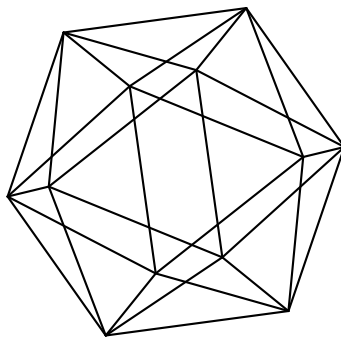


Max-Planck-Institut für Mathematik Bonn

On the Selmer group and rank of a family of elliptic
curves and curves of genus one violating the Hasse
Principle

by

Eleni Agathocleous



Max-Planck-Institut für Mathematik
Preprint Series 2024 (24)

Date of submission: September 12, 2024

On the Selmer group and rank of a family of elliptic curves and curves of genus one violating the Hasse Principle

by

Eleni Agathocleous

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

ON THE SELMER GROUP AND RANK OF A FAMILY OF ELLIPTIC CURVES AND CURVES OF GENUS ONE VIOLATING THE HASSE PRINCIPLE

ELENI AGATHOCLEOUS

Abstract ¹

We study an infinite family of j -invariant zero elliptic curves $E_D : y^2 = x^3 + 16D$ and their λ -isogenous curves $E_{D'} : y^2 = x^3 - 27 \cdot 16D$, where D and $D' = -3D$ are fundamental discriminants of a specific form, and λ is an isogeny of degree 3. A result of Honda guarantees that for our discriminants D , the quadratic number field $K_D = \mathbb{Q}(\sqrt{D})$ always has non-trivial 3-class group. We prove a series of results related to the set of rational points $E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$, and the $SL(2, \mathbb{Z})$ -equivalence classes of irreducible integral binary cubic forms of discriminant D . By assuming finiteness of the Tate-Shafarevich group, we derive a parity result between the rank of E_D and the rank of its 3-Selmer group, and we establish lower and upper bounds for the rank of our elliptic curves. Finally, we give explicit classes of genus-1 curves that correspond to irreducible integral binary cubic forms of discriminant $D = 48035713$, and we show that every curve in these classes violates the Hasse Principle.

AMS Mathematics Subject Classification: Primary: 11R29, 11G05.

Keywords: Elliptic Curves, Selmer group, Rank, Ideal Class Group, Tate-Shafarevich, Hasse Principle.

1. INTRODUCTION

The relation between the Selmer group of an elliptic curve and the ideal class group of an underlying quadratic number field is well known, and it is present in many papers related to descent on elliptic curves, such as [18], [19] and [42]. The particular relation between the 3-part of the ideal class group and the 3-Selmer group of elliptic curves, has received great attention by many authors, as one can see for example in the works of [4], [40], [41], [46], and more recently of [9] and [10]. Satgé [41] in particular, studied the Selmer group of the general family of elliptic curves of the form $y^2 = x^3 + k$, relative to a 3-isogeny, and showed that the rank of the Selmer group depends heavily on the decomposition of k .

In this paper, we study a specific family of such curves of j -invariant zero, namely the elliptic curves $E_D : y^2 = x^3 + 16D$, where D is a squarefree integer of the form $D = D_{m,n} = 4m^3 - 27n^2$ and the corresponding cubic polynomial $f_{m,n}(x) = x^3 - mx + n$ is irreducible. The curves E_D have an isogeny λ of degree 3, and their λ -isogenous curves $E_{D'}$ are of the form $E_{D'} : y^2 = x^3 - 27 \cdot 16D$, with $D' = -3D$ also a fundamental discriminant since $3 \nmid D$. A result of Honda [29] guarantees that for our discriminants D , the quadratic number field $K_D = \mathbb{Q}(\sqrt{D})$ has non-trivial 3-class group. Our main focus is the study of a direct relation between the elements of the quotient group $E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))$ under the Fundamental 3-descent map,

¹This work has been supported by the European Union's H2020 Programme under grant agreement number ERC-669891.

and the unramified cubic extensions of K_D , emerging from the so-called 3-virtual units of the quadratic resolvent $K_{D'} = \mathbb{Q}(\sqrt{-3D})$ of K_D . The topic and the main results of each section, are as follows:

In Section 2 we discuss known results related to the unramified cubic extensions of the quadratic number fields K_D , in relation to the so-called 3-virtual units of $K_{D'}$. Via a direct application of the results of Erdős [24], we show in Lemmas 2.4 and 2.6 that the set \mathcal{D} containing all these odd fundamental discriminants $D_{m,n}$ whose corresponding cubic polynomial $f_{m,n}(x) = x^3 - mx + n$ is irreducible, is an infinite set, and therefore the corresponding family of elliptic curves $\{E_D \mid D \in \mathcal{D}\}$ is infinite. At the same time, this result establishes the existence of an infinite family of cubic fields L , namely those generated by the polynomials $f_{m,n}$, which are monogenic; i.e. their maximal order \mathcal{O}_L is of the form $\mathbb{Z}[\alpha]$, for some $\alpha \in \mathcal{O}_L$. The property of monogenicity is directly related to the known Hasse's Problem from the 1960's regarding the classification of algebraic number fields (see for example [37] and [38]), and many authors study this problem in its own right (see for example [12], [30] and [31]). The study of cubic monogenic fields has recently gained increased attention, particularly due to their direct connection with the rank of the elliptic curves $y^2 = x^3 + k$ [1]. This direct connection is a key focus of our paper, and in the subsequent sections we explicitly demonstrate through a series of results, how monogenic orders of the cubic fields L of discriminant D , whose index in the maximal order \mathcal{O}_L is an integral cube, provide a lower bound for the rank of the elliptic curves E_D .

In Section 3, we discuss known results on descent by 3-isogeny adjusted to the elliptic curves E_D . In Proposition 3.4, via the results of Satgé [41], we prove that the ranks of the Selmer groups \mathcal{S}_λ and $\mathcal{S}_{\lambda'}$, with respect to the isogenies λ and λ' , are always equal to the following specific values associated with the 3-rank of the ideal class group of K_D , which we denote by $r_3(D)$:

Proposition 1.1. *For every $D \in \mathcal{D}$, the rank of the Selmer groups \mathcal{S}_λ and $\mathcal{S}_{\lambda'}$ of the curves E_D and $E_{D'}$ are as follows:*

$$\begin{aligned} d(\mathcal{S}_\lambda) &= d(\mathcal{S}_{\lambda'}) = r_3(D), & \text{if } D < -4 \\ d(\mathcal{S}_\lambda) &= r_3(D) \text{ and } d(\mathcal{S}_{\lambda'}) = r_3(D) + 1, & \text{if } D > 0. \end{aligned}$$

In the same section, by employing Voronoi's algorithm, we prove in Proposition 3.18 that a point P lies in the set $E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$ if and only if there exists an irreducible integral binary cubic form of discriminant D having leading coefficient an integral cube, and which defines the cubic field L associated to the point P . The direct relation between these points P and their associated cubic field L is proved in Lemma 3.17.

In Section 4, by assuming finiteness of the Tate-Shafarevich group and by employing Proposition 1.1 from above, we derive in Corollary 4.1 a parity result between the rank of E_D and the sum of the ranks of its two Selmer groups \mathcal{S}_λ and $\mathcal{S}_{\lambda'}$. This corollary, together with Proposition 4.2 that gives lower and upper bounds for the rank of our elliptic curves, constitute the

main result of the paper. The lower bound for the curves E_D is directly related to the rank of a group \mathcal{M} that is a subgroup of the 3-Selmer group of $K_{D'}$, by definition. This group \mathcal{M} is generated by all those 3-virtual units, which give rise to cubic fields that can be defined by a binary cubic form of discriminant D and with leading coefficient a cube. By our Proposition 3.18, the group \mathcal{M} corresponds to the so-called *solvable* part of the Selmer group \mathcal{S}_λ of the elliptic curve E_D , hence its rank \mathfrak{r}_3 as an \mathbb{F}_3 -vector space serves as a lower bound. In the proposition below we give the lower and upper bounds that we have established for the curves E_D . The third statement of the proposition is the parity result that we proved in the aforementioned Corollary 4.1.

Proposition 1.2. *For every $D \in \mathcal{D}$, by assuming finiteness of the 3-primary part $\text{III}(E_D)[3^\infty]$ of the Tate-Shafarevich group $\text{III}(E_D)$ of the elliptic curves E_D , the rank of the elliptic curves E_D , and therefore of $E_{D'}$, is bounded as follows:*

(1) *If \mathfrak{r}_3 is odd, then*

$$(i) \quad 2 \leq \mathfrak{r}_3 + 1 \leq \text{rank}(E_D) \leq 2r_3(D), \text{ if } D < -4$$

$$(ii) \quad 1 \leq \mathfrak{r}_3 \leq \text{rank}(E_D) \leq 2r_3(D) + 1, \text{ if } D > 4.$$

(2) *If \mathfrak{r}_3 is even, then*

$$(i) \quad 2 \leq \mathfrak{r}_3 \leq \text{rank}(E_D) \leq 2r_3(D), \text{ if } D < -4$$

$$(ii) \quad \mathfrak{r}_3 + 1 \leq \text{rank}(E_D) \leq 2r_3(D) + 1, \text{ if } D > 4.$$

(3) *$\text{rank}(E_D)$ is always even for negative $D \in \mathcal{D}$ and is always odd for positive $D \in \mathcal{D}$.*

The explicit knowledge of the elements of the Selmer group associated to the isogeny λ that we established in the previous sections, allows us in Section 5 to describe the classes of *solvable* Torsors associated with the Selmer group \mathcal{S}_λ . In Proposition 5.3 in particular, we give an if-and-only-if condition on whether specific genus-1 curves violate the Hasse Principle. In Section 5.1 we discuss the *solvable* part of \mathcal{S}_λ in relation to the open question of the uniform boundedness of the ranks of elliptic curves over \mathbb{Q} , and we address computational aspects related to the lower bound associated with \mathfrak{r}_3 .

Finally, in Section 6 we give explicit classes of genus-1 curves that correspond to irreducible integral binary cubic forms of discriminant $D = D_{229,3}$, and we show that every curve in these classes violates the Hasse Principle.

2. UNRAMIFIED CUBIC EXTENSIONS OF QUADRATIC NUMBER FIELDS

2.1. The 3-part of the ideal class group. Throughout the paper, $\text{disc}(\cdot)$ will denote the integer that is equal to the discriminant of the object in the parentheses.

Honda, in [29, Proposition 10], showed the following important result:

Proposition 2.1. [29, Proposition 10] *If the class number of a quadratic number field K is a multiple of 3, K must be of the form*

$$K_{\alpha,\beta} = \mathbb{Q}(\sqrt{4\alpha^3 - 27\beta^2}),$$

for some $\alpha, \beta \in \mathbb{Z}$. Conversely, if for integers α, β we have $\gcd(\alpha, 3\beta) = 1$ and if α cannot be represented by a form $(\beta + c^3)c^{-1}$ for any $c \in \mathbb{Z}$, then $3|h_{K_{\alpha,\beta}}$, the class number of $K_{\alpha,\beta}$. \square

We note that in Honda's result,

$$4\alpha^3 - 27\beta^2 = \mathfrak{f}^2 \text{disc}(K_{\alpha,\beta}),$$

where $\text{disc}(K_{\alpha,\beta})$ is the fundamental discriminant of $K_{\alpha,\beta}$ and $\mathfrak{f} \geq 1$ is some conductor. Furthermore, the condition on α not being represented by the given form, for any $c \in \mathbb{Z}$, implies that the cubic polynomial

$$f_{\alpha,\beta}(x) = x^3 - \alpha x + \beta \in \mathbb{Z}[x]$$

is irreducible. In the same paper [29], Honda showed that the irreducible cubic polynomials of the form $f_{\alpha,\beta}$ define the unramified cubic extensions of $K_{\alpha,\beta}$, and the Galois group of their Galois closure, equivalently, the Galois group of their splitting field

$$(1) \quad L_{\alpha,\beta} \supsetneq K_{\alpha,\beta} \supsetneq \mathbb{Q},$$

is isomorphic to the symmetric group \mathbb{S}_3 .

