FINAL REPORT

NIKOLAI KONOVALOV

Summary

I stayed at MPI for one year, starting on September 1, 2023 and ending on August 31, 2024. My research was focused mostly on algebraic topology (MSC 55), category theory (MSC 18), and a little bit on algebraic geometry (MSC 14).

During my stay, I have continued to work with Ivan Perunov and Artem Prikhodko (Scoltech) on the six-functor formalism for the category of *tempered sheaves*; that is the category assigned to a topological stack which controls its equivariant elliptic cohomology. Furthermore, I have started the project with Tobias Barthel (MPIM) and Kaif Hilman (MPIM) on endowing the category of d-excisive functors $\mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$ from spectra to spectra with a structure of an $\mathrm{Epi}_{\leq d}$*-parametrized category.* This project further extends the analogy between the category of excisive functors and the genuine stable homotopy theory observed firstly by S. Glasman and developed by G. Arone, T. Barthel, D. Heard, and B. Sanders. Finally, I did a few more computations in my old project related to *automorphism groups of complete intersections* in homogeneous varieties.

I. Tempered sheaves

I will describe each project in more details. Let G be a compact Lie group. The *equivariant elliptic cohomology* $\mathcal{E}ll_G$ is a G-equivariant cohomology theory which a generalization of the equivariant ordinary (Borel) cohomology (height zero) and the equivariant K-theory (height one) to *higher heights.* Let $\mathcal{E}ll_{G,E}$ be the equivariant elliptic cohomology associated with a fixed elliptic curve E. We note that since E is not affine, the natural target for the functor $\mathcal{E}ll_{G,E}$ are not abelian groups, but rather the category $\mathrm{QCoh}(E)$ of *quasi-coherent sheaves* over E.

Recently, if E is an oriented *spectral* elliptic curve over $\mathrm{Spec}(R)$, J. Lurie [5] (for finite groups) and D. Gepner and L. Meier [2] (for all compact Lie groups) proposed a construction for equivariant elliptic cohomology $\mathcal{E}ll_{G,E}$. The main gadget in [5] for working with $\mathcal{E}ll_{G,E}(X)$ is the category $\mathrm{LocSys}(X/G)$ of *tempered local system*, where X/G is the *stacky* quotient of a G-space X by G.

In our project, we are trying to show that the equivariant elliptic cohomology of D. Gepner and L. Meier satisfy standard properties of *geometric* nature. First, we additionally define equivariant elliptic cohomology *with compact support* $\mathcal{E}ll^c_G$ and equivariant elliptic *homology* (resp. *Borel-Moore homology*) $\mathcal{E}ll^G$ (resp. $\mathcal{E}ll^G_{BM}$); all four functors $\mathcal{E}ll_G$, $\mathcal{E}ll^c_G$, $\mathcal{E}ll^G$, and $\mathcal{E}ll^G_{BM}$ take values in the category $\mathrm{QCoh}(E)$ of quasi-coherent sheaves over E. Second, we attempt to obtain the *Thom isomorphism* and the *Poincaré duality* for these functors. This part is more problematic. In order to illustrate possible obstacles, we note that the Thom class for an S^1-equivariant vector bundle V over a point must be a *line bundle* $Th_V \in \mathrm{QCoh}(E)$ over E and it is difficult to construct rigorously line bundles over spectral elliptic curves only from definitions.

Inspired by [4], we are attacking the problem above in a more systematic way. Namely, for a topological stack $\mathcal{X}$, we define the category $\mathrm{Shv}^{\mathrm{temp}}(\mathcal{X})$ of *tempered sheaves* such that there is a natural fully faithful embedding

$$\mathrm{LocSys}(X/G) \hookrightarrow \mathrm{Shv}^{\mathrm{temp}}(X/G)$$

for a sufficiently good G-space X. Moreover, if X is a point, then the embedding above is an equivalence.

The category $\mathrm{Shv}^{\mathrm{temp}}(\mathcal{X})$ is functorial by $\mathcal{X}$; if $f\colon \mathcal{X} \to \mathcal{Y}$ is a map of topological stacks, then there is an adjoint pair

$$f^* : \mathrm{Shv}^{\mathrm{temp}}(\mathcal{Y}) \rightleftarrows \mathrm{Shv}^{\mathrm{temp}}(\mathcal{X}) : f_*.$$

Moreover, if f is of *finite type* (e.g. $\mathcal{X} = X/G$, $\mathcal{Y} = \mathrm{point}/G$, and G acts with finitely many orbit types on X), then there is a *shriek* adjoint pair

$$f_! : \mathrm{Shv}^{\mathrm{temp}}(\mathcal{X}) \rightleftarrows \mathrm{Shv}^{\mathrm{temp}}(\mathcal{Y}) : f^!$$

such that $f_* \simeq f_!$ for a *proper* map f. We use f_* (resp. $f_!$) to define the equivariant elliptic cohomology $\mathcal{E}ll_G$ (resp. cohomology with compact support $\mathcal{E}ll^c_G$).

During my stay, we showed that the functors $f^* \dashv f_*$ and $f_! \dashv f^!$ satisfy the standard axioms of six-functor formalism, i.e. proper base change, projection formula, purity theorem, etc. Using the six-formalism, we can derive desired properties of $\mathcal{E}ll_G$, $\mathcal{E}ll^c_G$, $\mathcal{E}ll^G$, and $\mathcal{E}ll^G_{BM}$. In order to show that the shriek-functor $f_!$ commutes with colimits and enjoys the base change, we developed and studied the (unexpected) functoriality of *oplax* limits with respect to *lax* natural transformations.

