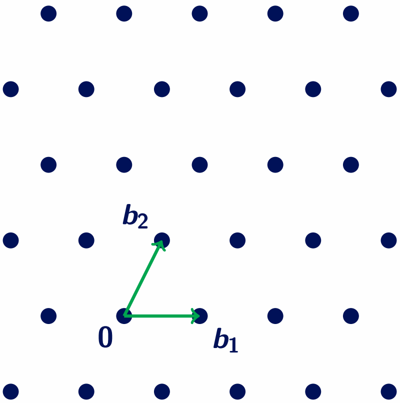
Dense (and smooth) lattices in any genus

Wessel van Woerden (Université de Bordeaux, IMB, Inria).



Lattice

$\mathbb{R}$-linearly independent $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$

$\mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$,

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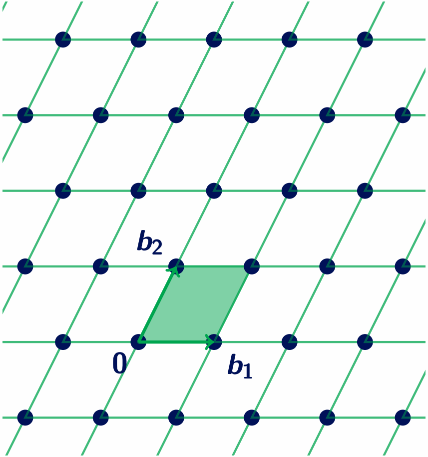

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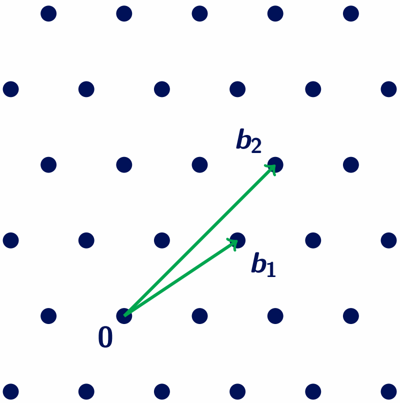

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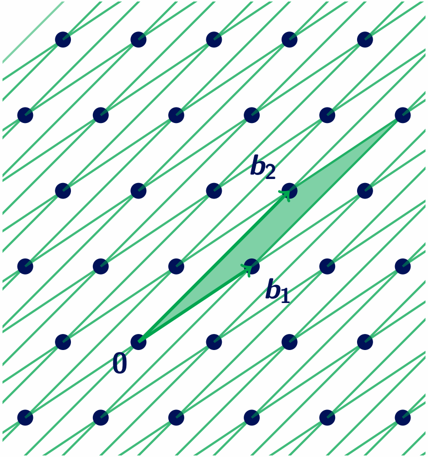

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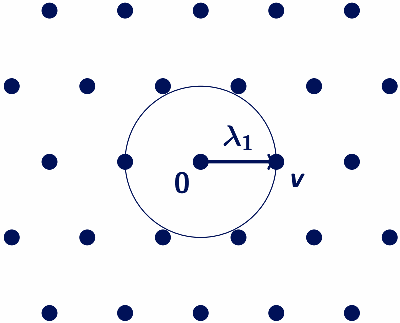
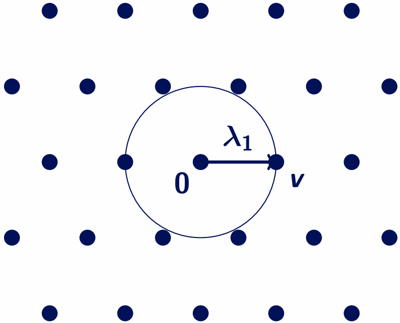


First minimum & theta series

$$\lambda_1(\mathcal{L}) := \min_{x \in \mathcal{L} \setminus \{0\}} \|x\|_2$$



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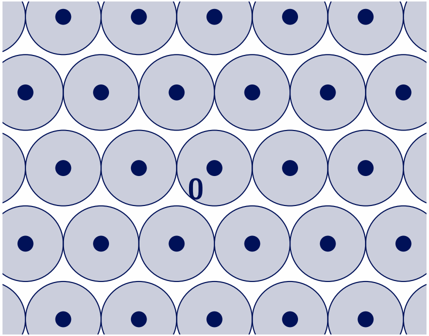

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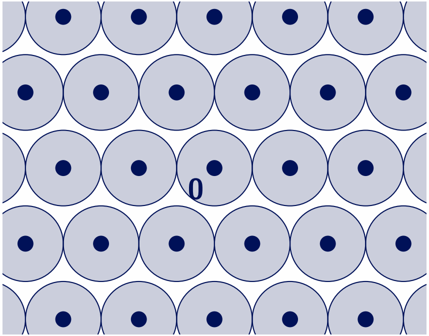

