FINAL RESEARCH REPORT IN MPIM

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My name is Xinchen Miao, a one-year postdoc in Max Planck Institute for Mathematics. The date of my stay is from September 1st 2023 to August 31st 2024. My mathematical research is in the general area of number theory, where the MathSciNet classification number is **11(F)**. More specifically, I am interested in both the local and global theory of automorphic forms and relative trace formulae during my stay in MPIM.

In the local aspect, I mainly focus on the study of Bessel functions over non-archimedean local fields and the local Kloosterman-type (orbital) integrals.

For the global theory, I am interested in studying the spectral reciprocity formulae for certain L-functions under the period integral and representation theoretic view. Moreover, I am also interested in the applications of spectral reciprocity formulae, for example, the estimations of subconvexity and moments for automorphic L-functions.

1. Global Theory: Spectral Reciprocity Formulae

1.1. **History and Background.** The most famous reciprocity law in the number theory is the quadratic reciprocity law first proved by F. Gauss. Let q and ℓ be two distinct odd prime numbers. The celebrated law of quadratic reciprocity expresses the quadratic Legendre symbol $\left(\frac{\ell}{q}\right)$ in terms of $\left(\frac{q}{\ell}\right)$.

Similar phenomenon will also appear in the theory of automorphic forms, which is known as the (automorphic) spectral reciprocity formula. As far as I know (from V. Blomer's Singapore lecture), motivated by the work of Motohashi [Mot93], Michel-Venkatesh [MV10] and many other related works in lower ranks (degree 2 or 4 L-functions), V. Blomer suggested to investigate spectral identities of the following shape:

$$\sum_{\pi \in \mathcal{F}} \mathcal{L}(\pi)\mathcal{H}(\pi) = \sum_{\pi \in \widetilde{\mathcal{F}}} \widetilde{\mathcal{L}}(\pi)\widetilde{\mathcal{H}}(\pi),$$

where $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are different families of automorphic representations, $\mathcal{L}(\pi)$ and $\widetilde{\mathcal{L}}(\pi)$ are certain (may be different) (products of) L-functions corresponding to automorphic representations. Moreover, $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are some (global) weight functions related to automorphic representation π. The map from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$ is given by an explicit integral transform.

Beyond the simplicity and beauty of spectral reciprocity formulae, these formulae are expected to have powerful applications to non-vanishing and subconvexity problems for the associated L-functions, which are very important in analytic number theory. This motivates an intension to understand the intrinsical symmetry behind such kind of identities. It is highly desirable to ask whether there exists a master formula of representation-theoretic nature to control all these reciprocity formulae related to L-functions.

1.2. **My work on Spectral Reciprocity I.** In [BK18], a non-symmetrical spectral reciprocity formula for degree 8 L-functions was established to attack GL(2) uniform subconvexity problem. The base field is $\mathbb{Q}$. Let F be an automorphic form (cusp form or Eisenstein series) for the group $\mathrm{SL}_3(\mathbb{Z})$ and f be a cusp form for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (f is ramifed at some finite places). A

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spectral reciprocity formula of the following shape was proved:

$$\sum_{\substack{f \text{ of level } q,\\ \text{unitary central character } \chi}} L(s, F\times f)L(w,\tilde{f})\lambda_f(\ell) \rightsquigarrow \sum_{\substack{f \text{ of level } \ell\\ \text{trivial central character}}} L(s',\tilde{F}\times f)L(w', f\times\bar{\chi}),$$

where q and ℓ are coprime integers and λ_f is the corresponding Hecke eigenvalue for the cusp form f and

$$s' = \tfrac{1}{2}(s+w), \quad w' = \tfrac{1}{2}(2+w-3s). \tag{1.1}$$

Here $\widetilde{F}$ and $\widetilde{f}$ are the contragredient automorphic forms of F and f. Furthermore, if F is a minimal Eisenstein series, we will also see that

$$\sum_{\substack{f \text{ of level } q,\\ \text{unitary central character } \chi}} |L(1/2,f)|^4\lambda_f(\ell) \rightsquigarrow \sum_{\substack{f \text{ of level } \ell\\ \text{trivial central character}}} (L(1/2,f))^3 L(1/2, f\times\bar{\chi}).$$

