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AN APPLICATION OF DYNNIKOV COORDINATES IN D_3

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Abstract. This work presents an application of Dynnikov coordinates to geometric group theory. In this work, we describe the orbits of reducible mapping classes of the pure mapping class group of a sphere with four punctures M with respect to the action of generating set $\{t_c, t_d\}$ via Dynnikov coordinates. We encode the vertices of the curve complex $C(M)$, taking as input their Dynnikov coordinates. Using these orbits, we describe an algorithm for reaching the set $\Delta = \{v_c, v_d, v_e\} \subset C(M)$ from any vertex in $C(M)$, where the vertices v_c, v_d and v_e are isotopy classes of simple closed curves c, d, and e such that $t_c t_d t_e = 1$. We present an algorithm that determines the actions of the pseudo-Anosov maps of M. Finally, we express the Dynnikov coordinates in M in terms of (p, q)-torus coordinates via a double branched cover over M.

1. Introduction

In geometric group theory, group actions have great importance. We can approach the group from a geometric perspective by looking at group actions. A group action can be dismantled into orbits, and conversely, by examining the orbits, we may also understand the structure of the group. We recall that if G is a group acting on a set C, the orbit of an element $c \in C$ with respect to this action is the set $\{g \cdot c |\, g \in G\}$. In this work, we determine the orbits of actions t_c and t_d of the pure mapping class group of a sphere with four punctures M via Dynnikov coordinates, where c and d are isotopy classes of nontrivial simple closed curves in M with geometric intersection number $i(c, d) = 2$ (see Figure 1). In this work, we explore some applications of these orbits, which are as follows:

In Section 3, we code the vertices of the curve complex $C(M)$, taking as input their Dynnikov coordinates. Let c, d, and e be simple closed curves such that $i(c, d) = 2 = i(c, e) = i(d, e)$ (see Figure 5). We define the set $\Delta = \{v_c, v_d, v_e\} \subset C(M)$, where the vertices v_c, v_d, and v_e are the isotopy classes of simple closed curves c, d, and e, respectively; such that the Dehn twists t_c, t_d and t_e satisfy $t_c t_d t_e = 1$. Then, we describe an algorithm for reaching the set Δ from an arbitrary vertex in $C(M)$. We also present alternative proofs of some classical results: the group generated by Dehn twists t_c and t_d is isomorphic to the free group of rank 2 in Section 4, where

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c and d are isotopy classes of simple closed curves in M with $i(c,d) = 2$ (see Figure 1). The other one is that the group generated by the Dehn twists t_c, t_d and $t_e = (t_c t_d)^{-1}$ is isomorphic to the free group of rank 3, in Section 5, where c, d, and e are isotopy classes of simple closed curves in M with $i(c,d) = 2 = i(c,e) = i(d,e)$ (see Figure 5). We also give an algorithm to determine the actions of the pseudo-Anosov maps $t_d t_c^{-1}$ and $t_d^{-1} t_c$ of M. This algorithm reveals how the iteration evolves geometrically, in Section 6. Finally, we state the Dynnikov coordinates in M in terms of (p,q)-torus coordinates via a double branched cover over M in Section 7.

This note is a ramification of our project in progress joint work with E. Dalyan, E. Medetoğulları, and Ö. Yurttaş, titled "Detecting free products generated by Dehn twists on $n-$ punctured discs," devoted to a special case $n = 3$.

2. Preliminaries

Let M be a sphere with four punctures. Let us take one of the punctures to lie at infinity, we can consider M as the thrice-punctured disc, where the punctures are aligned in the horizontal diameter of the disc. A simple closed curve c on M is called trivial if either it bounds a disc, once punctured disc or it is parallel to a boundary component. Otherwise, it is called nontrivial. Let c and d be distinct two isotopy classes of nontrivial simple closed curves on M such that each of c and d contains two punctures and that c intersects d precisely at two points and they intersect the diameter of the disc exactly twice (see Figure 1).

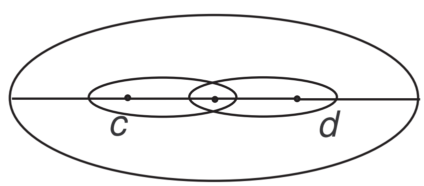


Figure 1. Two curves c and d on M

It is well known that the pure mapping class group $\mathrm{PMod}(M)$ of M is isomorphic to the free group F_2 and freely generated by the Dehn twists t_c and t_d about c and d, respectively. Since $\mathrm{PMod}(M)$ is isomorphic to F_2, the elements of $\mathrm{PMod}(M)$ different from the identity are either reducible or pseudo-Anosov. Moreover, conjugates of nonzero powers of t_c, t_d and $t_c t_d$ are the only reducible elements in $\mathrm{PMod}(M)$ (see Lemma 3.4 in [1]). Let us recall Thurston's classification of surface homeomorphisms. For a mapping class f not the identity, it says that one of the following holds:
(1) f is periodic, that is, $f^n = 1$ for some $n \geq 2$,
(2) f is reducible, i.e. there is a (closed) one-dimensional submanifold a of

a surface S such that $f(a) = a$,
(3) f is pseudo-Anosov if and only if f is neither periodic nor reducible.

Dynnikov introduced a coding for integral laminations on a sphere with $n+3$ punctures in [2]. Let C denote the set of isotopy classes of simple closed curves in M. Let α_i and β_i be arcs depicted in Figure 2. Let us take l in C intersecting each of these arcs minimally.

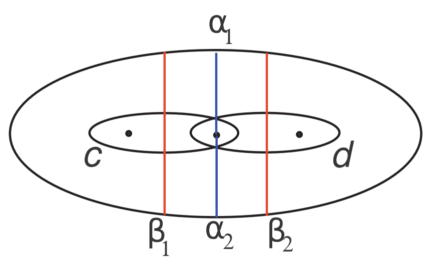


FIGURE 2. Two curves c and d on M

For convenience, we will also denote the number of intersections of l with each of the arcs α_i and β_i by the same symbols. For any isotopy class of simple closed curve l in M, the Dynnikov coordinates of l is given by $\rho(l) = (a, b)$, where $\rho : C \to \mathbb{Z}^2 \setminus \{0\}$ is bijective Dynnikov coordinate function (see [8]) such that

$$a = \frac{\alpha_2 - \alpha_1}{2} \ \ and \ \ b = \frac{\beta_1 - \beta_2}{2}.$$

PMod(M) acts on the set of isotopy classes of simple closed curves C. The update rules describe in [4] the action of the Artin braid generators σ_1 and σ_2 (and their inverses) on $\mathbb{Z}^2 \setminus \{0\}$. For any isotopy class of simple closed curve l in M, with $\rho(l) = (a, b)$, let $\rho(\sigma_i(l)) = (a', b')$ for $i = 1, 2$. By Lemma 4 (update rules for Artin generators) in [4], we have the following equations:

$$a' = a + b - max\{0, a, b\}, \quad b' = max\{b, 0\} - a \ for \ \sigma_1$$
$$a' = max\{a + max\{0, b\}, b\}, \quad b' = b - (a + max\{0, b\}) \ for \ \sigma_2$$

By Lemma 5 (update rules for inverse Artin generators) in [4], we have also the following equations:

$$a' = max\{0, a + max\{0, b\}\} - b, \quad b' = a + max\{0, b\} \ for \ \sigma_1^{-1}$$
$$a' = a - max\{a + b, 0, b\}, \quad b' = a + b - max\{0, b\} \ for \ \sigma_2^{-1}$$

Using these update rules, we can also obtain the Dynnikov matrices with respect to appropriate regions (see [9]).