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Minkowski's Theorem ($\delta(\mathcal{L}) \leq 1$)

$$\lambda_1(\mathcal{L}) \leq \underbrace{2\frac{\det(\mathcal{L})^{1/n}}{\mathrm{vol}(\mathcal{B}^n)^{1/n}}}_{\mathrm{Mk}(\mathcal{L})}$$

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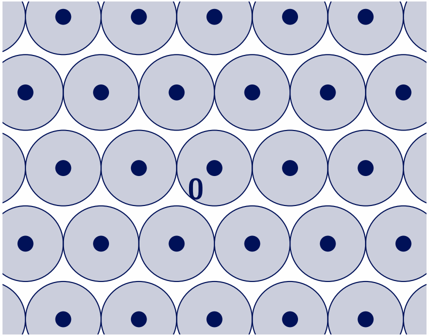

Minkowski-Hlawka Theorem

There exists a lattice $\mathcal{L} \subset \mathbb{R}^n$ with $\lambda_1(\mathcal{L}) > \mathrm{gh}(\mathcal{L}) := \frac{1}{2}\mathrm{Mk}(\mathcal{L})$.

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Definition (Siegel 1945): Haar measure

The Haar measure on $\mathcal{SL}_n(\mathbb{R})$ has finite mass on the quotient space of unit volume lattices $\mathcal{L}_{[n]} = \mathcal{SL}_n(\mathbb{R})/\mathcal{SL}_n(\mathbb{Z})$.

Average number of lattice points: Hlawka43, Siegel45

Let $\mathcal{L}_{[n]}$ be the space all lattices of dimension n and volume 1, then

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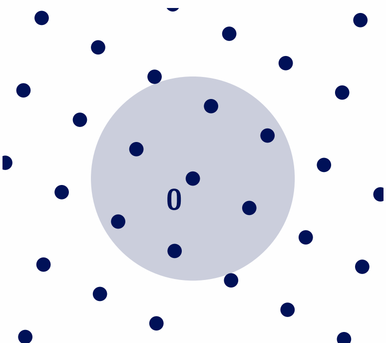

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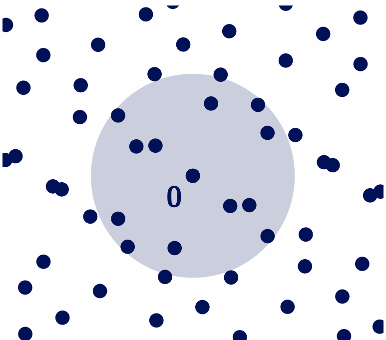

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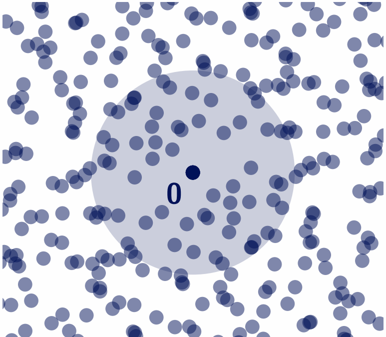

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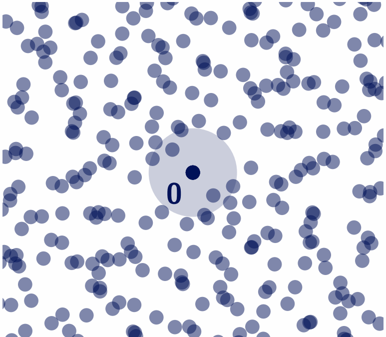

Proof: Minkowski-Hlawka Theorem

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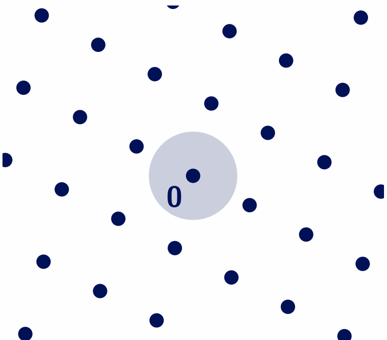

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$\Rightarrow \exists \mathcal{L} \in \mathcal{L}_{[n]}$ with $|\mathcal{L} \cap \lambda \cdot \mathcal{B}^n| \leq 2$,