In my preprint [Miao24d], we extend the method found by R. Nunes [Nun20] to this case, which is a non-symmetrical spectral reciprocity formula for the product of Rankin-Selberg L-functions of degree 8. We partially extend the result in [BK18] to the general number field. For the general number field, the spectral reciprocity formula expresses the following reciprocity relation:

$$\begin{aligned} &\sum_{\substack{\pi \text{ of level } \mathfrak{q},\\ \text{unitary central character } \chi}} L(s,\Pi\times\pi)L(w,\widetilde{\pi})H(\pi) \\ \rightsquigarrow &\sum_{\substack{\pi \text{ of level } \mathfrak{l}\\ \text{trivial central character}}} L(s',\widetilde{\Pi}\times\pi)L(w',\pi\times\bar{\chi})\widetilde{H}(\pi), \end{aligned} \tag{1.2}$$

with different unramified prime ideal $\mathfrak{q}$ and $\mathfrak{l}$, where $\widetilde{\pi}$ is the contragredient representation of π (π is the GL(2) automorphic form and χ is the Hecke character) and $\widetilde{\Pi}$ is the contragredient representation of Π. Here we further assume that Π is a cuspidal automorphic form on GL(3) which are unramified everywhere and have trivial central character. Moreover, we have

$$s' = \tfrac{1}{2}(s+w), \quad w' = \tfrac{1}{2}(2+w-3s). \tag{1.3}$$

The reciprocity of two coprime ideal $\mathfrak{q}$ and $\mathfrak{l}$ are given by the action of the combination of two Weyl elements $\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$ and $\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$ in GL(3).

1.3. **My work on spectral reciprocity II.** We note that the spectral reciprocity in above Section 1.2 connects different families of certain degree 8 L-functions. More precisely, it is a decomposition of $8 = 4\times 2 = (3+1)\times 2$. For another decomposition $8 = 4\times 2 = (2+2)\times 2$, we can also establish another type of spectral reciprocity formula for the second moment of $\mathrm{GL}(2)\times\mathrm{GL}(2)$ Rankin-Selberg L-functions (degree 8). After more work, we may study the fourth moment of standard GL(2) L-functions for different families in level aspect. A very rough idea is as follows: For unit vectors $v_1\in\pi_1$ and $v_2\in\pi_2$, we consider the following trivial inner product identity:

$$\langle v_1v_2, v_1v_2\rangle = \int_{\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A}_F)} |v_1v_2|^2(g)dg = \langle |v_1|^2, |v_2|^2\rangle.$$

By expanding each of these inner products over the L^2-spectrum of $L^2(\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A}_F))$ and applying the relation between the period integral and the central value of triple product L-functions, we obtain a spectral reciprocity formula of families of certain L-functions, roughly of the shape

$$\sum_\pi h(\pi)L(1/2,\pi\otimes\pi_1\otimes\pi_2) \approx 1 + \sum_\sigma \tilde{h}(\sigma)\sqrt{L(1/2,\pi_1\otimes\pi_1\otimes\sigma)L(1/2,\pi_2\otimes\pi_2\otimes\sigma)}.$$

Here π and σ run over cuspidal automorphic representations of PGL_2. This approach can be found in [MV10]. In [Zac20], the author followed this idea to study the twisted first moment of triple product

L-functions (degree 8) with the Hecke eigenvalue λ_π included. In my work [Miao24b] and [Miao24c], we will generalize the results in [Zac20] and obtain the explicit strong (hybrid) subconvex bound for the degree 8 triple product L-function in the level and depth aspect. These two papers are almost finished and will be uploaded soon. We use the estimations and results in [Hu18] [Hu20] and roughly have the following theorems.

