MPIM Report
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During my time as a postdoc at the Max Planck Institute for Mathematics, I advanced my work in two areas of research: harmonic mappings and isoperimetric inequalities. I identified several new questions in these areas, and in several related branches of differential geometry and geometric analysis, that I believe could be worthwhile goals for future work. I learned, through my conversations with people I met at the institute and other institutions in Europe, about several new topics and questions that I intend to pursue in future work. I also taught a graduate topics course at the University of Bonn and served as a referee for the Journal of Differential Geometry and a reviewer for Mathematical Reviews/MathSciNet. Papers written and planned during my stay at the institute appear in bold in the bibliography.

Research Activities. In work related to harmonic mappings during my stay at the institute, I showed that the infimum of the energy in any homotopy class of mappings from complex projective space to a Riemannian manifold is proportional to the infimal area in the homotopy class of mappings of the 2-sphere representing the induced homomorphism on the second homotopy group [Hois23]. This result gives a parallel, for complex projective space, to an important result about homotopy classes of maps of spheres which goes back to the foundation of the theory of harmonic mappings in the the work of Eells and Sampson [ES64]. It complements and builds on work Brian White [Wh86], Christopher Croke [Cr87], and others. It also leads to several natural questions at the intersection of complex geometry and the theory of harmonic mappings: several observations which enter into this result have natural extensions to any Kähler manifold biholomorphic to complex projective space, and it seems natural to ask whether the main result in [Hois23] extends to a result which gives information about all such manifolds.

Another natural goal is to establish a similar result determining or estimating the infimum of the energy in homotopy classes of mappings of real projective space. Note that, because of several more general results in the theory of harmonic mappings, real and complex projective space are an especially natural family of spaces to consider for this problem. In work at the institute [Hois24a], I established a two-sided estimate for the infimum of the energy in a homotopy class of mappings from real projective space to a Riemannian manifold, as well as a rigidity theorem for maps which realize the implied lower bound for energy. Another goal of ongoing work begun at the institute is to classify stable harmonic maps of real projective space. This may be done partly in collaboration with researchers I met during my time at the institute.

More speculatively, the parallels between the theory of harmonic mappings for surfaces and Yang-Mills fields for 4-manifolds suggest that there might be a parallel to the main result in [Hois23] for Yang-Mills fields over quaternionic projective space. I learned of these parallels between the theory of harmonic mappings and Yang-Mills theory during visits to other institutions in Germany, such as the University of Münster and the University of Marburg, while I was at the institute. In work before and during my time at the institute [Hois24c], I have also established estimates for a class of energy functionals for maps from quaternionic projective space to Riemannian manifolds, similar to the estimates for energy in [Hois23, Hois24a].

In work related to isoperimetric inequalities shortly before I arrived at the institute, I established a sharp, quantitative version of the linear isoperimetric inequality, an important inequality in the study of Riemannian manifolds with negative sectional curvature [Hois24b]. During my

time at the institute, I made progress extending this result from the classical setting in which it has been established to a broader range of settings where the linear isoperimetric inequality is of interest. I have also begun to develop an approach, based on work that enters into this result, to an important conjecture about the geometry of convex sets in spaces of nonpositive curvature. Through my discussions with other visitors and scientific staff at the institute, I learned more about the significance of isoperimetric inequalities in metric geometry and geometric group theory, and about several questions and possible directions for future work.

Collaboration. Throughout my stay at the institute, I discussed my research work and many questions and ideas with my mentor, Werner Ballmann. I talked regularly about my work with visitors and scientific staff at the institute, in particular Stephan Stadler, Richard Wentworth, and Sergio Zamora, and with faculty at the university, in particular Ursula Hamenstädt. These discussions have led to several questions and possible directions for future work, and I may collaborate with some of these people on them in the future.

Teaching. During the Spring 2023 term, I taught a graduate topics course at the university, called "Metric Inequalities in Geometry, Topology, and Analysis".

Refereeing and Service. During my stay at the institute, I served as a referee for the Journal of Differential Geometry, and I wrote several reviews for Mathematical Reviews/MathSciNet

References

[Cr87] Christopher B. Croke: *Lower bounds on the energy of maps*, Duke Math. J. 54.1 (1987): 901-908.

[ES64] James Eells and Joseph H. Sampson: *Harmonic mappings of Riemannian manifolds*, American Journal of Mathematics 86.1 (1964): 109-160.

[Hois23] **Joseph Hoisington: *Energy-minimizing mappings of complex projective spaces*, arXiv preprint arXiv:2311.08285 (2023).**

[Hois24a] **Joseph Hoisington: *Energy-minimizing mappings of real projective spaces*, arXiv preprint arXiv:2401.10201 (2024).**

[Hois24b] **Joseph Hoisington: *A quantitative refinement of the linear isoperimetric inequality*, in preparation.**

[Hois24c] **Joseph Hoisington: *Lower bounds for energy functionals in quaternionic projective space*, in preparation.**

[Wh86] Brian White: *Infima of energy functionals in homotopy classes of mappings*, Journal of Differential Geometry 23.2 (1986): 127-142.