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SUMS OF TWO SQUARES AND THE TAU-FUNCTION: RAMANUJAN'S TRAIL

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ABSTRACT. Ramanujan, in his famous first letter to Hardy, claimed a very precise estimate for the number of integers that can be written as a sum of two squares. Far less well-known is that he also made further claims of a similar nature for the non-divisibility of the Ramanujan tau-function for certain primes. In this survey, we provide more historical details and also discuss related later developments. These show that, as so often, Ramanujan was an explorer in a fascinating wilderness, leaving behind him a beckoning trail.

1. RAMANUJAN AND SUMS OF TWO SQUARES

G.H. Hardy called his collaboration with Ramanujan "the one romantic incident of my life" [66, p. 2]. The foundation for this collaboration was laid in 1913 when Hardy received a letter from Ramanujan,[1] in which the latter stated many results that he claimed to have proved. Of some of them, Hardy remarked that they could only have been written down by a mathematician of the highest class, and moreover that "They must be true because, if they were not true, no one would have had the imagination to invent them" [66, p. 9]. The claim in Ramanujan's first letter that is our entry point into Ramanujan's trail reads:

Claim 1.1. *The number of numbers between A and x which are either squares or sums of two squares is*

$$K\int_A^x \frac{dt}{\sqrt{\log t}} + \theta(x),$$

where $K = 0.764\ldots$ and $\theta(x)$ is very small compared with the previous integral. K and $\theta(x)$ have been exactly found, though complicated...

Answering an inquiry of Hardy, Ramanujan wrote in his second letter [27, p. 56]: "The order of $\theta(x)$ which you asked in your letter is $\sqrt{x/\log x}$." In his book *Ramanujan* [66, p. 61], Hardy informs us that Ramanujan also gave the exact value of K, namely,

$$K = \frac{1}{\sqrt{2}} \prod_{p\equiv 3 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{-1/2}. \tag{1.1}$$

(Here and in the remainder of the paper the letter p is exclusively used to indicate a prime number.) Using Euler's evaluation of $\zeta(2)$ and its product representation (now called *Euler product*) for it, i.e.,

$$\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \prod_p \frac{1}{1-p^{-2}} = \frac{\pi^2}{6},$$

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[1]The complete correspondence between Ramanujan and Hardy can be found in the book by the first author and Robert Rankin [27, Chapter 2]. For biographic information about Ramanujan, Kanigel's book [84] is our favorite.

we see that K can alternatively be written as

$$K = \frac{\pi}{4} \prod_{p \equiv 1 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{1/2}.$$

Ramanujan recorded the value of K twice in his notebooks [137, Vol. 2, pp. 307, 363]. In the first edition of [137], the entry on page 307 is very difficult to read. Moreover, in addition to its lack of clarity in the first edition, the value K (C in his notation on page 307) is incomplete. But in the second edition, the entry is much clearer; in the pagination of the second edition, the entry appears on page 350. In both editions, the entry on page 363 is clear.

Both Hardy and G.N. Watson expressed their wonder as to how Ramanujan, at such a young age and isolated from the centers of mathematics, came to discover such a result, which seems to require knowledge of complex analysis for its resolution, which he very likely did not have at that time. In particular, Hardy asserted [138, p. xxv]:

> The dominant term, viz. $KB(\log B)^{-1/2}$, in Ramanujan's notation, was first obtained by E. Landau in 1908. Ramanujan had none of Landau's weapons at his command; $\dots$. It is sufficiently marvellous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day.

Furthermore, Watson [180] wrote:

> The most amazing thing about this formula is that it was discovered, apparently independently, by Ramanujan in his early days in India, and it appears in its appropriate place in his manuscript note-books.

Among the 3200–3300 entries in his notebooks [137], there are only a few instances in which Ramanujan returns to a topic that he had discussed elsewhere. In his third notebook [137, Vol. 2, page 363, both editions], Ramanujan reconsiders Claim 1.1. Ramanujan's notebooks contain only a few indications of proofs, but more space is devoted to his argument here than to any other argument or proof in his notebooks! In particular, Ramanujan offers a *more general theorem* than the one that he employed to deduce Claim 1.1. In [18, pp. 62–66] the first author gave a detailed exposition of Ramanujan's argument and, moreover, attempted to reconstruct his reasoning.

Put $b(n) = 1$ if n can be written as a sum of two squares and $b(n) = 0$, otherwise. Since the time of Fermat, it has been well-known that $b(n) = 1$ precisely when in the canonical factorization of n, all primes $p \equiv 3 \pmod 4$ occur to an even exponent. Thus, in particular, $b(n)$ is a *multiplicative* function.

Following Landau, we put $B(x) = \sum_{n \le x} b(n)$. He proved in 1908 that asymptotically

$$B(x) \sim K \frac{x}{\sqrt{\log x}}, \quad x \to \infty, \tag{1.2}$$

where K is given by (1.1) (see [92] or [94, pp. 641–669]). In honor of the contributions of both Landau and Ramanujan, the constant K is called the *Landau–Ramanujan constant*. Since Ramanujan's time, the number of decimals of K that can be computed has increased dramatically. Indeed, D. Hare [68] claims to have computed 125,079 decimals; this result was recently superseded by A. Languasco [96] who reached 130,000 decimals. In particular,

$$K = 0.764223653589220662990698731250092328116790 54\ldots$$

For the latest on high precision evaluations of Euler products involving primes in arithmetic progressions, such as the one for K, see S. Ettahri et al. [49] or O. Ramaré [140]. A more leisurely account can be found in the book by Ramaré [141, Chap. 17].

The proof of (1.2) as given by Landau is nicely sketched by Hardy in his book on Ramanujan [66, pp. 60–63]. The quickest proof of (1.2) to date seems to be due to A. Selberg [157, pp. 183–185]. His method, however, seems to be hard to generalize; cf. M.R. Murty and N. Saradha [122]. It is based on analyzing a functional equation for $B(x)$ and is reminiscent of the elementary proofs of the prime number theorem based on Selberg's fundamental lemma. Landau's method on the other hand can be very widely used.

Both Landau and Hardy were aware that Landau's method can be extended to show that $B(x)$ satisfies an asymptotic series expansion in the sense of Poincaré:

$$B(x) = K\frac{x}{\sqrt{\log x}}\Big(1+\frac{c_0}{\log x}+\frac{c_1}{\log^2 x}+\cdots+\frac{c_{m-1}}{\log^m x}+O\Big(\frac{1}{\log^{m+1} x}\Big)\Big), \quad x\to\infty, \tag{1.3}$$

where m can be taken arbitrarily large and each c_j, $0 \le j \le m-1$, is a constant. Indeed, (1.3) became a folklore result, and a proof was finally written down by J-P. Serre [160] for the larger class of so called *Frobenian multiplicative functions*. These functions were subsequently considered by R.W.K. Odoni in a series of papers (see [125] for a survey). Serre [160] also gave several beautiful applications to coefficients of modular forms, the origin of which goes back to Ramanujan's Claim 5.1; see Section 7.4.

Put

$$R(x) = K\int_2^x \frac{dt}{\sqrt{\log t}}.$$

Note that Ramanujan's claim implies, by partial integration of $R(x)$, that

$$B(x) = K\frac{x}{\sqrt{\log x}}\Big(1+\frac{d_0}{\log x}+\frac{d_1}{\log^2 x}+\cdots+\frac{d_{m-1}}{\log^m x}+O\Big(\frac{1}{\log^{m+1} x}\Big)\Big), \quad x\to\infty,$$

where $d_j = (2j+1)!/(j!\,2^{2j+1})$. This seems promising, as asymptotically it is correct by (1.2) and the expansion follows the format (1.3). However, it turns out that $c_0 \neq d_0 = 1/2$, and thus Ramanujan's Claim 1.1 is *false*. This was first asserted by Gertrude Stanley [170], a Ph.D. student of Hardy. Unfortunately, as noted by D. Shanks [165], she made several errors (even after her Corrigenda [170] are taken into account). Stanley thought that $c_0 < 0$, and this led Hardy to the statement [66, p. 19] that "The integral is better replaced by the simpler function $Kx/\sqrt{\log x}$". This is false, as Shanks showed that $c_0 \approx 0.581948659$, which is very close to Ramanujan's predicted value $d_0 = 0.5$. Actually, Shanks [165] showed that Ramanujan's integral numerically approximates $B(x)$ closer than does $Kx/\sqrt{\log x}$ (for all x large enough). We will rederive his result (Theorem 4.10) as a particular case of something fitting in the more general framework discussed in Section 4.

Shanks' constant c_0 can be computed these days with much higher precision: the current record is by Languasco, who calculated it up to 130,000 decimal digits [96]. For a discussion of c_0 and related constants, see S.R. Finch's book [50, Section 2.3].

It might be appropriate to end this introduction with what Hardy wrote about Ramanujan in the context of Claim 1.1 being false [138, p. xxv]: "And yet I am not sure that, in some ways, his failure was not more wonderful than any of his triumphs."

2. First Interlude: Gauss's Circle Problem

Ramanujan's interest in sums of squares was not confined to Claim 1.1. Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares, where different signs and different orders are counted as distinct. For example, since

$$5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2,$$

$r_2(5) = 8$. By identifying each representation of n with a unit square, e.g., the square in which the lattice point lies in its southwest corner, it is easy to see that the number $R(x)$ of lattice points in a circle with radius $\sqrt{x}$ is approximately πx. In other words,

$$R(x) = \sum_{n \leq x} r_2(n) = \pi x + P(x), \tag{2.1}$$

where $r_2(0) = 1$ and $P(x)$ is the error made in this approximation. Observe that, from an examination of lattice points in circles of radii $\sqrt{x} + \sqrt{2}$ and $\sqrt{x} - \sqrt{2}$, we find that, respectively,

$$R(x) \leq (\sqrt{x} + \sqrt{2})^2 \qquad \text{and} \qquad R(x) \geq (\sqrt{x} - \sqrt{2})^2.$$

From these two inequalities, Gauss concluded that there exists a positive constant C such that

$$|P(x)| \leq C\sqrt{x}$$

for all $x \geq 0$. Finding the optimal power θ such that $|P(x)| \leq C' x^{\theta}$ for all $x > 0$ is known as the *Gauss Circle Problem*. Hardy in his famous paper [65] proved that

$$P(x) \neq O\big((x \log x)^{1/4}\big), \qquad x \to \infty, \tag{2.2}$$

and so in particular, $\theta > \frac{1}{4}$. In [65], Hardy offers a beautiful identity of Ramanujan that is not found elsewhere in Ramanujan's work, namely, if $a, b > 0$, then

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}}\, e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}}\, e^{-2\pi\sqrt{(n+b)a}}. \tag{2.3}$$

Hardy used an identity that can be derived from (2.3) to prove (2.2).

The error term $P(x)$ can be represented by an infinite series of Bessel functions. More precisely,

$$\sideset{}{'}\sum_{0 \leq n \leq x} r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}), \tag{2.4}$$

where the prime on the summation sign at the left indicates that if x is an integer, then only $\frac{1}{2} r(x)$ is counted, and where $J_1(x)$ is the ordinary Bessel function of order 1. The identity (2.4) apparently first appeared in Hardy's paper [65], where he wrote "The form of this equation was suggested to me by Mr. S. Ramanujan...." The identity (2.4) is an important key in deriving upper bounds for $P(x)$.