2.2. Cubic fields of fundamental discriminant. Denote by L_d the algebraic number fields of degree d whose splitting field Spl_d has Galois group isomorphic to the symmetric group \mathbb{S}_d . For degree $d \leq 5$, Kondo in [33, §4] and also Yamamura in [47, Remark on pg. 107] showed the following:

Theorem 2.2. ([33, §4] & [47, Remark on pg. 107]) *Assume that Spl_d/K is unramified at all finite primes, where K is a quadratic subfield. Then, if $d \leq 5$, the set of the non-conjugate algebraic number fields L_d equals the set of the non-conjugate algebraic number fields of degree d with discriminant equal to that of the quadratic number field K . \square*

For $d = 3$, the unramified cubic extensions Spl_d/K correspond to the unramified cubic extensions of Honda described in Proposition 2.1 above, with the underlying quadratic number fields K being of the form $K_{\alpha,\beta}$. The cubic fields L_3 have discriminant $\text{disc}(L_3) = \text{disc}(K_{\alpha,\beta})$, they can be defined by a cubic polynomial of the form $f_{\alpha,\beta}$, and they identify with the splitting fields $L_{\alpha,\beta}$ of Relation (1).

2.3. Our Discriminants D . For any pair of integers m, n we compute the quantity $D_{m,n} = 4m^3 - 27n^2$. Whenever $D_{m,n}$ is squarefree we include it in the set

$$\mathcal{D}' := \{D_{m,n} \mid m, n \in \mathbb{Z} \text{ and } D \text{ is squarefree}\},$$

and the corresponding pair of integers (m, n) we include it in the set

$$\mathcal{P}' := \{(m, n) \in \mathbb{Z}^2 \mid D_{m,n} \in \mathcal{D}'\}.$$

Remark 2.3. *Since $D_{m,n} \in \mathcal{D}'$ is squarefree, the following are straightforward observations:*

- (1) $\gcd(2m, 3n) = 1$, yielding in particular that n is odd, and therefore
- (2) $D_{m,n} = 4m^3 - 27n^2 \equiv 1 \pmod{4}$.

We show below, via a direct application of a theorem of Erdős, that the set \mathcal{D}' , and therefore the set \mathcal{P}' , is infinite.

Lemma 2.4. *The sets \mathcal{D}' and \mathcal{P}' are infinite.*

Proof. For any fixed odd integer n_0 , we consider the cubic polynomial

$$h_{n_0}(x) = 4x^3 - 27n_0^2.$$

It is straightforward to see that there are no linear polynomials g_1, g_2 with integral coefficients such that $h_{n_0} = g_1^2 g_2$. Therefore, following directly from Erdős' Theorem [24, Theorem, pg. 417], we have that there are infinitely many integers m for which the value $h_{n_0}(m)$ is squarefree. \square

Remark 2.5. *If we fix an integer m_0 coprime to 3, we can look instead at the quadratic polynomials $g_{m_0}(x) = 27x^2 - 4m_0^3$. Again, it is easy to see that g_{m_0} cannot be written as a square of a linear polynomial with integral coefficients. Then, as Erdős mentions in the same paper [24, pp. 416 - 417], it has been known in this case as well that there are infinitely many integers n for which $g_{m_0}(n)$ is squarefree. Moreover, in this case, the integers n for which $g_{m_0}(n)$ is squarefree, have positive density.*

Our discriminants $D_{m,n}$ are fundamental. Hence, the ones whose corresponding cubic polynomial

$$f_{m,n}(x) = x^3 - mx + n$$

is irreducible, are precisely the Honda discriminants with conductor $\mathfrak{f} = 1$ and with $\text{Gal}(L_{m,n}/\mathbb{Q}) \cong S_3$, where the fields $L_{m,n}$ are the splitting fields of Relation (1) corresponding to these fundamental discriminants $D_{m,n}$.

Lemma 2.6. *The set of cubic polynomials*

$$\mathcal{F} := \{f_{m,n} \mid f_{m,n} \text{ irreducible, } \text{disc}(f_{m,n}) \in \mathcal{D}', \text{Gal}(L_{m,n}/\mathbb{Q}) \cong S_3\}$$

is infinite.

Proof. For any pair $(m, n) \in \mathcal{P}'$, consider the corresponding polynomial $f_{m,n}$ and assume that it is reducible. Then, there is a monic linear integral polynomial such that

$$f_{m,n}(x) = (x + a)(x^2 + Ax + B).$$

Elementary computations show that in this case we should have

$$a = -A, \quad m = B - A^2, \quad \text{and} \quad n = -AB.$$

Since n is always odd, because $(m, n) \in \mathcal{P}'$, both A and B should also be odd, which implies that if $f_{m,n}$ is reducible, then m must be even.

Then, for any fixed m_0 odd, Remark 2.5 gives us the existence of infinitely many fundamental discriminants $D_{m_0,n}$ and infinitely many corresponding irreducible cubic polynomials $f_{m_0,n}$. \square

Given the set \mathcal{F} of Lemma 2.6, we define below the sets $\mathcal{P} \subseteq \mathcal{P}'$ and $\mathcal{D} \subseteq \mathcal{D}'$, which by Lemma 2.6, are also infinite:

$$\mathcal{D} := \{D_{m,n} \in \mathcal{D}' \mid f_{m,n} \in \mathcal{F}\} \subseteq \mathcal{D}',$$

$$\mathcal{P} := \{(m,n) \in \mathcal{P}' \mid f_{m,n} \in \mathcal{F}\} \subseteq \mathcal{P}'.$$

Our cubic polynomials $f_{m,n} \in \mathcal{F}$ define a subfamily of cubic fields, let us denote them by $L_{m,n}$, and since $\text{disc}(L_{m,n}) = D_{m,n} = \text{disc}(f_{m,n})$, this subfamily contains precisely those cubic fields L_3 of Theorem 2.2 which are monogenic.

From now on and throughout the paper, the letter D will always correspond to a discriminant $D_{m,n} \in \mathcal{D}$, for some pair of integers $(m,n) \in \mathcal{P}$, and with corresponding irreducible polynomial $f_{m,n} \in \mathcal{F}$. Whenever it is clear, we will lose the subscript and we will denote $D_{m,n}$ only by D .

We will always denote by K_D the quadratic number field $K_D = \mathbb{Q}(\sqrt{D})$, and by $K_{D'} = \mathbb{Q}(\sqrt{D'})$ its quadratic resolvent, where $D' = -3D$ is also a fundamental discriminant, since $3 \nmid D$. Furthermore, we always have that

$$D' = -3D \equiv 1 \pmod{4}.$$

Finally, we will say that a polynomial of the form $f_{\alpha,\beta}$ is in *standard form*, if there is no integer c such that $c^2 \mid \alpha$ and $c^3 \mid \beta$.

2.4. Unramified cubic extensions and 3-virtual units. In [15, Chapter 5], for a given prime l , number field K and ring of integers O_K , Cohen gives the definition of the group $V_l(K)$ of l -virtual units of K and proves a number of important properties. We cite below what we need, by adjusting them to our case, for $l = 3$.

Proposition 2.7. [15, Proposition 5.2.3] *Let $\mu \in K^\times$. The following two properties are equivalent:*

(1) *There exists an ideal \mathfrak{a} such that $\mu O_K = \mathfrak{a}^3$.*

(2) *The element μ belongs to the group generated by the units, the cube powers of elements of K^\times and the cube powers of nonprincipal ideals which become principal when raised to the cube power, hence they belong to the 3-torsion part $Cl(K)[3]$ of the ideal class group $Cl(K)$ of K .*

Definition 2.8. [15, Definition 5.2.4]

(1) *An element $\mu \in K^\times$ satisfying one of the two equivalent conditions of the above proposition will be called a 3-virtual unit.*

(2) *The set of 3-virtual units forms a multiplicative group which we denote by $V_3(K)$.*

(3) *The quotient group $\text{Sel}_3(K) := V_3(K)/K^{\times 3}$ will be called the 3-Selmer group of K .*

The following sequence [16, Section 5.2] shows the relation between the Selmer group $\text{Sel}_3(K) = V_3(K)/K^{\times 3}$ of K , the 3-part of its ideal class group, and the group of units $U_K \subset \mathcal{O}_K$:

$$(2) \quad 1 \rightarrow \frac{U_K}{U_K^3} \rightarrow \text{Sel}_3(K) \xrightarrow{\Phi} Cl(K)[3] \rightarrow 1.$$

We denote by $[\mu] := \mu \bmod (K^\times)^3$ the class of a 3-virtual unit $\mu = \mathfrak{a}^3$ in $\text{Sel}_3(K)$. The map Φ sends the class $[\mu] \in \text{Sel}_3(K)$, to the ideal class $[\mathfrak{a}] \in \text{Cl}(K)[3]$. The pre-image of Φ is unique, up to a unit. More specifically,

$$\Phi^{-1}([\mathfrak{a}]) = \{[\mu], \varepsilon[\mu], \varepsilon^2[\mu]\}, \text{ where } \varepsilon \text{ is the fundamental unit of } K.$$

Of course, if $\text{disc}(K) < -4$, there are no units other than ± 1 and Φ is an isomorphism.

We recall that an ideal of \mathcal{O}_K is called *primitive* if there exists no rational integer $k \neq \pm 1$ such that every element of the ideal is a multiple of k . A 3-virtual unit $\mu \in \mathcal{O}_K$ will be called *primitive* if $\mu\mathcal{O}_K = \mathfrak{m}$ for some primitive \mathcal{O}_K -ideal \mathfrak{m} .

It is known that every cubic field L of fundamental discriminant Δ arises from a 3-virtual unit in the quadratic resolvent $K_{\Delta'} = \mathbb{Q}(\sqrt{-3\Delta})$ of the quadratic number field $K_\Delta = \mathbb{Q}(\sqrt{\Delta})$, contained in the Galois closure of L . More specifically, [26, Lemma 4.2], every such cubic field L is generated by an irreducible cubic polynomial of the form

$$f_\mu(x) = x^3 - 3(\mu\bar{\mu})^{1/3}x + (\mu + \bar{\mu}) \in \mathbb{Z}[x],$$

where μ is a primitive 3-virtual unit $\mu \in \mathcal{O}_{\Delta'} \setminus \mathcal{O}_{\Delta'}^3$, with $\mathcal{O}_{\Delta'}$ being the maximal order of $K_{\Delta'}$. Two 3-virtual units $\mu_1 = \mathfrak{a}_1^3$ and $\mu_2 = \mathfrak{a}_2^3$ are generators of the same field, up to conjugation, if and only if μ_1 or $\bar{\mu}_1$ is equal to $\alpha^3\mu_2$, for some $\alpha \in K_{\Delta'}^\times$ [26, Theorem 4.3]. Equivalently, if and only if the two groups $\langle[\mu_1]\rangle$ and $\langle[\mu_2]\rangle$ generated by $[\mu_1], [\mu_2] \in \text{Sel}_3(K_{\Delta'})$, are equal. Therefore, a subgroup $\langle[\mu]\rangle \subseteq \text{Sel}_3(K)$ corresponds to a unique, up to conjugation, cubic field, which we will denote by $L_{[\mu]}$.

For fundamental discriminants Δ , a classical result of Hasse [27, Satz 7, pg.578] gives us that the number of non-conjugate cubic fields of discriminant Δ is equal to

$$\frac{(3^{r_3(\Delta)} - 1)}{2},$$

where $r_3(\Delta)$ is the rank of the 3-part of the ideal class group of K_Δ .

We keep in mind that $\text{Sel}_3(K)$ is 3-primary. Either every subgroup in $\text{Sel}_3(K)$, or all but one, corresponds to a cubic field of discriminant $\text{disc}(L) = \Delta$. Whether it is the first case or the second, it depends on the ranks of the 3-part of the ideal class groups of K_Δ and its quadratic resolvent $K_{\Delta'}$. For more details, the reader may refer to [26, §4.5].

3. THE ELLIPTIC CURVES E_D

3.1. Descent by 3-isogeny and the Selmer groups of E_D and $E_{D'}$.

For a detailed exposition of the theory and tools of this section, the reader may refer to [43, Chapter X.4] for the material related to the Selmer and Tate-Shafarevich groups, and to [43, Appendix B], [14, Chapter IV] and [39, Chapter IV, §3] for Galois Cohomology.