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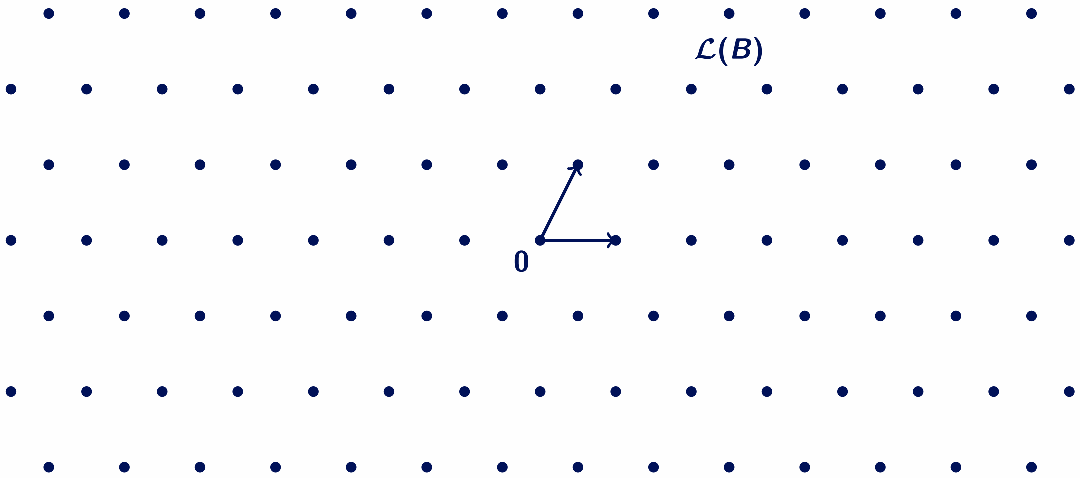
LIP and the genus of a lattice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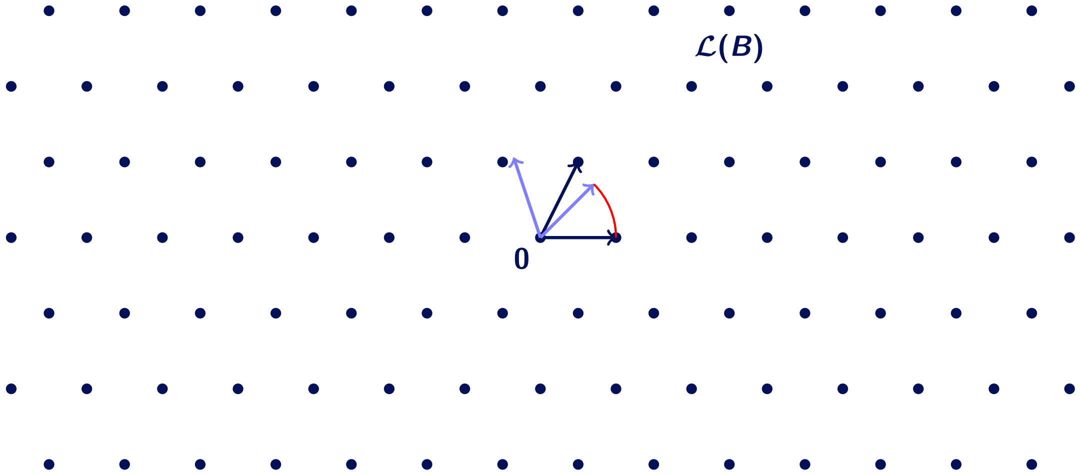
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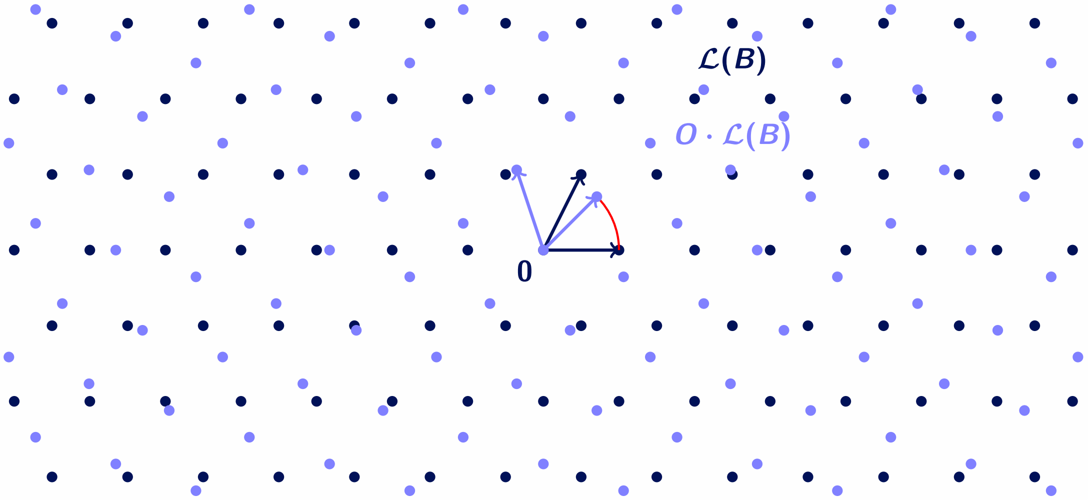
Lattice Isomorphism Problem (LIP)