II. Excisive functors

Let $\mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$ be the category of *d-excisive functors* from spectra to spectra. S. Glasman [3] showed that $\mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$ is equivalent to the category $\mathrm{Mack}(\mathrm{Epi}_{\leq d})$ of *spectral Mackey functors* indexed by the small category $\mathrm{Epi}_{\leq d}$ of *finite sets with cardinality at most d* and *surjections*. This result resembles the Guillou-May theorem stating that the *genuine G-equivariant stable homotopy theory* Sp^G (G is a finite group) is equivalent to the spectral Mackey functors $\mathrm{Mack}(\mathcal{O}_G)$ indexed by *finite G-sets with transitive G-action*. Therefore, the category $\mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$ must share a lot of properties with Sp^G.

In our project with Tobias Barthel and Kaif Hilman, we enhance the category $\mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$ to an $\mathrm{Epi}_{\leq d}$-*category*. More precisely, we construct a functor

$$\underline{\mathrm{Exc}}^d \colon \mathrm{Epi}^{op}_{\leq d} \to \mathcal{C}\mathrm{at}$$

such that $\underline{\mathrm{Exc}}^d([1]) \simeq \mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$, $\underline{\mathrm{Exc}}^d([d]) \simeq \mathrm{Exc}^{1,\dots,1}(\mathrm{Sp}^{\times d}, \mathrm{Sp})$ are multilinear functors of arity d, $\underline{\mathrm{Exc}}^d([r])$, $1 < r < d$ are certain intermediate functor categories, and the structure maps are given by *cross-effect* evaluations. We showed that the $\mathrm{Epi}_{\leq d}$-category $\underline{\mathrm{Exc}}^d$ is a presentable $\mathrm{Epi}_{\leq d}$-stable category. Moreover, it is $\mathrm{Epi}_{\leq d}$-presentably symmetric monoidal when $\mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$ is equipped with the monoidal product given by the Day convolution of two functors.

Using the last two results, we are trying to show that the category $\mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$ can be obtained from $\mathrm{Fun}(\mathrm{Epi}^{op}_{\leq d}, \mathcal{S})$ by inverting some sphere-like objects. Similarly, how the category Sp^G is obtained from $\mathrm{Fun}(\mathcal{O}^{op}_G, \mathcal{S})$ by inverting the representation spheres. Finally, we also show that the naive generalization of the Segal conjecture fails in $\mathrm{Exc}^d(\mathrm{Sp}, \mathrm{Sp})$, but certain (weaker) alterations seem to work.

III. List of papers

All titles are tentative.

(1) *Norms and universality in calculus*, in preparation, joint with Tobias Barthel and Kaif Hilman.
(2) *Elliptic Coulomb branches*, in preparation, joint with Ivan Perunov and Artem Prikhodko.
(3) *On the automorphism groups of smooth Fano threefolds* (2024), preprint, available at arXiv:2406.03584v2.

The next paper was prepared in 2022, but the revision before publication was made at MPI.

(1) *Koszul duality for simplicial restricted Lie algebras* (2022), preprint, available at arXiv:2209.03312. Preaccepted by High. Struct.

IV. Invited talks

(1) March 2024, *Algebraic Goodwillie spectral sequence*, Topology seminar, University of Haifa (online)
(2) March 2024, *Algebraic Goodwillie spectral sequence*, TopICS seminar, Utrecht University
(3) January 2024, *Tempered sheaves*, Oberseminar, MPIM, Bonn
(4) December 2023, *Algebraic Goodwillie spectral sequence*, Topology seminar, University of Bonn
(5) November 2023, *Algebraic Goodwillie spectral sequence*, Topology seminar, Ruhr University, Bochum

I also organized with Yuqing Shi the reading seminar on unstable Adams spectral sequence and I gave a one talk there. I participated in the reading seminar at MPI on the rational $K(n)$-local sphere and I also gave a one talk there.

V. Other collaborations

During July, I have started the project with Itamar Mor (MPIM) on the category of *synthetic spaces*. Intuitively, this category should be thought as the *total space* for a categorical *degeneration* (in a certain sense) with the *general fiber* to be the category $\mathcal{S}_p^\wedge$ of p-complete spaces and the *special fiber* $s\mathcal{L}$ to be simplicial restricted Lie algebras. We expect the *Bockstein spectral sequence* associated with such degeneration should coincide with the unstable Adams spectral sequence [1]. Informally, the ∞-category $\widehat{\mathcal{S}}$ should be considered as an unstable analog for the ∞-category of synthetic spectra constructed by P. Pstrągowski in [6].

References

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[2] D. Gepner and L. Meier. On equivariant topological modular forms. `https://arxiv.org/abs/2004.10254v4`, 2023. Preprint.
[3] S. Glasman. Goodwillie calculus and mackey functors. `https://arxiv.org/abs/1610.03127v3`, 2018. Preprint.
[4] H. Honkasalo. Sheaves on fixed point sets and equivariant cohomology. *Math. Scand.*, 78(1):37–55, 1996.
[5] J. Lurie. Elliptic Cohomology III: Tempered Cohomology. `https://www.math.ias.edu/~lurie/papers/Elliptic-III-Tempered.pdf`, 2019. Preprint.
[6] P. Pstrągowski. Synthetic spectra and the cellular motivic category. *Invent. Math.*, 232(2):553–681, 2023.