CLASSICAL ELLIPTIC HYPERGEOMETRIC FUNCTIONS AND THEIR APPLICATIONS

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To the memory of A. A. Bolibrukh

Abstract. General theory of elliptic hypergeometric series and integrals is outlined. Main attention is paid to the examples obeying properties of "classical" special functions. In particular, an elliptic analogue of the Gauss hypergeometric function and some of its properties are described. Present review is based on the author's habilitation thesis [Spi7] containing a more detailed account of the subject.

Contents

1. General definition of univariate elliptic hypergeometric series and integrals 2
2. An overview of classical hypergeometric functions 3
3. Elliptic functions versus balanced, well poised and very well poised hypergeometric functions 10
4. The univariate elliptic beta integral 12
5. An elliptic analogue of the ${}_2F_1$ function 15
6. Contiguous relations and the elliptic hypergeometric equation 17
7. Applications in mathematical physics 20
8. Biorthogonal functions 24
8.1. Difference equation and three term recurrence relation 24
8.2. Two-index biorthogonality 25
8.3. The unit circle case 26
9. Multiple elliptic beta integrals 28
9.1. General definition 28
9.2. Integrals for the root system C_n 29
9.3. Integrals for the root system A_n 30
10. Univariate integral Bailey chains 33

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11. Elliptic Fourier-Bailey type integral transformations on root systems 35
12. Applications to the Calogero-Sutherland type models 36
References 38

1. General definition of univariate elliptic hypergeometric series and integrals

<u>Broad definition</u> ($n = 1$, univariate case) [Spi2, Spi4].

Formal contour integrals $\int_C \Delta(u)du$ and series $\sum_{n\in\mathbb{Z}} c_n$ are called elliptic hypergeometric integrals and series, if there exist three constants $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$ such that

- $\Delta(u + \omega_1) = h(u)\Delta(u)$,
  where $h(u)$ is an elliptic function of u with periods ω_2, ω_3, i.e., $h(u)$ is meromorphic and
  $$h(u + \omega_2) = h(u + \omega_3) = h(u), \quad \mathrm{Im}(\omega_2/\omega_3) \neq 0;$$
- $c_{n+1} = h(n\omega_1)c_n$,
  where $h(n\omega_1)$ is an elliptic function of n with periods $\dfrac{\omega_2}{\omega_1}$, $\dfrac{\omega_3}{\omega_1}$.

There is a functional freedom in the definition of integrals: $\Delta(u) \to \varphi(u)\Delta(u)$, where $\varphi(u)$ is an arbitrary ω_1-periodic function, $\varphi(u+\omega_1) = \varphi(u)$ (such a freedom is not essential for series).

<u>Narrow definition of integrals.</u>

Formal contour integrals $\int_C \Delta(u)du$ are called elliptic hypergeometric integrals, if $\Delta(u)$ is a meromorphic solution of three linear first order difference equations

$$\Delta(u + \omega_i) = h_i(u)\Delta(u), \quad i = 1, 2, 3,$$

where $h_i(u)$ are elliptic functions with the periods ω_{i+1}, ω_{i+2} (we set $\omega_{i+3} = \omega_i$).

If all $h_i(u) \neq const$, then $\mathrm{Im}(\omega_i/\omega_j) \neq 0$, $i \neq j$. Interesting situations occur when one $h_i(u) = const$, in which case we can have either $\mathrm{Im}(\omega_i/\omega_{i+1}) = 0$ or $\mathrm{Im}(\omega_i/\omega_{i+2}) = 0$. For pairwise incommensurate ω_i, the functional freedom in the definition of $\Delta(u)$ is absent due to the non-existence of triply periodic functions.

Thus, we have in general three elliptic curves, but only two of them are independent. One can consider also elliptic hypergeometric functions in a more general context, when $h_i(u)$ are $N \times N$ matrices with elliptic function entries.

It is possible to abandon the requirement of double periodicity of $h(u)$ in favor of its double quasiperiodicity similar to the Jacobi theta or Weierstrass sigma functions. This leads to a more general family of theta hypergeometric series and integrals (theta analogs of the Meijer function) [Spi2, Spi4], but we skip their consideration. In the next section we describe certain "classical" special functions of hypergeometric type and their elliptic generalizations.

2. An overview of classical hypergeometric functions

The Euler's beta integral [AAR]

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}\,\alpha,\ \mathrm{Re}\,\beta > 0,$$

determines: i) the measure for Jacobi polynomials; ii) an integral representation for the ${}_2F_1$ series. Namely, Jacobi polynomials

$$P_n(x) = \frac{(\alpha)_n}{n!}\,{}_2F_1\left(\begin{matrix} -n,\ n+\alpha+\beta-1 \\ \gamma \end{matrix}; x\right),$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ is the Pochhammer symbol, satisfy the orthogonality relations

$$\begin{aligned} \langle P_n, P_m\rangle &= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}P_n(x)P_m(x)dx \\ &= \frac{\delta_{nm}}{2n+\alpha+\beta-1}\,\frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n+\alpha+\beta-1)n!}. \end{aligned}$$

The Gauss hypergeometric function has the form

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n}x^n \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(b-c)}\int_0^1 t^{b-1}(1-t)^{b-c-1}(1-xt)^{-a}dt, \end{aligned}$$

where we skip for brevity relevant constraints upon the parameters. It defines a solution of the hypergeometric equation

$$y''(x) + \left(\frac{c}{x} + \frac{a+b-c+1}{x-1}\right)y'(x) + \frac{ab}{x(x-1)}y(x) = 0,$$

which is analytical near the origin $x = 0$.

Two integrals described above fit into the general pattern $\int_C \Delta(x)dx$ with the kernel $\Delta(x) = \prod_{j=1}^k (x-x_j)^{\alpha_j}$ and some free parameters x_j

and α_j. It is characterized by the condition that its logarithmic derivative $\Delta'(x)/\Delta(x) = R(x)$ is a rational function of x. A very natural generalization of this criterion consists in the requirement that the kernel $\Delta(x)$ satisfies the first order linear finite difference equation $\Delta(x+\omega_1) = R(x)\Delta(x)$ with rational $R(x)$ (such a treatment is valid already for the ${}_2F_1$ function via the Mellin-Barnes integral representation). By definition, we obtain general *plain hypergeometric integrals* for which

$$\Delta(x) = \frac{\prod_{j=0}^{s-1}\Gamma(x/\omega_1+u_j)}{\prod_{j=0}^{r}\Gamma(x/\omega_1+v_j)}\,\varphi(x)\,y^{x/\omega_1}, \quad \varphi(x+\omega_1)=\varphi(x),$$

and $\Gamma(x)$ is the Euler's gamma function.

The Pochhammer series have the form $\sum_{n=0}^{\infty} c_n$ with

$$\frac{c_{n+1}}{c_n} = R(n) = \frac{\prod_{j=0}^{s-1}(n+u_j)}{(n+1)\prod_{j=1}^{r}(n+v_j)}\,y,$$

which leads automatically to the expression

$$\sum_{n=0}^{\infty} c_n = {}_sF_r\left(\begin{matrix} u_0,\ldots,u_{s-1} \\ v_1,\ldots,v_r\end{matrix};y\right) = \sum_{n=0}^{\infty}\frac{(u_0)_n\cdots(u_{s-1})_n}{n!(v_1)_n\cdots(v_r)_n}\,y^n.$$

These series admit confluence limits like ${}_sF_r(y) \sim {}_{s-1}F_r(u_0y)$ for $u_0 \to \infty$. Their q-generalization has by definition $c_{n+1}/c_n = R(q^n)$ for $q \in \mathbb{C}$ and some rational $R(x)$, which leads uniquely to the series

$${}_s\varphi_r\left(\begin{matrix} t_0,\ldots,t_{s-1} \\ w_1,\ldots,w_r\end{matrix};q;y\right) = \sum_{n=0}^{\infty}\frac{(t_0;q)_n\cdots(t_{s-1};q)_n}{(q;q)_n(w_1;q)_n\cdots(w_r;q)_n}\,y^n,$$

where $(t;q)_n = (1-t)(1-tq)\cdots(1-tq^{n-1})$ denotes the q-Pochhammer symbol. This definition differs from the one given in [AAR, GR] by the inversion $q \to q^{-1}$ and appropriate change of notation for parameters. For $t_i = q^{u_i}$, $w_i = q^{v_i}$ and $q \to 1^-$, we formally have ${}_s\varphi_r(y) \to {}_sF_r(\tilde{y})$ for some renormalized value of the argument $\tilde{y}$. In a similar way one can reconstruct the bilateral series ${}_sH_r$ and ${}_s\psi_r$.

<u>Elliptic hypergeometric series</u>, directly derived from the definition given in the first section, have the form (the unilateral case)

$${}_{r+1}E_r\left(\begin{matrix} t_0,t_1,\ldots,t_r \\ w_1,\ldots,w_r\end{matrix};q,p;y\right) = \sum_{n=0}^{\infty}\frac{(t_0)_n(t_1)_n\cdots(t_r)_n}{(w_0)_n(w_1)_n\cdots(w_r)_n}y^n,$$

where $w_0 = q$ (the canonical normalization) and

- $(t)_n = \theta(t,tq,\ldots,tq^{n-1};p) \equiv \prod_{j=0}^{n-1}\theta(tq^j;p)$,

  $$\boxed{\theta(t;p) = (t;p)_\infty(pt^{-1};p)_\infty,}$$

$(t;p)_\infty = \prod_{n=0}^\infty (1-tp^n), \quad |p|<1;$

- $\boxed{\prod_{j=0}^r t_j = \prod_{j=0}^r w_j,}$ the balancing condition.

The elliptic Pochhammer symbol $(t)_n$ (denoted also in some other places as $(t;q,p)_n$, $\theta(t)_n$, or $\theta(t;p;q)_n$) degenerates to $(t;q)_n$ for $p\to 0$, $(t)_n \to (t;q)_n$. Therefore for generic fixed t_j, w_j we have the termwise limiting relation ${}_{r+1}E_r \to {}_{r+1}\varphi_r$ with the balancing restriction indicated above (which does not coincide with the balancing condition usually accepted for q-hypergeometric series [GR]).

For $p = e^{2\pi i\tau}$, $\mathrm{Im}(\tau) > 0$, and any $\sigma, u \in \mathbb{C}$, $q = e^{2\pi i\sigma}$, we have the following relation between $\theta(t;p)$ and the Jacobi $\theta_1(x) \equiv \theta_1(x|\tau)$ function

$$\begin{aligned}\theta_1(\sigma u|\tau) &= -i\sum_{k=-\infty}^{\infty}(-1)^k p^{(2k+1)^2/8} q^{(k+1/2)u}\\ &= ip^{1/8}q^{-u/2}\,(p;p)_\infty\,\theta(q^u;p).\end{aligned}$$

Properties $\theta_1(x+1) = -\theta_1(x)$, $\theta_1(x+\tau) = -e^{-\pi i\tau-2\pi i x}\theta_1(x)$ and $\theta_1(-x) = -\theta_1(x)$ simplify to $\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p)$.

For ${}_{r+1}E_r$ series we have

$$\frac{c_{n+1}}{c_n} = y\prod_{j=0}^{r}\frac{\theta(t_jq^n;p)}{\theta(w_jq^n;p)} = h(n\omega_1),$$

an elliptic function of $n\in\mathbb{C}$ with periods ω_2/ω_1, ω_3/ω_1 for

$$\boxed{q = e^{2\pi i\omega_1/\omega_2}, \qquad p = e^{2\pi i\omega_3/\omega_2}.}$$

The integer $r+1$ is called the order of $h(x)$ and it counts the number of zeros or poles of $h(x)$ inside the fundamental parallelogram of periods. Due to the balancing condition, we have an interesting (and useful) property

$${}_{r+1}E_r\left({t_0, t_1, \ldots, t_r \atop w_1, \ldots, w_r}; q, p; y\right) = {}_{r+1}E_r\left({t_0^{-1}, t_1^{-1}, \ldots, t_r^{-1} \atop w_1^{-1}, \ldots, w_r^{-1}}; q^{-1}, p; y\right).$$

In the table below we describe known special functions with properties generalizing in a natural way the ${}_2F_1$ hypergeometric function features. It is rather sketchy and does not pretend on completeness. The ${}_2F_1$ series is the classical special function investigated by such giants as Euler, Gauss, Jacobi, Riemann and many other mathematicians. Its q-analogue has been proposed by Heine as far back as 1850. However, until the relatively recent time (landmarked by the appearance of quantum algebras from exactly solvable models of statistical mechanics) q-special functions did not attract much attention.

Chebyshev put forward general theory of orthogonal polynomials which played a major role in the search of classical special functions. Jacobi polynomials satisfy a three term recurrence relation and the hypergeometric equation. The general discrete set of ${}_3F_2$ polynomials was constructed by Chebyshev (I am indebted to Askey and Zhedanov for pointing this fact to me). Their continuous analogues and the ${}_3\phi_2$ series generalizations, known as Hahn polynomials, were proposed much later. These polynomials satisfy a second order difference equation (instead of the differential equation) in their argument lying on some nontrivial "grids". The next level of generalization is given by the Racah and Wilson polynomials described by special ${}_4F_3$ series. In 1985, Askey and Wilson have found [AW] the most general set of orthogonal polynomials with the self-duality property. They are expressed in terms of a special ${}_4\varphi_3$ series and their argument "lives" on the most general admissible grid for polynomials.