3. Orbits of the actions of t_c, t_d, and t_ct_d via Dynnikov coordinates

Let c and d be isotopy classes of nontrivial simple closed curves in M with $i(c,d)=2$ (see Figure 1). Using the update rules, ($(a,b)\neq(0,0)$), we have the following

$$\rho(\sigma_1^2(l))=\rho(t_c(l))=\begin{cases}(b-a,-b) & \text{if } 0\leq b\leq a\\ (b-a,b-2a) & \text{if } 0<a\leq b\leq 2a\\ (a,b-2a) & \text{if } 0<2a\leq b\\ (b-a,-b) & \text{if } a\geq 0, b\leq 0\\ (a+b,-2a-b) & \text{if } a\leq 0, b\leq 0\\ (a,b-2a) & \text{if } a\leq 0, b\geq 0\end{cases} \tag{1}$$

$$\rho(\sigma_2^2(l))=\rho(t_d(l))=\begin{cases}(a+b,-2a-b) & \text{if } a\geq 0, b\geq 0\\ (b-a,-b) & \text{if } a\leq 0, b\geq 0\\ (a,b-2a) & \text{if } a\geq 0, b\leq 0\\ (b-a,-b) & \text{if } a\leq b<0\\ (b-a,b-2a) & \text{if } 2a\leq b\leq a<0\\ (a,b-2a) & \text{if } b\leq 2a<0\end{cases} \tag{2}$$

We consider the actions of Dehn twists t_c and t_d on C given by Equation 1 and Equation 2, respectively. The orbit of the curve d is $\{(-1,2k-1),\ (1,2k-1)|k\in\mathbb{N}\}$, where $(-1,2k-1)=\rho(t_c^k(d))$ and $(1,2k-1)=\rho(t_c^{-k}(d))$. The orbit of the curve c is $\{(1,-(2k-1)),\ (-1,-(2k-1))|k\in\mathbb{N}\}$, where $(1,-(2k-1))=\rho(t_d^k(c))$ and $(-1,-(2k-1))=\rho(t_d^{-k}(c))$. The orbits of the actions of t_c and t_d on C are as shown in Figure 3.

Also, we have

$$\rho(\underbrace{t_ct_dt_ct_dt_c}_{k-times}(d))=(-k,1),$$
$$\rho(\underbrace{t_dt_ct_dt_ct_d}_{k-times}(c))=(k,-1),$$
$$\rho(\underbrace{t_c^{-1}t_d^{-1}t_c^{-1}t_d^{-1}t_c^{-1}}_{k-times}(d))=(k,1),$$
$$\rho(\underbrace{t_d^{-1}t_c^{-1}t_d^{-1}t_c^{-1}t_d^{-1}}_{k-times}(c))=(-k,-1)$$

for some integer $k\geq 1$.

Remark 3.1. The sequence of values generated by the iteration of t_c given by Equation 1 flows the second quadrant (see Figure 3 (a)). Similarly, the sequence of values generated by the iteration of t_d given by Equation 2 flows the fourth quadrant (see Figure 3 (b)).

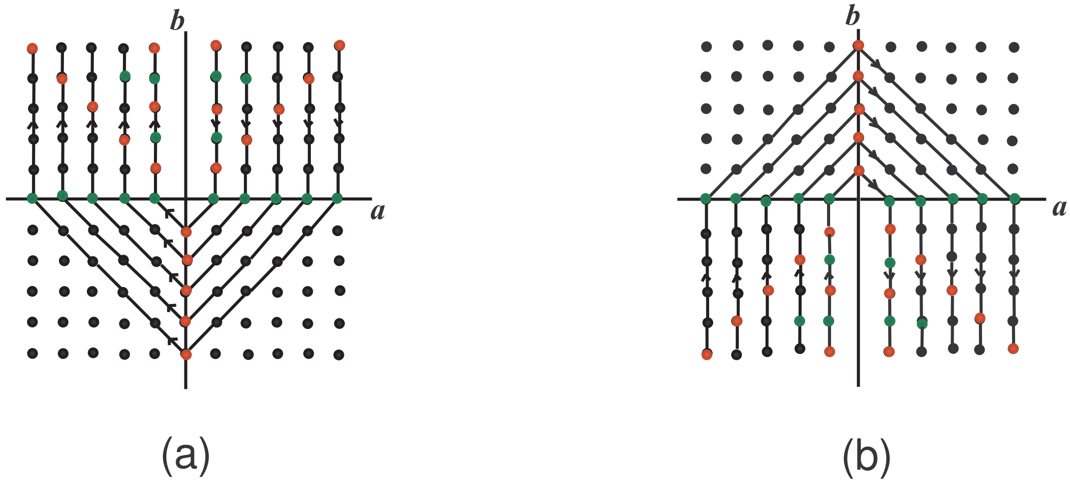


FIGURE 3. (a) Orbits of the action of t_c (b) Orbits of the action of t_d

$$
\rho(\sigma_1^2\sigma_2^2(l)) = \rho(t_c t_d(l)) = \begin{cases} (-3a-2b, 2a+b) & \text{if } a \geq 0, b \geq 0 \\ (-2b+a, b) & \text{if } a \leq 0, b \geq 0 \\ (b-3a, 2a-b) & \text{if } a \geq 0, b \leq 0 \\ (a-2b, b) & \text{if } a \leq 2b \leq 0 \\ (a-2b, 2a-3b) & \text{if } 4b \leq 2a \leq 3b < 0 \\ (b-a, 2a-3b) & \text{if } 3b \leq 2a < 2b < 0 \\ (b-a, -b) & \text{if } b \leq a < 0 \end{cases} \tag{3}
$$

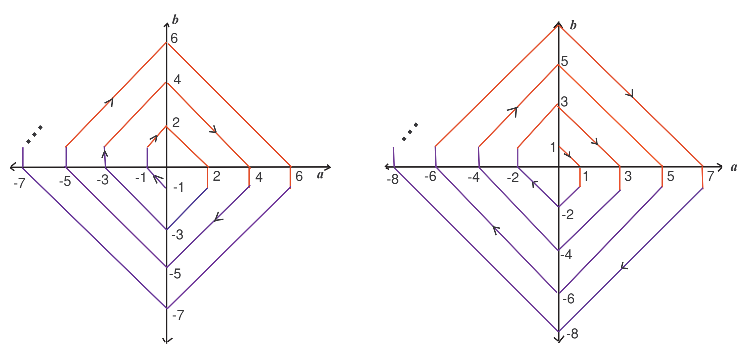


FIGURE 4. The blue (red) lines denote the action of t_c (t_d).