In his Lost Notebook [139, p. 335], Ramanujan offers a two-variable analogue of (2.4). This and an analogous identity on page 335 of [139] involving $d(n)$, the number of positive divisors of the integer n, are the focus of several papers by the first author, S. Kim, and A. Zaharescu [20, 21, 22, 23, 24]. These authors feel that Ramanujan derived the former identity to attack Gauss's *Circle Problem*, but, although the first author, Kim, and Zaharescu have proofs of Ramanujan's two identities, they have been unable to penetrate Ramanujan's thinking. For a discussion of these identities and an historical overview of the *Circle Problem*, see their survey article [25].

3. Second Interlude: Ramanujan's tau-function

In non-technical terms, a *modular form* is a holomorphic function on the Poincaré upper half plane having a lot of symmetry with respect to the modular group $SL_2(\mathbb{Z})$ or a congruence subgroup Γ of it. In particular, for some fixed k, called the *weight*, we have

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^k f(z) \quad \text{for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We also require moderate growth at the cusps of the group of symmetry. If the form vanishes at all of these cusps, it is said to be a *cusp form*. The lowest weight for cusp forms for the full modular group is 12, and it occurs for the *normalized discriminant function* $\Delta(z)$. As already known to C.G.J. Jacobi [82], it can be expressed as an infinite product:

$$\Delta(z) = q\prod_{j=1}^{\infty}(1-q^j)^{24} = \sum_{n\geq 1}\tau(n)q^n, \quad q = e^{2\pi i z}, \quad \mathrm{Im}(z) > 0, \tag{3.1}$$

called the modular discriminant. Its 24th root was extensively later studied by R. Dedekind and is appropriately named the *Dedekind η-function*. However, Ramanujan was the first to take a deep interest in arithmetic properties of $\tau(n)$ (now called *Ramanujan's tau-function*), and in 1916 published the important paper [136]. He calculated $\tau(n)$ for $n = 1, 2, \ldots, 30$; see Table 1. Note that the only odd values occur for $n = 1, 9$, and 25 (see §7.4.3 for a proof). Indeed, Ramanujan showed with ease that $\tau(n)$ is odd or even according as n is an odd square or not. He also made some observations that he was not able to prove:

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	11	534612	21	−4219488
2	−24	12	−370944	22	−12830688
3	252	13	−577738	23	18643272
4	−1472	14	401856	24	21288960
5	4830	15	1217160	25	−25499225
6	−6048	16	987136	26	13865712
7	−16744	17	−6905934	27	−73279080
8	84480	18	2727432	28	24647168
9	−113643	19	10661420	29	128406630
10	−115920	20	−7109760	30	−29211840

TABLE 1. The first 30 values of $\tau(n)$

Conjecture 3.1. *The following properties hold:*
(a) $\tau(n)$ is multiplicative; that is, $\tau(mn) = \tau(m)\tau(n)$ whenever $(m,n) = 1$;
(b) if p is prime, then $\tau(p^{e+1}) = \tau(p)\tau(p^e) - p^{11}\tau(p^{e-1})$ for any $e \geq 2$;
(c) $|\tau(n)| \leq d(n)n^{11/2}$, where $d(n)$ denotes the number of positive divisors of n.

The first property implies that the τ-values are determined by their values at prime arguments. Ramanujan already noted that the first two properties can be rephrased in terms of the L-function

$$L(\Delta, s) := \sum_{n\geq 1} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}, \qquad \mathrm{Re}(s) > \frac{13}{2}.$$

Conjecture 3.1(a) was proved by L.J. Mordell within the year, and years later it motivated E. Hecke to introduce the *Hecke operator* T_p at each prime p. In this set-up, the product expansion is equivalent to Δ being an eigenfunction of T_p with eigenvalue $\tau(p)$. Using his operators, Hecke [71] went on to prove a similar result for Fourier coefficients of modular forms for congruence subgroups of $SL_2(\mathbb{Z})$.

Conjecture 3.1(c), however, resisted proof attempts by the best minds for a long time. For primes p, it states that in the factorization

$$1 - \tau(p)p^{-s} + p^{11-2s} = \big(1 - \alpha(p)p^{-s}\big)\big(1 - \beta(p)p^{-s}\big),$$

we have $|\alpha(p)| = |\beta(p)| = p^{11/2}$, and hence

$$|\tau(p)| \leq |\alpha(p)| + |\beta(p)| \leq 2p^{11/2}. \tag{3.2}$$

Ramanujan himself showed that $\tau(n) = O(n^7)$ (we note that $d(n) = O(n^{\epsilon})$ for any $\epsilon > 0$). After a lot of intermediary work by many mathematicians. In 1971, P. Deligne [41] interpreted the numbers $\tau(p)$ as eigenvalues of the Frobenius automorphism on the cohomology of an appropriate 11-dimensional variety (generalization of the Eichler-Shimura isomorphism) to reduce the Ramanujan conjecture to the Weil conjectures for smooth projective varieties over finite fields, which he proved in 1974 [42] (see N.M. Katz [85] and E. Kowalski [89] for introductory accounts). This proof is one of the most important and highly regarded proofs in all of 20th century algebraic geometry.

Conjecture 3.1(c) can be extended (in a suitable manner depending on the weight) to all so-called normalised Hecke eigencuspforms. R.P. Langlands [95] reinterpreted the Hecke eigenforms in terms of automorphic representations for GL_2 over the rationals, so that both have the same associated L-functions. As such, the cusp forms correspond to cuspidal representations. The Ramanujan conjecture, in its general form, asserts that a generic cuspidal automorphic irreducible unitary representation of a reductive group over a global field should be locally tempered everywhere. It is largely unsettled, and research on it is part of the Langlands Program, a very active modern research area (see, S. Gelbart [58] for an introduction). For more on the mathematical road from Conjecture 3.1 to the Langlands Program, see the survey by W.-C.W. Li [106] or the book by the Murty brothers [120]. The three excellent survey articles highlighting different aspects of the tau-function [118, 146, 176] in the proceedings of the 1987 “Ramanujan Revisited" conference, focus more on the properties of the tau-function per se.

It is amazing to see that the properties Ramanujan uncovered and those that he conjectured initially formed an easily overlooked trail, but they are now major mathematical highways!

3.1. **Congruences for $\tau(n)$.** Ramanujan also discovered congruences for $\tau(n)$, mostly involving the *sum of divisors* function $\sigma_k(n)$, which is defined as $\sigma_k(n) = \sum_{d|n} d^k$. Here we give a sampling (taken from [176]), in which $(\frac{a}{b})$ denotes the *Legendre symbol*):

$$\tau(n) \equiv \begin{cases} \sigma_{11}(n) \pmod{2^8}, & \text{if } 2 \nmid n, \\ n^2\sigma_7(n) \pmod{3^3}, \\ n\sigma_9(n) \pmod{5^2}, \\ n\sigma_3(n) \pmod{7}, \\ \sigma_{11}(n) \pmod{691}, \\ 0 \pmod{23}, & \text{if } (\frac{n}{23}) = -1. \end{cases} \tag{3.3}$$

The latter congruence was refined in 1930 by J.R. Wilton [184]. If we restrict n to be a prime p, this refinement yields:

$$\tau(p) \equiv \begin{cases} 1 \pmod{23}, & \text{if } p = 23, \\ 0 \pmod{23}, & \text{if } (\frac{p}{23}) = -1, \\ 2 \pmod{23}, & \text{if } p = X^2 + 23Y^2 \text{ with } X \neq 0, \\ -1 \pmod{23}, & \text{otherwise.} \end{cases} \tag{3.4}$$

The primes satisfying the second congruence are, by quadratic reciprocity, primes in a union of arithmetic progressions. For the third congruence, this is no longer the case; it is inherently non-abelian. Wilton's starting point is the trivial observation that modulo 23 the n-th Fourier coefficient $t(n)$ of $\eta(z)\eta(23z)$ equals $\tau(n)$.

The congruence (3.4) was derived in a different way by F. van der Blij [179], who considered the number of the representations of the integer n by a form of class F_i, with F_i a reduced form of discriminant -23 (of which there are three, namely $F_1 = X^2 + XY + Y^2$, $F_2 = 2X^2 + XY + 3Y^2$ and $F_3 = 2X^2 - XY + 3Y^2$). He establishes the formulae

$$a(n, F_1) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{23}\right) + \frac{4}{3}t(n), \quad a(n, F_2) = a(n, F_3) = a(n, F_1) - 2t(n),$$

and deduces the congruences from this. By related arguments, cf. Serre [163], it can be shown that

$$\tau(p) \equiv N_p(X^3 - X - 1) - 1 \pmod{23},$$

where $N_p(f)$ denotes the number of distinct roots modulo p of a polynomial $f \in \mathbb{Z}[x]$. The crux is that the L-function of $\eta(z)\eta(23z)$ is closely related to the Dedekind zeta-function of the cubic field of $\mathbb{Q}[X]/(X^3 - X - 1)$ of discriminant -23, the Galois closure of which is the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$.

Another non-obvious result involves the *Padovan sequence* [127], which is defined by $B_0 = 0, B_1 = B_2 = 1$, and by $B_n = B_{n-2} + B_{n-3}$ for every $n \geq 3$. The second author and A. Noubissie [115] showed that a prime p divides B_{p-1} if and only if $\tau(p) \equiv 2 \pmod{23}$.

After Wilton's paper many further ones with congruences for the tau-function appeared. This whole rag-bag of results begged for a more theoretical explanation, which only became available in the early 1970s, and rests on the Serre–Deligne representation theorem and the theory of modular forms modulo ℓ. Thus the congruence modulo 691 in (3.3) is a consequence of the Δ-function reducing to the Eisenstein series E_{12} modulo 691. The Eisenstein series have sums of divisor functions as Fourier coefficients. For a detailed survey, we refer the reader to H.P.F. Swinnerton-Dyer's paper [176].

Many of the results proved in this setting eventually played a role in the work on Fermat's Last Theorem by Ribet, Wiles and others! Thus here again Ramanujan's trail would eventually become a highway.