For any number field M/\mathbb{Q} , denote by $E(M)$ the group of points of the elliptic curve E defined over M . Let \bar{M} denote the algebraic closure of M .

To every discriminant $D \in \mathcal{D}$ we attach the j -invariant zero elliptic curve E_D defined as

$$E_D : y^2 = x^3 + 16D.$$

The subgroup of order 3 of $E_D(\overline{\mathbb{Q}})$ invariant under the action of $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, is the torsion group

$$\mathcal{T}_D = \{O, (0, \pm 4\sqrt{D})\} \subseteq E_D(\overline{\mathbb{Q}}).$$

Therefore, the quotient of E_D over \mathcal{T}_D gives a family of isogenous elliptic curves, also defined over \mathbb{Q} , which we denote by $E_{D'}$, with $D' = -3D$. These curves $E_{D'}$ are of the form

$$(3) \quad E_{D'} : Y^2 = X^3 - 27 \cdot 16D,$$

and they also have a torsion subgroup of order 3 invariant under $G_{\mathbb{Q}}$, namely

$$\mathcal{T}_{D'} = \{O, (0, \pm 12\sqrt{D'})\} \subseteq E_{D'}(\overline{\mathbb{Q}}).$$

We remark that the subscripts D and $D' = -3D$ in our notation for E_D and $E_{D'}$, are consistent with the definition for the general equation of an elliptic curve with 3-isogeny, as given for example in [16, §8.4.2, pg. 558]. The discriminants D and D' constitute the squarefree part of the constant term of their respective equations.

We define by λ the rational 3-isogeny

$$\lambda : E_D \rightarrow E_{D'}$$

with kernel $\ker_{\lambda}(\overline{\mathbb{Q}}) = \mathcal{T}_D$, given by the quotient map:

$$X = (y^2 + 3 \cdot 16D)x^{-2} \text{ and } Y = y(x^3 - 8 \cdot 16D)x^{-3}.$$

The dual isogeny λ' is such that $\lambda\lambda' = \times 3$, where $\times 3$ is the multiplication-by-3 map. Since both $\ker_{\lambda}(\overline{\mathbb{Q}})$ and $\ker_{\lambda'}(\overline{\mathbb{Q}})$ contain no non-trivial rational point of order 3, we have that the rank of $E_D(\mathbb{Q})$ is

$$(4) \quad \text{rank}(E_D(\mathbb{Q})) = \dim_{\mathbb{F}_3}(E_D(\mathbb{Q})/3E_D(\mathbb{Q})).$$

Consider the short exact sequence, [43, Section X.4, Remark 4.7]:

$$(5) \quad 0 \rightarrow E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q})) \xrightarrow{\lambda'} E_D(\mathbb{Q})/3E_D(\mathbb{Q}) \rightarrow E_D(\mathbb{Q})/\lambda'(E_{D'}(\mathbb{Q})) \rightarrow 0.$$

Given fact (4) above, to compute the rank of E_D it suffices to compute the dimensions of the quotients $E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))$ and $E_D(\mathbb{Q})/\lambda'(E_{D'}(\mathbb{Q}))$.

Consider now the short exact sequence

$$(6) \quad 0 \rightarrow \mathcal{T}_D \rightarrow E_D(\overline{\mathbb{Q}}) \xrightarrow{\lambda} E_{D'}(\overline{\mathbb{Q}}) \rightarrow 0.$$

From this, we obtain the long exact cohomology sequence which gives in particular the following

$$(7) \quad 0 \rightarrow E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q})) \xrightarrow{\delta} H^1(G_{\mathbb{Q}}, \mathcal{T}_D) \rightarrow H^1(G_{\mathbb{Q}}, E_D(\overline{\mathbb{Q}}))[\lambda] \rightarrow 0.$$

By localising at each place p of \mathbb{Q} , we obtain the following commutative diagram, where res_p is the usual restriction map:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q})) & \xrightarrow{\delta} & H^1(G_{\mathbb{Q}}, \mathcal{T}_D) & \longrightarrow & H^1(G_{\mathbb{Q}}, E_D(\overline{\mathbb{Q}}))[\lambda] \longrightarrow 0 \\
 & & \downarrow & & \downarrow \prod_p \text{res}_p & & \downarrow \prod_p \text{res}_p \\
 0 & \rightarrow & \prod_p E_{D'}(\mathbb{Q}_p)/\lambda(E_D(\mathbb{Q}_p)) & \rightarrow & \prod_p H^1(G_{\mathbb{Q}_p}, \mathcal{T}_D) & \rightarrow & \prod_p H^1(G_{\mathbb{Q}_p}, E_D(\overline{\mathbb{Q}_p}))[\lambda] \rightarrow 0
 \end{array}$$

Definition 3.1. *The Selmer group of E_D relative to the isogeny λ is*

$$\mathcal{S}_\lambda = \{\xi \in H^1(G_{\mathbb{Q}}, \mathcal{T}_D) \mid \text{res}_p(\xi) \in \text{Im}(E_{D'}(\mathbb{Q}_p)/\lambda(E_D(\mathbb{Q}_p))) \text{ for every } p\}.$$

The Tate-Shafarevich group of E_D can now be defined as

$$\text{III}(E_D) = \{\xi \in H^1(G_{\mathbb{Q}}, E_D(\overline{\mathbb{Q}})) \mid \text{res}_p(\xi) = 0 \text{ for every } p\}.$$

These two groups are connected together as follows:

$$(8) \quad 0 \rightarrow E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q})) \rightarrow \mathcal{S}_\lambda \rightarrow \text{III}(E_D)[\lambda] \rightarrow 0.$$

Remark 3.2. *By considering the dual isogeny λ' instead, we get exact sequences analogous to (6) and (7) which in turn give us the analogous definitions for $\mathcal{S}_{\lambda'}$, $\text{III}(E_{D'})$ and $\text{III}(E_{D'})[\lambda']$.*

To study the quotient groups $E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))$ and $E_D(\mathbb{Q})/\lambda'(E_{D'}(\mathbb{Q}))$, an obvious approach is to study $H^1(G_{\mathbb{Q}}, \mathcal{T}_D)$ and $H^1(G_{\mathbb{Q}}, \mathcal{T}_{D'})$, which contain isomorphic copies of them. We will discuss here the results of Satgé [41], which are related to $H^1(G_{\mathbb{Q}}, \mathcal{T}_D)$ and which will be of use for us.

Let $G_{K_D} = \text{Gal}(\overline{\mathbb{Q}}/K_D)$. Since G_{K_D} acts trivially on \mathcal{T}_D , the inflation-restriction sequence gives:

$$(9) \quad 0 \rightarrow H^1(\text{Gal}(K_D/\mathbb{Q}), \mathcal{T}_D^{G_{K_D}}) \xrightarrow{\text{inf}} H^1(G_{\mathbb{Q}}, \mathcal{T}_D) \xrightarrow{\text{res}} H^1(G_{K_D}, \mathcal{T}_D)^{\text{Gal}(K_D/\mathbb{Q})} \rightarrow H^2(\text{Gal}(K_D/\mathbb{Q}), \mathcal{T}_D^{G_{K_D}}).$$

Since $|\text{Gal}(K_D/\mathbb{Q})| = 2$ and $|\mathcal{T}_D^{G_{K_D}}| = |\mathcal{T}_D| = 3$ we must have that the first term of the cohomology sequence (9) is trivial. For the same reason, the H^2 -term is also trivial and we therefore obtain the isomorphism

$$H^1(G_{\mathbb{Q}}, \mathcal{T}_D) \cong H^1(G_{K_D}, \mathcal{T}_D).$$

Fact 3.3. *It is shown by Satgé in [41, Lemma 1.3] that $H^1(G_{\mathbb{Q}}, \mathcal{T}_D)$ identifies with a subgroup of $\text{Hom}(G_{K_D}, \mathbb{Z}/3\mathbb{Z})$. In particular, the kernel of every homomorphism ξ in this subgroup fixes a cubic extension of K_D , which is non-abelian and Galois, over \mathbb{Q} . We add that therefore, the Galois group of this extension, over \mathbb{Q} , is isomorphic to \mathbb{S}_3 .*

Denote by $d(\mathcal{S}_\lambda)$ and $d(\mathcal{S}_{\lambda'})$ the rank of the Selmer groups \mathcal{S}_λ and $\mathcal{S}_{\lambda'}$, relative to the isogenies λ and λ' , of the curves E_D and $E_{D'}$ respectively, over \mathbb{Q} . Let $Cl(K_D)_3$ denote the 3-Sylow subgroup of the ideal class group $Cl(K_D)$ of K_D . We denote by $r_3(D)$ the rank of the 3-part of the ideal class group

$$r_3(D) := \dim_{\mathbb{F}_3}(Cl(K_D)_3/Cl(K_D)_3^3).$$

From [41, Section 3], we can compute the precise rank for the Selmer groups \mathcal{S}_λ and $\mathcal{S}_{\lambda'}$.

Proposition 3.4. *The rank of the Selmer groups \mathcal{S}_λ and $\mathcal{S}_{\lambda'}$ of the curves E_D and $E_{D'}$ are as follows:*

$$\begin{aligned} d(\mathcal{S}_\lambda) &= d(\mathcal{S}_{\lambda'}) = r_3(D), & \text{if } D < -4 \\ d(\mathcal{S}_\lambda) &= r_3(D) \text{ and } d(\mathcal{S}_{\lambda'}) = r_3(D) + 1, & \text{if } D > 0. \end{aligned}$$

Proof. Our elliptic curves E_D have $16D = 16(4m^3 - 27n^2)$ and with D squarefree we have that $2^4 \nmid 16D$. Therefore, Lemma 3.1 in [41] is vacuously true. We also see that $16D \equiv (4m)^3 \equiv \pm 1 \pmod{9}$, since $3 \nmid m$ by assumption. Therefore, Propositions 3.2 and 3.3 in [41] give us the desired result. \square

Remark 3.5. *Given Fact 3.3 and Proposition 3.4, we see that for our family of curves E_D (where $a = b = c = 0$ following the notation of [41, Definition 1.12]), each $\xi \in H^1(G_{\mathbb{Q}}, \mathcal{T}_D)$ belongs to the Selmer group \mathcal{S}_λ of the curves E_D over \mathbb{Q} , if and only if the cubic extension of K_D fixed by the kernel of ξ is unramified. Therefore, from Fact 3.3 and Theorem 2.2, we conclude that the corresponding cubic fields are of discriminant exactly D .*

Together with the discussion in Section 2.4, we have established the following bijections:

Bijection 3.6. *There is a bijection between the cocycle classes $[\xi]$ in \mathcal{S}_λ and the classes of 3-virtual units $[\mu] \in \text{Sel}_3(K_{D'})$ whose corresponding cubic polynomial f_μ defines an unramified cubic extension of K_D and hence, f_μ generates a cubic field of discriminant D .*

Bijection 3.7. *There is a bijection between the groups $\langle [\xi] \rangle \subseteq \mathcal{S}_\lambda$, and the groups $\langle [\mu] \rangle \subseteq \text{Sel}_3(K_{D'})$ that give rise to the cubic fields $L_{[\mu]}$, unique up to conjugation, of discriminant D .*

3.2. The Fundamental 3-Descent Map. In [16, Section 8.4.2], Cohen defines the Fundamental 3-descent map from $E_D(\mathbb{Q})$ to the multiplicative group $K_D^\times / (K_D^\times)^3$ where, by the explicit definition of the isogeny λ , each point $P = (x_P, y_P) \in E_D(\mathbb{Q})$ is mapped to $(y_P + 4\sqrt{D}) \cdot K_D^{\times 3}$. This map is a group homomorphism with kernel equal to $\lambda'(E_{D'}(\mathbb{Q}))$. Similarly, for the dual isogeny λ' , each point $P \in E_{D'}(\mathbb{Q})$ is mapped to $(y_P + 12\sqrt{D}) \cdot K_D^{\times 3}$ and the group homomorphism from $E_{D'}(\mathbb{Q})$ to the multiplicative group $K_{D'}^\times / (K_{D'}^\times)^3$ has kernel $\lambda(E_D(\mathbb{Q}))$. Given these 3-descent maps, let us call them ψ and ψ' , we obtain the injective group homomorphisms Ψ and Ψ' , which we will also call *Fundamental 3-descent maps*:

$$(10) \quad E_D(\mathbb{Q}) / \lambda'(E_{D'}(\mathbb{Q})) \xrightarrow{\Psi} \text{Ker}(N_{K_D} : K_D^\times / K_D^{\times 3} \rightarrow \mathbb{Q}^\times / \mathbb{Q}^{\times 3}),$$

$$(x, y) \mapsto (y + 4\sqrt{D}) \cdot K_D^{\times 3}$$

and

$$(11) \quad E_{D'}(\mathbb{Q}) / \lambda(E_D(\mathbb{Q})) \xrightarrow{\Psi'} \text{Ker}(N_{K_{D'}} : K_{D'}^\times / K_{D'}^{\times 3} \rightarrow \mathbb{Q}^\times / \mathbb{Q}^{\times 3}),$$

$$(X, Y) \mapsto (Y + 12\sqrt{D'}) \cdot K_{D'}^{\times 3}.$$

The way these two quotient groups fit together in a short exact sequence, is shown in (5).

