On the proper homotopy type of locally compact A_n^2-polyhedra

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ABSTRACT. In this paper we address the classification problem for locally compact $(n-1)$-connected CW-complexes with dimension $\leq n+2$ up to proper homotopy type. We obtain complete classification theorems in terms of purely algebraic data in those cases where the representation type of the involved algebra is finite. For this we define new quadratic functors in controlled algebra and new homotopy and cohomology invariants in proper homotopy theory.

Contents

Introduction

The program for the homotopy classification of polyhedra in terms of algebraic invariants was initiated by J. H. C. Whitehead in the late 1940s. He completely solved in a series of papers the homotopy classification problem for $(n-1)$-connected compact CW-complexes with dimension $\leq n+2$, the so called A_n^2-polyhedra, for $n \geq 2$, see [**Whi48**], [**Whi49**] and [**Whi50**]. Later progress was made by H.-J. Baues and his collaborators, see the survey article [**Bau95**] on the history and the development of Whitehead's program.

The proper homotopy type of a non-compact space is known to be a stronger topological invariant than the ordinary homotopy type. Proper maps, unlike ordinary continuous maps, keep track of the infinity behavior which arises naturally in non-compact spaces. Moreover, proper maps are still suitable to do homotopy theory with them. There are two main axiomatic approachs to proper homotopy theory in the literature. The Edwards-Hastings embedding of the proper category into the Quillen model category of pro-spaces ([**EH76**]) and the Baues-Quintero approach in terms of I-categories ([**BQ01**]) which is the one we follow in this paper since it is intrinsic.

So far Whitehead's classification program is at a very preliminary stage in the proper homotopy setting. The only class of non-compact spaces for which a nice algebraic model is available is given by the n-dimensional spherical objects under a locally compact tree T. Such a spherical object is obtained by pasting n-spheres on T in a locally finite way. As a consequence of results in [**BQ01**] and [**ACMQ03**] the proper homotopy category of these objects is equivalent to the category of finitely generated projective modules over the ring $\mathbb{Z}(\mathfrak{F}(T))$ for any $n \geq 2$, see Sections 1.2 and 1.3 below. This ring, up to isomorphism, only depends on the space of Freudenthal ends $\mathfrak{F}(T)$ of the tree T. For T a one-ended tree the ring $\mathbb{Z}(\mathfrak{F}(T))$ is the well-known ring of row-column-finite matrices, or locally finite matrices, over the integers. This ring plays also a role in algebraic K-theory, it is the cone of $\mathbb{Z}$ in the sense of [**Wag72**], see also [**Lod76**]. In general $\mathbb{Z}(\mathfrak{F}(T))$ is the endomorphism ring of a certain object in a category of free controlled modules, compare [**CP95**].

Spherical objects, despite being very simple, play a crucial role since they are the building blocks of any locally compact finite-dimensional CW-complex X, therefore the algebraic models for spherical objects allow the construction of a cellular homology

$$\mathcal{H}_*X$$

with values in $\mathbb{Z}(\mathfrak{F}(T))$-modules, see [**BQ01**]. This is the homology of the proper cellular chain complex $\mathcal{C}_*X$ which is a bounded complex of finitely generated projective $\mathbb{Z}(\mathfrak{F}(T))$-modules. The cohomology of this complex with coefficients in a

$\mathbb{Z}(\mathfrak{F}(T))$-module $\mathcal{M}$ is the cohomology of X

$$H^*(X, \mathcal{M}).$$

These are just abelian groups since $\mathbb{Z}(\mathfrak{F}(T))$ is non-commutative.

In this paper we consider locally compact A_n^2-polyhedra in proper homotopy theory. These spaces also arose early in the development of proper homotopy theory for independent reasons. The discovery of proper homology theories detecting proper homotopy equivalences led to the question of the existence and uniqueness of Moore spaces in proper homotopy, i. e. 1-connected locally compact CW-complexes with non-vanishing homology in a single dimension. It was noticed in [**ADMQ95**] that the proper homology of locally compact CW-complexes can have projective dimension ≥ 2, and actually no more than 2 by the main result in [**ACMQ03**]. As a consequence proper Moore spaces exist and they are locally compact A_n^2-polyhedra, but they are not unique, i. e. the proper homotopy type is not determined by the homology. The first explicit example of two different proper Moore spaces with the same homology was obtained in [**ADMQ95**]. At this point it was reasonable to wonder how many different proper Moore spaces are there with the same homology in the same dimension, or more generally, about the proper homotopy classification of proper Moore spaces in terms of purely algebraic data. When we addressed this problem we noticed that there was not substantial difference between considering only proper Moore spaces or more generally all locally compact A_n^2-polyhedra.

For the homotopy classification of compact A_n^2-polyhedra Whitehead used homology and the Pontrjagin-Steenrod invariant in cohomology. He also showed that one can also use as an alternative tool his "certain" exact sequence for the Hurewicz homomorphism, which can be deduced from the Pontrjagin-Steenrod invariant. A generalization of this sequence is also available in proper homotopy theory. However it is not useful since two proper Moore spaces with the same homology in the same degree have always the same exact sequence but they need not have the same proper homotopy type. We then turned to consider the existence of proper Pontrjagin-Steenrod invariants. When one tries to construct these invariants in proper cohomology by imitating the classical methods one finds again an obstacle coming from the projective dimension, which is not ≤ 1 like for abelian groups. Nevertheless the possibility of the existence of adequate proper Pontrjaging-Steenrod invariants can not be discarded just because of this. Actually we have been successful in answering the question on the existence of proper Pontrjagin-Steenrod invariants by placing the problem in the realm of cohomology of categories in the sense of Hochschild-Mitchell [**Mit72**] and Baues-Wirsching [**BW85**]. For instance, the obstruction to the existence of a proper Pontrjagin-Steenrod invariant suitable to classify proper homotopy types of A_n^2-polyhedra, $n \geq 3$, is a class

$$\{\theta\} \in H^1(\mathbf{chain}_n(\mathbb{Z}(\mathfrak{F}(T)))/\simeq, H^{n+2}(-, H_n \otimes \mathbb{Z}/2)) \tag{a}$$

in the first cohomology of the homotopy category $\mathbf{chain}_n(\mathbb{Z}(\mathfrak{F}(T)))/\simeq$ of bounded chain complexes of finitely generated projective $\mathbb{Z}(\mathfrak{F}(T))$-modules concentrated on dimensions $\geq n$, see Section 5.1 below. The use of these techniques in obstruction theory and homotopy classification problems goes back to [**Bau89**]. Cohomology of categories is strongly related to representation theory, therefore one can not expect to be able to carry out all necessary computations in the presence of a wild representation type. The representation theory needed to deal with proper Pontrjaging-Steenrod invariants was already developed in [**Mur04**]. Here we give a complete answer to the possible existence of proper Pontrjaging-Steenrod invariants in the finite representation type cases.

Chapter 1 is a basic introduction to the proper homotopy category and controlled algebra following [**BQ01**]. We include two new results on controlled algebra, Propositions 1.2.12 and 1.2.13, which are proved in the two appendices of this paper. We also recall from [**BQ01**] the definition of the proper homotopy and homology modules and proper cohomology groups, as well as their basic properties. In Chapter 2 we define new quadratic functors in controlled algebra. Among these new functors we have generalizations of the exterior square, the reduced tensor square, and Whitehead's quadratic functor

$$\wedge^2_T, \hat{\otimes}^2_T, \Gamma_T\colon \mathbf{mod}(\mathbb{Z}(\mathfrak{F}(T))) \longrightarrow \mathbf{mod}(\mathbb{Z}(\mathfrak{F}(T))).$$

These quadratic functors are applied in several computations of proper homotopy invariants, which are extensively used in Chapter 3 for the definition of new James-Hopf and cup-product invariants in proper homotopy theory, such as for example the reduced cup-product invariant of an $(n-1)$-connected locally compact finite-dimensional CW-complex X, $n \geq 3$,

$$\hat{\cup}_X \in H^{n+2}(X, \hat{\otimes}^2_T \mathcal{H}_n X).$$

The cup-product cohomology invariants are well-behaved with respect to obstruction theory. The necessary obstruction theory is recalled in the last section of Chapter 2 following the approach given by the "tower of categories" introduced in [**Bau89**] and generalized in [**Bau99**]. Among the different cup-product invariants defined in Chapter 3 we have the so called chain cup-product. This is not the cup-product invariant of a space but of a bounded chain complex $\mathcal{C}_*$ of finitely generated projective $\mathbb{Z}(\mathfrak{F}(T))$-modules concentrated in dimensions $\geq n$

$$\bar{\cup}_{\mathcal{C}_*} \in H^{n+2}(\mathcal{C}_*, \wedge^2_T H_n \mathcal{C}_*),$$

as for example the proper cellular chain complex $\mathcal{C}_*X$ of a locally compact A^2_n-polyhedron X. The chain cup-product $\cup_{\mathcal{C}_*}$ of a chain complex $\mathcal{C}_*$ concentrated in dimensions 2, 3 and 4 is the obstruction to the existence of an A^2_2-polyhedron admitting a co-H-multiplication with proper cellular chain complex $\mathcal{C}_*X = \mathcal{C}_*$. Such an obstruction does not arise in ordinary homotopy theory since ordinary Moore spaces are much better behaved. The collection of all chain cup-products assembles to a 0-dimensional class in cohomology of categories

$$\text{(b)} \qquad \bar{\cup} \in H^0(\mathbf{chain}_n(\mathbb{Z}(\mathfrak{F}(T)))/\simeq, H^{n+2}(-, \wedge^2_T H_n)).$$

The computation of this cohomology class is a major step towards the solution of the proper Prontrjagin-Steenrod invariant problem. This computation is addressed in Chapter 4, where we use techniques from homotopical algebra developed in [**Mur05**], as well as the localization theorem in cohomology of categories obtained in [**Mur06**]. We also use in this paper the representation theory of $\mathbb{Z}(\mathfrak{F}(T)) \otimes \mathbb{F}_2$ studied in [**Mur04**] and the homological computations in Chapter 6. As a consequence of the computations in Chapter 4 we obtain in this paper the first examples of proper Moore spaces in degree 2 which are not co-H-spaces, in contrast with the ordinary case, see Corollary 5.4.10 and Remark 5.4.11 below.

In Chapter 5 we finally attack the problem of the existence of proper Pontrjagin-Steenrod invariants. We show here that the cohomology class $\{\theta\}$ in (a) is the universal obstruction to the existence of proper Pontrjagin-Steenrod invariants, while the chain cup-product cohomology class (b) is the universal obstruction for the existence of proper Pontrjagin-Steenrod invariants which are in addition compatible with the cup-product invariants of CW-complexes in Chapter 3. This proves, together with the computations carried out in Chapter 4, that for spaces with 3 or more ends it is impossible to obtain proper Pontrjagin-Steenrod invariants which are well related to the cup-product invariants defined in Chapter 3. This is a surprising result which emphasizes the contrast between ordinary homotopy theory and proper homotopy theory. On the other hand we also show in Chapter 5 that for spaces with no more than 3 ends there exist Pontrjagin-Steenrod invariants which are suitable to classify proper homotopy types of locally compact A_n^2-polyhedra in the stable range $n \geq 3$. For this we use the computations with finitely presented $\mathbb{Z}(\mathfrak{F}(T)) \otimes \mathbb{F}_2$-modules in Chapter 6 in order to relate the cohomology classes (b) and (a) by a Bockstein exact sequence. In this way we deduce from the computation of (b) in Chapter 4 that (a) vanishes whenever the tree T has 3 or less ends. This leads to one of the main results of this paper which asserts that for a locally compact A_n^2-polyhedron X with less than 4 ends and $n \geq 3$ the pair

$$(\mathcal{C}_*X, \wp_n(X))$$

given by the proper cellular chain complex $\mathcal{C}_*X$ and the proper Steenrod invariant

$$\wp_n(X) \in H^{n+2}(X, \mathcal{H}_nX \otimes \mathbb{Z}/2)$$

constructed in Chapter 5 is an algebraic model for the proper homotopy type of X. See Theorem 5.3.2 in Section 5.3 for a precise statement of this result. The implications of these results for proper Moore spaces are considered in a separate section within Chapter 5.

The unstable case $n = 2$ and the case of spaces with more than 4 ends are out of any scope because of impediments coming from representation theory, in the sense explained above. In the remaining case, i. e. 4-ended locally compact A_n^2-polyhedra for $n \geq 3$, it should still be possible to answer the question on proper Pontrjaging-Steenrod invariants and maybe to extend the classification theorem proved here. Nevertheless the representation theory in this case is connected to the representation theory of the 4-subspace quiver, see [**Mur04**]. This quiver has tame representation type, as it was shown by Nazarova in [**Naz73**], but the classification of indecomposable 4-subspaces over the field with 2 elements is quite intricate. Therefore the algebraic computations carried out in the already lengthy Chapter 6 would have to be considerably expanded to include this special case. For this reason we decided not to deal with this case in this paper.

CHAPTER 1

Proper homotopy theory and controlled algebra

In this chapter we summarize the background on proper homotopy theory and controlled algebra which is necessary to understand the rest of this paper. We follow the axiomatic approach to proper homotopy theory in [**BQ01**]. In Section 1.2 we include two new results in controlled algebra, Propositions 1.2.12 and 1.2.13, which are crucial in this paper in order to extend some results for a tree with a certain number of Freudenthal ends to a tree with more ends. We also establish connections with the results in [**ACMQ03**] on the projective dimension in controlled algebra.

1.1. Basic aspects of proper homotopy theory

A continuous map $f\colon X \to Y$ is *proper* if it is closed and $f^{-1}(y)$ is compact for all $y \in Y$. An alternative definition says that f is proper provided $f^{-1}(K)$ is compact for any compact subspace $K \subset Y$. Both definitions are equivalent on locally compact Hausdorff spaces, in particular over locally compact CW-complexes, and this is the class of spaces on which we will concentrate soon. All maps in this paper are supposed to be proper, unless we explicitly state the contrary.

The ordinary *cylinder* functor $X \mapsto IX = [0,1] \times X$ restricts to the category **Topp** of spaces and proper maps. Moreover, the structural maps of a cylinder

$$X \overset{i_0}{\underset{i_1}{\rightrightarrows}} IX \xrightarrow{p} X \ ,$$

$i_k(x) = (k,x)$, $p(t,x) = x$, are proper, so one can define a homotopy natural equivalence relation in this category.

In fact this cylinder gives rise to an I-category structure on **Topp**, see [**BQ01**] I.3.6. *Cofibrations* $X \rightarrowtail Y$ in this category are maps with the proper homotopy extension property, in particular they are injective. Examples of cofibrations are inclusions of subcomplexes into finite-dimensional locally compact CW-complexes.

The axioms of an I-category, and the weaker ones of a cofibration category, were introduced in [**Bau89**]. There Baues develops the homotopy theory of these kind of categories, with emphasis on obstruction theory. He shows that avoiding the use of fibrations one can still generalize most of ordinary homotopy theory. These axioms are specially adequate for proper homotopy theory since **Topp** lacks of enough inverse limits, even the product of two spaces does not exist unless one of them is compact, so it does not fit into Quillen's model category axioms.

A *space X under A* is a map $i_X\colon A \to X$ in **Topp**. It is said to be *cofibrant* if this map $i_X\colon A \rightarrowtail X$ is a cofibration. The category $\mathbf{Topp}^A_c$ of cofibrant spaces and maps under A is also an I-category, see [**BQ01**] I.3.12. A map under A is a cofibration if it is a cofibration in **Topp**. The cylinder functor is the *relative cylinder* I_A. We shall write

$$[X,Y]^A$$

for the set of homotopy classes of maps $X \to Y$ in $\mathbf{Topp}_c^A$.

A *pair of spaces* (X, Y) *under* A is a map $Y \to X$ in $\mathbf{Topp}_c^A$. It is *cofibrant* if $Y \rightarrowtail X$ is a cofibration. The category of cofibrant pairs under A is also an I-category as a consequence of [**BQ01**] I.3.8 and I.3.6, see also the following remark. The set of homotopy classes of maps $(X, Y) \to (U, V)$ between cofibrant pairs under A is denoted by

$$[(X, Y), (U, V)]^A.$$

REMARK 1.1.1. In [**BQ01**] I.3.8 it is claimed that the category of pairs of an I-category with an extra condition satisfied by **Topp** is an I-category, but this is not correct since cofibrant pairs are exactly those pairs for which the map from the initial object is a cofibration, and this is a requirement of the I-category axioms. However the proof of [**BQ01**] I.3.8 is enough to check that the category of cofibrant pairs is an I-category.

A *based space under* A is a cofibrant space X under A together with a retraction $0_X\colon X \to A$ of i_X. If X is based the *trivial map* $0\colon X \to Y$ is the composite

$$X \xrightarrow{0_X} A \xrightarrow{i_Y} Y,$$

and $[X, Y]^A$ is a based set. A map $f\colon X \to Y$ under A between based spaces is a *based map* if $0_X = 0_Y f$. We say that f is *based up to homotopy* if 0_X is just homotopic to $0_Y f$. Based spaces and maps are used to define in the usual way cones CX, cofibers C_f of maps $f\colon X \to Y$, suspensions ΣX, and related constructions in $\mathbf{Topp}_c^A$ such as the long cofiber sequence of a based map

$$X \xrightarrow{f} Y \overset{i_f}{\rightarrowtail} C_f \xrightarrow{q} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \overset{\Sigma i_f}{\rightarrowtail} \Sigma C_f \xrightarrow{\Sigma q} \Sigma^2 X \xrightarrow{\Sigma^2 f} \cdots .$$

If the based map $f\colon X \rightarrowtail Y$ is already a cofibration the cofiber C_f is equivalent to the quotient space Y/X. One can check by using the "gluing lemma" [**Bau89**] II.1.2 that up to homotopy equivalence these constructions only depend on the homotopy class of the retraction $0_X : X \to A$ in $\mathbf{Topp}_c^A$. Moreover, up to homotopy equivalence the cofiber of a map f only depends on the homotopy class of f in $\mathbf{Topp}_c^A$. Furthermore, cylinders, cones and suspensions of based spaces and cofibers of based maps are canonically based. See [**BQ01**] I.6 and I.7 for further details.

A map $f\colon A \to B$ induces a "cobase change" functor

$$f_*\colon \mathbf{Topp}_c^A \longrightarrow \mathbf{Topp}_c^B$$

given by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] \arrow[dr, phantom, "\text{push}"] & B \arrow[d, tail] \\ X \arrow[r] & f_*X \end{tikzcd}$$

This functor preserves relative cylinders $f_* I_A X = I_B f_* X$, hence it induces a functor between the respective homotopy categories

$$f_*\colon \mathbf{Topp}_c^A/\simeq \longrightarrow \mathbf{Topp}_c^B/\simeq . \tag{1.1.2}$$

Moreover, up to natural equivalence this last functor only depends on the homotopy class of f in **Topp**. In particular it is an equivalence of categories if f is a homotopy equivalence. This can also be checked by using the "gluing lemma".

So far we have just considered the features of proper homotopy theory which are common to many other homotopy theories. The most basic purely proper homotopy invariant is the space of Freudenthal ends. The space of *Freudenthal ends* $\mathfrak{F}(X)$ of a connected locally compact finite-dimensional CW-complex X is the following inverse limit

$$\mathfrak{F}(X) = \lim_{\substack{K \subset X \\ \text{compact}}} \pi_0(X - K).$$

Here $\pi_0(X-K)$ denotes the finite set of connected components of $X-K$ with the discrete topology. This space is known to be homeomorphic to a closed subspace of the Cantor set. In fact any closed subspace of the Cantor set is homeomorphic to the space of Freudenthal ends of some tree. Recall that *trees* are contractible 1-dimensional CW-complexes. In addition in this paper we shall suppose that all trees are locally compact.

Taking space of Freudenthal ends defines a functor $\mathfrak{F}$ from the proper homotopy category of connected locally compact finite-dimensional CW-complexes. This functor determines an equivalence between the proper homotopy category of trees and the category of closed subspaces of the Cantor set, see [**BQ01**] II.1.10. Therefore, given a tree T, up to equivalence of categories $\mathbf{Topp}^T_c$ only depends on the space $\mathfrak{F}(T)$.

The *Freudenthal compactification* $\hat{X}$ of a connected locally compact finite-dimensional CW-complex X is, as a set, the disjoint union $\hat{X} = X \sqcup \mathfrak{F}(X)$. Given an open subset $U \subset X$ we define $U^{\mathfrak{F}}$ as the set of Freudenthal ends $\varepsilon \in \mathfrak{F}(X)$ such that U is the coordinate of ε in $\pi_0(X-K)$ for some compact subset $K \subset X$. The sets $U \sqcup U^{\mathfrak{F}}$ form a basis of the topology of $\hat{X}$. The space X is an open dense subspace of $\hat{X}$. Moreover, the Freudenthal compactification is a functor and the inclusions $X \subset \hat{X} \supset \mathfrak{F}(X)$ are natural.

Following [**BQ01**], we now describe the reduced and normalized models of connected locally compact CW-complexes in proper homotopy theory. The building blocks of these models are the so called spherical objects.

The n-dimensional *spherical object* S^n_α under a tree T associated to a proper map $\alpha\colon A \to T$ with A a discrete set is the space obtained by pasting an n-sphere S^n to $\alpha(a)$ by the base-point $* \in S^n$ for every $a \in A$

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, tail, "{(1,*)}"'] \arrow[dr, phantom, "\text{push}"] & T \arrow[d, tail] \\
A \times S^n \arrow[r] & T \cup_\alpha A \times S^n = S^n_\alpha
\end{tikzcd}$$

Spherical objects are based by the map $S^n_\alpha \to T$ which collapses each sphere to the point pasted to T. One can readily check that $\Sigma S^n_\alpha = S^{n+1}_\alpha$ ($n \geq 0$).

A *T-complex* is a finite-dimensional CW-complex X whose 1-skeleton is a 1-dimensional spherical object $X = S^1_{\alpha_1}$ and such that the $(n+1)$-skeleton X^{n+1} ($n \geq 1$) is the cofiber of a map $f_{n+1}\colon S^n_{\alpha_{n+1}} \to X^n$ in $\mathbf{Topp}^T_c$ which is called *attaching map* of $(n+1)$-cells. This is what Baues and Quintero call in [**BQ01**] a finite-dimensional, reduced and normalized CW-complex relative to T.

T-complexes X are connected, locally compact, and the inclusion $T \subset X$ induces a homeomorphism between the spaces of Freudenthal ends $\mathfrak{F}(T) \cong \mathfrak{F}(X)$

which will be regarded as an identification. Conversely for every connected finite-dimensional locally compact CW-complex Y there exists a subtree $T \subset Y$ which induces a homeomorphism on Freudenthal ends $\mathfrak{F}(T) \cong \mathfrak{F}(Y)$, and for any such a tree there exists a T-complex X and a homotopy equivalence $X \simeq Y$ in $\mathbf{Topp}_c^T$.

Any T-complex X can be regarded as a based space in a unique way up to homotopy since the set $[X, T]^T$ is a singleton, see [**BQ01**] IV.6.4. In particular all maps under T between T-complexes are based up to homotopy.

The elementary properties of T-complexes are similar to those of connected pointed CW-complexes, see [**BQ01**] for details. In particular there is a proper cellular approximation theorem, so if $\mathbf{CW}^T$ is the category of T-complexes and cellular maps under T the homotopy category $\mathbf{CW}^T/\simeq$ is a full subcategory of $\mathbf{Topp}_c^T/\simeq$. Also, as it happened with $\mathbf{Topp}_c^T/\simeq$, up to equivalence of categories $\mathbf{CW}^T/\simeq$ only depends on the proper homotopy type of the tree T, or equivalently on its space of Freudenthal ends $\mathfrak{F}(T)$. Therefore the first step in the study of proper homotopy types of connected finite-dimensional locally compact CW-complexes is to separate them attending to their most basic homotopy invariant, the space of Freudenthal ends, and then to concentrate on the study of the categories of those with a fixed space of Freudenthal ends, or equivalently on the categories $\mathbf{CW}^T$.

As it is well-known to experts, there are drastic differences between the homotopy theories of the categories $\mathbf{CW}^T$ when T is or not compact. In fact if T is any compact tree $\mathfrak{F}(T) = \emptyset$ is the empty set and $\mathbf{CW}^T/\simeq$ is equivalent to the ordinary homotopy category of connected compact pointed CW-complexes. In this paper we are concerned with the non-compact case so, unless we state the contrary, all trees considered will be non-compact.

1.2. Controlled algebra

Recall the category $\mathbf{mod}(\mathbf{A})$ of (right) *modules* over a small additive category $\mathbf{A}$ is the abelian category of additive functors $\mathcal{M}\colon \mathbf{A}^{op} \to \mathbf{Ab}$ and natural transformations. Here $\mathbf{Ab}$ is the category of abelian groups. There is a full Yoneda inclusion of categories

$$\text{Yoneda}\colon \mathbf{A} \hookrightarrow \mathbf{mod}(\mathbf{A}) \tag{1.2.1}$$

which sends an object X to the representable functor $\mathrm{Hom}_{\mathbf{A}}(-, X)$. These modules are said to be *finitely generated (f. g.) free* and constitute a set of small projective generators of $\mathbf{mod}(\mathbf{A})$. For simplicity we shall identify X with $\mathrm{Hom}_{\mathbf{A}}(-, X)$. An $\mathbf{A}$-module is *finitely generated (f. g.)* if it is the image of a f. g. free $\mathbf{A}$-module, and it is *finitely presented (f. p.)* if it is the cokernel of a morphism between f. g. free $\mathbf{A}$-modules.

If $\mathbf{B}$ is another small additive category an additive functor $\mathbb{F}\colon \mathbf{A} \to \mathbf{B}$ induces two "change of coefficient" functors

$$\mathbb{F}^*\colon \mathbf{mod}(\mathbf{B}) \longrightarrow \mathbf{mod}(\mathbf{A}),$$

$$\mathbb{F}_*\colon \mathbf{mod}(\mathbf{A}) \longrightarrow \mathbf{mod}(\mathbf{B}).$$

The first one is exact and preserves colimits, it is given by right composition with $\mathbb{F}$, $\mathbb{F}^*\mathcal{M} = \mathcal{M}\mathbb{F}$, and the second one is characterized by being left adjoint to $\mathbb{F}^*$, in particular $\mathbb{F}_*$ is right-exact and more generally colimit-preserving, see [**HS71**]

II.7.7. Moreover, $\mathbb{F}_*$ fits into the following commutative diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\mathbb{F}"] \arrow[d, hook, "\text{Yoneda}"'] & \mathbf{B} \arrow[d, hook, "\text{Yoneda}"] \\ \mathbf{mod}(\mathbf{A}) \arrow[r, "\mathbb{F}_*"] & \mathbf{mod}(\mathbf{B}) \end{tikzcd} \tag{1.2.2}$$

hence $\mathbb{F}_*$ restricts to the full subcategories of f. p. modules. Furthermore, if $\mathbb{F}$ is full and faithful $\mathbb{F}_*$ is too, and in this case $\mathbb{F}^*\mathbb{F}_*$ is naturally equivalent to the identity, see [**Bor94**] 3.4.1. This can be directly checked by using that any module has a projective resolution by (arbitrary) direct sums of f. g. free ones.

It is interesting to know when the category of modules over a small additive category $\mathbf{A}$ is equivalent as an abelian category to the category $\mathbf{mod}(R)$ of modules over a ring R, i. e. when $\mathbf{A}$ is Morita equivalent to a ring. The typical case in this paper will be the following: if there exists an object X of $\mathbf{A}$ such that any other object is a retract of X then $\mathbf{A}$ is Morita equivalent to the endomorphism ring of X, that is $\mathrm{End}_{\mathbf{A}}(X) = \mathrm{Hom}_{\mathbf{A}}(X, X)$. This follows from [**Mit72**] 8.1. The equivalence of categories is given by the evaluation functor

$$ev_X : \mathbf{mod}(\mathbf{A}) \longrightarrow \mathbf{mod}(\mathrm{End}_{\mathbf{A}}(X)) : \mathcal{M} \mapsto \mathcal{M}(X).$$

If Y is an object of $\mathbf{B}$ the morphism set $\mathrm{Hom}_{\mathbf{B}}(Y, \mathbb{F}X)$ is a left-$\mathrm{End}_{\mathbf{A}}(X)$-right-$\mathrm{End}_{\mathbf{B}}(Y)$-bimodule and one can check that the next diagram commutes up to natural equivalence

$$\begin{tikzcd} \mathbf{mod}(\mathbf{A}) \arrow[r, "\mathbb{F}_*"] \arrow[d, "ev_X"'] & \mathbf{mod}(\mathbf{B}) \arrow[d, "ev_Y"] \\ \mathbf{mod}(\mathrm{End}_{\mathbf{A}}(X)) \arrow[r, "- \underset{\mathrm{End}_{\mathbf{A}}(X)}{\otimes} \mathrm{Hom}_{\mathbf{B}}(Y, \mathbb{F}X)"'] & \mathbf{mod}(\mathrm{End}_{\mathbf{B}}(Y)) \end{tikzcd} \tag{1.2.3}$$

Let T be a tree and R a commutative ring. A *free T-controlled R-module* is a pair $R\langle A\rangle_\alpha$ formed by a free R module $R\langle A\rangle$ with basis a discrete set A together with a proper map $\alpha\colon A \to T$ which is called the *height function*. If we drop the properness condition for α we say that $R\langle A\rangle_\alpha$ is a *big free T-controlled R-module*. A *controlled homomorphism* $\varphi\colon R\langle A\rangle_\alpha \to R\langle B\rangle_\beta$ is a homomorphism between the underlying R-modules such that for any neighbourhood U of $\varepsilon \in \mathfrak{F}(T)$ in $\hat{T}$ there exists another one $V \subset U$ with $\varphi(\alpha^{-1}(V)) \subset R\langle\beta^{-1}(U)\rangle$. The category $\mathbf{M}_R^b(T)$ of possibly big free T-controlled R-modules and controlled homomorphisms is an additive category, and the full additive subcategory $\mathbf{M}_R(T)$ of (non-big) free T-controlled R-modules is small.

Remark 1.2.4. The category $\mathbf{M}_R(T)$ was introduced in [**BQ01**]. Here we give a slightly different but equivalent definition. For $R = \mathbb{Z}$ it is also isomorphic to the category of finitely generated free T-trees of abelian groups considered in [**ACMQ03**], this observation can be regarded as a particular case of [**BQ01**] VIII.3.5, see also [**BQ01**] V.3.10. In [**ACMQ03**] we only consider trees T with a base vertex v_0 and no leaves apart from possibly v_0, however this is not a restriction since any tree is proper homotopy equivalent to such a kind of tree. The representation theory of f. p. $\mathbf{M}_R(T)$-modules has been considered by us in [**Mur04**], see

Section 6.1 for a review on some of its results. The category $\mathbf{M}_R(T)$ corresponds to $\mathbf{M}_R(\bar{T})$ in [**Mur04**] for $\bar{T} = (\hat{T}, T, \mathfrak{F}(T))$.

The *support* of a free T-controlled R-module $R\langle A\rangle_\alpha$ is the derived set of $\alpha(A)$ in $\hat{T}$, that is the subset $\alpha(A)' \subset \hat{T}$ formed by those points which contain infinitely many points of $\alpha(A)$ in any neighborhood. One can readily check that the support is always formed by Freudenthal ends $\alpha(A)' \subset \mathfrak{F}(T)$. Moreover, the support of a finite direct sum is the union of the supports of the factors.

The isomorphism class of a free T-controlled R-module is completely determined by its support and the rank of the underlying R-module, see [**Mur04**] 3.2 and notice that we here assume that R is commutative. Notice also that the rank of the underlying R-module is $\aleph_0$ if and only if the support is non-empty.

Any free T-controlled R-module is a retract of any other one with support $\mathfrak{F}(T)$, see [**Mur04**] 3.5. The canonical free T-controlled R-module with support $\mathfrak{F}(T)$ is $R\langle T^0\rangle_\delta$ with $\delta\colon T^0 \hookrightarrow T$ the inclusion of the vertex set. In particular, as we observed in above, $\mathbf{M}_R(T)$ is Morita equivalent to the endomorphism ring of a free T-controlled R-module with support $\mathfrak{F}(T)$. Up to isomorphism this ring (in fact R-algebra) only depends on $\mathfrak{F}(T)$ and will be denoted $R(\mathfrak{F}(T))$, see [**Mur04**] 3.4. An explicit equivalence of module categories is given by the evaluation functor

$$ev_{R\langle T^0\rangle_\delta}\colon \mathbf{mod}(\mathbf{M}_R(T)) \longrightarrow \mathbf{mod}(R(\mathfrak{F}(T))). \tag{1.2.5}$$

A curious property of the rings $R(\mathfrak{F}(T))$ is stated in the following proposition.

Proposition 1.2.6. *Any finitely generated free $R(\mathfrak{F}(T))$-module is isomorphic to just one copy of $R(\mathfrak{F}(T))$.*

The proposition follows from the previous equivalence and the fact that the support of a finite direct sum of free T-controlled R-modules with support $\mathfrak{F}(T)$ has support $\mathfrak{F}(T)$ as well.

Proposition 1.2.7. *A proper map between trees $f\colon T \to T'$ induces a "change of tree" additive functor*

$$\mathbb{F}^f\colon \mathbf{M}_R^b(T) \longrightarrow \mathbf{M}_R^b(T'),\ \ \mathbb{F}^f R\langle A\rangle_\alpha = R\langle A\rangle_{f\alpha}.$$

This functor restricts to the full subcategories of non-big objects.

This proposition follows from the following sequential characterization of controlled homomorphisms.

Lemma 1.2.8. *Given two possibly big free T-controlled R-modules $R\langle A\rangle_\alpha$ and $R\langle B\rangle_\beta$, a homomorphism $\varphi\colon R\langle A\rangle \to R\langle B\rangle$ induces a controlled homomorphism $\varphi\colon R\langle A\rangle_\alpha \to R\langle B\rangle_\beta$ if and only if given a sequence $\{a_n\}_{n\geq 0}$ in A such that $\lim\limits_{n\to\infty} \alpha(a_n) = \varepsilon \in \mathfrak{F}(T)$ in $\hat{T}$, if b_n appears with non-trivial coefficient in the linear expansion of $\varphi(a_n)$ then $\lim\limits_{n\to\infty} \beta(b_n) = \varepsilon$.*

This lemma is an extension of [**BQ01**] III.4.14 to the possibly big case. The proof is analogous.

Moreover, as we show in the next proposition, up to natural equivalence $\mathbb{F}^f$ only depends on the proper homotopy class of f.

Proposition 1.2.9. *If two proper maps between trees $f, f'\colon T \to T'$ are homotopic then there is a natural equivalence $\mathbb{F}^f \simeq \mathbb{F}^{f'}$.*

PROOF. For any possibly big free T-controlled R-module $R\langle A\rangle_\alpha$ the identity in $R\langle A\rangle$ induces a controlled isomorphism $R\langle A\rangle_{f\alpha} \simeq R\langle A\rangle_{f'\alpha}$. This can be checked by using Lemma 1.2.8 and the fact that $\mathfrak{F}(f) = \mathfrak{F}(f')$. $\square$

This proposition shows that the categories $\mathbf{M}_R^b(T)$ and $\mathbf{M}_R(T)$ only depend, up to equivalence of additive categories, on the proper homotopy type of T, or equivalently on $\mathfrak{F}(T)$.

The functor $\mathbb{F}^f$ is obviously faithful. It is not full in general, but it is when f induces an injection in Freudenthal ends.

PROPOSITION 1.2.10. *If the proper map between trees $f\colon T \to T'$ induces an injection $\mathfrak{F}(f)\colon \mathfrak{F}(T) \hookrightarrow \mathfrak{F}(T')$ then $\mathbb{F}^f$ is full.*

This proposition follows from Lemma 1.2.8.

The fully faithful functor $\mathbf{M}_R(T) \hookrightarrow \mathbf{mod}(\mathbf{M}_R(T))$ in (1.2.1) can be extended to the whole category of possibly big free T-controlled R-modules

$$\mathbf{M}_R^b(T) \longrightarrow \mathbf{mod}(\mathbf{M}_R(T))\colon R\langle A\rangle_\alpha \mapsto \mathrm{Hom}_{\mathbf{M}_R^b(T)}(-, R\langle A\rangle_\alpha)_{|_{\mathbf{M}_R^{op}(T)}}. \tag{1.2.11}$$

In this way we can regard a big free T-controlled R-module $R\langle A\rangle_\alpha$ as an $\mathbf{M}_R(T)$-module. This extension is easily seen to be faithful but not necessarily full, and also satisfies the following property.

PROPOSITION 1.2.12. *Given a proper map between two trees $f\colon T \to T'$ the following diagram of functors is commutative up to natural equivalence*

$$\begin{tikzcd} \mathbf{M}_R^b(T) \arrow[r, "\mathbb{F}^f"] \arrow[d, hook] & \mathbf{M}_R^b(T') \arrow[d, hook] \\ \mathbf{mod}(\mathbf{M}_R(T)) \arrow[r, "\mathbb{F}^f_*"] & \mathbf{mod}(\mathbf{M}_R(T')) \end{tikzcd}$$

A proof of this proposition will be given in Appendix A.

Another crucial property of "change of tree" functors is stated in the next proposition.

PROPOSITION 1.2.13. *Given a proper map between two trees $f\colon T \to T'$ such that $\mathfrak{F}(f)\colon \mathfrak{F}(T) \hookrightarrow \mathfrak{F}(T')$ is injective, the change of coefficients*

$$\mathbb{F}^f_*\colon \mathbf{mod}(\mathbf{M}_R(T)) \longrightarrow \mathbf{mod}(\mathbf{M}_R(T'))$$

is exact.

The proof of this proposition is in Appendix B.

We will be specially interested in the case $R = \mathbb{Z}$ the integers. In order to keep the notation close to [**BQ01**] we will write $\mathbf{ab}(T)$ and $\mathbf{ab}^b(T)$ for $\mathbf{M}_\mathbb{Z}(T)$ and $\mathbf{M}_\mathbb{Z}^b(T)$ respectively.

A crucial property of the category of $\mathbf{ab}(T)$-modules is the following.

THEOREM 1.2.14. *The kernel of a morphism between f. g. free $\mathbf{ab}(T)$-modules is f. g. free.*

This follows from [**ACMQ03**] 3.2. A translation of the result in [**ACMQ03**] to the language used here for the special case $T = \mathbb{R}_+ = [0, +\infty)$ the half line appears in [**BQ01**] V.5.3. Some consequences of this theorem are the following.

COROLLARY 1.2.15. *F. p. $\mathbf{ab}(T)$-modules have projective dimension ≤ 2.*

Corollary 1.2.16. *If $\mathcal{M}$ is a f. p. $\mathbf{ab}(T)$-module the functors* $\mathrm{Ext}^n_{\mathbf{ab}(T)}(\mathcal{M}, -)$ *preserve colimits* ($n \geq 0$).

Corollary 1.2.17. *Every f. g. projective $\mathbf{ab}(T)$-module is f. g. free.*

This is a consequence of the standard fact that a f. g. projective $\mathbf{ab}(T)$-module is the kernel of an endomorphism of a f. g. free $\mathbf{ab}(T)$-module.

Corollary 1.2.18. $\mathbf{ab}(T)$ *is equivalent to the category of f. g. projective* $\mathbb{Z}(\mathfrak{F}(T))$*-modules.*

This is a consequence of the previous corollary and the equivalence of abelian categories in (1.2.5).

Another relevant ring for us will be the field with two elements $\mathbb{F}_2$. For simplicity we write $\mathbf{vect}^b(T)$ and $\mathbf{vect}(T)$ for the categories $\mathbf{M}^b_{\mathbb{F}_2}(T)$ and $\mathbf{M}_{\mathbb{F}_2}(T)$, respectively. The tensor product by $\mathbb{Z}/2$ induces a full functor

$$- \otimes \mathbb{Z}/2 \colon \mathbf{ab}^b(T) \longrightarrow \mathbf{vect}^g(T)$$

which restricts to the full subcategories of non-big objects. Moreover, if we denote $\mathbf{ab}^b(T) \otimes \mathbb{Z}/2$ and $\mathbf{ab}(T) \otimes \mathbb{Z}/2$ to the categories obtained from $\mathbf{ab}^b(T)$ and $\mathbf{ab}(T)$ by tensoring their morphism abelian groups by $\mathbb{Z}/2$, respectively, it is easy to check that the following proposition holds.

Proposition 1.2.19. *The functor $-\otimes\mathbb{Z}/2$ gives rise to isomorphisms $\mathbf{ab}^b(T) \otimes \mathbb{Z}/2 \simeq \mathbf{vect}^b(T)$ and $\mathbf{ab}(T) \otimes \mathbb{Z}/2 \simeq \mathbf{vect}(T)$.*

The functor $- \otimes \mathbb{Z}/2 \colon \mathbf{ab}(T) \to \mathbf{vect}(T)$ induces two "change of coefficient" functors

$$\begin{aligned} &(- \otimes \mathbb{Z}/2)^* \colon \mathbf{mod}(\mathbf{vect}(T)) \longrightarrow \mathbf{mod}(\mathbf{ab}(T)), \\ &(- \otimes \mathbb{Z}/2)_* \colon \mathbf{mod}(\mathbf{ab}(T)) \longrightarrow \mathbf{mod}(\mathbf{vect}(T)). \end{aligned} \tag{1.2.20}$$

The elementary properties of these functors are summarized in the following proposition.

Proposition 1.2.21. *The composite $(-\otimes\mathbb{Z}/2)_*(-\otimes\mathbb{Z}/2)^*$ is naturally equivalent to the identity, and $(-\otimes\mathbb{Z}/2)^*(-\otimes\mathbb{Z}/2)_*$ is naturally equivalent to the functor*

$$- \otimes \mathbb{Z}/2 \colon \mathbf{mod}(\mathbf{ab}(T)) \longrightarrow \mathbf{mod}(\mathbf{ab}(T))$$

given by left-composition with $- \otimes \mathbb{Z}/2 \colon \mathbf{Ab} \to \mathbf{Ab}$.

Notice that now we can identify $\mathbf{mod}(\mathbf{vect}(T))$ with the full subcategory of $\mathbf{mod}(\mathbf{ab}(T))$ formed by those $\mathbf{ab}(T)$-modules which take values in abelian groups with exponent 2. Moreover, as a consequence of Propositions 1.2.19 and 1.2.21 we obtain the following result.

Corollary 1.2.22. *For any possibly big free T-controlled $\mathbb{Z}$-module $\mathbb{Z}\langle A\rangle_\alpha$ we have a natural identification $\mathbb{Z}\langle A\rangle_\alpha \otimes \mathbb{Z}/2 = \mathbb{F}_2\langle A\rangle_\alpha$ as $\mathbf{ab}(T)$-modules.*

1.3. Elementary proper homotopy algebraic invariants

Let $\mathbf{S}^n(T)$ be the proper homotopy category of n-dimensional spherical objects under T. One of the main tools to construct algebraic invariants in proper homotopy theory is the following result.

PROPOSITION 1.3.1. *There are isomorphisms of categories* $(n \geq 2)$

$$\mathbf{S}^n(T) \xrightarrow{\simeq} \mathbf{ab}(T)\colon S^n_\alpha \mapsto \mathbb{Z}\langle A\rangle_\alpha$$

which are compatible with the suspension functors $\Sigma\colon \mathbf{S}^n(T) \to \mathbf{S}^{n+1}(T)$.

These isomorphisms are given by the ordinary homotopy groups $\pi_n(S^n_\alpha, T) = \mathbb{Z}\langle A\rangle$ in the level of underlying abelian groups. See [**BQ01**] II.4.15 and II.4.20 for a proof. From now on we shall use these isomorphisms as identifications.

DEFINITION 1.3.2. The *proper homotopy* $\mathbf{ab}(T)$*-modules* of a cofibrant space X under T are

$$\Pi_n X = [-, X]^T\colon \mathbf{S}^n(T)^{op} \longrightarrow \mathbf{Ab}, \ \ n \geq 2.$$

The proper homotopy modules of a cofibrant pair (X, Y) under T are also defined

$$\Pi_{n+1}(X, Y) = [(C-, -), (X, Y)]^T\colon \mathbf{S}^n(T)^{op} \longrightarrow \mathbf{Ab}, \ \ n \geq 2.$$

These proper homotopy modules are functors in the corresponding homotopy categories. Moreover, there is a natural long exact sequence

$$\cdots \to \Pi_{n+1}X \xrightarrow{j} \Pi_{n+1}(X, Y) \xrightarrow{\partial} \Pi_n Y \to \Pi_n X \to \cdots .$$

A cofibrant space X under T is 0*-connected* if the map $T \rightarrowtail X$ induces a surjection $\mathfrak{F}(T) \twoheadrightarrow \mathfrak{F}(X)$ between the spaces of Freudenthal ends, and n*-connected* $(n \geq 1)$ if in addition $[S^1_\alpha, X]^T = 0$ is a singleton for every 1-dimensional spherical object S^1_α and $\Pi_k X = 0$ for all $2 \leq k \leq n$.

Similarly, a cofibrant pair (X, Y) is 0*-connected* if the map $Y \rightarrowtail X$ induces an epimorphism $\mathfrak{F}(Y) \twoheadrightarrow \mathfrak{F}(X)$, 1*-connected* if $Y \rightarrowtail X$ also induces an epimorphism $[S^1_\alpha, Y]^T \twoheadrightarrow [S^1_\alpha, X]^T$ for all 1-dimensional spherical object S^1_α, and n-connected $(n \geq 2)$ if in addition $[(CS^1_\alpha, S^1_\alpha), (X, Y)]^T = 0$ is always a singleton and $\Pi_k(X, Y) = 0$ for all $3 \leq k \leq n$.