3.2. **Computation of τ-values.** As already mentioned, Ramanujan computed the first 30 tau-values in 1916 [136] (see Table 1). Watson [182], according to Rankin [145], as a pass-time during World War II, extended these computations to 1000 values. Around the same time D.H. Lehmer [101] computed $\tau(n)$ for similar ranges. The objectives of these early tabulators were the following: to check Conjecture 3.1(c), to find primes p for which p divides $\tau(p)$, and to find n for which $\tau(n) = 0$. Unsurprisingly, they never found a counter-example to Conjecture 3.1(c). It is now known that p divides $\tau(p)$ for $p = 2, 3, 5, 7, 2411$ and $p = 7758337633$, and that there are no further $p < 10^{10}$ with this property; see the paper by N. Lygeros and O. Rozier [107]. Using various congruences, Serre [158] proved that if $\tau(p) = 0$, then p lies in one of 33 congruence classes modulo $3488033912832000 = 2^{14} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 23 \cdot 691$. The conjecture that $\tau(n) \neq 0$ became famous as Lehmer's Conjecture. Any known computational approach to this makes use of the known congruences to discard many integers n. Lehmer [102] himself got to 3316798. The current record (see [43]) is $816212624008487344127999 \approx 8 \cdot 10^{23}$. This impressive result ultimately rests on sophisticated methods from algebraic geometry, which allow one to compute $\tau(p)$ in polynomial time. There is even an entire book [46] devoted to proving this result. The basic idea is to efficiently compute $\tau(p)$ modulo enough small primes ℓ such that the Chinese remainder theorem and the bound $|\tau(p)| \leq 2p^{11/2}$ allow one to uniquely determine $\tau(p)$. For example, 19 divides $\tau(10^{1000} + 46227)$. With the use of deep methods in analytic number theory, it was recently proved that the density of integers n with $\tau(n) = 0$ is at most $1.15 \cdot 10^{-12}$ [78].

A strengthening of Lehmer's conjecture, was proposed by A.O.L. Atkin and J-P. Serre [160], who conjectured that $|\tau(p)| \gg_\epsilon p^{9/2-\epsilon}$ for every $\epsilon > 0$. This conjecture implies, in particular, that given any fixed integer a, there are at most finitely many primes p for which $\tau(p) = a$. If a is odd, then Murty et al. [121] proved that there are at most finitely many integers n for which $\tau(n) = a$ (these n must be odd squares). More precisely, they demonstrated the existence of an effectively computable positive constant c such that if $\tau(n)$ is odd, then $|\tau(n)| > (\log n)^c$. M.A. Bennett et al. [16] proved results on the largest prime factor of odd $\tau(n)$ values. J.S. Balakrishnan et al. [10] showed that $\{\pm 1, \pm 3, \pm 5, \pm 7, \pm 691\}$ do not occur as τ-values, and in the follow-up paper [11] many other small odd values are excluded. Currently there is a flurry of activity excluding further values. Meanwhile some further *even* values can be excluded, e.g., $2p$, where p is a prime, $2 < p < 100$ or $-2p^j$ with $j \geq 1$ arbitrary and with p in a set containing eighteen primes [12].

Serre [161] initiated the general study of estimating the size of possible gaps in the Fourier expansions of modular forms, namely, he considered the gap function

$$i_f(n) = \max\{k : a_f(n+j) = 0 \text{ for all } 0 \leq j \leq k\}.$$

For Fourier coefficients of a newform f without complex multiplication, A. Balog and K. Ono [8] showed that

$$i_f(n) \ll_{f,\epsilon} n^{\frac{17}{41}+\epsilon}.$$

E. Alkan [1] improved the exponent $17/41$ to $51/34$ in case of newforms associated to elliptic curves without complex multiplication. In [2] he showed that for these newforms $i_f(n)^\lambda$ is bounded on average for $\lambda < 1/8$. Alkan and A. Zaharescu [3], making clever use of the Ramanujan-Wilton congruence (3.4), showed that $i_\Delta(n) \le 2\sqrt{46}n^{1/4} + 23$ for every $n \ge 1$. By a variation of their method S. Das and S. Ganguly [39] showed that $i_f(n) \ll_k n^{1/4}$ for any nonzero cusp form f of integral weight k on the full modular group. Interestingly, in the final step of their analysis they have to establish the existence of sums of two squares in short intervals.

4. Third Interlude: Euler-Kronecker constants of multiplicative sets

A set S of natural numbers is said to be *multiplicative* if for every pair m and n of co-prime integers in S, mn is also in S. Let i_S denote the *characteristic function* of S, i.e.,

$$i_S(n) = \begin{cases} 1, & n \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the set S is multiplicative if and only if i_S is a multiplicative function. A large class of multiplicative sets is provided by the sets

$$S_{f;q} := \{n : \ q \nmid f(n)\}, \tag{4.1}$$

where q is any prime and f is any integer-valued multiplicative function.

Let $\pi_S(x)$ and $S(x)$ denote the number of primes and, respectively, the number of integers in S not exceeding x. The following result is a special case of a theorem of E. Wirsing [185], with a reformulation following Finch et al. [51, p. 2732]. As usual, Γ denotes the gamma function.

Theorem 4.1. *Let S be a multiplicative set satisfying $\pi_S(x) \sim \delta x/\log x$, as $x \to \infty$, for some δ, $0 < \delta < 1$. Then*

$$S(x) \sim c_0(S) x \log^{\delta-1} x, \quad x \to \infty,$$

where

$$c_0(S) := \frac{1}{\Gamma(\delta)} \lim_{P\to\infty} \prod_{p<P} \Big(1 + \frac{i_S(p)}{p} + \frac{i_S(p^2)}{p^2} + \cdots\Big)\Big(1 - \frac{1}{p}\Big)^\delta$$

converges and hence is positive.

Recall that for any character χ, Dirichlet's L-function $L(s,\chi)$ is defined by

$$L(s,\chi) := \sum_{n=1}^\infty \frac{\chi(n)}{n^s}, \quad \text{Re}(s) > 1.$$

Example 4.2. *If S is the set of integers that can be written as a sum of two squares, Theorem 4.1 yields*

$$K = c_0(S) = \frac{\sqrt{2}}{\Gamma(1/2)} \Big(L(1,\chi_{-4}) \prod_{p\equiv 3 \,(\mathrm{mod}\, 4)} \frac{1}{1-p^{-2}}\Big)^{1/2},$$

where χ_{-4} denotes the nontrivial, quadratic Dirichlet character modulo 4, and where we used the identity

$$\lim_{P\to\infty}\prod_{\substack{p<P\\ p\equiv 1\,(\mathrm{mod}\,4)}}\Big(1-\frac{1}{p}\Big)^{-1}\prod_{\substack{p<P\\ p\equiv 3\,(\mathrm{mod}\,4)}}\Big(1+\frac{1}{p}\Big)^{-1}=L(1,\chi_{-4}).$$

Since $\Gamma(1/2)=\sqrt{\pi}$ and $L(1,\chi_{-4})=\pi/4$, we then obtain the expression (1.1) *for K.*

In this set-up, Ramanujan likely would have claimed that

$$S(x)=c_0(S)\int_2^x \log^{\delta-1} t\,dt+O(x\,\log^{-r} x), \tag{4.2}$$

where r is any positive number. We call

$$c_0(S)x\log^{\delta-1} x,\quad c_0(S)\int_2^x \log^{\delta-1} t\,dt,$$

respectively, the *Landau* and *Ramanujan approximations* to $S(x)$. If for all x sufficiently large,

$$\Big|\,S(x)-c_0(S)x\log^{\delta-1} x\,\Big|<\Big|\,S(x)-c_0(S)\int_2^x \log^{\delta-1} t\,dt\,\Big|,$$

we say that the Landau approximation is better than the Ramanujan approximation. If the reverse inequality holds for every x sufficiently large, we say that the Ramanujan approximation is better than the Landau approximation. We will now introduce a constant which can be used to decide which approximation is better.

For Re$(s)>1$, put

$$L_S(s):=\sum_{n\in S} n^{-s}.$$

If the limit

$$\gamma_S:=\lim_{s\to 1^+}\Big(\frac{L_S'(s)}{L_S(s)}+\frac{\alpha}{s-1}\Big) \tag{4.3}$$

exists for some $\alpha>0$, we say that the set S admits an *Euler–Kronecker constant* γ_S. In the case $S=\mathbb{N}$, we have $L_S(s)=\zeta(s)$, the *Riemann zeta function*, $\alpha=1$ and $\gamma_S=\gamma=0.5772156649\ldots$, the *Euler–Mascheroni constant*. Consult J. Lagarias's paper [91] for a beautiful survey, and G. Havil [69] for a popular account. If S is a set that in some sense is close to the set of all natural numbers, then γ_S will be close to γ. This will be, for example, the case if q in (4.1) is a large prime (see, e.g., Example 4.5).

As the following result shows, the Euler–Kronecker constant γ_S determines the second order behavior of $S(x)$. As usual $\pi(x)$ denotes the prime counting function.

Theorem 4.3. *Let S be a multiplicative set. If there exists $\rho>0$ and $0<\delta<1$ such that*

$$\pi_S(x)=\delta\pi(x)+O_S(x\log^{-2-\rho} x),\quad x\to\infty, \tag{4.4}$$

then $\gamma_S\in\mathbb{R}$ exists, and asymptotically, as $x\to\infty$,

$$S(x)=\sum_{\substack{n\le x\\ n\in S}}1=\frac{c_0(S)\,x}{\log^{1-\delta} x}\left(1+\frac{(1-\gamma_S)(1-\delta)}{\log x}\big(1+o_S(1)\big)\right). \tag{4.5}$$

In the case when the prime numbers belonging to S are, with finitely many exceptions, precisely those in a finite union of arithmetic progressions, we have, for arbitrary $j \geq 1$,

$$S(x) = \frac{c_0(S)\,x}{\log^{1-\delta} x}\Big(1 + \frac{c_1(S)}{\log x} + \frac{c_2(S)}{\log^2 x} + \cdots + \frac{c_j(S)}{\log^j x} + O_{j,S}\Big(\frac{1}{\log^{j+1} x}\Big)\Big), \quad x \to \infty, \tag{4.6}$$

where $c_1(S) = (1 - \gamma_S)(1 - \delta)$, and $c_2(S), \ldots, c_j(S)$ are further constants.

Proof. For the first assertion, see Moree [112, Theorem 4]; for the second, see Serre [160, Théorème 2.8]. □

By partial integration, as $x \to \infty$,

$$\int_2^x \log^{\delta-1} t\, dt = \frac{x}{\log^{1-\delta} x}\Big(1 + \frac{1-\delta}{\log x} + O\Big(\frac{1}{\log^2 x}\Big)\Big). \tag{4.7}$$

Thus, the Landau and Ramanujan approximations to $S(x)$ amount to taking $c_1(S) = 0$ and $c_1(S) = 1 - \delta$, respectively. This trivial observation leads to the following corollary of Theorem 4.3.

Corollary 4.4. *Suppose that the set S is multiplicative and satisfies* (4.4). *Then the Euler–Kronecker constant γ_S exists. Furthermore,*

- *The Ramanujan type claim* (4.2) *is true for every $r \leq 2 - \delta$, and, provided that $\gamma_S \neq 0$, is false for every $r > 2 - \delta$.*
- *If $\gamma_S > 1/2$, the Landau approximation is better. If $\gamma_S < 1/2$, the Ramanujan approximation is better.*

Example 4.5. *K. Ford et al.* [53] *studied the infinite family of sets $S_{\varphi;q} := \{n : q \nmid \varphi(n)\}$, where $q \geq 3$ is a prime and φ denotes Euler's totient function. They showed that the Ramanujan approximation is better if and only if $q \leq 67$. Further, they established that, as $q \to \infty$, $\gamma_{S_{\varphi;q}} = \gamma + O_\epsilon(q^{\epsilon-1})$, underlining the fact that for large q, the series $L_{S_{\varphi;q}}(s)$ starts to behave more like $\zeta(s)$. They also show that $\gamma_{S_{\varphi;q}}$ is intimately related to the Euler–Kronecker constant of the cyclotomic number field $\mathbb{Q}(\zeta_q)$. In general the Euler–Kronecker constant of a number field K is obtained on putting $\alpha = 1$ and $L_S(s) = \zeta_K(s)$ in* (4.3), *where $\zeta_K(s)$ denotes the Dedekind zeta function of K.*

4.1. **Abelian multiplicative subsets.** The sets in Example 4.5 and many others in the literature under the umbrella of Theorem 4.3 are abelian, which we now define.