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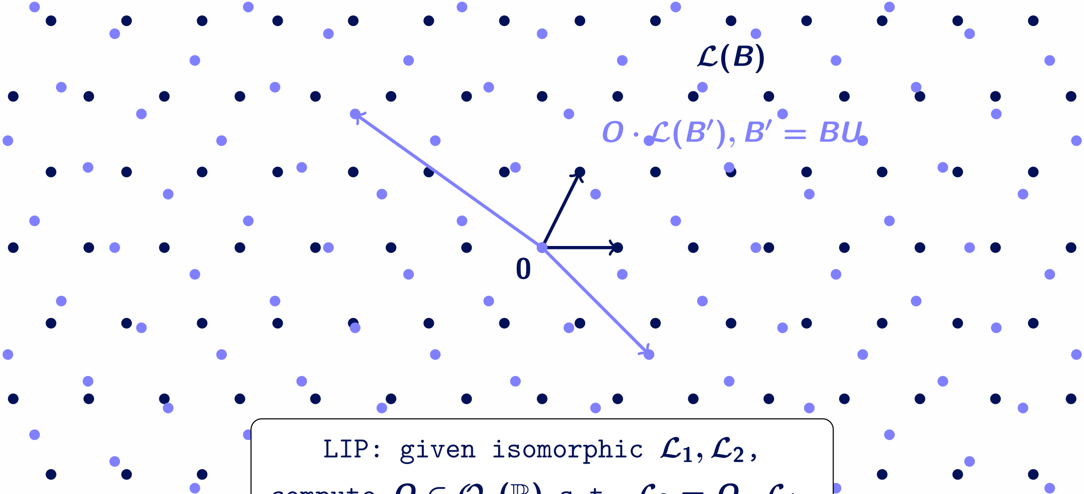

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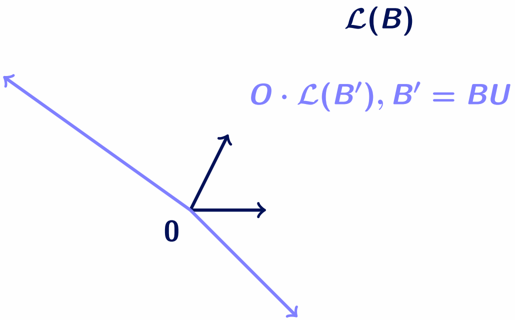

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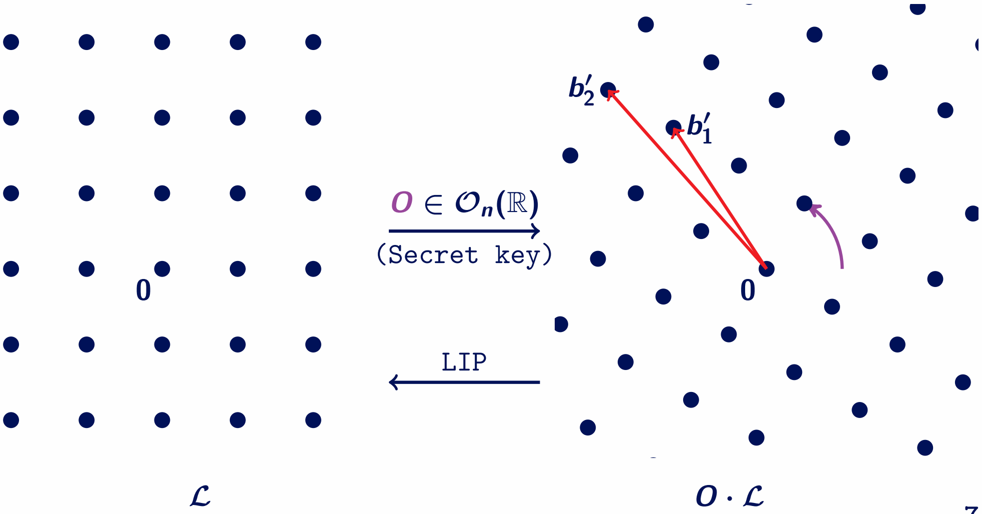
- We restrict to integer gram matrices $G := B^t B$.