The *Whitehead modules* of a T-complex X are

$$\Gamma_n X = \operatorname{Im}[\Pi_n X^{n-1} \to \Pi_n X^n], \ \ n \geq 2.$$

DEFINITION 1.3.3. The *cellular chain complex* $\mathcal{C}_* X$ of a T-complex X with $X^1 = S^1_{\alpha_1}$ and attaching maps $f_{n+1}\colon S^n_{\alpha_{n+1}} \to X^n$ $(n \geq 1)$ is the positive chain complex in $\mathbf{ab}(T)$ defined as $\mathcal{C}_n X = \mathbb{Z}\langle A_n\rangle_{\alpha_n}$ with lower differential $d_2\colon \mathcal{C}_2 X \to \mathcal{C}_1 X$ given by Σf_2 and higher differentials $d_{n+1}\colon \mathcal{C}_{n+1} X \to \mathcal{C}_n X$ given by the following composites $(n \geq 2)$

$$S^n_{\alpha_{n+1}} \xrightarrow{f_{n+1}} X^n \longrightarrow X^n/X^{n-1} = S^n_{\alpha_n}.$$

The cellular chain complex is a functor from the category of T-complexes to the category of bounded chain complexes in $\mathbf{ab}(T)$ concentrated in positive degrees

$$\mathcal{C}_*\colon \mathbf{CW}^T \longrightarrow \mathbf{chain}(\mathbf{ab}(T)).$$

The category $\mathbf{chain}(\mathbf{ab}(T))$ is also an I-category with the usual cylinder of chain complexes, compare [**Bau99**] III.9.2, and $\mathcal{C}_*$ preserves all the structure. In particular $\mathcal{C}_*$ factors trough the respective homotopy categories.

The homology of a chain complex in $\mathbf{ab}(T)$ is defined via the Yoneda inclusion (1.2.1), in particular one can define the *proper homology* $\mathbf{ab}(T)$*-modules* of X

$$\mathcal{H}_n X = H_n \mathcal{C}_* X.$$

The *proper cohomology groups* of X with coefficients in an $\mathbf{ab}(T)$-module $\mathcal{M}$ are

$$H^n(X,\mathcal{M}) = H^n \operatorname{Hom}_{\mathbf{ab}(T)}(\mathcal{C}_*X,\mathcal{M}).$$

One of the most relevant consequences of Theorem 1.2.14 in proper homotopy theory is the following.

PROPOSITION 1.3.4. *The proper homology modules of a T-complex are finitely presented.*

Several classical theorems generalize to the proper homotopy invariants defined above. In particular there is a homological Whitehead theorem ([**BQ01**] VI.6) and a Blakers-Massey's excision theorem ([**BQ01**] VI.7.3). This Blakers-Massey theorem can be used to prove a Freudenthal suspension theorem for proper homotopy modules. Nevertheless its main consequence is the fact that the category $\mathbf{Topp}^T$ is a homological cofibration category under $\mathbf{S}^1(T)$ in the sense of [**Bau99**] V.1.1., see [**Bau99**] V.1.2. Homological cofibration categories are cofibration categories with a certain class of objects, satisfying certain axioms, which are used, together with their suspensions, to construct the analogue of CW-complexes. This extra structure allows to develop a rich obstruction theory for these complexes in an abstract setting. We will review some of this obstruction theory in Section 2.5 for the particular example $\mathbf{Topp}^T$, see [**Bau99**] for further details.

A T-complex X is *n-reduced* ($n \geq 1$) if $X^n = T$, the relation of this concept with the connectivity is established in the next proposition.

PROPOSITION 1.3.5. *Given an n-connected T-complex X there exists a homotopy equivalent one Y which is n-reduced. Moreover, one can assume that the dimension of Y does not exceed the maximum between the dimension of X and $n+2$.*

The first part of the statement is a particular case of [**BQ01**] IV.7.2. In fact the techniques used there are go back to classical results by J. H. C. Whitehead adapted to proper homotopy theory. Following carefully the proof of [**BQ01**] IV.7.2 one notices that the second part of the proposition is also proven there.

COROLLARY 1.3.6. *If X is an $(n-1)$-connected T-complex $\Gamma_k X = 0$ for all $2 \leq k \leq n$.*

For any T-complex X there is a sequence of $\mathbf{ab}(T)$-modules

$$\cdots \xrightarrow{j} \Pi_{n+1}(X^{n+1},X^n) \xrightarrow{\partial} \Pi_n X^n \xrightarrow{j} \Pi_n(X^n,X^{n-1}) \xrightarrow{\partial} \cdots$$
$$\cdots \xrightarrow{j} \Pi_3(X^3,X^2) \xrightarrow{\partial} \Pi_2 X^2 \xrightarrow{=} \Pi_2 X^2 \to 0,$$

which is exact in the proper homotopy modules of all pairs and also in the second $\Pi_2 X^2$. If X is 1-reduced the cellular chain complex $\mathcal{C}_*X$ coincides with the following chain complex of $\mathbf{ab}(T)$-modules, see [**BQ01**] VI.5.5,

$$\cdots \to \Pi_{n+1}(X^{n+1},X^n) \xrightarrow{j\partial} \Pi_n(X^n,X^{n-1}) \to \cdots \to \Pi_3(X^3,X^2) \xrightarrow{\partial} \Pi_2 X^2 \to 0,$$

since in this case $\Pi_n(X^n,X^{n-1}) = \Pi_n X^n/X^{n-1} = \Pi_n S^n_{\alpha_n} = \mathcal{C}_n X$ for all $n \geq 2$ so one can construct a long exact sequence involving proper homotopy, homology and Whitehead modules as J. H. C. Whitehead did in [**Whi50**], see [**BQ01**] VI.5.4,

$$\cdots \to \mathcal{H}_{n+1}X \overset{b_{n+1}}{\to} \Gamma_n X \xrightarrow{i_n} \Pi_n X \xrightarrow{h_n} \mathcal{H}_n X \to \cdots \to \Pi_2 X \overset{h_2}{\twoheadrightarrow} \mathcal{H}_2 X.$$

The *Hurewicz morphisms* in proper homotopy theory are defined to be h_n ($n \geq 2$). Moreover, following Whitehead's terminology we call the b_n ($n \geq 3$) *secondary boundary operators*.

The -r6-er Hurewicz theorem follows fron Whitehead's long exact sequence in proper homotopy theory, Proposition 1.3.5 and Corollary 1.3.6.

1.4. 1-dimensional spherical objects

The proper homotopy category of 1-dimensional spherical objects $\mathbf{S}^1(T)$ has a purely algebraic model, as it happens in higher dimensions, see Proposition 1.3.1. It is given by the following non-abelian version of the free T-controlled modules introduced in Section 1.2.

DEFINITION 1.4.1. A *free T-controlled group* is a pair $\langle A\rangle_\alpha$ where $\langle A\rangle$ is the free group with basis the discrete set A and $\alpha\colon A \to T$ is a proper map, the *height function.* A *controlled homomorphism* $\alpha\colon \langle A\rangle_\alpha \to \langle B\rangle_\beta$ is a homomorphism between the underlying groups such that for any neighborhood U of $\varepsilon \in \mathfrak{F}(T)$ in $\hat{T}$ there exists another one $V \subset U$ with $\varphi(\alpha^{-1}(V)) \subset \langle\beta^{-1}(U)\rangle$. Let $\mathbf{gr}(T)$ be the category of free T-controlled groups and controlled homomorphisms. There is an obvious abelianization full functor

$$ab\colon \mathbf{gr}(T) \longrightarrow \mathbf{ab}(T)\colon \langle A\rangle_\alpha \mapsto \mathbb{Z}\langle A\rangle_\alpha.$$

PROPOSITION 1.4.2. *There is an isomorphism of categories*

$$\mathbf{S}^1(T) \xrightarrow{\simeq} \mathbf{gr}(T)\colon S^1_\alpha \mapsto \langle A\rangle_\alpha$$

which makes commutative the following diagram

$$\begin{tikzcd} \mathbf{S}^1(T) \arrow[r, "\Sigma"] & \mathbf{S}^2(T) \\ \mathbf{gr}(T) \arrow[u, "\simeq"] \arrow[r, "ab"] & \mathbf{ab}(T) \arrow[u, "\simeq"'] \end{tikzcd}$$

The isomorphism is induced by the ordinary homotopy groups $\pi_1(S^1_\alpha, T) = \langle A\rangle$ in the underlying groups, and the isomorphism in the left of the square is in Proposition 1.3.1. See [**BQ01**] 4.20 for a proof.

CHAPTER 2

Quadratic functors in controlled algebra and proper homotopy theory

In this chapter we first recall the basic definitions and elemetary examples of quadratic functors. In the Section 2.2 we construct quadratic functors in controlled algebra which play an important role in this paper. These functors are applied in Section 2.3 to the computation of proper homotopy invariants. In the Section 2.4 we use the controlled exterior square to describe the category of free controlled groups of nilpotency class 2 as a linear extension. Finally in Section 2.5 we give an account of the obstruction theory in proper homotopy theory developed in [**BQ01**] from the perspective of the tower of categories in [**Bau99**] and we establish the links with the cohomology of categories. For this we use the controlled quadratic functors defined in the Section 2.2 and the computations in Section 2.3.

2.1. General quadratic functors

DEFINITION 2.1.1. A function between abelian groups $f\colon A \to B$ is *quadratic* if the function

$$[-,-]\colon A \times A \to B\colon (a,b) \mapsto [a,b] = f(a+b) - f(a) - f(b)$$

is bilinear.

A functor $F\colon \mathbf{A} \to \mathbf{B}$ between additive categories is a *quadratic functor* if it induces quadratic functions between morphism abelian groups.

Suppose that $\mathbf{B}$ is abelian. Let X and Y be two objects in $\mathbf{A}$ and

(2.1.2)
$$X \underset{i_1}{\overset{p_1}{\rightleftarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

the projections and inclusions of the factors of the direct sum. The *quadratic crossed effect* $F(X|Y)$ is defined as

$$F(X|Y) = \mathrm{Im}[[i_1p_1, i_2p_2]\colon F(X \oplus Y) \to F(X \oplus Y)].$$

In particular $[i_1p_1, i_2p_2]$ factors as follows

$$[i_1p_1, i_2p_2]\colon F(X \oplus Y) \overset{r_{12}}{\twoheadrightarrow} F(X|Y) \overset{i_{12}}{\hookrightarrow} F(X \oplus Y),$$

and there is an isomorphism

$$F(X) \oplus F(X|Y) \oplus F(Y) \simeq F(X \oplus Y).$$

Since F is quadratic $F(-|-)$ is a biadditive functor. Moreover, $F(-|-)$ vanishes if and only if F is additive.

We are specially interested in the following classical quadratic functors from abelian groups to abelian groups:

- Whitehead's functor Γ,

- exterior square $\wedge^2$,
- tensor square $\otimes^2$,
- and reduced tensor square $\hat{\otimes}^2$.

The functor Γ is characterized by the existence of a natural function $\gamma\colon A \to \Gamma A$ which is universal among all quadratic functions $f\colon A \to B$ with $f(a) = f(-a)$ for all $a \in A$. The exterior square $\wedge^2$ and the tensor square $\otimes^2$ are fairly well-known, and the reduced tensor square is determined by the push-out square in the following commutative diagram with exact rows and column

(2.1.3)
$$\begin{tikzcd}
\otimes^2 A \arrow[d, "{[-,-]}"] & & \\
\Gamma A \arrow[r, "\tau"] \arrow[d, two heads, "\sigma"] \arrow[dr, phantom, "\text{push}"] & \otimes^2 A \arrow[r, two heads, "q"] \arrow[d, two heads, "\bar{\sigma}"] & \wedge^2 A \arrow[d, equal] \\
A \otimes \mathbb{Z}_2 \arrow[r, hook, "\bar{\tau}"] & \hat{\otimes}^2 A \arrow[r, two heads, "\bar{q}"] & \wedge^2 A
\end{tikzcd}$$

Here $\tau(\gamma(a)) = a \otimes a$, $q(a \otimes b) = a \wedge b$, $\sigma(\gamma(a)) = a \otimes 1$ and $\bar{\sigma}(a \otimes b) = a \hat{\otimes} b$. One can check that $\bar{\tau}$ is always a monomorphism by using its naturality, the natural projection $A \twoheadrightarrow A \otimes \mathbb{Z}/2$, and the fact that all vector spaces (over the field with 2 elements) have a basis, but it is not completely trivial, i. e. it is a not so easy consequence of its definition.

The quadratic crossed effect of Γ, $\wedge^2$ and $\hat{\otimes}^2$ is the tensor product $\otimes$ and the structure homomorphisms i_{12} and r_{12} are given as follows,

$$\Gamma(A \oplus B) \overset{r_{12}}{\twoheadrightarrow} A \otimes B \overset{i_{12}}{\hookrightarrow} \Gamma(A \oplus B),$$

$$r_{12}(\gamma(i_1 a + i_2 b)) = a \otimes b, \quad i_{12}(a \otimes b) = [i_1 a, i_2, b];$$

$$\wedge^2(A \oplus B) \overset{r_{12}}{\twoheadrightarrow} A \otimes B \overset{i_{12}}{\hookrightarrow} \wedge^2(A \oplus B),$$

$$r_{12}((i_1 a + i_2 b) \wedge (i_1 a' + i_2 b')) = i_1 a \otimes i_2 b' - i_1 a' \otimes i_2 b, \quad i_{12}(a \otimes b) = i_1 a \wedge i_2 b;$$

$$\hat{\otimes}^2(A \oplus B) \overset{r_{12}}{\twoheadrightarrow} A \otimes B \overset{i_{12}}{\hookrightarrow} \hat{\otimes}^2(A \oplus B),$$

$$r_{12}((i_1 a + i_2 b) \hat{\otimes} (i_1 a' + i_2 b')) = i_1 a \otimes i_2 b' - i_1 a' \otimes i_2 b, \quad i_{12}(a \otimes b) = i_1 a \hat{\otimes} i_2 b.$$

The tensor square $\otimes^2$ is the square of a biadditive functor, the tensor product. All quadratic functors arising in this way have a quite obvious quadratic crossed effect, in particular for $\otimes^2$ we have

$$\otimes^2(A|B) = A \otimes B \oplus B \otimes A.$$

One can notice that $[-,-] = \Gamma(1,1) i_{12}$ and $\tau = r_{12}\Gamma(i_1 + i_2)$, therefore the whole diagram (2.1.3) is determined by the functor Γ.

The functor $\otimes^2$ sends free abelian groups to free abelian groups, since the tensor product of two free abelian groups is free abelian $\mathbb{Z}\langle A\rangle \otimes \mathbb{Z}\langle B\rangle \simeq \mathbb{Z}\langle A \times B\rangle$. The functors $\wedge^2$ and Γ also preserve free abelian groups. In order to describe bases of

$\wedge^2\mathbb{Z}\langle E\rangle$ and $\Gamma\mathbb{Z}\langle E\rangle$ we choose a total ordering $\preceq$ in E. The bases of $\wedge^2\mathbb{Z}\langle E\rangle$ and $\Gamma\mathbb{Z}\langle E\rangle$ are the sets $\wedge^2 E$ and ΓE, respectively, defined as follows,

- $\Gamma E = \{\gamma(e), [e_1, e_2]\,;\, e, e_1, e_2 \in E, e_1 \succ e_2\}$,
- $\wedge^2 E = \{e_1 \wedge e_2\,;\, e_1 \prec e_2 \in E\}$.

2.2. Extension of classical quadratic functors to controlled algebra

In this section we define controlled generalizations of the classical quadratic functors considered in the previous section. The consistency of these new constructions depends on topological properties of trees that we now recall.

Definition 2.2.1. Recall that an *arc* in a topological space is a subspace homeomorphic to a compact interval. Any two points in a tree $u, v \in T$ can be joined by a unique arc denoted by $\overline{uv}$. Moreover, if $w \in \overline{uv}$ the union of $\overline{uw}$ and $\overline{wv}$ by their common point is $\overline{uv}$.

It is well known that there is a unique metric d in T such that de distance between two adjacent vertices is 1 and every arc $\overline{uv}$ is isometric to the interval $[0, d(u,v)] \subset \mathbb{R}$.

Fixed a base-point $v_0 \in T$ we define the map

$$\ell\colon T \times T \longrightarrow T$$

as the unique one which satisfies $\overline{uv} \cap \overline{v_0\ell(u,v)} = \{\ell(u,v)\}$ for any $u, v \in T$. This map is obviously symmetric $\ell(u,v) = \ell(v,u)$.

Lemma 2.2.2. *The map ℓ is well defined.*

Proof. Given $u, v \in T$ there exists at least one $w \in T$ such that $\overline{uv} \cap \overline{v_0 w} = \{w\}$. We can take w to be a point in the compact space $\overline{uv}$ where the function $\overline{uv} \to \mathbb{R}\colon x \mapsto d(v_0, x)$ reaches the minimum value. Moreover, w is the unique point with this property. Suppose by the contrary that there where two points $w_1, w_2 \in T$ with $\overline{uv} \cap \overline{v_0 w_i} = \{w_i\}$ $(i = 1, 2)$. Then $\overline{v_0 w_1} \cup \overline{w_1 w_2} \cup \overline{w_2 v_0} \subset T$ would be a subspace homeomorphic to a circle, so T would not be a tree. $\square$

Definition 2.2.3. Given a tree T we fix a base-point $v_0 \in T$ and a total ordering in all discrete sets so that the next definitions make sense.

For any map from a discrete set $\alpha\colon A \to T$ we define the maps

- $\Gamma\alpha\colon \Gamma A \to T$,
- $\wedge^2\alpha\colon \wedge^2 A \to T$,

as follows $(a, a_1, a_2 \in A; a_1 \succ a_2)$

- $(\Gamma\alpha)(\gamma(a)) = \alpha(a)$,
- $(\Gamma\alpha)([a_1, a_2]) = \ell(\alpha(a_1), \alpha(a_2))$,
- $(\wedge^2\alpha)(a_2 \wedge a_1) = \ell(\alpha(a_2), \alpha(a_1))$.

If $\beta\colon B \to T$ is another map we also define

- $\alpha \otimes \beta = \ell(\alpha \times \beta)\colon A \times B \to T$.

The *T-controlled Whitehead functor*

$$\Gamma_T\colon \mathbf{ab}^b(T) \longrightarrow \mathbf{ab}^b(T)$$

is defined as the unique functor with

$$\Gamma_T\mathbb{Z}\langle A\rangle_\alpha = \mathbb{Z}\langle\Gamma A\rangle_{\Gamma\alpha}$$

which coincides with the ordinary Whitehead functor Γ in underlying abelian groups.

Similarly the *T-controlled exterior square*

$$\wedge_T^2 \colon \mathbf{ab}^b(T) \longrightarrow \mathbf{ab}^b(T)$$

is the unique functor with

$$\wedge_T^2 \mathbb{Z}\langle A\rangle_\alpha = \mathbb{Z}\langle \wedge^2 A\rangle_{\wedge^2\alpha}$$

which coincides with the ordinary exterior square $\wedge^2$ in underlying abelian groups.

Finally the *T-controlled tensor product* is the functor

$$- \otimes_T - \colon \mathbf{ab}^b(T) \times \mathbf{ab}^b(T) \longrightarrow \mathbf{ab}^b(T)$$

defined as

$$\mathbb{Z}\langle A\rangle_\alpha \otimes_T \mathbb{Z}\langle B\rangle_\beta = \mathbb{Z}\langle A \times B\rangle_{\alpha\otimes\beta}$$

which coincides with the ordinary tensor product in underlying abelian groups.

It is not completely trivial that the functors Γ_T, $\wedge_T^2$ and $\otimes_T$ are well defined. One needs to check, for example for $\wedge_T^2$, that given a controlled homomorphism $\varphi \colon \mathbb{Z}\langle A\rangle_\alpha \to \mathbb{Z}\langle B\rangle_\beta$ the abelian group homomorphism $\wedge^2\varphi \colon\ \wedge^2 \mathbb{Z}\langle A\rangle \to \wedge^2\mathbb{Z}\langle B\rangle$ induces a controlled homomorphism $\wedge^2\varphi = \wedge_T^2\varphi \colon\ \wedge_T^2 \mathbb{Z}\langle A\rangle_\alpha \to \wedge_T^2\mathbb{Z}\langle B\rangle_\beta$.

PROPOSITION 2.2.4. *The functors Γ_T, $\wedge_T^2$ and $\otimes_T$ are well defined.*

In the proof of this proposition we shall use the following nice property of the map ℓ.

LEMMA 2.2.5. *Given two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in T, if $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = x \in \hat{T}$ then $\lim\limits_{n\to\infty} \ell(x_n, y_n) = x$. Moreover, if $x \in \mathfrak{F}(T)$ the converse also holds.*

REMARK 2.2.6. It is well known that there is a basis of neighborhoods of the space of Freudenthal ends of a tree $\mathfrak{F}(T)$ in the Freudenthal compactification $\hat{T}$ given by the sets $T_v \sqcup T_v^{\mathfrak{F}}$ $(v \in T)$ where $T_v \subset T$ is the subtree formed by those points $u \in T$ such that $v \in \overline{v_0 u}$, see [**BQ01**] II.1.14, III.1.2 and III.1.3. These subtrees have the following property.

LEMMA 2.2.7. *$x, y \in T_w$ if and only if $\ell(x,y) \in T_w$.*

PROOF. Clearly $\Rightarrow$ holds because $\ell(x,y) \in \overline{xy}$ and $\overline{xy} \subset T_w$ since T_w is a tree. On the other hand by Definition 2.2.1 $\ell(x,y) \in \overline{v_0 x} \cap \overline{v_0 y}$ hence if $\ell(x,y) \in T_w$ then $w \in \overline{v_0\ell(x,y)} \subset \overline{v_0 x} \cap \overline{v_0 y}$ so $x, y \in T_w$. $\square$

PROOF OF LEMMA 2.2.5. For $x \in \mathfrak{F}(T)$ Lemma 2.2.5 is an immediate consequence of Lemma 2.2.7. If $x \in T$ then $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$ and $d(x_n, y_n) = d(x_n, \ell(x_n, y_n)) + d(\ell(x_n, y_n), y_n)$ $(n \in \mathbb{N})$ so

$$\lim_{n\to\infty} d(x_n, \ell(x_n, y_n)) = 0 = \lim_{n\to\infty} d(y_n, \ell(x_n, y_n))$$

and therefore $\lim\limits_{n\to\infty} \ell(x_n, y_n) = x$. $\square$

PROOF OF PROPOSITION 2.2.4. We will make the proof for $\wedge^2_T$ and leave to the reader the cases Γ_T and $\otimes_T$.

Consider sequences $\left\{a_n^1 \wedge a_n^2\right\}_{n\geq 0}$ and $\left\{b_n^1 \wedge b_n^2\right\}_{n\geq 0}$ of elements in $\wedge^2 A$ and $\wedge^2 B$ respectively such that $\lim\limits_{n\to\infty} \ell(\alpha(a_n^1), \alpha(a_n^2)) = \varepsilon \in \mathfrak{F}(T)$ in $\hat{T}$ and $b_n^1 \wedge b_n^2$ appears with non-trivial coefficient in the linear expansion of $\wedge^2\varphi(a_n^1 \wedge a_n^2)$. We can therefore take permutations o_n of $\{1,2\}$ $(n \geq 0)$ such that $b_n^{o_n(i)}$ appears with non-trivial coefficient in the linear expansion of $\varphi(a_n^i)$. By Lemma 2.2.5 $\lim\limits_{n\to\infty} \alpha(a_n^i) = \varepsilon$ $(i = 1,2)$ so $\lim\limits_{n\to\infty} \beta(b_n^{o(i)}) = \varepsilon$ because φ is controlled, see Lemma 1.2.8. This implies that also $\lim\limits_{n\to\infty} \beta(b_n^i) = \varepsilon$ $(i = 1,2)$, hence by Lemma 2.2.5 $\lim\limits_{n\to\infty} \ell(\beta(b_n^1), \beta(b_n^2)) = \varepsilon$, so $\wedge^2\varphi$ is controlled by Lemma 1.2.8. □

The following proposition considers the compatibility of the functors Γ_T, $\wedge^2_T$ and $\otimes_T$ with the "change of tree" functors determined by a proper map.

PROPOSITION 2.2.8. *Given a proper map between two trees $f\colon T \to T'$ there are natural transformations*

- $\mathbb{F}^f\Gamma_T \to \Gamma_{T'}\mathbb{F}^f$,
- $\mathbb{F}^f\wedge^2_T \to \wedge^2_{T'}\mathbb{F}^f$,
- $\mathbb{F}^f(-\otimes_T -) \to \mathbb{F}^f(-)\otimes_{T'}\mathbb{F}^f(-)$,

which are natural equivalences provided $\mathfrak{F}(f)\colon \mathfrak{F}(T) \hookrightarrow \mathfrak{F}(T')$ is injective. Moreover, in any case

- $\mathbb{F}^f(-\otimes \mathbb{Z}/2) \simeq (\mathbb{F}^f -)\otimes \mathbb{Z}/2.$

PROOF. As in the proof of Proposition 2.2.4 we shall leave the cases Γ_T and $\otimes_T$ to the reader. The case $-\otimes\mathbb{Z}/2$ follows from Propositions 1.2.12 and 1.2.21 and the right-exactness of the additive functor $\mathbb{F}^f_*$, since $-\otimes\mathbb{Z}/2$ is the cokernel of the multiplication by 2.

Let $\ell'\colon T' \times T' \to T'$ be the function associated to a base-point $v_0' \in T'$ as in Definition 2.2.1. Notice that for the definition of $\wedge^2_{T'}$ we choose total orderings $\leq$ in discrete sets A which are possibly different from those orders $\preceq$ used to define $\wedge^2_T$. In order to distinguish the two possible definitions of the set $\wedge^2 A$ we will write $\wedge^2_{\leq} A$ and $\wedge^2_{\preceq} A$. Moreover, notice that the definition of $\wedge^2\alpha$ depends on whether the target of α is T or T'.

We claim that for any possibly big free T-controlled $\mathbb{Z}$-module $\mathbb{Z}\langle A\rangle_\alpha$ the identity in $\wedge^2\mathbb{Z}\langle A\rangle$ induces a controlled homomorphism

$$\varphi\colon \mathbb{F}^f \wedge^2_T \mathbb{Z}\langle A\rangle_\alpha = \mathbb{Z}\langle\wedge^2_{\preceq} A\rangle_{f(\wedge^2\alpha)} \longrightarrow \mathbb{Z}\langle\wedge^2_{\leq} A\rangle_{\wedge^2(f\alpha)} = \wedge^2_{T'}\mathbb{F}^f\mathbb{Z}\langle A\rangle_\alpha.$$

This morphism sends $a_1 \wedge a_2 \in \wedge^2_{\preceq} A$ to $a_1 \wedge a_2$ if $a_1 < a_2$ or to $-a_2 \wedge a_1$ in other case.

Suppose that $\left\{a_n^1 \wedge q_n^2\right\}_{n\geq 0}$ is a sequence of elements in $\wedge^2_{\preceq} A$ such that in $\hat{T}'$

$$\lim_{n\to\infty} f\ell(\alpha(a_n^1), \alpha(a_n^2)) = \varepsilon \in \mathfrak{F}(T').$$

If o_n is the permutation of $\{1,2\}$ such that $a_n^{o(1)} \wedge a_n^{o(2)}$ belongs to $\wedge^2_{\leq} A$ then we have to prove that $\lim\limits_{n\to\infty} \ell'((f\alpha)(a_n^{o(1)}), (f\alpha)(a_n^{o(2)})) = \varepsilon$. By the symmetry of ℓ' this is the same as showing that $\lim\limits_{n\to\infty} \ell'((f\alpha)(a_n^1), (f\alpha)(a_n^2)) = \varepsilon$. We have that the accumulation points of $\left\{\ell(\alpha(a_n^1), \alpha(a_n^2))\right\}_{n\geq 0}$ are contained in $\mathfrak{F}(f)^{-1}(\varepsilon)$, therefore by Lemma 2.2.5 the accumulation points of $\left\{\alpha(a_n^1)\right\}_{n\geq 0}$ and $\left\{\alpha(a_n^2)\right\}_{n\geq 0}$ are all in $\mathfrak{F}(f)^{-1}(\varepsilon)$, so $\lim_{n\to\infty}(f\alpha)(a_n^i) = \varepsilon$ $(i = 1,2)$. Again by Lemma 2.2.5 $\lim_{n\to\infty}\ell'((f\alpha)(a_n^1),(f\alpha)(a_n^2)) = \varepsilon$ so φ is controlled by Lemma 1.2.8.

If $\mathfrak{F}(f)$ is injective we claim that the identity in $\wedge^2\mathbb{Z}\langle A\rangle$ also induces a controlled homomorphism

$$\psi\colon\ \wedge^2_{T'}\mathbb{F}^f\mathbb{Z}\langle A\rangle_\alpha = \mathbb{Z}\langle\wedge^2_{\leq}A\rangle_{\wedge^2(f\alpha)} \longrightarrow \mathbb{Z}\langle\wedge^2_{\leq}A\rangle_{f(\wedge^2\alpha)} = \mathbb{F}^f\wedge^2_T\mathbb{Z}\langle A\rangle_\alpha.$$

Given a sequence $\left\{a_n^1\wedge a_n^2\right\}$ of elements in $\wedge^2_{\geq}A$ such that in $\hat{T}'$

$$\lim_{n\to\infty}\ell'((f\alpha)(a_n^1),(f\alpha)(a_n^2)) = \varepsilon\in\mathfrak{F}(T')$$

we now have to prove that $\lim_{n\to\infty}f\ell(\alpha(a_n^1),\alpha(a_n^2)) = \varepsilon$ as well. By Lemma 2.2.5 $\lim_{n\to\infty}(f\alpha)(a_n^i) = \varepsilon$ $(i = 1,2)$. Moreover, since $\mathfrak{F}(f)$ is injective $\lim_{n\to\infty}\alpha(a_n^i) = \varepsilon'$ is the unique $\varepsilon'\in\mathfrak{F}(T)$ such that $\mathfrak{F}(\varepsilon') = \varepsilon$ so again by Lemma 2.2.5

$$\lim_{n\to\infty}\ell(\alpha(a_n^1),\alpha(a_n^2)) = \varepsilon',$$

in particular $\lim_{n\to\infty}f\ell(\alpha(a_n^1),\alpha(a_n^2)) = \mathfrak{F}(f)(\varepsilon') = \varepsilon$ and ψ is controlled by Lemma 1.2.8. $\square$

In Definition 2.2.3 we have made several arbitrary choices to define three functors, however up to natural equivalence the functors do not depend on these choices, as the following corollary shows.

COROLLARY 2.2.9. *Up to natural equivalence the functors Γ_T, $\wedge^2_T$ and $\otimes_T$ do not depend on the choice of the base-point $v_0\in T$ and the total orderings on discrete sets.*

This corollary can be regarded as a special case of Proposition 2.2.8 for the identity map $f = 1\colon T\to T$.

The functors Γ_T and $\wedge^2_T$ are quadratic because they are quadratic in the underlying abelian groups. Similarly $\otimes_T$ is biadditive. We shall be interested in the composition of these functors with the full inclusion of non-big objects $\mathbf{ab}(T)\subset\mathbf{ab}^b(T)$ and the faithful functor $\mathbf{ab}^b(T)\hookrightarrow\mathbf{mod}(\mathbf{ab}(T))$ in (1.2.11) which we will regard as the inclusion of a subcategory. We shall use the same names for these composites

$$\begin{aligned}\Gamma_T\colon \mathbf{ab}(T)&\longrightarrow\mathbf{mod}(\mathbf{ab}(T)),\\ \wedge^2_T\colon \mathbf{ab}(T)&\longrightarrow\mathbf{mod}(\mathbf{ab}(T)),\\ -\otimes_T-\colon \mathbf{ab}(T)\times\mathbf{ab}(T)&\longrightarrow\mathbf{mod}(\mathbf{ab}(T)).\end{aligned}\tag{2.2.10}$$

These functors satisfy the properties stated in the following proposition where $\otimes^2_T$ denotes the square of the bifunctor $\otimes_T$.

PROPOSITION 2.2.11. *The functor $\otimes_T$ is the quadratic crossed effect of Γ_T and $\wedge^2_T$ in (2.2.10). Moreover, there are two exact sequences of natural transformations*

(1) $\Gamma_T\overset{\tau_T}{\hookrightarrow}\otimes^2_T\overset{q_T}{\twoheadrightarrow}\wedge^2_T$,
(2) $\otimes^2_T\xrightarrow{[-,-]_T}\Gamma_T\overset{\sigma_T}{\twoheadrightarrow}-\otimes\mathbb{Z}/2$.

Indeed (1) splits non-naturally; τ_T verifies $\tau_T = r_{12}\Gamma_T(i_1+i_2)$, where i_1 and i_2 are the inclusions of the factors of the coproduct; and $[-,-]_T = \Gamma_T(1,1)i_{12}$. Furthermore, the functors Γ_T, $\wedge^2_T$, $\otimes_T$ and $-\otimes\mathbb{Z}/2$ are compatible with the "change of tree" *functors in the sense of Proposition 2.2.8, as well as the natural transformations* τ_T, q_T, $[-,-]_T$ *and* σ_T.

PROOF. The last part of the statement follows from Propositions 1.2.12 and 2.2.8.

It is immediate to check that the structure maps of the quadratic crossed effects of Γ_T and $\wedge^2_T$ coincide with those of the ordinary functors Γ and $\wedge^2$ in the underlying abelian groups.

The exact sequence (1) is given by the upper row of (2.1.3) in underlying abelian groups. In particular the formula $\tau_T = r_{12}\Gamma_T(i_1, i_2)$ is satisfied. The splitting $s\colon \wedge^2_T \mathbb{Z}\langle A\rangle_\alpha \hookrightarrow \otimes^2_T\mathbb{Z}\langle A\rangle_\alpha$ of q_T and the retraction $r\colon \otimes^2_T \mathbb{Z}\langle A\rangle_\alpha \twoheadrightarrow \Gamma_T\mathbb{Z}\langle A\rangle_\alpha$ of τ_T depend on a given total order $\preceq$ in A, more precisely $s(a_1 \wedge a_2) = a_1 \otimes a_2$, $r(a_1 \otimes a_2) = 0$, $r(a_2 \otimes a_1) = [a_2, a_1]$ and $r(a \otimes a) = \gamma(a)$ $(a, a_1, a_2 \in A, a_1 \prec a_2)$.

Moreover, $[-,-]_T$ is also given by the homomorphism $[-,-]$ in (2.1.3) in underlying abelian groups, in particular the equality $[-,-]_T = \Gamma_T(1,1)i_{12}$ holds. In order to define σ_T we consider the non-natural controlled homomorphisms $\varsigma\colon \Gamma_T\mathbb{Z}\langle A\rangle_\alpha \to \mathbb{Z}\langle A\rangle_\alpha$ defined as $\varsigma(\gamma(a)) = a$ and $\varsigma([a_1, a_2]) = 0$ $(a, a_1, a_2 \in A, a_1 \succ a_2)$. Now we define the $\mathbf{ab}(T)$-module morphism $\sigma_T\colon \Gamma_T\mathbb{Z}\langle A\rangle_\alpha \to \mathbb{Z}\langle A\rangle_\alpha \otimes \mathbb{Z}/2$ as

$$\sigma_T = \hat{p}\varsigma_*\colon \mathrm{Hom}_{\mathbf{ab}^b(T)}(-, \Gamma_T\mathbb{Z}\langle A\rangle_\alpha) \to \mathrm{Hom}_{\mathbf{ab}^b(T)}(-, \mathbb{Z}\langle A\rangle_\alpha) \otimes \mathbb{Z}/2,$$

where $\hat{p}$ is the natural projection $\hat{p}\colon 1 \twoheadrightarrow - \otimes \mathbb{Z}/2$ in the category of abelian groups. This is an epimorphism since ς is an epimorphism which admits a section $j\colon \mathbb{Z}\langle A\rangle_\alpha \hookrightarrow \Gamma_T\mathbb{Z}\langle A\rangle_\alpha$ defined as $j(a) = \gamma(a)$ $(a \in A)$. The naturality of σ_T can be easily checked by using the naturality of σ in (2.1.3), therefore it is only left to check the exactness of (2) in the middle.

Given a controlled homomorphism $\varphi\colon \mathbb{Z}\langle B\rangle_\beta \to \Gamma_T\mathbb{Z}\langle A\rangle_\alpha$, $\sigma_T\varphi = 0$ if and only if $(\varsigma\varphi)\otimes\mathbb{Z}/2 = 0$. This means that the coefficient of $\gamma(a)$ in the linear expansion of $\varphi(b)$ is even for all $a \in A$ and $b \in B$, so we can define a controlled homomorphism $\psi\colon \mathbb{Z}\langle B\rangle_\beta \to \Gamma_T\mathbb{Z}\langle A\rangle_\alpha$ in such a way that the coefficient of $\psi(b)$ in $\gamma(a)$ is half the corresponding coefficient in $\varphi(b)$ and the coefficients of $[a_1, a_2]$ in $\psi(b)$ and $\varphi(b)$ coincide for all $a_1 \succ a_2$. In fact by construction φ is controlled if and only if ψ is. The controlled homomorphism $k\colon \Gamma_T\mathbb{Z}\langle A\rangle_\alpha \to \otimes^2_T\mathbb{Z}\langle A\rangle_\alpha$ defined by $k(\gamma(a)) = a\otimes a$ and $k([a_1, a_2]) = a_1 \otimes a_2$ $(a, a_1, a_2 \in A; a_1 \succ a_2)$ satisfies $([-,-]_Tk)(\gamma(a)) = 2\gamma(a)$ and $([-,-]_Tk)([a_1, a_2]) = [a_1, a_2]$, so $\varphi = [-,-]_Tk\psi$ as required. $\square$

DEFINITION 2.2.12. A quadratic functor $F\colon \mathbf{A} \to \mathbf{B}$ between abelian categories is *right-exact* if for any exact sequence in $\mathbf{A}$

$$X \xrightarrow{f} Y \overset{g}{\twoheadrightarrow} Z$$

the following sequence in $\mathbf{B}$ is also exact

$$F(X) \oplus F(X|Y) \xrightarrow{\zeta} F(Y) \overset{F(g)}{\twoheadrightarrow} F(Z), \quad \zeta = (F(f), F(f,1)i_{12}).$$

If F is right-exact its quadratic crossed effect $F(-|-)$ is exact in each variable in the usual additive sense.

PROPOSITION 2.2.13. *If* $F\colon \mathbf{A} \to \mathbf{B}$ *is a quadratic functor from a small additive category* $\mathbf{A}$ *to an abelian category* $\mathbf{B}$ *with exact filtered colimits, up to natural equivalence there exists a unique extension* $F\colon \mathbf{mod}(\mathbf{A}) \to \mathbf{B}$ *of* F *through the Yoneda inclusion* $\mathbf{A} \hookrightarrow \mathbf{mod}(\mathbf{A})$ *in (1.2.1) which is right-exact, quadratic, and preserves* *filtered colimits. Moreover, if* $G\colon \mathbf{A} \to \mathbf{B}$ *is another functor and* $\mu\colon F \to G$ *is a natural transformation* μ *extends uniquely to* $\mathbf{mod}(\mathbf{A})$*.*

REMARK 2.2.14. It is not difficult to obtain a direct proof of this proposition, however it is a tedious work to carry out all necessary verifications and details are not relevant for the rest of this paper, for this reason we leave them to the interested reader. The idea is to extend the functor $F\colon \mathbf{A} \to \mathbf{B}$ to f. p. $\mathbf{A}$-modules by right-exactness and to all $\mathbf{A}$-modules by using the fact that any $\mathbf{A}$-module is a filtered colimit of f. p. ones. The uniqueness up to natural equivalence follows immediately. The extended functor $F\colon \mathbf{mod}(\mathbf{A}) \to \mathbf{B}$ is quadratic since it is quadratic over f. g. free ones and f. g. free $\mathbf{A}$-modules are small projective generators of $\mathbf{mod}(\mathbf{A})$. The extension of the natural transformation μ can be constructed at the same time as the extensions of the functors F and G by using the same arguments.

Alternatively one can check that the extension of F is the André homology functor $H_0(-,F)\colon \mathbf{mod}(\mathbf{A}) \to \mathbf{B}$ in the sense of [**And67**]. In fact a quadratic functor $M\colon \mathbf{mod}(\mathbf{A}) \to \mathbf{B}$ is right-exact in the sense of Definition 2.2.12 if and only if its derived functor $L_0M(-,0)$ in the sense of Dold and Puppe [**DP61**] coincides with M. On the other hand by [**And67**] 1.1 and 3.3 $L_0H_0(-,F)(-,0) = H_0(-,H_0(-,F)_{|\mathbf{A}}) = H_0(-,F)$ therefore $H_0(-,F)$ is right-exact. The functor $H_0(-,F)$ preserves filtered colimits by [**Ulm69**] 14 (c). As an immediate consequence of this characterization of extensions in terms of André homology we observe that whenever we have an exact sequence of natural transformations

$$F \to G \twoheadrightarrow N$$

between functors $\mathbf{A} \to \mathbf{B}$ the extended sequence of functors $\mathbf{mod}(\mathbf{A}) \to \mathbf{B}$ is exact in the same way.

We use Proposition 2.2.13 to extend the quadratic functors Γ_T and $\wedge^2_T$ in (2.2.10) to the whole category of $\mathbf{ab}(T)$-modules.

$$\wedge^2_T, \Gamma_T\colon \mathbf{mod}(\mathbf{ab}(T)) \longrightarrow \mathbf{mod}(\mathbf{ab}(T)). \tag{2.2.15}$$

Now we can define the T-controlled tensor product of $\mathbf{ab}(T)$-modules

$$-\otimes_T -\colon \mathbf{mod}(\mathbf{ab}(T)) \times \mathbf{mod}(\mathbf{ab}(T)) \longrightarrow \mathbf{mod}(\mathbf{ab}(T)). \tag{2.2.16}$$

as the quadratic crossed effect of either Γ_T or $\wedge^2_T$. These two quadratic functors have the same quadratic crossed effect since their crossed effects are also right-exact and preserve filtered colimits in each variable, and both coincide with the T-controlled tensor product in (2.2.10) over f. g. free $\mathbf{ab}(T)$-modules by Proposition 2.2.11. In particular $\otimes_T$ in (2.2.16) is an extension of $\otimes_T$ in (2.2.10).

PROPOSITION 2.2.17. *There are two exact sequences of natural transformations involving the functors in (2.2.15) and (2.2.16)*

$$\Gamma_T \stackrel{\tau_T}{\to} \otimes^2_T \stackrel{q_T}{\twoheadrightarrow} \wedge^2_T,$$
$$\otimes^2_T \stackrel{[-,-]_T}{\longrightarrow} \Gamma_T \stackrel{\sigma_T}{\twoheadrightarrow} -\otimes \mathbb{Z}/2,$$

such that $\tau_T = r_{12}\Gamma_T(i_1+i_2)$ *and* $[-,-]_T = \Gamma_T(1,1)i_{12}$*. Moreover, the functors* Γ_T*,* $\wedge^2_T$*,* $\otimes_T$ *and* $-\otimes\mathbb{Z}/2$ *are compatible with the "change of tree" functors in the sense of Proposition 2.2.8, as well as the natural transformations* τ_T*,* q_T*,* $[-,-]_T$ *and* σ_T*.*

This proposition follows from Propositions 2.2.11 and 2.2.13 and Remark 2.2.14.

DEFINITION 2.2.18. The *T-controlled reduced tensor square*

$$\hat{\otimes}^2_T\colon \mathbf{mod}(\mathbf{ab}(T)) \longrightarrow \mathbf{mod}(\mathbf{ab}(T))$$

is defined by the following natural push-out

$$\begin{tikzcd} \Gamma_T \arrow[r, "\tau_T"] \arrow[d, "\sigma_T"'] \arrow[dr, phantom, "\text{push}"] & \otimes^2_T \arrow[d, "\bar{\sigma}_T"] \\ -\otimes \mathbb{Z}/2 \arrow[r, "\bar{\tau}_T"] & \hat{\otimes}^2_T \end{tikzcd}$$

In particular $\hat{\otimes}^2_T$ is quadratic, right-exact and preserves filtered colimits.

This definition completes the extension of (2.1.3) to the category of $\mathbf{ab}(T)$-modules, that is, we have a commutative diagram of natural transformations with exact rows and column

(2.2.19)
$$\begin{tikzcd} \otimes^2_T \arrow[d, "{[-,-]_T}"] & & \\ \Gamma_T \arrow[r, "\tau_T"] \arrow[d, two heads, "\sigma_T"] \arrow[dr, phantom, "\text{push}"] & \otimes^2_T \arrow[r, two heads, "q_T"] \arrow[d, two heads, "\bar{\sigma}_T"] & \wedge^2_T \arrow[d, equal] \\ -\otimes \mathbb{Z}_2 \arrow[r, "\bar{\tau}_T"] & \hat{\otimes}^2_T \arrow[r, two heads, "\bar{q}_T"] & \wedge^2_T \end{tikzcd}$$

Here τ_T and σ_T, and therefore the whole diagram, are determined by the functor Γ_T, see Proposition 2.2.17, as it happened in the ordinary case (2.1.3).