CLASSICAL SPECIAL FUNCTIONS OF HYPERGEOMETRIC TYPE

$$
\begin{tikzcd}
\boxed{{}_2F_1}\begin{pmatrix}\text{Euler}\\ \text{Gauss}\\ \text{Jacobi}\\ \text{Riemann}\end{pmatrix} \arrow[r] \arrow[d] & 2\varphi 1\begin{pmatrix}\text{Heine}\\ \text{1850}\end{pmatrix} \arrow[d] & \\
{}_3F_2\begin{pmatrix}\text{Chebyshev}\\ \text{1875}\\ \text{Hahn}\end{pmatrix} \arrow[r] \arrow[d] & 3\varphi 2\begin{pmatrix}\text{Hahn}\\ \text{1949}\end{pmatrix} \arrow[d] & \\
{}_4F_3\begin{pmatrix}\text{Racah}\\ \text{1942}\\ \text{Wilson}\\ \text{1978}\end{pmatrix} \arrow[r] \arrow[d, "\textit{self-dual orthogonal}"] & \boxed{4\varphi 3}\begin{pmatrix}\text{Askey,}\\ \text{Wilson}\\ \text{1985}\end{pmatrix} \arrow[d, "\textit{polynomials}"] & \\
{}_7F_6\begin{pmatrix}\text{Dougall}\\ \text{1907}\end{pmatrix} \arrow[r] \arrow[d, "\textit{summation formulas}"] & 8\varphi 7\begin{pmatrix}\text{Jackson}\\ \text{1921}\end{pmatrix} \arrow[r] \arrow[d] & {}_{10}E_9\begin{pmatrix}\text{Frenkel,}\\ \text{Turaev}\\ \text{1997}\end{pmatrix} \arrow[d] \\
{}_9F_8\begin{pmatrix}\text{Wilson}\\ \text{1978}\\ \text{Rahman}\\ \text{1986}\end{pmatrix} \arrow[r] & 10\varphi 9\begin{pmatrix}\text{Rahman}\\ \text{1986}\\ \text{Wilson}\\ \text{1991}\end{pmatrix} \arrow[r] & \boxed{{}_{12}E_{11}}\begin{pmatrix}\text{[SZ1]}\\ \text{2000}\\ \text{[Spi4]}\\ \text{2003}\end{pmatrix}
\end{tikzcd}
$$

self-dual biorthogonal rational functions

The next level of complexification of functions indicated in the table refers to the most general known summation formulas for terminating series of hypergeometric type. Sequentially, these are the Dougall's ${}_7F_6$ and Jackson's ${}_8\varphi_7$ sums going back to the first quarter of the last century, and the recent result by Frenkel and Turaev [FT] at the level of ${}_{10}E_9$ series to be described below.

Finally, until very recent time the most general set of known special functions satisfying some orthogonality relations and obeying other "classical" properties were given by biorthogonal rational functions related to the very well poised ${}_9F_8$ and ${}_{10}\varphi_9$ series. The discrete measure functions were discovered by Wilson [Wil] and their continuous measure generalizations were derived by Rahman [Rah]. An elliptic extension of the Wilson's biorthogonal functions with the key self duality property was constructed by Zhedanov and the author [SZ1]. The Rahman's family of rational functions was lifted to the elliptic level by the author [Spi4]. These functions "live" on the grids described by the second order elliptic functions—the most general type of grids for rational functions admitting a lowering divided difference operator [SZ3]. Moreover, in the elliptic case there appeared even more complicated objects existing only at this level [Spi4], which go beyond the space of rational functions of some argument and which satisfy unusual two index biorthogonality relations.

There exist also non-self-dual three parameter extension of the last row functions described by the very well poised ${}_9F_8$, ${}_{10}\varphi_9$, and ${}_{12}E_{11}$ series [SZ1], but many of their properties remain unknown.

In the following we restrict ourselves only to the elliptic hypergeometric functions and for further details concerning plain and q-hypergeometric objects we refer to the textbooks [AAR] and [GR], handbook [KS] and the original papers [AW, Rah, Wil]. For a description of general formal unilateral ${}_sE_r$ and bilateral ${}_sG_r$ theta hypergeometric series, see [GR, Spi2, Spi7].

Elliptic hypergeometric integrals are described with the help of the bases q, p and

$$\boxed{\tilde{q} = e^{-2\pi i\omega_2/\omega_1},\ \tilde{p} = e^{-2\pi i\omega_2/\omega_3},\ r = e^{2\pi i\omega_3/\omega_1},\ \tilde{r} = e^{-2\pi i\omega_1/\omega_3},}$$

where $\tilde{q}$, $\tilde{p}$, $\tilde{r}$ are modular transforms of q, p, and r.

Theorem 1. (An elliptic analogue of the Meijer function [Spi4]**)**
For incommensurate ω_i and $|p|, |q|, |r| < 1$ general solution of the equations

$$\Delta(u + \omega_i) = h_i(u)\Delta(u), \quad i = 1, 2, 3,$$

where

ell. fun-s	*periods*	*bases*	*moduli*
$h_1(u)$	$\omega_2,\ \omega_3$	p	$\tau_1=\omega_3/\omega_2$
$h_2(u)$	$\omega_1,\ \omega_3$	r	$\tau_2=\omega_3/\omega_1$
$h_3(u)$	$\omega_1,\ \omega_2$	q	$\tau_3=\omega_1/\omega_2$

$\tau_1=\tau_2\tau_3,$

is:

$$\Delta(u)=\prod_{j=0}^{m}\frac{\Gamma(t_je^{2\pi iu/\omega_2};p,q)}{\Gamma(w_je^{2\pi iu/\omega_2};p,q)}\prod_{j=0}^{m'}\frac{\Gamma(t_j'e^{-2\pi iu/\omega_1};r,\tilde{q})}{\Gamma(w_j'e^{-2\pi iu/\omega_1};r,\tilde{q})}\ e^{\gamma u}\times\text{constant},$$

where

$$\prod_{j=0}^{m}t_j=\prod_{j=0}^{m}w_j,\qquad \prod_{j=0}^{m'}t_j'=\prod_{j=0}^{m'}w_j',$$

and

$$\Gamma(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k},\quad |q|,|p|<1,$$

is the standard elliptic gamma function [Rui2].

The function $\Gamma(z;p,q)$ satisfies equations

$$\begin{aligned}\Gamma(z;p,q)&=\Gamma(z;q,p),\quad \Gamma(pq/z;p,q)=1/\Gamma(z;p,q),\\ \Gamma(qz;p,q)&=\theta(z;p)\Gamma(z;p,q),\\ \Gamma(pz;p,q)&=\theta(z;q)\Gamma(z;p,q).\end{aligned}$$

If we denote $f(u)=\Gamma(e^{2\pi iu/\omega_2};p,q)$, then this function solves uniquely (up to a multiplicative factor independent on u) the following system of three linear first order finite difference equations

$$\left\{\begin{array}{l}\boxed{f(u+\omega_1)=\theta(e^{2\pi iu/\omega_2};p)f(u),}\\ f(u+\omega_2)=f(u),\\ f(u+\omega_3)=\theta(e^{2\pi iu/\omega_2};q)f(u).\end{array}\right.$$

There are two choices of parameters with additional nice properties:

1) $\gamma=0$ and no t_k',w_k' (the "standard" case $|p|,|q|<1$);
2) $\gamma=0$ and $m'=m$, $t_j'=rt_j$, $w_j'=rw_j$ (the "unit circle" case).

In the second case, gamma function factors combine into the modified elliptic gamma function introduced in [Spi4]:

$$G(u;\omega_1,\omega_2,\omega_3)=\Gamma(e^{2\pi iu/\omega_2};p,q)\Gamma(re^{-2\pi iu/\omega_1};r,\tilde{q}).$$

This function solves uniquely another system of three equations:

$$\left\{\begin{array}{l}\boxed{f(u+\omega_1)=\theta(e^{2\pi iu/\omega_2};p)f(u),}\\ f(u+\omega_2)=\theta(e^{2\pi iu/\omega_1};r)f(u),\\ f(u+\omega_3)=e^{B_{2,2}(u)}f(u),\end{array}\right.$$

where

$$B_{2,2}(u) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.$$

These equations allow us to prove the representation [DS4]

$$G(u;\omega) = e^{-\pi i P(u)}\Gamma(e^{-2\pi i \frac{u}{\omega_3}};\tilde{r},\tilde{p}),$$

$$P\left(u + \sum_{n=1}^{3} \frac{\omega_n}{2}\right) = \frac{u(u^2 - \frac{1}{4}\sum_{k=1}^{3}\omega_k^2)}{3\omega_1\omega_2\omega_3},$$

related to modular transformations for the standard elliptic gamma function [FV]. From this representation it is easy to see that $G(u;\omega)$ is well defined for $|p|, |r| < 1$ and $|q| \leq 1$ (i.e., the $|q| = 1$ case is permitted in sharp difference from the $\Gamma(z;p,q)$ function!).

Permutations $\tilde{r} \leftrightarrow \tilde{p}$ and $\omega_1 \leftrightarrow \omega_2$ are equivalent. Therefore, we have

$$G(u;\omega_1,\omega_2,\omega_3) = G(u;\omega_2,\omega_1,\omega_3).$$

Due to the property $P(\sum_{k=1}^{3}\omega_k - u) = -P(u)$, we have the reflection equation

$$G(a;\omega)G(b;\omega) = 1, \quad a + b = \sum_{k=1}^{3}\omega_k.$$

In the limit $\omega_3 \to \infty$, taken in such a way that simultaneously $p, r \to 0$, the modified elliptic gamma function is reduced to the "unit circle" q-gamma function

$$\lim_{p,r\to 0} \frac{1}{G(u;\omega)} = S(u;\omega_1,\omega_2) = \frac{(e^{2\pi i u/\omega_2};q)_\infty}{(e^{2\pi i u/\omega_1}\tilde{q};\tilde{q})_\infty},$$

which remains well defined in the limit $|q| \to 1$. This function appeared in the modern time mathematics in the work of Shintani [Shi] as a ratio of Barnes' double gamma functions [Bar]; in the works of Faddeev and coauthors [Fad, FKV] on the modular double of quantum groups and quantum Liouville theory; in the work of Jimbo and Miwa [JM] on solutions of a q-difference equation and related correlation functions in statistical mechanics; in eigenfunctions of the q-Toda chain Hamiltonian [KLS]. In several independent studies it was named as the double sign function [Kur], or hyperbolic gamma function [Rui2, Rui3], or non-compact quantum dilogarithm [FKV]. For the operator algebra aspects of this nice function, see [Vol].

3. Elliptic functions versus balanced, well poised and very well poised hypergeometric functions

Some convenient terminology.

Theta functions: holomorphic functions $f(x)$ such that

$$f(x+\omega_2) = e^{ax+b} f(x), \quad f(x+\omega_3) = e^{cx+d} f(x),$$

for some $a, b, c, d \in \mathbb{C}$, with a finite number of zeros in the parallelogram of periods ω_2, ω_3, $\mathrm{Im}(\omega_2/\omega_3) \neq 0$. It is not difficult to deduce that

$$f(x) = e^{P_2(x)} \prod_{j=0}^{r} \theta_1(x+u_j), \quad u_j \in \mathbb{C},$$

for some polynomial of the second order $P_2(x)$.

Meromorphic theta functions: ratios of theta functions with different parameters r, u_j and $P_2(x)$.

Elliptic functions: balanced meromorphic theta functions

$$f(x) = \prod_{j=0}^{r} \frac{\theta_1(x+u_j)}{\theta_1(x+v_j)} = \prod_{j=0}^{r} \frac{\theta(t_j z; p)}{\theta(w_j z; p)},$$

where $p = e^{2\pi i\tau}$, $z = e^{2\pi i x}$, $t_j = e^{2\pi i u_j}$, $w_j = e^{2\pi i v_j}$ with the balancing constraint $\boxed{\prod_{j=0}^r t_j = \prod_{j=0}^r w_j,}$ or $\boxed{\sum_{j=0}^r u_j = \sum_{j=0}^r v_j \pmod 1}$ guaranteeing that $f(x+1) = f(x)$ and $f(x+\tau) = f(x)$. We can multiply these functions by arbitrary independent variable y which is omitted for brevity.

Modular invariant elliptic functions: elliptic functions invariant under the action of full $PSL(2;\mathbb{Z})$ group generated by the relations

$$f(x;\tau+1) = f(x;\tau), \qquad f(x/\tau; -1/\tau) = f(x;\tau).$$

Due to the symmetry properties

$$\begin{aligned}
&\theta_1(u|\tau+1) = e^{\pi i/4}\theta_1(u|\tau),\\
&\theta_1\Big(\frac{u}{\tau}\Big| -\frac{1}{\tau}\Big) = -i(-i\tau)^{1/2} e^{\pi i u^2/\tau}\theta_1(u|\tau),
\end{aligned}$$

elliptic functions are modular if $\boxed{\sum_{j=0}^r u_j^2 = \sum_{j=0}^r v_j^2 \pmod{2\tau}.}$ A useful form of the second transformation is

$$\frac{\theta(e^{-2\pi i \frac{u}{\omega_3}}; e^{-2\pi i \frac{\omega_2}{\omega_3}})}{\theta(e^{2\pi i \frac{u}{\omega_2}}; e^{2\pi i \frac{\omega_3}{\omega_2}})} = -i e^{\pi i \frac{\omega_2+\omega_3}{6\omega_2\omega_3}} e^{\pi i \frac{u^2 - u(\omega_2+\omega_3)}{\omega_2\omega_3}},$$

which indicates that the true modular transformation corresponds to the change $(\omega_2, \omega_3) \to (-\omega_3, \omega_2)$.

Totally elliptic functions: elliptic $f(x)$ which are elliptic also in u_j, v_j with the same periods. These are elliptic functions with the constraints $\boxed{v_j = -u_j (\bmod 1)}$ or $\boxed{w_j = t_j^{-1}}$ known in the theory of q-hypergeometric series as the well poisedness conditions. The balancing condition for such *well poised elliptic functions* $\prod_{j=0}^r t_j = \prod_{j=0}^r t_j^{-1}$ is reduced to $\boxed{\prod_{j=0}^r t_j = \pm 1,}$ i.e. we have a sign ambiguity! Totally elliptic functions are invariant under the shifts $t_j \to p t_j$ $(j = 0, 1, \ldots, r-1)$ and $z \to pz$. Moreover, they are automatically modular invariant and satisfy the relation $f(-x) = 1/f(x)$ (this relation reduces y, the arbitrary multiplier of $f(x)$, to $y = \pm 1$).