Definition 4.6. *A multiplicative set S is called* ***abelian*** *if it consists, with finitely many exceptions, of all the primes in a finite union of arithmetic progressions.*

For example, the set consisting of $5, 7$ and all primes p satisfying $p \equiv \pm 1 \pmod 7$ is abelian.

By the Chinese remainder theorem, we can find an integer d such that the primes are, with finitely many exceptions, those in a number of primitive residue classes modulo d.

If S_1 and S_2 are multiplicative sets such that any two elements of S_1 and S_2 are coprime, then $S_1 \cdot S_2 := \{m \cdot n : m \in S_1,\, n \in S_2\}$ is a multiplicative set and $L_{S_1 \cdot S_2}(s) = L_{S_1}(s)L_{S_2}(s)$. If both Euler–Kronecker constants γ_{S_1} and γ_{S_2} exist, then $\gamma_{S_1 \cdot S_2} = \gamma_{S_1} + \gamma_{S_1}$. Thus, in the case when S is multiplicatively abelian, the computation of γ_S can be reduced to the case where the primes in S are precisely those in *one* primitive residue class.

From now on we use the short-hand notation defined on the right-hand side below:

$$\frac{d}{ds}\log\{A(s)\} = \frac{A'(s)}{A(s)} =: \frac{A'}{A}(s).$$

Theorem 4.7 (Languasco and Moree [99]). *Let a and $d \geq 2$ be coprime integers. Let S be the set of integers including 1 and all integers composed only of primes $p \equiv a \pmod{d}$. Then S has an Euler–Kronecker constant $\gamma(d,a)$ given by*

$$\gamma(d,a) = \gamma_1(d,a) - \sum_{p\equiv a \,(\mathrm{mod}\, d)} \frac{\log p}{p(p-1)} + \sum_{n\equiv a \,(\mathrm{mod}\, d)} \frac{(1+\mu(n))\Lambda(n)}{n}, \tag{4.8}$$

where μ is the Möbius function, Λ is the von Mangoldt function,

$$\gamma_1(d,a) = \frac{1}{\varphi(d)}\Big(\gamma + \sum_{p|d}\frac{\log p}{p-1} + \sum_{\chi\neq\chi_0}\overline{\chi}(a)\frac{L'}{L}(1,\chi)\Big), \tag{4.9}$$

and χ_0 is the principal character modulo d.

It follows from this result that an abelian multiplicative set has an Euler–Kronecker constant involving Dirichlet L-series.

Remark 4.8. *In 1909, using the method with which he proved the asymptotic formula* (1.2)*, Landau* [93] *(see also* [94, §176–183]*) settled a question of D.N. Lehmer who, reformulated in our terminology, asked about the asymptotic behavior of*

$$x^{-1}\sum_{\substack{n\leq x,\\ n\in S}} 2^{\omega(n)},$$

where $\omega(n)$ denotes the number of different prime factors of n, and S is assumed to be abelian. In this context, he established (4.6)*, however, without identifying $c_1(S)$ as $(1-\gamma_S)(1-\delta)$.*

4.2. **The Euler–Kronecker constant for sums of two squares.** As an example, we will determine γ_S in the case when S is the set of sums of two squares. Since S is generated by the prime 2, the primes $\equiv 1 \pmod{4}$ and the squares of the primes $\equiv 3 \pmod{4}$, we obtain

$$L_S(s) = (1-2^{-s})^{-1}\prod_{p\equiv 1\,(\mathrm{mod}\,4)}(1-p^{-s})^{-1}\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1-p^{-2s})^{-1}. \tag{4.10}$$

Recall that

$$L(s,\chi_{-4}) = \prod_{p\equiv 1\,(\mathrm{mod}\,4)}(1-p^{-s})^{-1}\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1+p^{-s})^{-1}.$$

Comparing the Euler factors on both sides, we can verify that

$$L_S(s)^2 = \zeta(s)L(s,\chi_{-4})(1-2^{-s})^{-1}\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1-p^{-2s})^{-1}. \tag{4.11}$$

For the reader who is familiar with the *Dedekind zeta function* $\zeta_K(s)$, this identity is not so mysterious, and will realize that

$$\zeta_{\mathbb{Q}(i)}(s) = \zeta(s)L(s,\chi_{-4}) = (1-2^{-s})^{-1}\prod_{p\equiv 1\,(\mathrm{mod}\,4)}(1-p^{-s})^{-2}\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1-p^{-2s})^{-1},$$

which exhibits the relationship of $L_S(s)^2$ with $\zeta_{\mathbb{Q}(i)}(s)$.

Applying logarithmic differentiation to both sides of (4.11), we find that

$$2\frac{L_S'}{L_S}(s)+\frac{1}{s-1}=\frac{\zeta'}{\zeta}(s)+\frac{1}{s-1}+\frac{L'}{L}(s,\chi_{-4})-\frac{\log 2}{2^s-1}-\sum_{p\equiv 3\,(\mathrm{mod}\,4)}\frac{2\log p}{p^{2s}-1}.$$

Therefore, letting $s\to 1$, we deduce that

$$2\gamma_S=\gamma+\frac{L'}{L}(1,\chi_{-4})-\log 2-\sum_{p\equiv 3\,(\mathrm{mod}\,4)}\frac{2\log p}{p^2-1}. \tag{4.12}$$

4.2.1. *Rough numerical evaluation of* γ_S. The quotient

$$\frac{L'}{L}(1,\chi_{-4}) \tag{4.13}$$

in (4.12) is connected with two important ideas in the mathematical literature: the arithmetic-geometric mean (AGM) of Gauss and Lagrange, and the lemniscate integral.

First, we discuss the AGM. Let $a=a_0$ and $b=b_0$ be initial values, with $a_0,b_0>0$. Recursively define two sequences $\{a_n\}$ and $\{b_n\}$ for $n\geq 1$ by

$$a_{n+1}=\frac{a_n+b_n}{2}\quad\text{and}\quad b_{n+1}=\sqrt{a_nb_n}.$$

Then

$$\lim_{n\to\infty}a_n\quad\text{and}\quad\lim_{n\to\infty}b_n$$

both exist and are equal. The *arithmetic-geometric mean* of $\{a_n\}$ and $\{b_n\}$ is defined by

$$M(a,b):=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n. \tag{4.14}$$

To see how extensively the AGM appears in number theory and related analysis, consult J.M. and P.B. Borwein's fascinating treatise [30].

Second, the *lemniscate integral*, naturally arising in the calculation of the arc length of the lemniscate, is defined by

$$L:=\int_0^1\frac{dx}{\sqrt{1-x^4}}. \tag{4.15}$$

The lemniscate integral was initially studied by Count Giulio Fagnano and James Bernoulli. C.L. Siegel [168] considered the lemniscate integral so important that he began his series of lectures on elliptic functions with a thorough discussion of it. For an interesting historical account of the lemniscate integral, including the work of Fagnano and Bernoulli, consult R. Ayoub's paper [7]; see also a paper by G. Almkvist and the first author [4].

On pages 283, 285, and 286 in the unorganized portion of his second notebook [137], Ramanujan examined the lemniscate integral (4.15) and various extensions and analogues of it. In particular, he established several inversion formulas. We state one of his results, which was first proved by S. Bhargava and the first author [19].

Entry 4.9. *Let* θ, v *and* μ *be defined by*

$$\frac{\theta\mu}{\sqrt{2}}=\int_0^v\frac{dx}{\sqrt{1-x^4}}, \tag{4.16}$$

where $0 \leq \theta \leq \pi/2, 0 \leq v \leq 1$, *and* μ *is a constant defined by setting* $\theta = \pi/2$ *and* $v = 1$. *Then, for* $0 < \theta \leq \pi/2$,

$$\frac{\mu^2}{2v^2} = \csc^2\theta - \frac{1}{\pi} - 8\sum_{n=1}^{\infty}\frac{n\,\cos(2n\theta)}{e^{2\pi n}-1}. \tag{4.17}$$

We will deduce the value of the constant μ below. For proofs of Ramanujan's inversion formulas, see the first author's book [18, Chapter 26] or [19].

Returning to (4.12) and (4.13), we note the identity

$$\frac{L'}{L}(1,\chi_{-4}) = \log\big(M(1,\sqrt{2})^2e^{\gamma}/2\big), \tag{4.18}$$

where $M(1,\sqrt{2})$ is given by (4.14) with starting values $a_0 = 1$ and $b_0 = \sqrt{2}$. This formula appears to have been discovered independently at least by Berger (1883), Lerch (1897), de Séguier (1899) and Landau (1903) (for more complete references, see Shanks [165]). Gauss showed in his diary [63] that

$$G := \frac{1}{M(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1\frac{dx}{\sqrt{1-x^4}} = 0.8346268416740731862814297\ldots. \tag{4.19}$$

This constant is now named *Gauss's constant* and is also related to various other constants. Note that when we compare (4.16) with (4.19), we deduce that [57]

$$\mu = \frac{\sqrt{2}}{M(1,\sqrt{2})}.$$

We conclude from (4.18) and (4.19) that (4.12) can be rewritten as

$$\gamma_S = \gamma - \log G - \log 2 - \sum_{p\equiv 3\,(\mathrm{mod}\,4)}\frac{\log p}{p^2-1}. \tag{4.20}$$

The AGM algorithim is fast. Just taking a few decimals of the constants involved into account and only the prime $q = 3$ in the sum, we conclude that $\gamma_S < 0.578 - \log(5/6) - 0.693 - (\log 3)/8 < -0.07$, where we note that it takes only two steps in the AGM algorithm in order to conclude that $G > 5/6$. On applying Corollary 4.4 we obtain a new proof of the following result.

Theorem 4.10 (Shanks [165]). *Ramanujan's approximation* $K\int_2^x\frac{dt}{\sqrt{\log t}}$ *asymptotically better approximates* $B(x)$, *than does Landau's asymptotic* $K\frac{x}{\sqrt{\log x}}$. *However, Ramanujan's Claim 1.1 is false with any error term* $\theta(x)$ *satisfying* $\theta(x) = o(x\log^{-3/2}x)$.