Theorem 1.1. [Miao24b] *Let F be a number field with ring of integers $\mathcal{O}_F$. Let $\mathfrak{q}$ be an (not necessary squarefree) ideal of $\mathcal{O}_F$ of norm q and π a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_F)$ with finite conductor $\mathfrak{q}$. Let π_1, π_2 be fixed unitary tempered cuspidal automorphic representations with finite conductor $\mathfrak{m}$ and $\mathfrak{n}$. Assume that for all archimedean non-archimedean place $v|\infty\mathfrak{m}\mathfrak{n}$, either $\pi_{1,v}$ or $\pi_{2,v}$ is a principal series representation. If $(\mathfrak{q}, \mathfrak{m}\mathfrak{n}) = 1$, then for any $\epsilon > 0$, we have the following subconvex estimate*

$$L\left(\tfrac{1}{2}, \pi \otimes \pi_1 \otimes \pi_2\right) \ll_{\varepsilon, F, \pi_1, \pi_2, \pi_\infty} q^{1-\frac{1-2\theta}{6}+\epsilon}. \tag{1.4}$$

Theorem 1.2. [Miao24c] *Let F be a number field with ring of integers $\mathcal{O}_F$. Let $\mathfrak{q}$ be an (not necessary squarefree) ideal of $\mathcal{O}_F$ of norm q and π a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_F)$ with finite conductor $\mathfrak{q}$. Let π_1, π_2 be unitary tempered cuspidal automorphic representations with finite squarefull coprime conductor $\mathfrak{m}$ and $\mathfrak{n}$ (squarefull ideals are integral ideals for which all the prime ideal factors exponents are at least two), with m and n for their norms. Assume that for all archimedean places $v|\infty$, either $\pi_{1,v}$ or $\pi_{2,v}$ is a principal series representation. If $(\mathfrak{q}, \mathfrak{m}\mathfrak{n}) = 1$, then for any $\epsilon > 0$, we have the following subconvex estimate*

$$L\left(\tfrac{1}{2}, \pi \otimes \pi_1 \otimes \pi_2\right) \ll_{\varepsilon, F, \pi_{1,\infty}, \pi_{2,\infty}, \pi_\infty} m^{1-\frac{1-2\theta}{32}+\epsilon} n^{1-\frac{1-2\theta}{32}+\epsilon} q^{1-\frac{1-2\theta}{16}+\epsilon}. \tag{1.5}$$

Here we let the real number θ be the best exponent toward the Ramanujan-Petersson Conjecture for GL(2) over the number field F, we have $0 \le \theta \le \frac{7}{64}$.

2. Local Theory: Bessel functions

2.1. **History and Background.** The study of (classical) Bessel functions can be traced back to the 19th century. Bessel functions are first defined by D. Bernoulli and then generalized by F. Bessel. The classical Bessel functions are canonical solutions $y = y(x)$ of Bessel's differential equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

for an arbitrary complex number α, which is defined as the order of the Bessel function.

In the 20th century, more connections and applications of the Bessel functions and their generalizations were found in many other fields of mathematics, in particular, analytic number theory, automorphic forms and the Langlands Program.

For example, classical Bessel functions appears natrually in Voronoi's summation formula as well as Petersson's and Kuznetsov's trace formula for $\mathrm{GL}_2(\mathbb{R})$. These formulae have become fundamental analytic tools in attacking some deep problems in analytic number theory.

2.2. **My work on Bessel functions: local integrability.** My work mainly focus on the Bessel functions over non-archimedean local fields, which is an analogy of classical Bessel functions in finite places. The rigorous definition of Bessel functions depends on the study of stable integrals when the integral domain is non-compact and the famous multiplicity one and uniqueness theorem of non-degenerate Whittaker models proved by J. Shalika. Roughly speaking, by a theorem of E. M. Baruch and the results proved H. Jacquet and Y. Ye, the Bessel functions can be viewed locally as generalized Kloosterman sums.