We scale now $z \to t_0 z$ and replace parameters $t_j t_0$ by t_j (in particular, we change $t_0^2 \to t_0$). As a result, we obtain

$$f(z, \underline{t}) = \prod_{j=0}^r \frac{\theta(t_j z; p)}{\theta(t_j^{-1} z; p)} \to \boxed{\prod_{j=0}^r \frac{\theta(t_j z; p)}{\theta(t_0 t_j^{-1} z; p)}} \equiv h(z, \underline{t}).$$

The balancing condition takes now the form $\prod_{j=1}^r t_j = \pm t_0^{(r-1)/2}$. Let us take $r = 2k+1$ odd and resolve the sign ambiguity in favor of the relation $\boxed{\prod_{j=1}^{2k+1} t_j = +t_0^k.}$ Only for this case there are non-trivial summation and transformation formulas for series of hypergeometric type. In this case $h(z, \underline{t})$ is invariant under the shift $t_0 \to p t_0$ (accompanied by the compensating transformation $t_{2k+1} \to p^k t_{2k+1}$), i.e. it is an elliptic function of $\log t_0$ with the same periods as for the $\log z$ variable. Equivalently, we have

$$f(p^{1/2} z, p^{1/2} t_0, \ldots, p^{1/2} t_{r-1}, p^{-r/2} t_r) = f(z, t_0, \ldots, t_r),$$

i.e. there appears interesting symmetry playing with the half period shifts. We conclude that the total ellipticity requirement (in appropriate parametrization) fixes the correct form of the balancing condition in the most interesting case of odd r.

Another important structural constraint leading to interesting elliptic functions is called the *very well poisedness* condition. It consists in imposing on the well poised elliptic functions of the restrictions

$$\boxed{t_{r-3} = q\sqrt{t_0},\ t_{r-2} = -q\sqrt{t_0},\ t_{r-1} = q\sqrt{t_0/p},\ t_r = -q\sqrt{p t_0}}$$

related to the doubling of the $\theta_1(x)$ function argument.

We call elliptic hypergeometric series and integrals modular, well poised, or very well poised, if the ratios of their kernels c_{n+1}/c_n and $\Delta(u+\omega_1)/\Delta(u)$ are modular, well poised, or very well poised elliptic functions. It is convenient to introduce special notation for the very well poised elliptic hypergeometric series [Spi3]:

$$
\begin{aligned}
&{}_{r+1}E_r\left(\begin{matrix} t_0, t_1, \ldots, t_{r-4}, q\sqrt{t_0}, -q\sqrt{t_0}, q\sqrt{t_0/p}, -q\sqrt{pt_0} \\ qt_0/t_1, \ldots, qt_0/t_{r-4}, \sqrt{t_0}, -\sqrt{t_0}, \sqrt{pt_0}, -\sqrt{t_0/p}\end{matrix}; q, p; -y\right) \\
&\quad = \sum_{n=0}^{\infty} \frac{\theta(t_0 q^{2n}; p)}{\theta(t_0)} \prod_{m=0}^{r-4} \frac{(t_m)_n}{(qt_0 t_m^{-1})_n} (qy)^n \equiv {}_{r+1}V_r(t_0; t_1, \ldots, t_{r-4}; q, p; y),
\end{aligned}
$$

where $\prod_{k=1}^{r-4} t_k = \pm t_0^{(r-5)/2} q^{(r-7)/2}$ (for odd r we assume the positive sign, due to the property described above). All known applications of these series use a special value of the argument y, $y = 1$. Therefore, we shall drop y in the notation of ${}_{r+1}V_r$ series for this special case. For $p \to 0$, these series reduce to the very well poised ${}_{r-1}\varphi_{r-2}$ series denoted by the symbol ${}_{r-1}W_{r-2}$ in the monograph [GR]. Remarkably, the elliptic balancing condition coincides in this case with the usual balancing condition accepted for these particular basic hypergeometric series [GR, Spi2].

Various forms of the ellipticity requirement provide thus an explanation of the origin of the notions of balancing and very well poisedness for series of hypergeometric type [Spi2]. It is the clarification of these points that forced the author to change previous notation for elliptic hypergeometric series [Spi2, Spi3]. In particular, in this system of conventions accepted in [GR, Ros3, Spi4], etc the symbol ${}_{r+1}E_r$ used in the papers [DS1, KMNOY, SZ1] should read as ${}_{r+3}E_{r+2}$ or ${}_{r+3}V_{r+2}$.

If we take $r = 9$, $t_4 = q^{-N} (N \in \mathbb{N})$, $\prod_{m=1}^{5} t_m = qt_0^2$, $y = 1$, then

$$
\boxed{{}_{10}V_9(t_0; t_1, \ldots, t_5; q, p) = \frac{(qt_0)_N (\frac{qt_0}{t_1 t_2})_N (\frac{qt_0}{t_1 t_3})_N (\frac{qt_0}{t_2 t_3})_N}{(\frac{qt_0}{t_1 t_2 t_3})_N (\frac{qt_0}{t_1})_N (\frac{qt_0}{t_2})_N (\frac{qt_0}{t_3})_N}.}
$$

This is the Frenkel-Turaev summation formula [FT] (for its elementary proofs, see, e.g., [Ros3, SZ2]), which is reduced in the limit $p \to 0$ to the Jackson sum for terminating very well poised balanced ${}_8\varphi_7$ series.

4. The univariate elliptic beta integral

The elliptic beta integral is the simplest very well poised elliptic hypergeometric integral.

Theorem 2. (The standard elliptic beta integral [Spi1])
Let $t_1, \ldots, t_6 \in \mathbb{C}$, $|t_j| < 1$, $\prod_{j=1}^{6} t_j = pq$, and $|p|, |q| < 1$. Then

$$
\boxed{\kappa \int_{\mathbb{T}} \frac{\prod_{k=1}^{6} \Gamma(t_k z; p, q)\Gamma(t_k z^{-1}; p, q)}{\Gamma(z^2; p, q)\Gamma(z^{-2}; p, q)} \frac{dz}{z} = \prod_{1 \le j < k \le 6} \Gamma(t_j t_k; p, q),}
$$

where $\mathbb{T}$ is the positively oriented unit circle $|z| = 1$ and

$$\boxed{\kappa = \frac{(q;q)_\infty (p;p)_\infty}{4\pi i}.}$$

The first proof of this integration formula used an elliptic generalization of the Askey's method [Ask] which required some contiguous relations for the left-hand side expression and Bailey's ${}_2\psi_2$ summation formula. A very simple proof has been found later on in [Spi6].

The elliptic beta integral is the most general univariate beta type integral found so far. It serves as a measure in the biorthogonality relations for a particular system of functions to be described below. After taking the limit $p \to 0$, our integral is reduced to the Rahman's q-beta integral [Rah]

$$\frac{(q;q)_\infty}{4\pi i} \int_{\mathbb{T}} \frac{(z^2;q)_\infty (z^{-2};q)_\infty (Az;q)_\infty (Az^{-1};q)_\infty}{\prod_{m=1}^5 (t_m z;q)_\infty (t_m z^{-1};q)_\infty} \frac{dz}{z}$$
$$= \frac{\prod_{m=1}^5 (At_m^{-1};q)_\infty}{\prod_{1\le j<k\le 5} (t_j t_k;q)_\infty},$$

where $A = \prod_{m=1}^5 t_m$, $|t_m| < 1$. This integral determines the measure for Rahman's family of continuous biorthogonal rational functions [Rah].

If we take now the limit $t_5 \to 0$, then we obtain the celebrated Askey-Wilson integral

$$\frac{(q;q)_\infty}{4\pi i} \int_{\mathbb{T}} \frac{(z^2;q)_\infty (z^{-2};q)_\infty}{\prod_{m=1}^4 (t_m z;q)_\infty (t_m z^{-1};q)_\infty} \frac{dz}{z} = \frac{(t_1t_2t_3t_4;q)_\infty}{\prod_{1\le j<k\le 4} (t_j t_k;q)_\infty},$$

determining the measure in orthogonality relations for the most general set of classical orthogonal polynomials [AW].

Careful analysis of the structure of residues of the integrand's poles allows one to deduce the Frenkel-Turaev summation formula out of the elliptic beta integral [DS1]. We suppose that $|t_m| < 1$, $m = 1, \dots, 4$, $|pt_5| < 1 < |t_5|$, $|pq| < |A|$, $A = \prod_{s=1}^5 t_s$, and assume also that the arguments of all t_s, $s = 1, \dots, 5$, and p, q are linearly independent over $\mathbb{Z}$. We denote C a contour separating sequences of integrand's poles at $z = t_s q^j p^k$ and $A^{-1} q^{j+1} p^{k+1}$, from their reciprocals at $z = t_s^{-1} q^{-j} p^{-k}$, $A q^{-j-1} p^{-k-1}$, $j, k \in \mathbb{N}$. Then we obtain the following residue formula:

$$\kappa \int_C \Delta_E(z, \underline{t}) \frac{dz}{z} = \kappa \int_{\mathbb{T}} \Delta_E(z, \underline{t}) \frac{dz}{z} + c_0(\underline{t}) \sum_{\substack{n\ge 0 \\ |t_0 q^n|>1}} \nu_n(\underline{t}),$$

with

$$\Delta_E(z, \underline{t}) = \frac{\prod_{m=1}^5 \Gamma(t_m z^{\pm}; p, q)}{\Gamma(z^{\pm 2}; p, q) \Gamma(A z^{\pm}; p, q)},$$

$\Gamma(az^{\pm};p,q) \equiv \Gamma(az;p,q)\Gamma(az^{-1};p,q)$, and

$$c_0(\underline{t}) = \frac{\prod_{m=1}^4 \Gamma(t_m t_5^{\pm};p,q)}{\Gamma(t_5^{-2};p,q)\Gamma(At_5^{\pm};p,q)},$$

$$\nu_n(\underline{t}) = q^n\,\frac{\theta(t_5^2 q^{2n};p)}{\theta(t_5^2;p)}\prod_{m=0}^{5}\frac{(t_m t_5)_n}{(qt_m^{-1}t_5)_n},$$

where we have introduced a new parameter t_0 via the relation $\prod_{m=0}^5 t_m = q$. In the limit $t_5t_4 \to q^{-N}$, $N \in \mathbb{N}$, values of the integral on the left-hand side of this formula and of the factor $c_0(\underline{t})$ in front of the residues sum on the right-hand side blow up, but the integral over the unit circle $\mathbb{T}$ remains finite. Dividing all the terms by $c_0(\underline{t})$ and taking the limit, we obtain the summation formula presented in the end of the previous section.

Using the modified elliptic gamma function it is not difficult to deduce out of the standard elliptic beta integral its "unit circle" analogue remaining well defined for $|q| = 1$.

Theorem 3. (The modified elliptic beta integral [DS4]**)**
We suppose that $Im(\omega_1/\omega_2) \geq 0$ *and* $Im(\omega_3/\omega_1) > 0$, $Im(\omega_3/\omega_2) > 0$ *and* $g_j \in \mathbb{C}$, $j = 1,\ldots,6$, $Im(g_j/\omega_3) < 0$, *together with the constraint* $\sum_{j=1}^6 g_j = \sum_{k=1}^3 \omega_k$. *Then*

$$\tilde{\kappa}\int_{-\omega_3/2}^{\omega_3/2}\frac{\prod_{j=1}^6 G(g_j \pm u;\omega)}{G(\pm 2u;\omega)}\frac{du}{\omega_2} = \prod_{1\leq j<m\leq 6} G(g_j+g_m;\omega),$$

where

$$\tilde{\kappa} = -\frac{(q;q)_\infty (p;p)_\infty (r;r)_\infty}{2(\tilde{q};\tilde{q})_\infty}.$$

Here the integration is taken along the cut with the end points $-\omega_3/2$ *and* $\omega_3/2$. *We use also the convention that* $G(a\pm b;\omega) \equiv G(a+b;\omega)G(a-b;\omega)$.

If we take $\mathrm{Im}(\omega_3) \to \infty$ in such a way that $p, r \to 0$, then this integral reduces to a Mellin-Barnes type q-beta integral. More precisely, for $\omega_{1,2}$ such that $\mathrm{Im}(\omega_1/\omega_2) \geq 0$ and $\mathrm{Re}(\omega_1/\omega_2) > 0$, we substitute $g_6 = \sum_{k=1}^3 \omega_k - \mathcal{A}$, where $\mathcal{A} = \sum_{j=1}^5 g_j$ and apply the inversion formula for $G(u;\omega)$. Then we set $\omega_3 = it\omega_2$, $t \to +\infty$, and obtain formally

$$\int_{\mathbb{L}}\frac{S(\pm 2u, \mathcal{A}\pm u;\omega)}{\prod_{j=1}^5 S(g_j\pm u;\omega)}\frac{du}{\omega_2} = -2\frac{(\tilde{q};\tilde{q})_\infty}{(q;q)_\infty}\frac{\prod_{j=1}^5 S(\mathcal{A}-g_j;\omega)}{\prod_{1\leq j<m\leq 5} S(g_j+g_m;\omega)},$$

where the integration is taken along the line $\mathbb{L} \equiv i\omega_2\mathbb{R}$. Here parameters are subject to the constraints $\mathrm{Re}(g_j/\omega_2) > 0$ and $\mathrm{Re}((\mathcal{A} -$

$\omega_1)/\omega_2) < 1$. This integral was rigorously proven first in [Sto] and a quite simple proof was given in [Spi6] in a more general setting.