Encryption scheme from LIP (informal)

Decodable lattice Bad basis of rotation



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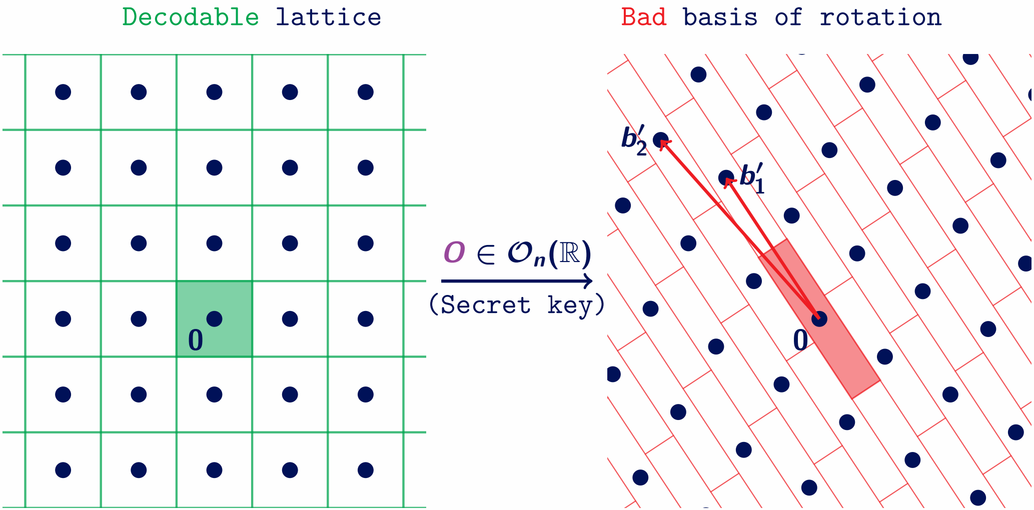

Hides (decoding) structure of $\mathcal{L}$

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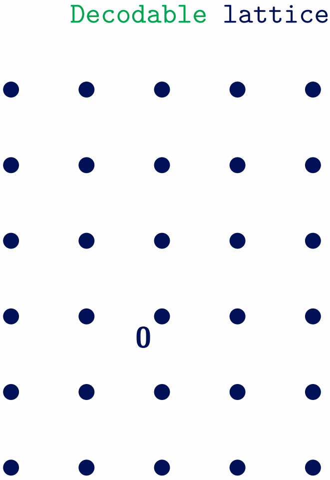

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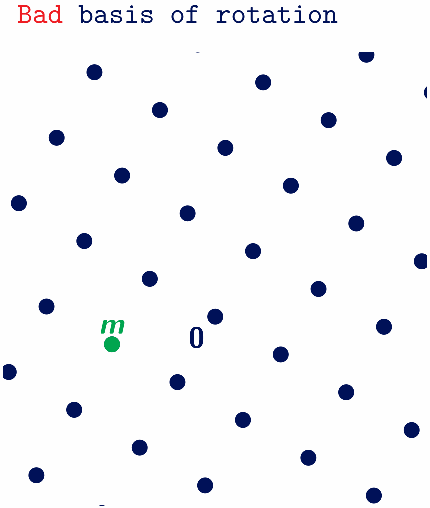

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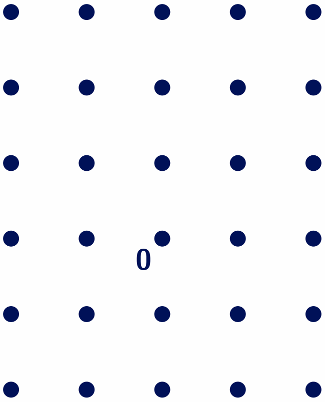

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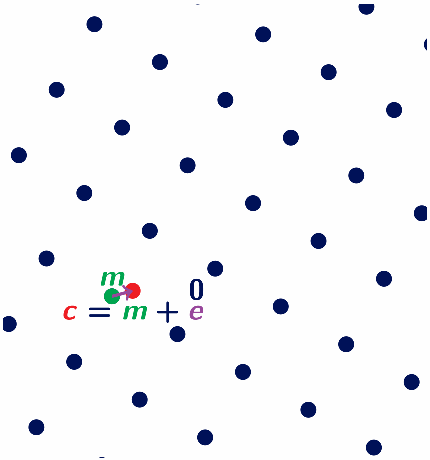

Encrypt by adding a small error

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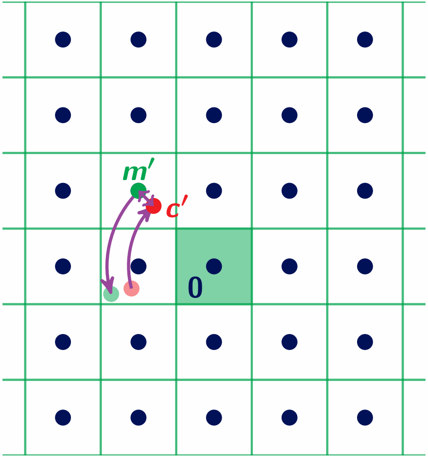