(4)

$$\rho((t_ct_d)^{-1}(l)) = \begin{cases} (-3a-b,-2a-b) & \text{if } a>0, b\geq 0 \\ (-3a+2b,-2a+b) & \text{if } a\geq 0, b<0 \\ (a+2b,b) & \text{if } a<0, b<0 \\ (a,4a+b) & \text{if } a<0, b=0 \\ (a+2b,b) & \text{if } 0<b\leq \left\lceil \frac{-a-1}{2} \right\rceil, a<-2 \\ (a+2b,-2a-3b) & \text{if } \frac{-a-1}{2}<b\leq \left\lfloor \frac{-2a-1}{3} \right\rfloor, a<-1 \\ (-a-b,-2a-3b) & \text{if } \frac{-2a-1}{3}<b\leq -a-1, a<-2 \\ (-a-b,-b) & \text{if } -a-1<b<-2a, a<-1 \\ (a,4a+b) & \text{if } -2a=b, a<0 \\ (-a-b,-b) & \text{if } -2a<b, a\leq 0 \end{cases}$$

3.1. **The curve complex $C(M)$ via Dynnikov Coordinates.** The curve complex $C(S)$ on a surface S is the abstract simplicial complex whose vertices are the isotopy classes of nontrivial simple closed curves, and a set of vertices $\{v_0, v_1, \ldots, v_k\}$ is defined to be a k-simplex if and only if $v_0, v_1, \ldots, v_k$ can be represented by pairwise disjoint curves. Since there is no disjoint nontrivial isotopy classes of simple closed curves in M, $C(M)$ is discrete. Since the Dynnikov coordinate function $\rho : C \to \mathbb{Z}^2 \setminus \{0\}$ is bijective(see [8]) and also, the nontrivial isotopy classes of oriented simple closed curves in M are in bijective correspondence with the set of primitive elements of $\mathbb{Z}^2 \setminus \{0\}$, we can encoding the vertices of $C(M)$ by the Dynnikov coordinates of the nontrivial isotopy classes of simple closed curves in M. Let us recall that an element (m,n) of $\mathbb{Z}^2$ is primitive if and only if $(m,n) = (0,\pm 1)$, $(m,n) = (\pm 1, 0)$, or $gcd(m,n) = 1$ (see [3]).

Let v_c, v_d and v_e be the isotopy classes of the curves c, d, and e, respectively; depicted in Figure 5. Let us code v_c, v_d and v_e by the Dynnikov coordinates $(0,1)$, $(0,-1)$ and $(-1,0)$ of c, d and e, respectively. Similarly, we can code any vertex of $C(M)$ by the set of primitive elements of $\mathbb{Z}^2 \setminus \{0\}$ via the Dynnikov coordinates. We define a set Δ consisting of the vertices v_c, v_d,and v_e.

Let v be any vertex in $C(M)$ with the Dynnikov coordinate (x,y) such that $gcd(x,y) = 1$. We want to find the minimum distance between a vertex v and the set Δ. From arbitrary vertex v, to reach the vertex v_c or v_d or v_e in Δ, we will use Dehn twists t_c and t_d, and their inverses.

Given a vertex $v \in C(M)$ with Dynnikov coordinates $\rho(v) = (x,y) \in \mathbb{Z}^2 \setminus \{0\}$ the following algorithm finds a mapping class ϕ such that $\phi(v)$ is reached the set Δ.

Reaching the set Δ Algorithm

Let $(x,y) \in \mathbb{Z}^2 \setminus \{0\}$ be the Dynnikov coordinates of v in $C(M)$ such that $gcd(x,y) = 1$. We will write $(x',y') \in \mathbb{Z}^2 \setminus \{0\}$ to denote the Dynnikov coordinates of $\psi(v)$, where ψ is is a generator of PMod(M).

Algorithm. Given $v \in C(M)$ let $\rho(v) = (x, y) \in \mathbb{Z}^2 \setminus \{0\}$, such that $gcd(x, y) = 1$.

Step 1: If $y > 0$, apply forward $2|x|$ units along $|x|^{th}$ level of orbits of the action of t_c, let $(x', y') = \rho(v')$, where $v' = t_c(v)$, if $x > 0$ and apply backward $2|x|$ units along $|x|^{th}$ level of orbits of the action of t_c, let $(x', y') = \rho(v')$, where $v' = t_c^{-1}(v)$ if $x < 0$. If $x' = 0$, input the pair (x', y') to Step 3. If $x' \neq 0$ and $y' > 0$, then input the pair (x', y') to Step 1. If $y' = 0$ and $x' > 0$, then input the pair (x', y') to Step 4. If $y' = 0$ and $x' < 0$, then input the pair (x', y') to Step 5.

Otherwise input the pair (x', y') to Step 2.

Step 2: If $y < 0$, apply backward $2|x|$ units along $|x|^{th}$ level of orbits of the action of t_d, let $(x', y') = \rho(v')$, where $v' = t_d^{-1}(v)$ if $x > 0$ and apply forward $2|x|$ units along $|x|^{th}$ level of orbits of the action of t_d, let $(x', y') = \rho(v')$ where $v' = t_d(v)$ if $x < 0$. If $x' = 0$ input the pair (x', y') to Step 3. If $x' \neq 0$ and $y' < 0$, then input the pair (x', y') to Step 2. If $y' = 0$ and $x' > 0$, then input the pair (x', y') to Step 4. If $y' = 0$ and $x' < 0$, then input the pair (x', y') to Step 5.

Otherwise input the pair (x', y') to Step 1.

Step 3: Since $x = 0$, v_c is reached if $b > 0$, and v_d is reached if $b < 0$. Write the generators used in Step 1 and Step 2 in order to express the mapping class ψ reaching v_c or v_d.

Step 4: If $y = 0$ and $x > 0$, apply forward $2|x|$ units along $|x|^{th}$ level of orbits of the action of t_c, let $(x', y') = \rho(v')$, where $v' = t_c(v)$ or apply backward $2|x|$ units along $|x|^{th}$ level of orbits of the action of t_d, let $(x', y') = \rho(v')$, where $v' = t_d^{-1}(v)$. Then input (x', y') to Step 5.

Step 5: Since $y = 0$ and $x < 0$, v_e is reached. Write the generators used in Step 1, Step 2, and Step 4 in order to express the mapping class ψ reaching v_e.