PROPOSITION 2.2.20. *Given a proper map between two trees $f\colon T \to T'$ there is a natural transformation*

$$\mathbb{F}^f \hat{\otimes}^2_T \longrightarrow \hat{\otimes}^2_{T'} \mathbb{F}^f$$

which is a natural equivalence provided $\mathfrak{F}(f)\colon \mathfrak{F}(T) \to \mathfrak{F}(T')$ is injective. The natural transformations $\bar{\tau}_T$, $\bar{\sigma}_T$ and $\bar{q}_T$ are also compatible with "change of tree" functors. Moreover, $\otimes_T$ is the quadratic crossed effect of $\hat{\otimes}^2_T$.

PROOF. The part concerning "change of tree" functors follows from Definition 2.2.18 and Proposition 2.2.17. The natural transformation $\bar{q}_T$ in (2.2.19) induces an equivalence between the crossed effects because $-\otimes \mathbb{Z}/2$ is additive. $\square$

The following proposition shows that $\otimes_T$, $\wedge^2_T$ and $\hat{\otimes}^2_T$ restrict to the full subcategory of $\mathbf{vect}(T)$-modules.

PROPOSITION 2.2.21. *There are natural equivalences*

(1) $(\Gamma_T-)\otimes \mathbb{Z}/2 \simeq (\Gamma_T(-\otimes \mathbb{Z}_2))\otimes \mathbb{Z}/2$,
(2) $(-\otimes_T -)\otimes \mathbb{Z}/2 \simeq (-)\otimes_T (-\otimes \mathbb{Z}/2)$,
(3) $(\wedge^2_T-)\otimes \mathbb{Z}/2 \simeq \wedge^2_T(-\otimes \mathbb{Z}/2)$,
(4) $(\hat{\otimes}^2_T-)\otimes \mathbb{Z}/2 \simeq \hat{\otimes}^2_T(-\otimes \mathbb{Z}/2)$.

In particular the functors $(\Gamma_T-)\otimes \mathbb{Z}/2$, $\otimes_T$, $\wedge^2_T$ and $\hat{\otimes}^2_T$ restrict to the category of $\mathbf{vect}(T)$-modules.

Proof. All these isomorphisms are induced by the natural projection $\hat{p}\colon 1 \twoheadrightarrow - \otimes \mathbb{Z}/2$. A free T-controlled $\mathbb{F}_2$-module $\mathbb{F}_2\langle A\rangle_\alpha = \mathbb{Z}\langle A\rangle_\alpha \otimes \mathbb{Z}/2$ is the cokernel of the multiplication-by-2 controlled homomorphism $2\colon \mathbb{Z}\langle A\rangle_\alpha \to \mathbb{Z}\langle A\rangle_\alpha$. Since Γ_T is right-exact and coincides on $\mathbf{ab}(T)$ with the ordinary Whitehead functor Γ in underlying abelian groups we obtain an exact sequence

$$\Gamma_T\mathbb{Z}\langle A\rangle_\alpha \oplus \mathbb{Z}\langle A\rangle_\alpha \otimes_T \mathbb{Z}\langle A\rangle_\alpha \xrightarrow{(4,2[-,-]_T)} \Gamma_T\mathbb{Z}\langle A\rangle_\alpha \overset{\Gamma_T\hat{p}}{\twoheadrightarrow} \Gamma_T\mathbb{F}_2\langle A\rangle_\alpha,$$

therefore we clearly obtain the desired isomorphism by applying $-\otimes\mathbb{Z}/2$

$$(\Gamma_T\hat{p})\otimes\mathbb{Z}/2\colon (\Gamma_T\mathbb{Z}\langle A\rangle_\alpha)\otimes\mathbb{Z}/2 \xrightarrow{\simeq} (\Gamma_T\mathbb{F}_2\langle A\rangle_\alpha)\otimes\mathbb{Z}/2.$$

The isomorphism (1) for f. p. $\mathbf{ab}(T)$-modules follows now from right-exactness, and for arbitrary $\mathbf{ab}(T)$-modules from the fact that they are filtered colimits of f. p. ones.

The isomorphism (2) follows easily from the biadditivity and right exactness properties of $\otimes_T$. The rest of isomorphisms follow from (1), (2) and the properties of (2.2.19). $\square$

The constructions of $\otimes_T$, $\wedge_T^2$ and Γ_T allow us to compute these functors from projective resolutions in a very convenient way. We do not have in general such a method for $\hat{\otimes}_T^2$, however as a consequence of the following proposition we can obtain an analogous method when we restrict to $\mathbf{vect}(T)$-modules. In its statement we use the following notation. Given a total ordering $\preceq$ on a discrete set A we define the set

$$\hat{\otimes}^2 A = \left\{a_1\hat{\otimes}a_2\,;\, a_1 \preceq a_2\right\},$$

and given a function $\alpha\colon A\to T$ we define another one

$$\hat{\otimes}^2\alpha\colon \hat{\otimes}^2 A \to T,\ \ (\hat{\otimes}^2\alpha)(a_1\hat{\otimes}a_2) = \ell(\alpha(a_1),\alpha(a_2)).$$

Proposition 2.2.22. *The restriction of $\hat{\otimes}_T^2$ to $\mathbf{vect}(T)$ factors through the functor $\hat{\otimes}_T^2\colon \mathbf{vect}(T)\to\mathbf{vect}^b(T)$ which is $\hat{\otimes}_T^2\mathbb{F}_2\langle A\rangle_\alpha = \mathbb{F}_2\langle\hat{\otimes}^2 A\rangle_{\hat{\otimes}^2\alpha}$ on objects and coincides with the ordinary reduced tensor square on underlying $\mathbb{F}_2$-vector spaces. Moreover, the natural transformations $\bar{\tau}_T$, $\bar{\sigma}_T$, $\bar{q}_T$ and the structure morphisms of the quadratic crossed effect $\otimes_T$ over $\mathbf{vect}(T)$ also coincides with those of the ordinary reduced tensor square on underlying $\mathbb{F}_2$-vector spaces.*

Proof. Since $-\otimes\mathbb{Z}/2$ is right-exact and $\mathbb{F}_2\langle A\rangle_\alpha = \mathbb{Z}\langle A\rangle_\alpha\otimes\mathbb{Z}/2$, by Proposition 2.2.21 $\hat{\otimes}_T^2\mathbb{F}_2\langle A\rangle_\alpha$ is also the following push-out

$$\begin{tikzcd}
(\Gamma_T\mathbb{Z}\langle A\rangle_\alpha)\otimes\mathbb{Z}/2 \arrow[r, "\tau_T\otimes\mathbb{Z}/2"] \arrow[d, "\sigma_T\otimes\mathbb{Z}/2"'] \arrow[dr, phantom, "\text{push}"] & (\otimes_T^2\mathbb{Z}\langle A\rangle_\alpha)\otimes\mathbb{Z}/2 \arrow[d, "\bar{\sigma}_T"] \\
\mathbb{Z}\langle A\rangle_\alpha\otimes\mathbb{Z}/2 \arrow[r, "\bar{\tau}_T"] & \hat{\otimes}_T^2\mathbb{F}_2\langle A\rangle_\alpha
\end{tikzcd}$$

By Propositions 1.2.22 and 2.2.11 we have that $(a, a_1, a_2 \in A; a_1 \succ a_2)$

$$\begin{aligned}
(\Gamma_T \mathbb{Z}\langle A\rangle_\alpha) \otimes \mathbb{Z}/2 &= \mathbb{F}_2\langle \Gamma A\rangle_{\Gamma\alpha},\\
(\otimes^2_T \mathbb{Z}\langle A\rangle_\alpha) \otimes \mathbb{Z}/2 &= \mathbb{F}_2\langle A\times A\rangle_{\alpha\otimes\alpha},\\
\mathbb{Z}\langle A\rangle_\alpha \otimes \mathbb{Z}/2 &= \mathbb{F}_2\langle A\rangle_\alpha,\\
(\tau_T \otimes \mathbb{Z}/2)(\gamma(a)) &= a\otimes a,\\
(\tau_T \otimes \mathbb{Z}/2)([a_1,a_2]) &= a_1\otimes a_2 + a_2\otimes a_1,\\
(\sigma_T \otimes \mathbb{Z}/2)(\gamma(a)) &= a,\\
(\sigma_T \otimes \mathbb{Z}/2)([a_1,a_2]) &= 0.
\end{aligned}$$

Now it is easy to see that the homomorphisms $\bar{\tau}_T\colon \mathbb{F}_2\langle A\rangle_\alpha \to \mathbb{F}_2\langle \hat{\otimes}^2 A\rangle_{\hat{\otimes}^2\alpha}$, $\bar{\tau}_T(a) = a\hat{\otimes}a$ $(a\in A)$ and $\bar{\sigma}_T\colon \mathbb{F}_2\langle A\times A\rangle_{\alpha\otimes\alpha} \to \mathbb{F}_2\langle \hat{\otimes}^2 A\rangle_{\hat{\otimes}^2\alpha}$ $\bar{\sigma}_T(a_1\otimes a_2) = a_1\hat{\otimes}a_2$ $(a_1,a_2\in A)$ are controlled homomorphisms and

$$\begin{tikzcd}
(\Gamma_T \mathbb{Z}\langle A\rangle_\alpha)\otimes\mathbb{Z}/2 \arrow[r, "\tau_T\otimes\mathbb{Z}/2"] \arrow[d, "\sigma_T\otimes\mathbb{Z}/2"'] \arrow[dr, phantom, "\text{push}"] & (\otimes^2_T\mathbb{Z}\langle A\rangle_\alpha)\otimes\mathbb{Z}/2 \arrow[d, "\bar{\sigma}_T"] \\
\mathbb{Z}\langle A\rangle_\alpha\otimes\mathbb{Z}/2 \arrow[r, "\bar{\tau}_T"] & \mathbb{F}_2\langle \hat{\otimes}^2 A\rangle_{\hat{\otimes}^2\alpha}
\end{tikzcd}$$

is a push-out.

The functors Γ_T, $\wedge^2_T$, $\otimes^2_T$ as well as the structure morphisms of their crossed effects and the natural transformations $[-,-]_T$, τ_T, q_T are known to agree with their ordinary analogues on underlying abelian groups when evaluated in free T-controlled $\mathbb{Z}$-modules, see Proposition 2.2.11. Moreover, we observe that in this case $\sigma_T\otimes\mathbb{Z}/2$ also coincides with $\sigma\otimes\mathbb{Z}/2$ on underlying $\mathbb{F}_2$-vector spaces. Furthermore, we also notice above that $\bar{\tau}_T$ and $\bar{\sigma}_T$ coincide with $\bar{\tau}$ and $\bar{\sigma}$ on underlying $\mathbb{F}_2$-vector spaces, therefore the properties of diagram (2.2.19) also imply this fact for $\bar{q}_T$, and hence for the structure morphisms of the quadratic crossed effect of $\hat{\otimes}_T$ because $\bar{q}_T$ induces an isomorphism between the quadratic crossed effects of $\hat{\otimes}^2_T$ and $\wedge^2_T$, see Proposition 2.2.20. □

2.3. The lower Whitehead module

The following theorem shows that the lower Whitehead module of a T-complex can be obtained from the homology by using some of the functors defined in the previous section.

THEOREM 2.3.1. *For any $(n-1)$-connected T-complex X there is a natural isomorphism*

$$\Gamma_{n+1}X \simeq \begin{cases} \Gamma_T\mathcal{H}_2X, & \text{if } n=2;\\ \mathcal{H}_nX\otimes\mathbb{Z}_2, & \text{if } n\geq 3.\end{cases}$$

These isomorphisms are compatible with the suspension morphisms in proper homology and Whitehead modules, that is, for $n=2$ there is a commutative diagram

$$\begin{tikzcd}
\Gamma_T\mathcal{H}_2X \arrow[r, two heads, "\sigma_T"] & \mathcal{H}_2X\otimes\mathbb{Z}_2 \arrow[r, "\Sigma\otimes\mathbb{Z}_2", "\simeq"'] & \mathcal{H}_3\Sigma X\otimes\mathbb{Z}_2 \\
\Gamma_3X \arrow[u, "\simeq"] \arrow[rr, "\Sigma"] & & \Gamma_4\Sigma X \arrow[u, "\simeq"']
\end{tikzcd}$$

and for $n \geq 3$

$$\begin{tikzcd} \mathcal{H}_n X \otimes \mathbb{Z}_2 \arrow[r, "\Sigma\otimes\mathbb{Z}_2", "\simeq"'] & \mathcal{H}_{n+1}\Sigma X \otimes \mathbb{Z}_2 \\ \Gamma_{n+1} X \arrow[u, "\simeq"] \arrow[r, "\Sigma"] & \Gamma_{n+2}\Sigma X \arrow[u, "\simeq"'] \end{tikzcd}$$

PROOF. Let us check that this proposition holds in the homotopy category category of n-dimensional spherical objects $\mathbf{S}^n(T)$, which are obviously $(n-1)$-connected. Notice that the functor Γ_{n+1} coincides with Π_{n+1} over this category.

By [**BQ01**] II.4.21 the isomorphism in the stable case $n \geq 3$ holds.

In [**BQ01**] II.2.15 the sets of proper homotopy classes $[S^3_\beta, S^2_\alpha]^T$ are computed as subsets of the sets of ordinary homotopy classes of pointed maps $[S^3_\alpha/T, S^2_\beta/T]^*_{\mathrm{ord}}$ by using the concepts of carrier and (T, α, β)-proper element in [**BQ01**] II.2.13 and II.2.14, respectively, however the equivalent description in terms of $(\bar{T}, \alpha, \beta)$-controlled elements given [**BQ01**] III.2.7 is better for our purposes, see [**BQ01**] III.2.8. It is well-known that $\pi_3(S^2_\beta/T) = \Gamma\mathbb{Z}\langle B\rangle$ where B is the source of β. Given a total ordering $\preceq$ in B the carrier of an element in the basis ΓB of $\Gamma\mathbb{Z}\langle B\rangle = [S^3, S^2_\alpha/T]^*_{\mathrm{ord}}$ is either $\mathrm{carr}(\gamma(b)) = \{\beta(b)\}$ or $\mathrm{carr}([b_1, b_2]) = \{\beta(b_1), \beta(b_2)\}$ for $b, b_1, b_2 \in B$ with $b_1 \succ b_2$. Now it is easy to see by using Lemma 2.2.7 that an element $\varphi \in [S^3_\alpha/T, S^2_\beta/T]^*_{\mathrm{ord}} = \mathrm{Hom}(\mathbb{Z}\langle A\rangle, \Gamma\mathbb{Z}\langle B\rangle)$, where A is the source of α, is $(\bar{T}, \alpha, \beta)$-controlled if and only if $\varphi\colon \mathbb{Z}\langle A\rangle_\alpha \to \Gamma_T\mathbb{Z}\langle B\rangle_\beta$ is a controlled homomorphism, so the isomorphism also holds in the unstable case $n = 2$.

It is known that the suspension $\Sigma\colon [S^3_\alpha/T, S^2_\beta/T]^*_{\mathrm{ord}} \to [S^4_\alpha/T, S^3_\beta/T]^*_{\mathrm{ord}}$ is equivalent to the following homomorphism

$$\mathrm{Hom}(\mathbb{Z}\langle A\rangle, \Gamma\mathbb{Z}\langle B\rangle) \stackrel{\mathrm{Hom}(1,\sigma)}{\longrightarrow} \mathrm{Hom}(\mathbb{Z}\langle A\rangle, \mathbb{Z}\langle B\rangle \otimes \mathbb{Z}/2) = \mathrm{Hom}(\mathbb{Z}\langle A\rangle, \mathbb{Z}\langle B\rangle) \otimes \mathbb{Z}/2.$$

Notice also that $\mathrm{Hom}(1, \sigma) = \hat{p}\,\mathrm{Hom}(1, \varsigma)$, where, ς is the homomorphism used to define σ_T in the proof of Proposition 2.2.11 and $\hat{p}$ is the natural projection $\hat{p}\colon 1 \twoheadrightarrow - \otimes \mathbb{Z}/2$, hence the first diagram of the statement commutes for spherical objects.

Once we have proved that the statement is true for spherical objects the general case can be checked as in [**Bau89**] IX.4.5 since a proper homotopy version of [**Bau89**] V.7.6 holds as a consequence of the proper Blakers-Massey theorem. □

We shall often use the isomorphisms in Theorem 2.3.1 as identifications. Below we give some propositions whose proofs follow by diagram chasing from Theorem 2.3.1, the properties of the quadratic functors defined in Section 2.2 and the elementary proper homotopy theory reviewed in Sections 1.1 and 1.3.

PROPOSITION 2.3.2. *Given two 1-connected T-complexes X and Y the following sequence is splitting short exact*

$$\begin{tikzcd} \mathcal{H}_2 X \otimes_T \mathcal{H}_2 Y \arrow[r, hook, "i_3 i_{12}"] & \Pi_3(X \vee Y) \arrow[r, two heads, "\xi"] & \Pi_3 X \oplus \Pi_3 Y \arrow[l, dashed, bend left, "\zeta"] \end{tikzcd} .$$

Here $\xi = \begin{pmatrix} p_{1*} \\ p_{2*} \end{pmatrix}$ and $\zeta = (i_{1*}, i_{2*})$, where i_k and p_k $(k = 1, 2)$ are the inclusions and retractions of the first and second factor of the coproduct $X \vee Y$; i_3 is one of *the morphisms in Whitehead's long exact sequence, and i_{12} is the inclusion of the quadratic crossed effect of Γ_T.*

Proof. One only has to consider the following commutative diagram with exact row and columns

$$\begin{tikzcd}
 & \Pi_4(X\vee Y) \arrow[r, two heads, "\xi"] \arrow[d, "h_4"'] & \Pi_4 X\oplus \Pi_4 Y \arrow[l, dashed, bend left, "\zeta"] \arrow[d, "h_4\oplus h_4"] \\
 & \mathcal{H}_4(X\vee Y) \arrow[r, equal] \arrow[d, "b_4"'] & \mathcal{H}_4 X\oplus \mathcal{H}_4 Y \arrow[d, "b_4\oplus b_4"] \\
\mathcal{H}_2X\otimes_T\mathcal{H}_2Y \arrow[r, hook, "i_{12}"] & \Gamma_T(\mathcal{H}_2X\oplus\mathcal{H}_2Y) \arrow[r, two heads] \arrow[d, "i_3"'] & \Gamma_T\mathcal{H}_2X\oplus\Gamma_T\mathcal{H}_2Y \arrow[d, "i_3\oplus i_3"] \\
 & \Pi_3(X\vee Y) \arrow[r, two heads, "\xi"] \arrow[d, two heads, "h_3"'] & \Pi_3 X\oplus \Pi_3 Y \arrow[l, dashed, bend left, "\zeta"] \arrow[d, two heads, "h_3\oplus h_3"] \\
 & \mathcal{H}_3(X\vee Y) \arrow[r, equal] & \mathcal{H}_3 X\oplus \mathcal{H}_3 Y
\end{tikzcd}$$

□

Proposition 2.3.3. *Given a 1-connected T-complex X the following sequence is exact*

$$\otimes_T^2\mathcal{H}_2X \overset{i_3[-,-]_T}{\longrightarrow} \Pi_3X \overset{\Sigma}{\twoheadrightarrow} \Pi_4\Sigma X.$$

Proof. It is enough to follow the next commutative diagram with exact rows and column where we use the suspension isomorphism in proper homology $\mathcal{H}_2X\simeq\mathcal{H}_3\Sigma X$ as an identification

$$\begin{tikzcd}
 & \otimes_T^2\mathcal{H}_2X \arrow[d, "{[-,-]_T}"'] & & \\
\mathcal{H}_4X \arrow[r, "b_4"] \arrow[d, "\Sigma"', "\simeq"] & \Gamma_T\mathcal{H}_2X \arrow[r, "i_3"] \arrow[d, two heads, "\sigma_T"'] & \Pi_3X \arrow[r, two heads, "h_3"] \arrow[d, "\Sigma"'] & \mathcal{H}_3X \arrow[d, "\Sigma"', "\simeq"] \\
\mathcal{H}_5\Sigma X \arrow[r, "b_5"] & \mathcal{H}_2X\otimes\mathbb{Z}_2 \arrow[r, "i_4"] & \Pi_4\Sigma X \arrow[r, two heads, "h_4"] & \mathcal{H}_4\Sigma X
\end{tikzcd}$$

□

Proposition 2.3.4. *Given two T-complexes X^2 and Z with* $\dim X^2\leq 2$ *there are natural central extensions* $(n\geq 1)$

$$H^{n+2}(\Sigma^nX^2,\Pi_{n+2}Z)\overset{j}{\hookrightarrow}[\Sigma^nX^2,Z]^T\twoheadrightarrow H^{n+1}(\Sigma^nX^2,\Pi_{n+1}Z).$$

Proof. The long cofiber sequence associated to the pasting map of 2-cells $f\colon S^1_{\alpha_2}\to S^1_{\alpha_1}$ in X^2 gives rise to exact sequences (in the left column) which fit into the following commutative diagrams ($n \geq 1$)

$$\begin{tikzcd}
{[S^{n+2}_{\alpha_1}, Z]^T} \arrow[r, equal] \arrow[d, "(\Sigma^{n+1} f)^*"'] & (\Pi_{n+2} Z)(\mathcal{C}_{n+1}\Sigma^n X^2) \arrow[d, "d^*_{n+2}"] \\
{[S^{n+2}_{\alpha_2}, Z]^T} \arrow[r, equal] \arrow[d, "(\Sigma^n q)^*"'] & (\Pi_{n+2} Z)(\mathcal{C}_{n+2}\Sigma^n X^2) \\
{[\Sigma^n X^2, Z]^T} \arrow[d] & \\
{[S^{n+1}_{\alpha_1}, Z]^T} \arrow[r, equal] \arrow[d, "(\Sigma^n f)^*"'] & (\Pi_{n+1} Z)(\mathcal{C}_{n+1}\Sigma^n X^2) \arrow[d, "d^*_{n+2}"] \\
{[S^{n+1}_{\alpha_2}, Z]^T} \arrow[r, equal] & (\Pi_{n+1} Z)(\mathcal{C}_{n+2}\Sigma^n X^2)
\end{tikzcd}$$

Here $(\Sigma^n q)^*$ is always central, see [**Bau89**] II.8.26. □

PROPOSITION 2.3.5. *Given three T-complexes X^2, Y and Z with $\dim X^2 \leq 2$ and Y and Z 1-connected, we have a natural central extension*

$$H^3(\Sigma X^2, \mathcal{H}_2 Y \otimes_T \mathcal{H}_2 Z) \overset{j(i_3 i_{12})_*}{\hookrightarrow} [\Sigma X^2, Y \vee Z]^T \overset{(p_{1*}, p_{2*})}{\twoheadrightarrow} [\Sigma X^2, Y]^T \times [\Sigma X^2, Z]^T\ .$$

PROOF. By evaluating the central extension in Proposition 2.3.4 for $n = 1$ on the retractions of the coproduct $p_1 \colon Y \vee Z \to Y$, $p_2 \colon Y \vee Z \to Z$ and using Hurewicz's isomorphism as an identification we obtain the following commutative diagram whose columns are central extensions

$$\begin{tikzcd}
H^3(\Sigma X^2, \Pi_3(Y \vee Z)) \arrow[r] \arrow[d, hook, "j"] & H^3(\Sigma X^2, \Pi_3 Y) \oplus H^3(\Sigma X^2, \Pi_3 Y) \arrow[d, hook, "j"] \\
{[\Sigma X^2, Y \vee Z]^T} \arrow[r, "(p_{1*}, p_{2*})"] \arrow[d, two heads] & {[\Sigma X^2, Y]^T \times [\Sigma X^2, Z]^T} \arrow[d, two heads] \\
H^2(\Sigma X^2, \mathcal{H}_2(Y \vee Z)) \arrow[r, equal] & H^2(\Sigma X^2, \mathcal{H}_2 Y) \oplus H^2(\Sigma X^2, \mathcal{H}_2 Z)
\end{tikzcd}$$

Moreover, if we evaluate this functor in the splitting short exact sequence in Proposition 2.3.2 we obtain another one

$$H^3(\Sigma X^2, \mathcal{H}_2 Y \otimes_T \mathcal{H}_2 Z) \overset{(i_3 i_{12})_*}{\hookrightarrow} H^3(\Sigma X, \Pi_3(Y \vee Z)) \twoheadrightarrow H^3(\Sigma X, \Pi_3 Y \oplus \Pi_3 Z).$$

Now the proposition follows from the diagram obtained by fitting this short exact sequence in the previous diagram. □

PROPOSITION 2.3.6. *Given two T-complexes X^2 and Z with $\dim X^2 \leq 2$ and Z 1-connected, the following natural sequence is exact*

$$H^3(\Sigma X^2, \otimes^2_T \mathcal{H}_2 Z) \overset{j(i_3[-,-]_T)_*}{\longrightarrow} [\Sigma X^2, Z]^T \overset{\Sigma}{\twoheadrightarrow} [\Sigma^2 X^2, \Sigma Z]^T.$$

Moreover, the suspension operator induces isomorphisms ($n \geq 2$)

$$\Sigma \colon [\Sigma^n X^2, \Sigma^{n-1} Z]^T \simeq [\Sigma^{n+1} X^2, \Sigma^n Z]^T.$$

PROOF. The central extensions in Proposition 2.3.4, the suspension homomorphisms in proper homotopy modules and (co)homology, and the Hurewicz isomorphism give rise to the following commutative diagram whose columns are central extensions

$$\begin{tikzcd}
H^{n+2}(\Sigma^n X^2, \Pi_{n+2}\Sigma^{n-1}Z) \arrow[r] \arrow[d, hook, "j"] & H^{n+3}(\Sigma^{n+1}X^2, \Pi_{n+3}\Sigma^n Z) \arrow[d, hook, "j"] \\
{[\Sigma^n X^2, \Sigma^{n-1}Z]^T} \arrow[r] \arrow[d, two heads] & {[\Sigma^{n+1}X^2, \Sigma^n Z]^T} \arrow[d, two heads] \\
H^{n+1}(\Sigma^n X^2, \mathcal{H}_{n+1}\Sigma^{n-1}Z) \arrow[r, "\simeq"] & H^{n+2}(\Sigma^{n+1}X^2, \mathcal{H}_{n+2}\Sigma^n Z)
\end{tikzcd}$$

Moreover, by Freudenthal's suspension theorem in proper homotopy theory the homomorphism $\Sigma\colon \Pi_{n+2}\Sigma^{n-1}Z \to \Pi_{n+3}\Sigma^n Z$ is an isomorphism for $n \geq 2$, hence the upper row in the diagram is an isomorphism within this range, and the second part of the statement follows. On the other hand for $n = 1$ the functor $H^3(\Sigma X^2, -)$ is right-exact since $\dim \Sigma X^2 \leq 3$, in particular by applying it to the exact sequence in Proposition 2.3.3 we obtain another exact sequence

$$H^3(\Sigma X^2, \otimes_T^2 \mathcal{H}_2 Z) \overset{(i_3[-,-]_T)_*}{\longrightarrow} H^3(\Sigma X^2, \Pi_3 Z) \twoheadrightarrow H^3(\Sigma X^2, \Pi_4 \Sigma Z).$$

The first part of the proposition follows from the diagram obtained by inserting this exact sequence into the previous diagram for $n = 1$. $\square$

2.4. Free controlled groups of nilpotency class 2

Recall that a free group with basis A in the variety of groups of nilpotency class 2 is the quotient

$$\langle A\rangle^{nil} = \frac{\langle A\rangle}{\Gamma_3\langle A\rangle},$$

where $\Gamma_3\langle A\rangle \subset \langle A\rangle$ is the third term of the lower central series of the free group $\langle A\rangle$, that is the subgroup generated by commutators of weight three, i. e. $[x,[y,z]]$ $(x, y, z \in \langle A\rangle)$. Here $[x, y] = -x - y + x + y$.

DEFINITION 2.4.1. A *free T-controlled group of nilpotency class* 2 is a pair $\langle A\rangle_\alpha^{nil}$ formed by a free group of nilpotency class 2 $\langle A\rangle^{nil}$ with basis the discrete set A and a proper map $\alpha\colon A \to T$. A *controlled homomorphism* $\varphi\colon \langle A\rangle_\alpha^{nil} \to \langle B\rangle_\beta^{nil}$ is a homomorphism between the underlying groups such that for any neighbourhood U of $\varepsilon \in \mathfrak{F}(T)$ in $\hat{T}$ there exists another one $V \subset U$ such that $\varphi(\alpha^{-1}(V)) \subset \langle\beta^{-1}(U)\rangle^{nil}$.

Notice that this definition is completely analogous to that of free T-controlled R-module in Section 1.2 and free T-controlled group in Definition 1.4.1. We will denote $\mathbf{nil}(T)$ to the category of free T-controlled groups of nilpotency class 2 and controlled homomorphisms. This category admitis finite coproducts. More precisely, the coproduct of two objects is

$$\langle A\rangle_\alpha^{nil} \vee \langle B\rangle_\beta^{nil} = \langle A \sqcup B\rangle_{(\alpha,\beta)}^{nil}.$$

There are full abelianization and nilization functors

$$ab\colon \mathbf{nil}(T) \longrightarrow \mathbf{ab}(T)\colon \langle A\rangle_\alpha^{nil} \mapsto \mathbb{Z}\langle A\rangle_\alpha,$$

$$nil\colon \mathbf{gr}(T) \longrightarrow \mathbf{nil}(T)\colon \langle A\rangle_\alpha \mapsto \langle A\rangle_\alpha^{nil}.$$

Recall that there exists a natural central extension

$$\wedge^2\mathbb{Z}\langle A\rangle \overset{i}{\hookrightarrow} \langle A\rangle^{nil} \overset{p}{\twoheadrightarrow} \mathbb{Z}\langle A\rangle, \quad i(a\wedge b)=[a,b], \quad (a,b\in A).$$

By using this central extension we can define a (right) action of the abelian group $\mathrm{Hom}_{\mathbf{ab}(T)}(\mathbb{Z}\langle B\rangle_\beta, \wedge_T^2\mathbb{Z}\langle A\rangle_\alpha)$ on the set $\mathrm{Hom}_{\mathbf{nil}(T)}(\langle B\rangle_\beta^{nil}, \langle A\rangle_\alpha^{nil})$ as

$$\varphi+\zeta\colon \langle B\rangle_\beta^{nil} \longrightarrow \langle A\rangle_\alpha^{nil}\colon x\mapsto \varphi(x)+i\zeta p(x).$$

It is easy to check that $\varphi+\zeta$ is controlled by using that φ and ζ are.

Proposition 2.4.2. *The previous action determines a linear extension of categories in the sense of Definition 2.5.1*

$$\mathrm{Hom}_{\mathbf{ab}(T)}(-,\wedge_T^2) \xrightarrow{+} \mathbf{nil}(T) \xrightarrow{ab} \mathbf{ab}(T).$$

Proof. We only need to check that given two controlled homomorphisms $\varphi,\psi\colon \langle B\rangle_\beta^{nil} \to \langle A\rangle_\alpha^{nil}$ with $\varphi^{ab}=\psi^{ab}$ the unique homomorphism $\zeta\colon \mathbb{Z}\langle B\rangle_\beta \to \mathbb{Z}\langle A\rangle_\alpha$ with $\varphi+\zeta=\psi$ is indeed controlled.

Let $T_v \sqcup T_v^{\mathfrak{F}}$, as defined in the paragraph preceding Lemma 2.2.7, be a neighbourhood of $\varepsilon\in\mathfrak{F}(T)$ in $\hat{T}$. Since φ and ψ are controlled we can take $T_w\subset T_v$ such that $T_w\sqcup T_w^{\mathfrak{F}}$ is also another neighborhood of ε in $\hat{T}$ and $\varphi(\beta^{-1}(T_w)), \psi(\beta^{-1}(T_w)) \subset \langle\alpha^{-1}(T_v)\rangle^{nil}$. Given $b\in B$, $\zeta(b)$ is a linear combination of elements $a_1\wedge a_2$ $(a_1,a_2\in A, a_1\prec a_2)$ such that a_i appears with non-trivial coefficient in the linear expansion of either $\varphi(b)$ or $\psi(b)$ $(i=1,2)$, in particular by Lemma 2.2.7 $\zeta(b)\in\mathbb{Z}\langle(\wedge^2\alpha)^{-1}(T_v)\rangle$ provided $\beta(b)\in T_w$, so $\zeta(\beta^{-1}(T_w))\subset\mathbb{Z}\langle(\wedge^2\alpha)^{-1}(T_v)\rangle$ and therefore ζ is a controlled homomorphism. □

2.5. Obstruction theory and cohomology of categories

Given a small category $\mathbf{C}$ a $\mathbf{C}$-*bimodule* D is a functor

$$D\colon \mathbf{C}^{op}\times\mathbf{C} \longrightarrow \mathbf{Ab}.$$

For simplicity given two morphisms φ and ψ in $\mathbf{C}$ we write $D(\varphi,1)=\varphi^*$ and $D(1,\psi)=\psi_*$. The cohomology groups $H^*(\mathbf{C},D)$ were first defined in [**Mit72**]. They are also the cohomology of the cochain complex $F^*(\mathbf{C},D)$ in [**Bau89**] IV.5.

In order to describe this cohomology as a functor we consider the category $\mathbf{Nat}$ whose objects are pairs $(\mathbf{C},D)$ given by a small category $\mathbf{C}$ and a $\mathbf{C}$-bimodule D. The *pull-back* of a $\mathbf{C}$-bimodule D along a functor $\lambda\colon \mathbf{B}\to\mathbf{C}$ is $\lambda^*D=D(\lambda^{op}\times\lambda)$. Similarly one defines the pull-back of a natural transformation $t\colon D\to D'$ between $\mathbf{C}$-bimodules as $\lambda^*t=t(\lambda^{op}\times\lambda)$. A morphism $(\lambda,t)\colon(\mathbf{C},D)\to(\mathbf{C}',D')$ is a functor $\lambda\colon\mathbf{C}'\to\mathbf{C}$ together with a natural transformation $t\colon\lambda^*D\to D'$. The composition in $\mathbf{Nat}$ is given by the formula $(\mu,u)(\lambda,t)=(\lambda\mu,u(\mu^*t))$, in particular $(\lambda,t)=(1_{\mathbf{C}'},t)(\lambda,1_{\lambda^*D})$.

The cohomology of categories induces functors $(n\geq 0)$

$$H^n\colon \mathbf{Nat}\longrightarrow\mathbf{Ab}, \quad H^n(\lambda,t)=t_*\lambda^*.$$

Here we denote $t_*=H^n(1,t)$ and $\lambda^*(\lambda,1)$.

As usual a short exact sequence of $\mathbf{C}$-bimodules

$$D\overset{t}{\hookrightarrow} D'\overset{u}{\twoheadrightarrow} D''$$

induces a Bockstein long exact sequence in cohomology

$$\cdots \to H^n(\mathbf{C}, D) \xrightarrow{t_*} H^n(\mathbf{C}, D') \xrightarrow{u_*} H^n(\mathbf{C}, D'') \xrightarrow{\beta} H^{n+1}(\mathbf{C}, D) \to \cdots .$$

A 0-*cochain* of $\mathbf{C}$ with coefficients in D is a function c which sends an object A of $\mathbf{C}$ to an element $c_A \in D(A, A)$. It is a 0-cocycle if for any $\sigma\colon A \to B$ in $\mathbf{C}$ the equation $\sigma_* c_A = \sigma^* c_B$ holds.

Suppose that $\mathbf{A}$ is a small additive category and D an $\mathbf{A}$-bimodule which is additive in the first variable, that is, given a direct sum diagram as in (2.1.2)

$$X \underset{i_1}{\overset{p_1}{\leftrightarrows}} X \oplus Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

the following homomorphism is an isomorphism

$$(p_1^*, p_2^*)\colon D(X, Z) \oplus D(Y, Z) \xrightarrow{\simeq} D(X \oplus Y, Z).$$

We say that a 0-cochain f is *additive* if the following equality always holds

$$f(X \oplus Y) = p_1^* i_{1*} f(X) + p_2^* i_{2*} f(Y) \in D(X \oplus Y, X \oplus Y).$$

It is not difficult to check that all 0-cocycles are additive. This is also a consequence of much more general results proved in [**BT96**] which point out a strong connection between cohomology of categories and representation theory.

Definition 2.5.1. Given two $\mathbf{C}$-bimodules H and D, an *exact sequence for a functor* $\lambda\colon \mathbf{B} \to \mathbf{C}$

$$H \xrightarrow{+} \mathbf{B} \xrightarrow{\lambda} \mathbf{C} \xrightarrow{\theta} D$$

consists of the following:

(1) For any morphism $f\colon X \to Y$ in $\mathbf{B}$ the abelian group $H(\lambda X, \lambda Y)$ acts transitively on the right of the set $\lambda^{-1}(\lambda f) \subset \mathrm{Hom}_{\mathbf{B}}(X, Y)$.
(2) The previous action satisfies the formula $(f + \alpha)(g + \beta) = fg + (\lambda f)_* \beta + (\lambda g)^* \alpha$.
(3) Given two objects X and Y in $\mathbf{B}$ and a morphism $f\colon \lambda X \to \lambda Y$ there is a well-defined element $\theta_{X,Y}(f) \in D(\lambda X, \lambda Y)$ which vanishes if and only if $f = \lambda g$ for some $g\colon X \to Y$.
(4) The obstruction operator θ is a derivation, that is, the following formula holds
$$\theta_{X,Z}(gf) = g_* \theta_{X,Y}(f) + f^* \theta_{Y,Z}(g).$$
(5) Given an object X in $\mathbf{B}$ and $\alpha \in D(\lambda X, \lambda X)$ there exists another one Y such that $\lambda X = \lambda Y$ and $\theta_{X,Y}(1_{\lambda X}) = \alpha$. One can check by using the previous axioms that the object Y is well defined up to isomorphism by X and α, and also X is determined up to isomorphism by Y and α. In these conditions we write $X = Y + \alpha$.

Any functor fitting into an exact sequence reflects isomorphisms, see [**Bau89**] IV.4.11.

A *linear extension of categories*

$$H \xrightarrow{+} \mathbf{B} \xrightarrow{\lambda} \mathbf{C}$$

is an exact sequence $H \xrightarrow{+} \mathbf{B} \xrightarrow{\lambda} \mathbf{C} \to 0$ such that λ induces a bijection between the sets of objects and the action in (1) is effective.

REMARK 2.5.2. Suppose that the functor λ fitting into an exact sequence as above induces a surjection between object sets $\lambda\colon Ob\mathbf{B} \twoheadrightarrow Ob\mathbf{C}$. In this case the obstruction operator θ determines a well defined cohomology class

$$\{\theta\} \in H^1(\mathbf{C}, D)$$

in the following way, compare [**Bau89**] IV.7. Take a splitting function $s\colon Ob\mathbf{C} \hookrightarrow Ob\mathbf{B}$ of λ. The function $\tilde{s}$ sending a morphism $\sigma\colon A \to B$ in $\mathbf{C}$ to $\tilde{s}(\sigma) = \theta_{sA,sB}(\sigma)$ is a 1-cocycle representing $\{\theta\}$. Moreover any representative cocycle of this cohomology class can be obtained as $\tilde{s}$ for an adequate splitting s. Furthermore, $\{\theta\} = 0$ if and only if there exists a 0-cocycle c of $\mathbf{B}$ with coefficients in λ^*D such that $\theta_{X,Y}(f) = f_*(c_X) - f^*(c_Y)$. Indeed if such a 0-cocycle c exists the 1-cocycle $\tilde{s}$ representing $\{\theta\}$ is the coboundary of the 0-cochain sending an object A in $\mathbf{C}$ to c_{sA}. On the other hand if $\{\theta\} = 0$ there is a splitting function s such that $\tilde{s} = 0$ and we can define the 0-cocycle c as $c_X = \theta_{X,s\lambda X}(1_{\lambda X})$.

DEFINITION 2.5.3. An $(n-1)$-reduced T-*homotopy system* $(n \geq 2)$ is a pair $(\mathcal{C}_*, f_{n+2})$ given by a bounded chain complex $\mathcal{C}_*$ in $\mathbf{ab}(T)$ concentrated in degrees $\geq n$ and an $\mathbf{ab}(T)$-module morphism $f_{n+2}\colon \mathcal{C}_{n+2} \to \Pi_{n+1}C_{d_{n+1}}$ such that $f_{n+2}d_{n+3} = 0$ and d_{n+2} coincides with the following composite

$$\mathcal{C}_{n+2} \xrightarrow{f_{n+2}} \Pi_{n+1}C_{d_{n+1}} \xrightarrow{j} \Pi_{n+1}(C_{d_{n+1}}, S^n_{\alpha_n}) = \mathcal{C}_{n+1}.$$

Here we identify the differential $d_{n+1}\colon \mathcal{C}_{n+1} \to \mathcal{C}_n$ with a map between n-dimensional spherical objects $d_{n+1}\colon S^n_{\alpha_{n+1}} \to S^n_{\alpha_n}$ by using Proposition 1.3.1 so that we can take its cofiber $C_{d_{n+1}}$.

A *morphism* $(\xi, \eta)\colon (\mathcal{C}_*, f_{n+2}) \to (\mathcal{C}'_*, g_{n+2})$ of $(n-1)$-reduced T-homotopy systems is given by a chain complex morphism $\xi\colon \mathcal{C}_* \to \mathcal{C}'_*$ and a map $\eta\colon C_{d_{n+1}} \to C_{d'_{n+1}}$ such that $\mathcal{C}_*\eta$ coincides with ξ in degrees n and $n+1$, and the following square commutes

$$\begin{array}{ccc} \mathcal{C}_{n+2} & \xrightarrow{\xi_{n+2}} & \mathcal{C}'_{n+2} \\ {\scriptstyle f_{n+2}}\downarrow & & \downarrow{\scriptstyle g_{n+2}} \\ \Pi_{n+1}C_{d_{n+1}} & \xrightarrow[\eta_*]{} & \Pi_{n+1}C_{d'_{n+1}} \end{array} \tag{2.5.4}$$

that is

$$\eta f_{n+2} = g_{n+2}\xi_{n+2} \in [S^{n+1}_{\alpha_{n+2}}, C'_{d_{n+1}}]^T.$$

Here $S^{n+1}_{\alpha_{n+2}}$ is the $(n+1)$-dimensional spherical object corresponding to $\mathcal{C}_{n+2}$. Let $\mathbf{H}_n(T)$ be the category of $(n-1)$-reduced T-homotopy systems.

Two morphisms $(\xi, \eta), (\xi', \eta')\colon (\mathcal{C}_*, f_{n+2}) \to (\mathcal{C}'_*, g_{n+2})$ of $(n-1)$-reduced T-homotopy systems are *homotopic* if there exists a chain homotopy $\xi \simeq \xi'$ given by morphisms $\omega_m\colon \mathcal{C}_m \to \mathcal{C}'_{m+1}$ such that

$$\eta + g_{n+2}\omega_{n+1} = \eta' \in [C_{d_{n+1}}, C_{d'_{n+1}}]^T.$$

Here we use the action of the group $[S^{n+1}_{\alpha_{n+1}}, C_{d'_{n+1}}]^T = (\Pi_{n+1}C_{d'_{n+1}})(\mathcal{C}_{n+1})$ on the set $[C_{d_{n+1}}, C_{d'_{n+1}}]^T$ induced by the long cofiber sequence

$$S^n_{\alpha_{n+1}} \xrightarrow{d_{n+1}} S^n_{\alpha_n} \rightarrowtail C_{d_{n+1}} \to S^{n+1}_{\alpha_{n+1}} \to \cdots,$$

see [**Bau89**] II.8.8.

The homotopy relation is a natural equivalence relation in he category of $(n-1)$-reduced T-homotopy systems so the quotient category $\mathbf{H}_n(T)/\simeq$ is well defined.

The general definition of homotopy systems in homological cofibration categories appears in [**Bau99**] VI.1. Using the terminology of [**Bau99**] we have defined above $(n-1)$-reduced twisted homotopy systems of order $n+2$ satisfying the cocycle condition in $\mathbf{Topp}^T$. The homotopy systems we will use here are closer to those used in [**Bau89**] IX, since we will always be in the 1-reduced case.

Let $\mathbf{CW}_n^T$ be the category of $(n-1)$-reduced T-complexes. There is a canonical functor

$$r\colon \mathbf{CW}_n^T \longrightarrow \mathbf{H}_n(T)$$

which sends a T-complex X to the pair $rX = (\mathcal{C}_*X, f_{n+2})$ where $f_{n+2}\colon \mathcal{C}_{n+2}X \to \Pi_{n+1}X^{n+1}$ is induced by the pasting map of $(n+2)$-cells $S^{n+1}_{\alpha_{n+2}} \to X^{n+1}$. Notice that $C_{d_{n+1}} = X^{n+1}$. We will not discuss here the cylinders in $\mathbf{H}_n(T)$ which give rise to the homotopy relation defined above, however we point out that the functor r preserves cylinders so it factors through the homotopy categories

$$r\colon \mathbf{CW}_n^T/\simeq \longrightarrow \mathbf{H}_n(T)/\simeq, \tag{2.5.5}$$

see [**Bau99**] VI.1.7 for further details.