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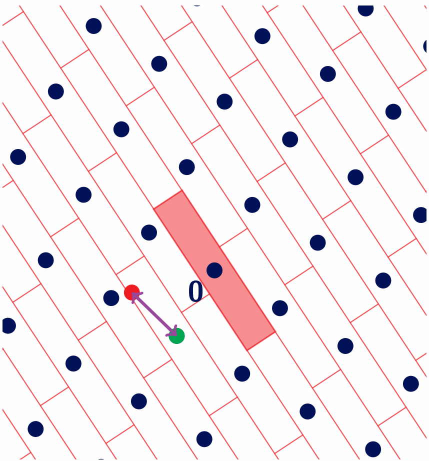

Decrypt using decoding algorithm

Cryptography from LIP

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DvW, EC2022: On LIP, QFs, Remarkable Lattices, and Cryptography

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- Many other works using LIP appeared recently

Distinghuish LIP

Definition: distinguish LIP ($\boldsymbol{\Delta}$-LIP)

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Usual security assumption:

Given:

1. some remarkable lattice $\mathcal{L}_1$
2. an auxiliary lattice $\mathcal{L}_2$ with certain (good) geometric properties

Then: cryptographic scheme is secure if Δ-LIP on $\mathcal{L}_1, \mathcal{L}_2$ is hard.

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- $\mathcal{L}$ is represented by a random $U^t G_b U \leftarrow \mathcal{D}([G_b])$ (worst-case)

Usual security assumption:

Given:

1. some remarkable lattice $\mathcal{L}_1$
2. an auxiliary lattice $\mathcal{L}_2$ with certain (good) geometric properties

Then: cryptographic scheme is secure if Δ-LIP on $\mathcal{L}_1, \mathcal{L}_2$ is hard.

Goal: find an auxiliary lattice with the right geometric properties
Example: good packing, smoothing, covering..

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$\Longrightarrow$ auxiliary lattice must have same invariants

p-adic integers:

For a prime p the p-adic integers $\mathbb{Z}_p$ are given by formal series, i.e.,

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- How restricting is the genus invariant?

Dense lattices in any genus

Application

BGPSD, EC 2023: Just how hard are rotations of $\mathbb{Z}^n$?

Do there exist lattices $\mathcal{L} \in \mathbf{gen}(\mathbb{Z}^n)$ with

- $\lambda_1(\mathcal{L}) \geq \Omega(\sqrt{n/\log(n)})$, or
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Conjecture: for $n \geq 85$ there exists a lattice $\mathcal{L} \in \mathbf{gen}(\mathbb{Z}^n)$ with

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DvW, EC 2022: On LIP, QFs, Remarkable Lattices, and Cryptography

For any lattice $\mathcal{L}_1$, does there exist a lattice $\mathcal{L}_2 \in \mathbf{Gen}(\mathcal{L}_1)$ such that

- $\mathrm{gh}(\mathcal{L})/\lambda_1(\mathcal{L}) = O(1)$ for $\mathcal{L} = \mathcal{L}_2, \mathcal{L}_2^*$?

(would significantly improve general instantiation)

Theorem: Smith-Minkowski-Siegel mass formula (Siegel, 1935)

Any genus $\mathcal{G}$ contains a finite number of isom. classes and its mass

$$M(\mathcal{G}) := \sum_{[\mathcal{L}]\in\mathcal{G}} \frac{1}{|\mathrm{Aut}(\mathcal{L})|},$$

is efficiently computable given the prime factorization of $\det(\mathcal{G})^2$.

Mass formula and the size of a genus

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- Question: do these behave like random lattices?

Random distribution over genus

Definition: distribution over Genus

Let $w(\mathcal{L}) =: 1/|\mathrm{Aut}(\mathcal{L})|$. For a genus $\mathcal{G}$ let $\mathcal{D}(\mathcal{G})$ be the distribution such that each class $[\mathcal{L}] \in \mathcal{G}$ is sampled with probability $\frac{w(\mathcal{L})}{M(\mathcal{G})}$.

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  (limit distribution of randomized Kneser's p-neighbouring method)
- Comes with similar average point counting results!
  ($\implies$ Minkowski-Hlawka like theorem?)

Kneser p-neighbouring (1957) and sampling

- Two integral lattices $\mathcal{L}_1, \mathcal{L}_2$ are p-neighbours $\mathcal{L}_1 \sim_p \mathcal{L}_2$ if

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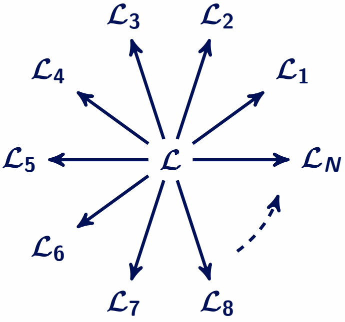

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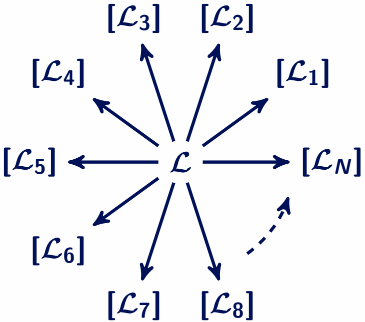

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- For large enough p, a random walk has limit distribution $\mathcal{D}(\mathcal{G})$.
 $\implies$ efficient sampling algorithm for $\mathcal{D}(\mathcal{G})$.