Example 1. Let us take the vertex v in $C(M)$ with the Dynnikov coordinate $(x, y) = (10, 3)$. Then, since $y = 3 > 0$ and $x = 10 > 0$, by Step 1, we apply forward 20 units along 10^{th} level of orbits of the action of t_c, we have $\rho(v') = \rho(t_c(v)) = (-7, -3)$. Since $x < 0$, $y < 0$, by Step 2, we apply forward $2|x| = 2|-7| = 14$ units along 7^{th} level of orbits of the action of t_d, we get $\rho(v'') = \rho(t_d(v')) = (4, 3)$. Since $y = 3 > 0$ and $x = 4 > 0$, by Step 1, we apply forward 8 units along 4^{th} level of orbits of the action of t_c, we have $\rho(v''') = \rho(t_c(v'')) = (-1, -3)$. Since $x < 0$, $y < 0$, by Step 2, we apply forward 2 units along 1^{st} level of orbits of the action of t_d, we get $\rho(v^{(4)}) = \rho(t_d(v''')) = (-1, -1)$. Again, we apply forward 2 units along 1^{st} level of orbits of the action of t_d, we conclude that $\rho(v^{(5)}) = \rho(t_d(v^{(4)})) = (0, 1)$, as desired. Hence, we obtain that

$$(10,3) \xrightarrow{t_c} (-7,-3) \xrightarrow{t_d} (4,3) \xrightarrow{t_c} (-1,-3) \xrightarrow{t_d} (-1,-1) \xrightarrow{t_d} (0,1)$$

Example 2. Let us take the vertex v in $C(M)$ with the Dynnikov coordinate $(x,y) = (3,10)$. Then, since $y = 3 > 0$ and $x = 10 > 0$, by Step 1, we apply forward 6 units along 3^{th} level of orbits of the action of t_c, we have $\rho(v') = \rho(t_c(v)) = (3,4)$. Since $y = 4 > 0$ and $x = 3 > 0$, by Step 1, we apply forward 6 units along 3^{th} level of orbits of the action of t_c, we have $\rho(v'') = \rho(t_c(v')) = (1,-2)$. Since $x = 1 > 0$, $y = -2 < 0$, apply backward 2 units along 1^{st} level of orbits of the action of t_d, we have $\rho(v''') = \rho(t_d^{-1}(v'')) = (1,0)$. By Step 4, $y = 0$ and $x = 1 > 0$, we can apply forward 2 units along 1^{st} level of orbits of the action of t_c, we get $\rho(v^{(4)}) = \rho(t_c(v''')) = (-1,0)$, as desired. Hence, we have

$$(3,10) \xrightarrow{t_c} (3,4) \xrightarrow{t_c} (1,-2) \xrightarrow{t_d^{-1}} (1,0) \xrightarrow{t_c} (-1,0)$$

4. An alternative proof for the free group F_2 generated by two Dehn twists

In this subsection, we give alternative proof for the classical result: Let c and d be isotopy classes of simple closed curves in M with $i(c,d) = 2$. Then, the group generated by Dehn twists t_c and t_d is isomorphic to the free group of rank 2. This theorem, which plays a crucial role in understanding the relationships between Dehn twists, was proved by Ishida [5] and Hamidi-Tehrani [6]. For non-orientable surfaces, Stukow provided the proof in [7].

Proof. Let G be a group generated by Dehn twists t_c and t_d. Let C be the set of isotopy classes of simple closed curves in M. Let (a_l, b_l) denote the Dynnikov coordinates of $l \in C$. We will use the following the ping pong lemma:

Lemma 4.1. *(Ping pong lemma). Let G be a group which acts on a set X. Let $g_1, \ldots, g_k$ be elements of G. Suppose that there are nonempty, disjoint subsets $X_1, \ldots, X_k$ of X with the property that, for each i and each $j \neq i$, we have $g_i^p(X_j) \subset X_i$ for every integer $p \neq 0$. Then the group generated by the g_m is a free group of rank k.*

Let us define sets C_1 and C_2:

$$C_1 = \{l \in C | b_l > 0\} \text{ and } C_2 = \{l \in C | b_l < 0\}.$$

Since the Dynnikov coordinates of the curves c and d in the Figure 1 are $(0,1)$ and $(0,-1)$, respectively, that is, since $b_c = 1 > 0$ and $b_d = -1 < 0$, $c \in C_1$ and $d \in C_2$. So, the sets C_1 and C_2 are nonempty and disjoint. By applying the ping pong lemma, the proof reduces to verifying that $t_c^p(C_2) \subset C_1$ and $t_d^p(C_1) \subset C_2$ for $p \neq 0$. Because of symmetry, it is enough to verify the former inclusion.

If $l \in C_2$ then $b_l < 0$. Let $t_c^p(l) = l'$, $p \neq 0$. We must show that $t_c^p(l) \in C_1$, $p \neq 0$. We must show that b_l' is positive. Since the update rules for t_c^{-1} can be derived by symmetry, we may assume that $p > 0$. By Equation 1, in the case $b_l < 0$, we have two cases:

Case 1. For $a_l \leq 0$, $\rho(t_c(l)) = (a_l + b_l, -(2a_l + b_l))$.

Case 2. For $a_l \geq 0$, $\rho(t_c(l)) = (-a_l + b_l, -b_l)$.

If we investigate the sign of the components of $\rho(t_c(l))$ in both cases:
In the Case 1, since $a_l \leq 0$ and $b_l < 0$, we have $a_l + b_l < 0$ and $-(2a_l + b_l) > 0$.

In the Case 2, since $a_l \geq 0$ and $b_l < 0$, we have $-a_l + b_l < 0$ and $-b_l > 0$.
So, in both cases, we obtain that the $a-$coordinate is negative and the $b-$coordinate is positive. Then, by Eqation 1,

Case 1. For $a_l \leq 0$, $\rho(t_c^2(l)) = (a_l + b_l, -(4a_l + 3b_l))$.

Case 2. For $a_l \geq 0$, $\rho(t_c^2(l)) = (-a_l + b_l, 2a_l - 3b_l)$.

The sign of the components of $\rho(t_c(l))$ in both cases: In the Case 1, since $a_l \leq 0$ and $b_l < 0$, we have $a_l + b_l < 0$ and $-(4a_l + 3b_l) > 0$.

In the Case 2, since $a_l \geq 0$ and $b_l < 0$, we have $-a_l + b_l < 0$ and $2a_l - 3b_l > 0$.
So, in both cases, we obtain that the $a-$coordinate is negative and the $b-$coordinate is positive again. Then, if we continue this process to find the image of $t_c^p(l)$, by Equation 1,

Case 1. For $a_l \leq 0$, $\rho(t_c^p(l)) = (a_l + b_l, -(2pa_l + (2p-1)b_l))$, $p \geq 2$. Since $a_l \leq 0$ and $p \geq 2$, we have $-(2pa_l + (2p-1)b_l) > 0$.

Case 2. For $a_l \geq 0$, $\rho(t_c^p(l)) = (-a_l + b_l, 2(p-1)a_l - (2p-1)b_l)$, $p \geq 2$. Similarly, we can see $2(p-1)a_l - (2p-1)b_l) > 0$.

Hence, for $p \geq 2$, we always conclude that the $b-$coordinate is positive for any a_l. Therefore, since $b_l' > 0$, $t_c^p(l)$ is in C_1, as desired.
This completes the proof. □

5. An alternative proof for the free group F_3 generated by three Dehn twists

Using the lantern relation, there is a unique simple closed curve e on M separating M into two pairs of pants and intersecting both c and d twice such that t_c, t_d and t_e satisfy $t_c t_d t_e = 1$ (see Figure 5). Then, we have $t_e = (t_c t_d)^{-1}$. In this section, we provide an alternative proof that the group generated by the Dehn twists t_c, t_d and $(t_c t_d)^{-1}$ is isomorphic to the free group of rank 3.