Let us write $\mathbf{A}_n^2(T)$ for the homotopy category of $(n-1)$-reduced T-complexes of dimension $\leq n+2$ and $\mathbf{H}_n^2(T)$ for the category $(n-1)$-reduced T-homotopy systems whose chain complex is concentrated in dimensions $\leq n+2$.

Proposition 2.5.6. *The functor r restricts to a full functor*

$$r\colon \mathbf{A}_n^2(T) \longrightarrow \mathbf{H}_n^2(T)/\simeq$$

which induces a bijection between the sets of isomorphisms classes of objects in both categories.

This proposition can be easily checked by using principal maps in the sense of [**Bau89**] V.2 and the proper homological Whitehead theorem.

If $\mathbf{chain}_n(\mathbf{ab}(T))$ is the category of bounded chain complexes in $\mathbf{ab}(T)$ concentrated in degrees $\geq n$ there is a forgetful functor

$$\lambda\colon \mathbf{H}_n(T) \longrightarrow \mathbf{chain}_n(\mathbf{ab}(T)),\ \ \lambda(\mathcal{C}_*, f_{n+2}) = \mathcal{C}_*.$$

This functor also factors through the homotopy categories

$$\lambda\colon \mathbf{H}_n(T)/\simeq \longrightarrow \mathbf{chain}_n(\mathbf{ab}(T))/\simeq .$$

Moreover $\lambda r = \mathcal{C}_*$ is the proper cellular chain complex functor.

In the rest of this section we will concentrate in describing the properties the functor λ. We refer the reader to [**Bau99**] VI for the details.

There are exact sequences for the functors λ in the sense of Definition 2.5.1 as follows $(n \geq 3)$

(2.5.7)

$$H^3(-,\Gamma_T H_2) \xrightarrow{+} \mathbf{H}_2(T)/\simeq \xrightarrow{\lambda} \mathbf{chain}_2(\mathbf{ab}(T))/\simeq \xrightarrow{\theta^2} H^4(-,\Gamma_T H_2),$$

$$H^{n+1}(-, H_n \otimes \mathbb{Z}/2) \xrightarrow{+} \mathbf{H}_n(T)/\simeq \xrightarrow{\lambda} \mathbf{chain}_n(\mathbf{ab}(T))/\simeq \xrightarrow{\theta^n} H^{n+2}(-, H_n \otimes \mathbb{Z}/2).$$

These exact sequences are defined in [**Bau99**] VI in a much more general setting, here we will translate some of their properties to proper homotopy theory, mainly the definition of the obstruction operators θ^n. Here we use Theorem 2.3.1

to give a convenient description of the bimodules involved, otherwise these bimodules would be defined directly in terms of Whitehead modules.

Given two $(n-1)$-reduced homotopy systems $(\mathcal{C}_*, f_{n+2})$, $(\mathcal{C}'_*, g_{n+2})$ and a chain homotopy class $\xi\colon \mathcal{C}_* \to \mathcal{C}'_*$ we can take a principal map $\eta\colon C_{d_{n+1}} \to C_{d'_{n+1}}$ in the sense of [**Bau89**] V.2 such that $\mathcal{C}_*\eta$ coincides with ξ in dimensions n and $n+1$, but the commutativity of (2.5.4) is not guaranteed, that is the pair (ξ, η) need not be a morphism of homotopy systems. However the lower square and the two triangles in the following diagram do commute by the properties of η and the definition of homotopy systems, respectively,

$$\begin{tikzcd}
 & \mathcal{C}_{n+2} \arrow[r, "\xi_{n+2}"] \arrow[d, "f_{n+2}"] \arrow[ddl, "d_{n+2}"'] & \mathcal{C}'_{n+2} \arrow[d, "g_{n+2}"'] \arrow[ddr, "d_{n+2}"] & \\
 & \Pi_{n+1}C_{d_{n+1}} \arrow[r, "\eta_*"] \arrow[d] & \Pi_{n+1}C_{d'_{n+1}} \arrow[d, "j"] & \\
\mathcal{C}_{n+1} \arrow[r, equal] & \Pi_{n+1}(C_{d'_{n+1}}, S^n_{\alpha_n}) \arrow[r, "\xi_{n+1}"'] & \Pi_{n+1}(C_{d'_{n+1}}, S^n_{\alpha'_n}) \arrow[r, equal] & \mathcal{C}'_{n+1}
\end{tikzcd}$$

so the failure $-g_{n+2}\xi_{n+2}+\eta f_{n+2}$ in the commutativity of the upper square, which is (2.5.4), factors through the kernel of j which coincides with the Whitehead module $\Gamma_{n+1}C_{d'_{n+1}}$ by the exactness of the sequence of the pair $(C_{d'_{n+1}}, S^n_{\alpha'_n})$ in proper homotopy modules,

$$\begin{tikzcd}
\mathcal{C}_{n+2} \arrow[rr, "-g_{n+2}\xi_{n+2}+\eta f_{n+2}"] \arrow[dr, "\beta"'] & & \Pi_{n+1}C_{d'_{n+1}} \\
 & \Gamma_{n+1}C_{d'_{n+1}} \arrow[ur, hook] &
\end{tikzcd}$$

By Theorem 2.3.1 $\Gamma_{n+1}C_{d'_{n+1}}$ is either $\Gamma_T H_2\mathcal{C}'_*$ if $n=2$ or $H_n\mathcal{C}'_*\otimes\mathbb{Z}/2$ if $n \geq 3$, and finally the cohomology class

$$\theta^n_{(\mathcal{C}_*, f_{n+2}),(\mathcal{C}'_*, g_{n+2})}(\xi) \in \begin{cases} H^4(\mathcal{C}_*, \Gamma_T H_2\mathcal{C}'_*), & \text{if } n=2; \\ H^{n+2}(\mathcal{C}_*, H_n\mathcal{C}'_*\otimes\mathbb{Z}_2), & \text{if } n\geq 3; \end{cases}$$

is represented by the cocycle β.

Proposition 2.5.8. *The functors λ are surjective on objects.*

Proof. Let $\mathcal{C}_*$ be a bounded chain complex concentrated in dimensions $\geq n$ for some $n \geq 2$. Since $C_{d_{n+1}}$ is 1-connected one can check by using the exact sequence of the pair $(C_{d_{n+1}}, S^n_{\alpha_n})$ in proper homotopy modules that the image of $j\colon \Pi_{n+1}C_{d_{n+1}} \to \Pi_{n+1}(C_{d_{n+1}}, S^n_{\alpha_n}) = \mathcal{C}_{n+1}$ is $\operatorname{Ker} d_{n+1}$ which is a f. g. free $\mathbf{ab}(T)$-module by Theorem 1.2.14, so there exists a section $s\colon \operatorname{Ker} d_{n+1} \hookrightarrow \Pi_{n+1}C_{d_{n+1}}$ of the natural projection onto the image $p\colon \Pi_{n+1}C_{d_{n+1}} \twoheadrightarrow \operatorname{Ker} d_{n+1}$. The differential d_{n+2} factors through the inclusion $i\colon \operatorname{Ker} d_{n+1} \hookrightarrow \mathcal{C}_{n+1}$, that is $d_{n+2} = i\bar{d}_{n+2}$. We claim that $(\mathcal{C}_*, s\bar{d}_{n+2})$ is an $(n-1)$-reduced homotopy system. On one hand $js\bar{d}_{n+2} = ips\bar{d}_{n+2} = i\bar{d}_{n+2} = d_{n+2}$, as required, and on the other hand $\bar{d}_{n+2}d_{n+3} = 0$ since $i\bar{d}_{n+2}d_{n+3} = d_{n+2}d_{n+3} = 0$ and i is a monomorphism, hence $s\bar{d}_{n+2}d_{n+3} = 0$ is also trivial. Now the proof is finished. $\square$

By this proposition and Remark 2.5.2 the exact sequences of functors in (2.5.7) define classes in cohomology of categories

$$\{\theta^n\} \in \begin{cases} H^1(\mathbf{chain}_2(\mathbf{ab}(T))/\simeq, H^4(-,\Gamma_T H_2)), & \text{if } n=2; \\ H^1(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-,H_n\otimes\mathbb{Z}/2)), & \text{if } n\geq 3; \end{cases} \tag{2.5.9}$$

The suspension of chain complexes

$$\Sigma\colon \mathbf{chain}_n(\mathbf{ab}(T))/\simeq \longrightarrow \mathbf{chain}_{n+1}(\mathbf{ab}(T))/\simeq \tag{2.5.10}$$

and the suspension of T-complexes induce suspension functors

$$\Sigma\colon \mathbf{H}_n(T)/\simeq \longrightarrow \mathbf{H}_{n+1}(T)/\simeq,\ \ \Sigma(\mathcal{C}_*, f_{n+2}) = (\Sigma\mathcal{C}_*, \Sigma f_{n+2}),$$

such that $\Sigma r = r\Sigma$ and $\Sigma\lambda = \lambda\Sigma$. It is easy to see by using the description of θ^n given above and Theorem 2.3.1 that these suspension functors are also compatible with the obstruction operators, that is the following equality holds in $H^{n+3}(\Sigma\mathcal{C}_*, H_{n+1}\Sigma\mathcal{C}'_*\otimes\mathbb{Z}/2) = H^{n+2}(\mathcal{C}_*, H_n\mathcal{C}'_*\otimes\mathbb{Z}/2)$

$$\theta^{n+1}_{\Sigma(\mathcal{C}_*,f_{n+2}),\Sigma(\mathcal{C}'_*,g_{n+2})}(\Sigma\xi) = \begin{cases} (\sigma_T)_*\theta^2_{(\mathcal{C}_*,f_4),(\mathcal{C}'_*,g_4)}(\xi), & \text{if } n=2; \\ \theta^n_{(\mathcal{C}_*,f_{n+2}),(\mathcal{C}'_*,g_{n+2})}(\xi), & \text{if } n\geq 3. \end{cases}$$

Therefore the cohomology classes defined by the exact sequences in (2.5.7) are compatible with the suspension functor in the sense that

$$(\sigma_T)_*\left\{\theta^2\right\} = \Sigma^*\left\{\theta^3\right\} \text{ and } \left\{\theta^n\right\} = \Sigma^*\left\{\theta^{n+1}\right\} \text{ for } n\geq 3. \tag{2.5.11}$$

Notice that the suspension of chain complexes (2.5.10) is an isomorphism of categories, therefore the induced homomorphisms Σ^* above are abelian group isomorphisms, in particular $\left\{\theta^2\right\}$ determines $\{\theta^n\}$ for all $n\geq 3$ and all cohomology classes θ^n $(n\geq 3)$ correspond to each other. From now on we shall simply write θ for the obstruction operator θ^n, or θ^T if we want to specify the base tree.

Given a proper map $f\colon T\to T'$ between trees the "change of tree" functors $\mathbb{F}^f\colon \mathbf{ab}(T)\to\mathbf{ab}(T')$ and $f_*\colon \mathbf{Topp}^T_c/\simeq\to \mathbf{Topp}^{T'}_c/\simeq$ in Proposition 1.2.7 and (1.1.2), respectively, give rise to maps between exact sequences for functors in the sense of [**Bau89**] IV.4.13 $(n\geq 3)$

(2.5.12)

$$\begin{tikzcd} H^3(-,\Gamma_T H_2) \arrow[r, "+"] \arrow[d, "\bar{\mathbb{F}}^f"'] & \mathbf{H}_2(T)/\simeq \arrow[r, "\lambda"] \arrow[d, "(\mathbb{F}^f,f_*)"] & \mathbf{chain}_2(\mathbf{ab}(T))/\simeq \arrow[r, "\theta^T"] \arrow[d, "\mathbb{F}^f"] & H^4(-,\Gamma_T H_2) \arrow[d, "\bar{\mathbb{F}}^f"] \\ H^3(-,\Gamma_{T'} H_2) \arrow[r, "+"] & \mathbf{H}_2(T')/\simeq \arrow[r, "\lambda"] & \mathbf{chain}_2(\mathbf{ab}(T'))/\simeq \arrow[r, "\theta^{T'}"] & H^4(-,\Gamma_{T'} H_2) \end{tikzcd}$$

$$\begin{tikzcd} H^{n+1}(-,H_n\otimes\mathbb{Z}/2) \arrow[r, "+"] \arrow[d, "\bar{\mathbb{F}}^f"'] & \mathbf{H}_n(T)/\simeq \arrow[r, "\lambda"] \arrow[d, "(\mathbb{F}^f,f_*)"] & \mathbf{chain}_n(\mathbf{ab}(T))/\simeq \arrow[r, "\theta^T"] \arrow[d, "\mathbb{F}^f"] & H^{n+2}(-,H_n\otimes\mathbb{Z}/2) \arrow[d, "\bar{\mathbb{F}}^f"] \\ H^{n+1}(-,H_n\otimes\mathbb{Z}/2) \arrow[r, "+"] & \mathbf{H}_n(T')/\simeq \arrow[r, "\lambda"] & \mathbf{chain}_n(\mathbf{ab}(T'))/\simeq \arrow[r, "\theta^{T'}"] & H^{n+2}(-,H_n\otimes\mathbb{Z}/2) \end{tikzcd}$$

Here the natural transformations $\bar{\mathbb{F}}^f\colon H^m(-,\Gamma_T H_2)\to H^m(\mathbb{F}^f,\Gamma_{T'}H_2\mathbb{F}^f)$ between bimodules over $\mathbf{chain}_2(\mathbf{ab}(T))/\simeq$ are induced by taking H^m on the following composition of cochain homomorphisms

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathbf{ab}(T)}(\mathcal{C}_*, \Gamma_T H_2\mathcal{C}_*) \arrow[r, "\mathbb{F}^f_*"] & \mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, \mathbb{F}^f_*\Gamma_T H_2\mathcal{C}_*) \arrow[d] \\
\mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, \Gamma_{T'} H_2\mathbb{F}^f\mathcal{C}_*) \arrow[r, equal] & \mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, \Gamma_{T'}\mathbb{F}^f_* H_2\mathcal{C}_*)
\end{tikzcd}
\tag{2.5.13}
$$

Here the first arrow is given by the functor $\mathbb{F}^f_*$ extending $\mathbb{F}^f$ to the category of all $\mathbf{ab}(T)$-modules in the sense of (1.2.2), the second one is given by the natural transformation $\mathbb{F}^f_*\Gamma_T \to \Gamma_{T'}\mathbb{F}^f_*$, see Proposition 2.2.17, and for the equality we use the facts that $\mathbb{F}^f_*$ is right-exact and $\mathcal{C}_*$ is concentrated in dimensions ≥ 2. Similarly the natural transformations $\bar{\mathbb{F}}^f\colon H^m(-, H_n \otimes \mathbb{Z}/2) \to H^m(\mathbb{F}^f, (H_n\mathbb{F}^f) \otimes \mathbb{Z}/2)$ between bimodules over $\mathbf{chain}_n(\mathbf{ab}(T))/\simeq$ are given by the cochain homomorphism

(2.5.14)

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathbf{ab}(T)}(\mathcal{C}_*, H_n\mathcal{C}_* \otimes \mathbb{Z}/2) \arrow[r, "\mathbb{F}^f_*"] & \mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, \mathbb{F}^f_*(H_n\mathcal{C}_* \otimes \mathbb{Z}/2)) \arrow[d, equal] \\
\mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, H_n(\mathbb{F}^f\mathcal{C}_*) \otimes \mathbb{Z}/2) \arrow[r, equal] & \mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, (\mathbb{F}^f_* H_n\mathcal{C}_*) \otimes \mathbb{Z}/2)
\end{tikzcd}
$$

As a consequence of this we have the following result.

Proposition 2.5.15. *Given a proper map $f\colon T \to T'$ between trees the cohomology classes in (2.5.9) are compatible with the induced "change of tree" functor $\mathbb{F}^f\colon \mathbf{ab}(T) \to \mathbf{ab}(T')$, i. e. with the notation above*

$$(\mathbb{F}^f)^*\left\{\theta^{T'}\right\} = \bar{\mathbb{F}}^f_*\left\{\theta^T\right\}.$$

Remark 2.5.16. We point out that $\bar{\mathbb{F}}^f_*$ is an isomorphism provided f induces an injection $\mathfrak{F}(f)\colon \mathfrak{F}(T) \hookrightarrow \mathfrak{F}(T')$. In fact (2.5.13) and (2.5.14) are cochain isomorphisms in this case, see Propositions 1.2.10 and 2.2.11. Therefore the cohomology class $\left\{\theta^{T'}\right\}$ determines $\left\{\theta^T\right\}$ for all trees T such that $\mathfrak{F}(T)$ is homeomorphic to a subspace of $\mathfrak{F}(T')$.

CHAPTER 3

James-Hopf and cohomology invariants in proper homotopy theory

In this chapter we introduce new interrelated proper homotopy and cohomology invariants: James-Hopf invariants, interior cup-producs, cup-products for T-homotopy systems, T-complexes and chain complexes, and reduced or stable versions of them. These invariants are constructed by using the controlled quadratic functors from Chapter 2. We show that the (reduced) cup-product invariant for T-complexes is related to obstruction theory through a change of coefficient given by natural transformations between controlled quadratic functors. The cup-product for chain complexes measures the obstruction to the existence of a T-complex with a co-H-multiplication and a prescribed proper cellular chain complex, as we show in the last section of this chapter. Such an onstruction is trivial in classical homotopy theory since classical Moore spaces are much better behaved than in proper homotopy theory. In proper homotopy theory the cup-product for chain complexes need not be trivial since we are in projective dimension 2, and not just 1 as in the classical case. Indeed the computation of the cup-product for chain complexes is one of the main problems that we tackle in this paper, see Chapter 4. This computation is the basis of the main results of this paper on the proper homotopy classification of locally compact A_n^2-polyhedra, see Chapter 5.

3.1. James-Hopf invariants

Definition 3.1.1. Recall that a *co-H-space* in $\mathbf{Topp}_c^T$ is an object X endowed with a proper map $\mu\colon X \to X \vee X$ under T, the *co-H-multiplication*, such that the composition with the projections $p_1, p_2\colon X \vee X \to X$ are homotopic to the identity $p_i\mu \simeq 1_X$ $(i = 1, 2)$. Spherical objects of dimension ≥ 1 are co-H-spaces. More generally, it is well known that all suspensions are canonically co-H-spaces, see [**Bau89**] II.6.16.

Definition 3.1.2. The *James-Hopf invariant* of a T-complex X is the unique $\mathbf{ab}(T)$-module morphism

$$\gamma_2\colon \Pi_3\Sigma X \longrightarrow \otimes_T^2 \mathcal{H}_2\Sigma X$$

such that if $i_1, i_2, \mu\colon \Sigma X \to \Sigma X \vee \Sigma X$ are the inclusions of the factors of the coproduct and the co-H-multiplication of the suspension ΣX, respectively, then

$$i_3 i_{12}\gamma_2 = \mu_* - i_{2*} - i_{1*}\colon \Pi_3\Sigma X \longrightarrow \Pi_3(\Sigma X \vee \Sigma X).$$

The morphism γ_2 exists and is unique by Proposition 2.3.2, and it is natural in X.

Proposition 3.1.3. *For any T-complex X the following diagram is commutative*

$$\begin{tikzcd} \Gamma_T\mathcal{H}_2\Sigma X \arrow[rr, "\tau_T"] \arrow[dr, "i_3"'] & & \otimes^2_T\mathcal{H}_2\Sigma X \\ & \Pi_3\Sigma X \arrow[ur, "\gamma_2"'] & \end{tikzcd}$$

Proof. On one hand we have that

$$\begin{aligned} i_3 i_{12}\gamma_2 i_3 &= (\Pi_3\mu - \Pi_3 i_2 - \Pi_3 i_1) i_3 \\ &= i_3(\Gamma_T\mathcal{H}_2\mu - \Gamma_T\mathcal{H}_2 i_2 - \Gamma_T\mathcal{H}_2 i_1) \\ &= i_3(\Gamma_T(i_1+i_2) - \Gamma_T i_2 - \Gamma_T i_1). \end{aligned}$$

For the first equality we use the Definition 3.1.2, for the second one we use the naturality of Whitehead's long exact sequence and Theorem 2.3.1, for the third one we use the natural identifications $\mathcal{H}_2(\Sigma X \vee \Sigma X) = \mathcal{H}_2\Sigma X \oplus \mathcal{H}_2\Sigma X$, $\mathcal{H}_2\mu = i_1 + i_2$ and $\mathcal{H}_2 i_k = i_k$ $(k = 1, 2)$.

On the other hand we have that

$$\text{(a)} \qquad \begin{aligned} i_3 i_{12}\tau_T &= i_3 i_{12} r_{12}\Gamma_T(i_1+i_2) \\ &= i_3(\Gamma_T(i_1p_1 + i_2p_2) - \Gamma_T(i_1p_1) - \Gamma_T(i_2p_2))\Gamma_T(i_1+i_2) \\ &= i_3(\Gamma_T(i_1+i_2) - \Gamma_T i_1 - \Gamma_T i_2). \end{aligned}$$

Here for the first equality we use the Proposition 2.2.17, for the second one we use the definitions of i_{12} and r_{12} in Definition 2.1.1, and finally for the third one we use the properties of the structural morphisms of a direct sum.

Now that we have proved the equality $i_3 i_{12}\gamma_2 i_3 = i_3 i_{12}\tau_T$ the proposition follows from the fact that $i_3 i_{12}$ is a monomorphism, see Proposition 2.3.2. □

Definition 3.1.2. Given a T-complex X and $n \geq 2$ we define the *stable James-Hopf invariant* as the unique $\mathbf{ab}(T)$-module morphism γ_2^n that fits into the following commutative diagram

$$\begin{tikzcd} \Pi_{n+2}\Sigma^n X \arrow[r, "\gamma_2^n"] & \hat{\otimes}^2_T\mathcal{H}_{n+1}\Sigma^n X \\ \Pi_3\Sigma X \arrow[u, two heads, "\Sigma^{n-1}"] \arrow[r, "\gamma_2"] & \otimes^2_T\mathcal{H}_2\Sigma X \arrow[u, two heads, "\bar{\sigma}_T"'] \\ \otimes^2_T\mathcal{H}_2\Sigma X \arrow[u, "{i_3[-,-]_T}"] \arrow[r, equal] & \otimes^2_T\mathcal{H}_2\Sigma X \arrow[u, "{\tau_T[-,-]_T}"'] \end{tikzcd}$$

Here we use the suspension isomorphism in proper homology $\mathcal{H}_2\Sigma X = \mathcal{H}_{n+1}\Sigma^n X$ as an identification. The commutativity of the lower square follows from Proposition 3.1.3. Moreover, the morphism γ_2^n exists and is unique since the column in the left is exact by Proposition 2.3.3 and Freudenthal's suspension theorem, and $\bar{\sigma}_T\tau_T[-,-]_T = \bar{\tau}_T\sigma_T[-,-]_T = 0$, see (2.2.19).

As a consequence of Proposition 3.1.3, the properties of diagram (2.2.19) and Theorem 2.3.1 we have the following result.

PROPOSITION 3.1.3. *The following diagram commutes for any T-complex X and $n \geq 2$*

$$\begin{tikzcd}
\mathcal{H}_{n+1}\Sigma^n X \otimes \mathbb{Z}_2 \arrow[rr, "\bar{\tau}_T"] \arrow[dr, "i_{n+2}"'] & & \hat{\otimes}^2_T \mathcal{H}_{n+1}\Sigma^n X \\
& \Pi_{n+2}\Sigma^n X \arrow[ur, "\gamma_2^n"'] &
\end{tikzcd}$$

DEFINITION 3.1.4. Given two T-complexes X^2 and Y with $\dim X^2 \leq 2$ we define the *James-Hopf invariant* of a map $f\colon \Sigma X^2 \to \Sigma Y$ as the unique element

$$\gamma_2(f) \in H^3(\Sigma X^2, \otimes^2_T \mathcal{H}_2 \Sigma Y)$$

such that the following equality holds in $[\Sigma X^2, \Sigma Y \vee \Sigma Y]^T$,

$$\mu f = i_1 f + i_2 f + j(i_3 i_{12})_* \gamma_2(f).$$

Here $i_1, i_2, \mu\colon \Sigma X^2 \to \Sigma X^2 \vee \Sigma X^2$ are the inclusions of the factors of the coproduct and the co-H-multiplication of the suspension ΣX^2. Moreover, $\gamma_2(f)$ exists and is unique by Proposition 2.3.5.

If $X = S^2_\alpha$ the function

$$\gamma_2\colon [\Sigma X^2, \Sigma Y] \longrightarrow H^3(\Sigma X^2, \otimes^2_T \mathcal{H}_2 \Sigma Y)$$

coincides with the $\mathbf{ab}(T)$-module morphism in Definition 3.1.2 evaluated at the object in $\mathbf{ab}(T)$ corresponding to the spherical object S^2_α. However in general this function γ_2 need not be a homomorphism but a quadratic function in the sense of Definition 2.1.1. In order to check this fact we define the following new proper homotopy operation.

DEFINITION 3.1.5. Given three T-complexes X^2, Y and Z with $\dim X^2 \leq 2$ and two maps $f\colon \Sigma X^2 \to \Sigma Y$ and $g\colon \Sigma X^2 \to \Sigma Z$ we define their *interior cup-product*

$$f \cup g \in H^3(\Sigma X^2, \mathcal{H}_2 \Sigma Y \otimes_T \mathcal{H}_2 \Sigma Z)$$

as the unique element satisfying the following equality in $[\Sigma X^2, \Sigma Y \vee \Sigma Z]^T$,

$$i_2 f + i_1 g = i_1 g + i_2 f + j(i_3 i_{12})_*(f \cup g).$$

The element $f \cup g$ exists and is unique by Proposition 2.3.5. Moreover, by Proposition 2.3.4 the proper homotopy group $[\Sigma X^2, \Sigma Y \vee \Sigma Z]^T$ has nilpotency class 2, so the interior cup-product is bilinear and factors through the cokernel of j in Proposition 2.3.4 by a homomorphism

$$H^2(\Sigma X^2, \mathcal{H}_2 \Sigma Y) \otimes H^2(\Sigma X^2, \mathcal{H}_2 \Sigma Z) \xrightarrow{\cup} H^3(\Sigma X^2, \mathcal{H}_2 \Sigma Y \otimes_T \mathcal{H}_2 \Sigma Z).$$

Here we use the Hurewicz isomorphism as an identification. The suspension isomorphisms in proper (co)homology allow us to regard this homomorphism as follows

$$H^1(X^2, \mathcal{H}_1 Y) \otimes H^1(X^2, \mathcal{H}_1 Z) \xrightarrow{\cup} H^2(X^2, \mathcal{H}_1 Y \otimes_T \mathcal{H}_1 Z).$$

Furthermore, by using [**Ark64**] one can check that this is a generalization of the cup-product in ordinary homotopy theory. We point out that a complete definition of a cup-product operation for the proper cohomology in Chapter 1.3 is not known.

PROPOSITION 3.1.6. *Given two T-complexes X^2 and Y with $\dim X^2 \leq 2$ and two maps $f, g\colon \Sigma X^2 \to \Sigma Y$ the following equality is satisfied*

$$\gamma_2(f + g) = \gamma_2(f) + \gamma_2(g) + f \cup g \in H^3(\Sigma X^2, \otimes^2_T \mathcal{H}_2 \Sigma Y).$$

PROOF. The result follows from the equalities in $[\Sigma X^2, \Sigma Y \vee \Sigma Y]^T$

$$\begin{aligned}
& i_1(f+g)+i_2(f+g) \\
+j(i_3 i_{12})_*(\gamma_2(f)+\gamma_2(g)+f\cup g) &= i_1 f + i_1 g + i_2 f + i_2 g \\
&\quad + j(i_3 i_{12})_*(\gamma_2(f)+\gamma_2(g)+f\cup g) \\
&= i_1 f + i_2 f + i_1 g + i_2 g \\
&\quad + j(i_3 i_{12})_*(\gamma_2(f)+\gamma_2(g)) \\
&= \mu f + \mu g \\
&= \mu(f+g).
\end{aligned}$$

Here we use that j in Proposition 2.3.4 is central and Definitions 3.1.4 and 3.1.5. □

This result together with [**Bau96**] A.10.2 (f) can be used to prove that the interior cup-product that we have defined in proper homotopy theory generalizes the corresponding one in ordinary homotopy theory, which appears for example in [**Bau96**] A.1.18.

PROPOSITION 3.1.7. *Given two T-complexes X^2 and Y with $\dim X^2 \leq 2$ the following diagram commutes*

$$\begin{tikzcd}
H^3(\Sigma X^2, \Gamma_T \mathcal{H}_2 \Sigma Y) \arrow[r, "i_{3*}"] \arrow[d, "\tau_{T*}"'] & H^3(\Sigma X^2, \Pi_3 \Sigma Y) \arrow[d, hook, "j"] \\
H^3(\Sigma X^2, \otimes_T^2 \mathcal{H}_2 \Sigma Y) & {[\Sigma X^2, \Sigma Y]^T} \arrow[l, "\gamma_2"]
\end{tikzcd}$$

PROOF. Given $a \in H^3(\Sigma X^2, \Gamma_T \mathcal{H}_2 \Sigma Y)$ we have the equalities

$$\begin{aligned}
j(i_3 i_{12})_* \gamma_2(j i_{3*}(a)) &= -i_2 j i_{3*}(a) - i_1 j i_{3*}(a) + \mu j i_{3*}(a) \\
&= j[i_3(-\Gamma_T i_2 - \Gamma_T i_1 + \Gamma_T(i_1+i_2))]_*(a) \\
&= j(i_3 i_{12} \tau_T)_*(a) \\
&= j(i_3 i_{12})_*(\tau_T)_*(a).
\end{aligned}$$

For the first one we use Definition 3.1.4; for the second one we use the naturality of j in Proposition 2.3.4 and Whitehead's long exact sequence, as well as Theorem 2.3.1; finally for the third one we use equalities (a) in the proof of Proposition 3.1.3.

The homomorphism j is injective by Proposition 2.3.4 and $(i_3 i_{12})_*$ is injective because $i_3 i_{12}$ is a split monomorphism, see Proposition 2.3.2, therefore the proposition follows. □

DEFINITION 3.1.8. Given two T-complexes X^2 and Y with $\dim X^2 \leq 2$ we define the *stable James-Hopf invariant* γ_2^n $(n \geq 2)$ as the unique function which fits into the following commutative diagram

$$\begin{tikzcd}
{[\Sigma^n X^2, \Sigma^n Y]^T} \arrow[r, "\gamma_2^n"] & H^{n+2}(\Sigma^n X^2, \hat{\otimes}_T^2 \mathcal{H}_{n+1}\Sigma^n Y) \\
{[\Sigma X^2, \Sigma Y]^T} \arrow[u, two heads, "\Sigma^{n-1}"] \arrow[r, "\gamma_2"] & H^3(\Sigma X^2, \otimes_T^2 \mathcal{H}_2 \Sigma Y) \arrow[u, two heads, "(\bar{\sigma}_T)_*"'] \\
H^3(\Sigma X^2, \otimes_T^2 \mathcal{H}_2 \Sigma Y) \arrow[u, "j(i_3[-,-]_T)_*"] \arrow[r, equal] & H^3(\Sigma X^2, \otimes_T^2 \mathcal{H}_2 \Sigma Y) \arrow[u, "(\tau_T[-,-]_T)_*"']
\end{tikzcd}$$

The commutativity of the lower square follows from Proposition 3.1.7. Here we use the suspension isomorphisms in (co)homology as identifications.

It is not as clear as in Definition 3.1.2 that γ_2^n exists and is unique, but we check this in the following lemma.

Lemma 3.1.9. *Indeed the function γ_2^n is well defined.*

Proof. The column in the left is exact by Proposition 2.3.6, therefore we just need to check that always $(\bar{\sigma}_T)_*\gamma_2(f + j(i_3[-,-]_T)_*(\alpha)) = (\bar{\sigma}_T)_*\gamma_2(f)$. The properties of the interior cup-product defined in Definition 3.1.5 and Proposition 2.3.4 show that $f \cup j(i_3[-,-]_T)_*(\alpha) = 0$, hence by Proposition 3.1.6 we have that $\gamma_2(f + j(i_3[-,-]_T)_*(\alpha)) = \gamma_2(f) + \gamma_2(j(i_3[-,-]_T)_*(\alpha))$, but using the commutativity of the lower square we get $(\bar{\sigma}_T)_*\gamma_2(j(i_3[-,-]_T)_*(\alpha)) = (\bar{\sigma}_T)_*(\tau_T[-,-]_T)_*(\alpha) = 0$ by (2.2.19), hence we are done. $\square$

Again γ_2^n $(n \geq 2)$ in Definition 3.1.8 generalizes the stable James-Hopf invariant in Definition 3.1.2 when $X^2 = S_\alpha^2$ is a 2-dimensional spherical object, and this new γ_2^n is not in general a homomorphism but a quadratic map. To check this last statement we define a new stable proper homotopy operation.

Definition 3.1.10. Given three T-complexes X^2, Y and Z with $\dim X^2 \leq 2$ and maps $f\colon \Sigma^n X^2 \to \Sigma^n Y$, $g\colon \Sigma^n X^2 \to \Sigma^n Z$ $(n \geq 2)$ then by using the suspension isomorphisms in proper (co)homology we define their *interior cup-product*

$$f \cup g \in H^{n+2}(\Sigma^n X^2, \mathcal{H}_{n+1}\Sigma^n Y \otimes_T \mathcal{H}_{n+1}\Sigma^n Z)$$

as the interior cup-product $f' \cup g'$ in the sense of Definition 3.1.5 of two maps $f'\colon \Sigma X^2 \to \Sigma Y$ and $g'\colon \Sigma X^2 \to \Sigma Z$ with $\Sigma^{n-1} f' = f$ and $\Sigma^{n-1} g' = g$.

The properties of the interior cup-product in Definition 3.1.5 together with Proposition 2.3.6 guarantee that this new interior cup-product is well defined, bilinear, and factors through the cokernel of j in Proposition 2.3.4 by a homomorphism

$$\begin{tikzcd}
H^{n+1}(\Sigma^n X^2, \mathcal{H}_{n+1}\Sigma^n Y) \otimes H^{n+1}(\Sigma^n X^2, \mathcal{H}_{n+1}\Sigma^n Z) . \arrow[d, "\cup"] \\
H^{n+2}(\Sigma^n X^2, \mathcal{H}_{n+1}\Sigma^n Y \otimes_T \mathcal{H}_{n+1}\Sigma^n Z)
\end{tikzcd}$$

Moreover, by Proposition 3.1.6 and Definition 3.1.8 we get the following result.

Proposition 3.1.11. *Given two T-complexes X^2 and Y with $\dim X^2 \leq 2$ and two maps $f, g\colon \Sigma^n X^2 \to \Sigma^n Y$ $(n \geq 2)$ the following equality is satisfied*

$$\gamma_2^n(f+g) = \gamma_2^n(f) + \gamma_2^n(g) + (\bar{\sigma}_T)_*(f \cup g) \in H^{n+2}(\Sigma^n X, \hat{\otimes}_T^2 \mathcal{H}_{n+1}\Sigma^n Y).$$

Finally we will show the behaviour of the James-Hopf invariants in Definitions 3.1.4 and 3.1.8 with respect to compositions of maps.

PROPOSITION 3.1.12. *Given three T-complexes X^2, Y^2 and Z with dimensions $\dim X^2$, $\dim Y^2 \leq 2$ and maps $f\colon \Sigma X^2 \to \Sigma Y^2$, $g\colon \Sigma Y^2 \to \Sigma Z$ the following equality holds*

$$\gamma_2(gf) = f^*\gamma_2(g) + g_*\gamma_2(f) \in H^3(\Sigma X^2, \otimes_T^2 \mathcal{H}_2 \Sigma Z).$$

Here $f^* = H^3(f, \otimes_T^2 \mathcal{H}_2 \Sigma Z)$ and $g_* = H^3(\Sigma X^2, \otimes_T^2 \mathcal{H}_2 g)$.

PROOF. It follows from the equalities

$$\begin{array}{llcl}
\text{(a)} & (i_1+i_2)gf & = & (i_1g + i_2g + j(i_3i_{12})_*\gamma_2(g))f \\
\text{(b)} & & = & (i_1g+i_2g)f + j(i_3i_{12})_*f^*\gamma_2(g) \\
 & & = & (g\vee g)(i_1+i_2)f + j(i_3i_{12})_*f^*\gamma_2(g) \\
\text{(c)} & & = & (g\vee g)(i_1f+i_2f+j(i_3i_{12})_*\gamma_2(f)) + j(i_3i_{12})_*f^*\gamma_2(g) \\
\text{(d)} & & = & i_1gf + i_2gf + j(i_3i_{12})_*g_*\gamma_2(f) + j(i_3i_{12})_*f^*\gamma_2(g).
\end{array}$$

For (a) and (c) we use Definition 3.1.4, and for (b) and (d) we use Proposition 2.3.4 and the naturality of j and i_3i_{12} in Propositions 2.3.4 and 2.3.2, respectively. $\square$

The following corollary follows directly from Definitions 3.1.8 and 3.1.10.

COROLLARY 3.1.2. *Given three T-complexes X^2, Y^2 and Z with dimensions $\dim X^2$, $\dim Y^2 \leq 2$ and maps $f\colon \Sigma^n X^2 \to \Sigma^n Y^2$, $g\colon \Sigma^n Y^2 \to \Sigma^n Z$ the following equality holds ($n \geq 2$)*

$$\gamma_2^n(gf) = f^*\gamma_2^n(g) + g_*\gamma_2^n(f) \in H^{n+2}(\Sigma^n X^2, \hat{\otimes}_T^2 \mathcal{H}_{n+1}\Sigma^n Z).$$

Here $f^* = H^{n+2}(f, \hat{\otimes}_T^2 \mathcal{H}_{n+1}\Sigma^n Z)$ and $g_* = H^{n+2}(\Sigma^n X^2, \hat{\otimes}_T^2 \mathcal{H}_{n+1} g)$.

3.2. Cup-product invariants

DEFINITION 3.2.1. The *cup-product invariant* of a 1-reduced homotopy system $(\mathcal{C}_*, f_4)$ is the cohomology class

$$\cup_{(\mathcal{C}_*, f_4)} \in H^4(\mathcal{C}_*, \otimes_T^2 H_2 \mathcal{C}_*)$$

represented by the cocycle

$$\mathcal{C}_4 \xrightarrow{f_4} \Pi_3 C_{d_3} \xrightarrow{\gamma_2} \otimes_T^2 H_2 \mathcal{C}_*.$$

This is indeed a cocycle since f_4 already is. Moreover, notice that in order to define the cup-product invariant we have to choose a suspension structure in the cofiber C_{d_3}, which exists by Freudental's suspension theorem, otherwise the James-Hopf invariant γ_2 would not be defined.

The cup product invariant is a 0-cocycle in $\mathbf{H}_2(T)/\simeq$.

PROPOSITION 3.2.2. *Given a morphism $(\xi, \eta)\colon (\mathcal{C}_*, f_4) \to (\mathcal{C}'_*, g_4)$ in $\mathbf{H}_2(T)/\simeq$ the following equality holds*

$$\xi_*\cup_{(\mathcal{C}_*, f_4)} = \xi^*\cup_{(\mathcal{C}'_*, g_4)} \in H^4(\mathcal{C}_*, \otimes_T^2 H_2 \mathcal{C}'_*).$$

Here $\xi^* = H^4(\xi, \otimes_T^2 H_2 \mathcal{C}'_*)$ and $\xi_* = H^4(\mathcal{C}_*, \otimes_T^2 H_2 \xi)$.

PROOF. Recall from Definition 2.5.3 that, by using the isomorphisms in Proposition 1.3.1 as identifications, we have the equality $\eta f_4 = g_4\xi_4 \in [S^3_{\alpha_4}, C_{d'_{n+1}}]^T$ where $S^3_{\alpha_4}$ corresponds to $\mathcal{C}_4$, hence $\gamma_2(\eta f_4) = \gamma_2(g_4\xi_4)$. Moreover, by Proposition 3.1.12 we have that

$$\gamma_2(\eta f_4) = f_4^*\gamma_2(\eta) + \eta_*\gamma_2(f_4) \overset{\text{(a)}}{=} f_4^*\gamma_2(\eta) + (\otimes^2_T H_2\xi)\gamma_2 f_4,$$

$$\gamma_2(g_4\xi_4) = \xi_4^*\gamma_2(g_4) + g_{4*}\gamma_2(\xi_4) \overset{\text{(b)}}{=} \gamma_2 g_4\xi_4.$$

For (a) we use that η induces ξ in dimensions 2 and 3, and for (b) we use that $\gamma_2(\xi_4) = 0$ since the target of ξ_4 is a 3-dimensional spherical object, whose 2-dimensional proper homology module is trivial.

Moreover, $f_4^*\gamma_2(\eta)$ is a coboundary since by Definition 2.5.3 the morphism induced in cohomology $H^*(-, \otimes^2_T H_2\mathcal{C}'_*)$ by the composite

$$S^3_{\alpha_4} \xrightarrow{f_4} C_{d_3} \xrightarrow{q} S^3_{\alpha_3}$$

coincides with

$$(\otimes^2_T H_2\mathcal{C}'_*)(d_4)\colon (\otimes^2_T H_2\mathcal{C}'_*)(\mathcal{C}_3) \overset{q^*}{\twoheadrightarrow} H^3(C_{d_3}, \otimes^2_T H_2\mathcal{C}'_*) \xrightarrow{f_4^*} (\otimes^2_T H_2\mathcal{C}'_*)(\mathcal{C}_4),$$

and q^* is surjective because q is part of the long cofiber sequence

$$S^2_{\alpha_3} \overset{d_3}{\to} S^2_{\alpha_2} \rightarrowtail C_{d_3} \overset{q}{\to} S^3_{\alpha_3} \overset{d_3}{\to} S^3_{\alpha_2} \to \cdots.$$

Therefore the cocycle $(\otimes^2_T H_2\xi)\gamma_2 f_4$ represents the same class as $\gamma_2 g_4\xi_4$ in cohomology, and this is what we exactly wanted to prove. □

COROLLARY 3.2.3. *The cup-product invariant of a* 1*-reduced homotopy system* $(\mathcal{C}_*, f_4)$ *does not depend on the suspension structure chosen in* C_{d_3} *for its definition.*

This can be regarded as a special case of Proposition 3.2.2 for the identity morphism in $(\mathcal{C}_*, f_4)$.

DEFINITION 3.2.4. The *reduced cup-product invariant* of an $(n-1)$-reduced homotopy system $(\mathcal{C}_*, f_{n+2})$ $(n \geq 3)$ is the cohomology class

$$\hat{\cup}_{(\mathcal{C}_*, f_{n+2})} \in H^{n+2}(\mathcal{C}_*, \hat{\otimes}^2_T H_n\mathcal{C}_*)$$

represented by the cocycle

$$\mathcal{C}_{n+2} \xrightarrow{f_{n+2}} \Pi_{n+1}C_{d_{n+1}} \xrightarrow{\gamma_2^{n-1}} \hat{\otimes}^2_T H_n\mathcal{C}_*.$$

This is a cocycle because f_{n+2} is a cocycle. Notice that in order to define the reduced cup-product we have to choose an $(n-1)$-fold suspension structure in the cofiber $C_{d_{n+1}}$, this is possible by Freudental's suspension theorem, otherwise the reduced James-Hopf invariant γ_2^{n-1} would not be defined.

The reduced cup-product invariant of a 1-reduced homotopy system $(\mathcal{C}_*, f_4)$ is the following change of coefficients of the cup-product invariant

$$\hat{\cup}_{(\mathcal{C}_*, f_4)} = (\bar{\sigma}_T)_*\cup_{(\mathcal{C}_*, f_4)} \in H^4(\mathcal{C}_*, \hat{\otimes}^2_T H_2\mathcal{C}_*).$$

One can check that the reduced cup-product is a 0-cocycle in $\mathbf{H}_n(T)/\simeq$, for $n = 2$ it follows directly from Proposition 3.2.2 and for $n \geq 3$ one argues as in the proof of that proposition but using Proposition 3.1.2 instead of Proposition 3.1.12.

PROPOSITION 3.2.5. *Given a morphism* $(\xi,\eta)\colon (\mathcal{C}_*, f_{n+2}) \to (\mathcal{C}'_*, g_{n+2})$ *in the category* $\mathbf{H}_n(T)/\simeq$ $(n \geq 2)$ *the following equality holds*

$$\xi_*\hat{\cup}_{(\mathcal{C}_*,f_{n+2})} = \xi^*\hat{\cup}_{(\mathcal{C}'_*,g_{n+2})} \in H^{n+2}(\mathcal{C}_*, \hat{\otimes}^2_T H_n\mathcal{C}'_*).$$

Here $\xi^* = H^{n+2}(\xi, \hat{\otimes}^2_T H_n\mathcal{C}'_*)$ *and* $\xi_* = H^{n+2}(\mathcal{C}_*, \hat{\otimes}^2_T H_n\xi)$.

COROLLARY 3.2.6. *The reduced cup-product invariant of an* $(n-1)$*-reduced homotopy system* $(\mathcal{C}_*, f_{n+2})$ $(n \geq 2)$ *does not depend on the* $(n-1)$*-fold suspension structure chosen in* $C_{d_{n+1}}$ *for its definition.*

By Definition 3.1.2 the reduced cup-product invariant is preserved by suspensions in the sense of the statement of the following proposition.