5. An elliptic analogue of the ${}_2F_1$ function

We consider the double integral

$$\kappa \int_{C^2} \frac{\prod_{j=1}^3 \Gamma(a_j z^{\pm}, b_j w^{\pm})\, \Gamma(c z^{\pm} w^{\pm})}{\Gamma(z^{\pm 2}, w^{\pm 2}, c^2 A z^{\pm}, c^2 B w^{\pm})} \frac{dz}{z} \frac{dw}{w},$$

where $a_j, b_j, c \in \mathbb{C}$, $A = a_1 a_2 a_3$, $B = b_1 b_2 b_3$, and C is a contour separating converging to zero sequences of poles in z and w from the diverging ones, and

$$\Gamma(t_1, \ldots, t_k) \equiv \Gamma(t_1; p, q) \cdots \Gamma(t_k; p, q).$$

Applying the elliptic beta integral formula to integrations with respect to z or w (permutation of integrations is allowed since the integrand is bounded), we obtain a symmetry transformation for a pair of elliptic hypergeometric integrals [Spi4]

$$\prod_{j=1}^3 \frac{\Gamma(A/a_j)}{\Gamma(c^2 A/a_j)} \int_C \frac{\prod_{j=1}^3 \Gamma(c a_j z^{\pm}, b_j w^{\pm})}{\Gamma(z^{\pm 2}, c A z^{\pm}, c^2 B w^{\pm})} \frac{dz}{z}$$
$$= \prod_{j=1}^3 \frac{\Gamma(B/b_j)}{\Gamma(c^2 B/b_j)} \int_C \frac{\prod_{j=1}^3 \Gamma(a_j z^{\pm}, c b_j w^{\pm})}{\Gamma(z^{\pm 2}, c^2 A z^{\pm}, c B w^{\pm})} \frac{dz}{z}.$$

This is an elliptic analogue of the four term Bailey transformation for non-terminating ${}_{10}\varphi_9$ series. It cannot be written yet as some relation for infinite ${}_{12}V_{11}$ elliptic hypergeometric series due to the severe problems with their convergence at the boundary values of the argument $|y| = 1$.

We denote $t_{1,2,3} = c a_{1,2,3}$, $t_4 = pq/cA$, $t_{5,6,7} = b_{1,2,3}$, $t_8 = pq/c^2 B$ and introduce the elliptic hypergeometric function—an elliptic analogue of the Gauss hypergeometric function

$$\boxed{V(\underline{t}; p, q) = \kappa \int_C \frac{\prod_{j=1}^8 \Gamma(t_j z^{\pm})}{\Gamma(t z^{\pm 2})} \frac{dz}{z}, \qquad \prod_{j=1}^8 t_j = p^2 q^2.}$$

Due to the reflection equation for $\Gamma(z; p, q)$ function, we have

$$V(\underline{t}; p, q)\Big|_{t_7 t_8 = pq} = \prod_{1 \le j < k \le 6} \Gamma(t_j t_k; p, q),$$

which is the elliptic beta integration formula (evidently, in this relation t_7 and t_8 can be replaced by any other pair of parameters).

In the notation $V(\underline{t}) = V(\underline{t}; p, q)$, the transformation derived above reads

$$\text{(i)} \quad \boxed{V(\underline{t}) = \prod_{1\leq j<k\leq 4} \Gamma(t_jt_k, t_{j+4}t_{k+4})\, V(\underline{s}),}$$

where

$$\begin{cases} s_j = \varepsilon^{-1}t_j & (j = 1,2,3,4) \\ s_j = \varepsilon t_j & (j = 5,6,7,8) \end{cases} \qquad \varepsilon = \sqrt{\frac{t_1t_2t_3t_4}{pq}} = \sqrt{\frac{pq}{t_5t_6t_7t_8}}.$$

We repeat this transformation with s_3, s_4, s_5, s_6 playing the role of t_1, t_2, t_3, t_4 and permute parameters t_3, t_4 with t_5, t_6 in the result. This yields

$$\text{(ii)} \quad \boxed{V(\underline{t}) = \prod_{j,k=1}^{4} \Gamma(t_jt_{k+4})\; V(T^{\frac{1}{2}}/t_1, \ldots, T^{\frac{1}{2}}/t_4, U^{\frac{1}{2}}/t_5, \ldots, U^{\frac{1}{2}}/t_8),}$$

where $T = t_1t_2t_3t_4$ and $U = t_5t_6t_7t_8$.

We equate now the right-hand sides of relations (i) and (ii), express t_j parameters in terms of s_j and obtain

$$\text{(iii)} \quad \boxed{V(\underline{s}) = \prod_{1\leq j<k\leq 8} \Gamma(s_js_k)\, V(\sqrt{pq}/\underline{s}),}$$

where $\sqrt{pq}/\underline{s} = (\sqrt{pq}/s_1, \ldots, \sqrt{pq}/s_8)$.

Transformations (ii) and (iii) were proven by Rains [Rai1] in a straightforward manner using evaluations of determinants of theta functions on a dense set of parameters. However, as we just have seen [Spi7], they are mere repetitions of the key transformation (i).

It is convenient to set temporarily $t_j = e^{2\pi i x_j}(pq)^{1/4}$. We take vectors $x \in \mathbb{R}^8$ and denote as $x = \sum_{i=1}^8 x_ie_i$ their standard decomposition in the orthonormal basis e_i, $\langle e_i, e_j\rangle = \delta_{ij}$. Then the balancing condition implies $\sum_{i=1}^8 x_i = 0$ which defines a hyperplane Y orthogonal to the vector $e_1 + \ldots + e_8$. Considering reflections $x \to x - 2\langle v, x\rangle\, v/\langle v, v\rangle$ with respect to the hyperplane normal to some vector $v \in Y$, it is not difficult to see that the transformation of coordinates in (i) corresponds to the reflection with respect to the vector $v = (\sum_{i=5}^8 e_i - \sum_{i=1}^4 e_i)/2$, which has the canonical normalization of the length $\langle v, v\rangle = 2$.

The elliptic hypergeometric function $V(\underline{t})$ appeared for the first time in our paper [Spi4] together with the transformation (i). However, it was not recognized there that (i) and permutations of parameters $t_i \leftrightarrow t_j$ generate the exceptional E_7 Weyl group of symmetries: the function $V(\underline{t})/\prod_{1\leq k<l\leq 8}\sqrt{\Gamma(t_kt_l)}$ is simply invariant under these transformations. This fact was understood at the level of series in [KMNOY]

(where, actually, only the E_6 group is valid since one of the parameters is fixed to terminate the series) and for general function $V(\underline{t})$ in [Rai1].

For elliptic hypergeometric functions it is convenient to keep two systems of notation—the "multiplicative" system, described above, and the "additive" one [GR, Spi2, Spi3]. Therefore we define the function

$$v(\underline{g};\omega_1,\omega_2,\omega_3) \equiv V(e^{2\pi i g_1/\omega_2},\ldots,e^{2\pi i g_8/\omega_2};e^{2\pi i\omega_1/\omega_2},e^{2\pi i\omega_3/\omega_2}),$$

where $\sum_{j=1}^{8} g_k = 2\sum_{k=1}^{3}\omega_k$. It will be useful for a description of elliptic hypergeometric equation solutions.

6. Contiguous relations and the elliptic hypergeometric equation

The fundamental addition formula for elliptic theta functions can be written in the following form

$$\boxed{\theta\left(xw,\frac{x}{w},yz,\frac{y}{z};p\right)-\theta\left(xz,\frac{x}{z},yw,\frac{y}{w};p\right)=\frac{y}{w}\theta\left(xy,\frac{x}{y},wz,\frac{w}{z};p\right),}$$

where w,x,y,z are arbitrary complex variables. If we denote $y = t_1, w = t_2$, and $x = q^{-1}t_8$, then this identity for theta functions is equivalent to the following q-difference equation

$$\begin{aligned}\Delta(z,t_1,\ldots,t_5,qt_6,t_7,q^{-1}t_8)-\frac{\theta(t_6t_7^{\pm};p)}{\theta(q^{-1}t_8t_7^{\pm};p)}\Delta(z,\underline{t})\\ =\frac{t_6}{t_7}\frac{\theta(q^{-1}t_8t_6^{\pm};p)}{\theta(q^{-1}t_8t_7^{\pm};p)}\Delta(z,t_1,\ldots,t_6,qt_7,q^{-1}t_8),\end{aligned}$$

where $\Delta(z,\underline{t})=\prod_{k=1}^{8}\Gamma(t_kz^{\pm})/\Gamma(z^{\pm 2})$ is the V-function integrand. Integrating now this equality over z along the contour C, we derive the first contiguous relation

$$\begin{aligned}t_7\theta\left(t_8t_7/q,t_8/qt_7;p\right)V(qt_6,q^{-1}t_8)-(t_6\leftrightarrow t_7)\\ =t_7\theta\left(t_6t_7,t_6/t_7;p\right)V(\underline{t}),\end{aligned}$$

which was used in the first proof of the elliptic beta integral [Spi1]. Here $V(qt_6,q^{-1}t_8)$ denotes $V(\underline{t})$ with the parameters t_6 and t_8 replaced by qt_6 and $q^{-1}t_8$ respectively and $(t_6\leftrightarrow t_7)$ means permutation of the parameters in the preceding expression.

In the same way as in the case of series [SZ2], we can substitute symmetry transformation (iii) of the previous section into this equation and obtain the second contiguous relation

$$t_6\theta(t_7/qt_8;p)\prod_{k=1}^5\theta(t_6t_k/q;p)V(q^{-1}t_6,qt_8)-(t_6\leftrightarrow t_7)$$
$$=t_6\theta(t_7/t_6;p)\prod_{k=1}^5\theta(t_8/t_k;p)V(\underline{t}).$$

An appropriate combination of these two equations yields

$$b(\underline{t})\Big(U(qt_6,q^{-1}t_7)-U(\underline{t})\Big)+(t_6\leftrightarrow t_7)+U(\underline{t})=0,$$

where

$$U(\underline{t})=\frac{V(\underline{t})}{\prod_{k=1}^7\Gamma(t_kt_8,t_k/t_8)}$$

and the potential

$$b(\underline{t})=\frac{\theta(t_6/qt_8,t_6t_8,t_8/t_6;p)}{\theta(t_6/t_7,t_7/qt_6,t_6t_7/q;p)}\prod_{k=1}^5\frac{\theta(t_7t_k/q;p)}{\theta(t_8t_k;p)}$$
$$=\frac{\theta(qt_0/t_6,t_0t_6,t_0/t_6;p)}{\theta(t_6/t_7,qt_6/t_7,q/t_6t_7;p)}\prod_{k=1}^5\frac{\theta(q/t_7t_k;p)}{\theta(t_0t_k;p)}$$

(the second expression is obtained after setting $t_8=p^2t_0$) is a modular invariant elliptic function of variables $g_1,\ldots,g_7$ ($t_j=e^{2\pi i g_j/\omega_2}$).

If we substitute $t_6=az,t_7=a/z$ and replace $U(\underline{t})$ by some unknown function $f(z)$, then we obtain a q-difference equation of the second order called the *elliptic hypergeometric equation*:

$$\frac{\theta(az/qt_8,at_8z,t_8/az;p)}{\theta(z^2,1/qz^2;p)}\prod_{k=1}^5\theta(at_k/qz;p)\,(f(qz)-f(z))$$
$$+\frac{\theta(a/qt_8z,at_8/z,t_8z/a;p)}{\theta(1/z^2,z^2/q;p)}\prod_{k=1}^5\theta(at_kz/q;p)\,\big(f(q^{-1}z)-f(z)\big)$$
$$+\theta(a^2/q;p)\prod_{k=1}^5\theta(t_kt_8;p)\,f(z)=0,$$

where $t_8=p^2q^2/a^2\prod_{k=1}^5t_k$. We have found already one functional solution of this equation $U(\underline{t})$ in the restricted region of parameters. The second independent solution can be obtained after scaling any of the parameters $a,t_1,\ldots,t_5$ or z by p. We can replace also the standard elliptic gamma functions in the definition of $U(\underline{t})$ by the modified elliptic gamma functions and get new solutions of the elliptic hypergeometric equation. Indeed, we can rewrite the elliptic hypergeometric equation in the "additive" notation $t_j = e^{2\pi i g_j/\omega_2}$. Then the function

$$v^{mod}(\underline{g};\omega) = \int_{-\omega_3/2}^{\omega_3/2} \frac{\prod_{j=1}^8 G(g_j \pm x;\omega)}{G(\pm 2x;\omega)} \frac{dx}{\omega_2},$$

where $\sum_{j=1}^8 g_j = 2\sum_{k=1}^3 \omega_k$, defines its solution linearly independent from $V(\underline{t})$, provided we impose appropriate restrictions upon the parameters. Namely, we should line up sequences of the integrand's poles to the left or right of the line passing through the points $-\omega_3/2$ and $\omega_3/2$. Evidently, E_7 symmetry remains intact which follows from the fact that in the derivation of relevant properties of the $V(\underline{t})$ function we used only the first (boxed) equation for the elliptic gamma function $\Gamma(z;q,p)$ which coincides with one of the equations for $G(u;\omega)$. Simple computations yield the relation

$$v^{mod}(\underline{g};\omega) = \frac{2\omega_3 e^{2\pi i(P(0)-\sum_{j=1}^8 P(g_j))}}{\omega_2(\tilde{p};\tilde{p})_\infty(\tilde{r};\tilde{r})_\infty}\, v(\underline{g};\omega_1,-\omega_3,\omega_2),$$

showing that this solution is proportional to the modular transformation of the function $v(\underline{g};\omega_1,\omega_2,\omega_3)$.

Now we shift $g_{7,8} \to g_{7,8} + \sum_{k=1}^3 \omega_k$ and take the limit $\mathrm{Im}(\omega_3) \to \infty$ in such a way that $p, r \to 0$. Then our v^{mod}-function is reduced to

$$s(\underline{g};\omega_1,\omega_2) = \int_{\mathbb{L}} \frac{S(\pm 2u, -g_7 \pm u, -g_8 \pm u;\omega)}{\prod_{j=1}^6 S(g_j \pm u;\omega)} \frac{du}{\omega_2},$$

where $\sum_{j=1}^8 g_j = 0$. This is a q-hypergeometric function which is well defined for $|q| = 1$ and which provides a functional solution of the $p = 0$ degeneration of the elliptic hypergeometric equation.

It should be noticed that $V(\underline{t})$ satisfies not one, but much more equations of the derived type due to the permutational symmetry in all its parameters, including the equation obtained after the permutation of q and p. Most probably there is only one function satisfying all of them, since the linearly independent solutions break one of its symmetries, E_7 or $p \leftrightarrow q$.