4.2.2. *Precise numerical evaluation of* γ_S. We first describe the computation of Shanks [165]. His starting point is formula (4.12) and the observation that, for $\mathrm{Re}(s) > 1/2$,

$$\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1-p^{-2s})^{-2} = \zeta(2s)(1-2^{-2s})L(2s,\chi_{-4})^{-1}\prod_{p\equiv 3\,(\mathrm{mod}\,4)}(1-p^{-4s})^{-2}. \tag{4.21}$$

Using the logarithmic differentiated form of this repeatedly, he obtains

$$\sum_{p\equiv 3\,(\mathrm{mod}\,4)}\frac{2\log p}{p^2-1} = \sum_{k=1}^{\infty}\Big(\frac{L'}{L}(2^k,\chi_{-4}) - \frac{\zeta'}{\zeta}(2^k) - \frac{\log 2}{2^{2^k}-1}\Big), \tag{4.22}$$

which in combination with (4.12) and (4.18) then yields $\gamma_S = -0.1638973186\ldots$.

These days, one can do much better. To compute $\frac{L'}{L}(j,\chi)$ for $j \geq 2$, first recall that $L(s,\chi)$ can be expressed as a linear combination of Hurwitz zeta functions $\zeta(s,x)$. An efficient algorithm to compute $\zeta(s)$, $\zeta(s,x)$, and $\frac{d\zeta}{ds}(s,x)$ for $s > 1$, has been devised by Languasco [98]. Invoking (4.12) and (4.22) he determined 130,000 decimals for γ_S in [96].

We note that the typical prime sums can be naively estimated by computing all terms up to some large value x and estimating the tail by

$$\sum_{p>x} \frac{\log p}{p^k - 1} \leq \frac{x}{x^k - 1}\Big(-0.98 + 1.017\frac{k}{k-1}\Big), \quad k \in \mathbb{R}_{>1} \text{ and } x \geq 7481,$$

which follows easily on using the estimate $0.98x \leq \sum_{p\leq x} \log p \leq 1.017x$ for $x \geq 7481$ due to J.B. Rosser and L. Schoenfeld [149]. The same bound can be used if the primes p are restricted to some arithmetic progression.

4.3. **Connection with Cilleruelo's constant.** The result mentioned in the previous section allows us to compute *Cilleruelo's constant*

$$J := \gamma - 1 - \frac{\log 2}{2} - \sum_{p>2} \frac{(\frac{-1}{p})\log p}{p-1} = -0.0662756342\ldots,$$

with many more decimals.[2]

Given an integer valued polynomial $f(X)$, we let $L(N)$ denote the least common multiple of $f(1),\ldots,f(N)$. In case $f(X) = X$ an equivalent formulation of the Prime Number Theorem states that $\log L(N) \sim N$. It is a natural question to generalize this to other polynomials. For example, in case f is a product of linear terms, we have $\log L(N) \sim c_f N$ for some positive constant c_f; see S. Hong et al. [73]. The main ingredient of the proof is Dirichlet's theorem on primes in arithmetic progressions. J. Cilleruelo [33] considers the case where f is an irreducible polynomial of degree 2 and shows that $\log L(N) \sim N\log N$ and conjectures that if f is an irreducible polynomial of degree $d > 2$, then

$$\log L(N) \sim (d-1)N\log N.$$

Z. Rudnick and S. Zehavi [150] showed that the conjecture is true on average for $f(X) - a$, with a taken in a sufficiently large range. J. Maynard and Z. Rudnick [108] proved that

$$(1/d + o(1))N\log N \leq \log L(N) \leq (d - 1 + o(1))N\log N.$$

A. Sah [164] later established the lower bound with $1/d$ replaced by 1.

In case $f(X) = X^2 + 1$, Cilleruelo established the more precise result

$$\log L(N) = N\log N + JN + o(N). \tag{4.23}$$

It is easy to see that ([113, Prop. 4])

$$-\sum_{p>2} \frac{(\frac{-1}{p})\log p}{p-1} = \frac{L'}{L}(1,\chi_{-4}) + \sum_{p\equiv 3\,(\mathrm{mod}\,4)} \frac{2\log p}{p^2-1}.$$

[2] In the first formula given in Moree [113] for J read $p > 2$ instead of $p > 3$.

This, in combination with the identities (4.18) and (4.20), then yields

$$J = 4\gamma - 1 - (7/2)\log 2 - 4\log G - 2\gamma_S,$$

revealing that from a computational point of view there is almost no difference between J and γ_S, and also 130,000 decimals precision for J can be obtained. It is an open problem to determine whether or not there exists a multiplicative set C such that $\gamma_C = J$. It is believed, see Cilleruelo et al. [34, p. 104], that a result of the form (4.23) also holds for other irreducible quadratic monic polynomials with the analogue of J being non-zero.

5. Ramanujan and the non-divisibility of $\tau(n)$

Let us return to Ramanujan's unpublished, highly influential manuscript on $\tau(n)$ and the partition function $p(n)$, the number of ways of representing a positive integer n as a sum of positive integers, irrespective of their order. Portions of the manuscript were likely written in the years 1917–1919, prior to Ramanujan's return to India in 1919, while other parts might have been written after he returned to India. Most likely, the manuscript was sent to Hardy by Francis Dewsbury, Registrar at the University of Madras, in 1923. This shipment of papers contained several unpublished manuscripts and fragments of Ramanujan, including what was later to be called, *Ramanujan's Lost Notebook* [139]. The manuscript is in two parts. The first is 43 pages long and is in Ramanujan's handwriting; the second is 6 pages long and is in the handwriting of Watson. Unfortunately, the second portion in Ramanujan's handwriting has never been found. (There are several manuscripts of Ramanujan that exist only in Watson's handwriting. Evidently, he copied them for his own use and then, sadly, discarded Ramanujan's original manuscripts.) Supplying details when needed, the first author and Ono [26] made a thorough examination of the manuscript. A revised version of their study appears in [6, Chapter 5]. Congruences for $\tau(n)$ and $p(n)$ are highlights of the manuscript.

In this manuscript, Ramanujan considers, for the primes $q = 3, 5, 7, 23, 691$, appearing in his congruences (3.3), the quantity

$$\sum_{n\le x,\, q\nmid\tau(n)} 1 \tag{5.1}$$

and makes claims similar to Claim 1.1. He defines

$$t_n = \begin{cases} 1, & \text{if } q \nmid \tau(n), \\ 0, & \text{otherwise,} \end{cases}$$

and then typically writes:

Claim 5.1. *It is easy to prove by quite elementary methods that $\sum_{k=1}^n t_k = o(n)$, as $n \to \infty$. It can be shown by transcendental methods that*

$$\sum_{k=1}^n t_k \sim \frac{C_q n}{(\log n)^{\delta_q}}; \tag{5.2}$$

and

$$\sum_{k=1}^n t_k = C_q \int_1^n \frac{dx}{(\log x)^{\delta_q}} + O\Big(\frac{n}{(\log n)^r}\Big), \tag{5.3}$$

where r is any positive number.

Ramanujan used the notation C and δ, but for us, to indicate their dependence on q, it is more convenient to use C_q and δ_q. Certain values of δ_q are given in the final column of Table 3. Note that the truth of $\sum_{k=1}^{n} t_k = o(n)$ would imply that $q|\tau(n)$ for almost all n.

It appears from Stanley's paper [170], which we discussed earlier in Section 1, that Hardy planned to have this manuscript published under Ramanujan's name after some editing. Although he published some parts of it (see [26]), unfortunately, he never published Ramanujan's full manuscript. Proofs of some of Ramanujan's assertions in his unpublished manuscript were further worked out by Stanley [170]. She asserted (5.3) to be false for $q = 5$ and any $r > 1 + \delta_5$ with $\delta_5 = 1/4$ (cf. Table 3). Corrections to her paper have been given by the second author [111, Section 5].

In 1928, Hardy passed on the unpublished manuscript to Watson, who unfortunately kept it hidden away from the mathematical community. (F.J. Dyson colorfully described Watson's penchant for keeping things to himself [45].) Watson wrote approximately 30 papers devoted to Ramanujan's work; see Rankin [143] for an overview.

Note that Ramanujan's Claims 1.1 and 5.1 are very reminiscent of the Prime Number Theorem in the form

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\big(\sqrt{x}\log^2 x\big), \tag{5.4}$$

where we have given an error term that can be established if the Riemann Hypothesis holds true. By partial integration, (5.4) yields an asymptotic series expansion in the sense of Poincaré, with initial term $x/\log x$. The integral turns out to give a much better numerical approximation to $\pi(x)$, than does $x/\log x$. Thus, perhaps in arriving at Claim 5.1 Ramanujan was deceived by a false analogy with the problem of the distribution of the primes. Some evidence for this is provided by his statement that the proof of $\sum_{k=1}^{n} t_k = o(n)$ in the case $q = 5$ is "quite elementary and very similar to that for showing that $\pi(x) = o(x)$" [6, p. 97].

Rankin [146, p. 263] was of the opinion that Hardy must have informed Ramanujan of Landau's method after receiving his first letter and that Ramanujan had a sufficient understanding of the method to be able to make the informed Claim 5.1. About his first impressions, Hardy remarked, "he ... had indeed but the vaguest idea of what a function of a complex variable was [138, p. xxx]." However, Hardy further remarked, "In a few years' time he had a very tolerable knowledge of the theory of functions ... [138, p. xxxi]." Moreover, on the next-to-last page of Ramanujan's third notebook, which was likely written either shortly before he returned to India or shortly after his arrival home, several integral evaluations are recorded. Next to one of them appear the words, "contour integration" [137, pp. 391].

5.1. **The leading term in Claim 5.1.** In this subsection, in greater detail, we consider Claim 5.1 for $q \in \{3, 5, 7, 23, 691\}$ in the unpublished manuscript [26]. Still further details can be found in the article by A. Ciolan et al. [35, Sec. 5]. These claims are recorded in Table 2, and all involve the tau-function. Not listed are those cases where Ramanujan only claimed an estimate of the form $O(n \log^{-\delta} n)$.

The " $+$ " entry indicates a correct claim, the " $-$ " entry indicates a false one, and no entry indicates that no claim was made. The first column concerns the value of δ_q (recorded in Table 3), the second the Euler product (E.P.) of the generating series, the third the value of

q	δ_q	E.P.	C_q	pp.	Sec.
3	$+$	$+$	$+$	22–23	11
5	$+$	$+$		06–08	2
7	$+$	$+$	$+$	11–12	6
23	$+$	$-$	$-$	36–37	17
691	$+$			24–25	12

TABLE 2. Non-divisibility claims of Ramanujan

the constant C_q, and the two remaining columns give the page numbers, respectively, section numbers in [139], where the specific claims can be found.

Using his ideas from [142], Rankin confirmed the correctness of C_3 and C_7, and the signs in the δ_q column [144, p. 10]. However, C_{23} needs a minor correction, as first pointed out by the second author [111], namely, Ramanujan omitted a factor $(1-23^{-s})^{-1}$ in the generating function (17.6). He calculated correctly the asymptotic constant associated to his Euler product (17.6), but it has to be multiplied by $23/22$ in order to obtain the correct value of C_{23}.