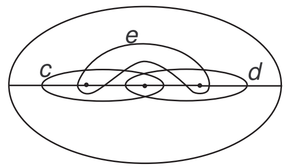


FIGURE 5

Proof. Let G be a group generated by Dehn twists t_c, t_d and $(t_c t_d)^{-1}$. Let C be the set of isotopy classes of simple closed curves in M.

Let us define sets C_1, C_2 and C_3:

$$C_1 = \{l \in C | a_l \leq 0,\ \ b_l > 0, 2b_l + a_l > -1\},\ C_2 = \{l \in C | a_l \geq 0,\ \ b_l < 0\}$$

and $C_3 = \{l \in C | a_l < 0,\ b_l \leq 0, 2a_l < b_l\}$, where (a_l, b_l) denotes the Dynnikov coordinates of $l \in C$. Since the Dynnikov coordinates of the curves c, d, and e in the Figure 5 are $(0,1)$, $(0,-1)$, and $(-1,0)$, respectively. In other words, since $a_c = 0,\ b_c = 1 > 0$, and $2b_c + a_c = 2 > -1$, $c \in C_1$ and since $a_d = 0,\ b_d = -1 < 0, d \in C_2$. We have also $e \in C_3$, as $a_e = -1,\ b_e = 0$, and $2a_e = -2 < 0$. So, the sets C_i are nonempty and disjoint for all $i = 1, 2, 3$.

By the ping pong lemma, the proof reduces to verifying that $t_c^p(C_2) \subset C_1$, $t_d^p(C_1) \subset C_2$, $t_c^p(C_3) \subset C_1$, $t_d^p(C_3) \subset C_2$, $((t_c t_d)^{-1})^p(C_1) \subset C_3$, and $((t_c t_d)^{-1})^p(C_2) \subset C_3$, for $p \neq 0$. For brevity, let us denote $(t_c t_d)^{-1}$ by g.

$g^p(\boldsymbol{C}_1) \subset \boldsymbol{C}_3$:

If $l \in C_1$, then $a_l \leq 0$, $b_l > 0$, and $2b_l + a_l > -1$. Then, we must show that $g^p(l) \in C_3$, $p \neq 0$. Let $g^p(l) = l'$, $p \neq 0$. In other words, we need to show that $a_{l'} < 0$, $b_{l'} \leq 0$ and $2a_{l'} < b_{l'}$. Since the update rules for g^{-1} can be derived by symmetry, we may assume that $p > 0$. In the case $a_l \leq 0$, $b_l > 0$ and $2b_l + a_l > -1$, we will consider the following five cases:

Case 1. For $-2a_l < b_l$, $a_l \leq 0$. By Equation 4, we have $\rho(g(l)) = (-a_l - b_l, -b_l)$. We obtain that $-a_l - b_l < 0$ and $-b_l < 0$ since $-2a_l < b_l$ and $a_l \leq 0$. Then, by Equation 4, we find $\rho(g^2(l)) = (-a_l - 3b_l, -b_l)$. In this case, since $-a_l - 3b_l < 5a_l \leq -5$, we have $-a_l - 3b_l < 0$ and $-b_l < 0$. Then, similarly, we find the p-th iterate of g,

$$\rho(g^p(l)) = (-a_l - (2p-1)b_l, -b_l),$$

$p \geq 2$. By similar argument in the above, we have

$$-a_l - (2p-1)b_l < (4p-3)a_l \leq 3 - 4p,$$

for $p \geq 2$. Hence, we obtain that $-a_l - (2p-1)b_l < 0$, $-b_l < 0$ and

$$\begin{aligned} -b_l - 2(-a_l - (2p-1)b_l) &= \\ 2a_l + (4p-3)b_l > -b_l + (4p-3)b_l &= \\ 4(p-1)b > 0, \end{aligned}$$

since $2a_l > -b_l$. So, $g^p(l)$ is in C_3.

Case 2. For $-2a_l = b_l$, $a_l < 0$, by Equation 4, we have $\rho(g(l)) = (a_l, 2a_l)$. Since $a_l < 0$, neither component is positive. Then, we will apply a similar process, we obtain that $\rho(g^p(l)) = ((4p-3)a_l, 2a_l)$, for $a_l < 0$, $p \geq 2$. Therefore, $(4p-3)a_l < 0$, $2a_l < 0$ and

$$2a_l - 2(4p-3)a_l = -2(4p-4)a = (4p-4)b > 0$$

since $-2a = b$. It follows that $g^p(l)$ is in C_3.

Case 3. For $-a_l - 1 < b_l < -2a_l$, $a_l < -1$, by Equation 4, $\rho(g(l)) = (-a_l - b_l, -b_l)$. Since $-a_l - 1 < b_l < -2a_l$, $a_l < -1$, we have $-a_l - b_l < 0$ and $-b_l < 0$. In this case, to find $g^p(l)$, again applying a similar process,

$$\rho(g^p(l)) = (-a_l - (2p-1)b_l, -b_l),$$

$p \geq 2$. Since

$$(4p-3)a_l < -a_l - (2p-1)b_l < 1 + (a_l+1)(2p-2),$$

for $p \geq 2$, $a \leq -2$. Hence, we obtain that $-a_l - (2p-1)b_l < 0$, $-b_l < 0$ and

$$\begin{aligned} -b_l - 2(-a_l - (2p-1)b_l) &= \\ 2a_l + (4p-3)b > 2a_l + (4p-3)(-a-1) \geq 4p - 7 &\geq 1, \end{aligned}$$

since $-a_l - 1 < b_l < -2a_l$, $a_l \leq -2$, $p \geq 2$. So, $g^p(l)$ is in C_3.

Case 4. For $\frac{-2a_l-1}{3} < b \leq -a_l - 1$, $a_l < -2$, by Equation 4, $\rho(g(l)) = (-a_l - b_l, -2a_l - 3b_l)$. We have $-a_l - b_l > 0$, $-2a_l - 3b_l < 0$, since $\frac{-2a_l-1}{3} < b \leq -a_l - 1$, $a_l \leq -3$. Now, by Equation 4, it follows that $\rho(g^2(l)) = (-a_l - 3b_l, -b_l)$. We have $-a_l - 3b_l < 0$, $-b_l < 0$. In this case, by Equation 4, we again conclude that the first and the second components are negative. So, continuing this process, we get

$$\rho(g^p(l)) = (-a_l - (2p-1)b_l, -b_l)$$

for $p \geq 3$. Then, $-a_l - (2p-1)b_l < 0$, $-b_l < 0$, and

$$\begin{aligned} -b_l - 2(-a_l - (2p-1)b_l) &= \\ 2a + (4p-3)b > 2a_l + (4p-3)\frac{(-2a_l-1)}{3} &= \\ \frac{(8p-12)(-a_l) + (3-4p)}{3} \geq 9 > 0, \end{aligned}$$

since $\frac{-2a_l-1}{3} < b \leq -a_l - 1$, $a_l \leq -3$, $p \geq 3$. Hence, $g^p(l)$ is in C_3.