At the level of q-hypergeometric functions, in the limit $p \to 0$ we obtain the equation investigated in detail by Gupta and Masson [GM]. They derived its functional solutions in the form of special combinations of non-terminating ${}_{10}\varphi_9$ series, the integral representation for which has been found earlier by Rahman [Rah] and to which our representation for $V(\underline{t})$ is reduced in the limit $p \to 0$.

In a similar way one can construct contiguous relations for elliptic ${}_{12}V_{11}$ series with $y = 1$. Denoting $\mathcal{E}(\underline{t}) \equiv {}_{12}V_{11}(t_0;t_1,\ldots,t_7;q,p)$, where

$\prod_{m=1}^7 t_m = t_0^3 q^2$ and $t_m = q^{-n}$, $n \in \mathbb{N}$, for some m, we have the first relation [SZ1, SZ2]

$$\mathcal{E}(\underline{t}) - \mathcal{E}(q^{-1}t_6, qt_7) = \frac{\theta(qt_0, q^2t_0, qt_7/t_6, t_6t_7/qt_0; p)}{\theta(qt_0/t_6, q^2t_0/t_6, t_0/t_7, t_7/qt_0; p)} \times \prod_{r=1}^5 \frac{\theta(t_r; p)}{\theta(qt_0/t_r; p)}\, \mathcal{E}(q^2t_0; qt_1, \ldots, qt_5, t_6, qt_7),$$

and the second one

$$\frac{\theta(t_7; p)\prod_{r=1}^5 \theta(t_rt_6/qt_0; p)}{\theta(t_6/qt_0, t_6/q^2t_0, t_6/t_7; p)}\, \mathcal{E}(q^2t_0; qt_1, \ldots, qt_5, t_6, qt_7) + \frac{\theta(t_6; p)\prod_{r=1}^5 \theta(t_rt_7/qt_0; p)}{\theta(t_7/qt_0, t_7/q^2t_0, t_7/t_6; p)}\, \mathcal{E}(q^2t_0; qt_1, \ldots, qt_6, t_7) = \frac{\prod_{r=1}^5 \theta(qt_0/t_r; p)}{\theta(qt_0, q^2t_0; p)}\, \mathcal{E}(\underline{t}).$$

These relations can also be obtained after application of the residue calculus similar to the one described above. For this it is necessary to take one of the parameters of $V(\underline{t})$ outside of the contour C and represent this elliptic hypergeometric function as a sum of an integral over C and of the residues picked up during this procedure. An accurate limit for one of the parameters converting the sum of residues into the terminating ${}_{12}V_{11}$ series brings in the needed contiguous relations, which take the described form after changing notation.

An appropriate combination of these two relations yields

$$\frac{\theta(t_6, t_0/t_6, qt_0/t_6; p)}{\theta(qt_6/t_7, t_6/t_7; p)} \prod_{r=1}^5 \theta(qt_0/t_7t_r; p) \left(\mathcal{E}(qt_6, q^{-1}t_7) - \mathcal{E}(\underline{t})\right) + \frac{\theta(t_7, t_0/t_7, qt_0/t_7; p)}{\theta(qt_7/t_6, t_7/t_6; p)} \prod_{r=1}^5 \theta(qt_0/t_6t_r; p) \left(\mathcal{E}(q^{-1}t_6, qt_7) - \mathcal{E}(\underline{t})\right) + \theta(qt_0/t_6t_7; p) \prod_{r=1}^5 \theta(t_r; p)\, \mathcal{E}(\underline{t}) = 0,$$

which is another form of the elliptic hypergeometric equation.

7. Applications in mathematical physics

The theory outlined above did not emerge from scratch. It appeared from long time developments in mathematical physics related to classical and quantum completely integrable systems. Below we list some of the known applications of elliptic hypergeometric series and integrals.

(1) Elliptic solutions of the Yang-Baxter equation (elliptic $6j$-symbols) sequentially derived by Baxter [Bax], Andrews, Baxter and Forrester [ABF], Date, Jimbo, Kuniba, Miwa, and Okado [DJKMO] appear to combine into terminating ${}_{12}V_{11}$ series with special discrete values of parameters, as it was shown by Frenkel and Turaev in their profound paper [FT]. For a recent work in this direction including the algebraic aspects of the elliptic $6j$-symbols, see [Kon, Rai2, Ros3, Ros4]. Since solvable two-dimensional statistical mechanics models are related to the conformal field theory [BM, Zub], it is natural to expect that elliptic hypergeometric functions will emerge there as well.

(2) In a joint work with Zhedanov [SZ1], the terminating ${}_{12}V_{11}$ series with arbitrary continuous parameters were discovered as solutions of the linear problem for some classical integrable system. More precisely, these series emerged from self-similar solutions of the discrete time chain associated with biorthogonal rational functions which generalizes ordinary and relativistic discrete-time Toda chains.

(3) As shown by Kajiwara, Masuda, Noumi, Ohta, and Yamada [KMNOY], Sakai's elliptic Painlevé equation [Sak] has a solution expressed in terms of the terminating ${}_{12}V_{11}$ series. This observation follows from the reduction of corresponding nonlinear second order finite difference equation to the elliptic hypergeometric equation. Therefore, $V(\underline{t})$ also provides its solution. Moreover, the function $v(\underline{g};\omega_1,-\omega_3,\omega_2)$, well defined in the $|q|=1$ region, plays a similar role [Spi7] since it defines an independent solution of the elliptic hypergeometric equation with the E_7 symmetry. More complicated solutions of this equation expressed in terms of the multiple elliptic hypergeometric integrals were presented by Rains at this workshop [Rai3].

(4) Elliptic hypergeometric functions provide particular solutions of the finite difference (relativistic) analogues of the elliptic Calogero-Sutherland type models [Spi7]. This application is outline below and in the last section.

The original investigations of completely integrable many particles systems on the line (or circle) by Calogero, Sutherland and Moser were continued by Olshanetsky and Perelomov [OP] who showed that such models are naturally associated with the root systems. Relativistic (or finite-difference) generalizations of these models have been discovered by Ruijsenaars [Rui1] who worked out the A_n root system case in detail. The corresponding eigenvalue problem is also known to be related to the Macdonald polynomials [Mac]. Inozemtsev [Ino] has investigated the most general BC_n root system extension of the Heun equation absorbing previously derived differential operator models. In a further step, van Diejen [Die] has unified Inozemtsev and Ruijsenaars models by coming up with even more general integrable model, which was investigated in detail by Komori and Hikami [KH]. A special degeneration of this model to the trigonometric level corresponds to the Koornwinder polynomials [Koo].

The Hamiltonian of the van Diejen model has the form

$$\mathcal{H} = \sum_{j=1}^{n} \Big(A_j(\underline{z}) T_j + A_j(\underline{z}^{-1}) T_j^{-1} \Big) + u(\underline{z}),$$

where $u(\underline{z})$ is some complicated explicit combination of theta functions, $T_j f(\ldots, z_j, \ldots) = f(\ldots, q z_j, \ldots)$, and

$$A_j(\underline{z}) = \frac{\prod_{m=1}^{8} \theta(t_m z_j; p)}{\theta(z_j^2, q z_j^2; p)} \prod_{\substack{k=1 \\ \neq j}}^{n} \frac{\theta(t z_j z_k, t z_j z_k^{-1}; p)}{\theta(z_j z_k, z_j z_k^{-1}; p)}.$$

If we impose the constraint $t^{2n-2}\prod_{m=1}^{8} t_m = p^2 q^2$, then the operator $\mathcal{H}$ can be rewritten in the form

$$\mathcal{D} = \sum_{j=1}^{n} \Big(A_j(\underline{z})(T_j - 1) + A_j(\underline{z}^{-1})(T_j^{-1} - 1) \Big)$$

up to some additive constant independent on variables z_j (for details, see [Die, KH, Rui4]).

The standard eigenvalue problem, $\mathcal{D} f(\underline{z}) = \lambda f(\underline{z})$, in the univariate case $n = 1$ looks like

$$\begin{aligned} &\frac{\prod_{j=1}^{8} \theta(t_j z; p)}{\theta(z^2, q z^2; p)} (f(qz) - f(z)) \\ &\qquad + \frac{\prod_{j=1}^{8} \theta(t_j z^{-1}; p)}{\theta(z^{-2}, q z^{-2}; p)} (f(q^{-1} z) - f(z)) = \lambda f(z). \end{aligned}$$

Comparing it with the elliptic hypergeometric equation in the form derived in [Spi4], which will be described in the next section, we see that they coincide for a restricted choice of parameters $t_6 = t_5/q$ and a special eigenvalue for the Hamiltonian $\mathcal{D}$, $\lambda = -\kappa_\mu$ (a similar observation has been done by Komori).

However, connections between the elliptic hypergeometric functions and Calogero-Sutherland type models are deeper than it is just indicated. Let us introduce the inner product

$$\langle \varphi, \psi \rangle = \kappa \int_C \frac{\prod_{m=1}^8 \Gamma(t_m z^{\pm})}{\Gamma(z^{\pm 2})}\, \varphi(z)\, \psi(z)\, \frac{dz}{z},$$

where contour C separates sequences of the kernel's poles converging to $z = 0$ from those diverging to infinity. Additionally, we impose restrictions upon values of t_j and functions $\varphi(z), \psi(z)$, such that we can scale the contour C by q and q^{-1} with respect to the point $z = 0$ without crossing any poles. Under these conditions, the operator $\mathcal{D}$ formally becomes hermitian with respect to the taken inner product: $\langle \varphi, \mathcal{D}\psi \rangle = \langle \mathcal{D}\varphi, \psi \rangle$. However, this property is not unique—the weight function in the inner product can be multiplied by any elliptic function $\rho(z)$, $\rho(qz) = \rho(z)$, with an accompanying change of the contour of integration.

In a trivial way, $f(z) = 1$ is an eigenfunction of $\mathcal{D}$ with the eigenvalue $\lambda = 0$ (actually, it solves simultaneously two such equations—the second equation is obtained by permutation of q and p). Evidently, the norm of this eigenfunction equals to the elliptic hypergeometric function, $\|1\|^2 = V(\underline{t})$. This relation holds for $|p|, |q| < 1$. If we change the integration variable in the taken inner product $z = e^{2\pi i u/\omega_2}$, then, instead of $f(z) = 1$, we could have chosen as an $\lambda = 0$ eigenfunction of $\mathcal{D}$ (where the operator T is acting now as a shift, $Tv(u) = v(u+\omega_1)$) any function $h(u)$ with the property $h(u+\omega_1) = h(u)$, but then the normalization of this function would not be related to $V(\underline{t})$ in a simple way. For a special choice of this $h(u)$, we can obtain $\|h\|^2 = v^{mod}(\underline{g}; \omega)$, the modified elliptic hypergeometric function for which we can take $|q| = 1$. Equivalently, we could have changed the inner product by replacing the standard elliptic gamma functions by their modified version and considering the pair of equations $\mathcal{D}v(u) = 0$ and its $\omega_1 \leftrightarrow \omega_2$ permuted partner. Similar picture holds in the multivariable case considered in the end of this paper.

Because of these relations between $V(\underline{t})$ and the Calogero-Sutherland type models, it is natural to expect that elliptic hypergeometric functions will play a major role in the solution of the standard eigenvalue problem for the operator $\mathcal{D}$. In particular, we conjecture that the E_7 group of symmetries of $V(\underline{t})$ can be lifted to E_8 at the level of unconstrained Hamiltonian $\mathcal{H}$ and that there is some direct relation of this model with the elliptic Painlevé equation (for this it would be desirable to understand an analogue of the Painlevé-Calogero correspondence principle [LO, Man] at the level of finite difference equations).

8. Biorthogonal functions

8.1. Difference equation and three term recurrence relation. For $n = 0, 1, \ldots,$ we define a sequence of functions [Spi4]

$$R_n(z; q, p) = {}_{12}V_{11}\left(\frac{t_3}{t_4}; \frac{q}{t_0 t_4}, \frac{q}{t_1 t_4}, \frac{q}{t_2 t_4}, t_3 z, \frac{t_3}{z}, q^{-n}, \frac{Aq^{n-1}}{t_4}; q, p\right),$$

where $A = \prod_{m=0}^4 t_m$. They solve the elliptic hypergeometric equation rewritten in the form

$$\mathcal{D}_\mu f(z) = 0, \quad \mathcal{D}_\mu = V_\mu(z)(T-1) + V_\mu(z^{-1})(T^{-1}-1) + \kappa_\mu,$$

where $Tf(z) = f(qz)$ and

$$\begin{aligned} V_\mu(z) &= \theta\Big(\frac{pq\mu z}{t_4}, \frac{pq^2 z}{A\mu}, \frac{t_4 z}{q}; p\Big) \frac{\prod_{r=0}^4 \theta(t_r z; p)}{\theta(z^2, qz^2; p)}, \\ \kappa_\mu &= \theta\Big(\frac{A\mu}{q t_4}, \mu^{-1}; p\Big) \prod_{r=0}^{3} \theta(t_r t_4; p), \end{aligned}$$

provided we quantize one of the parameters $\boxed{\mu = q^n}$ ("the spectrum"). Equivalently, this equation can be rewritten as a generalized eigenvalue problem

$$\mathcal{D}_\eta f(z) = \lambda \mathcal{D}_\xi f(z)$$

with the spectral variable lying on the elliptic curve

$$\lambda = \frac{\theta(\frac{\mu A \eta}{q t_4}, \frac{\mu}{\eta}; p)}{\theta(\frac{\mu A \xi}{q t_4}, \frac{\mu}{\xi}; p)}, \quad \xi, \eta \in \mathbb{C}, \quad \xi \neq \eta p^n, \frac{q t_4}{A\eta} p^n,\ n \in \mathbb{Z},$$

where ξ and η are gauge parameters. Out of this representation one obtains formal biorthogonality $\langle T_n, R_m \rangle = 0$ for $n \neq m$, where $\langle \cdot, \cdot \rangle$ is some inner product and $T_n(z; q, p)$ is a solution of a dual generalized eigenvalue problem.