Theorem (good packing): Minkowski-Hlawka theorem for fixed genus

Let $\mathcal{G}$ be any genus of dimension $n \geq 6$ such that $\mathrm{rk}_{\mathbb{F}_p}(\mathcal{G}) \geq 6$ for all primes p. Let $C = \frac{7\zeta(3)}{9\zeta(2)} \approx 0.57$. Then there exists a $\mathcal{L} \in \mathcal{G}$ with

$$\lambda_1(\mathcal{L})^2 \geq \left\lceil (C \cdot \det(\mathcal{L})/\omega_n)^{2/n} \right\rceil \approx n/2\pi e \cdot \det(\mathcal{L})^{2/n} = \mathrm{gh}(\mathcal{L})^2.$$

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- Similar result for simultaneous good primal and dual packing.
- For a constant $0 < c \leq 1$ we get that

$$\mathbb{P}\left[\lambda_1(\mathcal{L}) \geq \left\lceil c^2 \cdot (C \cdot \det(\mathcal{L})/\omega_n)^{2/n} \right\rceil\right] > 1 - c^n.$$

- Similar result for smoothing parameter and covering radius.

Definition: average theta series

For a genus $\mathcal{G}$ its average theta series is given by

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- Recall that computing (coeff. of) $\theta_{\mathcal{L}}(q)$ is usually extremely hard

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$$\Theta_{\mathcal{G}}(q) = \mathop{\mathbb{E}}_{[\mathcal{L}] \leftarrow \mathcal{D}(\mathcal{G})} [\theta_{\mathcal{L}}(q)] = \frac{\sum_{[\mathcal{L}] \in \mathcal{G}} w(\mathcal{L}) \cdot \theta_{\mathcal{L}}(q)}{M(\mathcal{G})}$$

Theorem: Siegel-Weil mass formula (informal)

The coefficient N_m of q^m in $\Theta_{\mathcal{G}}(q)$ can be efficiently computed given the prime factorization of $m \det(\mathcal{G})^2$.

- Recall that computing (coeff. of) $\theta_{\mathcal{L}}(q)$ is usually extremely hard
- The average is efficient to compute

Example: even unimodular case (1)

Definition: even unimodular lattices

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Lemma: mass formula

For $n = 8k \geq 8$, B_i the i-th Bernoulli number, and $\sigma_z(m) = \sum_{d|m} d^z$ is the sum of positive divisors function, we have

$$\Theta_{\mathcal{G}_{8k,e}}(q) = E_{4k}(q^2) = 1 + \frac{-8k}{B_{4k}} \sum_{m=1}^{\infty} \sigma_{4k-1}(m) q^{2m}.$$

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- $\Theta_{\mathcal{G}_{128,e}}(q) \approx 1 + 6.11 \cdot 10^{-37} q^2 + 5.64 \cdot 10^{-18} q^4 + 7.00 \cdot 10^{-7} q^6 + 52.01 q^8 + 6.63 \cdot 10^7 q^{10} + O(q^{12})$

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Lemma: existence of good packing

Let $\mathcal{G}$ be a genus with average theta series $\Theta_{\mathcal{G}}(q) = 1+\sum_{m=1}^{\infty} N_m q^m$. If $\sum_{m=1}^{\lambda} N_m < 2$, then there exists a lattice $\mathcal{L}\in\mathcal{G}$ s.t. $\lambda_1(\mathcal{L})^2 > \lambda$.

Example: even unimodular case (2)

Lemma: even packing (Milnor, Serre, 73)

Let $n = 8k \geq 8$ with $k \in \mathbb{N}$, then there exists an n-dimensional even unimodular lattice $\mathcal{L}$ with $\lambda_1(\mathcal{L})^2 \geq 2 \cdot \left\lceil \frac{1}{2}(\frac{3}{5}\omega_n)^{-2/n} \right\rceil \approx n/2\pi e$.