We have the following, immediate corollary.

Corollary 3.8. *The ranks of*

$$E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q})) \text{ and } \Psi'(E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))),$$

as \mathbb{F}_3 -vector spaces, are equal.

Proof. We observe that $\Psi'(E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))) \subseteq \text{Sel}_3(K_{D'})$, and the result now follows directly from the fact that the map Ψ' induces an isomorphism between these two \mathbb{F}_3 -vector spaces. \square

From the definition of $D = D_{m,n}$, the curve $E_{D'}(\mathbb{Q})$ has the integral point $P_{m,n} = (12m, 108n)$.

Lemma 3.9. *The integral point $P_{m,n} = (12m, 108n)$ belongs to*

$$P_{m,n} \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q})).$$

Proof. Define the $K_{D'}$ -integral element

$$\mu_{m,n} = \frac{27n + 3\sqrt{D'}}{2} \in \mathcal{O}_{D'}.$$

We compute its trace and norm

$$\text{Norm}_{K_{D'}/\mathbb{Q}}(\mu_{m,n}) = (3m)^3,$$

$$\text{Trace}_{K_{D'}/\mathbb{Q}}(\mu_{m,n}) = 27n.$$

As usual, we form the corresponding cubic polynomial

$$(12) \quad \begin{aligned} f_{\mu_{m,n}}(x) &= x^3 - 3(\text{Norm}(\mu_{m,n}))^{1/3}x + \text{Trace}(\mu_{m,n}) = \\ &= x^3 - 3(3m)x + 27n = x^3 - Mx + N. \end{aligned}$$

Since $3^2|M$ and $3^3|N$, we consider the polynomial $f_{m,n}$ below, which is in standard form:

$$f_{m,n}(x) = x^3 - mx + n.$$

Since $f_{m,n}$ is irreducible (by assumption, since $D \in \mathcal{D}$), this implies that $\mu_{m,n}$ is not a cube in $K_{D'}^\times$. This can be explicitly verified as follows:

Assume by way of contradiction that $\mu = a^3, a \in K_{D'}^\times$. Then

$$-\text{Trace}_{K_{D'}/\mathbb{Q}}(a) = a + \bar{a} \in \mathbb{Z} \text{ and } \text{Norm}_{K_{D'}/\mathbb{Q}}(a) = (\mu\bar{\mu})^{1/3} = 3m.$$

Then one can immediately see that $-\text{Trace}_{K_{D'}/\mathbb{Q}}(a)$ is an integral root of $f_{\mu_{m,n}}(x)$ of Equation (12), which in turn gives us that

$$(13) \quad \begin{aligned} 0 &= f_{\mu_{m,n}}(-\text{Trace}_{K_{D'}/\mathbb{Q}}(a)) = -(a + \bar{a})^3 + 9m(a + \bar{a}) + 27n \\ &\iff \\ 0 &= \left(\frac{-(a + \bar{a})}{3}\right)^3 - m\left(\frac{-(a + \bar{a})}{3}\right) + n = f_{m,n}\left(\frac{-(a + \bar{a})}{3}\right). \end{aligned}$$

We also see that

$$(14) \quad \begin{aligned} 27n = \mu + \bar{\mu} &= a^3 + \bar{a}^3 = (a + \bar{a})^3 - 3a\bar{a}(a + \bar{a}) \\ &\implies \frac{-(a + \bar{a})}{3} \in \mathbb{Z}. \end{aligned}$$

Equations (13 & 14) now imply that $f_{m,n}$ has an integral root, which is a contradiction since the corresponding discriminant $D_{m,n}$ belongs to the set \mathcal{D} .

Let us look now at the image of $P_{m,n}$ under the Fundamental 3-descent map:

$$\Psi'(P_{m,n}) = 108n + 12\sqrt{D'} \equiv \left(\frac{1}{2}\right)^3 (108n + 12\sqrt{D'}) = \mu_{m,n} \pmod{(K_{D'}^\times)^3}.$$

Since, as we have just seen, $\mu_{m,n}$ is not a cube in $K_{D'}^\times$, this implies that $P_{m,n}$ is not in the kernel of Ψ' . \square

We conclude this section with the following lemma.

Lemma 3.10. *Two points $P, Q \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$ satisfy $P = kQ$, for some integer k , if and only if their corresponding cubic polynomials f_P and f_Q define the same cubic field, under conjugation.*

Proof. We have $P, Q \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$ with $P = kQ$,

if and only if $P \equiv rQ + \lambda(E_D(\mathbb{Q}))$ where $k = 3k' + r$ and $r = 1, 2$,

if and only if under the Fundamental 3-descent Map Ψ' , the corresponding groups $\langle[\mu_P]\rangle$ and $\langle[\mu_Q]\rangle$ are equal,

if and only if $L_{[\mu_P]} = L_{[\mu_Q]}$. \square

3.3. Binary Cubic Forms: Known Facts and Definitions. For a more detailed exposition of the theory and facts of this section, the reader may refer to [9], [10], [15] and [26].

We will always consider here the irreducible cubic polynomials

$$f(x) = ax^3 + bx^2 + cx + d \in \mathbb{Z}[x],$$

whose splitting field has Galois group isomorphic to the Symmetric group \mathbb{S}_3 , and whose discriminant equals the discriminant of the cubic field they define. Each such irreducible cubic polynomial, defines a class of conjugate cubic fields

$$\{L_i\}_{0 \leq i \leq 2} = \{\mathbb{Q}(\delta_i)\}_{0 \leq i \leq 2}, \text{ where } f(\delta_i) = 0 \text{ for all } 0 \leq i \leq 2,$$

and in our case, $\text{disc}(f) = \text{disc}(L_i)$, for all $0 \leq i \leq 2$. Unless otherwise stated, the cubic field L_f will refer to any one of the cubic fields L_i .

By ‘homogenizing’ f , we obtain the irreducible integral binary cubic form

$$F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

which for brevity we will also denote by

$$F = [a, b, c, d].$$

The discriminant of F is given by the formula

$$\text{disc}(F) = \text{disc}(f) = b^2c^2 + 18abcd - 27a^2d^2 - 4ac^3 - 4b^3d.$$

We remind the reader that a binary cubic form is called *irreducible* if and only if $F(x, 1)$ is an irreducible polynomial over \mathbb{Q} .

Given any matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{Q}),$$

it is straightforward to see that

$$(15) \quad \text{disc}(MF(x, y)) := \text{disc}(F(\alpha x + \beta y, \gamma x + \delta y)) = \det(M)^6 \text{disc}(F(x, y)).$$

We deduce that the discriminant of a binary cubic form is $GL(2, \mathbb{Z})$ -invariant, and therefore $SL(2, \mathbb{Z})$ -invariant.

We denote by $[F]$ the $SL(2, \mathbb{Z})$ -equivalence class of the irreducible integral binary cubic forms F , of discriminant $\text{disc}(F)$. Under the Davenport-Heilbronn correspondence [20] and [21], every such $SL(2, \mathbb{Z})$ -equivalence class $[F]$ corresponds to a triple of conjugate cubic fields $\mathbb{Q}(\delta_i)$ of discriminant $\text{disc}(F)$ and with $F(\delta_i, 1) = f(\delta_i) = 0$, where F is the ‘homogenization’ of f ; i.e. $f(x) = F(x, 1)$.

For simplicity, and where it is clear, L will refer to the cubic field $L_f \equiv L_F$. Following the terminology in [26], we will call any irreducible integral binary cubic form F with discriminant $\text{disc}(F) = \text{disc}(L)$, an *Index Form* for L . We will denote the equivalence class of Index Forms of a cubic field L , by $IF_L(\text{disc}(L))$. Every $F \in IF_L(\text{disc}(L))$ corresponds to the maximal order \mathcal{O}_L of L , under the Delone-Fadeev correspondence.

It is well known that given any Index Form, there is a known algorithm that reduces it to the *Reduced Index Form* of the same cubic field L . Each $SL(2, \mathbb{Z})$ -equivalence class $[F]$ contains a *unique* Reduced Index Form; see for example [26, §3.2 & §3.3] and [7]. For brevity, we will denote the Reduced Index Form of $IF_L(\text{disc}(L))$ by RIF_L . As we see below, the definition of the reduced form differs slightly, depending on whether the discriminant is positive or negative. Before giving the definition, we remind the reader of the Hessian H and the Jacobian J associated with a binary cubic form F .

Definition 3.11. *Let $F = [a, b, c, d]$ be an integral binary cubic form. The Hessian of F is the binary quadratic form defined as*

$$H = [A, B, C] = [b^2 - 3ac, bc - 9ad, c^2 - 3bd]$$

and the Jacobian of F is another binary cubic form defined as

$$J = [3aB - 2Ab, 9aC - 3Ac, 3bC - 9Ad, 2cC - 3Bd].$$

Definition 3.12. *Let $F = [a, b, c, d]$ be an integral binary cubic form.*

- (A) *If F is of positive discriminant Δ , the Hessian has negative discriminant equal to $B^2 - 4AC = -3\Delta$, and we say that F is reduced if the following conditions hold:*
- (a) $|B| \leq A \leq C$.
 - (b) $a > 0, b \geq 0$.
 - (c) *If $b = 0$, then $d < 0$.*
 - (d) *If $A = B$, then $b < |3a - b|$.*
 - (e) *If $A = C$, then $a < |d|$, or $a = |d|$ and $b < |c|$.*
- (B) *If F is of negative discriminant Δ , let $\delta \in \mathbb{R}$ satisfy $F(\delta, 1) = 0$, $\rho_1 = a\delta$ and $\rho_2 = a\delta^2 + b\delta$. Let $\bar{A} = a, \bar{B} = b + \rho_1$, and $\bar{C} = c + \rho_2$. We say that F is reduced if*

$$|\bar{B}| < \bar{A} < \bar{C},$$

and either $b > 0$ or $b = 0$ and $d > 0$.

The Hessian H and the Jacobian J of a binary cubic form F are known as the quadratic, and respectively the cubic, covariants of F since these forms are covariant with respect to the action of $GL(2, \mathbb{Z})$. Specifically

$$(16) \quad H_{MF} = MH_F \text{ and } J_{MF} = MJ_F, \quad \forall M \in GL(2, \mathbb{Z}),$$

where the action of M on H and J is defined the same way as we defined the action of M on F in (15) above.

Given an integral binary cubic form F of discriminant Δ , its quadratic and cubic covariants H and J , satisfy the following equality, known as *Cayley's Syzygy*:

$$(17) \quad J(x, y)^2 = 4H(x, y)^3 - 27\Delta F(x, y)^2.$$

We see that if F has a point $P = (x_P, y_P) \in \mathbb{Q}^2$ such that $F(x_P, y_P) = 1$, then this point can be easily transformed into a rational point on our elliptic curve $E_{D'}(\mathbb{Q})$.

We will discuss Cayley's Syzygy further, later in the paper, in the context of Homogeneous Spaces for our elliptic curves, that we introduce in Section 5.

3.4. The group of rational points $E_{D'}(\mathbb{Q})$. In Lemma 3.13 below, the first statement regarding the form of every point $P \in E_{D'}(\mathbb{Q})$ is already proved in [45, pp.71-72], for arbitrary curves of the form $y^2 = x^3 + ax^2 + bx + c$. We include our proof of the first statement here, for completeness. The second statement of the lemma is specific to the curves $E_{D'}$.