The associated Dirichlet series

$$T_q(s) = \sum_{q\nmid\tau(n)} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1,$$

with $q \in \{3, 7, 23\}$, are the easiest in the sense that the most complicated function they involve is $L(s, \chi_{-q})$, where χ_{-q} denotes the nontrivial, quadratic Dirichlet character modulo q. In these three cases, we have $\delta_q = 1/2$. As $\delta_q = q/(q^2-1)$ for $q \notin \{2, 3, 5, 7, 23, 691\}$ (see, e.g., Serre [160, p. 229]), it follows that for the tau-function there are no further primes with $\delta_q = 1/2$. For $\delta_q < 1/2$, the constants C_q become far more difficult to evaluate, and so this may explain why Ramanujan did not venture to write down C_5 and C_{691}.

5.2. **The error term in Claim 5.1.** In order to deal with Claim 5.1 in its sharpest form, we proceed as in our determination of Shanks' constant in Section 4.2. The L-series $T_q(s)^{1/\delta_q}$ has a simple pole at $s = 1$. Employing the product representation for $T_q(s)^{1/\delta_q}$ and taking logarithmic derivatives, we arrive at a closed expression for the associated Euler–Kronecker constant (similar to (4.12)). It involves terms of the form $\frac{L'}{L}(1, \chi)$, which can be evaluated to a high numerical precision with the methods in the papers of Languasco [97] and A. Languasco and L. Righi [100]. This then leads to the following result.

Theorem 5.2 (Moree [111]). *For $q \in \{3, 5, 7, 23, 691\}$, the set $S_{\tau;q} = \{n : q \nmid \tau(n)\}$ has an Euler–Kronecker constant given in Table 3.*

Corollary 5.3. *Ramanujan's claim* (5.3) *is false if $r > 1 + \delta_q$.*

The key to the work described above is that by the congruences described in Section 3.1, the relevant sets $S_{\tau;q}$ are multiplicative abelian, and hence we can apply Theorem 4.7.

The Euler–Kronecker constants in Table 3 have been confirmed and determined with slightly higher accuracy by Ciolan et al. [35]; further computed digits are indicated in parentheses.

set	$\gamma_{S_{\tau;q}}$	winner	δ_q
$3 \nmid \tau(n)$	$0.5349(21)\ldots$	Landau	$1/2$
$5 \nmid \tau(n)$	$0.3995(47)\ldots$	Ramanujan	$1/4$
$7 \nmid \tau(n)$	$0.2316(40)\ldots$	Ramanujan	$1/2$
$23 \nmid \tau(n)$	$0.2166(91)\ldots$	Ramanujan	$1/2$
$691 \nmid \tau(n)$	$0.5717(14)\ldots$	Landau	$1/690$

TABLE 3. Euler–Kronecker constants related to Claim 5.1

6. FOURTH INTERLUDE: SQUARES GALORE!

Let $r_k(n)$ denote the number of representations of n as a sum of k squares, where k is a positive integer. Squares of positive numbers, negative numbers and zero are all allowed, and the ordering of the squares of the numbers that occur in this summation also counts. We have, for example, $r_{24}(2) = 1104$. In Section 2, we considered $r_2(n)$. Focusing, for simplicity, on $r_k(p)$ with p prime, one finds that this is a polynomial in p for several smaller values of k. For example, Jacobi in the early part of the 19th century found that $r_4(p) = 8p + 8$ and $r_8(p) = 16p^3 + 16$. However, for larger k there is very often no polynomial formula for $r_k(p)$. Here, Fourier coefficients of cusp forms come to the rescue. A particular charming example, as it involves both sums of squares and Ramanujan's tau-function, is

$$r_{24}(p) = \frac{16}{691}(p^{11} + 1) + \frac{33152}{691}\tau(p).$$

By Deligne's bound (3.2) for $\tau(p)$, we have $r_{24}(p) = \frac{16}{691}p^{11} + O(p^{\frac{11}{2}})$. For a very readable elementary introduction on the behavior of $r_{24}(p)$, see B. Mazur [109].

The book by E. Grosswald [64] and the monograph by S. Milne [110] are perhaps the two primary sources on $r_k(n)$. A shorter read is Chapter 9 of Hardy [66], which ends with a discussion of $r_{24}(n)$.

7. SOME GENERALIZATIONS OF RAMANUJAN'S CLAIMS

Generalizing Ramanujan's Claims 1.1 and 5.1 in Section 4, we presented the theory of multiplicative sets. Here we consider analogues of Claim 1.1 involving the representations of integers by binary quadratic forms, other than $X^2 + Y^2$, and analogues of Claim 5.1 for other moduli and/or other Fourier coefficients of modular forms. Of the many relevant papers, we will discuss only a modest selection. We start by discussing the relevance of a variant of the original problem.

7.1. **Easy numerical approximation of $B(x)$.** Shanks [165] asserted that "An unsolved problem of interest is to find an approximation to $B(x)$ that could be computed without undue difficulty by a convergent process, and which would be accurate to $O(x \log^{-m} x)$ for all m." Perhaps he would have regarded the following result as giving an adequate answer (with $L_S(s)$ as in (4.10)).

Theorem 7.1. *Let* $0 < \epsilon < 1/2$. *There exists a constant* $c > 0$ *such that*

$$B(x) = \frac{1}{\pi}\int_{1/2+\epsilon}^{1} \sqrt{L_S(\sigma)}\frac{x^\sigma}{\sigma}\,d\sigma + O(x\,e^{-c\sqrt{\log x}}).$$

If we assume RH for both $\zeta(s)$ *and* $L(s,\chi_{-4})$, *then we can replace the error term by* $O(x^{\frac{1}{2}+\epsilon})$.

This result can be regarded as a corrected version of Claim 1.1! It can be used in practice to numerically approximate $B(x)$. We refer to O. Gorodetsky and B. Rodgers [61, Appendix B] or C. David et al. [40] for a proof. The authors note that this result (known to experts before) seems to go back to a 1976 paper of K. Ramachandra [135].

7.2. **A variant of Claim 1.1.** Let $d \geq 1$ be a squarefree integer. By the Hasse principle the *negative Pell equation* $X^2 - dY^2 = -1$ has a solution with X and Y rational if and only if d has no prime factor $p \equiv 3 \pmod 4$. A minor variation of Landau's proof of (1.2) gives that the number of such $d \leq x$ is asymptotically $\sim cx\log^{-1/2} x$ for some $c > 0$. P. Stevenhagen [171] conjectured that a similar asymptotic holds if we ask for *integer* solutions X and Y and indicated an explicit value of c. Recently P. Koymans and C. Pagano [90] established this conjecture, after earlier deep work by É. Fouvry and J. Klüners [54] placing c in some interval.

7.3. **Generalizations of Claim 1.1.** Let $s_1, s_2, \ldots$ be a sequence of integers that can be written as sums of two squares. Generalizations mostly concern other related sequences of integers in the following Sections 7.3.1 and 7.3.2, or they focus on $s_1, s_2, \ldots$, but ask for more refined distributional properties (Section 7.3.3). Indeed, the sequence $s_1, s_2, \ldots$ is among those most intensively studied.

Several of the generalizations discussed here have also been considered in the *function field* setting; see, for example, the paper of Gorodetsky [59].

7.3.1. *Generalizations to other (binary) quadratic forms.* C. Stewart and Y. Xiao [173] showed that if F is a binary form of degree $d \geq 3$ with integer coefficients and non-zero discriminant, then

$$\{n : |n| \leq x,\ F(X,Y) = n \text{ for some } X, Y \in \mathbb{Z}\} = C_F x^{2/d} + O_{F,\epsilon}(x^{\beta_F+\epsilon}),$$

where C_F and the rational number $\beta_F < 2/d$ can be obtained explicitly. It remains to discuss what happens for arbitrary positive definite binary quadratic forms. Let us first discuss the easier problem in which integers can be represented by *some* primitive binary form of a prescribed discriminant D. A less difficult variant of this arises when one restricts oneself to counting integers that are co-prime only to D. In this direction, R.D. James [83] showed, by Landau's method, that the number $B_D(x)$ of positive integers $m \leq x$, co-prime to D, which are represented by some primitive binary form of discriminant $D \leq -3$, satisfies

$$B_D(x) = b_D\frac{x}{\sqrt{\log x}}\Big(1 + O\Big(\frac{1}{\sqrt{\log x}}\Big)\Big), \quad x \to \infty, \tag{7.1}$$

where b_D is some explicit positive value. The algebraic fact on which the proof relies is that an integer m that is co-prime to D is represented by a primitive binary quadratic form of discriminant D, if and only if the primes p that occur with odd exponent in m satisfy $(D/p) = 1$, where (D/p) denotes the Kronecker symbol. In particular, (D/p) equals the

Legendre symbol if p is an odd prime. G. Pall [129] showed that (7.1) remains valid, but with a different explicit constant b_D, if the restriction that m is co-prime to D is dropped. The second coefficient in the Poincaré series of Pall's theorem was obtained by W. Heupel [72], who also obtained the second coefficient in the Poincaré series of $B_D(x)$. K.S. Williams [183] gave an elementary proof of (7.1) with an error term $O\big((\log\log x)^{-1}\big)$.

Now we return to the problem of representations by a given binary quadratic form. In his 1912 Ph.D. thesis, P. Bernays [17] proved that the counting functions for the integers represented by a reduced binary form of discriminant D are asymptotically identical; specifically, he showed that they behave asymptotically as

$$C(D)\frac{x}{\sqrt{\log x}},$$

where $C(D)$ is a positive constant depending only on D. Note that $C(-4)$ is the Landau–Ramanujan constant. Odoni [123] presented his own proof of Bernays' result in the more precise form

$$C(D)\frac{x}{\sqrt{\log x}}\Big(1+O\Big(\frac{1}{\log^{e(f)}x}\Big)\Big), \quad e(f)>0, \tag{7.2}$$

but it seems that his method does not permit the calculation of the constant $C(D)$ and gives no information about $e(f)$. Fomenko [52] gave a nice proof of (7.2), which enables one to calculate $C(D)$ and to estimate $e(f)$.

Bernays left number theory for logic (in which he was to become famous) and did not publish his thesis. Thus, unfortunately many later researchers were unaware of it, or did not have access to it. Had they been, this would have profoundly altered what subsequently happened, as Bernays' methods are more powerful than those employed by many later authors. As it is, Bernays' result was only improved and generalized after roughly 60 years by Odoni, who wrote a long series of papers generalizing this type of result, eventually resulting in his theory of *Frobenian multiplicative functions*; see [124] and [125] for a short, respectively, longer survey.

7.3.2. *Generalizations with a computational aspect.* P. Shiu [167] adapted the Meissel–Lehmer method, initially developed for calculating $\pi(x)$, to $B(x)$. This allows one to compute $B(x)$ more efficiently, but at the expense of explicitly knowing which integers $\le x$ can be written as a sum of two squares.

In 1966, focusing on the family of forms X^2+kY^2, Shanks and L.P. Schmid [166] considered the problem of determining $C(-4k)$ in which negative values of k were also considered. They expected that $C(-8)$ would be the largest such constant. However, D. Brink et al. [31] showed that $C(-4k)$ is unbounded as k runs through both the positive and negative integers.

