varepsilon}$ and $J$ be the maximal integer such that $(x/2^J)^{1-\varepsilon} \ge q$. Using Brun-Titchmarsh Theorem (Classical Theorem 1), Lemma 7 and equation (35) with for $x^{\varepsilon} \le q \le x^{1-2\varepsilon}$, $Z = x/2^j$, $j = 0,\dots,J$, we have
$$\begin{aligned}\psi(x;q,a) - \frac{x}{\varphi(q)} &= \sum_{j=0}^{J}\Big(\psi\big(\frac{x}{2^j};q,a\big) - \psi\big(\frac{x}{2^{j+1}};q,a\big) - \frac{x}{2^{j+1}\varphi(q)}\Big) + \psi\big(\frac{x}{2^{J+1}};q,a\big) - \frac{x}{2^{J+1}\varphi(q)} \\ &\ll \sum_{j=0}^{J} \sqrt{\frac{x}{2^j q}} x^{\varepsilon} + \frac{x}{2^{J+1}\varphi(q)} \ll \sqrt{\frac{x}{q}} x^{\varepsilon}.\end{aligned}$$

Theorem 4 now follows on recalling that in the remaining interval $1 \le q \le x^{\varepsilon}$ the result follows from GRH only. □

*Proof of Theorem 6.* The proof of Theorem 4 and Lemma 7 remains valid if we replace the Conjecture 2 with Conjecture 6. These changes result in $x^{1/2}q^{-1/2}x^{\varepsilon}$ being replaced by $(xg(q))^{1/2}q^{-1/2}x^{\varepsilon}$ in the final estimates. □

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