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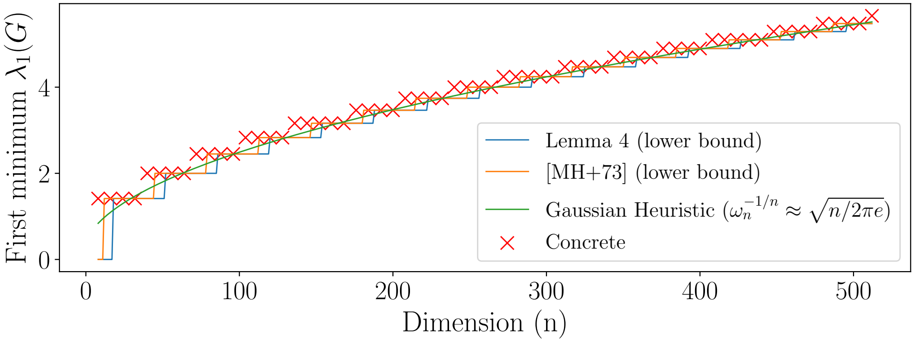

General case: compute mass formula

- We want to count the average number of solutions N_m to $f(x) := x^t G_{\mathcal{L}} x = m$ with $x \in \mathbb{Z}^n$ when $[\mathcal{L}] \leftarrow \mathcal{D}(\mathcal{G})$.

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Theorem: Siegel-Weil mass formula

For any genus $\mathcal{G}$ of dimension ≥ 2 and average theta series $\Theta_{\mathcal{G}}(q) = 1 + \sum_{y=1}^{\infty} N_m q^m$ we have

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- Local-global principle
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- Can even be generalized to matrix equations!

  (mass formula from $M(\mathcal{G})$ follows from equation $U^t G U = G$)

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- Conjecture: remove conditions $\implies$ extra factor $\mathbf{poly}(m)$

(but rather tedious to work out)

Open Questions

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Can we construct stronger efficiently computable invariants?

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WC-AC reductions:

- the random case $[\mathcal{L}] \leftarrow \mathcal{D}(\mathcal{G})$ is heuristically the hardest.
- from any class $[\mathcal{L}] \in \mathcal{G}$ we can efficiently step to a random class.

Can we make a worst-case to average-case reduction within a genus?

Example: SVP, SIVP, LIP

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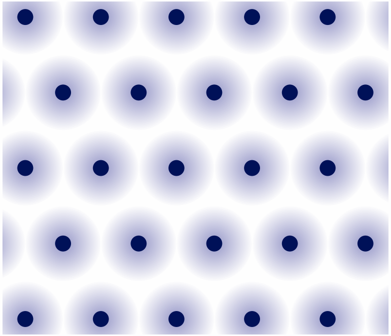
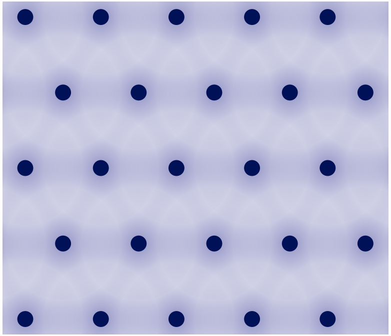
Thanks!



Smoothing parameter

'minimum $s > 0$ such that centered Gaussian with width s is ϵ-close to uniform over $\mathbb{R}^n/\mathcal{L}$'



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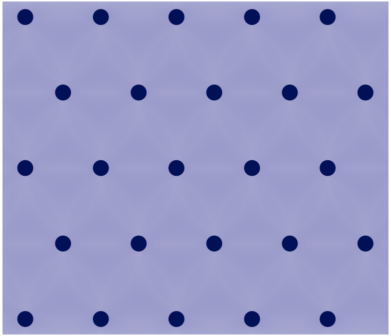

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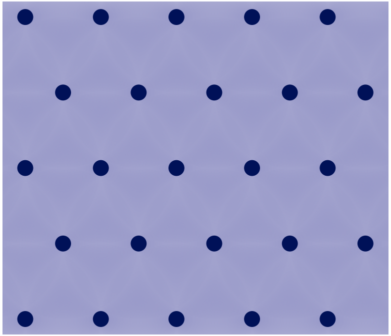

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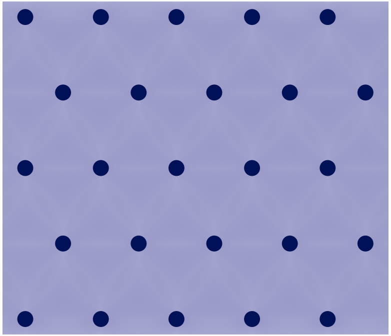

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Good smoothing: $\epsilon \in (e^{-n}, 1]$

For a random lattice $\mathcal{L}^*$, $\theta_{\mathcal{L}^*}(\exp(-\pi s^2)) \le 1 + O(ns^{-n}\det(\mathcal{L}))$

$\Rightarrow$ there exists a lattice with $\eta_\varepsilon(\mathcal{L}) \le (\det(\mathcal{L})/\varepsilon)^{1/n}$.