Let $F_3(x)$ denote the number of integers $n \le x$ represented by X^2+XY+Y^2. Since $C(-4) > C(-3)$, we have $B(x) \ge F_3(x)$ for all x sufficiently large. In connection with his work on lattices, P. Schmutz Schaller [153] conjectured that actually $B(x) \ge F_3(x)$ for *every* $x \ge 1$. This conjecture was proven by the second author and H.J.J. te Riele [117]. É. Fouvry et al. [55] determined an asymptotic for the integers $n \le x$ represented by *both* X^2+Y^2 and X^2+XY+Y^2. Languasco and the first author [99, Sec. 7] noticed that this, in essence, was already done by Serre [161, pp. 185–187] who counted asymptotically the number of non-zero Fourier coefficients a_n of $q\prod_{n\ge1}(1-q^{12n})^2$. Namely, we have $a_n \neq 0$ if and only if $n \equiv 1 (\mathrm{mod}\ 12)$ and n is representable by both X^2+Y^2 and X^2+XY+Y^2.

Note that $B(x)$ and $F_3(x)$ are related to the quadratic, respectively, hexagonal lattices. A quantity associated with an n-dimensional lattice L is its *Erdős number*[3] and is given by $E_L = F_L\, d^{1/n}$, where d is the determinant of the lattice and F_L its *population fraction*, which is given by

$$F_L = \lim_{x\to\infty} \frac{N_L(x)\sqrt{\log x}}{x} \quad \text{if } n=2, \quad F_L = \lim_{x\to\infty} \frac{N_L(x)}{x} \quad \text{if } n \ge 3,$$

where $N_L(x)$ is the *population function* associated with the corresponding quadratic form, i.e., the number of values not exceeding x taken by the form. The *Erdős number* is the population fraction when the lattice is normalized to have covolume 1. In the case of a two dimensional lattice, it is related to the constant b_D appearing in (7.1) (see e.g., [38]). In 2006, the second author and R. Osburn [116] showed, relying on the work of many others, that the Erdős number is minimal for the hexagonal lattice. Also, relying on the work of many others, J.H. Conway and N.J.A. Sloane [38] solved the analogous problem for dimensions 3 to 8 in 1991.

7.3.3. *Other distributional aspects of sums of two squares.* A pair $(n, n+1)$ is said to be B*-twin* if both n and $n+1$ can be written as a sum of two squares. In 1965, G.J. Rieger [147] proved with the one-dimensional sieve the upper bound $\ll x/\log x$ for the number of such pairs with $n \le x$. The analogous lower bound was independently proved in 1974 by C. Hooley [76] using the asymptotic estimate

$$\sum_{n\le x} r(n)r(n+1) = 8x + O(x^{5/6+\epsilon})$$

due to T. Estermann [48], and by K.-H. Indlekofer [80] with the sieve method. It is a very challenging problem to prove an asymptotic estimate for $r(n)r(n+h)r(n+k)$ with $n \le x$ and $h \ne k$ fixed positive integers. Hooley [75] made some progress by showing that this function, for fixed h and k, is infinitely often positive (thus answering a question of J.E. Littlewood).

Given any two positive integers k and l, we let $B(x,k,l)$ denote the number of integers $s_i \le x$ with $s_i \equiv l\,(\mathrm{mod}\ k)$. It is sufficient to assume that k and l are co-prime. We will assume that $l \equiv 1\,(\mathrm{mod}\ 4)$ in the case $4 \mid k$, for otherwise $B(x,k,l) = 0$ for every x. K. Prachar [132] showed that

$$B(x,k,l) \sim B_k \frac{x}{\sqrt{\log x}}, \quad x \to \infty, \tag{7.3}$$

where B_k is a positive constant depending only on k. His Ph.D. student, H. Bekić, [14] proved that this asymptotic holds uniformly in the range k up to $e^{c\sqrt{\log x}}$, where c is a positive constant, a result subsequently extended by Prachar [133] to $e^{(\log x)^{2/3-\epsilon}}$. H. Iwaniec [81], using the half dimensional sieve, strengthened this to

$$B(x,k,l) = B_k \frac{x}{\sqrt{\log x}}\Big(1 + O\Big(\Big(\frac{\log k}{\log x}\Big)^{1/5}\Big), \quad x \to \infty,$$

where the implicit constant is absolute.

[3]Not to be mixed up with the celebrated Erdős collaborative distance!

Gorodetsky [60] showed that $B(n,k,l_1) > B(n,k,l_2)$, with l_1, l_2 a quadratic, respectively, non-quadratic residue modulo k, for a density-1 set of integers n, provided some standard number theoretic conjectures including GRH hold true.

J. Schlitt [152] generalized (7.3) to other positive definite binary quadratic forms. However, although asymptotically there is equidistribution, at the level of the second order constant, this is not always the case. For example, if $k \equiv 1 \,(\mathrm{mod}\, 4)$ is a prime, then there is a preponderance for residue classes $0 \,(\mathrm{mod}\, k)$ over the others. Numerically, this effect is clearly detectable.

Let $r_p(n)$ be the number of representations of n as a sum of two prime squares. It is known that asymptotically $\sum_{n\le x} r_p(n)^j \sim 2^{j-1}\pi x \log^{-2} x$ for $j = 1, 2, 3$, where the case $j \le 2$ is fairly standard and $j = 3$ is due to V. Blomer and J. Brüdern [29]. Recently, partial results for $j \ge 4$ have been obtained by C. Sabuncu [151].

P. Lévy [105] gave a simple heuristic derivation of (1.2), however without determing K. His argument also led him to conjecture that $R_k(x)$, the number of $n \le x$ for which $r_2(n) = k$, asymptotically satisfies

$$R_k(x) \sim \frac{Kx}{\sqrt{\log x}} \frac{e^{-\theta}\,\theta^k}{k!}, \qquad \text{where } \theta = c\sqrt{\log x}.$$

In probabilistic terms, this means roughly that the integers m for which $r_2(m)$ has a specified value, have a Poisson distribution with parameter θ. W.J. LeVeque [104] showed that the conjecture is wrong and that in fact the asymptotic behavior of $R_k(x)$ not only depends on the size of k, but also its arithmetic structure.

W.D. Banks et al. [13] showed that every integer $n > 720$ that can be written as a sum of two squares satisfies Robin's inequality

$$\sum_{d|n} \frac{1}{d} < e^{\gamma} \log\log n. \tag{7.4}$$

G. Robin [148] famously proved that the Riemann Hypothesis is true if and only if (7.4) holds for all $n > 5040$. We leave it as a (not so difficult) challenge for the reader to show that (7.4) is satisfied for all *odd* integers, and thus, to wit, prove half of the Riemann Hypothesis!

A. Balog and T.D. Wooley [9] found "unexpected irregularities" in the distribution of the squares s_i in short intervals. To be precise, they showed that there are infinitely many short intervals containing considerably more integers s_i than expected, and infinitely many intervals containing considerably fewer than expected. Hooley [74, 75, 76, 77] wrote four papers on the distribution of the gaps $s_{i+1} - s_i$.

C. David et al. [40] studied how frequently (s_i, s_{i+1}) belongs to a prescribed congruence class pair modulo q. Certain congruence pairs appear more frequently than others, which they heuristically explain. A similar phenomenon was found earlier by R.J. Lemke Oliver and K. Soundararajan for consecutive prime numbers [103] and caused quite a sensation. N. Kimmel and V. Kuperberg [86] considered the situation where given a fixed integer $m \ge 0$, one requires $s_i, s_{i+1}, \ldots, s_{i+m}$ to be in prescribed progressions (which are allowed to be different).

The energy levels of generic, integrable systems are conjectured to be Poisson distributed in the semiclassical limit. The square billiard, although completely integrable, is non-generic in this respect. Its levels, when suitably scaled, are the numbers s_i. This led to some interest in theoretical mathematical physics in the distribution of the numbers s_i [36, 37, 56]. For example, T. Freiberg et al. [56] formulate an analog of the Hardy–Littlewood prime k-tuple conjecture for sums of two squares, and show that it implies that the spectral gaps, after removing degeneracies and rescaling, are Poisson distributed.

A positive integer n is called *square-full* if p^2 divides n, whenever p is a prime divisor of n. It was shown in 1935 by P. Erdős and G. Szekeres [47] that the number of square-full integers $\leq x$ is equal to $\zeta(3/2)\sqrt{x}/\zeta(3) + O(x^{1/3})$. Thus the square-full numbers are not much more abundant than perfect squares. Thus one might conjecture, as did Erdős, that $V(x)$ the number of integers $\leq x$ that can be written as a sum of two square-full integers grows asymptotically like $cx/\sqrt{\log x}$. However, V. Blomer [28] showed that asymptotically

$$V(x) = x \log^{-\alpha+o(1)} x \ \text{ with } \ \alpha = 1 - 2^{-1/3} = 0.2062994740\ldots.$$

It is a famous result of C.-F. Gauss that n can be written as a sum of three integer squares if and only if $n \neq 4^a(8b+7)$. D.R. Heath-Brown [70] showed that there is an effectively computable constant n_0 such that every $N \geq n_0$ is a sum of at most three square-full integers.

7.4. **Generalizations of Claim 5.1.** In this section we assume some familiarity with the theory of modular forms, the reader can consult, for example, the books by F. Diamond and J. Shurman [44], N. Koblitz [87] or the relatively short classic book by Serre [159]. More advanced is the book by Ono [128], which has some focus on Fourier coefficient congruences. Ram Murty [119] gives a nice warm-up for what is going on in this section, where both the algebraic and analytic side are kept at a very accessible level.

7.4.1. *Non-divisibility of sums of divisors functions.* For the primes $q = 2, 3, 5, 7, 691$, the Ramanujan congruences (3.3) relate the non-divisibility of certain Fourier coefficients to those of $n^a\sigma_k(n)$ for an appropriate a. This suggests considering, for an *arbitrary* integer $k \geq 1$ and a prime q,

$$S_{k,q}(x) := S_{\sigma_k;q}(x) = \sum_{\substack{n \leq x, \\ q \nmid \sigma_k(n)}} 1. \tag{7.5}$$

By a minor variation, the general case $q \nmid n^a\sigma_k(n)$ can be handled. It is therefore natural that Ramanujan was interested in the asymptotic behavior of $S_{k,q}(x)$, and he seems to have been the first to do so. He discusses this function in his unpublished manuscript [26, Sec. 19], and made three claims (also reproduced by Rankin [144]), which were proved in 1935 by Watson [181]. One of these claims asserted that, for odd k,

$$S_{k,q}(x) = O(x \log^{-1/(q-1)} x), \tag{7.6}$$

and it is discussed by Hardy in his Harvard lectures on Ramanujan's mathematics [66, §10.6]. Since $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$, this claim implies Watson's estimate (**??**). However, as $S_{\sigma_k;q}$ is an abelian multiplicative set, Landau's results from 1909 (see Remark 4.8) already imply Watson's estimate.

The precise asymptotic behavior of $S_{k,q}(x)$ was first determined by Rankin [142]. His Ph.D. student, Eira Scourfield, [154] generalized his work by establishing an asymptotic formula in the case where a prime power exactly divides $\sigma_k(n)$.