Case 5. For $\frac{-a_l-1}{2} < b_l \leq \lfloor \frac{-2a_l-1}{3} \rfloor$, $a_l < -1$, by Equation 4, we have $\rho(g(l)) = (a_l + 2b_l, -2a_l - 3b_l)$. So, we see that $a_l + 2b_l \geq 0$ and $-2a_l - 3b_l > 0$, for $\frac{-a_l-1}{2} < b_l \leq \lfloor \frac{-2a_l-1}{3} \rfloor$, $a_l \leq -2$. Then, using Equation 4, we get $\rho(g^2(l)) = (-(a_l + 3b_l), -b_l)$. It can be seen that both components are negative. By a similar process, to find the p-th iteration of g, we get

$$\rho(g^p(l)) = (-(a_l + (2p-1)b_l), -b_l),$$

for $p \geq 3$. Since $\frac{-a_l-1}{2} < b_l \leq \lfloor \frac{-2a_l-1}{3} \rfloor$, $a_l \leq -2$, and $p \geq 3$, we have $-(a_l + (2p-1)b_l) < 0$, $-b_l < 0$ and

$$\begin{aligned} -b_l + 2(a_l + (2p-1)b_l) &= \\ 2a_l + (4p-3)b > 2a_l + (4p-3)(\frac{-a_l-1}{2}) &= \\ \frac{(7-4p)a + (3-4p)}{2} &\geq \frac{1}{2} > 0. \end{aligned}$$

Hence, $g^p(l)$ is in C_3.

$g^p(\boldsymbol{C}_2) \subset \boldsymbol{C}_3$:

If $l \in C_2$ then $a_l \geq 0$, $b_l < 0$. Then, we must show that $g^p(l) \in C_3$, $p \neq 0$. In this case, we will use by Equation 4, we have $\rho(g(l)) = (-3a_l + 2b_l, -2a_l + b_l)$. Since both of the components are negative, similarly as above, to get the p-th iteration g, using Equation 4, we get

$$\rho(g^p(l)) = ((1-4p)a_l + 2pb_l, -2a_l + b_l),$$

for $p \geq 2$. Since $a_l \geq 0$, $b_l < 0$, and $p \geq 2$, we conclude that both of the components are negative and

$$(-2a_l + b_l) - 2((1-4p)a_l + 2pb_l) = (8p-4)a + (1-4p)b > 0.$$

So, $g^p(l)$ is in C_3.

$t_c^p(\boldsymbol{C}_3) \subset \boldsymbol{C}_1$:

If $l \in C_3$ then $a_l < 0$, $b_l \leq 0$, $2a_l < b_l$. In this case, by using Equation 1, we have $\rho(t_c(l)) = (a_l + b_l, -2a_l - b_l)$. The first component is negative, the second component is positive. Then,

$$\rho(t_c^2(l)) = (a_l + b_l, -2(a_l + b_l) - (2a_l + b_l)).$$

Again, we obtain that the first component is negative, the second component is positive. Then, similarly, the p-th iteration of t_c,

$$\rho(t_c^p(l)) = (a_l + b_l, -2(p-1)(a_l + b_l) - (2a_l + b_l)),$$

$p \geq 2$. We can easily see that the first component is negative, the second component is positive. Also,

$$2[-2(p-1)(a_l + b_l) - (2a_l + b_l)] + (a_l + b_l) = (1-4p)a_l + (3-4p)b_l.$$

since $p \geq 2$, we obtain that $(1-4p)a_l + (3-4p)b_l > 0 > -1$. Hence, $t_c^p(l)$ is in C_1.

$t_c^p(\boldsymbol{C}_2) \subset \boldsymbol{C}_1$:

If $l \in C_2$ then $a_l \geq 0$, $b_l < 0$. By using Equation 1, we have $\rho(t_c(l)) = (-a_l + b_l, -b_l)$. The first component is negative, the second component is positive. By a similar process in above, we get

$$\rho(t_c^p(l)) = (-a_l + b_l, -2(p-1)(-a_l+b_l) - b_l),$$

$p \geq 2$. It can be seen that the first component is negative, the second component is positive. Moreover,

$$2[-2(p-1)(-a_l+b_l) - b_l] + (-a_l+b_l) = (4p-5)a_l + (3-4p)b_l > 0 > -1,$$

since $p \geq 2$. Hence, $t_c^p(l)$ is in C_1.

$t_d^p(\boldsymbol{C}_3) \subset \boldsymbol{C}_2$:

If $l \in C_3$ then $a_l < 0$, $b_l \leq 0$, $2a_l < b_l$. In this case, there are two subcases:

Subcase (i): $a < b$**.** By using Equation 2, we have $\rho(t_d(l)) = (-a_l + b_l, -b_l)$. Since both of the components are positive as $a < b$. In this case, again applying t_d, we get $\rho(t_d^2(l)) = (-a_l, 2a_l - b_l)$ by Equation 2. Now, we have the first component $-a_l > 0$, the second component $2a_l - b_l < 0$. Then, similarly, p-th iteration of t_d, we get $\rho(t_d^p(l)) = (-a_l, 2(p-1)a_l - b_l)$, $p \geq 3$. The first component is positive, but the second one is negative as $a < b$. Thus, $t_d^p(l)$ is in C_2.

Subcase (ii): $2a < b \leq a$**.** By using Equation 2, we have $\rho(t_d(l)) = (-a_l + b_l, -2a_l + b_l)$. Since $b - a \leq 0$ and $0 < b - 2a$, the first component is not positive but the second one is positive. Now, by using Equation 2, we get $\rho(t_d^2(l)) = (-a_l, 2a_l - b_l)$. We have $-a_l > 0$ and $2a_l - b_l < 0$. Then, if we apply similar process to get $t_d^p(l)$, then we get $\rho(t_d^p(l)) = (-a_l, 2(p-1)a_l - b_l)$, $p \geq 3$. Since the first component is positive and the second component is negative as $2a < b$. Therefore, $t_d^p(l)$ is in C_2.

By using argument of the proof in Section 4, it can be easily seen that $t_d^p(C_1) \subset C_2$.

This finishes the proof. □

6. The action of pseudo-Anosov map $t_d t_c^{-1}$ and $t_d^{-1} t_c$

In this section, we will present an algorithm that determines the actions of $t_d t_c^{-1}$ and $t_d^{-1} t_c$ of M making use of orbits of the actions t_c and t_d. This algorithm will demonstrate how the iteration changes geometrically.

Let $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$ be the Dynnikov coordinates of a simple closed curve l in M. We will write $(a', b') \in \mathbb{Z}^2 \setminus \{0\}$ to denote the Dynnikov coordinates of $\psi(l)$, where ψ is a generator of PMod(M).

Main Algorithm for $t_d t_c^{-1}$. If $b \geq 0$ apply Algorithm 1 otherwise apply Algorithm 2.