Lemma 3.13. *Every point $P \in E_{D'}(\mathbb{Q})$ is of the form $P = (x/z^2, y/z^3)$ for some $x, y \in \mathbb{Z}^\times$ and $z \in \mathbb{Z}^+$, with $\gcd(x, z) = \gcd(y, z) = 1$. Furthermore, $d := \gcd(x, y) | 6$ and if $2 | d$, then $4 | |y$ and $4 | |x$, and if $3 | d$, then $3 | |x$ and $9 | |y$.*

Proof. Let us write $P = (x/z_1, y/z_2)$, with $z_1, z_2, x, y \in \mathbb{Z}$. Without loss of generality, we may assume that z_1, z_2 are always positive and that $\gcd(x, z_1) = \gcd(y, z_2) = 1$, with x and y arbitrary integers. Plugging P into the equation of $E_{D'}$ we obtain

$$(18) \quad x^3 z_2^2 - y^2 z_1^3 = 27 \cdot 16 z_1^3 z_2^2 D.$$

Since $\gcd(x, z_1) = \gcd(y, z_2) = 1$, we must have that

$$z_1^3 | z_2^2, \text{ and } z_2^2 | z_1^3.$$

Since z_1, z_2 are both positive integers, we conclude that there is a positive integer z such that

$$(19) \quad z_1 = z^2, \quad z_2 = z^3, \quad \gcd(xy, z) = 1.$$

Equation(18) now becomes

$$(20) \quad x^3 - y^2 = 27 \cdot 16 D z^6.$$

Let $d = \gcd(x, y)$ denote the greatest common divisor of x and y . Then, from relation(19), we should have that $\gcd(d, z) = 1$. From equation(20)

and from the fact that D is squarefree and $6 \nmid D$, we conclude that $d|6$ and it is now straightforward to see that:

$$(21) \quad \begin{cases} \text{if } 2|d, \text{ then } 4||y \text{ and } 4|x, \\ \text{if } 3|d, \text{ then } 3||x \text{ and } 27|y. \end{cases}$$

□

Lemma 3.14. *Every cubic polynomial $f(x) = x^3 + ax^2 + bx + c$, with $a, b, c \in \mathbb{Z}$, can be written in the form $F_{m,n}(X) = X^3 - mX + n$, where $X = x + \frac{a}{3}$, $m = \frac{a^2}{3} - b$ and $n = c + \frac{2a^3}{27} - \frac{ab}{3}$.*

Proof. Straightforward. □

Proposition 3.15. *The set of points $E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$ is non-empty if and only if there exists an irreducible, integral, binary cubic form F_3 of discriminant $\text{disc}(F_3) = k^6 D$, where k is some nonzero integer, and F_3 has leading coefficient an integral cube, i.e. F_3 is of the form $F_3 = [a^3, b, c, d]$, for some $a, b, c, d \in \mathbb{Z}$.*

Proof. (\Rightarrow) Assume that $E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q})) \neq \emptyset$. By Lemma 3.13, P is of the form $P = (x/z^2, y/z^3)$. Under the Fundamental 3-Descent map, this point corresponds to the non-trivial element

$$\Psi'(P) = y/z^3 + 12\sqrt{D'} \in K_{D'}^\times / (K_{D'}^\times)^3.$$

In other words, $\Psi'(P)$ is a rational element of $K_{D'}^\times$, that is not a cube in $K_{D'}$.

We assume first that $\gcd(6, xy) = 1$ and we form the polynomial

$$(22) \quad f_P(X) = X^3 - \frac{x}{12z^2}X + \frac{y}{108z^3} \in \mathbb{Q}[x].$$

We see that its discriminant is

$$\text{disc}(f_P) = 4\left(\frac{x}{12z^2}\right)^3 - 27\left(\frac{y}{108z^3}\right)^2 = D.$$

By ‘homogenizing’ f_P we obtain the following (rational) binary cubic form, also of discriminant D ,

$$(23) \quad \begin{aligned} F_P(X, Y) &= X^3 - \frac{x}{12z^2}XY^2 + \frac{y}{108z^3}Y^3 = \\ X^3 - 3xX\left(\frac{Y}{6z}\right)^2 + 2y\left(\frac{Y}{6z}\right)^3 &= F_3(X, Y/6z) = MF_3(u, v), \end{aligned}$$

where

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1/(6z) \end{pmatrix} \in GL(2, \mathbb{Q}), \text{ and} \\ F_3(u, v) = u^3 - 3xuv^2 + 2yv^3$$

is an integral binary cubic form with leading coefficient an integral cube. Furthermore, from (23) we deduce that

$$\text{disc}(F_3) = (\det(M)^6)^{-1} \text{disc}(F_P) = (6z)^6 D.$$

It remains to show that F_3 is irreducible. We observe that the element

$$\mu_P := y + 12z^3\sqrt{D'} \equiv y/z^3 + 12\sqrt{D'} = \Psi'(P) \pmod{(K_{D'}^\times)^3},$$

is not a cube in $K_{D'}$ and therefore neither is $\nu_P = -y + 12z^3\sqrt{D'} = -\overline{\mu_P}$, where the bar denotes the Galois conjugate of an element. By employing Cardano's classical formula for the zero's of cubic polynomials, [26, §1.4, pg.25], we verify via direct computation that the zero's of $F_3(u, 1)$ are given by

$$\delta_P^{(i)} = w^i(\nu_P)^{(1/3)} + w^{-i}(\overline{\nu_P})^{(1/3)},$$

for $0 \leq i \leq 2$, and w a cube root of unity.

With ν_P not being a cube in $K_{D'}^\times$, this proves that $F_3(u, 1)$ cannot have a quadratic factor, and consequently $F_3(u, 1)$, and therefore $F_3(u, v)$, are irreducible.

Finally, if either 2 or 3 divide x or y , then from Lemma 3.13(21) we know exactly the powers of 2 or 3 that divide x and y . Therefore, the general form of the binary cubic form F_3 and the corresponding transformation matrix M , are:

$$F_3 = [1, 0, -3x/(9^i 4^j), 2y/(27^i 4^j)],$$

$$\text{with } \text{disc}(F_3) = (3^{(1-i)} 2^{(1-j)} z)^6 D,$$

and both $3x/(9^i 4^j)$ and $2y/(27^i 4^j)$ are integers, by Lemma 3.13(21),

$$\text{and } M = \begin{pmatrix} 1 & 0 \\ 0 & 1/(3^{(1-i)} 2^{(1-j)} z) \end{pmatrix} \in GL(2, \mathbb{Q}),$$

with $i, j \in \{0, 1\}$ and

$i = 1$ if and only if $3|x$,

$j = 1$ if and only if $2|x$.

The elements μ_P will of course change accordingly and we observe that in all cases, μ_P is an integral primitive 3-virtual unit and equivalently, $F_3(u, 1)$ is always in standard form.

(\Leftarrow) Assume now that there exists an integral, irreducible, binary cubic form $F_3 = [a^3, b, c, d]$, of discriminant equal to $\text{disc}(F_3) = k^6 D$, for some nonzero integer C . We observe that

$$(24) \quad \begin{aligned} F_3(u, v) &= a^3 u^3 + bu^2 v + cuv^2 + dv^3 = \\ &= (au)^3 + b(au)^2 \left(\frac{v}{a^2}\right) + ca^3(zu) \left(\frac{v}{a^2}\right)^2 + da^6 \left(\frac{v}{a^2}\right)^3 = \\ &= F\left(au, \frac{v}{a^2}\right) = MF(u, v), \end{aligned}$$

where

$$F(u, v) = u^3 + bu^2 v + ca^3 uv^2 + da^6 v^3,$$

$$\text{and } M = \begin{pmatrix} a & 0 \\ 0 & 1/a^2 \end{pmatrix} \in GL(2, \mathbb{Q}).$$

The discriminant of the new, also irreducible, integral binary cubic form is equal to $\text{disc}(F) = \det(M)^{-1} \text{disc}(F_3) = (ak)^6 D$. Consider the polynomial $F(u, 1) = u^3 + bu^2 + ca^3 u + da^6$. By employing Lemma 3.14, we obtain the irreducible polynomial

$$F_{M,N}(U) = U^3 - MU + N,$$

$$\text{with } M = \frac{b^2 - 3ca^3}{3}, N = \frac{27da^6 + 2b^3 - 9bca^3}{27}, \text{ and}$$

$$(25) \quad \text{disc}(F_{M,N}) = 4M^3 - 27N^2 = (ak)^6 D.$$

But now, relation(25), together with the fact that $F_{M,N}$ is irreducible, proves that the point $P_{M,N} = (\frac{12M}{(ak)^2}, \frac{108N}{(ak)^3})$ belongs to $E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$, via the same proof of Lemma 3.9. \square

Proposition 3.15 establishes a correspondence between the irreducible integral binary cubic forms of discriminant equal to $k^6 D$, for some nonzero integer k , and the points of $E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$. Under the Delone-Fadeev correspondence [23], each such irreducible integral binary cubic form corresponds to an order $\mathcal{O}_P \subseteq \mathcal{O}_L$ of conductor $[\mathcal{O}_L : \mathcal{O}_P]$ equal to an integral cube, where L , as in the introduction, is a cubic field of discriminant D and whose Galois closure has Galois group isomorphic to S_3 . We have therefore established the following:

Bijection 3.16. *There exists a bijection between the points*

$$P \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$$

and suborders

$$\mathcal{O}_P \subseteq \mathcal{O}_L, \text{ with conductor a cube,}$$

where L is as above.

3.5. From IBCF's to Index Forms. In this section we will show that such an order $\mathcal{O}_P \subseteq \mathcal{O}_L$ exists, if and only if there exists an Index Form in $IF_L(D)$ with leading coefficient a cube. Under the Delone-Fadeev correspondence, this Index Form corresponds to the maximal order \mathcal{O}_L .

Note that Proposition 3.15 only implies the existence of such a form of discriminant $k^6 D$ for some nonzero integer k .

We will first need the following lemma:

Lemma 3.17. *To each point $P = (X_P/Z^2, Y_P/Z^3) \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$ there corresponds an order $\mathcal{O}_P \subseteq \mathcal{O}_L$ of conductor equal to k^3 , for some non-zero integer k , and with basis $\mathcal{B}_P = \{1, \delta, \delta^2 - A\}$, where δ is a root of the monic irreducible polynomial*

$$f_P(x) = x^3 - Ax + B := x^3 - 3X_P/(9^i 4^j)x + 2Y_P/(27^i 4^j) \in \mathbb{Z}[x].$$

The cubic field L corresponds to the one under Bijection (3.16).

Proof. To each point $P = (X_P/Z^2, Y_P/Z^3) \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$, Proposition 3.15 gives us the irreducible integral monic binary cubic form

$$F_P = [1, 0, -3X_P/(9^i 4^j), 2Y_P/(27^i 4^j)],$$

$$\text{with } \text{disc}(F_P) = (3^{(1-i)} 2^{(1-j)} Z)^6 D,$$

which under the Delone-Fadeev Correspondence, it corresponds to an order \mathcal{O}_P of conductor equal to the integral cube $(3^{(1-i)} 2^{(1-j)} Z)^3$. To ease notation, we will denote this form by

$$F_P =: [1, 0, -A, B].$$

We denote by δ any one of the roots of the monic irreducible polynomial

$$F_P(x, 1) := f_P(x) = x^3 - Ax + B,$$

of discriminant $4A^3 - 27B^2 = k^6D$, and we let

$$w_1 = \delta \text{ and } w_2 = \delta^2 - A.$$

To compute a basis of the corresponding order \mathcal{O}_P , we first observe that

$$(26) \quad \begin{aligned} w_1^2 &= w_2 + A \\ w_1 w_2 &= -B \\ w_2^2 &= -Bw_1 - Aw_2. \end{aligned}$$

Relations (26) imply that indeed the set

$$\mathcal{B}_P = \{1, w_1, w_2\}$$

is a basis for the order \mathcal{O}_P [9, Theorem 9]. This basis is called *normal*. \square

In the next proposition, we will employ Voronoi's algorithm (see for example [26, Theorem 1.6] or [3, Theorems 2 & 3]), and we will compute from \mathcal{B}_P , an explicit basis \mathcal{B}_L for the maximal order \mathcal{O}_L of L . We will then show how the corresponding Index Form must have leading coefficient an integral cube.