From the elliptic hypergeometric equation one can derive also the three-term recurrence relation

$$\begin{aligned} &(\gamma(z) - \alpha_{n+1})\rho(Aq^{n-1}/t_4)\Big(R_{n+1}(z; q, p) - R_n(z; q, p)\Big) \\ &\quad + \ (\gamma(z) - \beta_{n-1})\rho(q^{-n})\Big(R_{n-1}(z; q, p) - R_n(z; q, p)\Big) \\ &\quad\quad + \ \delta\big(\gamma(z) - \gamma(t_3)\big) R_n(z; q, p) = 0, \end{aligned}$$

with the initial conditions $R_{-1}=0,\ R_0=1$ and

$$\rho(x)=\frac{\theta(x,\frac{t_3}{t_4x},\frac{qt_3}{t_4x},\frac{qx}{t_0t_1},\frac{qx}{t_0t_2},\frac{qx}{t_1t_2},\frac{q^2\eta x}{A},\frac{q^2x}{A\eta};p)}{\theta(\frac{qt_4x^2}{A},\frac{q^2t_4x^2}{A};p)},$$

$$\delta=\theta\left(\frac{q^2t_3}{A},\frac{q}{t_0t_4},\frac{q}{t_1t_4},\frac{q}{t_2t_4},t_3\eta,\frac{t_3}{\eta};p\right),$$

$$\gamma(z)=\frac{\theta(z\xi,z/\xi;p)}{\theta(z\eta,z/\eta;p)},\quad \alpha_n=\gamma(q^n/t_4),\quad \beta_n=\gamma(q^{n-1}A).$$

Since the whole z-dependence in this relation is concentrated in the $\gamma(z)$ function, $R_n(z;q,p)$ are rational functions of $\gamma(z)$ with poles at $\gamma(z)=\alpha_1,\dots,\alpha_n$.

From the general theory of biorthogonal rational functions [Zhe1] it follows that $R_n(z;q,p)$ can be orthogonal to a rational function $T_n(z;q,p)$ with poles at $\gamma(z)=\beta_1,\dots,\beta_n$. The involution $t_4\to pq/A$ permutes α_n and β_n, therefore the dual functions are obtained after an application of this transformation to $R_n(z;q,p)$:

$$T_n(z;q,p)={}_{12}V_{11}\left(\frac{At_3}{q};\frac{A}{t_0},\frac{A}{t_1},\frac{A}{t_2},t_3z,\frac{t_3}{z},q^{-n},\frac{Aq^{n-1}}{t_4};q,p\right),$$

where the p-dependence in parameters drops out due to the total ellipticity property (in particular, we have $R_n(pz;q,p)=R_n(z;q,p)$).

8.2. **Two-index biorthogonality.** Let us denote the operator $\mathcal{D}_\mu$ introduced above as $\mathcal{D}_\mu^{q,p}$. Then the product $R_{nm}(z)\equiv R_n(z;q,p)\cdot R_m(z;p,q)$ solves two generalized eigenvalue problems

$$\mathcal{D}_\mu^{q,p}f(z)=0,\qquad \mathcal{D}_\mu^{p,q}f(z)=0$$

with the spectrum $\boxed{\mu=q^np^m.}$ Similar property is valid for the dual product $T_{nm}(z)\equiv T_n(z;q,p)\cdot T_m(z;p,q)$ for a different choice of parameters in $\mathcal{D}_\mu^{q,p}$.

Theorem 4. (Two-index biorthogonality [Spi4]**)**

If we denote

$$\Delta(z,\underline{t})=\frac{(q;q)_\infty(p;p)_\infty}{4\pi i}\frac{\prod_{m=0}^4\Gamma(t_mz,t_mz^{-1})}{\Gamma(z^2,z^{-2},Az,Az^{-1})},$$

$$\mathcal{N}(\underline{t})=\frac{\prod_{0\le m<k\le 4}\Gamma(t_mt_k)}{\prod_{m=0}^4\Gamma(At_m^{-1})},$$

where $|q|,|p|<1,\ |t_m|<1,\ |pq|<|A|$, *then*

$$\boxed{\int_{C_{mn,kl}}T_{nl}(z)R_{mk}(z)\Delta(z,\underline{t})\frac{dz}{z}=h_{nl}\,\mathcal{N}(\underline{t})\,\delta_{mn}\,\delta_{kl},}$$

where $C_{mn,kl}$ is a contour separating points

$$t_j p^a q^b \, (j=0,1,2,3), \;\; t_4 p^{a-k} q^{b-m}, p^{a+1-l} q^{b+1-n}/A, \quad a,b \in \mathbb{N},$$

from their $z \to z^{-1}$ reciprocals and normalization constants

$$\begin{aligned} h_{nl} &= h_n(q,p)\cdot h_m(p,q), \\ h_n(q,p) &= \frac{\theta(A/qt_4;p)(q,qt_3/t_4,t_0t_1,t_0t_2,t_1t_2,At_3)_n\, q^{-n}}{\theta(Aq^{2n}/t_4q;p)(1/t_3t_4,t_0t_3,t_1t_3,t_2t_3,A/qt_3,A/qt_4)_n}. \end{aligned}$$

Only for $k=l=0$ there exists the $p \to 0$ limit and functions $R_n(z;q,0)$ and $T_n(z;q,0)$ coincide with the Rahman's family of continuous ${}_{10}\varphi_9$ biorthogonal rational functions [Rah]. Note also that only for $k=l=0$ or $n=m=0$ we have rational functions of some argument depending on z; the general functions $R_{nm}(z)$ and $T_{nm}(z)$ should be considered as some meromorphic functions of z with essential singularities at $z=0$ and $z=\infty$.

For some quantized values of z and one of the parameters t_j the functions $R_n(z;q,p)$ and $T_n(z;q,p)$ are reduced to the finite dimensional set of biorthogonal rational functions constructed by Zhedanov and the author in [SZ1]. They generalize to the elliptic level Wilson's family of discrete very well poised ${}_{10}\varphi_9$ biorthogonal functions [Wil]. As described by Zhedanov at this workshop [Zhe2], these functions have found nice applications within the general Padé interpolation scheme.

Functional solutions of the elliptic hypergeometric equation open the road to construction of the associated biorthogonal functions following the procedure described in [IR] and this is one of the interesting open problems for the future. A terminating continued fraction generated by the three term recurrence relation described above has been calculated in [SZ2]. It is expressed in terms of a terminating ${}_{12}V_{11}$ series and, again, the function $V(\underline{t})$ is expected to appear in the description of non-terminating convergent continued fractions generalizing q-hypergeometric examples of [GM].

8.3. **The unit circle case.** In order to describe biorthogonal functions for which the measure is defined by the modified elliptic beta integral, we parametrize $t_j = e^{2\pi i g_j/\omega_2}$ and introduce new notation for the functions $R_n(z;q,p)$:

$$\begin{aligned} r_n(u;\omega_1,\omega_2,\omega_3) = {}_{12}V_{11}\Big(& e^{2\pi i(g_3-g_4)/\omega_2}; e^{2\pi i(\omega_1-g_0-g_4)/\omega_2}, \\ & e^{2\pi i(\omega_1-g_1-g_4)/\omega_2}, e^{2\pi i(\omega_1-g_2-g_4)/\omega_2}, e^{2\pi i(\mathcal{A}+(n-1)\omega_1-g_4)/\omega_2}, \\ & e^{-2\pi i n\omega_1/\omega_2}, e^{2\pi i(g_3+u)/\omega_2}, e^{2\pi i(g_3-u)/\omega_2}; e^{2\pi i\omega_1/\omega_2}, e^{2\pi i\omega_3/\omega_2}\Big), \end{aligned}$$

where $\mathcal{A} = \sum_{j=0}^{4} g_j$. Similarly, we redenote the functions $T_n(z; q, p)$ as $s_n(u; \omega_1, \omega_2, \omega_3)$.

The $q \leftrightarrow p$ symmetric situation (the standard set of biorthogonal functions with $|p|, |q| < 1$) is defined as the $\omega_1 \leftrightarrow \omega_3$ symmetric product of these functions:

$$R_{nm}(e^{2\pi i u/\omega_2}) = r_n(u; \omega_1, \omega_2, \omega_3) \cdot r_m(u; \omega_3, \omega_2, \omega_1),$$

with a similar relation for $T_{nm}(e^{2\pi i u/\omega_2})$. As described above, we have the biorthogonality relations $\langle T_{nl}, R_{mk}\rangle = h_{nl}\delta_{mn}\delta_{kl}$, where $\langle 1, 1\rangle = 1$ coincides with the normalized standard elliptic beta integral with a special contour of integration $C_{mn,kl}$ and

$$h_{nl} = h_n(\omega_1, \omega_2, \omega_3) h_l(\omega_3, \omega_2, \omega_1)$$

with $h_n(\omega_1, \omega_2, \omega_3) \equiv h_n(q, p)$. These functions are modular invariant: $r_n(u; \omega_1, \omega_2, \omega_3) = r_n(u; \omega_1, -\omega_3, \omega_2)$, $h_n(\omega_1, \omega_2, \omega_3) = h_n(\omega_1, -\omega_3, \omega_2)$.

In the unit circle case we define functions

$$\begin{aligned} r^{mod}_{nm}(u) &= r_n(u; \omega_1, \omega_2, \omega_3) \cdot r_m(u; \omega_2, \omega_1, \omega_3), \\ s^{mod}_{nm}(u) &= s_n(u; \omega_1, \omega_2, \omega_3) \cdot s_m(u; \omega_2, \omega_1, \omega_3), \end{aligned}$$

which are now symmetric with respect to the permutations $\omega_2 \leftrightarrow \omega_1$ and $n \leftrightarrow m$. These functions satisfy the biorthogonality relations $\langle s^{mod}_{nl}, r^{mod}_{mk}\rangle = h^{mod}_{nl}\delta_{mn}\delta_{kl}$, where $\langle 1, 1\rangle = 1$ coincides with the normalized modified elliptic beta integral with the integration contour $\tilde{C}_{mn,kl}$ chosen in an appropriate way and

$$h^{mod}_{nl} = h_n(\omega_1, \omega_2, \omega_3) \cdot h_l(\omega_2, \omega_1, \omega_3).$$

In sharp difference from the previous case, the limit $p \to 0$ (taken in such a way that simultaneously $r \to 0$, i.e. $\mathrm{Im}(\omega_3/\omega_1)$, $\mathrm{Im}(\omega_3/\omega_2) \to +\infty$) exists for all values of indices n, l, k, m and we obtain:

$$\begin{aligned} r_{nm}(u; \omega_1, \omega_2) = {}_{10}W_9&\Big(e^{2\pi i(g_3-g_4)/\omega_2}; \ldots, e^{2\pi i(g_3-u)/\omega_2}; q, q\Big) \\ &\times {}_{10}W_9\Big(e^{2\pi i(g_3-g_4)/\omega_1}; \ldots, e^{2\pi i(g_3-u)/\omega_1}; \tilde{q}^{-1}, \tilde{q}^{-1}\Big). \end{aligned}$$

Their partners from the dual space $s_{nm}(u; \omega_1, \omega_2)$ are defined in a similar way. These functions $r_{nm}(u; \omega_1, \omega_2)$ and $s_{nm}(u; \omega_1, \omega_2)$ are not rational functions of some particular combination of the variable u for $n, m \neq 0$. They satisfy the two-index biorthogonality relations

$$\langle r_{nl}, s_{mk}\rangle = \nu_{nl}\delta_{mn}\delta_{kl},$$

where ν_{nl} are obtained from h^{mod}_{nl} after setting $\mathrm{Im}(\omega_3/\omega_1)$, $\mathrm{Im}(\omega_3/\omega_2) \to +\infty$ and $\langle 1, 1\rangle = 1$ coincides with the normalized "unit circle" partner of the Rahman's integral [Sto] with a special contour of integration.

Further simplification of these relations to the Askey-Wilson polynomials level is highly non-trivial due to some problems with the convergence of the integral and requires a thorough investigation. In a similar way it is possible to define unit circle partners of the Rains' multivariable generalization of the author's univariate biorthogonal functions [Rai2] as well as their limiting two-index q-biorthogonal functions.

9. Multiple elliptic beta integrals

9.1. General definition. Multiple integrals

$$\int_D \Delta(u_1,\ldots,u_n)\,du_1\cdots du_n,$$

where $D\subset\mathbb{C}^n$ are some n-dimensional cycles, are called elliptic hypergeometric integrals if $\Delta(u_1,\ldots,u_n)$ are meromorphic functions of $u_1,\ldots,u_n$ satisfying the following system of equations

$$\Delta(u_1,\ldots,u_k+\omega_1,\ldots,u_n)=h^{(k)}(u_1,\ldots,u_n)\,\Delta(u_1,\ldots,u_n),$$

where $h^{(k)}(\underline{u})$, $k=1,\ldots,n$, are elliptic functions of all u_j, i.e.,

$$h^{(k)}(u_j+\omega_2)=h^{(k)}(u_j+\omega_3)=h^{(k)}(u),\quad \mathrm{Im}(\omega_2/\omega_3)\neq 0.$$

This is a "broad" definition of the integrals introduced in [Spi4]; one can make it "narrow" by tripling the number of equations for $\Delta(\underline{u})$ using the shifts by all quasiperiods ω_i.

In order to describe general possible forms of the integrand, we need an elliptic extension of the Ore-Sato theorem on the general form of terms in plain hypergeometric series (see, e.g., [GGR]). For all "good" known elliptic hypergeometric integrals, the kernels $\Delta(\underline{u})$ are equal to ratios of elliptic gamma functions $\Gamma(z;q,p)$ with an integer power dependence on the variables $z_j=e^{2\pi i u_j/\omega_2}$. However, in general case we can multiply the integrands by elliptic functions of all u_j's with the periods ω_2,ω_3 which do not have such a representation.

Multiple elliptic hypergeometric series are defined in a similar way. It is simply necessary to replace integrals by discrete sums over some sublattices of $u_1,\ldots,u_n\in\mathbb{Z}^n$ keeping other properties of $\Delta(\underline{u})$. We shall not consider them in the present review.