PROPOSITION 3.2.7. *The following equality holds for any* $(n-1)$*-reduced homotopy system* $(\mathcal{C}_*, f_{n+2})$ $(n \geq 2)$

$$\hat{\cup}_{(\mathcal{C}_*,f_{n+2})} = \hat{\cup}_{\Sigma(\mathcal{C}_*,f_{n+2})} \in H^{n+2}(\mathcal{C}_*, \hat{\otimes}^2_T H_n\mathcal{C}_*) = H^{n+3}(\Sigma\mathcal{C}_*, \hat{\otimes}^2_T H_{n+1}\Sigma\mathcal{C}_*).$$

We can now can define cup-product invariants for T-complexes by using the functors r in (2.5.5).

DEFINITION 3.2.8. The *cup-product invariant* of a 1-reduced T-complex X is

$$\cup_X = \cup_{rX} \in H^4(X, \otimes^2_T \mathcal{H}_2 X).$$

Moreover, the *reduced cup-product invariant* of an $(n-1)$-reduced T-complex X $(n \geq 2)$ is

$$\hat{\cup}_X = \hat{\cup}_{rX} \in H^{n+2}(X, \hat{\otimes}^2_T \mathcal{H}_n X).$$

These invariants are 0-cocycles in $\mathbf{CW}^T_n/\simeq$ by Propositions 3.2.2 and 3.2.5, as we state in the following proposition.

PROPOSITION 3.2.9. *Given a proper map* $f\colon X \to Y$ *between* 1*-reduced* T*-complexes the following equality holds*

$$f_*\cup_X = f^*\cup_Y \in H^4(X, \otimes^2_T \mathcal{H}_2 Y).$$

Here $f^* = H^4(f, \otimes^2_T \mathcal{H}_2 Y)$ *and* $f_* = H^4(X, \otimes^2_T \mathcal{H}_2 f)$.

PROPOSITION 3.2.10. *Given a proper map* $f\colon X \to Y$ *between* $(n-1)$*-reduced* T*-complexes* $(n \geq 2)$ *the following equality holds*

$$f_*\hat{\cup}_X = f^*\hat{\cup}_Y \in H^{n+2}(X, \hat{\otimes}^2_T \mathcal{H}_n Y).$$

Here $f^* = H^{n+2}(f, \hat{\otimes}^2_T \mathcal{H}_n Y)$ *and* $f_* = H^{n+2}(X, \hat{\otimes}^2_T \mathcal{H}_n f)$.

3.3. Cup-product and obstruction theory

The following theorem shows the significance of the cup-product invariant in obstruction theory. In the statement we use the natural transformations τ_T and $\bar{\tau}_T$ in (2.2.19).

THEOREM 3.3.1. *Given two* $(n-1)$*-reduced homotopy systems* $(\mathcal{C}_*, f_{n+2})$ *and* $(\mathcal{C}'_*, g_{n+2})$ *and a chain homotopy class* $\xi\colon \mathcal{C}_* \to \mathcal{C}'_*$ *the following equalities hold: if* $n = 2$

$$(\tau_T)_*\theta_{(\mathcal{C}_*,f_4),(\mathcal{C}'_*,g_4)}(\xi) = \xi_* \cup_{(\mathcal{C}_*,f_4)} - \xi^*\cup_{(\mathcal{C}'_*,g_4)} \in H^4(\mathcal{C}_*, \otimes^2_T H_2\mathcal{C}'_*),$$

and if $n \geq 3$

$$(\bar{\tau}_T)_* \theta_{(\mathcal{C}_*, f_{n+2}), (\mathcal{C}'_*, g_{n+2})}(\xi) = \xi_* \hat{\cup}_{(\mathcal{C}_*, f_{n+2})} - \xi^* \hat{\cup}_{(\mathcal{C}'_*, g_{n+2})} \in H^{n+2}(\mathcal{C}_*, \hat{\otimes}^2_T H_n \mathcal{C}'_*).$$

Here θ is the obstruction operator in one of the exact sequences for functors appearing in (2.5.7).

Proof of Theorem 3.3.1. Let us prove the case $n = 2$. With the same notation as in Section 2.5 recall that the cohomology class $\theta_{(\mathcal{C}_*, f_4), (\mathcal{C}'_*, g_4)}(\xi)$ is represented by a cocycle β fitting into the following commutative diagram

$$\begin{tikzcd} \mathcal{C}_4 \arrow[rr, "-g_4\xi_4+\eta f_4"] \arrow[dr, "\beta"'] & & \Pi_3 C_{d'_3} \\ & \Gamma_T H_2 \mathcal{C}'_* \arrow[ur, hook, "i_3"'] & \end{tikzcd}$$

where $\eta\colon C_{d_3} \to C_{d'_3}$ is a map inducing ξ in dimensions 2 and 3. By Proposition 3.1.3 the change of coefficients $(\tau_T)_* \theta_{(\mathcal{C}_*, f_4), (\mathcal{C}'_*, g_4)}(\xi)$ is represented by the cocycle

$$\begin{aligned} \gamma_2(-g_4\xi_4 + \eta f_4) &= -\gamma_2(g_4\xi_4) + \gamma_2(\eta f_4) \\ &= -\xi_4^* \gamma_2(g_4) + f_4^* \gamma_2(\eta) + \xi_* \gamma_2(f_4) \\ &= -\gamma_2 g_4 \xi_4 + f_4^* \gamma_2(\eta) + \xi_* \gamma_2(f_4). \end{aligned}$$

For the second equality we use Proposition 3.1.12 and the fact that $\gamma_2(\xi_4) = 0$, see the proof of Proposition 3.2.2. Moreover, one can check, also as in the proof of Proposition 3.2.2, that $f_4^* \gamma_2(\eta)$ is a coboundary and the case $n = 2$ follows since $\gamma_2 g_4 \xi_4$ and $\xi_* \gamma_2(f_4)$ are cocycles representing $\xi^* \cup_{(\mathcal{C}'_*, g_4)}$ and $\xi_* \cup_{(\mathcal{C}_*, f_4)}$, respectively.

The case $n \geq 3$ can be obtained analogously by using Propositions 3.1.3 and 3.1.2 instead of Propositions 3.1.3 and 3.1.12. □

The following corollary is a vanishing result on the change of coefficients of the cohomology classes $\{\theta\}$ in (2.5.9). It is a consequence of Theorem 3.3.1 and Propositions 3.2.2 and 3.2.5, see Remark 2.5.2.

Corollary 3.3.2. *For* $n = 2$

$$0 = (\tau_T)_* \{\theta\} \in H^1(\mathbf{chain}_2(\mathbf{ab}(T))/\simeq, H^4(-, \otimes^2_T H_2)),$$

and for any $n \geq 3$

$$0 = (\bar{\tau}_T)_* \{\theta\} \in H^1(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \hat{\otimes}^2_T H_n)).$$

In the second corollary of Theorem 3.3.1 we use the notation introduced in Definition 2.5.1 (5).

Corollary 3.3.3. *Let* $(\mathcal{C}_*, f_{n+2})$ *be an* $(n-1)$*-reduced homotopy system, for* $n = 2$ *if* $\alpha \in H^4(\mathcal{C}_*, \Gamma_T H_2 \mathcal{C}_*)$ *then*

$$\cup_{(\mathcal{C}_*, f_4)+\alpha} = \cup_{(\mathcal{C}_*, f_4)} + (\tau_T)_*(\alpha),$$

and for any $n \geq 3$ *if* $\alpha \in H^{n+2}(\mathcal{C}_*, H_n \mathcal{C}_* \otimes \mathbb{Z}/2)$ *then*

$$\hat{\cup}_{(\mathcal{C}_*, f_{n+2})+\alpha} = \hat{\cup}_{(\mathcal{C}_*, f_{n+2})} + (\bar{\tau}_T)_*(\alpha).$$

3.4. The chain cup-product and cohomology of categories

This section contains the definition and elementary properties of the chain cup-produc invariant. We carry out a thorough study of this invariant in the next chapter. This invariant plays a crucial role in the main results obtained in this paper concerning the proper homotopy classification problem for locally compact A_n^2-polyhedra.

DEFINITION 3.4.1. The *chain cup-product invariant* of a bounded chain complex $\mathcal{C}_*$ in $\mathbf{ab}(T)$ concentrated in degrees $\geq n$ ($n \geq 2$) is defined as

$$\bar{\cup}_{\mathcal{C}_*} = (\bar{q}_T)_* \hat{\cup}_{(\mathcal{C}_*, f_{n+2})} \in H^{n+2}(\mathcal{C}_*, \wedge_T^2 H_n \mathcal{C}_*),$$

where $(\mathcal{C}_*, f_{n+2})$ is an $(n-1)$-reduced homotopy system with $\lambda(\mathcal{C}_*, f_{n+2}) = \mathcal{C}_*$. Such a homotopy system exists by Proposition 2.5.8.

PROPOSITION 3.4.2. *The chain cup-product is a* 0*-cocycle of* $\mathbf{chain}_n(\mathbf{ab}(T))/\simeq$ *with coefficients in* $H^{n+2}(-, H_n \otimes \mathbb{Z}_2)$ ($n \geq 2$)*, i. e. given a chain homotopy class* $\xi\colon \mathcal{C}_* \to \mathcal{C}'_*$ *between bounded chain complexes in* $\mathbf{ab}(T)$ *concentrated in degrees* $\geq n$ *the following equality holds*

$$\xi_* \bar{\cup}_{\mathcal{C}_*} = \xi^* \bar{\cup}_{\mathcal{C}'_*} \in H^{n+2}(\mathcal{C}_*, \wedge_T^2 H_n \mathcal{C}'_*).$$

PROOF. Let $(\mathcal{C}_*, f_{n+2})$ and $(\mathcal{C}'_*, g_{n+2})$ be $(n-1)$-reduced homotopy systems. By Proposition 3.3.1 and (2.2.19) we have for all $n \geq 3$

$$\begin{array}{rcl} 0 & = & (\bar{q}_T)_*(\bar{\tau}_T)_*\theta_{(\mathcal{C}_*, f_{n+2}),(\mathcal{C}'_*, g_{n+2})}(\xi) \\ & = & (\bar{q}_T)_*(\xi_*\hat{\cup}_{(\mathcal{C}_*, f_{n+2})} - \xi^*\hat{\cup}_{(\mathcal{C}'_*, g_{n+2})}) \\ & = & \xi_*\bar{\cup}_{\mathcal{C}_*} - \xi^*\bar{\cup}_{\mathcal{C}'_*}, \end{array}$$

and for $n = 2$

$$\begin{array}{rcl} 0 & = & (\bar{q}_T)_*(\bar{\tau}_T)_*(\sigma_T)_*\theta_{(\mathcal{C}_*, f_4),(\mathcal{C}'_*, g_4)}(\xi) \\ & = & (\bar{q}_T)_*(\bar{\sigma}_T)_*(\tau_T)_*\theta_{(\mathcal{C}_*, f_4),(\mathcal{C}'_*, g_4)}(\xi) \\ & = & (\bar{q}_T)_*(\bar{\sigma}_T)_*(\xi_* \cup_{(\mathcal{C}_*, f_4)} - \xi^*\cup_{(\mathcal{C}'_*, g_4)}) \\ & = & (\bar{q}_T)_*(\xi_*\hat{\cup}_{(\mathcal{C}_*, f_4)} - \xi^*\hat{\cup}_{(\mathcal{C}'_*, g_4)}) \\ & = & \xi_*\bar{\cup}_{\mathcal{C}_*} - \xi^*\bar{\cup}_{\mathcal{C}'_*}. \end{array}$$

□

We will denote

$$\bar{\cup}_n \in H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge_T^2 H_n))$$

to the element in cohomology defined by the chain cup-product ($n \geq 2$).

COROLLARY 3.4.3. *The chain cup-product invariant of a bounded chain complex* $\mathcal{C}_*$ *in* $\mathbf{ab}(T)$ *does not depend on the* $(n-1)$*-reduced homotopy system* $(\mathcal{C}_*, f_{n+2})$ *chosen for its definition.*

This can be regarded as a special case of Proposition 3.4.2 for the identity morphism in $\mathcal{C}_*$.

As a consequence of the stability of the reduced cup-product, see Proposition 3.2.7, we have that the chain cup-product invariant is also stable.

Proposition 3.4.4. *Given a bounded chain complex $\mathcal{C}_*$ in $\mathbf{ab}(T)$ concentrated in degrees $\geq n$ ($n \geq 2$) we have that*

$$\bar{\cup}_{\mathcal{C}_*} = \bar{\cup}_{\Sigma\mathcal{C}_*} \in H^{n+2}(\mathcal{C}_*, \wedge^2_T H_n\mathcal{C}_*) = H^{n+3}(\Sigma\mathcal{C}_*, \wedge^2_T H_{n+1}\Sigma\mathcal{C}_*),$$

in particular

$$\bar{\cup}_n = \Sigma^*\bar{\cup}_{n+1}.$$

From now one we shall simply write $\bar{\cup}$ for the cohomology element $\bar{\cup}_n$, or $\bar{\cup}^T$ if we want to specify the tree.

Given a proper map $f\colon T \to T'$ between trees the "change of tree" functor $\mathbb{F}^f\colon \mathbf{ab}(T) \to \mathbf{ab}(T')$ in Proposition 1.2.7 induces an obvious functor

$$\mathbb{F}^f\colon \mathbf{chain}_n(\mathbf{ab}(T))/\simeq \longrightarrow \mathbf{chain}_n(\mathbf{ab}(T'))/\simeq,$$

already considered in (2.5.12), and a natural transformation between bimodules over $\mathbf{chain}_n(\mathbf{ab}(T))/\simeq$

$$\bar{\mathbb{F}}^f\colon H^{n+2}(-, \wedge^2_T H_n) \longrightarrow H^{n+2}(\mathbb{F}^f, \wedge^2_T H_n\mathbb{F}^f)$$

by taking H^{n+2} in the following composition of cochain homomorphisms

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{ab}(T)}(\mathcal{C}_*, \wedge^2_T H_n\mathcal{C}_*) \arrow[r, "\mathbb{F}^f_*"] & \mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, \mathbb{F}^f_* \wedge^2_T H_n\mathcal{C}_*) \arrow[d] \\ \mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, \wedge^2_{T'} H_n\mathbb{F}^f\mathcal{C}_*) \arrow[r, equal] & \mathrm{Hom}_{\mathbf{ab}(T')}(\mathbb{F}^f\mathcal{C}_*, \wedge^2_{T'}\mathbb{F}^f_* H_n\mathcal{C}_*) \end{tikzcd} \tag{3.4.5}$$

defined as (2.5.13) and (2.5.14).

Proposition 3.4.6. *Given a proper map $f\colon T \to T'$ between trees the chain cup-product in cohomology of categories is compatible with the "change of tree" functor $\mathbb{F}^f\colon \mathbf{ab}(T) \to \mathbf{ab}(T')$, i. e. with the notation above*

$$(\mathbb{F}^f)^*\bar{\cup}^T = \bar{\mathbb{F}}^f_*\bar{\cup}^T.$$

Proof. By the stability of the chain cup-product, see Proposition 3.4.4, it is enough to make the proof of $n = 2$. Recall that we also have a topological "change of tree" functor $f_*\colon \mathbf{Topp}^T_c/\simeq \to \mathbf{Topp}^{T'}_c/\simeq$, see (1.1.2). If Z is any T complex this functor induces an $\mathbf{ab}(T)$-module morphism

$$\Pi_3 Z \longrightarrow (\mathbb{F}^f)^*\Pi_3 f_* Z,$$

given by

$$(\Pi_3 Z)(\mathbb{Z}\langle A\rangle_\alpha) = [S^3_\alpha, Z]^T \xrightarrow{f_*} [f_*S^3_\alpha, f_*Z]^{T'} = ((\mathbb{F}^f)^*\Pi_3 Z)(\mathbb{Z}\langle A\rangle_\alpha).$$

This morphism has an adjoint which is an $\mathbf{ab}(T')$-module morphism

$$\mathbb{F}^f_*\Pi_3 Z \longrightarrow \Pi_3 f_* Z.$$

The functor f_* commutes up to natural equivalence with the suspension, and it is easy to check that for $Z = \Sigma Y$ this adjoint $\mathbf{ab}(T')$-module morphism above is compatible with the James-Hopf invariant in Definition 3.1.2, i. e. there is a commutative diagram

$$\begin{tikzcd} \mathbb{F}^f_*\Pi_3\Sigma Y \arrow[rr, "\mathbb{F}^f_*\gamma_2"] \arrow[d] & & \mathbb{F}^f_* \otimes^2_T \mathcal{H}_2\Sigma Y \arrow[d] \\ \Pi_3\Sigma f_*Y \arrow[r, "\gamma_2"] & \otimes^2_{T'}\mathcal{H}_2\Sigma f_*Y \arrow[r, equal] & \otimes^2_{T'}\mathbb{F}^f_*\mathcal{H}_2\Sigma Y \end{tikzcd}$$

The vertical arrow in the right is given by Proposition 2.2.17 and $\mathbb{F}^f_*\mathcal{H}_2\Sigma Y = \mathcal{H}_2\Sigma f_*Y$ because $\mathcal{C}_*\Sigma f_*Y = \mathbb{F}^f\mathcal{C}_*\Sigma Y$ is concentrated in dimensions ≥ 2 and $\mathbb{F}^f_*$ is right-exact.

This implies that if $(\mathcal{C}_*, f_4)$ is a 1-reduced T-homotopy system then $(\mathbb{F}^f\mathcal{C}_*, f_*f_4)$ is also a 1-reduced T'-homotopy system and

$$\bar{\mathbb{F}}^f\cup_{(\mathcal{C}_*,f_4)} = \cup_{(\mathbb{F}^f\mathcal{C}_*,f_*f_4)} \in H^4(\mathbb{F}^f\mathcal{C}_*, \otimes^2_T H_2\mathbb{F}^f\mathcal{C}_*),$$

in particular

$$\bar{\mathbb{F}}^f\bar{\cup}_{\mathcal{C}_*} = \bar{\cup}_{\mathbb{F}^f\mathcal{C}_*},$$

as we wanted to show. □

REMARK 3.4.7. Remark 2.5.16 also applies in this case, i. e. if $\mathfrak{F}(f)\colon \mathfrak{F}(T) \hookrightarrow \mathfrak{F}(T')$ is injective then $\bar{\mathbb{F}}^f$ is a natura isomorphism because (3.4.5) is a cochain isomorphism. In particular if $\bar{\cup}^T \neq 0$ for a tree T with a finite number of ends then $\bar{\cup}^{T'} \neq 0$ provided T' has more ends that T, compare Corollary 4.1.3.

3.5. Cup-product and co-H-multiplications

In this section we characterize under some assumptions the existence of co-H-multiplications by using the chain cup-product. As a consequence of this and the computations in the next chapter we obtain in Corollary 5.4.10 examples of degree 2 proper Moore spaces which do not admit co-H-multiplications.

PROPOSITION 3.5.1. *A* 1*-connected* T*-reduced* X *with* $\dim X \leq 4$ *possesses a co-H-multiplication if and only if* $0 = \cup_X \in H^4(X, \otimes^2_T\mathcal{H}_2X)$.

PROOF. Let $f_n\colon S^{n-1}_{\alpha_n} \to X^{n-1}$ be the attaching map of n-cells on X ($n \geq 2$), and $\mu\colon X^3 \to X^3 \vee X^3$ the co-H-multiplication associated to the suspension structure chosen over X^3 for the definition of the James-Hopf invariant $\gamma_2\colon \Pi_3X^3 \to \otimes^2_T\mathcal{H}_2X$. Suppose that $\cup_X = 0$, then there exists $\alpha \in (\otimes^2_T\mathcal{H}_2X)(\mathcal{C}_3X)$ such that $\gamma_2(f_4) = \alpha d_4$. If we write $\tilde{\alpha}$ for the image of α under the natural projection

$$(\otimes^2_T\mathcal{H}_2X)(\mathcal{C}_3X) \twoheadrightarrow H^3(X^3, \otimes^2_T\mathcal{H}_2X),$$

the equality $\gamma_2(f_4) = \alpha d_4$ is equivalent to say that $\gamma_2(f_4) = f_4^*(\tilde{\alpha})$ since $\alpha d_4 = f_4^*\tilde{\alpha}$, compare with the proof of Proposition 3.2.2. By Proposition 2.3.5 $-j(i_3i_{12})_*(\tilde{\alpha})+\mu$ is also a co-H-multiplication in X^3. In addition if μ' is the co-H-multiplication of

$S^3_{\alpha_4}$, which is unique up to homotopy, then

$$\begin{aligned}(-j(i_3i_{12})_*(\tilde{\alpha})+\mu)f_4 &= -j(i_3i_{12})_*f_4^*(\tilde{\alpha})+\mu f_4\\ &= -j(i_3i_{12})_*\gamma_2(f_4)+\mu f_4\\ &= i_1f_4+i_2f_4-\mu f_4+\mu f_4\\ &= i_1f_4+i_2f_4\\ &= (f_4\vee f_4)\mu'.\end{aligned}$$

Here we use in the first equality the naturality of the central extension in Proposition 2.3.5, and in the third one Definition 3.1.2. The previous equalities show that the following diagram commutes up to homotopy,

$$\begin{tikzcd} X^3 \arrow[r, "-j(i_3i_{12})_*(\tilde{\alpha})+\mu"] & X^3\vee X^3 \\ S^3_{\alpha_4} \arrow[u, "f_4"] \arrow[r, "\mu'"'] & S^3_{\alpha_4}\vee S^3_{\alpha_4} \arrow[u, "f_4\vee f_4"'] \end{tikzcd}$$

This diagram induces a map (a principal map in the sense of [**Bau89**] V.2) between the cofibers of the vertical arrows which is a co-H-multiplication in X.

Suppose now that X has a co-H-multiplication $\bar{\mu}\colon X\to X\vee X$. In the following chains of equalities we will use the inclusions $i_1,i_2\colon X\to X\vee X$ and the projections $p_1,p_2\colon X\vee X\to X$ of the factors of the coproduct; the inclusions $i_1,i_2\colon \mathcal{H}_2X\to\mathcal{H}_2X\oplus\mathcal{H}_2X$ and the projections $p_1,p_2\colon\mathcal{H}_2X\oplus\mathcal{H}_2X\to\mathcal{H}_2X$ of the factors of the direct sum; the natural identifications $\mathcal{H}_2(X\vee X)=\mathcal{H}_2X\oplus\mathcal{H}_2X$, $\mathcal{H}_2i_1=i_1$, $\mathcal{H}_2i_2=i_2$, $\mathcal{H}_2\bar{\mu}=i_1+i_2$, $\mathcal{H}_2p_1=p_1$ and $\mathcal{H}_2p_1=p_2$; and Proposition 3.2.2. We have that

$$\begin{aligned}\cup_{X\vee X} &= p_1^*i_{1*}\cup_X+p_2^*i_{2*}\cup_X\\ &= p_1^*(\otimes^2_Ti_1)_*\cup_X+p_2^*(\otimes^2_Ti_2)_*\cup_X,\end{aligned}$$

Moreover, by using these equalities

$$\begin{aligned}(\otimes^2_T(i_1+i_2))_*\cup_X &= \mu_*\cup_X\\ &= \mu^*\cup_{X\vee X}\\ &= (\otimes^2_Ti_1)_*\cup_X+(\otimes^2_Ti_2)_*\cup_X,\end{aligned}$$

i. e.

$$0=(i_1\otimes_T i_2)_*\cup_X+(i_2\otimes_T i_1)_*\cup_X\in H^4(X,\otimes^2_T(\mathcal{H}_2X\oplus\mathcal{H}_2X)),$$

therefore

$$\begin{aligned}0 &= (p_1\otimes_T p_2)_*((i_1\otimes_T i_2)_*\cup_X+(i_2\otimes_T i_1)_*\cup_X)\\ &= (p_1i_1\otimes_T p_2i_2)_*\cup_X+(p_1i_2\otimes_T p_2i_1)_*\cup_X\\ &= \cup_X.\end{aligned}$$

Now the proof is finished. $\square$

Proposition 3.5.2. *Let $\mathcal{C}_*$ be a chain complex in $\mathbf{ab}(T)$ concentrated in degrees 2, 3 and 4. There exists a 1-reduced co-H-space X with $\dim X\le 4$ and a proper cellular chain complex $\mathcal{C}_*X=\mathcal{C}_*$ if and only if the chain cup-product of $\mathcal{C}_*$ vanishes $0=\bar{\cup}_{\mathcal{C}_*}\in H^4(\mathcal{C}_*,\wedge^2_TH_2\mathcal{C}_*)$.*

PROOF. Since $\mathcal{C}_*$ is concentrated in dimensions ≤ 4 the following sequence is exact, see (2.2.19),

$$H^4(\mathcal{C}_*, \Gamma_T H_2\mathcal{C}_*) \xrightarrow{(\tau_T)_*} H^4(\mathcal{C}_*, \otimes_T^2 H_2\mathcal{C}_*) \overset{(q_T)_*}{\twoheadrightarrow} H^4(\mathcal{C}_*, \wedge_T^2 H_2\mathcal{C}_*).$$

Let $(\mathcal{C}_*, f_4)$ be a 1-reduced homotopy system. By Definition 3.4.1

$$(q_T)_* \cup_{(\mathcal{C}_*, f_4)} = (\bar{q}_T)_*(\bar{\sigma}_T)_* \cup_{(\mathcal{C}_*, f_4)} = (\bar{q}_T)_* \hat{\cup}_{(\mathcal{C}_*, f_4)} = \bar{\cup}_{\mathcal{C}_*},$$

therefore $\bar{\cup}_{\mathcal{C}_*} = 0$ if and only if there exists $\alpha \in H^4(\mathcal{C}_*, \Gamma_T H_2\mathcal{C}_*)$ such that $(\tau_T)_*(\alpha) = \cup_{(\mathcal{C}_*, f_4)}$. If there is such an α, by using Corollary 3.3.3 we see that $\cup_{(\mathcal{C}_*, f_4)-\alpha} = 0$, so if X is a T-complex with $rX = (\mathcal{C}_*, f_4) - \alpha = (\mathcal{C}_*, f_4')$, for example $X = C_{f_4'}$ the cofiber of the map $S^3_{\alpha_4} \to C_{d_2}$ determined by f_4', then by Proposition 3.5.1 this X is a co-H-space, and $\mathcal{C}_*X = \lambda rX = \lambda(\mathcal{C}_*, f_4) = \mathcal{C}_*$.

On the other hand if X is a co-H-space with $\mathcal{C}_*X = \mathcal{C}_*$ then $\cup_X = 0$ by Proposition 3.5.1, hence

$$\bar{\cup}_{\mathcal{C}_*} = (\bar{q}_T)_* \hat{\cup}_{rX} = (\bar{q}_T)_*(\bar{\sigma}_T)_* \cup_X = 0.$$

$\square$

CHAPTER 4

The computation of the chain cup-product

In the first section of this chapter we give a purely algebraic formula for the chain cup-product invariant which was already defined in a homotopical way in Chapter 3. In the second section we compute the chain cup-product mod 2 by using the algebraic description in the first section, the localization theorem in cohomology of categories ([**Mur06**]), and the algebrain computations in Chapter 6, which are based on the representation theory in controlled algebra developed in [**Mur04**]. The computations in this chapter are crucial for the main homotopical results of this paper.

4.1. A purely algebraic description of the chain cup-product

In the statement of the following theorem we use the linear extension of categories in Proposition 2.4.2.

THEOREM 4.1.1. *Let $\mathcal{C}_*$ be a chain complex in $\mathbf{ab}(T)$ concentrated in degrees $\geq n$,*

$$\langle A_{n+2}\rangle^{nil}_{\alpha_{n+2}} \stackrel{\partial_{n+2}}{\longrightarrow} \langle A_{n+1}\rangle^{nil}_{\alpha_{n+1}} \stackrel{\partial_{n+1}}{\longrightarrow} \langle A_n\rangle^{nil}_{\alpha_n}$$

a sequence of morphisms in $\mathbf{nil}(T)$ whose abelianization is

$$\mathcal{C}_{n+2} \stackrel{d_{n+2}}{\longrightarrow} \mathcal{C}_{n+1} \stackrel{d_{n+1}}{\longrightarrow} \mathcal{C}_n,$$

$\vartheta\colon \mathcal{C}_{n+2} \to \wedge^2_T\mathcal{C}_n$ the unique morphism such that $\partial_{n+1}\partial_{n+2} = 0 + \vartheta$, and $\tilde{p}\colon \mathcal{C}_n \twoheadrightarrow H_n\mathcal{C}_*$ the natural projection. Then the composite

$$\mathcal{C}_{n+2} \stackrel{\vartheta}{\longrightarrow} \wedge^2_T\mathcal{C}_n \stackrel{\wedge^2_T\tilde{p}}{\twoheadrightarrow} \wedge^2_T H_n\mathcal{C}_*$$

is a cocycle representing chain cup-product

$$\bar{\cup}_{\mathcal{C}_*} \in H^{n+2}(\mathcal{C}_*, \wedge^2_T H_n\mathcal{C}_*).$$

In the proof of this theorem we use an extended technique in proper homotopy theory which however we only use here. It consists of lifting natural properties in ordinary homotopy theory to proper homotopy theory by using the naturality with respect to the inclusions of a basis of neighbourhoods of the space of Freudenthal ends into the Freudenthal compactification. We also use without explicit mention the fact that the ordinary homotopy theory of pointed spaces coincides with the ordinary homotopy theory of spaces with a base tree, since trees are all contractible from the ordinary homotopy viewpoint.

PROOF OF THEOREM 4.1.1. By the stability of the chain cup-product, see Proposition 3.4.4, it is enough to consider the case $n = 2$. Suppose first that the result is already known for a collection of length-two sequences in $\mathbf{nil}(T)$ whose abelianizations are resolutions of all f. p. $\mathbf{ab}(T)$-modules. Let

$$\text{(a)} \qquad \langle B_4\rangle^{nil}_{\beta_4} \xrightarrow{\partial'_4} \langle B_3\rangle^{nil}_{\beta_3} \xrightarrow{\partial'_3} \langle B_2\rangle^{nil}_{\beta_2}$$

be such a sequence whose abelianization is a resolution of $H_2\mathcal{C}_*$. Then there is a diagram in $\mathbf{nil}(T)$

$$\begin{tikzcd}
\langle A_4\rangle^{nil}_{\alpha_4} \arrow[r, "\partial_4"] \arrow[d, "f_4"'] & \langle A_3\rangle^{nil}_{\alpha_3} \arrow[r, "\partial_3"] \arrow[d, "f_3"'] & \langle A_2\rangle^{nil}_{\alpha_2} \arrow[d, "f_2"'] \\
\langle B_4\rangle^{nil}_{\beta_4} \arrow[r, "\partial'_4"'] & \langle B_3\rangle^{nil}_{\beta_3} \arrow[r, "\partial'_3"'] & \langle B_2\rangle^{nil}_{\beta_2}
\end{tikzcd}$$

whose abelianization if a chain morphism inducing the identity on H_2. There are well-defined controlled homomorphisms $\vartheta_2\colon \mathbb{Z}\langle A_4\rangle_{\alpha_4} \to \wedge^2_T\mathbb{Z}\langle B_3\rangle_{\beta_3}$, $\vartheta_1\colon \mathbb{Z}\langle A_3\rangle_{\alpha_3} \to \wedge^2_T\mathbb{Z}\langle B_2\rangle_{\beta_2}$ and $\vartheta'\colon \mathbb{Z}\langle B_4\rangle_{\beta_4} \to \wedge^2_T\mathbb{Z}\langle B_2\rangle_{\beta_2}$ such that

$$\begin{aligned}
\partial'_4 f_4 + \vartheta_2 &= f_3\partial_4,\\
\partial'_3 f_3 + \vartheta_1 &= f_2\partial_3,\\
\partial'_3\partial'_4 &= 0 + \vartheta'.
\end{aligned}$$

Moreover,

$$\begin{aligned}
0 + (f_n^{ab})_*\vartheta &= f_2\partial_3\partial_4\\
&= \partial'_3 f_3\partial_4 + d_4^*\vartheta_1\\
&= \partial'_3\partial'_4 f_4 + d_4^*\vartheta_1 + (\partial'_3)^{ab}_*\vartheta_2\\
&= 0 + (f_4^{ab})^*\vartheta' + d_4^*\vartheta_1 + (\partial'_3)^{ab}_*\vartheta_2
\end{aligned}$$

so

$$(\wedge^2_T f_2^{ab})\vartheta = \vartheta' f_4^{ab} + \vartheta_1 d_4 + (\wedge^2_T(\partial'_3)^{ab})\vartheta_2.$$

If $\tilde{p}'\colon \mathbb{Z}\langle B_2\rangle_{\beta_2} \twoheadrightarrow H_2\mathcal{C}_*$ is the natural projection then $\tilde{p} = \tilde{p}' f_2^{ab}$, therefore

$$(\wedge^2_T\tilde{p})\vartheta = (\wedge^2_T\tilde{p}')\vartheta' f_4^{ab} + (\wedge^2_T\tilde{p}')\vartheta_1 d_4,$$

i. e. $(\wedge^2_T\tilde{p})\vartheta$ represents the same cohomology class as $(\wedge^2_T\tilde{p}')\vartheta' f_4^{ab}$ in the cohomology group $H^4(\mathcal{C}_*, \wedge^2_T H_2\mathcal{C}_*)$, but by hypothesis $(\wedge^2_T\tilde{p}')\vartheta'$ represents the chain cup of the abelianization of (a), hence by Proposition 3.2.10 $(\wedge^2_T\tilde{p}')\vartheta' f_4^{ab}$ represents the chain cup-product of $\mathcal{C}_*$, and the theorem would follow.

Now we are going to prove the the theorem for a particular

$$\text{(b)} \qquad \langle A_4\rangle^{nil}_{\alpha_4} \xrightarrow{\partial_4} \langle A_3\rangle^{nil}_{\alpha_3} \xrightarrow{\partial_3} \langle A_2\rangle^{nil}_{\alpha_2}$$

whose abelianization $\mathcal{C}_*$ is a resolution of an arbitrary f. p. $\mathbf{ab}(T)$-module $\mathcal{M} = H_2\mathcal{C}_*$.

The construction in [**ACMQ03**] of the kernel of a morphism between free T-controlled $\mathbb{Z}$-modules shows that we can choose the resolution $\mathcal{C}_*$ in such a way that there are countable bases of connected neighborhoods $\left\{T_{v_n^i} \sqcup T^{\mathfrak{F}}_{v_n^i}\right\}_{n\geq 0}$ as in Remark 2.2.6 ($i =, 2, 3, 4$) of the points of $\mathfrak{F}(T)$ in the Freudenthal compactification $\hat{T}$ such that for all $n \geq 0$ $T_{v_n^4} \subset T_{v_n^3} \subset T_{v_n^2}$ and the restriction of $\mathcal{C}_*$

$$\text{(c)} \qquad \mathbb{Z}\langle\alpha_4^{-1}(T_{v_n^4})\rangle \overset{d_4^n}{\hookrightarrow} \mathbb{Z}\langle\alpha_3^{-1}(T_{v_n^3})\rangle \xrightarrow{d_3^n} \mathbb{Z}\langle\alpha_2^{-1}(T_{v_n^2})\rangle$$

is defined an it is an exact sequence of abelian groups.

The resolution $\mathcal{C}_*$ is the abelianization of a sequence in $\mathbf{gr}(T)$

$$\langle A_4\rangle_{\alpha_4} \xrightarrow{\bar{\partial}_4} \langle A_3\rangle_{\alpha_3} \xrightarrow{\bar{\partial}_3} \langle A_2\rangle_{\alpha_2}$$

such that the restrictions

$$\text{(d)} \qquad \langle \alpha_4^{-1}(T_{v_n^4})\rangle \overset{\bar{\partial}_4^n}{\hookrightarrow} \langle \alpha_3^{-1}(T_{v_n^3})\rangle \xrightarrow{\bar{\partial}_3^n} \langle \alpha_2^{-1}(T_{v_n^2})\rangle$$

are also defined for all $n \geq 0$. We define (b) as $\partial_i = \partial_i^{nil}$ $(i = 3, 4)$.

In order to compute the chain cup-product $\bar{\cup}_{\mathcal{C}_*}$ we are going to choose a convenient 1-reduced T-homotopy system $(\mathcal{C}_*, f_4)$, see Definition 3.4.1. Consider the T-complex $Y = C_{\bar{\partial}_3}$ which is the cofiber of the map $S^1_{\alpha_3} \to S^1_{\alpha_2}$ induced by $\bar{\partial}_3$, see Proposition 1.4.2. Notice that $\Sigma Y = C_{d_3}$. Let $\alpha_i^n\colon \alpha_i^{-1}(T_{v_n^i}) \to T_{v_n^2}$ be the restriction of α_i $(i = 2, 3, 4)$, and $Y_n = C_{\bar{\partial}_3^n}$ the cofiber of the map $S^1_{\alpha_3^n} \to S^1_{\alpha_2^n}$ induced by the controlled homomorphism $\bar{\partial}_3^n$ in (d).

(e) The collection $\left\{\Sigma Y_n \sqcup T^{\mathfrak{F}}_{v_n^2}\right\}_{n\geq 0}$ is a basis of connected neighborhoods of $\mathfrak{F}(T) = \mathfrak{F}(\Sigma Y)$ in the Freudenthal compactification $\widehat{\Sigma Y}$,

i. e. any point of $\mathfrak{F}(T)$ has an "arbitrary small" neighborhood in $\widehat{\Sigma Y}$ belonging to this collection, despite $\Sigma Y_n \sqcup T^{\mathfrak{F}}_{v_n^2}$ need not be a neighborhood of all points in $T^{\mathfrak{F}}_{v_n^2}$, but only of those $T^{\mathfrak{F}}_{v_n^3} \subset T^{\mathfrak{F}}_{v_n^2}$.

By the exactness of (c) the ordinary homology group H_3 of ΣY_n is

$$H_3 \Sigma Y_n = \mathbb{Z}\langle \alpha_4^{-1}(T_{v_n^4})\rangle, \quad n \geq 0.$$

As we remarked in [**Mur05**] 6.2 in the proof of [**Mur05**] 6.1 we construct sections ζ_{Y_n} of the ordinary Hurewicz homomorphism $\pi_3\Sigma Y_n \twoheadrightarrow H_3\Sigma Y_n$ which are compatible with the inclusion of subcomplexes, therefore by (e) they determine a proper map $f_4\colon S^3_{\alpha_4} \to \Sigma Y$ which restricts to maps $S^3_{\alpha_4^n} \to \Sigma Y_n$ inducing ζ_{Y_n} on π_3 $(n \geq 0)$. Now it is immediate to notice that $(\mathcal{C}_*, f_4)$ is a well-defined 1-reduced T-homotopy system.

The homomorphism η defined at the beginning of the proof of [**Mur05**] 6.1 determines a controlled homomorphism $\eta\colon\ \wedge^2_T \mathbb{Z}\langle A_2\rangle_{\alpha_2} \to \otimes^2_T \mathbb{Z}\langle A_2\rangle_{\alpha_2}$. It satisfies $q_T\eta = 1$. If i_1 and i_2 are the inclusions of the factors of the coproduct $\langle A_3\rangle^{nil}_{\alpha_3} \vee \langle A_3\rangle^{nil}_{\alpha_3}$ and i_{12} is the inclusion of the quadratic crossed effect of $\wedge^2_T$ then there is a unique controlled homomorphism $\nabla\partial_4\colon \mathbb{Z}\langle A_4\rangle_{\alpha_4} \to \otimes^2_T\mathbb{Z}\langle A_3\rangle_{\alpha_3}$ with

$$(i_2 + i_1)\partial_4 = (i_2\partial_4 + i_1\partial_4) + i_{12}(\nabla\partial_4).$$

By Proposition 2.3.2 the controlled homomorphism

$$\eta(\vartheta - (\wedge^2_T d_2) q_T(\nabla\partial_4))\colon \mathbb{Z}\langle A_4\rangle_{\alpha_4} \to \otimes^2_T \mathbb{Z}\langle A_2\rangle_{\alpha_2}$$

can be identified with the proper homotopy class of a map $\kappa\colon S^3_{\alpha_4} \to S^2_{\alpha_2} \vee S^2_{\alpha_2}$ under T which becomes null-homotopic in we project onto any of the factors of the target.

Let $i_1, i_2, \mu\colon \Sigma Y \to \Sigma Y \vee \Sigma Y$ be now the inclusion of the factors and the co-H-multiplication of the suspension of Y. The homotopy class of

$$S^3_{\alpha_4} \xrightarrow{\kappa} S^2_{\alpha_2} \vee S^2_{\alpha_2} \subset \Sigma Y \vee \Sigma Y$$

represents

$$\mu f_4 - i_2 f_4 - i_1 f_4 \in [S^3_{\alpha_4}, \Sigma Y \vee \Sigma Y]^T.$$

This follows from (d) and the fact that the corresponding formula in ordinary homotopy theory holds for all restrictions

$$S^3_{\alpha_4^n} \xrightarrow{\kappa} S^2_{\alpha_2^n} \vee S^2_{\alpha_2^n} \subset \Sigma Y_n \vee \Sigma Y_n.$$

In particular by Definition 3.1.2

$$\gamma_2(f_4) = (\otimes^2_T \tilde{p})\eta(\vartheta - (\wedge^2_T d_2) q_T(\nabla d_4)),$$

and by Definitions 3.2.4 and 3.4.1 the chain cup-product $\bar{\cup}_{\mathcal{C}_*}$ in indeed represented by

$$\begin{array}{rcl} \bar{q}_T \bar{\sigma}_T \gamma_2(f_4) & = & q_T \gamma_2(f_4) \\ & = & q_T(\otimes^2_T \tilde{p})\eta(\vartheta - (\wedge^2_T d_2) q_T(\nabla d_4)) \\ & = & (\wedge^2_T \tilde{p}) q_T \eta(\vartheta - (\wedge^2_T d_2) q_T(\nabla d_4)) \\ & = & (\wedge^2_T \tilde{p})(\vartheta - (\wedge^2_T d_2) q_T(\nabla d_4)) \\ & = & (\otimes^2_T \tilde{p})\vartheta. \end{array}$$

□

In the statement of the following proposition we consider the $\mathbf{vect}(T_3)$-module $\mathbb{M}\underline{V}^{(3,5)}$ in Theorem 6.1.2, see also Proposition 6.1.3.

PROPOSITION 4.1.2. *If $\mathcal{C}_*$ is a finite-type resolution of the f. p. $\mathbf{vect}(T_3)$-module $\mathbb{M}\underline{V}^{(3,5)}$ as an $\mathbf{ab}(T_3)$-module then*

$$0 \neq \bar{\cup}_{\mathcal{C}_*} \in H^{n+2}(\mathcal{C}_*, \wedge^2_{T_3} H_n \mathcal{C}_*) = \mathbb{Z}/2.$$

PROOF. By Proposition 6.3.3 and Lemma 6.3.9 we have that

$$\begin{array}{rcl} H^{n+2}(\mathcal{C}_*, \wedge^2_{T_3} H_n \mathcal{C}_*) & = & \mathrm{Ext}^2_{\mathbf{ab}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \wedge^2_{T_3} \mathbb{M}\underline{V}^{(3,5)}) \\ & = & \mathrm{Ext}^1_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \wedge^2_{T_3} \mathbb{M}\underline{V}^{(3,5)}) \\ & = & \mathbb{Z}/2. \end{array}$$

Let us check now that the cup-product invariant of $\mathcal{C}_*$ is not zero. For this we will make a convenient choice of $\mathcal{C}_*$. In the proof of [**Mur04**] 9.1 we construct a finite presentation of $\mathbb{M}\underline{V}^{(3,5)}$ as a $\mathbf{vect}(T_3)$-module

$$\mathbb{F}_2\langle D\rangle_\delta \overset{\rho}{\hookrightarrow} \mathbb{F}_2\langle C\rangle_\gamma \overset{\tilde{p}}{\twoheadrightarrow} \mathbb{M}\underline{V}^{(3,5)}.$$

Here $D = \{{}_m w^i\,;\, 1 \leq i \leq 3, m \geq 1\}$, $C = D \sqcup \{w_1, w_2\}$, $\delta({}_m w^i) = \gamma({}_m w^i) = v^i_m$, $\gamma(w_j) = v_0$ and

$$\rho({}_m w^i) = \begin{cases} {}_m w^i - {}_{m-1} w^i, & \text{for all } i = 1,2,3 \text{ provided } m > 1; \\ {}_1 w^1 - w_1, & \text{if } i = 1 \text{ and } m = 1; \\ {}_1 w^2 - w_2 - w_1, & \text{if } i = 2 \text{ and } m = 1; \\ {}_1 w^3 - w_2, & \text{if } i = 3 \text{ and } m = 1. \end{cases}$$

Moreover the projection $\tilde{p}\colon \mathbb{F}_2\langle C\rangle_\gamma \twoheadrightarrow \mathbb{M}\underline{V}^{(3,5)}$ is determined by the vector space homomorphism $\tilde{p}_0\colon \mathbb{F}_2\langle C\rangle \twoheadrightarrow \mathbb{F}_2\langle x,y\rangle$ with

$$\tilde{p}_0({}_mw^i) = \begin{cases} x, & i=1,\, m\geq 1;\\ x+y, & i=2,\, m\geq 1;\\ y, & i=3,\, m\geq 1;\end{cases}$$

$$\tilde{p}_0(w_1) = x;$$

$$\tilde{p}_0(w_2) = y.$$

If $\bar{\varphi}\colon \mathbb{Z}\langle D\rangle_\delta \hookrightarrow \mathbb{Z}\langle C\rangle_\gamma$ is the controlled homomorphism defined by the same formulas as ρ then

$$\mathbb{Z}\langle D\rangle_\delta \overset{(2,-\bar{\varphi})}{\hookrightarrow} \mathbb{Z}\langle D\rangle_\delta \oplus \mathbb{Z}\langle C\rangle_\gamma \xrightarrow{(\bar{\varphi},2)} \mathbb{Z}\langle C\rangle_\gamma$$

is a resolution of $\mathbb{M}\underline{V}^{(3,5)}$ as an $\mathbf{ab}(T_3)$-module. We take $\mathcal{C}_*$ as the translation of this resolution to degree n.