Proposition 3.18. *The set $E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$ is not empty, if and only if there exists an Index Form $\in IF_L(D)$ with leading coefficient a cube, where the cubic field L corresponds to the one under Bijection (3.16).*

Proof. (\Leftarrow) Denote by $F = [a^3, b, c, d] \in IF_L(D)$ the given Index Form that corresponds to one of our cubic fields L . Then $F(1/a, 0) = 1$ and therefore we have the point

$$P = (4J(1/a, 0), 4H(1/a, 0)) \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q})),$$

via Cayley's Syzygy (17) and the fact that F is irreducible.

(\Rightarrow) Let $P = (X_P/Z, Y_P/Z) \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$ and let

$$F_P = [1, 0, -A, B] \in \mathbb{Z}[x, y]$$

be the corresponding irreducible integral binary cubic form, as defined in Lemma 3.17, with corresponding irreducible polynomial

$$f_P(x) = x^3 - Ax + B,$$

and δ being any one of its roots. We know from Proposition 3.15 and [26, §1.4, pg.20] that

$$\text{disc}(F_P) = (3^{(1-i)}2^{(1-j)}Z)^6D = D[\mathcal{O}_L : \mathbb{Z}[\delta]]^2.$$

The index of $\mathbb{Z}[\delta]$ in the maximal order is therefore equal to

$$i(\delta) = [\mathcal{O}_L : \mathbb{Z}[\delta]] = (3^{(1-i)}2^{(1-j)}Z)^3.$$

Since $D = 4m^3 - 27n^2 \equiv 1 \pmod{4}$ is squarefree and $3 \nmid D$, straightforward computations by following Voronoi's algorithm via the proof of [3], will give us an integral basis for \mathcal{O}_L via the normal basis of \mathcal{O}_P established in Lemma 3.17. As we see below, the basis for \mathcal{O}_L differs, depending on whether $3|A$ or not. We recall from Proposition 3.15 that $3|A$ if and only if

$3 \nmid X_P$. Furthermore, for each prime p , we easily see that the exponent r_p defined in [3, pg. 164] and which decides the exponent of k below, in our case is always 0 or 3. More specifically,

$$(27) \quad \begin{cases} r_2 \text{ and } r_3 \text{ are either 0 or 3,} \\ r_p = 3 \text{ for } p|Z \text{ and } p|D \\ r_p = 3 \text{ for } p|Z \text{ and } p \nmid D \\ r_p = 0 \text{ in all other cases.} \end{cases}$$

The basis \mathcal{B}_L is as follows:

(1) $3|A$. In this case, [3, Theorem 2] gives us the following basis for \mathcal{O}_L :

$$\mathcal{B}_L = \left\{ 1, \delta, \frac{t^2 - A + t\delta + \delta^2}{k^3} \right\},$$

where $k = 3 \cdot 2^{(1-j)}Z$, and $j = 1$ if and only if $2|X_P$.

(2) $3 \nmid A$. In this case, [3, Theorem 3] gives us the following basis:

$$\mathcal{B}_L = \left\{ 1, \frac{-t + \delta}{3}, \frac{t^2 - A + t\delta + \delta^2}{9k^3} \right\},$$

where $k = 2^{(1-j)}Z$, and $j = 1$ if and only if $2|X_P$.

We recall that

$$\mathcal{B}_P = \{1, \delta, \delta^2 - A\} = \{1, w_1, w_2\}.$$

We let

$$\mathcal{B}_L = \{1, u_1, u_2\}$$

and we see that in the first case above,

$$w_1 = \delta = u_1, \text{ and}$$

$$w_2 = k^3 u_2 - t u_1 - t^2,$$

which gives us the following matrix

$$(28) \quad \mathcal{U}_1 = \begin{pmatrix} 1 & 0 & -t^2 \\ 0 & 1 & -t \\ 0 & 0 & k^3 \end{pmatrix} \in GL(3, \mathbb{Q}),$$

such that

$$\begin{pmatrix} 1 & u_1 & u_2 \end{pmatrix} \mathcal{U}_1 = \begin{pmatrix} 1 & w_1 & w_2 \end{pmatrix}.$$

In the second case, we have that

$$w_1 = 3u_1 + t, \text{ and}$$

$$w_2 = -3t u_1 + 9k^3 u_2 - 2t^2.$$

The corresponding matrix in this case is

$$(29) \quad \mathcal{U}_2 = \begin{pmatrix} 1 & t & -2t^2 \\ 0 & 3 & -3t \\ 0 & 0 & 9k^3 \end{pmatrix} \in GL(3, \mathbb{Q}).$$

From the matrices $\mathcal{U}_1, \mathcal{U}_2 \in GL(3, \mathbb{Q})$ we can extract the 2×2 transformation matrices, M_1 and M_2 , which transform, in either case, the irreducible

integral binary cubic form F_P to the Index Form that corresponds to the basis \mathcal{B}_L . These matrices are:

$$(30) \quad M_1 = \begin{pmatrix} 1 & -t \\ 0 & k^3 \end{pmatrix}, M_2 = \begin{pmatrix} 3 & -3t \\ 0 & 9k^3 \end{pmatrix} \in GL(2, \mathbb{Q}).$$

Given these matrices M_1, M_2 , from the form $F_P = [1, 0, -A, B]$ and by following [26, §3.4, Corollary 3.4 & Remark 3.2], we obtain the following Index Form for each case:

$$(31) \quad F_1 = [k^3, 3t, \frac{3t^2 - A}{k^3}, \frac{t^3 - At + B}{k^6}], \quad F_2 = [k^3, t, \frac{3t^2 - A}{9k^3}, \frac{t^3 - At + B}{27k^6}].$$

Direct computation of the discriminant gives us in each case

$$\text{disc}(F_1) = \frac{4A^3 - 27B^2}{k^6} = D, \quad \text{disc}(F_2) = \frac{4A^3 - 27B^2}{(3k)^6} = D,$$

verifying that indeed, F_1 and F_2 are Index Forms and have leading coefficient a rational cube. \square

4. ON THE RANK OF E_D/\mathbb{Q}

4.1. On the parity of the rank. The following commutative diagram contains all the important groups related to the elliptic curves E_D . Since the curves E_D and $E_{D'}$ do not have any non-trivial rational 3-torsion points, the exact sequences below are a more simplified version of the standard sequences that emerge from the method of *descent by a rational 3-isogeny*, as found for example in [32], [43] and [46].

$$(32) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q})) & \longrightarrow & \mathcal{S}_\lambda(E_D) & \longrightarrow & \text{III}(E_D)[\lambda] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_D(\mathbb{Q})/3(E_D(\mathbb{Q})) & \longrightarrow & \mathcal{S}_3(E_D) & \longrightarrow & \text{III}(E_D)[3] \longrightarrow 0 \\ (32) & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_D(\mathbb{Q})/\lambda'(E_{D'}(\mathbb{Q})) & \longrightarrow & \mathcal{S}_{\lambda'}(E_{D'}) & \longrightarrow & \text{III}(E_{D'})[\lambda'] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & \frac{\text{III}(E_{D'})[\lambda']}{\lambda(\text{III}(E_D)[3])} & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Given the non-degenerate alternating pairing defined by Cassels in [13], by assuming finiteness of the 3-primary part $\text{III}(E_D)[3^\infty]$, the torsion group $\text{III}(E_D)[3]$ becomes even dimensional. Furthermore, the term $\frac{\text{III}(E_{D'})[\lambda']}{\lambda(\text{III}(E_D)[3])}$ is

also even-dimensional as an \mathbb{F}_3 -vector space [10, Prop. 4.2]. Therefore, from the diagram above we conclude:

$$(33) \quad \text{rank}(E_{D'}(\mathbb{Q})) \equiv d(\mathcal{S}_3(E_D)) \equiv d(\mathcal{S}_\lambda(E_D)) + d(\mathcal{S}_{\lambda'}(E_{D'})) \pmod{2}.$$

Corollary 4.1. *By assuming finiteness of the 3-primary part $\text{III}(E_D)[3^\infty]$ of the Tate-Shafarevich group $\text{III}(E_D)$, the rank of the elliptic curves E_D , and consequently of $E_{D'}$, is always even for negative $D \in \mathcal{D}$ and is always odd for positive $D \in \mathcal{D}$.*

Proof. The result follows directly from Proposition 3.4, Diagram (32), and relation (33). \square

4.2. Lower bounds. As we have already mentioned in Section 2.4, for fundamental discriminants such as D , Hasse gives the number of non-conjugate cubic fields of discriminant D to be equal to $\frac{(3r_3(D)-1)}{2}$. Given $D \in \mathcal{D}$, let us denote by $\#(\text{cubic})$ the number of non-conjugate cubic fields $L_{[\mu]}$ of discriminant D whose $SL(2, \mathbb{Z})$ -equivalence class of Index Forms contains an Index Form with leading coefficient a rational cube. Let

$$\mathcal{M} = \langle [\mu]_i \mid 1 \leq i \leq \#(\text{cubic}) \rangle$$

denote the subgroup of $\text{Sel}_3(K_{D'})$ generated by the corresponding classes of 3-virtual units. We denote by \mathfrak{r}_3 its rank as an \mathbb{F}_3 -vector space. Then the rank of E_D , and therefore of $E_{D'}$, is bounded as follows:

Proposition 4.2. *By assuming finiteness of the 3-primary part $\text{III}(E_D)[3^\infty]$ of the Tate-Shafarevich group $\text{III}(E_D)$ of the elliptic curves E_D , the rank of the elliptic curves E_D , and therefore of $E_{D'}$, is bounded as follows:*

(1) *If \mathfrak{r}_3 is odd, then*

$$(i) \quad 2 \leq \mathfrak{r}_3 + 1 \leq \text{rank}(E_D) \leq 2r_3(D), \text{ if } D < -4$$

$$(ii) \quad 1 \leq \mathfrak{r}_3 \leq \text{rank}(E_D) \leq 2r_3(D) + 1, \text{ if } D > 4.$$

(2) *If \mathfrak{r}_3 is even, then*

$$(i) \quad 2 \leq \mathfrak{r}_3 \leq \text{rank}(E_D) \leq 2r_3(D), \text{ if } D < -4$$

$$(ii) \quad \mathfrak{r}_3 + 1 \leq \text{rank}(E_D) \leq 2r_3(D) + 1, \text{ if } D > 4.$$

Proof. From the definition of \mathcal{M} , and via the results of Proposition 3.18, every element of \mathcal{M} corresponds, under the Fundamental 3-descent map Ψ' , to a point in $E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$. Therefore, from Corollary 3.8, Lemma 3.9 and the short exact sequence (5), we have that

$$1 \leq \mathfrak{r}_3 = \dim_{\mathbb{F}_3}(E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))) \leq \text{rank}(E_D(\mathbb{Q})).$$

This, together with the parity result of Corollary 4.1, establish the lower bounds.

We now recall that $\text{rank}(E_D(\mathbb{Q})) \leq \dim_{\mathbb{F}_3}(\mathcal{S}_3)$ and therefore, from the second vertical exact sequence of Diagram (32) and from Proposition 3.4, we obtain the upper bounds, again together with the parity result of Corollary 4.1. \square

An immediate corollary of Proposition 4.2 is the following:

Corollary 4.3. *For the negative discriminants $D \in \mathcal{D}$ and their corresponding imaginary quadratic fields K_D , if $r_3(D) = 1$ then, by assuming finiteness of $\text{III}(E_D)[3^\infty]$, we always have that $\dim_{\mathbb{F}_3}(\text{III}(E_D))[3] = 0$. \square*

5. HOMOGENEOUS SPACES

We recall the definition of a cover map between two curves (see for example [10, Remark 24]), adjusted to our notation and context.