Put $h = \frac{q-1}{(q-1,k)}$. Ciolan et al. [35] went beyond determining an asymptotic formula, and showed that when h is even and $x \to \infty$,

$$S_{k,q}(x) = \frac{c_0\, x}{\log^{1/h} x}\Big(1 + \frac{1-\gamma_{k,q}}{h \log x} + \frac{c_2}{\log^2 x} + \cdots + \frac{c_j}{\log^j x} + O_{j,k,q}\Big(\frac{1}{\log^{j+1} x}\Big)\Big), \tag{7.7}$$

where $\gamma_{k,q}$ is the Euler–Kronecker constant for $\sigma_k(n)$, and can be explicitly given. The case when h is odd is rather trivial, and there we have $S_{k,q}(x) \sim c_{k,q}x$, where $c_{k,q}$ is a positive constant [35, Sect. 3.8].

7.4.2. *Analogues of Claim 5.1 for cusp forms of higher weight for the full modular group.* The space of cusp forms of weight k of the full modular group is one-dimensional if and only if $k \in \{12, 16, 18, 20, 22, 26\}$. For these weights the 'elementary' congruences of prime modulus were completely classified using ℓ-adic representations by the efforts of Serre and Swinnerton-Dyer, cf. [176]. Ciolan et al. [35], using the results on the non-divisibility of the the sum of divisor functions alluded to in §7.4.1, determined the associated leading constant and Euler-Kronecker constant (cf. (4.5)) for all of these, with one exception. The exception is the congruence (rather, congruential restriction)

$$a_p(\Delta E_4)^2 \equiv 0, p^{15}, 2p^{15}, 4p^{15} \pmod{59} \quad (p \neq 59), \tag{7.8}$$

where ΔE_4 is the unique normalized cusp of weight 16 for the full modular group, which was recently dealt with by S. Charlton et al. [32]. In this case the associated generating series turns out to be expressible in terms of Dedekind zeta functions of some non-abelian number fields, and one cannot just do with Dirichlet L-series, as in the other cases[4]. In order for the associated Euler-Kronecker constant of $S = \{n : 59 \nmid a_n(\Delta E_4)\}$ to be obtained with moderate precision, the authors had to assume the Riemann Hypothesis for the fields involved and used a method of Y. Ihara [79].

7.4.3. *Parity of Fourier coefficients and the partition function.* Let us consider the parity of $\tau(n)$ first. Trivially $(1-q^n)^8 \equiv 1 - q^{8n} \pmod 2$. By the Jacobi Triple Product Identity (see [5, Thm. 2.8]),

$$\eta(8z)^3 = q\prod_{j=1}^{\infty}(1-q^{8j})^3 = \sum_{k\ge 0}(-1)^k(2k+1)q^{(2k+1)^2},$$

which one can then use to deduce that

$$\Delta(z) \equiv \eta(8z)^3 \equiv \sum_{k\ge 0} q^{(2k+1)^2} \pmod 2. \tag{7.9}$$

It follows that $\tau(n)$ is odd if and only if n is an odd square. We will now see that this behavior is quite exceptional among modular forms f of level one with integer Fourier coefficients a_n. J.-Bellaîche and J.-L. Nicolas [15], improving on an earlier result of Serre [160, §6.6], prove that for such f one has the asymptotic estimate

$$\#\{n \le x : 2 \nmid a_n\} = C_f \frac{x}{\log x}(\log\log x)^{g(f)-2}\Big(1 + O\Big(\frac{1}{\log\log x}\Big)\Big),$$

[4] Recall that the Dedekind zeta function of an abelian number field factorizes in Dirichlet L-series; for a non-abelian number field Artin L-series arise as factors.

if $g(f) \geq 2$. Here $g(f)$ is the order of nilpotency of a certain Hecke algebra. They show that $g(f) = 1$ only if $f = \Delta$, which by (7.9) leads to $\#\{n \leq x : 2 \nmid \tau(n)\} = \sqrt{x}/2 + O(1)$. They first establish their asymptotic estimate for $f = \Delta^k$ with k arbitrary and odd. They then try to write a general f (taken modulo two) in a favorable way as linear combinations of powers of Δ, and in this way obtain the general result. They use here the simple result that the graded algebra of modular forms modulo 2 on $\mathrm{SL}_2(\mathbb{Z})$ is $\mathbb{F}_2[\Delta]$ (see Swinnerton-Dyer [175, Thm. 3]). For some further remarks see, for example, Ono [128, §2.7].

Although it is a bit tangential, we will briefly discuss the parity of the partition function $p(n)$, since it is attracting a lot of research interest, and much remains to be done. The parity of $p(n)$ seems to be quite random, and it is widely believed that the partition function is "equally often" even and odd. More precisely, T.R. Parkin and Shanks [130] made the conjecture that

$$\#\{n \leq x : 2 \nmid p(n)\} \sim \frac{x}{2}.$$

This is far from being proved: at the moment of writing it cannot be excluded that there exists an $\epsilon > 0$ such that the latter counting function is $O(x^{1/2+\epsilon})$ for some $\epsilon > 0$. Bellaîche and Nicolas [15], using their approach involving powers of Δ, showed that, for $x \geq 2$,

$$\#\{n \leq x : 2 \mid p(n)\} \geq 0.069\sqrt{x}\log\log x, \quad \#\{n \leq x : 2 \nmid p(n)\} \geq \frac{0.048\sqrt{x}}{\log^{7/8} x}.$$

S. Radu [134] proved that every arithmetic progression $r \pmod{t}$ contains infinitely many integers N for which $p(N)$ is even, and infinitely for which it is odd (this was a conjecture of M. Subbarao [174]). It is an open problem to determine an upper bound for the smallest such N.

7.4.4. *Non-divisibility of integer value multiplicative functions.* Analogues of Claim 5.1 for $S = \{n : q \nmid f(n)\}$ can be made, where q is any prime and f is any multiplicative function. In Example 4.5 we already discussed the case where f is the Euler totient function and q is any prime. Scourfield [156] considered the case where $f(n) = r_k(n)$, with $r_k(n)$ defined as in Section 6 and the more general case in [155]. In case $f(p)$ is a polynomial the relevant generating series factorizes in terms of Dirichlet L-series. A more interesting class is that of the Frobenian multiplicative functions, where f is Frobenian (relative to an extension K of the rationals). This entails that for all primes p and q coprime to some prescribed number that have the same Frobenius symbol we have $f(p^n) = f(q^n)$ for every $n \geq 1$. Now the relevant generating series will factorize as a product of Artin L-series. The density of primes with a prescribed Frobenius symbol is given by the Chebotarev density theorem, see P. Stevenhagen and H.W. Lenstra [172] for an introduction.

7.4.5. *Lacunarity.* If f is any multiplicative function, then clearly the set $S = \{n : a_n \neq 0\}$ is multiplicative. The set S is lacunary if it has natural density zero. Serre [162] classified all lacunary even powers $\eta(z)^r$ of the eta-function with $r > 0$. He showed that $\eta(mz)^r$ (with m chosen such that only integer powers of q appear in the Fourier series) are lacunary if and only if $r \in \{2, 4, 6, 8, 10, 14, 26\}$. In the middle of Section 7.3.2 we mentioned $\eta(12z)^2$ (so the case $r = 2$). The equivalence given there immediately shows it is lacunary. Serre also writes down asymptotics for the associated sets S of indices of non-zero Fourier coefficients, except for $r = 26$, where determining the asymptotics is an open problem to this day. In that case Serre showed that $c_1 \frac{x}{\sqrt{\log x}} \leq S(x) \leq c_2 \frac{x}{\sqrt{\log x}}$ for every $x \geq 2$ and some constants $0 < c_1 < c_2$.

More generally, Serre [161] proved that every holomorphic weight one cusp form for $\Gamma_1(N)$ is lacunary, whereas holomorphic modular forms of higher positive integral weight are lacunary if and only if they are linear combinations of CM-cusp forms.

7.5. **Averaging multiplicative functions.** In Theorem 4.1 we considered the counting function $S(x) = \sum_{n \leq x} i_S(n)$, where i_S assumes non-negative values only and is multiplicative. Likewise, we can ask if a similar result can be obtained if we replace i_S by any non-negative real valued multiplicative function.

Most authors who have been working in this direction, e.g., B.M. Bredikhin, H. Delange, A.S. Fainleib, H. Halberstam, B.V. Levin, R.W.K. Odoni, J.M. Song and E. Wirsing, have concentrated on finding conditions on $\sum_{p \leq x} f(p)$ or $\sum_{p \leq x} f(p)/p$ that are as weak as possible, so that they could prove an asymptotic formula for $\sum_{n \leq x} f(n)$. In particular, see the book by A.G. Postnikov [131]. The next result is a famous example of this, and is due to Wirsing [185].

Theorem 7.2. *Let $f(n)$ be a multiplicative function such that $f(n) \geq 0$, for $n \geq 1$. Suppose that there exist constants γ_1 and γ_2, with $\gamma_2 < 2$, such that for every prime p and every $\nu \geq 2$, $f(p^\nu) \leq \gamma_1 \gamma_2^\nu$. Assume that, as $x \to \infty$,*

$$\sum_{p \leq x} f(p) \sim \delta \frac{x}{\log x}, \tag{7.10}$$

where δ is a positive constant. Then, as $x \to \infty$,

$$\sum_{n \leq x} f(n) \sim \frac{e^{-\gamma \delta}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \leq x} \Big(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \frac{f(p^3)}{p^3} + \cdots \Big).$$

Using a theorem of F. Mertens, see for example [67, p. 466, Theorem 429],

$$\prod_{p \leq x} (1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log x}, \quad x \to \infty,$$

we see that this result implies Theorem 4.1.

In a follow-up paper Wirsing [186] proved variant of Theorem 7.2, where the PNT type condition (7.10) is replaced by the weaker Mertens type condition

$$\sum_{p \leq x} \frac{\log p}{p} f(p) \sim \delta \log x.$$

In case $f(n) \in \{0, 1\}$ for every n, this variant is actually stronger than Theorem 7.2. Another variant was established by Song [169] (Theorem A of part I). She also extended this type of result to the counting of integers having largest prime factor $\leq y$ (so-called *y-friable integers*). See also a series of a papers by G. Tenenbaum and J. Wu, culminating in [178].

In many situations one can prove a much more precise estimate for $\sum_{p \leq x} f(p)$, and this can be used to obtain a much stronger result than that given in the conclusion of Wirsing's theorem. There are several methods in that direction. One can be regarded as a considerable extension of the method of contour integration that Landau used in his paper [92] in

1908, and is commonly called the Selberg–Delange method. It applies, in principle, to any nonnegative multiplicative function, but requires the analytic continuation of the associated Dirichlet series, a condition that can be difficult to confirm in practice. In view of our story up to this point, the reader might be inclined to think that Landau–Selberg–Delange (LSD) is a more appropriate name, and is indeed not alone in this; see the article by A. Granville and D. Koukoulopoulos [62]. For introductory material to the LSD method we refer to the books by Koukoulopoulos [88] and G. Tenenbaum [177]. For a brief exposition, see the article by Finch et al. [51].

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