Let $\varphi\colon \langle D\rangle_\delta^{nil} \hookrightarrow \langle C\rangle_\gamma^{nil}$ be the controlled homomorphism defined as ρ and $\bar{\varphi}$. The sequence

$$\langle D\rangle_\delta^{nil} \xrightarrow{\partial_{n+2}} \langle D\rangle_\delta^{nil} \vee \langle C\rangle_\gamma^{nil} \xrightarrow{\partial_{n+1}} \langle C\rangle_\gamma^{nil}$$

with $\partial_{n+2}(d) = i_1(d)+i_1(d)-i_2\varphi(d)$ and $\partial_{n+1}(d,c) = (\varphi(d), c+c)$ $(c\in C, d\in D)$ satisfies the hypothesis of Theorem 4.1.1.

Given $1\leq i\leq 3$ and $m>1$ we have that

$$\begin{aligned}
\partial_{n+1}\partial_{n+2}({}_mw^i) &= \partial_{n+1}(i_1({}_mw^i)+i_1({}_mw^i)-i_2({}_mw^i-{}_{m-1}w^i))\\
&= \partial_{n+1}(i_1({}_mw^i)+i_1({}_mw^i)+i_2({}_{m-1}w^i)-i_2({}_mw^i))\\
&= {}_mw^i-{}_{m-1}w^i+{}_mw^i-{}_{m-1}w^i\\
&\quad +{}_{m-1}w^i+{}_{m-1}w^i-{}_mw^i-{}_mw^i\\
&= {}_mw^i+[{}_{m-1}w^i,-{}_mw^i]-{}_mw^i\\
&= [{}_mw^i,{}_{m-1}w^i].
\end{aligned}$$

Moreover

$$\begin{aligned}
\partial_{n+1}\partial_{n+2}({}_1w^1) &= \partial_{n+1}(i_1({}_1w^1)+i_1({}_1w^1)-i_2({}_1w^1-w_1))\\
&= \partial_{n+1}(i_1({}_1w^1)+i_1({}_1w^1)+i_2(w_1)-i_2({}_1w^1))\\
&= {}_1w^1-w_1+{}_1w^1-w_1\\
&\quad +w_1+w_1-{}_1w^1-{}_1w^1\\
&= {}_1w^1+[w_1,-{}_1w^1]-{}_1w^1\\
&= [{}_1w^1,w_1],
\end{aligned}$$

$$\begin{aligned}
\partial_{n+1}\partial_{n+2}({}_1w^2) &= \partial_{n+1}(i_1({}_1w^2)+i_1({}_1w^2)-i_2({}_1w^2-w_2-w_1))\\
&= \partial_{n+1}(i_1({}_1w^2)+i_1({}_1w^2)+i_2(w_1)+i_2(w_2)-i_2({}_1w^2))\\
&= {}_1w^2-w_2-w_1+{}_1w^2-w_2-w_1\\
&\quad +w_1+w_1+w_2+w_2-{}_1w^2-{}_1w^2\\
&= {}_1w^2+[w_1+w_2,w_2-{}_1w^2]-{}_1w^2\\
&= [{}_1w^2,w_1]+[{}_1w^2,w_2]+[w_1,w_2],
\end{aligned}$$

$$\begin{aligned}
\partial_{n+1}\partial_{n+2}({}_1w^3) &= \partial_{n+1}(i_1({}_1w^3)+i_1({}_1w^3)-i_2({}_1w^3-w_2)) \\
&= \partial_{n+1}(i_1({}_1w^3)+i_1({}_1w^3)+i_2(w_2)-i_2({}_1w^3)) \\
&= {}_1w^3-w_2+{}_1w^3-w_2 \\
&\quad +w_2+w_2-{}_1w^3-{}_1w^3 \\
&= {}_1w^3+[w_2,-{}_1w^3]-{}_1w^3 \\
&= [{}_1w^3,w_2],
\end{aligned}$$

therefore the controlled homomorphism $\vartheta\colon \mathbb{Z}\langle D\rangle_\delta \to \wedge^2_{T_3}\mathbb{Z}\langle C\rangle_\gamma$ satisfying the equation $\partial_{n+1}\partial_{n+2} = 0+\vartheta$ is given by

$$\vartheta({}_mw^i) = \begin{cases} {}_mw^i \wedge {}_{m-1}w^i, & \text{for all } i=1,2,3 \text{ if } m>1; \\ {}_1w^1\wedge w_1, & \text{if } i=1 \text{ and } m=1; \\ {}_1w^2\wedge w_1+{}_1w^2\wedge w_2+w_1\wedge w_2, & \text{if } i=2 \text{ and } m=1; \\ {}_1w^3\wedge w_2, & \text{if } i=3 \text{ and } m=1. \end{cases}$$

For any $\mathbf{ab}(T_3)$-module $\mathcal{M}$ we write $\hat{p}\colon \mathcal{M} \twoheadrightarrow \mathcal{M}\otimes\mathbb{Z}/2$ for the natural projection. By Theorem 4.1.1 $\bar{\cup}_{\mathcal{C}_*}$ is represented by the cocycle

$$\mathbb{Z}\langle D\rangle_\delta \xrightarrow{\vartheta} \mathbb{Z}\langle C\rangle_\gamma \xrightarrow{\wedge^2_{T_3}(\tilde{p}\hat{p})} \wedge^2_{T_3}\mathbb{M}\underline{V}^{(3,5)}.$$

By Proposition 6.4.9 and [**Mur04**] 7.7 $\wedge^2_{T_3}\mathbb{M}\underline{V}^{(3,5)} = \mathbb{F}_2\langle x\wedge y\rangle_\phi$ is a free T_3-controlled $\mathbb{F}_2$-module with only one generator $x\wedge y$. Under this identification the cocycle $(\wedge^2_{T_3}(\tilde{p}\hat{p}))\vartheta\colon \mathbb{Z}\langle D\rangle_\delta \to \mathbb{F}_2\langle x\wedge y\rangle_\phi$ corresponds to the controlled homomorphism $(\wedge^2\tilde{p}_0)\vartheta\colon \mathbb{F}_2\langle D\rangle_\delta \to \mathbb{F}_2\langle x\wedge y\rangle_\phi$.

For any $m>1$ and $i=1,2,3$

$$\begin{aligned}
(\wedge^2\tilde{p}_0)\vartheta({}_mw^i) &= (\wedge^2\tilde{p}_0)({}_mw^i\wedge {}_{m-1}w^i) \\
&= \left\{ \begin{array}{ll} x\wedge x, & \text{if } i=1; \\ (x+y)\wedge(x+y), & \text{if } i=2; \\ y\wedge y, & \text{if } i=3; \end{array} \right\} \\
&= 0; \\
(\wedge^2\tilde{p}_0)\vartheta({}_1w^1) &= (\wedge^2\tilde{p}_0)({}_1w^1\wedge w_1) \\
&= x\wedge x \\
&= 0; \\
(\wedge^2\tilde{p}_0)\vartheta({}_1w^2) &= (\wedge^2\tilde{p}_0)({}_1w^2\wedge w_1+{}_1w^2\wedge w_2+w_1\wedge w_2) \\
&= (x+y)\wedge x+(x+y)\wedge y+x\wedge y \\
&= x\wedge y \\
(\wedge^2\tilde{p}_0)\vartheta({}_1w^3) &= (\wedge^2\tilde{p}_0)({}_1w^3\wedge w_2) \\
&= y\wedge y \\
&= 0.
\end{aligned}$$

If $(\wedge^2_{T_3}(\tilde{p}p))\vartheta$ represented the trivial cohomology class then there would be a controlled homomorphism

$$(\xi_1,\xi_2)\colon \mathbb{F}_2\langle D\rangle_\delta \oplus \mathbb{F}_2\langle C\rangle_\gamma \longrightarrow \mathbb{F}_2\langle x\wedge y\rangle_\phi$$

such that

$$\begin{aligned}(\wedge^2\tilde{p}_0)\vartheta &= (\xi_1,\xi_2)((2,-\bar{\varphi})\otimes\mathbb{Z}/2)\\ &= (\xi_1,\xi_2)(0,\rho)\\ &= (0,\xi_2\rho).\end{aligned}$$

In particular for all $m\geq 1$ and $i=1,2,3$ the following equalities should hold

$$\begin{aligned}0 &= (\wedge^2\tilde{p}_0)\vartheta({}_{m+1}w^i)\\ &= \xi_2\rho({}_{m+1}w^i)\\ &= \xi_2({}_{m+1}w^2-{}_mw^i),\end{aligned}$$

i. e.

$$\xi_2({}_mw^i)=\xi_2({}_{m+1}w^i),\quad m\geq 1, i=1,2,3.$$

Since ξ_2 is supposed to be controlled and $\gamma({}_mw^i)=v^i_m$ then $\xi_2({}_mw^i)$ should vanish for m big enough and therefore for all m by using the previous equality

$$\xi_2({}_mw^i)=0,\quad m\geq 1, i=1,2,3;$$

hence

$$\begin{aligned}0 &= (\wedge^2\tilde{p}_0)\vartheta({}_1w^1)\\ &= \xi_2\rho({}_1w^1)\\ &= \xi_2({}_1w^1-w_1)\\ &= \xi_2(w_1),\end{aligned}$$

and

$$\begin{aligned}0 &= (\wedge^2\tilde{p}_0)\vartheta({}_1w^3)\\ &= \xi_2\rho({}_1w^3)\\ &= \xi_2({}_1w^3-w_2)\\ &= \xi_2(w_2),\end{aligned}$$

so

$$\begin{aligned}x\wedge y &= (\wedge^2\tilde{p}_0)\vartheta({}_1w^2)\\ &= \xi_2\rho({}_1w^2)\\ &= \xi_2({}_1w^2-w_2-w_1)\\ &= \xi_2(w_1)+\xi_2(w_2)\\ &= 0,\end{aligned}$$

and we reach a contradiction derived from supposing that the cohomology class $\bar{\cup}_{\mathcal{C}_*}$ represented by the cocycle $(\wedge^2_{T_3}(\tilde{p}p))\vartheta$ was trivial. $\square$

The following corollary follows from this last proposition and Remark 3.4.7.

COROLLARY 4.1.3. *If T has more than 2 ends then the chain cup-product in cohomology of categories does not vanish*

$$0\neq\bar{\cup}\in H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-,\wedge^2_T H_n)).$$

We do not know wether the chain cup-product vanishes for trees with 1 or 2 ends, however in the following section we completely compute its mod 2 version for trees with less than four ends.

4.2. The computation of the mod 2 chain cup-product element in cohomology of categories

Recall from Section 3.4 that the chain cup-product can be regarded as an element

$$\bar{\cup} \in H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge^2_T H_n)).$$

This element determines obstructions to the existence of co-H-spaces with a given proper cellular chain complex, see Proposition 3.5.2.

The best we can do to compute this cohomology group is to apply the localization theorem in [**Mur06**]. By Theorem 1.2.14 the homology functor

$$H_n\colon \mathbf{chain}_n(\mathbf{ab}(T))/\simeq \longrightarrow \mathbf{fp}(\mathbf{ab}(T))$$

has a right adjoint determined by the choice of a finite-type resolution in degree n for any f. p. $\mathbf{ab}(T)$-module, therefore by [**Mur06**] 6.5

$$H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge^2_T H_n)) \simeq H^0(\mathbf{fp}(\mathbf{ab}(T)), \mathrm{Ext}^2_{\mathbf{ab}(T)}(-, \wedge^2_T)).$$

The image of $\bar{\cup}$ under this isomorphism is the universal obstruction to the existence of co-H-structures in degree 2 proper Moore spaces.

As we mentioned in Section 2.5 the cohomology of $\mathbf{fp}(\mathbf{ab}(T))$ is strongly related to its representation theory, i. e. the representation theory of $\mathbb{Z}(\mathfrak{F}(T))$. As a consequence of the results in [**Mur04**] 4 we can not expect to obtain satisfactory classification theorems for f. p. $\mathbf{ab}(T)$-modules because such a result would imply a classification of all countable abelian groups, therefore we are unable to compute the previous cohomology group of $\mathbf{fp}(\mathbf{ab}(T))$. For this reason we will consider the image of $\bar{\cup}$ under the change of coefficients

$$\begin{array}{c} H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge^2_T H_n)) \\ \Big\downarrow{\scriptstyle \hat{p}_*} \\ H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge^2_T H_n \otimes \mathbb{Z}_2)) \end{array}$$

induced by the natural projection $\hat{p}\colon \mathcal{M} \twoheadrightarrow \mathcal{M} \otimes \mathbb{Z}_2$. The element $\hat{p}_*\bar{\cup}$ will be termed *chain cup-product mod* 2. It is the universal obstruction to the existence of co-H-structures in degree 2 proper Moore spaces with exponent 2 in proper homology.

Once again the localization theorem in [**Mur06**] yields an isomorphism

$$H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge^2_T H_n \otimes \mathbb{Z}/2)) \simeq H^0(\mathbf{fp}(\mathbf{ab}(T)), \mathrm{Ext}^2_{\mathbf{ab}(T)}(-, \wedge^2_T \otimes \mathbb{Z}/2)).$$

Moreover, the adjoint "change of coefficients" functors in (1.2.20) together with Proposition 6.3.3 and the localization theorem in [**Mur06**] yield another isomorphism

$$H^0(\mathbf{fp}(\mathbf{ab}(T)), \mathrm{Ext}^2_{\mathbf{ab}(T)}(-, \wedge^2_T \otimes \mathbb{Z}/2)) \simeq H^0(\mathbf{fp}(\mathbf{vect}(T)), \mathrm{Ext}^1_{\mathbf{vect}(T)}(-, \wedge^2_T)),$$

and the representation theory of $\mathbf{fp}(\mathbf{vect}(T))$ is not as complicated as that of $\mathbf{fp}(\mathbf{ab}(T))$. More precisely, as we proved in [**Mur04**] $\mathbf{fp}(\mathbf{vect}(T))$ has finite representation type if (and only if) T has less than four ends and we have explicit classification theorems for f. p. $\mathbf{vect}(T)$-modules in these cases, see Section 6.1, therefore we can expect to be able to compute the previous cohomology group of $\mathbf{fp}(\mathbf{vect}(T))$ for trees with less than four ends. Indeed we will do it in the following theorem with the help of the computations carried out in Sections 6.3 and 6.4.

THEOREM 4.2.1. *If T has 1 or 2 ends then*

$$H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge_T^2 H_n \otimes \mathbb{Z}_2)) = 0,$$

so $\hat{p}_*\bar{\cup} = 0$. Moreover, if T has 3 ends

$$0 \neq \hat{p}_*\bar{\cup} \in H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge_T^2 H_n \otimes \mathbb{Z}_2)) = \mathbb{Z}_2.$$

PROOF. By Propositions 2.2.21 and 4.1.2 $\bar{\cup} \neq 0$ for T a tree with 3 ends, therefore in this case the cohomology group in the statement is non-trivial and all we have to do now is to compute it for trees with less than four ends. As we have seen above it is isomorphic to

$$H^0(\mathbf{fp}(\mathbf{vect}(T)), \mathrm{Ext}^1_{\mathbf{vect}(T)}(-, \wedge_T^2))$$

by the localization theorem in cohomology of categories, see [**Mur06**].

Recall from Section 2.5 that any 0-cocycle c is additive, i. e. it is completely determined by its values

$$c_{\mathcal{M}} \in \mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathcal{M}, \wedge_T^2 \mathcal{M})$$

where $\mathcal{M}$ runs along all elementary f. p. $\mathbf{vect}(T)$-modules, see Section 6.1. This fact and Lemmas 6.3.7, 6.3.8 and 6.3.9 prove that the cohomology group

$$H^0(\mathbf{fp}(\mathbf{vect}(T)), \mathrm{Ext}^1_{\mathbf{vect}(T)}(-, \wedge_T^2))$$

is trivial if T has 1 or 2 ends, while it can be trivial or $\mathbb{Z}/2$ if T has 3 ends, but as we have noticed before it is non-trivial in this case, hence we are done. □

The isomorphism observed above and given by the localization theorem for cohomology of categories ([**Mur06**] 6.5)

$$H^0(\mathbf{fp}(\mathbf{vect}(T)), \mathrm{Ext}^1_{\mathbf{vect}(T)}(-, \wedge_T^2)) \xrightarrow{\simeq} H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge_T^2 H_n \otimes \mathbb{Z}_2))$$

is $\kappa_*(H_n \otimes \mathbb{Z}/2)^*$, where $H_n \otimes \mathbb{Z}/2$ is the homology functor

$$H_n \otimes \mathbb{Z}/2\colon \mathbf{chain}_n(\mathbf{ab}(T))/\simeq \longrightarrow \mathbf{fp}(\mathbf{vect}(T))$$

and κ is the following composite of natural transformations between bimodules over $\mathbf{chain}_n(\mathbf{ab}(T))$

$$\begin{array}{c} \mathrm{Ext}^1_{\mathbf{vect}(T)}(H_n \otimes \mathbb{Z}/2, \wedge_T^2 H_n \otimes \mathbb{Z}/2) \\ \downarrow{\simeq} \\ \mathrm{Ext}^2_{\mathbf{ab}(T)}(H_n \otimes \mathbb{Z}/2, \wedge_T^2 H_n \otimes \mathbb{Z}/2) \\ \downarrow{\hat{p}^*} \\ \mathrm{Ext}^2_{\mathbf{ab}(T)}(H_n, \wedge_T^2 H_n \otimes \mathbb{Z}/2) \\ \downarrow \\ H^{n+2}(-, \wedge_T^2 H_n \otimes \mathbb{Z}_2) \end{array}$$

which is induced by Proposition 6.3.3, the natural projection $\hat{p}\colon \mathcal{M} \twoheadrightarrow \mathcal{M} \otimes \mathbb{Z}_2$, and the universal coefficients spectral sequence for the computation of the cohomology of a chain complex $\mathcal{C}_*$ in $\mathbf{ab}(T)$

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathbf{ab}(T)}(H_q\mathcal{C}_*, \mathcal{M}) \Rightarrow H^{p+q}(\mathcal{C}_*, \mathcal{M}). \tag{4.2.2}$$

Therefore Proposition 4.1.2, Lemma 6.3.9 and the additivity of 0-cocycles in cohomology of categories, see Section 2.5, prove more than Theorem 4.2.1 for a tree with three ends.

THEOREM 4.2.3. *Given a tree T with 3 ends and a bounded chain complex $\mathcal{C}_*$ in $\mathbf{ab}(T)$ concentrated in dimensions $\geq n$, if $H_n\mathcal{C}_* \otimes \mathbb{Z}/2$ contains $\mathbb{M}\underline{V}^{(3,5)}$ as a direct summand the cup-product mod 2 $\hat{p}_*\bar{\cup}_{\mathcal{C}_*}$ is the image of the generator*

$$\begin{array}{c} \mathbb{Z}/2 = \mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathbb{M}\underline{V}^{(3,5)}, \wedge^2_T \mathbb{M}\underline{V}^{(3,5)}) \\ \big\downarrow \\ \mathrm{Ext}^1_{\mathbf{vect}(T)}(H_n\mathcal{C}_* \otimes \mathbb{Z}/2, \wedge^2_T H_n\mathcal{C}_* \otimes \mathbb{Z}/2) \\ \big\downarrow {\scriptstyle \kappa} \\ H^{n+2}(\mathcal{C}_*, \wedge^2_T H_n\mathcal{C}_* \otimes \mathbb{Z}/2) \end{array}$$

Otherwise $\hat{p}_*\bar{\cup}_{\mathcal{C}_*} = 0$.

CHAPTER 5

Pontrjagin-Steenrod invariants and proper homotopy types

This is the central chapter of this paper. Here we obtain our main proper homotopy classification results for A_n^2-polyhedra when the involved representation theory is of finite type. For this we answer the questions on proper Pontrjagin-Steenrod invariants raised in the introduction. In order to obtain these results we use the computations carried out in Chapters 4 and 6.

5.1. On the existence of Pontrjagin-Steenrod invariants in proper homotopy theory

The Pontrjagin-Steenrod invariant in ordinary homotopy theory is one of the tools used by J. H. C. Whitehead to classify homotopy types of A_n^2-polyhedra in terms of algebraic data, see [**Whi48**], [**Whi49**] and [**Whi50**]. The alternative tool was the lower part of Whitehead's exact sequence for the Hurewicz homomorphism. In fact the lower part of this sequence can be derived from the Pontrjagin-Steenrod invariant and the universal coefficients exact sequence. As we mentioned in Section 1.3 there is a proper analogue of Whitehead's exact sequence, however it can not be used to classify proper homotopy types of A_n^2-polyhedra because there are proper Moore spaces with the same homology but a different proper homotopy type, see [**ADMQ95**] or Remarks 5.4.5 and 5.4.9 below. For this reason it becomes more interesting to study the existence of Pontrjagin-Steenrod invariants in proper homotopy theory.

A Pontrjagin-Steenrod invariant in proper homotopy theory should be a cohomology invariant (0-cocycle) in the category of $(n-1)$-reduced T-homotopy systems

$$\wp_n(\mathcal{C}_*, f_{n+2}) \in \begin{cases} H^4(\mathcal{C}_*, \Gamma_T H_2\mathcal{C}_*), & n = 2 \text{ (Pontrjagin invariant)}; \\ \\ H^{n+2}(\mathcal{C}_*, H_n\mathcal{C}_* \otimes \mathbb{Z}/2), & n > 2 \text{ (Steenrod invariant)}. \end{cases}$$

This cohomology invariant should determine the obstruction operator θ in the exact sequence of functors (2.5.7), i. e. given a chain morphism $\xi\colon \mathcal{C}_* \to \mathcal{C}'_*$ the following equality should hold

$$\theta_{(\mathcal{C}_*, f_{n+2}),(\mathcal{C}'_*, g_{n+2})}(\xi) = \xi_*\wp_n(\mathcal{C}_*, f_{n+2}) - \xi^*\wp_n(\mathcal{C}'_*, g_{n+2}).$$

Therefore Remark 2.5.2 shows that the existence of this invariant is equivalent to the vanishing of the characteristic class in cohomology of categories determined by the obstruction operator, see (2.5.9),

$$\{\theta\} \in H^1(\mathbf{chain}_2(\mathbf{ab}(T))/\simeq, H^4(-, \Gamma_T H_2)), \qquad \text{if } n = 2;$$

$$\{\theta\} \in H^1(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, H_n \otimes \mathbb{Z}/2)), \quad \text{if } n > 2.$$

If the proper Pontrjagin-Steenrod invariant were completely analogous to the ordinary one then it should be compatible with the (reduced) cup-product invariant defined in Section 3.2, i. e.

$$(\tau_T)_*\wp_2(\mathcal{C}_*, f_4) = \cup_{(\mathcal{C}_*, f_4)}, \qquad \text{if } n = 2;$$

$$(\bar{\tau}_T)_*\wp_n(\mathcal{C}_*, f_{n+2}) = \hat{\cup}_{(\mathcal{C}_*, f_{n+2})}, \quad \text{if } n > 2;$$

see also (2.2.19). The existence of a proper Pontrjagin-Steenrod invariant compatible with the (reduced) cup-product would immediately imply the vanishing of the chain cup-product, see (2.2.19) and Definition 3.4.1, therefore we can already discard this possibility for trees with more than two ends, see Corollary 4.1.3. However for our purposes, the classification of proper homotopy types in terms of algebraic data, this compatibility is not necessary, therefore we will now concentrate in the study of $\{\theta\}$.

5.2. Steenrod invariants for spaces with less than 4 ends

Suppose that T is a tree such that the natural transformation in (2.2.19)

$$\bar{\tau}_T \colon \mathcal{M} \longrightarrow \hat{\otimes}^2_T \mathcal{M} \tag{5.2.1}$$

is a split monomorphism for any f. p. $\mathbf{vect}(T)$-module $\mathcal{M}$. In this case we would have a short exact sequence of natural transformations between bimodules over $\mathbf{chain}_n(\mathbf{ab}(T))/\simeq$

$$H^{n+2}(-, H_n \otimes \mathbb{Z}_2) \overset{(\bar{\tau}_T)_*}{\hookrightarrow} H^{n+2}(-, \hat{\otimes}^2_T H_n) \overset{(\bar{q}_T)_*}{\twoheadrightarrow} H^{n+2}(-, \wedge^2_T H_n), \tag{5.2.2}$$

which would give rise to a Bockstein long exact sequence in cohomology of categories

$$\begin{array}{c}
\vdots \\
\downarrow \\
H^k(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, H_n \otimes \mathbb{Z}_2)) \\
\Big\downarrow {\scriptstyle (\bar{\tau}_T)_*} \\
H^k(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \hat{\otimes}^2_T H_n \otimes \mathbb{Z}_2)) \\
\Big\downarrow {\scriptstyle (\bar{q}_T)_*} \\
H^k(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge^2_T H_n \otimes \mathbb{Z}_2)) \\
\Big\downarrow {\scriptstyle \beta} \\
H^{k+1}(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, H_n \otimes \mathbb{Z}_2)) \\
\downarrow \\
\vdots
\end{array} \tag{5.2.3}$$

Otherwise the sequence of bimodules would only be exact in the middle and the Bockstein exact sequence would not even be defined.

THEOREM 5.2.4. *In the previous conditions the characteristic class $\{\theta\}$ represented by the obstruction operator θ in (2.5.7) $(n > 2)$ is the image of the chain cup-product mod 2 $\hat{p}_*\bar{\cup}$ in Section 4.2 under the Bockstein homomorphism*

$$\{\theta\} = \beta\hat{p}_*\bar{\cup} \in H^1(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, H_n \otimes \mathbb{Z}_2)).$$

In particular this is satisfied for T a tree with less than four ends.

The first part of the statement is an immediate consequence of Definition 3.4.1 and Theorem 3.3.1. The second part follows from Proposition 6.4.7.

The weaker fact that $\{\theta\}$ is in the image of the Bockstein homomorphism can also be deduced from Corollary 3.3.2.

Moreover, by Remark 6.4.8 Theorem 5.2.4 can not hold for trees with more than six ends because, as one can readily see, the splitability of (5.2.1) is equivalent to the exactness of (5.2.2). For trees with 4, 5 or 6 ends nothing is known.

The following corollary follows from Theorems 5.2.4 and 4.2.1.

COROLLARY 5.2.5. *If T has 1 or 2 ends the characteristic class $\{\theta\}$ represented by the obstruction operator θ in (2.5.7) $(n > 2)$ is trivial*

$$0 = \{\theta\} \in H^1(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, H_n \otimes \mathbb{Z}_2)),$$

therefore Steenrod invariants in the sense of Section 5.1 do exist for trees with 1 or 2 ends.

An explicit definition of these Steenrod invariants will be given later in this section. Before we consider the case of trees with 3 ends, which is a bit more complicated because the cup-product mod 2 does not vanish in this case, see Theorem 4.2.1.

THEOREM 5.2.6. *If T has 3 ends then*

$$\begin{array}{c} H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \hat{\otimes}^2_T H_n \otimes \mathbb{Z}_2)) \\ \simeq \Big\downarrow (\bar{q}_T)_* \\ H^0(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, \wedge^2_T H_n \otimes \mathbb{Z}_2)) = \mathbb{Z}/2 \end{array}$$

is an isomorphism. In particular the characteristic class $\{\theta\}$ represented by the obstruction operator θ in (2.5.7) $(n > 2)$ is also trivial

$$0 = \{\theta\} \in H^1(\mathbf{chain}_n(\mathbf{ab}(T))/\simeq, H^{n+2}(-, H_n \otimes \mathbb{Z}_2)),$$

therefore Steenrod invariants in the sense of Section 5.1 does exist for trees with 3 ends as well.

PROOF. We will use the following Bockstein long exact sequence

$$\begin{array}{c}\vdots\\ \downarrow\\ H^k(\mathbf{fp}(\mathbf{vect}(T)),\mathrm{Ext}^1_{\mathbf{vect}(T)})\\ \downarrow{\scriptstyle(\bar\tau_T)_*}\\ H^k(\mathbf{fp}(\mathbf{vect}(T)),\mathrm{Ext}^1_{\mathbf{vect}(T)}(-,\hat\otimes^2_T))\\ \downarrow{\scriptstyle(\bar q_T)_*}\\ H^k(\mathbf{fp}(\mathbf{vect}(T)),\mathrm{Ext}^1_{\mathbf{vect}(T)}(-,\wedge^2_T))\\ \downarrow{\scriptstyle\beta}\\ H^{k+1}(\mathbf{fp}(\mathbf{vect}(T)),\mathrm{Ext}^1_{\mathbf{vect}(T)})\\ \downarrow\\ \vdots\end{array}$$

which is isomorphic to (5.2.3) by the localization theorem for cohomology of categories in [**Mur06**].

We showed in Theorem 4.2.1 that

$$H^0(\mathbf{fp}(\mathbf{vect}(T)),\mathrm{Ext}^1_{\mathbf{vect}(T)}(-,\wedge^2_T))\simeq\mathbb{Z}/2$$

generated by the 0-cocycle c corresponding to the chain cup-product mod 2. This 0-cocycle c is additive, hence it is determined by its values over elementary f. p. $\mathbf{vect}(T)$-modules, see Theorem 6.1.2,

$$c_{\mathcal{M}}\in\mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathcal{M},\wedge^2_T\mathcal{M}).$$

In fact $c_{\mathcal{M}}=0$ unless $\mathcal{M}=\mathbb{M}\underline{V}^{(3,5)}$, and in this case

$$0\neq c_{\mathbb{M}\underline{V}^{(3,5)}}\in\mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathbb{M}\underline{V}^{(3,5)},\wedge^2_T\mathbb{M}\underline{V}^{(3,5)})=\mathbb{Z}/2,$$

see Lemma 6.3.9.

Let

$$\varepsilon\colon\ \wedge^2_T\,\mathbb{M}\underline{V}^{(3,5)}\twoheadrightarrow\hat\otimes^2_T\mathbb{M}\underline{V}^{(3,5)}$$

be a splitting of $\bar q_T$. Such a splitting exists by Proposition 6.4.7. We define the 0-cochain d of $\mathbf{fp}(\mathbf{vect}(T))$ with coefficients in $\mathrm{Ext}^1_{\mathbf{vect}(T)}(-,\hat\otimes^2_T)$ as

$$\begin{array}{ll} d_{\mathcal{M}}=0, & \text{if } \mathcal{M} \text{ is an elementary } \mathbf{vect}(T)\text{-module } \mathcal{M}\neq\mathbb{M}\underline{V}^{(3,5)};\\ d_{\mathbb{M}\underline{V}^{(3,5)}}=\varepsilon_*(c_{\mathbb{M}\underline{V}^{(3,5)}}). & \end{array}$$

If we manage to prove that d is a cocycle then immediately $(\bar q_T)_*d=c$ and the surjectivity will follow.

The elementary $\mathbf{vect}(T)$-module $\mathbb{M}\underline{V}^{(3,5)}$ has only the identity and the trivial endomorphisms, see Proposition 6.3.5 (4), therefore all we have to do to prove that d is a cocycle is to check that given an an elementary $\mathbf{vect}(T)$-module $\mathcal{M}\neq\mathbb{M}\underline{V}^{(3,5)}$

and arbitrary morphisms $f\colon \mathcal{M} \to \mathbb{M}\underline{V}^{(3,5)}$ and $g\colon \mathbb{M}\underline{V}^{(3,5)} \to \mathcal{M}$ the following equalities hold

$$0 = f^*\varepsilon_* c_{\mathbb{M}\underline{V}^{(3,5)}} \in \mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathcal{M}, \hat{\otimes}^2_T \mathbb{M}\underline{V}^{(3,5)}),$$

$$0 = g_*\varepsilon_* c_{\mathbb{M}\underline{V}^{(3,5)}} \in \mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathbb{M}\underline{V}^{(3,5)}, \hat{\otimes}^2_T \mathcal{M}).$$

The first one follows easily from the equality $f^*\varepsilon_* = \varepsilon_* f^*$ and the fact that c is a cocycle. In order to prove the second one it is enough to check that for any elementary $\mathbf{vect}(T)$-module $\mathcal{M} \neq \mathbb{M}\underline{V}^{(3,5)}$ either

$$\mathrm{Hom}_{\mathbf{vect}(T)}(\mathbb{M}\underline{V}^{(3,5)}, \mathcal{M}) = 0$$

or

$$\mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathbb{M}\underline{V}^{(3,5)}, \hat{\otimes}^2_T \mathcal{M}) = 0.$$

This is proved in Lemma 6.3.10 below.

In order to finish the proof we just have to check that

$$H^0(\mathbf{fp}(\mathbf{vect}(T)), \mathrm{Ext}^1_{\mathbf{vect}(T)}) = 0.$$

This follows from the additivity of 0-cocycles and the fact that elementary f. p. $\mathbf{vect}(T)$-modules have no self-extensions, see Proposition 6.3.2. □

In the proof of Theorem 5.2.6 it is also implicitly contained the proof of the following one, compare with Theorem 4.2.3. In the statement we use the natural transformation $\hat{\kappa}$ which is the composite

$$\begin{array}{c}
\mathrm{Ext}^1_{\mathbf{vect}(T)}(H_n \otimes \mathbb{Z}/2, \hat{\otimes}^2_T H_n \otimes \mathbb{Z}/2) \\
\big\downarrow{\scriptstyle\simeq} \\
\mathrm{Ext}^2_{\mathbf{ab}(T)}(H_n \otimes \mathbb{Z}/2, \hat{\otimes}^2_T H_n \otimes \mathbb{Z}/2) \\
\big\downarrow{\scriptstyle\hat{p}^*} \\
\mathrm{Ext}^2_{\mathbf{ab}(T)}(H_n, \hat{\otimes}^2_T H_n \otimes \mathbb{Z}/2) \\
\big\downarrow \\
H^{n+2}(-, \hat{\otimes}^2_T H_n \otimes \mathbb{Z}_2)
\end{array}$$

given by Proposition 6.3.3, the natural projection $\hat{p}\colon \mathcal{M} \twoheadrightarrow \mathcal{M} \otimes \mathbb{Z}_2$, and the universal coefficients spectral sequence (4.2.2).

THEOREM 5.2.7. *Given a tree T with 3 ends and a bounded chain complex $\mathcal{C}_*$ in $\mathbf{ab}(T)$ concentrated in dimensions $\geq n$, if $H_n\mathcal{C}_* \otimes \mathbb{Z}/2$ contains $\mathbb{M}\underline{V}^{(3,5)}$ as a direct summand $((\bar{q}_T)^{-1}_*\hat{p}_*\bar{\cup})_{\mathcal{C}_*}$ then is the image of the generator*

$$\begin{array}{c}
\mathbb{Z}/2 = \mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathbb{M}\underline{V}^{(3,5)}, \hat{\otimes}^2_T \mathbb{M}\underline{V}^{(3,5)}) \\
\big\downarrow \\
\mathrm{Ext}^1_{\mathbf{vect}(T)}(H_n\mathcal{C}_* \otimes \mathbb{Z}/2, \hat{\otimes}^2_T H_n\mathcal{C}_* \otimes \mathbb{Z}/2) \\
\big\downarrow{\scriptstyle\hat{\kappa}} \\
H^{n+2}(\mathcal{C}_*, \hat{\otimes}^2_T H_n\mathcal{C}_* \otimes \mathbb{Z}/2)
\end{array}$$

Otherwise $((\bar{q}_T)^{-1}_*\hat{p}_*\bar{\cup})_{\mathcal{C}_*} = 0$.

Now that we have proved the existence of Steenrod invariants for trees T with less than 4 ends we proceed to define them explicitly.

DEFINITION 5.2.8. Let $(\mathcal{C}_*, f_{n+2})$ be an $(n-1)$-reduced T-homotopy system $(n>2)$. If T has 1 or 2 ends its *Steenrod invariant*

$$\wp_n(\mathcal{C}_*, f_{n+2}) \in H^{n+2}(\mathcal{C}_*, H_n\mathcal{C}_* \otimes \mathbb{Z}/2)$$

is the unique element such that

$$(\bar{\tau}_T)_*\wp_n(\mathcal{C}_*, f_{n+2}) = \hat{p}_*\hat{\cup}_{(\mathcal{C}_*, f_{n+2})} \in H^{n+2}(\mathcal{C}_*, \hat{\otimes}^2_T H_n\mathcal{C}_* \otimes \mathbb{Z}/2),$$

and if T has 3 ends it is the unique element satisfying

$$(\bar{\tau}_T)_*\wp_n(\mathcal{C}_*, f_{n+2}) = \hat{p}_*\hat{\cup}_{(\mathcal{C}_*, f_{n+2})} - ((\bar{q}_T)_*^{-1}\hat{p}_*\bar{\cup})_{\mathcal{C}_*} \in H^{n+2}(\mathcal{C}_*, \hat{\otimes}^2_T H_n\mathcal{C}_* \otimes \mathbb{Z}/2).$$

THEOREM 5.2.9. *If T has less than 4 ends the Steenrod invariant of an $(n-1)$-reduced T-homotopy system $(n>2)$ is well defined and determines the obstruction operator θ in (2.5.7) in the following sense, given a chain homomorphism $\xi\colon \mathcal{C}_* \to \mathcal{C}'_*$ then*

$$\theta_{(\mathcal{C}_*, f_{n+2}),(\mathcal{C}'_*, g_{n+2})}(\xi) = \xi_*\wp_n(\mathcal{C}_*, f_{n+2}) - \xi^*\wp_n(\mathcal{C}'_*, g_{n+2}).$$

PROOF. The existence is derived from the fact that (5.2.2) is short exact in this case, see Proposition 6.4.7, and the equalities

$$(\bar{q}_T)_*\hat{p}_*\hat{\cup}_{(\mathcal{C}_*, f_{n+2})} = \hat{p}_*\bar{\cup}_{\mathcal{C}_*} = 0$$

if T has 1 or 2 ends, see Definition 3.4.1 and Theorem 4.2.1; or

$$(\bar{q}_T)_*(\hat{p}_*\hat{\cup}_{(\mathcal{C}_*, f_{n+2})} - ((\bar{q}_T)_*^{-1}\hat{p}_*\bar{\cup})_{\mathcal{C}_*}) = \hat{p}_*\bar{\cup}_{\mathcal{C}_*} - \hat{p}_*\bar{\cup}_{\mathcal{C}_*} = 0$$

if T has 3, see Definition 3.4.1 and Theorem 5.2.6.

The second part of the statement follows from Theorem 3.3.1 and the facts that $(\bar{q}_T)^{-1}\hat{p}_*\bar{\cup}$ is a 0-cocycle and $(\bar{\tau}_T)_*$ is a monomorphism in this case, see Proposition 6.4.7. □

Steenrod invariants for T-complexes are defined by using the functor r in Section 2.5.

DEFINITION 5.2.10. Let T be a tree with less than 4 ends, the *Steenrod invariant* of an $(n-1)$-reduced T-complex X $(n>2)$ is defined as

$$\wp_n(X) = \wp_n(rX) \in H^{n+2}(X, \mathcal{H}_nX \otimes \mathbb{Z}/2).$$

5.3. The proper homotopy classification of A^2_n-polyhedra with less than 4 ends $(n>2)$

Given a fixed tree T an A^2_n-*polyhedron* is an $(n-1)$-reduced T-complex X of dimension $\leq n+2$, the proper homotopy category of A^2_n-polyhedra was already considered in Section 2.5 and is denoted by $\mathbf{A}^2_n(T)$. This category is equivalent to the proper homotopy category of $(n-1)$-connected T-complexes of dimension $\leq n+2$, see Proposition 1.3.5.

DEFINITION 5.3.1. The objects of the category $\mathbf{P}^2_n(T)$ are pairs $(\mathcal{C}_*, \wp)$ where $\mathcal{C}_*$ is a chain complex in $\mathbf{ab}(T)$ concentrated in dimensions n, $n+1$ and $n+2$, together with an element

$$\wp \in H^{n+2}(\mathcal{C}_*, H_n\mathcal{C}_* \otimes \mathbb{Z}/2).$$

Morphisms $\xi\colon (\mathcal{C}_*, \wp) \to (\mathcal{C}'_*, \wp')$ are chain homotopy classes $\xi\colon \mathcal{C}_* \to \mathcal{C}'_*$ such that

$$\xi_*\wp = \xi^*\wp' \in H^{n+2}(\mathcal{C}_*, H_n\mathcal{C}'_* \otimes \mathbb{Z}/2).$$

THEOREM 5.3.2. *If T has less than 4 ends and $n > 2$ the functor*

$$\mathbf{A}^2_n(T) \longrightarrow \mathbf{P}^2_n(T)\colon X \mapsto (\mathcal{C}_*X, \wp_n(X))$$

induces a bijection between the sets of isomorphism classes of objects.

PROOF. Recall from Section 2.5 that $\mathbf{H}^2_n(T)$ denotes the category of $(n-1)$-reduced T-homotopy systems $(\mathcal{C}_*, f_{n+2})$ such that the chain complex $\mathcal{C}_*$ is concentrated in dimensions n, $n+1$ and $n+2$. By proposition 2.5.6 this theorem is equivalent to prove that the functor

$$\mathbf{H}^2_n(T)/\simeq \longrightarrow \mathbf{P}^2_n(T)\colon (\mathcal{C}_*, f_{n+2}) \mapsto (\mathcal{C}_*, \wp_n(\mathcal{C}_*, f_{n+2}))$$

induces a bijection between the sets of isomorphism classes of objects in both categories.

By Proposition 2.5.8 given $(\mathcal{C}_*, \wp)$ in $\mathbf{P}^2_n(T)$ there exists an $(n-1)$-reduced T-homotopy system $(\mathcal{C}_*, f_{n+2})$. With the notation in Section 2.5 it is easy to see by applying Proposition 5.2.9 that the new homotopy system $(\mathcal{C}_*, f_{n+1}) + (\wp - \wp_n(\mathcal{C}_*, f_{n+1}))$ has Steenrod invariant $\wp$.

Suppose now that we have two homotopy systems $(\mathcal{C}_*, f_{n+2})$, $(\mathcal{C}'_*, g_{n+2})$ and an isomorphism in $\mathbf{P}^2_n(T)$

$$\xi\colon (\mathcal{C}_*, \wp_n(\mathcal{C}_*, f_{n+2})) \longrightarrow (\mathcal{C}'_*, \wp_n(\mathcal{C}'_*, g_{n+2})).$$

By Proposition 5.2.9 there exists a morphism of homotopy systems

$$(\xi, \eta)\colon (\mathcal{C}_*, f_{n+2}) \longrightarrow (\mathcal{C}'_*, g_{n+2}).$$

This morphism is an isomorphism in the homotopy category because the functor $(\mathcal{C}_*, f_{n+2}) \mapsto \mathcal{C}_*$ fits into an exact sequence in the sense of Section 2.5. $\square$

COROLLARY 5.3.3. *If T has less than 4 ends, given a chain complex $\mathcal{C}_*$ in $\mathbf{ab}(T)$ concentrated in dimensions n, $n+1$ and $n+2$ for some $n > 2$ the set of proper homotopy classes of A^2_n-polyhedra X with proper cellular chain complex $\mathcal{C}_*X$ homotopy equivalent to $\mathcal{C}_*$ is in bijective correspondence with the orbit set*

$$H^{n+2}(\mathcal{C}_*, H_n\mathcal{C}_* \otimes \mathbb{Z}/2)/\operatorname{Aut}(\mathcal{C}_*)$$

of the right action of the group $\operatorname{Aut}(\mathcal{C}_*)$ of self-homotopy equivalences of $\mathcal{C}_*$ on the cohomology group $H^{n+2}(\mathcal{C}_*, H_n\mathcal{C}_* \otimes \mathbb{Z}/2)$ given by

$$a^\xi = \xi^*\xi_*^{-1}(a).$$

PROOF. One can readily check that the function sending X to the class of $\wp_n(X)$ in the orbit set induces the desired bijection. $\square$

5.4. Moore spaces in proper homotopy theory

DEFINITION 5.4.1. Given a fixed tree T a Moore space of degree n is a 1-connected T-complex X whose unique non-trivial proper homology $\mathbf{ab}(T)$-module is $\mathcal{H}_nX$.

By Theorem 1.2.14 and Proposition 2.5.8 there are Moore spaces in degree $n \geq 2$ for any finitely presented $\mathbf{ab}(T)$-module. Moreover, the proper homotopy category $\mathbf{M}_n(T)$ of Moore spaces of degree n can be regarded as a full subcategory of $\mathbf{A}_n^2(T)$. In order to specify the results in the previous section to this subcategory we give the following definition.