Definition 5.1. *A λ -covering for our elliptic curve $E_{D'}/\mathbb{Q}$, is a rational map between curves $C \rightarrow E_{D'}$ which is a twist of λ . By descent, the group $H^1(G_{\mathbb{Q}}, \mathcal{T}_D)$ is in bijection with isomorphism classes of such λ -coverings.*

We saw earlier in the paper, via Cayley's Syzygy (17), that there is an explicit degree-3 map φ from the genus-1 curves

$$C : F(x, y) = z^3, \quad F \in IF_L(D)$$

to the elliptic curve $E_{D'}$, as follows:

$$\begin{aligned} \varphi : C &\longrightarrow E_{D'} \\ (x : y : z) &\longmapsto (4J(x/z, y/z), 4H(x/z, y/z)). \end{aligned}$$

These curves C/\mathbb{Q} are called Homogeneous Spaces or Torsors for our elliptic curve $E_{D'}/\mathbb{Q}$.

Given Bijection 3.7, and under the Davenport-Heilbronn correspondence, we obtain the following bijection:

Bijection 5.2. *Each $SL(2, \mathbb{Z})$ -equivalence class of Index Forms $[F]$ of discriminant $D \in \mathcal{D}$, gives an equivalence class $[C]$ of Torsors for $E_{D'}$, to which there corresponds a subgroup $\langle [\xi] \rangle \subseteq \mathcal{S}_\lambda$ that fixes an unramified cubic extension $L_{[\mu]}$ of K_D , where $\langle [\xi] \rangle$ corresponds to $\langle [\mu] \rangle$ under bijections 3.6 and 3.7.*

Proposition 5.3. *A cocycle $\xi \in \mathcal{S}_\lambda$ corresponds to the trivial class in $\text{III}(E_D)[\lambda]$ if and only if the group $\langle [\xi] \rangle$ it generates, corresponds to a class of Index Forms $IF_L(D)$ that contains a form with leading coefficient an integral cube.*

Proof. A cocycle $\xi \in \mathcal{S}_\lambda$ corresponds to the trivial class in $\text{III}(E_D)[\lambda]$,

if and only if it belongs to the image of the connecting homomorphism δ of the short exact sequence (8),

if and only if there is a rational point $P \in E_{D'}(\mathbb{Q}) \setminus \lambda(E_D(\mathbb{Q}))$ such that $\delta(P) = \xi$,

if and only if by Proposition 3.18 and under Bijection 5.2 there is a corresponding Index Form in $IF_L(D)$ with leading coefficient an integral cube. \square

5.1. A Remark regarding Solvability and Uniform Boundedness.

We call a cocycle $\xi \in \mathcal{S}_\lambda$ *solvable* if it is in the image of the connecting homomorphism δ of (8). From Proposition 5.3, ξ is solvable if and only if any (and therefore every) Torsor in the corresponding class $[C]$, under Bijection (5.2), has a rational point. We will also call such a class of Torsors,

solvable. Let $C_1 : F_1(x, y) = z^3$ and $C_2 : F_2(x, y) = z^3$ be two Torsors that belong to a solvable class $[C]$, and let $Q_1 = (x_1 : y_1 : z)$ be a rational point on C_1 . The transformation matrix in $SL(2, \mathbb{Z})$ that transforms F_1 into F_2 , also transforms Q_1 into a rational point Q_2 of C_2 , but we notice that it leaves the z -coordinate unchanged. Therefore, Q_2 is of the form $Q_2 = (x_2 : y_2 : z)$. Furthermore, since by Proposition 5.3 there is always an Index Form of the form

$$F_a = [a^3, b, c, d], \text{ for some } a, b, c, d \in \mathbb{Z},$$

the point $Q_a = (1, 0, a)$ is always a point that corresponds to a solvable class of Torsors, and every other Torsor in $[C]$ must have a point of the form $Q = (x : y : a)$. More specifically, if $C : F(x, y) = z^3 \in [C]$ is such that $F(x, y) = MF_a(x, y)$, for some $M \in SL(2, \mathbb{Z})$, then the x - and y -coordinates of the point Q on C , are determined by M as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The rank of the solvable part of \mathcal{S}_λ is equal to \mathfrak{r}_3 , which we defined in Section 4.2. The non-conjugate *monogenic* cubic fields of discriminant D form a subgroup of \mathcal{M} of rank $\leq \mathfrak{r}_3$. We recall that we always have $1 \leq \mathfrak{r}_3$ since we always have at least one monogenic cubic field of discriminant D , namely the one defined by $f_{m,n}$. To determine which of the $\binom{3^{r_3(D)}-1}{2}$ -many non-conjugate cubic fields L of discriminant D are monogenic, is enough to compute their RIF_L and observe whether it has leading coefficient 1. The RIF_L has rather small coefficients bounded by the discriminant, and there are known algorithms to compute both the cubic field L and their corresponding RIF_L ; see for example [7] and [17] respectively.

To compute the whole rank \mathfrak{r}_3 now, by the discussion in the first paragraph of this section, amounts to deciding whether the Thue equations $F(x, y) = \pm 1$ have *rational* solutions, say of the form $(x/a, y/a)$ for any nonzero integer a , where $F \in IF_L(D)$ is any representative. The problem of determining only the *integral* solutions can be solved efficiently [25, pp.1-2]. When looking for rational solutions, this problem is equivalent, as we can see from Lemma 3.13 and Cayley's Syzyzygy (17), to finding *integral* points of infinite order on the elliptic curves $E_a : y^2 = x^3 - 27 \cdot 16Da^6$. These curves E_a are all isomorphic to each other and to E_D .

For any irreducible integral binary cubic form $f(x, y)$, denote by $N_f(m)$ the number of *integral* solutions of the Thue equation $C_{f,m} : f(x, y) = m$. Silverman in [44, Theorem A] gave an upper bound for $N_f(m)$ in terms of the rank of the elliptic curve $E_{f,m} : y^2 = x^3 + m^2 \text{disc}(f)$, $E_{f,m}$ being the Jacobian of $C_{f,m}$. In the same paper, Silverman asked the question [44, pg.396]:

$$(34) \quad \text{Does } \sup_{m \text{ cube-free}} N_f(m) = \infty ?$$

Obviously if $N_f(m)$ could get arbitrarily large this would imply that the rank of the corresponding elliptic curves $E_{f,m}$ would also get arbitrarily large, providing at the same time a negative answer to the open question

of whether the ranks of elliptic curves over \mathbb{Q} are uniformly bounded. In Silverman's case however, $N_f(m)$ can be large only very rarely. But he only considered the case where m is *cube-free*, which is a case that never occurs for our elliptic curves $E_{D'}$. We remark for clarity that the elliptic curves $E_{D'}$ are the Jacobians of the curves $F(x, y) = a^3, F \in IF_L(D)$. Specifically for the case of m being non-cube-free, Silverman cites Mahler's result [35], which gives that in this case there are infinitely many m for which $N_f(m) > (\log |m|)^{1/4}$.

It would be very interesting, within the context of Uniform Boundedness, to ask Silverman's question (34) again, specifically for the curves $E_{D'}$ and with the condition that m is a cube.

In the same spirit, it also makes sense to consider infinite families of quadratic number fields with ideal class group of large 3-rank, and with a large number of corresponding cubic monogenic fields. To the best of our knowledge, the authors of [34] give the current highest lower bound for the 3-rank of such families, which is 5. Moreover, the authors of [22] construct an infinite family of triples of polynomials of the same discriminant, defining distinct monogenic cubic fields. Even though their discriminant is never squarefree, it does raise the question whether similar constructions could yield infinite families of such tuples of polynomials, which have the same squarefree discriminant.

Along these lines let us add here that the three famous elliptic curves of Quer [40] of rank 12, are elliptic curves of the form E_D with $D = D_{m,n}$ and $(m_1, n_1) = (-363179, 89599201)$, $(m_2, n_2) = (-669454, 264029105)$ and $(m_3, n_3) = (-502270, 313570727)$. The 3-rank of the ideal class group of the corresponding imaginary quadratic number fields K_D is exactly 6, implying that in this case the whole part of the corresponding Selmer group \mathcal{S}_λ is solvable.

We close this section with a lemma that gives us some information on whether the point $Q_a = (1 : 0 : a)$ is a point on the Reduced Index Form, in the case of a solvable Torsor.

We denote by $C_{RIF_L} \in [C]$ the Torsor whose corresponding binary cubic form is the RIF_L of $IF_L(D)$. In the case of positive discriminants $D > 0$, and by recalling from Section 3.3 that the Reduced Index Form is unique in its equivalence class, we can say the following regarding the possible existence of a rational point on the corresponding Torsor.

Corollary 5.4. *Assume F is an Index Form of discriminant $0 < D \in \mathcal{D}$, whose corresponding Torsor $C : F(x, y) = z^3$ defines a solvable class. Then the Reduced Index Form RIF_L , with $C_{RIF_L} \in [C]$, is either of the form $RIF_L = [a^3, b, c, d]$, and therefore the point $Q_a = (1, 0, a)$ is a point on the corresponding Torsor C_{RIF_L} , or every rational point $Q = (x_a : y_a : a)$ of C_{RIF_L} , evaluated at the Hessian H of RIF_L , should satisfy the inequality*

$$H(x_a/a, y_a/a) \geq \frac{1}{2a^2} \sqrt{3D},$$

equivalently

$$H(x_a, y_a) \geq \frac{1}{2}\sqrt{3D}.$$

Proof. The proof is almost identical to that of [8, Lemma 5.1] and so we omit it. \square

6. EXAMPLE OF CURVES VIOLATING THE HASSE PRINCIPLE

The real quadratic number field $K_{48035713} = K_{229,3}$ has ideal class group with 3-rank $r_3(D) = 2$, therefore, Proposition 3.4 gives us that $d(\mathcal{S}_\lambda) = 2$. On the other hand, the rank of the elliptic curve E_D was computed in MAGMA and PARI/GP and was found equal to 1. Hence, the short exact sequence (5) and Lemma 3.9 imply that the rank of the quotient group $E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))$ is exactly equal to 1, and the short exact sequence (8) implies that the rank of $\text{III}(E_D)[\lambda]$ should also be equal to 1. Denote by $\langle[\xi_1]\rangle \subseteq \mathcal{S}_\lambda$ the subgroup corresponding to the solvable part of \mathcal{S}_λ . The group $\langle[\xi_1]\rangle$ defines the only cubic field, up to conjugation, with an Index Form in its $SL(2, \mathbb{Z})$ -equivalence class whose leading coefficient is a cube. If any one of the other three, $3 = \left(\frac{3^{r_3(D)}-1}{2} - 1\right)$, remaining non-conjugate cubic fields would have such a form, then the subgroup of \mathcal{S}_λ corresponding to this field must also have a form with leading coefficient a cube, implying that the rank of $E_{D'}(\mathbb{Q})/\lambda(E_D(\mathbb{Q}))$ and therefore of E_D must be at least 2, by Lemma 3.10, a contradiction.

Let us denote by $L^i, 1 \leq i \leq 4$, the 4 non-conjugate cubic fields of discriminant D . Let L^1 be the one with defining polynomial the monic polynomial $f_{229,3}(x) = x^3 - 229x + 3$, equivalently with Index Form $[1, 0, -m, n]$. Apart from $f_{229,3}(x)$, as expected, the remaining three cubic polynomials that we computed in PARI/GP were monic but of discriminant t^2D for some $0 \neq t \in \mathbb{N} \setminus \mathbb{N}^3$. We followed the procedure outlined in [6, Proposition 2.1] and we computed the four binary cubic forms $F_i(x, y)$ corresponding to L^i :

$$\begin{aligned} F_1(x, y) &= x^3 - 229xy^2 + 3y^3, \\ F_2(x, y) &= -134x^3 + 45yx^2 + 41y^2x - 2y^3, \\ F_3(x, y) &= -19x^3 + 16yx^2 + 83y^2x - 7y^3, \\ F_4(x, y) &= 23x^3 + 20yx^2 - 75y^2x - 17y^3. \end{aligned}$$

The curve $C_1 : F_1(x, y) = z^3$ has at least two rational points, namely $Q_1 = (1 : 0 : 1)$ and $Q_{m/n} = (1 : m/n : 1)$. The point Q_1 is the point Q_a we defined above, with $a = 1$ since F_1 is monic.