In [Miao24a], we study and prove one of the important property of Bessel functions, which is the local integrability of Bessel functions for $\mathrm{GL}_n(\mathbb{Q}_p)$. This result was known previously only for $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{GL}_3(\mathbb{Q}_p)$. **The paper was revised and finalized during my stay in MPIM and**

was published in Journal of Functional Analysis (See [Miao24a]**).** This is the main part of my PhD Thesis. In the proof, although the Bessel function $j_\pi(g)$ depends on the local smooth irreducible generic representations, however, instead of using the classification of local representations and Bessel functions, we use a uniform method from E. M. Baruch, H. Jacquet and Y. Ye to reduce the estimation of Bessel functions to certain $\mathrm{GL}_n(\mathbb{Q}_p)$ Kloosterman sums. A nontrivial upper bound for the $\mathrm{GL}_n(\mathbb{Q}_p)$ Kloosterman sums will be enough to prove the local integrability of Bessel functions.

2.3. **My work on Bessel functions: certain refinement.** By [Miao24a, Corollary 7.4], it is known that the Bessel functions determine the representations in a unique way, that is to say if $j_{\pi_1}(g) = cj_{\pi_2}(g)$ for a non-zero constant c, then $\pi_1 \cong \pi_2$. A nature question is as follows: Understand the property (asymptotic behaviour) of Bessel functions over $\mathrm{GL}_n(\mathbb{Q}_p)$ for different types of smooth irreducible generic representations (principal series, special representations and supercuspidal representations). In our proof ([Miao24a]), such kind of problem is not well understood, since we tranfer the local integrability problem which is a local problem in a uniform way by a result proved by E. M. Baruch. Finally, the problem reduced to estimating a non-trivial upper bound for $\mathrm{GL}(n)$ generalized Kloosterman sums. Therefore, the local integrability problem for Bessel functions is proved for all smooth irreducible generic representations. In this project, we plan to understand certain specific Bessel functions on $\mathrm{GL}_n(\mathbb{Q}_p)$ for level zero or simple supercuspidal representations, which are both highly ramified. In this two special cases of representations, we can expect to get the asymptotic behaviour of Bessel functions, which should be better than the uniform bound for $\mathrm{GL}(n)$ Kloosterman sums in [Miao24a]. This is an ongoing project.

3. Papers, Lectures and courses

During my stay in MPIM, I mainly work on the Bessel functions (generalized Kloosterman sums), spectral reciprocity formulae and their applications (See above Section 1 and 2). The following is a list of all papers planned, written, or published during my visit in MPIM.

(1) X. Miao, *Bessel Functions and Kloosterman Integrals on* $\mathrm{GL}(n)$*.* Journal of Functional Analysis, Volume 286, Issue 4, 2024. [Miao24a]
(2) X. Miao, *Spectral Reciprocity for the first moment of triple product L-functions and applications.* preprint, 2024. [Miao24b]
(3) X. Miao, *Spectral Reciprocity and Hybrid subconvexity bound for triple product L-functions.* preprint, 2024. [Miao24c]
(4) X. Miao, *A non-symmetrical spectral reciprocity formula for degree* 8 *L-functions.* preprint, 2024. [Miao24d]

Moreover, during my stay in MPIM, I am also a member of the research group on Analytic Number Theory and Automorphic Forms leading by Professor V. Blomer in University of Bonn. I discussed mathematics with all the group members. Hoperfully we will have some cooperation in the near future. Now I am a postdoc in the University of Bonn supervised by Professor V. Blomer and will continue to discuss mathematics with mathematicians in Bonn. The following is a list of talks I gave during my visit in MPIM and University of Bonn.

(1) 24.10.2023, One talk about Linnik's basic lemma in Number theory learning seminar on the modern ergodic method for proving equidistribution results.
(2) 30.10.2023, One talk about Local Integrability of Bessel functions on $\mathrm{GL}(n)$ in MPI-Oberseminar (See Section 2.2).
(3) 21.12.2023. One talk about Local Integrability of Bessel functions on $\mathrm{GL}(n)$ in University of Bonn Oberseminar Analytic Number Theory and Automorphic Forms (See Section 2.2).
(4) 24.05.2024. One talk about Local Integrability of Bessel functions on $\mathrm{GL}(n)$ in 36-th Automorphic Forms Workshop, Oklahoma State University, Stillwater (See Section 2.2).
(5) 27.06.2024. One talk about Spectral reciprocity formulae for certain L-functions in University of Bonn Oberseminar Analytic Number Theory and Automorphic Forms (See Section 1.2).

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