Definition 5.4.2. Objects in the category $\mathbf{PM}_n^2(T)$ are pairs $(\mathcal{M}, \wp)$ given by a finitely presented $\mathbf{ab}(T)$-module $\mathcal{M}$ together with an element

$$\wp \in \operatorname{Ext}^2_{\mathbf{ab}(T)}(\mathcal{M}, \mathcal{M} \otimes \mathbb{Z}/2).$$

Morphisms $\xi\colon (\mathcal{M}, \wp) \to (\mathcal{M}', \wp')$ are $\mathbf{ab}(T)$-module morphisms $\varphi\colon \mathcal{M} \to \mathcal{M}'$ such that

$$\varphi_* \wp = \varphi^* \wp' \in \operatorname{Ext}^2_{\mathbf{ab}(T)}(\mathcal{M}, \mathcal{M}' \otimes \mathbb{Z}/2).$$

Theorem 5.4.3. *If T has less than 4 ends and $n > 2$ the functor*

$$\mathbf{M}_n^2(T) \longrightarrow \mathbf{PM}_n^2(T)\colon X \mapsto (\mathcal{H}_n X, \wp_n(X))$$

induces a bijection between the sets of isomorphism classes of objects.

Corollary 5.4.4. *If T has less than 4 ends, given a f. p. $\mathbf{ab}(T)$-module $\mathcal{M}$ the set of proper homotopy classes of Moore spaces X of degree n with proper cellular homology $\mathcal{H}_n X$ isomorphic to $\mathcal{M}$ is in bijective correspondence with the orbit set*

$$\operatorname{Ext}^2_{\mathbf{ab}(T)}(\mathcal{M}, \mathcal{M} \otimes \mathbb{Z}/2)/\operatorname{Aut}(\mathcal{M})$$

of the right action given by

$$a^{\varphi} = \varphi^* \varphi_*^{-1}(a).$$

Remark 5.4.5. The easiest explicit computation one can do with the previous corollary is the following. Consider the $\mathbf{vect}(T_1)$-module $\mathcal{A} \oplus \mathcal{C}$, see Theorem 6.1.2. By Proposition 6.3.3 and [**Mur04**] 7.17 and A.3

$$\operatorname{Ext}^2_{\mathbf{ab}(T)}(\mathcal{A} \oplus \mathcal{C}, (\mathcal{A} \oplus \mathcal{C}) \otimes \mathbb{Z}/2) = \mathbb{Z}/2,$$

in particular the action of $\operatorname{Aut}(\mathcal{A} \oplus \mathcal{C})$ on this group is trivial and there are exactly 2 proper Moore space of degree $n > 2$ with proper homology $\mathcal{A} \oplus \mathcal{C}$.

The first example of non homotopy equivalent Moore spaces with the same proper homology $\mathbf{ab}(T_1)$-module $\mathcal{S}$ was discovered in [**ADMQ95**]. Although they used the pro-categorical approach to proper homology one can check by using the Brown-Grossman functor, see [**BQ01**] V.3.10, and the projective resolution for $\mathcal{S}$ in [**ADMQ95**] Appendix A that $\mathcal{S}$ in our algebraic context is $\mathcal{S} = \mathcal{R} \oplus \mathcal{B}$. By Proposition 6.3.3, [**Mur04**] 7.12 and A.3

$$\dim_{\mathbb{F}_2} \operatorname{Ext}^2_{\mathbf{ab}(T)}(\mathcal{R} \oplus \mathcal{B}, (\mathcal{R} \oplus \mathcal{B}) \otimes \mathbb{Z}/2) = 2^{\aleph_0},$$

An immediate consequence of Corollary 5.4.4 is the following.

Corollary 5.4.6. *If T has less than 4 ends and $n > 2$ there is a unique Moore space of degree n with homology $\mathcal{M}$ if and only if* $\operatorname{Ext}^2_{\mathbf{ab}(T)}(\mathcal{M}, \mathcal{M} \otimes \mathbb{Z}/2) = 0$.

In the next corollary we completely determine the uniqueness of proper Moore spaces of degree $n > 2$ with only one end and exponent 2 in proper homology. It follows from the previous corollary, Proposition 6.3.3 and the homological computations in the appendix of [**Mur04**].

COROLLARY 5.4.7. *If T has only 1 end and $\mathcal{M}$ is a f. p.* $\mathbf{vect}(T)$*-module then there exists a unique Moore space X of degree $n > 2$ with homology $\mathcal{M}$ if and only if this* $\mathbf{vect}(T)$*-module satisfies one of the following conditions:*

(1) *it does not contain $\mathcal{A}$ as a direct summand,*
(2) *it does not contain either $\mathcal{R}$ or $\mathcal{C}$ as a direct summand,*
(3) *it does not contain either $\mathcal{B}$ or $\mathcal{C}$ as a direct summand.*

Now by Theorem 6.1.2 we get the following result.

COROLLARY 5.4.8. *It T has only one end there are infinite f. p.* $\mathbf{vect}(T)$*-modules $\mathcal{M}$ such that there exist different Moore spaces in degree $n > 2$ with homology $\mathcal{M}$.*

REMARK 5.4.9. The lower part of the Whitehead long exact sequence

$$\mathcal{H}_{n+2}X \overset{b_{n+2}}{\to} \Gamma_{n+1}X \overset{i_{n+1}}{\to} \Pi_{n+1}X \overset{h_{n+1}}{\twoheadrightarrow} \mathcal{H}_{n+1}X, \;\; \mathcal{H}_nX,$$

does not classify proper homotopy types of A_n^2-polyhedra. By Theorem 2.3.1 the sequence is the same for any two Moore spaces of degree n with the same homology. However, as we can see for example in the previous corollary, the homology does not determine the proper homotopy type.

The following result on the existence of co-H-structures on degree 2 Moore spaces is a consequence of Theorem 4.2.3 and Proposition 3.5.2.

COROLLARY 5.4.10. *Let $\mathcal{M}$ be a f. p.* $\mathbf{vect}(T)$*-module. If T has 1 or 2 ends all Moore spaces in degree 2 with homology $\mathcal{M}$ are co-H-spaces. If T has three ends such a Moore space is a co-H-space if and only if $\mathcal{M}$ does not contain $\mathbb{M}\underline{V}^{(3,5)}$ as a direct summand.*

REMARK 5.4.11. The module $\mathbb{M}\underline{V}^{(3,5)}$ has the curious property that there exists a unique Moore space of degree $n > 2$ with that homology, but all possible Moore spaces with homology $\mathbb{M}\underline{V}^{(3,5)}$ in degree 2 are not co-H-spaces.

CHAPTER 6

Computations in controlled algebra

In this section we perform technical algebraic computations leading to the main homotopical results of this paper already presented in the previous chapter. For this we use the representation theory of the algebras $k(\mathfrak{F}(T))$ for k a field considered in [**Mur04**].

6.1. Review of the representation theory of the algebras $k(\mathfrak{F}(T))$

Let k be an arbitrary field. Recall from Section 1.2 that $k(\mathfrak{F}(T))$ is a k-algebra Morita equivalent to the small additive category $\mathbf{M}_k(T)$.

In [**Mur04**] 1.1 we determined the representation type of the k-algebras $k(\mathfrak{F}(T))$ for k an arbitrary field in terms of the number of Freudenthal ends of T.

THEOREM 6.1.1. *The representation type of $k(\mathfrak{F}(T))$ is*

$\operatorname{card}\mathfrak{F}(T)$	*type*
< 4	*finite*
$= 4$	*tame*
> 4	*wild*

In the finite and tame cases we give in [**Mur04**] explicit classification theorems for f. p. $\mathbf{M}_k(T)$-modules. In order to state them we fix an explicit tree T_n with a finite number n of Freudenthal ends, see Section 3.1 in [**Mur04**]. The vertex set of T_n is

$$T_n^0 = \{v_0\} \cup \{v_m^1, \dots v_m^n\}_{m\geq 1}$$

and there are edges joining v_0 with v_1^i and v_m^i with v_{m+1}^i $(1 \leq i \leq n, m \geq 1)$. Notice that we can identify T_1 with the half-line $\mathbb{R}_+$ and T_1^0 with the non-negative integers $\mathbb{N}_0$. Consider the $\mathbb{N}_0 \times \mathbb{N}_0$ matrices A and B with entries in k given by

- $\mathsf{a}_{i+1,i} = 1$ $(i \in \mathbb{N}_0)$ and $\mathsf{a}_{ij} = 0$ in other cases,
- $\mathsf{b}_{\frac{n(n+1)}{2}+i,\frac{(n-1)n}{2}+i} = 1$ for any $n > 0$ and $0 \leq i < n$, and $\mathsf{b}_{ij} = 0$ otherwise.

Moreover, let I be the identity matrix. The f. p. $\mathbf{M}_k(\mathbb{R}_+)$-modules $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{B}_\infty$ and $\mathcal{C}_\infty$ are defined as the cokernels of the endomorphisms of the free T-controlled k-module $k\langle\mathbb{N}_0\rangle_\delta$, where $\delta\colon \mathbb{N}_0 \hookrightarrow \mathbb{R}_+$ is the inclusion, given by the matrices A, $\mathsf{I}-\mathsf{A}$, $\mathsf{I}-\mathsf{A}^t$, $\mathsf{I}-\mathsf{B}$ and $\mathsf{I}-\mathsf{B}^t$, respectively. Here $(-)^t$ denotes transposition. Furthermore, we call $\mathcal{R}$ to the f. g. free $\mathbf{M}_k(\mathbb{R}_+)$-module $k\langle\mathbb{N}_0\rangle_\delta$.

There are n proper homotopy classes of maps $f_i\colon \mathbb{R}_+ \to T_n$ $(1 \leq i \leq n)$ corresponding to the inclusions of the n different Freudenthal ends of T_n. We can take f_i as the simplicial map with $f_i(0) = v_0$ and $f_i(j) = w_n^i$ $(j \geq 1)$. As we know from Section 1.2 these maps induce "change of tree" functors

$$\mathbb{F}^i = \mathbb{F}^{f_i}\colon \mathbf{M}_k(\mathbb{R}_+) \longrightarrow \mathbf{M}_k(T_n).$$

For trees with more than one Freudenthal end the classification theorems depend on the classification of what we call finite-dimensional rigid n-subspaces, see Section 8 of [**Mur04**], which are special representations of the n-subspace quiver Q_n

$$\begin{tikzcd}
1 \arrow[drr] & 2 \arrow[dr] & \cdots\cdots\cdots & n \arrow[dl] \\
 & & 0 &
\end{tikzcd}$$

Recall that a representation of Q_n is a diagram of k-vector spaces indexed by Q_n, i. e. $n+1$ vector spaces $V_0, V_1, \dots, V_n$ together with homomorphisms $V_i \to V_0$ $(0 \leq i \leq n)$. These representations form an abelian category which coincides with the category of modules over the path algebra kQ_n. An n-subspace is a representation such that all the morphisms $V_i \to V_0$ are inclusions of subspaces. There is an exact, full and faithful functor between the categories of finite-dimensional n-subspaces and f. p. $\mathbf{M}_k(T_n)$-modules, see Section 9 of [**Mur04**],

$$\mathbb{M}\colon \mathbf{sub}_n^{\mathrm{fin}} \longrightarrow \mathbf{fp}(\mathbf{M}_k(T_n)).$$

The following classification theorem is proved in [**Mur04**] 10.5 for arbitrary fields. In the statement we use the following terminology: a solution to the *decomposition problem* in a small additive category $\mathbf{A}$ consists of a set of objects (which we call *elementary objects*) and of a set of isomorphisms (*elementary isomorphisms*) between finite direct sums of elementary objects. These sets must satisfy that any object in $\mathbf{A}$ is isomorphic to a finite direct sum of elementary ones, and any isomorphism relation between two such direct sums can be derived from the elementary isomorphisms.

THEOREM 6.1.2. *Let*

$$\left\{\underline{V}^{(n,j)}\right\}_{j\in J_n}$$

be the set of indecomposable rigid n-subspaces $(n \geq 1)$, see Proposition 6.1.3. There is a solution to the decomposition problem in the category of f. p. $\mathbf{M}_k(T_n)$-modules given by the following $1 + 5n + \operatorname{card} J_n$ elementary modules $(1 \leq i \leq n, j \in J_n)$

$$\mathbb{F}_*^1\mathcal{A},\ \mathbb{F}_*^i\mathcal{R},\ \mathbb{F}_*^i\mathcal{B},\ \mathbb{F}_*^i\mathcal{B}_\infty,\ \mathbb{F}_*^i\mathcal{C},\ \mathbb{F}_*^i\mathcal{C}_\infty,\ \mathbb{M}\underline{V}^{(n,j)},$$

and $6n$ elementary isomorphisms $(1 \leq i \leq n)$

$$\mathbb{F}_*^1\mathcal{A} \oplus \mathbb{F}_*^i\mathcal{R} \simeq \mathbb{F}_*^i\mathcal{R},\ \mathbb{F}_*^i\mathcal{R} \oplus \mathbb{F}_*^i\mathcal{R} \simeq \mathbb{F}_*^i\mathcal{R},\ \mathbb{F}_*^i\mathcal{B} \oplus \mathbb{F}_*^i\mathcal{B}_\infty \simeq \mathbb{F}_*^i\mathcal{B}_\infty,$$

$$\mathbb{F}_*^i\mathcal{B}_\infty \oplus \mathbb{F}_*^i\mathcal{B}_\infty \simeq \mathbb{F}_*^i\mathcal{B}_\infty,\ \mathbb{F}_*^i\mathcal{C} \oplus \mathbb{F}_*^i\mathcal{C}_\infty \simeq \mathbb{F}_*^i\mathcal{C}_\infty,\ \mathbb{F}_*^i\mathcal{C}_\infty \oplus \mathbb{F}_*^i\mathcal{C}_\infty \simeq \mathbb{F}_*^i\mathcal{C}_\infty.$$

For $n = 1, 2$ and 3 indecomposable rigid n-subspaces are listed in the following proposition, see [**Mur04**] 8.7.

PROPOSITION 6.1.3. *The following are complete lists of (representatives of the isomorphism classes of) indecomposable rigid n-subspaces for $n < 4$*

- $n = 1$, *none,*
- $n = 2$, $\underline{V}^{(2,1)} = (k \to k \leftarrow k)$,
- $n = 3$,

$$\underline{V}^{(3,1)} = \left(\begin{array}{ccccc} & & k & & \\ & & \downarrow & & \\ k & \to & k & \leftarrow & 0 \end{array}\right),\quad \underline{V}^{(3,2)} = \left(\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ k & \to & k & \leftarrow & k \end{array}\right),$$

$$\underline{V}^{(3,3)} = \left(\begin{array}{ccccc} & & k & & \\ & & \downarrow & & \\ 0 & \to & k & \leftarrow & k \end{array}\right), \quad \underline{V}^{(3,4)} = \left(\begin{array}{ccccc} & & k & & \\ & & \downarrow & & \\ k & \to & k & \leftarrow & k \end{array}\right),$$

$$\underline{V}^{(3,5)} = \left(\begin{array}{ccccc} & & k\langle x+y\rangle & & \\ & & \downarrow & & \\ k\langle x\rangle & \to & k\langle x,y\rangle & \leftarrow & k\langle y\rangle \end{array}\right).$$

For $n = 4$ the list of indecomposable rigid n-subspaces is infinite and very complicated to describe. It can be extracted from [**Naz73**], see the remark in [**Mur04**] 8.8.

6.2. Some "change of tree" computations

PROPOSITION 6.2.1. *For all $n \geq 1$ and $1 \leq i \leq n$ we have that $\mathbb{F}^1_*\mathcal{A} = \mathbb{F}^i_*\mathcal{A}$.*

This proposition follows either from [**Mur04**] 7.7 or from [**Mur04**] 10.3.

LEMMA 6.2.2. *Let $k\langle A\rangle_\alpha$ be a T_1-controlled k-module and let $k\langle B\rangle_\beta$ be a T_n-controlled k-module such that there exists a neighbourhood U of the i^{th} Freudenthal end of T_n in $\hat{T}_n$ with $\beta^{-1}(U) = \emptyset$. Then there is a natural abelian group isomorphism*

$$\mathrm{Hom}_{\mathbf{M}_k(T_n)}(\mathbb{F}^i_* k\langle A\rangle_\alpha, k\langle B\rangle_\beta) \simeq \bigoplus_A k\langle B\rangle \subset \prod_A k\langle B\rangle = \mathrm{Hom}_k(k\langle A\rangle, k\langle B\rangle).$$

PROOF. It is enough to check that any controlled homomorphism $\varphi\colon \mathbb{F}^i_* k\langle A\rangle_\alpha = k\langle A\rangle_{f_i\alpha} \to k\langle B\rangle_\beta$ vanishes in almost all A. Since φ is controlled there exists another neighbourhood $V \subset U$ of the i^{th} Freudenthal end of T_n in $\hat{T}_n$ such that $\varphi(\alpha^{-1}(f_i^{-1}(U))) = \varphi((f_i\alpha)^{-1}(U)) \subset k\langle\beta^{-1}(U)\rangle = 0$. The closure of the set $T_1 - f_i^{-1}(U) \subset T_1$ is compact because $(f_i\alpha)^{-1}(U)$ is a neighbourhood of the unique Freudenthal end of T_1 hence $\alpha^{-1}(T_1 - f_i^{-1}(U))$ is finite and the lemma follows. $\square$

COROLLARY 6.2.3. *Given two T_1-controlled k-modules $k\langle A\rangle_\alpha$ and $k\langle B\rangle_\beta$, if $i \neq j$ there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{M}_k(T_1)}(k\langle A\rangle_\alpha, (\mathbb{F}^i)^*\mathbb{F}^j_* k\langle B\rangle_\beta) \simeq \bigoplus_A k\langle B\rangle \subset \prod_A k\langle B\rangle = \mathrm{Hom}_k(k\langle A\rangle, k\langle B\rangle).$$

This corollary follows from (1.2.2), Lemma 6.2.2 and the fact that $(\mathbb{F}^i)^*$ is right adjoint to $\mathbb{F}^i_*$.

COROLLARY 6.2.4. *If $\mathcal{M}$ a $\mathbf{M}_k(T_1)$-module f. p. by the controlled homomorphism $\varphi\colon k\langle B\rangle_\beta \to k\langle C\rangle_\gamma$ and $i \neq j$ then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{M}_k(T_1)}(k\langle A\rangle_\alpha, (\mathbb{F}^i)^*\mathbb{F}^j_*\mathcal{M}) \simeq \bigoplus_A \frac{k\langle C\rangle}{\varphi(k\langle B\rangle)} \subset \prod_A \frac{k\langle C\rangle}{\varphi(k\langle B\rangle)} = \mathrm{Hom}_k\left(k\langle A\rangle, \frac{k\langle C\rangle}{\varphi(k\langle B\rangle)}\right).$$

This follows from Corollary 6.2.3 and the right-exactness of $(\mathbb{F}^i)^*$ and $\mathbb{F}^j_*$.

LEMMA 6.2.5. *There is an isomorphism natural in the T_1-controlled k-module $k\langle A\rangle_\alpha$*

$$\mathrm{Hom}_{\mathbf{M}_k(T_1)}(k\langle A\rangle_\alpha, \bigoplus_B \mathcal{A}) \simeq \bigoplus_A k\langle B\rangle \subset \prod_A k\langle B\rangle = \mathrm{Hom}_k(k\langle A\rangle, k\langle B\rangle).$$

PROOF. Since

$$\mathrm{Hom}_{\mathbf{M}_k(T_1)}(k\langle A\rangle_\alpha, \bigoplus_B \mathcal{A}) = \bigoplus_B \mathrm{Hom}_{\mathbf{M}_k(T_1)}(k\langle A\rangle_\alpha, \mathcal{A})$$

it is enough to prove the lemma for B a singleton. By [**Mur04**] 7.7 we have to prove that if $k\langle e\rangle_\phi$ is a T_1-controlled k-module with one generator e and $\phi(e) = v_0$ there is an isomorphism natural in $k\langle A\rangle_\alpha$

$$\mathrm{Hom}_{\mathbf{M}_k(T_1)}(k\langle A\rangle_\alpha, k\langle e\rangle_\phi) \subset \bigoplus_A k\langle e\rangle \subset \prod_A k\langle e\rangle = \mathrm{Hom}_k(k\langle A\rangle, k\langle e\rangle),$$

i. e. any controlled homomorphism $k\langle A\rangle_\alpha \to k\langle e\rangle_\phi$ vanishes in almost all A. This can be easily checked by using the definition of controlled homomorphism as in the proof of Lemma 6.2.2. In fact it can also be derived from Proposition 6.2.1 and Corollary 6.2.3 by using the fact that $\mathbb{F}^i$ is fully faithful, see Proposition 1.2.10, and hence $(\mathbb{F}^i)^*\mathbb{F}^i_* \simeq 1$, see Section 1.2. $\square$

PROPOSITION 6.2.6. *Given $i \neq j$ there are isomorphisms*

1. $(\mathbb{F}^i)^*\mathbb{F}^j_*\mathcal{B} \simeq \mathcal{A}$,
2. $(\mathbb{F}^i)^*\mathbb{F}^j_*\mathcal{B}_\infty \simeq \bigoplus_0^\infty \mathcal{A}$,
3. $(\mathbb{F}^i)^*\mathbb{F}^j_*\mathcal{C} = 0$,
4. $(\mathbb{F}^i)^*\mathbb{F}^j_*\mathcal{C}_\infty = 0$.

PROOF. By Corollary 6.2.4 and Lemma 6.2.5 it is enough to notice that

$$\begin{aligned} \dim k\langle\mathbb{N}_0\rangle/(\mathsf{I}-\mathsf{A})k\langle\mathbb{N}_0\rangle &= 1, \\ \dim k\langle\mathbb{N}_0\rangle/(\mathsf{I}-\mathsf{B})k\langle\mathbb{N}_0\rangle &= \aleph_0, \\ \dim k\langle\mathbb{N}_0\rangle/(\mathsf{I}-\mathsf{A}^t)k\langle\mathbb{N}_0\rangle &= 0, \\ \dim k\langle\mathbb{N}_0\rangle/(\mathsf{I}-\mathsf{B}^t)k\langle\mathbb{N}_0\rangle &= 0. \end{aligned}$$

See the proof of [**Mur04**] 7.3 $\square$

PROPOSITION 6.2.7. *Let $\underline{W}^i$ $(0 \le i \le n)$ be the n-subspace with* $\dim W^i_0 = \dim W^i_i = 1$ *(if $i > 0$) and* $\dim W^i_j = 0$ *otherwise, then* $(1 \le i \le n)$

1. $\mathbb{M}\underline{W}^0 = \mathbb{F}^i_*\mathcal{A}$,
2. $\mathbb{M}\underline{W}^i = \mathbb{F}^i_*\mathcal{B}$.

PROOF. By using the commutativity of (1.2.2) and the right-exactness of $\mathbb{F}^i_*$ one obtains a finite presentation of $\mathbb{F}^i_*\mathcal{A}$ and $\mathbb{F}^i_*\mathcal{B}$ as an $\mathbf{M}_k(T_n)$-module from [**Mur04**] 7.7 and the very definition $\mathcal{B}$. It is easy to se that this presentation coincides with the presentation of $\mathbb{M}\underline{W}^0$ or $\mathbb{M}\underline{W}^i$, respectively, given in the proof of [**Mur04**] 9.1. $\square$

6.3. Computation of some Hom and Ext groups

In this section we calculate some Hom and Ext groups of $\mathbf{ab}(T)$-modules and $\mathbf{vect}(T)$-modules which play a role in computations of cohomology groups of categories carried out in previous sections. The main tools are contained in the appendix of [**Mur04**].

PROPOSITION 6.3.1. *The following equalities hold*

1. $\dim \mathrm{Hom}_{\mathbf{vect}(T_1)}(\mathcal{B}, \mathcal{A}) = 0$,

(2) $\dim \mathrm{Hom}_{\mathbf{vect}(T_1)}(\mathcal{A}, \mathcal{B}) = 1$,
(3) $\dim \mathrm{Hom}_{\mathbf{vect}(T_n)}(\mathbb{F}^i_* \mathcal{B}, \mathbb{F}^j_* \mathcal{B}_\infty) = 0$ *for* $i \neq j$,
(4) $\dim \mathrm{Hom}_{\mathbf{vect}(T_1)}(\mathcal{C}, \mathcal{B}_\infty) = 0$,
(5) $\dim \mathrm{Ext}^1_{\mathbf{vect}(T_n)}(\mathbb{F}^j_* \mathcal{B}, \mathbb{F}^i_* \mathcal{B}_\infty) = 0$.

PROOF. The equalities (1) and (2) follow easily from Proposition 6.2.7. Moreover

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{vect}(T_n)}(\mathbb{F}^i_* \mathcal{B}, \mathbb{F}^j_* \mathcal{B}_\infty) &= \mathrm{Hom}_{\mathbf{vect}(T_1)}(\mathcal{B}, (\mathbb{F}^i)^* \mathbb{F}^j_* \mathcal{B}_\infty) \\
&= \mathrm{Hom}_{\mathbf{vect}(T_1)}(\mathcal{B}, \bigoplus_0^\infty \mathcal{A}) \\
&= \bigoplus_0^\infty \mathrm{Hom}_{\mathbf{vect}(T_1)}(\mathcal{B}, \mathcal{A}) \\
&= 0.
\end{aligned}$$

Here we use that $(\mathbb{F}^i)^*$ is right-adjoint to $\mathbb{F}^i_*$, Proposition 6.2.6 (2), and the fact that all finitely presented $\mathbf{vect}(T_1)$-modules are small as a consequence of [**Mur04**] 7.15.

By using [**Mur04**] 7.11 one readily sees that $\mathrm{Hom}_{\mathbf{vect}(T_1)}(\mathcal{M}, \mathcal{B}_\infty) = 0$ whenever $\mathcal{M}$ is the cokernel of a controlled homomorphism between free T_1-controlled $\mathbb{F}_2$-modules $\varphi\colon \mathbb{F}_2\langle A\rangle_\alpha \to \mathbb{F}_2\langle B\rangle_\beta$ whose underlying vector space homomorphism is surjective $\varphi\colon \mathbb{F}_2\langle A\rangle \twoheadrightarrow \mathbb{F}_2\langle B\rangle$, and this happens with $\mathcal{C}$, see the proof of [**Mur04**] 7.3.

Finally we have that

$$\begin{aligned}
\mathrm{Ext}^1_{\mathbf{vect}(T_n)}(\mathbb{F}^j_* \mathcal{B}, \mathbb{F}^i_* \mathcal{B}_\infty) &= \mathrm{Ext}^1_{\mathbf{vect}(T_1)}(\mathcal{B}, (\mathbb{F}^j)^* \mathbb{F}^i_* \mathcal{B}_\infty) \\
(\text{if } i = j) &= \mathrm{Ext}^1_{\mathbf{vect}(T_1)}(\mathcal{B}, \mathcal{B}_\infty) = 0, \\
(\text{if } i \neq j) &= \mathrm{Ext}^1_{\mathbf{vect}(T_1)}(\mathcal{B}, \bigoplus_0^\infty \mathcal{A}) = \bigoplus_0^\infty \mathrm{Ext}^1_{\mathbf{vect}(T_1)}(\mathcal{B}, \mathcal{A}) = 0.
\end{aligned}$$

Here we use Propositions 1.2.13, 6.2.6 (2), [**Mur04**] 7.12 and A.3 (5), and the fact that f. p. $\mathbf{vect}(T_1)$-modules are small, as noticed above. □

PROPOSITION 6.3.2. *If* T *has less than* 4 *ends the elementary finitely presented* $\mathbf{vect}(T)$*-modules in Theorem 6.1.2 have no self-extensions.*

PROOF. For T with only one end this follows from [**Mur04**] 7.12, 7.17 and the fact that $\mathcal{A}$ and $\mathcal{R}$ are projective, see [**Mur04**] 7.7. For trees with 2 or 3 ends it follows in addition from Proposition 1.2.13, [**Mur04**] A.4 and the well-known fact from representation theory that indecomposable representations over Dynkin diagrams have no self-extensions. □

We are interested in computing Ext^2 groups of $\mathbf{vect}(T)$-modules regarded as $\mathbf{ab}(T)$-modules. For this we use the change-of-rings spectral sequence associated to the natural projection $\mathbb{Z}(\mathfrak{F}(T)) \twoheadrightarrow \mathbb{F}_2(\mathfrak{F}(T))$, compare [**McC85**] 10.2 (c), or equivalently the change-of-ringoids spectral sequence associated to the functor $-\otimes \mathbb{Z}/2\colon \mathbf{ab}(T) \to \mathbf{vect}(T)$, see (1.2.5).

PROPOSITION 6.3.3. *For any pair of* $\mathbf{vect}(T)$*-modules* $\mathcal{M}$, $\mathcal{N}$*, the first one finitely presented, there is a natural isomorphism*

$$\mathrm{Ext}^2_{\mathbf{ab}(T)}(\mathcal{M},\mathcal{N}) \simeq \mathrm{Ext}^1_{\mathbf{vect}(T)}(\mathcal{M},\mathcal{N}).$$

Moreover, finitely presented $\mathbf{vect}(T)$*-modules have projective dimension* ≤ 1.

PROOF. For the proof of this proposition we prefer to work with modules over rings. To simplify we write $\mathfrak{F} = \mathfrak{F}(T)$. As a $\mathbb{Z}(\mathfrak{F})$-module $\mathbb{F}_2(\mathfrak{F})$ is the cokernel of the multiplication by 2 homomorphism $2\colon \mathbb{Z}(\mathfrak{F}) \hookrightarrow \mathbb{Z}(\mathfrak{F})$ therefore the E_2-term of the change-of-rings spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^q_{\mathbb{F}_2(\mathfrak{F})}(\mathrm{Tor}_p^{\mathbb{Z}(\mathfrak{F})}(\mathcal{M},\mathbb{F}_2(\mathfrak{F})),\mathcal{N}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbb{Z}(\mathfrak{F})}(\mathcal{M},\mathcal{N})$$

is concentrated in $0 \leq p \leq 1$. Moreover,

$$\mathrm{Tor}_p^{\mathbb{Z}(\mathfrak{F})}(\mathcal{M},\mathbb{F}_2(\mathfrak{F})) = \mathcal{M}, \quad p = 0, 1.$$

F. p. $\mathbb{F}_2(\mathfrak{F})$-modules are also f. p. as $\mathbb{Z}(\mathfrak{F})$-modules so by Corollary 1.2.15 they have projective dimension ≤ 2 and hence the E_∞-term is concentrated in $0 \leq p+q \leq 2$, in particular for any $q \geq 2$

$$\mathrm{Ext}^q_{\mathbb{F}_2(\mathfrak{F})}(\mathcal{M},\mathcal{N}) = E_2^{1,q} = E_\infty^{1,q} \hookrightarrow \mathrm{Ext}^{1+q}_{\mathbb{Z}(\mathfrak{F})}(\mathcal{M},\mathcal{N}) = 0,$$

therefore the $\mathbb{F}_2(\mathfrak{F})$-module $\mathcal{M}$ has projective dimension ≤ 1 and the E_2-term is concentrated in $0 \leq p, q \leq 1$, so

$$\mathrm{Ext}^1_{\mathbb{F}_2(\mathfrak{F})}(\mathcal{M},\mathcal{N}) = E_2^{1,1} = E_\infty^{1,1} = \mathrm{Ext}^2_{\mathbb{Z}(\mathfrak{F})}(\mathcal{M},\mathcal{N}).$$

□

By [**Mur04**] A.4 the computation of Ext^1 groups of $\mathbf{vect}(T_n)$-modules coming from finite-dimensional n-subspaces can be reduced to the computation of the corresponding Ext^1 groups in the category of representations of Q_n. In this category we have a powerful tool to carry out this kind of computations, the bilinear form $\langle -,- \rangle_{Q_n}$. This is a homomorphism

$$\langle -,- \rangle_{Q_n} \colon (\oplus_0^n \mathbb{Z}) \otimes_{\mathbb{Z}} (\oplus_0^n \mathbb{Z}) \longrightarrow \mathbb{Z},$$

which satisfies the following formula

$$\langle \underline{\dim}\, \underline{V}, \underline{\dim}\, \underline{W} \rangle_{Q_n} = \dim \mathrm{Hom}_{kQ_n}(\underline{V},\underline{W}) - \dim \mathrm{Ext}^1_{kQ_n}(\underline{V},\underline{W}) \tag{6.3.4}$$

for any pair of finite-dimensional n-subspaces $\underline{V}$ and $\underline{W}$. Here

$$\underline{\dim}\, \underline{V} = (\dim V_0, \dots, \dim V_n) \in \oplus_0^n \mathbb{Z}$$

is the *dimension vector* of the n-subspace $\underline{V}$. The bilinear form is defined by the following formula

$$\langle \underline{a}, \underline{b} \rangle_{Q_n} = \sum_{i=0}^n a_i b_i - \sum_{i=1}^n a_i b_0,$$

where $\underline{a} = (a_0, \dots, a_n)$ and $\underline{b} = (b_0, \dots, b_n)$ are elements in $\oplus_0^n \mathbb{Z}$.

The following result can be easily checked by hand.

PROPOSITION 6.3.5. *Consider the* 3*-subspaces defined in Propositions 6.1.3 and 6.2.7. We have that*

(1) $\dim \mathrm{Hom}_{kQ_3}(\underline{V}^{(3,5)}, \underline{W}^i) = 0$ $(0 \leq i \leq 3)$,
(2) $\dim \mathrm{Hom}_{kQ_3}(\underline{V}^{(3,5)}, \underline{V}^{(3,i)}) = 1$ $(1 \leq i \leq 3)$,
(3) $\dim \mathrm{Hom}_{kQ_3}(\underline{V}^{(3,5)}, \underline{V}^{(3,4)}) = 2$,

(4) $\dim \mathrm{Hom}_{kQ_3}(\underline{V}^{(3,5)}, \underline{V}^{(3,5)}) = 1$.

The following proposition follows immediately from the previous one by using formula (6.3.4).

PROPOSITION 6.3.6. *For the* 3*-subspaces defined in Propositions 6.1.3 and 6.2.7 we have that*

(1) $\dim \mathrm{Ext}^1_{kQ_3}(\underline{V}^{(3,5)}, \underline{W}^0) = 1$,
(2) $\dim \mathrm{Ext}^1_{kQ_3}(\underline{V}^{(3,5)}, \underline{W}^i) = 0$ $(1 \leq i \leq 3)$,
(3) $\dim \mathrm{Ext}^1_{kQ_3}(\underline{V}^{(3,5)}, \underline{V}^{(3,i)}) = 0$ $(0 \leq i \leq 4)$.

Finally we include some computations of Hom and Ext groups related to some other calculations with controlled quadratic functors in the following section.

LEMMA 6.3.7. $\mathrm{Ext}^1_{\mathbf{vect}(T_1)}(\mathcal{M}, \wedge^2_{T_1}\mathcal{M}) = 0$ *for any elementary f. p.* $\mathbf{vect}(T_1)$*-module* $\mathcal{M}$ *in the sense of Theorem 6.1.2.*

PROOF. This is true for $\mathcal{A}$ and $\mathcal{R}$ because they are projective. For $\mathcal{B}$ and $\mathcal{B}_\infty$ it follows from Proposition 6.4.9 and [**Mur04**] 7.12. Since $\wedge^2_{T_1}$ is right-exact, by Proposition 6.4.9 (1) and [**Mur04**] 7.10 (1) we have that $\wedge^2_{T_1}\mathcal{C} = 0$, hence the case $\mathcal{M} = \mathcal{C}$ follows. By [**Mur04**] 7.10 (2) and 7.16 there is an epimorphism

$$\mathrm{Ext}^1_{\mathbf{vect}(T_1)}(\mathcal{C}_\infty, \wedge^2_{T_1}\mathcal{B}_\infty) \twoheadrightarrow \mathrm{Ext}^1_{\mathbf{vect}(T_1)}(\mathcal{C}_\infty, \wedge^2_{T_1}\mathcal{C}_\infty),$$

hence the case of $\mathcal{C}_\infty$ follows from Proposition 6.4.9 (2) and [**Mur04**] 7.12. □

LEMMA 6.3.8. *The equality* $\mathrm{Ext}^1_{\mathbf{vect}(T_2)}(\mathcal{M}, \wedge^2_{T_2}\mathcal{M}) = 0$ *also holds for any elementary f. p.* $\mathbf{vect}(T_2)$*-module* $\mathcal{M}$ *in the sense of Theorem 6.1.2.*

PROOF. By Proposition 1.2.13 and Lemma 6.3.7 it is enough to check it for the elementary module $\mathbb{M}\underline{V}^{(2,1)}$, see Theorem 6.1.2 and Proposition 6.1.3, and it is easy to prove by using Proposition 6.4.2 that $\wedge^2_{T_2}\mathbb{M}\underline{V}^{(2,1)} = 0$. □

LEMMA 6.3.9. *For an elementary f. p.* $\mathbf{vect}(T_3)$*-module* $\mathcal{M}$ *in the sense of Theorem 6.1.2 the equality* $\mathrm{Ext}^1_{\mathbf{vect}(T_3)}(\mathcal{M}, \wedge^2_{T_3}\mathcal{M}) = 0$ *is satisfied unless* $\mathcal{M} = \mathbb{M}\underline{V}^{(3,5)}$. *In this case*

$$\mathrm{Ext}^1_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \wedge^2_{T_3}\mathbb{M}\underline{V}^{(3,5)}) = \mathbb{Z}/2.$$

PROOF. By Proposition 1.2.13 and Lemma 6.3.7 in order to prove the first part of the statement it is enough to check the equality $\mathrm{Ext}^1_{\mathbf{vect}(T_3)}(\mathcal{M}, \wedge^2_{T_3}\mathcal{M}) = 0$ for the elementary modules $\mathbb{M}\underline{V}^{(3,i)}$ for $1 \leq i \leq 4$, see Theorem 6.1.2 and Proposition 6.1.3, and by using Proposition 6.4.2 one readily checks that $\wedge^2_{T_3}\mathbb{M}\underline{V}^{(3,i)} = 0$ in these cases.

The equality

$$\mathrm{Ext}^1_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \wedge^2_{T_3}\mathbb{M}\underline{V}^{(3,5)}) = \mathbb{Z}/2$$

follows easily from Propositions 6.4.2, 6.3.6 and [**Mur04**] A.4. □

LEMMA 6.3.10. *For any elementary f. p.* $\mathbf{vect}(T_3)$*-module* $\mathcal{M} \neq \mathbb{F}^1_*\mathcal{A}$, $\mathbb{F}^i_*\mathcal{R}$, $\mathbb{M}\underline{V}^{(3,5)}$ $(1 \leq i \leq 3)$ *in the sense of Theorem 6.1.2*

$$\mathrm{Ext}^1_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \hat{\otimes}^2_{T_3}\mathcal{M}) = 0.$$

Moreover if $\mathcal{M} = \mathbb{F}^1_*\mathcal{A}$ *or* $\mathbb{F}^i_*\mathcal{R}$ *then*

$$\operatorname{Hom}_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \mathcal{M}) = 0.$$

PROOF. If $\mathcal{M} \neq \mathbb{F}^i_*\mathcal{B}_\infty, \mathbb{F}^i_*\mathcal{C}, \mathbb{F}^i_*\mathcal{C}_\infty$ $(1 \leq i \leq 3)$ the natural transformation $\bar{\tau}_{T_3}\colon \mathcal{M} \to \hat{\otimes}^2_{T_3}\mathcal{M}$ is an isomorphism. This can be easily checked by using for example Proposition 6.2.7 and 6.4.2. Hence the lemma for these modules follows from Proposition 6.2.7, 6.3.5 and 6.3.6. For $\mathcal{M} = \mathbb{F}^i_*\mathcal{C}$ $(1 \leq i \leq 3)$ since $\hat{\otimes}^2_{T_3}$ is right-exact and all $\mathbf{vect}(T_3)$-modules have projective dimension 1 by Proposition 6.3.3, then the epimorphism in [**Mur04**] 7.10 (1) induces a surjection

$$\operatorname{Ext}^1_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \hat{\otimes}^2_{T_3}\mathbb{F}^i_*\mathcal{B}) \twoheadrightarrow \operatorname{Ext}^1_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \hat{\otimes}^2_{T_3}\mathbb{F}^i_*\mathcal{C}),$$

hence this case follows from the case $\mathcal{M} = \mathbb{F}^i_*\mathcal{B}$ already checked.

The cases $\mathcal{M} = \mathbb{F}^i_*\mathcal{B}_\infty$ $(1 \leq i \leq 3)$ are a bit more complicated. Let $k = \mathbb{F}_2$ be the field with 2 elements and let

$$\varphi\colon (0 \to k) \longrightarrow (k \to k)$$

be the unique non-trivial morphism of 1-subspaces and

$$\mathbb{F}^i\colon \mathbf{sub}^{\mathrm{fin}}_1 \to \mathbf{sub}^{\mathrm{fin}}_3, \;\; 1 \leq i \leq 3,$$

the functors considered in [**Mur04**] 8.4, i. e. $\mathbb{F}^i\underline{V} = \underline{W}$ with $W_0 = V_0$, $W_i = V_1$ and $W_j = 0$ otherwise. One can easily check that the cokernel of the monomorphism

$$\phi = \begin{pmatrix} \mathbb{F}^1\varphi \\ \mathbb{F}^2\varphi \\ \mathbb{F}^3\varphi \end{pmatrix} : \mathbb{F}^1(0 \to k) = \mathbb{F}^2(0 \to k) = \mathbb{F}^3(0 \to k) \hookrightarrow \begin{array}{c} \mathbb{F}^1(k \to k) \\ \oplus \\ \mathbb{F}^2(k \to k) \\ \oplus \\ \mathbb{F}^3(k \to k) \end{array}$$

is $\underline{V}^{(3,5)}$. By applying the functor $\mathbb{M}$ we obtain a momomorphism of $\mathbf{vect}(T_3)$-modules

$$\mathbb{M}\phi = \begin{pmatrix} \mathbb{M}\mathbb{F}^1\varphi \\ \mathbb{M}\mathbb{F}^2\varphi \\ \mathbb{M}\mathbb{F}^3\varphi \end{pmatrix} : \mathbb{F}^1_*\mathcal{A} = \mathbb{F}^2_*\mathcal{A} = \mathbb{F}^3_*\mathcal{A} \hookrightarrow \begin{array}{c} \mathbb{F}^1_*\mathcal{B} \\ \oplus \\ \mathbb{F}^2_*\mathcal{B} \\ \oplus \\ \mathbb{F}^3_*\mathcal{B} \end{array},$$

see Proposition 6.2.7, whose cokernel is $\mathbb{M}\underline{V}^{(3,5)}$. We are now going to prove that

$$\operatorname{Hom}_{\mathbf{vect}(T_3)}(\mathbb{M}\phi, \mathbb{F}^i_*\mathcal{B}_\infty)$$

is an isomorphism. By Propositions 6.2.6 (1) and 6.3.1 (2) and the fact that $\mathbb{F}^i_*$ is a fully faithful left-adjoint of $(\mathbb{F}^i)^*$ it is enough to see that

$$\text{(a)} \qquad \operatorname{Hom}_{\mathbf{vect}(T_1)}(\mathbb{M}\varphi, \mathcal{B}_\infty)$$

is an isomorphism. Moreover, since $\varphi \neq 0$ by Proposition 6.3.1 (2) $\mathbb{M}\varphi\colon \mathcal{A} \hookrightarrow \mathcal{B}$ coincides with the monomorphism in [**Mur04**] 7.10 (1), hence (a) is an isomorphism by Proposition 6.3.1 (4) and [**Mur04**] 7.12. Now the case $\mathcal{M} = \mathbb{F}^i_*\mathcal{B}_\infty$ follows from Propositions 6.3.1 (5) and 6.4.9 (7) by applying the long exact sequence for the derived functors of $\operatorname{Hom}_{\mathbf{vect}(T_3)}(-, \mathbb{F}^i_*\mathcal{B}_\infty)$ to the short exact sequence

$$\mathbb{F}^1_*\mathcal{A} \overset{\mathbb{M}\phi}{\hookrightarrow} \mathbb{F}^1_*\mathcal{B} \oplus \mathbb{F}^2_*\mathcal{B} \oplus \mathbb{F}^3_*\mathcal{B} \twoheadrightarrow \mathbb{M}\underline{V}^{(3,5)}.$$

Moreover, this also proves that

$$\mathrm{Hom}_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \mathbb{F}_*^i\mathcal{B}_\infty) = 0,$$

therefore the case $\mathcal{M} = \mathbb{F}_*^i\mathcal{R}$ follows from Proposition 1.2.13 and the existence of a monomorphism $\mathcal{R} \hookrightarrow \mathcal{B}_\infty$, see [**Mur04**] 7.10 (2).

For $\mathcal{M} = \mathbb{F}_*^i\mathcal{C}_\infty$ $(1 \leq i \leq 3)$ by Proposition 6.3.3 and [**Mur04**] 7.10 (2) there is an epimorphism

$$\mathrm{Ext}^1_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \hat{\otimes}^2_{T_3}\mathbb{F}_*^i\mathcal{B}_\infty) \twoheadrightarrow \mathrm{Ext}^1_{\mathbf{vect}(T_3)}(\mathbb{M}\underline{V}^{(3,5)}, \hat{\otimes}^2_{T_3}\mathbb{F}_*^i\mathcal{C}_\infty),$$

so it follows from the case $\mathcal{M} = \mathbb{F}_*^i\mathcal{B}_\infty$ already proved. □

6.4. Some computations with the controlled quadratic functors

In this section the ground field will be $k = \mathbb{F}_2$. One of the main results of this section is the following theorem.