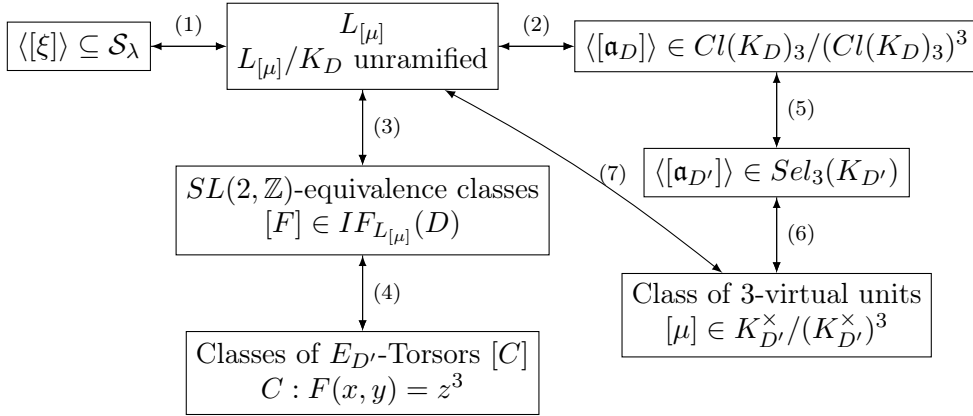
The other three curves $C_i : F_i(x, y) = z^3, 2 \leq i \leq 4$, as we already mentioned above, cannot have any rational points but since they correspond to elements in \mathcal{S}_λ , they violate the Hasse principle.

We remark that any other Torsor $C_F : F(x, y) = z^3$, with F belonging to the $SL(2, \mathbb{Z})$ -equivalence class of either F_2, F_3 , or F_4 , cannot have any rational points either, and therefore violates the Hasse Principle as well. We also note that these three classes $[C_2], [C_3]$, and $[C_4]$, even though they

correspond to non-conjugate cubic fields, they cannot belong to three independent classes in $\text{III}(E_D)[\lambda]$, since the dimension of this \mathbb{F}_3 -vector space is 1. The reason two distinct classes of Torsors might be mapped to the same class in $\text{III}(E_D)[\lambda]$, is because they might become equivalent modulo the solvable part of \mathcal{S}_λ , as one can infer from the short exact sequence (8).

7. FINAL REMARKS

In the diagram below, we summarize some of the bijections that were discussed or established in the previous sections. Only bijections (2) and (5) were not discussed before, but these follow directly from Class Field Theory and Kummer Theory respectively.



Our final remark concerns the form of the discriminants. Let us consider instead of our discriminants $D_{m,n}$, the general case of the Honda discriminants $\Delta = \mathfrak{f}^2 \text{disc}(K_{\alpha,\beta})$ discussed in Section 2.3. We can easily see that the subfamily of these discriminants satisfying $6 \nmid \Delta$ and $\Delta \equiv \pm 4 \pmod{9}$, would still satisfy Propositions 3.4, 3.15 and 3.18, Lemma 3.13, the equivalence result (33), and Corollary 4.1. We notice that in this case, and for $\Delta < 0$, the rank of the elliptic curves E_Δ might be zero, whereas for $\Delta > 0$, we must always have at least one rational point, always by assuming finiteness of the Tate-Shafarevich group. Furthermore, there is no monogenic field amongst the unramified cubic extensions of K_Δ , equivalently $\Delta \notin \mathcal{D}$, if and only if $E_{\Delta'}$ has no *integral* points. And therefore, in the case of $\Delta > 0$, the existence of no monogenic fields amongst the unramified cubic extensions of a quadratic number field K_Δ , would yield elliptic curves $E_{\Delta'}$ with nontrivial rank, yet no integral points.

ACKNOWLEDGEMENTS

The author is grateful to the anonymous Referee for their careful review of the paper, and for providing valuable comments and suggestions that improved the quality of this manuscript. The author would like to thank Antoine Joux and the CISPA Helmholtz Center for Information Security, as well as the Max Planck Institute for Mathematics in Bonn, for their hospitality and support during the writing of this paper. The author would also

like to thank Evis Ieronymou for interesting discussions, and the University of Cyprus where she spent two weeks as a research visitor while working on this paper.

REFERENCES

- [1] L. Alpöge, M. Bhargava, A. Shnidman: A positive proportion of quartic fields are not monogenic yet have no local obstruction to being so, *Mathematische Annalen* **388**(4) (2024), 4037-4052.
- [2] S. Akhtari and M. Bhargava: A Positive Proportion of Thue Equations Fail the Integral Hasse Principle, *American Journal of Mathematics* Johns Hopkins University Press, **141**(2) (2019), 283-307.
- [3] L. Alaca and K. S. Williams: On Voronoi's method for finding an integral basis of a cubic field, *Util. Math.*, **65** (2004), 163-166.
- [4] A. Bandini: Three-Descent and the Birch and Swinnerton-Dyer Conjecture, *Rocky Mountain Journal of Mathematics* **34**(1), (2004), 13-27.
- [5] C. Batut, K. Belabas, D. Bernardi, H. Cohen, M. Olivier: User's Guide to PARI-GP (version 2.12.0-ALPHA), Université Bordeaux I, Bordeaux 2019.
- [6] K. Belabas, H. Cohen: Binary Cubic Forms and Cubic Number Fields, <http://www.cecm.sfu.ca/organics/papers/hcohen/paper/html/node2.html>.
- [7] K. Belabas, A fast algorithm to compute cubic fields, *Math.Comp.* **66**(219) (1997), 1213-1237.
- [8] M. A. Bennett: On the representation of unity by binary cubic forms, *Transactions of the American Mathematical Society* **353**(4) (2000), 1507-1534.
- [9] M. Bhargava, A. Shankar and J. Tsimerman: On the Davenport-Heilbronn theorems and second-order terms, *Invent. Math.*, **193**(2) (2013), 439-499.
- [10] M. Bhargava, N. Elkies, and A. Shnidman: The average size of the 3-isogeny Selmer groups of elliptic curves $y^2 = x^3 + k$, *Journal of the London Mathematical Society*, **101**(1), (2020) 299-327.
- [11] W. Bosma, J. Cannon and C. Playoust: The Magma algebra system. I. The user language, *J. Symbolic Comput.*, **24** (1997), 235-265.
- [12] D. W. Boyd, G. Martin, and M. Thom: Squarefree values of trinomial discriminants, *LMS J. Comput. Math.* **18**(1) (2015), 148-169.
- [13] J. W. S. Cassels, Arithmetic on curves of genus 1, IV, Proof of the Hauptvermutung, *J. Reine Angew. Math.* **211**, (1962), 95-112.
- [14] J.W.S. Cassels and A. Fröhlich: Algebraic Number Theory Proceedings of an Instructional Conference Organized by the London Mathematical Society (a NATO Advanced Study Institute) with the Support of the International Mathematical Union, London Mathematical Society, 2010.
- [15] H. Cohen: *Advanced Topics in Computational Number Theory*, Springer-Verlag New York, 2000.
- [16] H. Cohen: *Number Theory Volume I: Tools and Diophantine Equations*, Springer-Verlag, 2007.

- [17] J.E. Cremona, Reduction of Binary Cubic and Quartic Forms, *LMS J. Comput. Math.* **2** (2001), 62-92.
- [18] J.E. Cremona, T.A. Fisher, C. O'Neil, D. Simon and M. Stoll: Explicit n -descent on elliptic curves, I Algebra, *Journal für die reine und angewandte Mathematik* **2008**(615) (2008), 121-155.
- [19] J.E. Cremona, T.A. Fisher, C. O'Neil, D. Simon and M. Stoll: Explicit n -descent on elliptic curves, III Algorithms, *Math. Comp.* **84** (2015), 895-922.
- [20] H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields, *Bulletin of The London Mathematical Society* **1** (1969), 345-348.
- [21] H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields (II), *Proc. Roy. Soc. Lond. A* **322** (1971), 405-420.
- [22] C. T. Davis, B. K. Spearman, and J. Yoo: Cubic polynomials defining monogenic fields with the same discriminant, *Journal de Théorie des Nombres de Bordeaux* **30** (2018), 991-996.
- [23] B. N. Delone and D. K. Faddeev, *The theory of irrationalities of the third degree*, AMS Translations of Mathematical Monographs **10**, 1964.
- [24] P. Erdős, Arithmetical properties of polynomials, *J. London Math. Soc.* **28**, (1953), 146-425.
- [25] J.-H. Evertse, A Survey on Monogenic Orders, *Publicationes Mathematicae Debrecen* **79**, (2011), 411-422.
- [26] S. Hambleton and H. Williams: *Cubic Fields with Geometry*, Springer Nature Switzerland Ag, 2018.
- [27] H. Hasse: Arithmetische Theorie der kubischen Zahlkörper auf klassenkörpertheoretischer Grundlage, *Mathematische Zeitschrift* **31**, (1930), 565-582.
- [28] T. Honda, On Real Quadratic Fields whose Class Numbers are Multiples of 3, *Journal für die reine und angewandte Mathematik* **233** (1968), 101-102.
- [29] T. Honda, Isogenies, rational points and section points of group varieties: *Japanese J. Math.* **30** (1960), 84-101.
- [30] R. Ibarra, H. Lembeck, M. Ozaslan, H. Smith and K. E. Stange: Monogenic Fields arising from Polynomials, *Experimental Mathematics* **15**(2) (2022), 299-317.
- [31] K. S. Kedlaya: A construction of polynomials with squarefree discriminants, *Proc. Amer. Math. Soc.* **140**(9) (2012), 3025-3033.
- [32] R. Kloosterman and E. Schaefer, Selmer groups of elliptic curves that can be arbitrarily large, *J. Number Theory* **99**(1) (2003), 148-163.
- [33] T. Kondo, Algebraic Number Fields with the discriminant equal to that of a quadratic number field, *J. Math. Soc. Japan*, **47**(1), (1995), 31-36.
- [34] A. Levin, Y. Shengkuan and L. Wiljanen, Quadratic fields with a class group of large 3-rank, *Acta Arithmetica* **197**(3) (2021), 275-292.
- [35] K. Mahler: On the lattice points on curves of genus 1, *Proc. London Math. Soc. Second Series* **39** (1935), 431-466.
- [36] J.S. Milne, *Class field theory (v4.03)*, www.jmilne.org/math/, 2020.

- [37] Y. Motoda, T. Nakahara, S. I. Ali Shah, and T. Uehara: *On a problem of Hasse*, Algebraic number theory and related topics 2007, RIMS Kôkyûroku Bessatsu, B12, pp.209-221. Res. Inst. Math. Sci. (RIMS), Kyoto, 2009.
- [38] T. Nakahara, Hasse's Problem for Monogenic Fields, **16**(1) (1009), 47-56.
- [39] J. Neukirch, *Algebraic Number Theory, Translated from the German by Norbert Schappacher*, Springer-Verlag, Berlin Heidelberg New York, 1999.
- [40] J. Quer, Corps quadratiques de 3-rang 6 et courbes elliptiques de rang 12, *C.R. Acad. Sci. Paris*, **305** (1987), 215–218.
- [41] Ph. Satgé: Groupes de Selmer et corps cubiques, *J.NumberTheory* , **23** (1986), 294–317.
- [42] E. F. Schaefer and M. Stoll: How to Do a p -Descent on an Elliptic Curve, *Transactions of the American Mathematical Society* **356**(3) (2004), 1209-1231.
- [43] J. Silverman: *The Arithmetic of Elliptic Curves, Graduate Texts in Mathematics (2nd Edition)*, Springer-Verlag, New York, 2009.
- [44] J. Silverman, Integer Points and the Rank of Thue Elliptic Curves, *Invent. math.* **66** (1982), 395-404.
- [45] J. Silverman and J. Tate: *Rational Points on Elliptic Curves (Second Edition)*, Springer, 2015.
- [46] J. Top, *Descent by 3-isogeny and the 3-rank of quadratic fields*, Advances in Number Theory: The Proceedings of the Third Conference of the Canadian Number Theory Association (1993), 303–317.
- [47] K. Yamamura, Some analogue of Hilbert's irreducibility theorem and the distribution of algebraic number fields, *J. Fac. Sci. Univ. Tokyo*, **38** (1991), 99 -135.

MAX-PLANCK-INSTITUT FÜR MATHEMATIK,
VIVATSGASSE 7, D-53111 BONN, GERMANY.
Email address: agathocleous@mpim-bonn.mpg.de