The most interesting elliptic hypergeometric integrals are related to multiple generalizations of the elliptic beta integral, which are split formally into three different groups. Type I integrals contain $2n+3$ free parameters and bases p and q and their proofs use in one or another way analytical continuation procedure over discrete values of parameters. Type II integrals contain less than $2n+3$ free parameters and they can be proved by purely algebraic means on the basis of type I integrals.

Finally, type III elliptic beta integrals arise through computations of n-dimensional determinants with entries composed of one-dimensional integrals. It goes without saying that all these integrals have their partners expressed in terms of the modified elliptic gamma function.

9.2. **Integrals for the root system C_n.** In order to define n-dimensional type I elliptic beta integral for the root system C_n (abbreviated as the C_I integral), we take bases $|p|, |q| < 1$ and parameters $t_1, \ldots, t_{2n+4} \in \mathbb{C}$ such that $\prod_{j=1}^{2n+4} t_j = pq$ and $|t_1|, \ldots, |t_{2n+4}| < 1$.

Theorem 5. (Type I C_n elliptic beta integral [DS2]**)**

$$\boxed{\kappa_n^C \int_{\mathbb{T}^n} \prod_{j=1}^n \frac{\prod_{i=1}^{2n+4} \Gamma(t_i z_j^{\pm})}{\Gamma(z_j^{\pm 2})} \prod_{1\leq i<j\leq n} \frac{1}{\Gamma(z_i^{\pm} z_j^{\pm})} \frac{dz}{z} = \prod_{1\leq i<j\leq 2n+4} \Gamma(t_i t_j),}$$

where $\Gamma(z) \equiv \Gamma(z; q, p)$ *and*

$$\boxed{\kappa_n^C = \frac{(p;p)_\infty^n (q;q)_\infty^n}{(2\pi i)^n 2^n n!}.}$$

Different complete proofs of this formula were given by Rains [Rai1] and the author [Spi6]. In the limit $p \to 0$ it is reduced to one of the Gustafson results [Gus1]. Its modified elliptic gamma function partner has been established by the author [Spi6] together with its q-degeneration valid for $|q| \leq 1$ (which we skip for brevity).

Type II integral for this root system (abbreviated as the C_{II} integral) depends on seven parameters t and t_m, $m = 1, \ldots, 6$, and bases q, p constrained by one relation. It can be derived as a consequence of the C_I integral.

Theorem 6. (Type II C_n elliptic beta integral [DS1]**)**

Let nine complex parameters $t, t_m (m = 1, \ldots, 6), p$ and q be constrained by the conditions $|p|, |q|, |t|, |t_m| < 1$, and $t^{2n-2} \prod_{m=1}^6 t_m = pq$. Then,

$$\boxed{\begin{aligned} \kappa_n^C \int_{\mathbb{T}^n} \prod_{1\leq j<k\leq n} \frac{\Gamma(t z_j^{\pm} z_k^{\pm})}{\Gamma(z_j^{\pm} z_k^{\pm})} \prod_{j=1}^n \frac{\prod_{m=1}^6 \Gamma(t_m z_j^{\pm})}{\Gamma(z_j^{\pm 2})} \frac{dz}{z} \\ = \prod_{j=1}^n \left(\frac{\Gamma(t^j)}{\Gamma(t)} \prod_{1\leq m<s\leq 6} \Gamma(t^{j-1} t_m t_s) \right). \end{aligned}}$$

This is an elliptic analogue of the Selberg integral which appears after a number of reductions, the first step being the $p \to 0$ limit leading to one of the Gustafson's integrals [Gus2]. In order to take this limit it is necessary to express t_6 in terms of other parameters and remove the multipliers pq by the inversion formula for $\Gamma(z;q,p)$ (see [DS2]). During this procedure the integral takes a less symmetric form—in the given form it has the explicit S_6 symmetry in parameters (see [Rai1]). For the modified version of this integration formula valid for $|q|\leq 1$ and its q-degeneration, see [DS4].

Presently the author knows only one type III elliptic beta integral [Spi4]. It is ascribed to the C_n root system (we abbreviate it as the C_{III} integral) and it is computed by evaluation of a determinant of the univariate elliptic beta integrals which is reduced to the computation of the Warnaar's determinant [War1]. We skip it for brevity, but it is expected that there are much more such integrals due to the universality of the method used for their derivation (see, e.g., [TV]) and existence of several nice exact determinant evaluations for elliptic theta functions.

9.3. **Integrals for the root system A_n.** Classification of the A_n elliptic beta integrals follows the same line as in the C_n case. We start from the description of the simplest type I integral introduced by the author in [Spi4], which we symbolize as $A_I^{(1)}$.

Theorem 7. (The $A_I^{(1)}$ integral [Spi4]**)**

$$\boxed{\begin{aligned}\kappa_n^A\int_{\mathbb{T}^n}\prod_{1\leq j<k\leq n+1}\frac{1}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j)}\prod_{j=1}^{n+1}\prod_{m=1}^{n+2}\Gamma(s_mz_j,t_mz_j^{-1})\,\frac{dz}{z}\\ =\prod_{m=1}^{n+2}\Gamma(Ss_m^{-1},Tt_m^{-1})\prod_{k,m=1}^{n+2}\Gamma(s_kt_m),\end{aligned}}$$

where $\boxed{z_1z_2\cdots z_{n+1}=1}$ *and*

$$\boxed{\kappa_n^A=\frac{(p;p)_\infty^n(q;q)_\infty^n}{(2\pi i)^n(n+1)!}}$$

with the parameters satisfying the constraints $|t_m|,|s_m|<1$, $m=1,\ldots,n+2$, *and* $\boxed{ST=pq,}$ $S=\prod_{m=1}^{n+2}s_m$, $T=\prod_{m=1}^{n+2}t_m$.

For complete proofs of this formula, see [Rai1, Spi6]. Here we have a split of $2n+4$ parameters (homogeneous in the C_n case) with one constraint into two homogeneous groups with $n+2$ entries in each group. The $p\to 0$ limiting value of this integral was derived by Gustafson [Gus1]. The unit circle analogue together with the appropriate q-degeneration valid for $|q|\leq 1$ were derived in [Spi6]. Another type I A_n integral is described below.

There are several type II integrals on the A_n root system, the first of which we abbreviate as $A_{II}^{(1)}$. For its description we define the kernel

$$\boxed{\Delta_{II}^{(1)}(\underline{z}) = \prod_{1\leq i<j\leq n+1} \frac{\Gamma(tz_iz_j)}{\Gamma(z_iz_j^{-1}, z_i^{-1}z_j)} \prod_{j=1}^{n+1}\prod_{k=1}^{n+1}\Gamma(t_kz_j^{-1})\prod_{i=1}^{4}\Gamma(s_iz_j),}$$

where $t^{n-1}\prod_{k=1}^{n+1}t_k\prod_{i=1}^{4}s_i = pq$ and $\prod_{j=1}^{n+1}z_j = 1$.

Theorem 8. (The $A_{II}^{(1)}$ integral [Spi4]**)**

As a consequence of the C_I and $A_I^{(1)}$ integration formulas, we have for odd n

$$\boxed{\begin{aligned}\kappa_n^A\int_{\mathbb{T}^n}\Delta_{II}^{(1)}(\underline{z})\frac{dz}{z} &= \frac{\Gamma(t^{\frac{n+1}{2}}, A)}{\Gamma(t^{\frac{n+1}{2}}A)}\prod_{k=1}^{n+1}\prod_{i=1}^{4}\Gamma(t_ks_i)\\ &\times\prod_{1\leq j<k\leq n+1}\Gamma(tt_jt_k)\prod_{1\leq i<m\leq 4}\Gamma(t^{\frac{n-1}{2}}s_is_m).\end{aligned}}$$

where $A=\prod_{k=1}^{n+1}t_k$.

For even n, we have

$$\boxed{\begin{aligned}\kappa_n^A\int_{\mathbb{T}^n}\Delta_{II}^{(1)}(\underline{z})\frac{dz}{z} &= \Gamma(A)\prod_{k=1}^{n+1}\prod_{i=1}^{4}\Gamma(t_ks_i)\\ &\times\prod_{1\leq j<k\leq n+1}\Gamma(tt_jt_k)\prod_{i=1}^{4}\frac{\Gamma(t^{\frac{n}{2}}s_i)}{\Gamma(t^{\frac{n}{2}}As_i)}.\end{aligned}}$$

These formulas contain only $n+5$ free parameters. In the $p\to 0$ limit they are reduced to the main result of [GuR].

We abbreviate the second type II A_n integral as $A_{II}^{(2)}$. For its description we need the kernel

$$\boxed{\Delta_{II}^{(2)}(\underline{z}) = \prod_{1\leq i<j\leq n+1}\frac{\Gamma(tz_iz_j, sz_i^{-1}z_j^{-1})}{\Gamma(z_iz_j^{-1}, z_i^{-1}z_j)}\prod_{j=1}^{n+1}\prod_{k=1}^{3}\Gamma(t_kz_j, s_kz_j^{-1}),}$$

where ten variables $p, q, t, s, t_1, t_2, t_3, s_1, s_2, s_3\in\mathbb{C}$ satisfy one constraint $(ts)^{n-1}\prod_{k=1}^{3}t_ks_k = pq$.

Theorem 9. (The $A_{II}^{(2)}$ integral [Spi4]**)**

As a consequence of the $A_I^{(1)}$, C_I, and C_{II} integration formulas, we have an additional type II elliptic beta integral for the A_n root system. For odd n, we have

$$\boxed{\begin{aligned}\kappa_n^A\int_{\mathbb{T}^n}\Delta_{II}^{(2)}(\underline{z})\frac{dz}{z}=\Gamma(t^{\frac{n+1}{2}},s^{\frac{n+1}{2}})\prod_{1\leq i<k\leq 3}\Gamma(t^{\frac{n-1}{2}}t_it_k,s^{\frac{n-1}{2}}s_is_k)\\ \times\prod_{j=1}^{(n+1)/2}\prod_{i,k=1}^{3}\Gamma((ts)^{j-1}t_is_k)\\ \times\prod_{j=1}^{(n-1)/2}\left(\Gamma((ts)^j)\prod_{1\leq i<k\leq 3}\Gamma(t^{j-1}s^jt_it_k,t^js^{j-1}s_is_k)\right).\end{aligned}}$$

For even n, we have

$$\boxed{\begin{aligned}\kappa_n^A\int_{\mathbb{T}^n}\Delta_{II}^{(2)}(\underline{z})\frac{dz}{z}=\prod_{i=1}^{3}\Gamma(t^{\frac{n}{2}}t_i,s^{\frac{n}{2}}s_i)\\ \times\Gamma(t^{\frac{n}{2}-1}t_1t_2t_3,s^{\frac{n}{2}-1}s_1s_2s_3)\prod_{j=1}^{n/2}\Bigg(\Gamma((ts)^j)\\ \times\prod_{i,k=1}^{3}\Gamma((ts)^{j-1}t_is_k)\prod_{1\leq i<k\leq 3}\Gamma(t^{j-1}s^jt_it_k,t^js^{j-1}s_is_k)\Bigg).\end{aligned}}$$

In this and previous theorems we assume constraints on the parameters guaranteeing that all sequences of integrands' poles converging to zero (or their reciprocals) lie within (or outside) of $\mathbb{T}$.

This theorem formulas contain only seven free parameters. In the $p\to 0$ limit we obtain one of the integrals in [Gus2].

Recently, Warnaar and the author have found a complementary type I elliptic beta integral for the A_n root system.

Theorem 10. (The $A_I^{(2)}$ integral [SW]**)**

$$\boxed{\begin{aligned}\kappa_n^A\int_{\mathbb{T}^n}\prod_{1\leq i<j\leq n+1}\frac{\Gamma(Sz_i^{-1}z_j^{-1})}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j)}\prod_{j=1}^{n+1}\frac{\prod_{k=1}^n\Gamma(t_kz_j)\prod_{m=1}^{n+3}\Gamma(s_mz_j^{-1})}{\prod_{k=1}^n\Gamma(St_kz_j^{-1})}\frac{dz}{z}\\ =\prod_{k=1}^{n}\prod_{m=1}^{n+3}\frac{\Gamma(t_ks_m)}{\Gamma(St_ks_m^{-1})}\prod_{1\leq l<m\leq n+3}\Gamma(Ss_l^{-1}s_m^{-1}),\end{aligned}}$$

where $|t_k|<1$ ($k=1,2,\dots,n$), $|s_m|<1$ ($m=1,2,\dots,n+3$), $|pq|<|t_jS|$, $S=\prod_{m=1}^{n+3}s_m$, and $z_1\cdots z_{n+1}=1$.

Here we have a split of $2n+3$ independent parameters into two groups with n and $n+3$ homogeneous entries. This integration formula appeared to be new even in the $p\to 0$ limit as well as in its further degeneration to the plain hypergeometric level $q\to 1$. Its unit circle analogue valid for $|q|\leq 1$ is constructed in [Spi6].

For each of the described integrals we can apply the residue calculus similar to the one described above in the univariate case and derive summation formulas for particular multiple elliptic hypergeometric series on root systems generalizing the Frenkel-Turaev sum. For the C_I integral, the corresponding formula was derived by van Diejen and the author [DS3] and its recursive proof was given by Rosengren [Ros2]. For the C_{II} integral, the corresponding sum was conjectured first by Warnaar [War1], it was deduced from the residue calculus by van Diejen and the author [DS1] and proven recursively by Rosengren [Ros1]. The $A_I^{(1)}$ resides sum was deduced by the author [Spi4], leading to an elliptic generalization of the Milne's sum [Mil]. Residue calculus for the $A_I^{(2)}$ integral performed by Warnaar and the author [SW] leads to an elliptic generalization of the Bhatnagar-Schlosser "D_n" summation formula [BS]. These elliptic $A_I^{(1)}$ and $A_I^{(2)}$ summation formulas were proven first inductively by Rosengren [Ros2]. A summation formula following from the $A_{II}^{(1)}$ integral was conjectured by the author [Spi4], but it still remains unproven. Residue calculus for the C_{III} integral is expected to lead to a Warnaar's sum [War1], but this question was not investigated either.