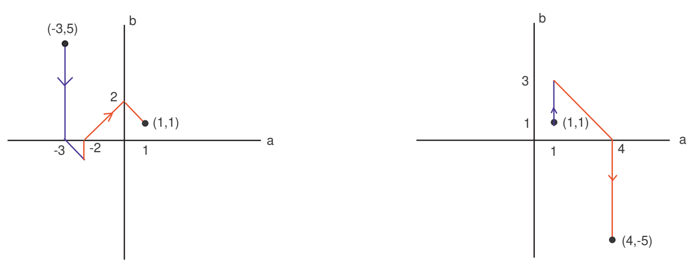
Algorithm 1. Given $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$ the Dynnikov coordinates of a simple closed curve l in M.

Step 1: Apply backward $2|a|$ units along $|a|^{th}$ level of orbits of the action of t_c, and input the new coordinates (a', b') to Step 2.

Step 2: If $b' > 0$, apply forward $2(|a'| + |b'|)$ units along $(|a'| + |b'|)^{th}$ level of orbits of the action of t_d, and input the new coordinates (a'', b'') to Step 4. Otherwise, input (a', b') to Step 3.

Step 3: If $b' \leq 0$, apply forward $2|a'|$ units along $|a'|^{th}$ level of orbits of the action of t_d, and input the new coordinates (a'', b'') to Step 4. Otherwise, input (a', b') to Step 2.

Step 4: Since (a'', b'') is the Dynnikov coordinates of $t_d(l')$, where $l' = t_c^{-1}(l)$, and (a', b') is the Dynnikov coordinates of $t_c^{-1}(l)$, (a'', b'') is the Dynnikov coordinates of $t_d t_c^{-1}(l)$.



Backward 6 units along 3rd level of the orbits of the action t_c and forward 4 units along 2nd level of the orbits of the action t_d

Backward 2 units along 1st level of the orbits of the action of t_c and forward 8 units along 4th level of the orbits of the action of t_d

FIGURE 6

The following algorithm works for case $b < 0$.

Algorithm 2. Given $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$ the Dynnikov coordinates of a simple closed curve l in M.

Step 1: Apply backward $2(|a| + |b|)$ units along $(|a| + |b|)^{th}$ level of orbits of the action of t_c, and input the new coordinates (a', b') to Step 2.

Step 2: If $b' > 0$, apply forward $2(|a'| + |b'|)$ units along $(|a'| + |b'|)^{th}$ level of orbits of the action of t_d, and input the new coordinates (a'', b'') to Step 4.

Otherwise, input (a', b') to Step 3.

Step 3: If $b' \leq 0$, apply forward $2|a'|$ units along $|a'|^{th}$ level of orbits of the action of t_d, and input the new coordinates (a'', b'') to Step 4. Otherwise, input (a', b') to Step 2.

Step 4: Since (a'', b'') is the Dynnikov coordinates of $t_d(l')$, where $l' = t_c^{-1}(l)$, and (a', b') is the Dynnikov coordinates of $t_c^{-1}(l)$, (a'', b'') is the Dynnikov coordinates of $t_d t_c^{-1}(l)$.

Main Algorithm for $t_d^{-1} t_c$. If $b \geq 0$ apply Algorithm 1 otherwise apply Algorithm 2.

Algorithm 1. Given $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$ the Dynnikov coordinates of a simple closed curve l in M.

Step 1: Apply forward $2|a|$ units along $|a|^{th}$ level of orbits of the action of t_c, and input the new coordinates (a', b') to Step 2.

Step 2: If $b' > 0$, apply backward $2(|a'| + |b'|)$ units along $(|a'| + |b'|)^{th}$ level of orbits of the action of t_d, and input the new coordinates (a'', b'') to Step 4. Otherwise, input (a', b') to Step 3.

Step 3: If $b' \leq 0$, apply backward $2|a'|$ units along $|a'|^{th}$ level of orbits of the action of t_d, and input the new coordinates (a'', b'') to Step 4. Otherwise, input (a', b') to Step 2.

Step 4: Since (a'', b'') is the Dynnikov coordinates of $t_d^{-1}(l')$, where $l' = t_c(l)$, and (a', b') is the Dynnikov coordinates of $t_c(l)$, (a'', b'') is the Dynnikov coordinates of $t_d^{-1} t_c(l)$.

The following algorithm works for case $b < 0$.

Algorithm 2. Given $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$ the Dynnikov coordinates of a simple closed curve l in M.

Step 1: Apply forward $2(|a| + |b|)$ units along $(|a| + |b|)^{th}$ level of orbits of the action of t_c, and input the new coordinates (a', b') to Step 2.

Step 2: If $b' > 0$, apply backward $2(|a'| + |b'|)$ units along $(|a'| + |b'|)^{th}$ level of orbits of the action of t_d, and input the new coordinates (a'', b'') to Step 4. Otherwise, input (a', b') to Step 3.

Step 3: If $b' \leq 0$, apply backward $2|a'|$ units along $|a'|^{th}$ level of orbits of the action of t_d, and input the new coordinates (a'', b'') to Step 4. Otherwise, input (a', b') to Step 2.

Step 4: Since (a'', b'') is the Dynnikov coordinates of $t_d^{-1}(l')$, where $l' = t_c(l)$, and (a', b') is the Dynnikov coordinates of $t_c(l)$, (a'', b'') is the Dynnikov coordinates of $t_d^{-1} t_c(l)$.

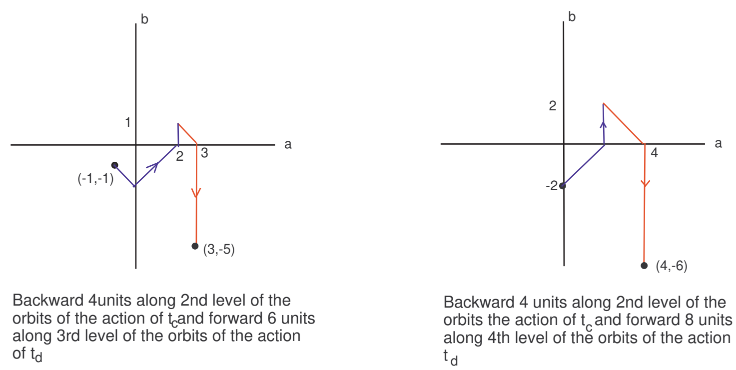


FIGURE 7

Now, we will give the sequences of the Dynnikov coordinates of values of $(t_d t_c^{-1})^{10}(c)$ and $(t_d t_c^{-1})^{10}(d)$, where c $(\rho(c) = (0,1))$ and d $(\rho(d) = (0,-1))$ are as in Figure 1.