THEOREM 6.4.1. *If T has less than four ends the morphism $\bar{\tau}_T\colon \mathcal{M} \otimes \mathbb{Z}/2 \to \hat{\otimes}^2_T\mathcal{M}$ is a split monomorphism for any f. p.* $\mathbf{ab}(T)$*-module $\mathcal{M}$.*

PROOF. We will see that it is enough to prove the theorem for $\mathbf{vect}(T)$-modules, hence it will follow from Proposition 6.4.7 below.

Since $-\otimes\mathbb{Z}/2$ is additive and right-exact $\mathcal{M}\otimes\mathbb{Z}/2$ is f. p. as a $\mathbf{vect}(T)$-module. The natural projection $\hat{p}\colon \mathcal{M} \twoheadrightarrow \mathcal{M} \otimes \mathbb{Z}/2$ induces a commutative diagram

$$\begin{tikzcd}
\mathcal{M} \otimes \mathbb{Z}/2 \arrow[r, "\bar{\tau}_T"] \arrow[d, "\hat{p}\otimes\mathbb{Z}/2"', "\simeq"] & \hat{\otimes}^2\mathcal{M} \arrow[d, "\hat{\otimes}^2_T\hat{p}"] \\
(\mathcal{M} \otimes \mathbb{Z}/2) \otimes \mathbb{Z}/2 \arrow[r, "\bar{\tau}_T"] & \hat{\otimes}^2_T(\mathcal{M} \otimes \mathbb{Z}/2)
\end{tikzcd}$$

If s is a left-inverse of the lower horizontal row then $(\hat{p} \otimes \mathbb{Z}/2)^{-1}s(\hat{\otimes}^2_T\hat{p})$ is a left-inverse of the upper one. □

All endofunctors of the category of finite-dimensional vector spaces extend to n-subspaces in the obvious way. The same happens with natural transformations, in particular diagram (2.1.3) tensored by $\mathbb{Z}/2$ gives rise to a natural diagram with exact rows and column in $\mathbf{sub}_n^{\mathrm{fin}}$,

$$\begin{tikzcd}
\otimes^2\underline{V} \arrow[d, "{[-,-]\otimes\mathbb{Z}/2}"'] & & \\
(\Gamma\underline{V}) \otimes \mathbb{Z}/2 \arrow[r, "\tau\otimes\mathbb{Z}/2"] \arrow[d, two heads, "\sigma\otimes\mathbb{Z}/2"'] \arrow[dr, phantom, "\text{push}"] & \otimes^2\underline{V} \arrow[r, two heads, "q"] \arrow[d, two heads, "\bar{\sigma}"] & \wedge^2\underline{V} \arrow[d, equal] \\
\underline{V} \arrow[r, hook, "\bar{\tau}"] & \hat{\otimes}^2\underline{V} \arrow[r, two heads, "\bar{q}"] & \wedge^2\underline{V}
\end{tikzcd}$$

The category $\mathbf{sub}_n^{\mathrm{fin}}$ is not abelian, but it is fully included into the abelian category of representations of Q_n. It is in this last category where the previous sequences of n-subspaces are exact.

As it happened with (2.1.3) and (2.2.19) this diagram is completely determined by the functor $(\Gamma-) \otimes \mathbb{Z}/2$.

PROPOSITION 6.4.2. *The following diagram of* $\mathbf{vect}(T_n)$*-modules*

$$\begin{tikzcd}
\mathbb{M}\otimes^2\underline{V} \arrow[d, "\mathbb{M}([-,-]\otimes\mathbb{Z}/2)"'] & & \\
\mathbb{M}((\Gamma\underline{V})\otimes\mathbb{Z}/2) \arrow[r, "\mathbb{M}(\tau\otimes\mathbb{Z}/2)"] \arrow[d, two heads, "\mathbb{M}(\sigma\otimes\mathbb{Z}/2)"'] \arrow[dr, phantom, "\text{push}"] & \mathbb{M}\otimes^2\underline{V} \arrow[r, two heads, "\mathbb{M}q"] \arrow[d, two heads, "\mathbb{M}\bar{\sigma}"'] & \mathbb{M}\wedge^2\underline{V} \arrow[d, equal] \\
\mathbb{M}\underline{V} \arrow[r, hook, "\mathbb{M}\bar{\tau}"] & \mathbb{M}\hat{\otimes}^2\underline{V} \arrow[r, two heads, "\mathbb{M}\bar{q}"] & \mathbb{M}\wedge^2\underline{V}
\end{tikzcd}$$

is naturally isomorphic to

$$\begin{tikzcd}
\otimes^2_{T_n}\mathbb{M}\underline{V} \arrow[d, "{[-,-]_{T_n}\otimes\mathbb{Z}/2}"'] & & \\
(\Gamma_{T_n}\mathbb{M}\underline{V})\otimes\mathbb{Z}/2 \arrow[r, "\tau_{T_n}\otimes\mathbb{Z}/2"] \arrow[d, two heads, "\sigma_{T_n}\otimes\mathbb{Z}/2"'] \arrow[dr, phantom, "\text{push}"] & \otimes^2_{T_n}\mathbb{M}\underline{V} \arrow[r, two heads, "q_{T_n}"] \arrow[d, two heads, "\bar{\sigma}_{T_n}"'] & \wedge^2_{T_n}\mathbb{M}\underline{V} \arrow[d, equal] \\
\mathbb{M}\underline{V} \arrow[r, "\bar{\tau}_{T_n}"] & \hat{\otimes}^2_{T_n}\mathbb{M}\underline{V} \arrow[r, two heads, "\bar{q}_{T_n}"] & \wedge^2_{T_n}\mathbb{M}\underline{V}
\end{tikzcd}$$

PROOF. Since the first diagram is determined by $\mathbb{M}((\Gamma-)\otimes\mathbb{Z}/2)$ and the second one by $(\Gamma_{T_n}-)\otimes\mathbb{Z}/2$ it will be enough to construct a natural isomorphism

$$(\Gamma_{T_n}\mathbb{M}\underline{V})\otimes\mathbb{Z}/2\simeq\mathbb{M}((\Gamma\underline{V})\otimes\mathbb{Z}/2).$$

In [**Mur04**] 9.4 we constructed a finite presentation

$$\mathbb{F}_2\langle D\rangle_\delta\xrightarrow{\rho}\mathbb{F}_2\langle C\rangle_\gamma\overset{p}{\twoheadrightarrow}\mathbb{M}\underline{V}.$$

Here p is defined by a certain vector space homomorphism $p_0\colon\mathbb{F}_2\langle C\rangle\twoheadrightarrow V_0$, see the definition of $\mathbb{M}$ in [**Mur04**] (9.a). By right-exactness we have an exact sequence

$$(\Gamma_{T_n}\mathbb{F}_2\langle D\rangle_\delta)\otimes\mathbb{Z}/2\oplus\mathbb{F}_2\langle D\rangle_\delta\otimes_{T_n}\mathbb{F}_2\langle C\rangle_\gamma\xrightarrow{\xi}(\Gamma_{T_n}\mathbb{F}_2\langle C\rangle_\gamma)\otimes\mathbb{Z}/2\overset{(\Gamma_{T_n}p)\otimes\mathbb{Z}/2}{\twoheadrightarrow}(\Gamma_{T_n}\mathbb{M}\underline{V})\otimes\mathbb{Z}/2$$

with $\xi=((\Gamma_{T_n}\rho)\otimes\mathbb{Z}/2,([-,-]_{T_n}\otimes\mathbb{Z}/2)(\rho\otimes_{T_n}1))$.

We leave to the reader to check that the vector space homomorphism $(\Gamma p_0)\otimes\mathbb{Z}/2\colon(\Gamma\mathbb{F}_2\langle C\rangle)\otimes\mathbb{Z}/2\to(\Gamma V_0)\otimes\mathbb{Z}/2$ determines a morphism $p'\colon(\Gamma_{T_n}\mathbb{F}_2\langle C\rangle_\gamma)\otimes\mathbb{Z}/2\to\mathbb{M}((\Gamma\underline{V})\otimes\mathbb{Z}/2)$ such that the sequence

$$(\Gamma_{T_n}\mathbb{F}_2\langle D\rangle_\delta)\otimes\mathbb{Z}/2\oplus\mathbb{F}_2\langle D\rangle_\delta\otimes_{T_n}\mathbb{F}_2\langle C\rangle_\gamma\xrightarrow{\xi}(\Gamma_{T_n}\mathbb{F}_2\langle C\rangle_\gamma)\otimes\mathbb{Z}/2\overset{p'}{\twoheadrightarrow}\mathbb{M}((\Gamma\underline{V})\otimes\mathbb{Z}/2)$$

is also exact, hence we get the desired isomorphism.

$\square$

Recall from [**Mur04**] Section 4 that there is an exact, full and faithful functor from the category of vector spaces with countable dimension

$$\mathfrak{i}\colon\mathbf{mod}_{\aleph_0}(\mathbb{F}_2)\longrightarrow\mathbf{fp}(\mathbf{vect}(\mathbb{R}_+)).$$

PROPOSITION 6.4.3. *Let V be a vector space with $\dim V \leq \aleph_0$, the following diagram of $\mathbf{vect}(\mathbb{R}_+)$-modules*

$$\begin{tikzcd}
\mathfrak{i} \otimes^2 V \arrow[d, "\mathfrak{i}([-,-]\otimes\mathbb{Z}/2)"'] & & \\
\mathfrak{i}((\Gamma V)\otimes\mathbb{Z}/2) \arrow[r, "\mathfrak{i}(\tau\otimes\mathbb{Z}/2)"] \arrow[d, two heads, "\mathfrak{i}(\sigma\otimes\mathbb{Z}/2)"'] \arrow[dr, phantom, "\text{push}"] & \mathfrak{i} \otimes^2 V \arrow[r, two heads, "\mathfrak{i}q"] \arrow[d, two heads, "\mathfrak{i}\bar{\sigma}"'] & \mathfrak{i} \wedge^2 V \arrow[d, equal] \\
\mathfrak{i}V \arrow[r, hook, "\mathfrak{i}\bar{\tau}"] & \mathfrak{i}\hat{\otimes}^2 V \arrow[r, two heads, "\mathfrak{i}\bar{q}"] & \mathfrak{i} \wedge^2 V
\end{tikzcd}$$

is naturally isomorphic to

$$\begin{tikzcd}
\otimes^2_{\mathbb{R}_+} \mathfrak{i}V \arrow[d, "{[-,-]_{\mathbb{R}_+}\otimes\mathbb{Z}/2}"'] & & \\
(\Gamma_{\mathbb{R}_+}\mathfrak{i}V)\otimes\mathbb{Z}/2 \arrow[r, "\tau_{\mathbb{R}_+}\otimes\mathbb{Z}/2"] \arrow[d, two heads, "\sigma_{\mathbb{R}_+}\otimes\mathbb{Z}/2"'] \arrow[dr, phantom, "\text{push}"] & \otimes^2_{\mathbb{R}_+} \mathfrak{i}V \arrow[r, two heads, "q_{\mathbb{R}_+}"] \arrow[d, two heads, "\bar{\sigma}_{\mathbb{R}_+}"'] & \wedge^2_{\mathbb{R}_+} \mathfrak{i}V \arrow[d, equal] \\
\mathfrak{i}V \arrow[r, "\bar{\tau}_{\mathbb{R}_+}"] & \hat{\otimes}^2_{\mathbb{R}_+} \mathfrak{i}V \arrow[r, two heads, "\bar{q}_{\mathbb{R}_+}"] & \wedge^2_{\mathbb{R}_+} \mathfrak{i}V
\end{tikzcd}$$

PROOF. As in the proof of Proposition 6.4.2 it will de enough to construct a natural isomorphism

$$(\Gamma_{\mathbb{R}_+}\mathfrak{i}V)\otimes\mathbb{Z}/2 \simeq \mathfrak{i}((\Gamma V)\otimes\mathbb{Z}/2).$$

An explicit finite presentation

$$\mathbb{F}_2\langle D\rangle_\delta \xrightarrow{\rho} \mathbb{F}_2\langle C\rangle_\gamma \overset{p}{\twoheadrightarrow} \mathfrak{i}V$$

can be obtained from the proof of [**Mur04**] 4.3. Here p is determined by a vector space homomorphism $p_0\colon \mathbb{F}_2\langle C\rangle \to V$. By right exactness the following sequence is exact

$$(\Gamma_{\mathbb{R}_+}\mathbb{F}_2\langle D\rangle_\delta)\otimes\mathbb{Z}/2\oplus\mathbb{F}_2\langle D\rangle_\delta\otimes_{\mathbb{R}_+}\mathbb{F}_2\langle C\rangle_\gamma \xrightarrow{\xi} (\Gamma_{\mathbb{R}_+}\mathbb{F}_2\langle C\rangle_\gamma)\otimes\mathbb{Z}/2 \overset{(\Gamma_{\mathbb{R}_+}p)\otimes\mathbb{Z}/2}{\twoheadrightarrow} (\Gamma_{\mathbb{R}_+}\mathfrak{i}V)\otimes\mathbb{Z}/2,$$

where $\xi = ((\Gamma_{\mathbb{R}_+}\rho)\otimes\mathbb{Z}/2, ([-,-]_{\mathbb{R}_+}\otimes\mathbb{Z}/2)(\rho\otimes_{\mathbb{R}_+}1))$.

Now the reader can check that the morphism $p'\colon (\Gamma_{\mathbb{R}_+}\mathbb{F}_2\langle C\rangle_\gamma)\otimes\mathbb{Z}/2 \to \mathfrak{i}((\Gamma V)\otimes\mathbb{Z}/2)$ induced by the vector space homomorphism $(\Gamma p_0)\otimes\mathbb{Z}/2\colon (\Gamma\mathbb{F}_2\langle C\rangle)\otimes\mathbb{Z}/2 \to (\Gamma V)\otimes\mathbb{Z}/2$ fits into an exact sequence

$$(\Gamma_{\mathbb{R}_+}\mathbb{F}_2\langle D\rangle_\delta)\otimes\mathbb{Z}/2\oplus\mathbb{F}_2\langle D\rangle_\delta\otimes_{\mathbb{R}_+}\mathbb{F}_2\langle C\rangle_\gamma \xrightarrow{\xi} (\Gamma_{\mathbb{R}_+}\mathbb{F}_2\langle C\rangle_\gamma)\otimes\mathbb{Z}/2 \overset{p'}{\twoheadrightarrow} \mathfrak{i}((\Gamma V)\otimes\mathbb{Z}/2).$$

This determines the desired isomorphism. □

PROPOSITION 6.4.4. *The morphism $\bar{\tau}_{\mathbb{R}_+}\colon \mathcal{C}_\infty\otimes\mathbb{Z}/2 \to \hat{\otimes}^2_{\mathbb{R}_+}\mathcal{C}_\infty$ is indeed a split monomorphism.*

PROOF. Consider the free $\mathbb{R}_+$-controlled $\mathbb{F}_2$-module $\mathbb{F}_2\langle A\rangle_\alpha$ with

$$A = \{a_{mn}\,;\, 0 < n \leq m\}$$

and $\alpha(a_{mn}) = m$, and its endomorphism φ defined as

$$\varphi(a_{mn}) = \begin{cases} a_{mm}, & \text{if } n = m; \\ a_{mn} + a_{m-1,n}, & \text{if } n < m. \end{cases}$$

By [**Mur04**] 7.21 $\mathcal{C}_\infty = \operatorname{Coker}\varphi$. Moreover by right exactness we have that $\hat{\otimes}^2_{\mathbb{R}_+}\mathcal{C}_\infty$ is the cokernel of the controlled homomorphism

$$(\hat{\otimes}^2_{\mathbb{R}_+}\varphi, (\hat{\otimes}^2_{\mathbb{R}_+}(\varphi|1))i_{12})\colon \mathbb{F}_2\langle \hat{\otimes}^2 A\rangle_{\hat{\otimes}^2\alpha} \oplus \mathbb{F}_2\langle A\times A\rangle_{\alpha\otimes\alpha} \longrightarrow \mathbb{F}_2\langle \hat{\otimes}^2 A\rangle_{\hat{\otimes}^2\alpha},$$

see Proposition 2.2.22. We consider the following total order in A for the definition of $\hat{\otimes}^2 A$

$$a_{mn} \preceq a_{pq} \Leftrightarrow \begin{cases} m < p, \\ \text{or} \\ m = p \text{ and } n \leq q. \end{cases}$$

There is a commutative diagram of solid arrows

$$\begin{tikzcd}
\mathbb{F}_2\langle A\rangle_\alpha \arrow[r, "(\bar\tau_{\mathbb{R}_+},0)"] \arrow[d, "\varphi"'] & \mathbb{F}_2\langle \hat{\otimes}^2 A\rangle_{\hat{\otimes}^2\alpha} \oplus \mathbb{F}_2\langle A\times A\rangle_{\alpha\otimes\alpha} \arrow[l, dashed, bend left, "s_1"] \arrow[d, "(\hat{\otimes}^2_{\mathbb{R}_+}\varphi,(\hat{\otimes}^2_{\mathbb{R}_+}(\varphi|1))i_{12})"] \\
\mathbb{F}_2\langle A\rangle_\alpha \arrow[r, "\bar\tau_{\mathbb{R}_+}"] & \mathbb{F}_2\langle \hat{\otimes}^2 A\rangle_{\hat{\otimes}^2\alpha} \arrow[l, dashed, bend left, "s_0"]
\end{tikzcd}$$

One can check that the controlled homomorphisms s_1 and s_0 given by

$$s_1(a_{mn}\hat{\otimes}a_{pq}) = \begin{cases} a_{mn}, & \text{if } p = m \geq n = q; \\ 0, & \text{otherwise}; \end{cases}$$

$$s_0(a_{mn}\hat{\otimes}a_{pq}) = s_1(a_{mn}, a_{pq}) = \begin{cases} a_{mn}, & \text{if } p \geq m \geq n = q; \\ 0, & \text{otherwise}; \end{cases}$$

are retractions of the horizontal solid arrows and satisfy

$$\varphi s_1 = s_0(\hat{\otimes}^2_{\mathbb{R}_+}\varphi, (\hat{\otimes}^2_{\mathbb{R}_+}(\varphi|1))i_{12}),$$

hence we obtain the desired retraction $\hat{\otimes}^2_{\mathbb{R}_+}\mathcal{C}_\infty \twoheadrightarrow \mathcal{C}_\infty \otimes \mathbb{Z}/2$ by taking cokernels. $\square$

The following corollary follows from the fact that $\mathcal{C}$ is a retract of $\mathcal{C}_\infty$, see Theorem 6.1.2.

Corollary 6.4.5. *The morphism $\bar\tau_{\mathbb{R}_+}\colon \mathcal{C}\otimes\mathbb{Z}/2 \to \hat{\otimes}^2_{\mathbb{R}_+}\mathcal{C}$ is a split monomorphism.*

The following proposition is a consequence of the right exactness of $-\otimes\mathbb{Z}/2$, Propositions 2.2.11 and 2.2.22 and Lemma 2.2.21.

Proposition 6.4.6. *The morphism $\bar\tau_T\colon \mathbb{F}_2\langle A\rangle_\alpha \to \hat{\otimes}^2_T\mathbb{F}_2\langle A\rangle_\alpha$ is indeed a split monomorphism for any free T-controlled $\mathbb{F}_2$-module $\mathbb{F}_2\langle A\rangle_\alpha$.*

Proposition 6.4.7. *Given a tree T with* $\operatorname{card}\mathfrak{F}(T) \leq 3$ *the morphism $\bar\tau_T\colon \mathcal{M} \to \hat{\otimes}^2_T\mathcal{M}$ is a split monomorphism for any f. p.* $\mathbf{vect}(T)$*-module $\mathcal{M}$.*

Proof. It is enough to prove the proposition for elementary $\mathbf{vect}(T)$-modules. For trees with less than four ends a complete list of elementary f. p. $\mathbf{vect}(T)$-modules can be found in Theorem 6.1.2, see also Proposition 6.1.3.

If T has one end the proposition follows from 6.4.6 and [**Mur04**] 7.7 for the elementary modules $\mathcal{A}$ and $\mathcal{R}$, from Proposition 6.4.3 and [**Mur04**] 7.11 for $\mathcal{B}$ and $\mathcal{B}_\infty$, from Proposition 6.4.4 for $\mathcal{C}_\infty$, and from Corollary 6.4.5 for $\mathcal{C}$.

For trees with $n = 2$ or 3 ends by Proposition 2.2.22 it is only left to prove this proposition for elementary f. p. modules coming from rigid n-subspaces, but this is easy to check by using Proposition 6.4.2 and 6.1.3. $\square$

REMARK 6.4.8. Proposition 6.4.7 does hot hold for trees with more than 6 ends. Indeed by Propositions 2.2.22 and 6.4.2 it will be enough to show a 7-subspace $\underline{V}$ such that $\bar{\tau}\colon \underline{V} \hookrightarrow \hat{\otimes}^2\underline{V}$ does not split. We have found the following example,

$$\begin{array}{rcl} V_0 &=& \mathbb{F}_2\langle x, y, z\rangle \\ V_1 &=& \mathbb{F}_2\langle x, y\rangle, \\ V_2 &=& \mathbb{F}_2\langle x, z\rangle, \\ V_3 &=& \mathbb{F}_2\langle y, z\rangle, \\ V_4 &=& \mathbb{F}_2\langle x+y, z\rangle, \\ V_5 &=& \mathbb{F}_2\langle x+z, y\rangle, \\ V_6 &=& \mathbb{F}_2\langle x, y+z\rangle, \\ V_7 &=& \mathbb{F}_2\langle x+y, x+z\rangle. \end{array}$$

The 7-subspace $\hat{\otimes}^2\underline{V}$ is given by

$$\begin{array}{rcl} \hat{\otimes}^2 V_0 &=& \mathbb{F}_2\langle x\hat{\otimes}x, x\hat{\otimes}y, x\hat{\otimes}z, y\hat{\otimes}y, y\hat{\otimes}z, z\hat{\otimes}z\rangle \\ \hat{\otimes}^2 V_1 &=& \mathbb{F}_2\langle x\hat{\otimes}x, x\hat{\otimes}y, y\hat{\otimes}y\rangle, \\ \hat{\otimes}^2 V_2 &=& \mathbb{F}_2\langle x\hat{\otimes}x, x\hat{\otimes}z, z\hat{\otimes}z\rangle, \\ \hat{\otimes}^2 V_3 &=& \mathbb{F}_2\langle y\hat{\otimes}y, y\hat{\otimes}z, z\hat{\otimes}z\rangle, \\ \hat{\otimes}^2 V_4 &=& \mathbb{F}_2\langle x\hat{\otimes}x + y\hat{\otimes}y, x\hat{\otimes}z + y\hat{\otimes}z, z\hat{\otimes}z\rangle, \\ \hat{\otimes}^2 V_5 &=& \mathbb{F}_2\langle x\hat{\otimes}x + z\hat{\otimes}z, x\hat{\otimes}y + y\hat{\otimes}z, y\hat{\otimes}y\rangle, \\ \hat{\otimes}^2 V_6 &=& \mathbb{F}_2\langle x\hat{\otimes}x, x\hat{\otimes}y + x\hat{\otimes}z, y\hat{\otimes}y + z\hat{\otimes}z\rangle, \\ \hat{\otimes}^2 V_7 &=& \mathbb{F}_2\langle x\hat{\otimes}x + x\hat{\otimes}y + x\hat{\otimes}z + y\hat{\otimes}z, x\hat{\otimes}x + y\hat{\otimes}y, x\hat{\otimes}x + z\hat{\otimes}z\rangle. \end{array}$$

If there where a retraction $s\colon \hat{\otimes}^2\underline{V} \twoheadrightarrow \underline{V}$ of $\bar{\tau}$ it should satisfy

(a) $$\begin{array}{rcl} s(x\hat{\otimes}x) &=& x, \\ s(y\hat{\otimes}y) &=& y, \\ s(z\hat{\otimes}z) &=& z, \end{array}$$

(b) $$s(x\hat{\otimes}y) \in V_1,$$

(c) $$s(x\hat{\otimes}z) \in V_2,$$

(d) $$s(y\hat{\otimes}z) \in V_3,$$

(e) $$s(x\hat{\otimes}z + y\hat{\otimes}z) \in V_4,$$

(f) $$s(x\hat{\otimes}y + y\hat{\otimes}z) \in V_5,$$

(g) $$s(x\hat{\otimes}y + x\hat{\otimes}z) \in V_6,$$

(h) $$s(x\hat{\otimes}x + x\hat{\otimes}y + x\hat{\otimes}z + y\hat{\otimes}z) \in V_7.$$

By using (b), (c) and (d) we see that there would exist $a, b, c, d, e, f \in \mathbb{F}_2$ such that

$$\begin{aligned} s(x\hat{\otimes}y) &= ax + by, \\ s(x\hat{\otimes}z) &= cx + dz, \\ s(y\hat{\otimes}z) &= ey + fz. \end{aligned}$$

By combining this with (e), (f) and (g) we obtain that the following equalities should hold

$$\begin{aligned} e &= c, \\ f &= a, \\ d &= b. \end{aligned}$$

Finally (h) together with (a) and the previous equalities would imply that

$$(1 + a + c)x + (b + c)y + (a + b)z \in V_7 = \{0, x + y, x + z, y + z\}.$$

That is, at least one of the following systems of linear equations with three indeterminacies (a, b, c) should have a solution over $\mathbb{F}_2$,

$$\left\{\begin{aligned} 1 + a + c &= 0, \\ b + c &= 0, \\ a + b &= 0, \end{aligned}\right. \qquad \left\{\begin{aligned} 1 + a + c &= 1, \\ b + c &= 1, \\ a + b &= 0, \end{aligned}\right.$$

$$\left\{\begin{aligned} 1 + a + c &= 1, \\ b + c &= 0, \\ a + b &= 1, \end{aligned}\right. \qquad \left\{\begin{aligned} 1 + a + c &= 0, \\ b + c &= 1, \\ a + b &= 1. \end{aligned}\right.$$

It is easy to see that all these systems are incompatible, therefore we would reach a contradiction, so the retraction can not exist.

We do not know whether Proposition 6.4.7 is true for trees with 4, 5 or 6 ends. One could try to check it for trees with 4 ends by using Theorem 6.1.2 and Nazarova's classification of finite-dimensional 4-subspaces in [**Naz73**]. We have not carried out such a computation because the infinite list of indecomposable 4-subspaces over $\mathbb{F}_2$ is quite complicate to describe and we have already obtained very satisfactory results for trees with less than 4 ends. For trees T with 5 or 6 ends there is not a direct method to check wether Proposition 6.4.7 is true since the category of finitely presented $\mathbf{vect}(T)$-modules has wild representation type, see Theorem 6.1.1.

We conclude this section with explicit computations of values of the controlled quadratic functors over elementary f. p. $\mathbf{vect}(T)$-modules.

PROPOSITION 6.4.9. *We have the following identifications*

(1) $\wedge^2_{T_1}\mathcal{B} = 0$,
(2) $\wedge^2_{T_1}\mathcal{B}_\infty = \mathcal{B}_\infty$,
(3) $\wedge^2_{T_2}\mathbb{M}\underline{V}^{(2,1)} = 0$,
(4) $\wedge^2_{T_3}\mathbb{M}\underline{V}^{(3,i)} = 0$ *for* $i = 1, 2, 3$ *and* 4,
(5) $\wedge^2_{T_3}\mathbb{M}\underline{V}^{(3,5)} = \mathcal{A}$,
(6) $\hat{\otimes}^2_{T_1}\mathcal{B} = \mathcal{B}$,
(7) $\hat{\otimes}^2_{T_1}\mathcal{B}_\infty = \mathcal{B}_\infty$,
(8) $\hat{\otimes}^2_{T_2}\mathbb{M}\underline{V}^{(2,1)} = \mathbb{M}\underline{V}^{(2,1)}$,
(9) $\hat{\otimes}^2_{T_3}\mathbb{M}\underline{V}^{(3,i)} = \mathbb{M}\underline{V}^{(3,i)}$ *for* $i = 1, 2, 3$ *and* 4.

PROOF. The identifications (1), (2), (6) and (7) follow from Proposition 6.4.3 and [**Mur04**] 7.11; (3), (4), (8) and (9) can be easily derived from Proposition 6.4.2. Moreover, if $\underline{W}$ is the 3-subspace with $W_0 = \mathbb{F}_2$ and $W_i = 0$ for $1 \leq i \leq 3$ one readily checks by using Proposition 6.4.2 that $\wedge^2_{T_3}\mathbb{M}\underline{V}^{(3,5)} = \mathbb{M}\underline{W}$. The finite presentation of $\mathbb{M}\underline{W}$ as a $\mathbf{vect}(T_3)$-module constructed in the proof of [**Mur04**] 9.1 shows that $\mathbb{M}\underline{W}$ is a free T_3-controlled $\mathbb{F}_2$-module with only one generator, hence by [**Mur04**] 7.7 $\mathbb{M}\underline{W} = \mathcal{A}$ and (5) holds. $\square$

APPENDIX A

Proof of Proposition 1.2.12

For an arbitrary tree T we will consider the following commutative diagram of functors

$$\begin{tikzcd}
\mathbf{M}_R(T) \arrow[rr, "Y"] \arrow[dr, "J"'] & & \mathbf{mod}(\mathbf{M}_R(T)) \\
& \mathbf{M}_R^b(T) \arrow[ur, "Y'"'] &
\end{tikzcd}$$

Here Y is the Yoneda full inclusion in (1.2.1), J is the inclusion of the full subcategory, and Y' is the faithful extension of Y along J in (1.2.11).

The functors J and Y are natural with respect to "change of tree" functors induced by a proper map $f\colon T \to T'$, i. e. the following two diagrams commute

(A.0.10)
$$\begin{tikzcd}
\mathbf{M}_R(T) \arrow[r, "\mathbb{F}^f"] \arrow[d, hook, "J"'] & \mathbf{M}_R(T') \arrow[d, hook, "J"] \\
\mathbf{M}_R^b(T) \arrow[r, "\mathbb{F}^f"] & \mathbf{M}_R^b(T')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{M}_R(T) \arrow[r, "\mathbb{F}^f"] \arrow[d, hook, "Y"'] & \mathbf{M}_R(T') \arrow[d, hook, "Y"] \\
\mathbf{mod}(\mathbf{M}_R(T)) \arrow[r, "\mathbb{F}^f_*"] & \mathbf{mod}(\mathbf{M}_R(T'))
\end{tikzcd}$$

The first one commutes by the very definition of $\mathbb{F}^f$ in Section 1.2, the second one is a particular example of (1.2.2). Proposition 1.2.12 establishes that Y' is also natural with respect to "change of tree" functors.

In this appendix we will make an extensive use of comma categories, see [**Mac71**] II.6 for basic definitions. In particular given a functor $F\colon \mathbf{C} \to \mathbf{D}$ we shall consider the comma categories $c \downarrow F$ and $F \downarrow d$, where c and d are objects of $\mathbf{C}$ and $\mathbf{D}$, respectively. There is a canonical functor $P\colon F \downarrow d \to \mathbf{C}$. If we have a commutative diagram of functors

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "M"] \arrow[d, "F"'] & \mathbf{C}' \arrow[d, "G"] \\
\mathbf{D} \arrow[r, "N"] & \mathbf{D}'
\end{tikzcd}$$

there is also an induced functor $M \downarrow N\colon F \downarrow d \to G \downarrow N(d)$ which satisfies

(A.0.11) $$P(M \downarrow N) = MP\colon F \downarrow d \to \mathbf{D}.$$

It is well-known that for any $\mathbf{M}_R(T)$-module $\mathcal{M}$

(A.0.12) $$\operatorname{colim}[YP\colon Y \downarrow \mathcal{M} \to \mathbf{mod}(\mathbf{M}_R(T))] = \mathcal{M}.$$

The crucial step in the proof of Proposition 1.2.12 is the next lemma.

Lemma A.0.13. *For any object $R\langle A\rangle_\alpha$ in $\mathbf{M}^b_R(T)$ the functor*

$$\mathbb{F}^f \downarrow \mathbb{F}^f \colon J \downarrow R\langle A\rangle_\alpha \to J \downarrow \mathbb{F}^f R\langle A\rangle_\alpha$$

is final.

See the first definition of [**Mac71**] IX.3 for a definition of *final* functor. In the proof of Lemma A.0.13 we shall use the following technical result.

Lemma A.0.14. *If $R\langle A\rangle_\alpha$ is an object of $\mathbf{M}^b_R(T)$ and $R\langle B'\rangle_{\beta'}$ is an object of $\mathbf{M}_R(T')$, given a controlled homomorphism $\varphi\colon R\langle B'\rangle_{\beta'} \to \mathbb{F}^f R\langle A\rangle_\alpha$ there exists a subset $\bar{A} \subset A$ such that if $\bar{\alpha}$ is the restriction of α to $\bar{A}$ then $R\langle \bar{A}\rangle_{\bar{\alpha}}$ is an object of $\mathbf{M}_R(T)$ and if $i\colon R\langle \bar{A}\rangle_{\bar{\alpha}} \hookrightarrow R\langle A\rangle_\alpha$ is the inclusion, which is obviously splitting, φ factors through the monomorphism $\mathbb{F}^f i$.*

Proof. If we define $\bar{A} \subset A$ as the least subset such that $\varphi(B') \subset R\langle \bar{A}\rangle$ it is enough to check that the object $R\langle \bar{A}\rangle_{\bar{\alpha}}$ belongs to $\mathbf{M}_R(T)$. Suppose by the contrary that there exists a compact subset $K \subset T$ such that $\bar{\alpha}^{-1}(K)$ contains an infinite countable subset $\{a_n\}_{n\geq 0} \subset A$, then we can take a sequence $\{b_n\}_{n\geq 0} \subset B'$ such that a_n appears in the linear expansion of $\varphi(b_n)$ with non-trivial coefficient $(n \geq 0)$. Since $\beta'\colon B' \to T'$ is proper we can suppose without loss of generality, taking subsequences if necessary, that $\lim_{n\to\infty} \beta'(b_n) = \varepsilon \in \mathfrak{F}(T')$. By using the definition of controlled homomorphism, see Section 1.2, it is easy to see that this implies that $\lim_{n\to\infty} f\alpha(a_n) = \varepsilon \in \mathfrak{F}(T')$, but for all $n \geq 0$ $f\alpha(a_n) \subset f(K) \in T'$ compact, and we reach a contradiction. ☐

Proof of Lemma A.0.13. An object of $J \downarrow \mathbb{F}^f R\langle A\rangle_\alpha$ is a morphism

$$\varphi\colon R\langle B'\rangle_{\beta'} \to \mathbb{F}^f R\langle A\rangle_\alpha$$

as in the statement of Lemma A.0.14. We have to check that the category $\varphi \downarrow (\mathbb{F}^f \downarrow \mathbb{F}^f)$ is non-empty and connected. It is non-empty since by Lemma A.0.14 there is a commutative diagram

$$\begin{tikzcd}
R\langle B'\rangle_{\beta'} \arrow[rr] \arrow[dr, "\varphi"'] & & \mathbb{F}^f R\langle \bar{A}\rangle_{\bar{\alpha}} \arrow[dl, hook', "\mathbb{F}^f i"] \\
& \mathbb{F}^f R\langle A\rangle_\alpha &
\end{tikzcd}$$

where $i\colon R\langle \bar{A}\rangle_{\bar{\alpha}} \hookrightarrow R\langle A\rangle_\alpha$ is an object of $J \downarrow R\langle A\rangle_\alpha$.

Let us check now that $\varphi \downarrow (\mathbb{F}^f \downarrow \mathbb{F}^f)$ is connected. Two objects a and b in this category are exactly two commutative triangles of controlled homomorphisms as follows

$$\begin{tikzcd}
& R\langle B'\rangle_{\beta'} \arrow[dl, "\tau_1"'] \arrow[d, "\varphi"] \arrow[dr, "\tau_2"] & \\
\mathbb{F}^f R\langle B_1\rangle_{\beta_1} \arrow[r, "\mathbb{F}^f \psi_1"'] & \mathbb{F}^f R\langle A\rangle_\alpha & \mathbb{F}^f R\langle B_2\rangle_{\beta_2} \arrow[l, "\mathbb{F}^f \psi_2"]
\end{tikzcd}$$

Here $R\langle B_1\rangle_{\beta_1}$ and $R\langle B_2\rangle_{\beta_2}$ are not big. Since $\mathbb{F}^f$ is additive this diagram can be extended commutatively in the following way

$$\begin{tikzcd}
\mathbb{F}^f R\langle A\rangle_\alpha & & \mathbb{F}^f R\langle B_1\rangle_{\beta_1} \arrow[ll, "\mathbb{F}^f\psi_1"'] \\
& R\langle B'\rangle_{\beta'} \arrow[ul, "\varphi"'] \arrow[ur, "\tau_1"] \arrow[dl, "\tau_2"] \arrow[dr, "\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}"] & \\
\mathbb{F}^f R\langle B_2\rangle_{\beta_2} \arrow[uu, "\mathbb{F}^f\psi_2"'] & & \mathbb{F}^f(R\langle B_1\rangle_{\beta_1}\oplus R\langle B_2\rangle_{\beta_2}) \arrow[ll, "\mathbb{F}^f p_2"'] \arrow[uu, "\mathbb{F}^f p_1"'] \arrow[uull, bend right=60, "\mathbb{F}^f(\psi_1,\psi_2)"'] \arrow[d, equal] \\
& & \mathbb{F}^f R\langle B_1\rangle_{\beta_1}\oplus\mathbb{F}^f R\langle B_2\rangle_{\beta_2} \arrow[uuull, bend left=60, "\mathbb{F}^f(\psi_1,\psi_2)"]
\end{tikzcd}$$

This diagram represents two morphisms $a \leftarrow c \rightarrow b$ in $\varphi \downarrow (\mathbb{F}^f \downarrow \mathbb{F}^f)$, where c is the object given by φ, $\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}$, and any of the outer arrows in the diagram. Hence $\varphi \downarrow (\mathbb{F}^f \downarrow \mathbb{F}^f)$ is connected and the proof is finished. □

Now we are ready to prove Proposition 1.2.12.

Proof of Proposition 1.2.12. Let $R\langle A\rangle_\alpha$ be a possibly big T-controlled R-module. There is a chain of equalities and natural isomorphisms

$$\begin{aligned}
Y'\mathbb{F}^f R\langle A\rangle_\alpha &\overset{(a)}{=} \operatorname{colim}[YP\colon Y\downarrow Y'\mathbb{F}^f R\langle A\rangle_\alpha \to \mathbf{mod}(\mathbf{M}_R(T'))]\\
&\overset{(b)}{=} \operatorname{colim}[YP(1\downarrow Y') = YP\colon J\downarrow \mathbb{F}^f R\langle A\rangle_\alpha \to \mathbf{mod}(\mathbf{M}_R(T'))]\\
&\overset{(c)}{\simeq} \operatorname{colim}[YP(\mathbb{F}^f\downarrow\mathbb{F}^f) = Y\mathbb{F}^f P\colon J\downarrow R\langle A\rangle_\alpha \to \mathbf{mod}(\mathbf{M}_R(T'))]\\
&\overset{(d)}{=} \operatorname{colim}[\mathbb{F}^f_* YP\colon J\downarrow R\langle A\rangle_\alpha \to \mathbf{mod}(\mathbf{M}_R(T'))]\\
&\overset{(e)}{=} \mathbb{F}^f_* \operatorname{colim}[YP = YP(1\downarrow Y')\colon J\downarrow R\langle A\rangle_\alpha \to \mathbf{mod}(\mathbf{M}_R(T))]\\
&\overset{(f)}{=} \mathbb{F}^f_* \operatorname{colim}[YP\colon Y\downarrow Y'R\langle A\rangle_\alpha \to \mathbf{mod}(\mathbf{M}_R(T))]\\
&\overset{(g)}{=} \mathbb{F}^f_* R\langle A\rangle_\alpha.
\end{aligned}$$

Here (a) is a particular case of (A.0.12); (b) holds since $(1\downarrow Y')\colon J\downarrow\mathbb{F}^f R\langle A\rangle_\alpha \to Y\downarrow Y'\mathbb{F}^f R\langle A\rangle_\alpha$ is obviously an isomorphism of categories by definition of Y', here 1 is the identity functor and we also use (A.0.11); for (c) we use Lemma A.0.13, [**Mac71**] IX.3 Theorem 1, and (A.0.11); in (d) we use (A.0.11) and the commutativity of the second diagram in (A.0.10); in (e) we use that $\mathbb{F}^f_*$ is a left-adjoint and hence preserves colimits, as well as (A.0.11); (f) is analogous to (b); and finally (g) is again a special case of (A.0.12). □

APPENDIX B

Proof of Proposition 1.2.13

Proof of Proposition 1.2.13. One can suppose without loss of generality that $\mathfrak{F}(f)\colon \mathfrak{F}(T) \subset \mathfrak{F}(T')$ is in fact an inclusion and $f\colon T \subset T'$ is an inclusion of a subtree. Moreover, since $\mathbb{F}^f$ is full and faithful in this case, up to equivalence of categories, we can regard $\mathbf{M}_R(T)$ as the full subcategory of $\mathbf{M}_R(T')$ formed by those free T'-controlled R-modules with support contained in $\mathfrak{F}(T)$. Let $R\langle {T'}^0\rangle_{\delta'}$ be the canonical free T'-controlled R-module with support $\mathfrak{F}(T)$, $R\langle T^0\rangle_\delta$ the free T'-controlled R-module with T^0 the vertex set of T and $\delta\colon T^0 \subset T'$ the inclusion, and $\mathcal{M}$ the left-$R(\mathfrak{F}(T))$-right-$R(\mathfrak{F}(T'))$-module $\mathcal{M} = \mathrm{Hom}_{\mathbf{M}_R(T')}(R\langle {T'}^0\rangle_{\delta'}, R\langle T^0\rangle_\delta)$. By (1.2.3) there is a diagram of functors which commutes up to natural equivalence

$$\begin{tikzcd}
\mathbf{mod}(\mathbf{M}_R(T)) \arrow[r, "\mathbb{F}^f_*"] \arrow[d, "ev_{R\langle T^0\rangle_\delta}"', "\sim"] & \mathbf{mod}(\mathbf{M}_R(T')) \arrow[d, "\sim"', "ev_{R\langle {T'}^0\rangle_{\delta'}}"] \\
\mathbf{mod}(R(\mathfrak{F}(T))) \arrow[r, "- \underset{R(\mathfrak{F}(T))}{\otimes} \mathcal{M}"'] & \mathbf{mod}(R(\mathfrak{F}(T')))
\end{tikzcd}$$

Also recall that the vertical arrows are equivalences of categories, see (1.2.5), hence we only need to prove that $\mathcal{M}$ is flat as a left-$R(\mathfrak{F}(T))$-module. We shall use the planarity test ([**BK00**] 3.2.9) which ensures that it is enough to check that for any finitely generated right-ideal $\mathfrak{a} \subset R(\mathfrak{F}(T))$ the epimorphism induced by multiplication

$$m\colon \mathfrak{a} \underset{R(\mathfrak{F}(T))}{\otimes} \mathcal{M} \twoheadrightarrow \mathfrak{a}\mathcal{M},$$

is in fact an isomorphism.

By Proposition 1.2.6 he ideal $\mathfrak{a}$ is necessarily principal. Let $\psi\colon R\langle T^0\rangle_\delta \to R\langle T^0\rangle_\delta$ be a generator. An arbitrary element in $\mathfrak{a} \underset{R(\mathfrak{F}(T))}{\otimes} \mathcal{M}$ has the form $\psi \otimes \varphi$ for some controlled homomorphism $\varphi\colon R\langle {T'}^0\rangle_{\delta'} \to R\langle T^0\rangle_\delta$. Suppose that $m(\psi\otimes\varphi) = \psi\varphi = 0$. Consider the set

$$A = \left\{ t \in {T'}^0\,;\, \varphi(t) \neq 0 \right\}.$$

Since φ is controlled is easy to see that $\delta'(A)' \subset \mathfrak{F}(T)$, in particular if α is the restriction of δ' to A then $R\langle A\rangle_\alpha$ belongs to $\mathbf{M}_R(T)$ so it is a retract of $R\langle T^0\rangle_\delta$,

$$R\langle T^0\rangle_\delta \underset{i}{\overset{r}{\rightleftarrows}} R\langle A\rangle_\alpha\ , \quad ri = 1.$$

There is also a retraction diagram

$$R\langle {T'}^0\rangle_{\delta'} \underset{j}{\overset{p}{\rightleftarrows}} R\langle A\rangle_\alpha\ , \quad pj = 1,$$

where p vanishes over the elements in ${T'}^0 - A$ and j is induced by the inclusion $A \subset {T'}^0$.

Clearly φ coincides with the following composite

$$R\langle {T'}^0\rangle_{\delta'} \overset{p}{\twoheadrightarrow} R\langle A\rangle_\alpha \overset{i}{\hookrightarrow} R\langle T^0\rangle_\delta \overset{r}{\twoheadrightarrow} R\langle A\rangle_\alpha \overset{j}{\hookrightarrow} R\langle {T'}^0\rangle_{\delta'} \xrightarrow{\varphi} R\langle T^0\rangle_\delta,$$

so

$$\begin{aligned} \psi \otimes \varphi &= \psi \otimes \varphi jrip \\ &= \psi\varphi jr \otimes ip \\ &= 0. \end{aligned}$$

Therefore m is injective and the proof is finished. □

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