All the described integrals are expected to serve as measures in the orthogonality relations for some biorthogonal functions. A program of searching multivariable analogues of the ${}_{12}V_{11}$ biorthogonal functions was put forward in [DS1, Spi2]. The first example of a multivariable extension of the author's two-index continuous biorthogonal functions was found by Rains [Rai2] on the basis of the C_{II} elliptic beta integral (these functions generalize also the Okounkov's interpolating polynomials [Oko]).

The notion of root systems provides the main guiding principle in the construction of multiple elliptic beta integrals. Although this connection is not straightforward, it is natural to expect that there exist other such integrals attached, in particular, to the exceptional Lie algebras. In this respect it is worth analyzing whether all multiple Askey-Wilson type integrals classified by Ito [Ito] admit a further lift up to the levels of Rahman's q-beta integral and the author's elliptic beta integral.

10. Univariate integral Bailey chains

The Bailey chains techniques is well known as a powerful tool for derivation of infinite sequences of identities for series of hypergeometric type [AAR]. The most general known q-hypergeometric Bailey chain was proposed by Andrews [And]. It is related to the Bressoud's matrix inverse [Bre] and has at the bottom the original constructions by Rogers and Bailey used for proving the Rogers-Ramanujan identities [BM]. It was generalized to the elliptic hypergeometric series by the author [Spi3] (for some further developments in this direction, see [War2]). We shall not describe these chains here, although they have quite interesting consequences. Instead, we present Bailey chains for integrals discovered in [Spi5].

<u>Definition.</u> Two functions $\alpha(z,t)$ and $\beta(z,t)$ form an elliptic integral Bailey pair with respect to the parameter t, if

$$\boxed{\beta(w,t) = \kappa \int_{\mathbb{T}} \Gamma(tw^{\pm}z^{\pm})\alpha(z,t)\frac{dz}{z}.}$$

Theorem 11. (First integral Bailey lemma [Spi5]**)**

For a given integral Bailey pair $\alpha(z,t)$, $\beta(z,t)$ with respect to t, the functions

$$\alpha'(w,st) = \frac{\Gamma(tuw^{\pm})}{\Gamma(ts^2uw^{\pm})}\alpha(w,t),$$

$$\beta'(w,st) = \kappa\frac{\Gamma(t^2s^2, t^2suw^{\pm})}{\Gamma(s^2, t^2, suw^{\pm})}\int_{\mathbb{T}} \frac{\Gamma(aw^{\pm}x^{\pm}, ux^{\pm})}{\Gamma(x^{\pm 2}, t^2s^2ux^{\pm})}\beta(x,t)\frac{dx}{x},$$

where $w \in \mathbb{T}$, form a new Bailey pair with respect to the parameter st.

The proof is quite simple, it is necessary to substitute the key relation for $\beta(x,t)$ into the definition of $\beta'(w,st)$, to change the order of integrations, and to apply the elliptic beta integral (under some mild restrictions upon parameters). Note that these substitution rules introduce two new parameters u and s into the Bailey pairs at each step of their iterative application.

Theorem 12. (Second integral Bailey lemma [Spi5]**)**

For a given integral Bailey pair $\alpha(z,t)$, $\beta(z,t)$ with respect to the parameter t, the functions

$$\alpha'(w,t) = \kappa\frac{\Gamma(s^2t^2, uw^{\pm})}{\Gamma(s^2, t^2, w^{\pm 2}, t^2s^2uw^{\pm})}\int_{\mathbb{T}} \frac{\Gamma(t^2sux^{\pm}, sw^{\pm})}{\Gamma(sux^{\pm})}\alpha(x,st)\frac{dx}{x},$$

$$\beta'(w,t) = \frac{\Gamma(tuw^{\pm})}{\Gamma(ts^2uw^{\pm})}\beta(w,st)$$

form a new Bailey pair with respect to t.

It appears that these two lemmas are related to each other by inversion of the integral operator figuring in the definition of integral Bailey pairs [SW]. Application of these lemmas is algorithmic: one should take the initial $\alpha(z,t)$ and $\beta(z,t)$ defined by the elliptic beta integral and apply to them described transformations in all possible ways, which yields a binary tree of identities for multiple elliptic hypergeometric integrals of different dimensions. In particular, the very first step yields the key transformation (i) for the elliptic hypergeometric function $V(\underline{t})$. The residue calculus is supposed to recover elliptic Bailey chains for the ${}_{r+1}V_r$ series [Spi3]. We can take the limit $p \to 0$ and reduce all elliptic results to the level of standard q-hypergeometric integrals which admit further simplification down to identities generated by the plain hypergeometric beta integrals.

As to the unit circle case, we can start from the relation

$$\tilde{\beta}(v,g) = \tilde{\kappa} \int_{-\omega_3/2}^{\omega_3/2} G(g \pm v \pm u; \omega)\, \tilde{\alpha}(u,g) \frac{du}{\omega_2}$$

and apply the modified elliptic beta integral for building needed analogues of the Bailey lemmas. In this case, the $p, r \to 0$ limit brings in identities for q-hypergeometric integrals defined over the non-compact contour $\mathbb{L}$ with the kernels well defined for $|q| = 1$.

11. Elliptic Fourier-Bailey type integral transformations on root systems

Similar to the situation with elliptic beta integrals, the univariate integral transformation of the previous section has been generalized by Warnaar and the author to root systems [SW]. It appears that in the multivariable setting the original space of functions and its image can belong to different root systems.

For the (A, A) pair of root systems, we take the space of meromorphic functions $f_A(\underline{z}; t)$ with A_n symmetry in its variables $z_1, \ldots, z_{n+1}$, $\prod_{j=1}^{n+1} z_j = 1$, and define its image space by setting

$$\boxed{\widehat{f}_A(\underline{w}; t) = \kappa_n^A \int_D \rho(\underline{z}, \underline{w}; t^{-1}) f_A(\underline{z}; t) \frac{dz}{z},}$$

where the kernel has the form

$$\boxed{\rho(\underline{z}, \underline{w}; t) = \frac{\prod_{i,j=1}^{n+1} \Gamma(t w_i^{-1} z_j^{-1})}{\Gamma(t^{n+1}) \prod_{1 \le i < j \le n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j)}.}$$

In a relatively general situation this map can be inverted explicitly.

Theorem 13. (The (A, A) transform inversion [SW])

For a suitable n-dimensional cycle D, the inverse of the (A, A) transform is given by the map

$$\boxed{f_A(\underline{x}; t) = \kappa_n^A \int_{\mathbb{T}^n} \rho(\underline{w}^{-1}, \underline{x}^{-1}; t) \widehat{f}_A(\underline{w}; t) \frac{dw}{w},}$$

where it is assumed that functions $f_A(\underline{x};t)$ are analytical in a sufficiently wide annulus encircling $\mathbb{T}$.

The proof consists in a quite tedious residue calculus with the use of the $A_I^{(1)}$ integration formula.

In the (A,C)-case, we map functions $f_A(\underline{z};t)$ to its image space belonging to the C_n root system:

$$\boxed{\widehat{f}_C(\underline{w};t) = \kappa_n^A \int_D \delta_A(\underline{z},\underline{w};t^{-1}) f_A(\underline{z};t)\frac{dz}{z},}$$

where the kernel has the form

$$\boxed{\delta_A(\underline{z},\underline{w};t) = \frac{\prod_{i=1}^n \prod_{j=1}^{n+1} \Gamma(tw_i^{\pm} z_j^{\pm})}{\prod_{1\leq i<j\leq n+1} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j, t^{-2} z_i z_j, t^2 z_i^{-1} z_j^{-1})}.}$$

Theorem 14. (The (A,C) transform inversion [SW])

For a suitable n-dimensional cycle D, the inverse of the (A,C) integral transform looks as follows

$$\boxed{f_A(\underline{x};t) = \kappa_n^C \int_{\mathbb{T}^n} \delta_C(\underline{w},\underline{x};t)\widehat{f}_C(\underline{w};t)\frac{dw}{w},}$$

with the kernel

$$\boxed{\delta_C(\underline{w},\underline{x};t) = \frac{\prod_{i=1}^n \prod_{j=1}^{n+1} \Gamma(tw_i^{\pm} x_j)}{\prod_{i=1}^n \Gamma(w_i^{\pm 2}) \prod_{1\leq i<j\leq n} \Gamma(w_i^{\pm} w_j^{\pm})},}$$

where it is assumed that functions $f_A(\underline{x};t)$ are analytical in a sufficiently wide annulus containing $\mathbb{T}$.

Corollary 15. *If we choose $\widehat{f}_C(\underline{w};t)$ such that the product $\delta_C(\underline{w},\underline{x};t)\cdot \widehat{f}_C(\underline{w};t)$ is equal to the C_I elliptic beta integral kernel, then the original relation $\widehat{f}_C \sim \int_D \delta_A\cdot f_A\, dz/z$ defines the $A_I^{(2)}$ integration formula.*

There are more such Fourier-Bailey type integral transforms with explicit inversions some of which still are in the conjectural form. All of them can be put into the integral Bailey chains setting yielding many infinite sequences of transformations for the elliptic hypergeometric integrals on root systems.

12. Applications to the Calogero-Sutherland type models

After discussing multiple elliptic beta integrals, we would like to return to applications of elliptic hypergeometric functions to the Calogero-Sutherland type models [Spi7].

First, we define the inner product

$$\langle\varphi,\psi\rangle^{I,II}=\kappa_n^C\int_{\mathbb{T}^n}\Delta^{I,II}(\underline{z},\underline{t})\varphi(\underline{z})\psi(\underline{z})\,\frac{dz}{z}.$$

Let us take the Hamiltonian of the van Diejen model [Die] with the restriction $t^{2n-2}\prod_{m=1}^8 t_m=p^2q^2$

$$\mathcal{D}_{II}=\sum_{j=1}^n\Big(A_j(\underline{z})(T_j-1)+A_j(\underline{z}^{-1})(T_j^{-1}-1)\Big),$$

$$A_j(\underline{z})=\frac{\prod_{m=1}^8\theta(t_mz_j;p)}{\theta(z_j^2,qz_j^2;p)}\prod_{\substack{k=1\\ \neq j}}^n\frac{\theta(tz_jz_k,tz_jz_k^{-1};p)}{\theta(z_jz_k,z_jz_k^{-1};p)}.$$

Under some relatively mild restrictions upon parameters, this operator is formally hermitian with respect to the above inner product, $\langle\varphi,\mathcal{D}_{II}\psi\rangle^{II}=\langle\mathcal{D}_{II}\varphi,\psi\rangle^{II}$, for the weight function

$$\Delta^{II}(\underline{z},\underline{t})=\prod_{1\leq j<k\leq n}\frac{\Gamma(tz_j^{\pm}z_k^{\pm})}{\Gamma(z_j^{\pm}z_k^{\pm})}\prod_{j=1}^n\frac{\prod_{k=1}^8\Gamma(t_kz_j^{\pm})}{\Gamma(z_j^{\pm2})}.$$

Evidently, $f(\underline{z})=1$ is a $\lambda=0$ solution of the standard eigenvalue problem $\mathcal{D}_{II}f(\underline{z})=\lambda f(\underline{z})$. The norm of this eigenfunction

$$\boxed{\|1\|^2=V(\underline{t};C_{II})=\kappa_n^C\int_{\mathbb{T}^n}\Delta^{II}(\underline{z},\underline{t})\,\frac{dz}{z}}$$

is a multivariable analogue of the elliptic hypergeometric function $V(\underline{t})$ for the type II C_n elliptic beta integral.

We conjecture that with all multiple elliptic beta integrals one can associate Calogero-Sutherland type models in the described fashion. Let us take the weight function

$$\Delta^I(\underline{z},\underline{t})=\frac{1}{\prod_{1\leq i<j\leq n}\Gamma(z_i^{\pm}z_j^{\pm})}\prod_{j=1}^n\frac{\prod_{k=1}^{2n+6}\Gamma(t_kz_j^{\pm})}{\Gamma(z_j^{\pm2})}.$$

We associate with it the Hamiltonian

$$\mathcal{D}_I=\sum_{j=1}^n\Big(A_j(z)(T_j-1)+A_j(z^{-1})(T_j^{-1}-1)\Big),$$

$$A_j(z)=\frac{\prod_{k=1}^{2n+6}\theta(t_kz_j;p)}{\theta(z_j^2,qz_j^2;p)}\prod_{\substack{k=1\\ \neq j}}^n\frac{1}{\theta(z_jz_k^{\pm};p)},\qquad\prod_{k=1}^{2n+6}t_k=p^2q^2,$$

which is formally hermitian with respect to the taken inner product, $\langle\varphi,\mathcal{D}_I\psi\rangle^I=\langle\mathcal{D}_I\varphi,\psi\rangle^I$, for some mild restrictions upon the parameters.

Again, $f(\underline{z}) = 1$ is a $\lambda = 0$ eigenfunction of the operator $\mathcal{D}_I$ and its normalization

$$\boxed{\|1\|^2 = V(\underline{t}; C_I) = \kappa_n^C \int_{\mathbb{T}^n} \Delta^I(\underline{z}, \underline{t}) \, \frac{dz}{z}}$$

defines type I generalization of the elliptic hypergeometric function for the root system C_n. The functions $V(\underline{t}; C_{I,II})$ were considered first by Rains [Rai1] in the context of symmetry transformations for multiple elliptic hypergeometric integrals. It is not difficult to define their unit circle analogues which also play similar role in the context of Calogero-Sutherland type models.

One can construct analogues of the $V(\underline{t})$ function for multiple elliptic beta integrals on the A_n root system and build corresponding Hamiltonians (all of which coincide in the rank 1 case). Although all these models are degenerate—their particles' pairwise coupling constant is fixed in one or another way, it would be interesting to clarify whether these models define new completely integrable quantum systems.

Acknowledgments

These lectures are dedicated to the memory of a prominent russian mathematician A. A. Bolibrukh whom I knew for a decade and who has untimely passed away one year before this workshop. He considered elliptic beta integrals as some universal mathematical objects and presented the paper [Spi1] for publication as a communication of the Moscow mathematical society.

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