$(0,1) \overset{t_d t_c^{-1}}{\longrightarrow} (1,-1) \overset{t_d t_c^{-1}}{\longrightarrow} (5,-7) \overset{t_d t_c^{-1}}{\longrightarrow} (29,-41) \overset{t_d t_c^{-1}}{\longrightarrow} (169,-239) \overset{t_d t_c^{-1}}{\longrightarrow} (985,-1393) \overset{t_d t_c^{-1}}{\longrightarrow} (5741,-8119) \overset{t_d t_c^{-1}}{\longrightarrow} (33461,-47321) \overset{t_d t_c^{-1}}{\longrightarrow} (195025,-275807) \overset{t_d t_c^{-1}}{\longrightarrow} (1136689,-1607521) \overset{t_d t_c^{-1}}{\longrightarrow} (6625109,-9369319) \overset{t_d t_c^{-1}}{\longrightarrow} (38613965,-54608393)$

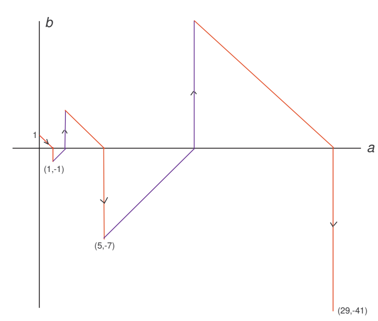


FIGURE 8

$(0,-1) \overset{t_d t_c^{-1}}{\longrightarrow} (2,-3) \overset{t_d t_c^{-1}}{\longrightarrow} (12,-17) \overset{t_d t_c^{-1}}{\longrightarrow} (70,-99) \overset{t_d t_c^{-1}}{\longrightarrow} (408,-577) \overset{t_d t_c^{-1}}{\longrightarrow} (2378,-3363) \overset{t_d t_c^{-1}}{\longrightarrow} (13860,-19601) \overset{t_d t_c^{-1}}{\longrightarrow} (80782,-114243) \overset{t_d t_c^{-1}}{\longrightarrow} (470832,-665857) \overset{t_d t_c^{-1}}{\longrightarrow} (2744210,-3880899) \overset{t_d t_c^{-1}}{\longrightarrow} (15994428,-22619537) \overset{t_d t_c^{-1}}{\longrightarrow} (93222358,-131836323)$

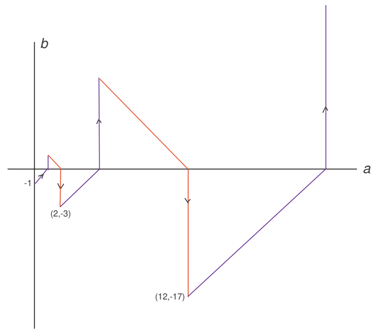


FIGURE 9

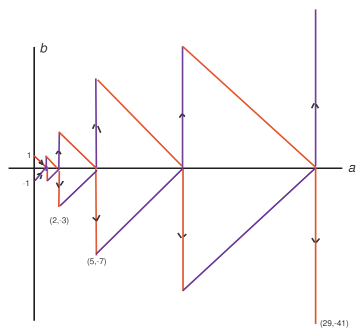


FIGURE 10

Remark 6.1. 1) If the pairs in the sequence of values generated by the iteration of $t_d t_c^{-1}$, starting from the point $(0, 1)$, are combined with the pairs in the sequence of values generated by the iteration of $t_d t_c^{-1}$, starting from the point $(0, -1)$, respectively; we obtain a sequence of isosceles triangles (see Figure 10. The areas of these triangles are 1, 1, 9, 49, 289, and so on.

2) The pairs in the sequence of values generated by the iteration of $t_d t_c^{-1}$, starting from the point $(0, 1)$ satisfy the equation $2a^2 - b^2 = 1$, except for the pair $(0, 1)$. These are the NSW numbers (named after Newman, Shanks, and Williams) integers m that solve the Diophantine equation $2n^2 = m^2 + 1$. As the starting points change, it is observed that the pairs in the value sequence of values generated by the iteration of $t_d t_c^{-1}$ satisfy the equation $2a^2 - b^2 = Det(A)$, where $A = \begin{bmatrix} 2a & b \\ b & a \end{bmatrix}$.

7. The Dynnikov coordinates in M in terms of (p, q)-torus coordinates via a double branched cover

We consider a double branched cover over a sphere M with four marked points by a torus T with one boundary component $p : T \to M$(the marked points are the branch points). The deck transformation is a hyperbolic involution τ switching the two sheets. τ induces a bijection between the set of homotopy classes of nontrivial simple closed curves in T and the set of homotopy classes of nontrivial simple closed curves in M (see Proposition 2.6 in [3]). Let γ and δ be two nontrivial simple closed curves in T intersecting each other in one point. Let us encode γ and δ with $(1, 0)$-torus coordinate and $(0, 1)$-torus coordinate, respectively. Up to homotopy, we can assume that γ and δ project to c and d in M via τ, as in Figure 1. We recall that the Dynnikov coordinates of c and d in M are $(0, 1)$ and $(0, -1)$, respectively. If we consider the torus with one hole as a square with opposite sides identified, as shown in the Figure 11, we can determine the Dynnikov coordinates in M of a curve whose torus coordinates are given in T using the formula below:

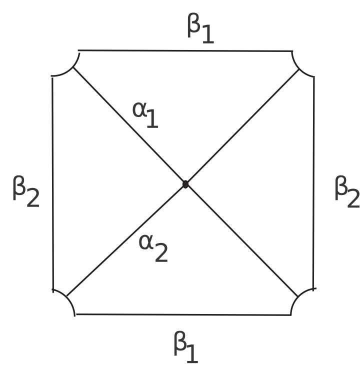


Figure 11

$$a = \frac{\alpha_2 - \alpha_1}{2} = \frac{|p-q| - |p+q|}{2},$$

$$b = \frac{\beta_1 - \beta_2}{2} = \frac{2|p| - 2|q|}{2} = |p| - |q|.$$

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References

[1] F. Atalan and M. Korkmaz, Number of pseudo–Anosov elements in the mapping class group of a four–holed sphere, Turk J Math **34** (2010) 585-592.

[2] I.A. Dynnikov, On a Yang-Baxter mapping and the Dehornoy ordering. Russ Math Surveys **57(3)** (2002) 592-594.

[3] B. Farb and D. Margalit, A primer on mapping class groups Princeton University Press, New Jersey, 2012.

[4] T. Hall and S.O. Yurttas, On the topological entropy of families of braids, Topology and its Applications, **156** (2009) 1554–1564.

[5] A. Ishida, The structure of subgroup of mapping class groups generated by two Dehn twists, Proc. Japan Acad. Ser. A Math. Sci., **72**(10) (1996) 240–241.

[6] H. Hamidi-Tehrani, Groups generated by positive multi-twists and the fake lantern problem, Algebr. Geom. Topol., 2:1155–1178, 2002 (electronic).

[7] M. Stukow, Subgroups generated by two Dehn twists on a nonorientable surface, Topology Proc. **50** (2017) 151-201.

[8] S.O. Yurttas, Geometric intersection of curves on punctured disks,J. Math. Soc. Japan **65(4)** (2013) 1153-1168.

[9] S.O. Yurttas, Dynnikov and train track transition matrices of pseudo-Anosov braids, Discrete and continuous dynamical systems **36(1)** (2016) 541-570.

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