Max-Planck-Institut für Mathematik
Bonn

Quantum KdV hierarchy and shifted symmetric functions

by

Jan-Willem M. van Ittersum
Giulio Ruzza

Max-Planck-Institut für Mathematik
Preprint Series 2024 (15)

Date of submission: July 5, 2024

Quantum KdV hierarchy and shifted symmetric functions

by

Jan-Willem M. van Ittersum
Giulio Ruzza

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics and
Computer Science
University of Cologne
Weyertal 86-90
50931 Cologne
Germany

Departamento de Matemática
Faculdade de Ciências
da Universidade de Lisboa
Campo Grande Edifício C6
1749-016 Lisboa
Portugal

Grupo de Física Matemática
Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais
1049-001 Lisboa
Portugal

MPIM 24-15

QUANTUM KDV HIERARCHY AND SHIFTED SYMMETRIC FUNCTIONS

JAN-WILLEM M. VAN ITTERSUM AND GIULIO RUZZA

Abstract. We study spectral properties of the quantum Korteweg–de Vries hierarchy defined by Buryak and Rossi. We prove that eigenvalues to first order in the dispersion parameter are given by shifted symmetric functions. The proof is based on the boson-fermion correspondence and an analysis of quartic expressions in fermions. As an application, we obtain a new closed evaluation of certain double Hodge integrals on the moduli spaces of curves. Finally, we provide an explicit formula for the eigenvectors to first order in the dispersion parameter. In particular, we show that its Schur expansion is supported on partitions for which the Hamming distance is minimal.

1. Introduction and results

1.1. **Generating functions of double ramification intersection numbers.** The quantum double ramification hierarchy is a remarkable construction by Buryak and Rossi [10], also inspired by the symplectic field theory program of Eliashberg, Givental and Hofer [16], which associates a quantum integrable hierarchy to an arbitrary cohomological field theory [26]. This construction was an important step forward in studying relations between the geometry of moduli spaces of curves and the theory of integrable systems, initiated with Witten's seminal conjecture [35]. Double ramification hierarchies have been extensively studied in recent years, cf. [4, 5, 8, 9, 6]. In this paper, we are interested in the spectral problem for this hierarchy.

Let $\widetilde{\mathcal{B}}$ be the $\mathbb{Q}$-algebra with generators ω_a $(a \in \mathbb{Z})$ and $\hbar$ and commutation relations

$$[\hbar, \omega_a] = 0, \quad [\omega_a, \omega_b] \;=\; -a\,\delta_{a,-b}\,\hbar \qquad (a, b \in \mathbb{Z}), \tag{1.1}$$

where we denote by $[A, B] = AB - BA$ the commutator. Let us introduce the normal ordering

$$:\!\omega_{a_1} \dots \omega_{a_n}\!: = \prod_{a_i \geq 0} \omega_{a_i} \prod_{a_i < 0} \omega_{a_i}. \tag{1.2}$$

In [10], for $k \geq 0$, Buryak and Rossi consider generating functions

$$g_k(\epsilon; z) \;=\; \sum_{\substack{g,n\geq 0\\ 2g-1+n>0}} \frac{\hbar^g}{n!} \sum_{a_1,\dots,a_n\in\mathbb{Z}} I_{g,k;a_1,\dots,a_n}\left(\frac{\epsilon}{\hbar}\right) z^{\sum_\ell a_\ell} \;:\!\omega_{a_1} \dots \omega_{a_n}\!: \tag{1.3}$$

of intersection numbers

$$I_{g,k;a_1,\dots,a_n}(y) \;=\; \int_{\mathrm{DR}_g(-\sum_\ell a_\ell, a_1,\dots,a_n)} \psi_1^k \left(1 + y\lambda_1 + y^2\lambda_2 + \dots + y^g\lambda_g\right) \tag{1.4}$$

of ψ and λ classes over the double ramification cycle $\mathrm{DR}_g(-\sum_l a_l, a_1, \dots, a_n)$ in the moduli spaces of curves $\overline{\mathcal{M}}_{g,n+1}$. (Here, y is a formal variable.) Note that $g + n \leq k + 2$ in the right-hand side of (1.3) for dimensional reasons. For more details, we refer the reader to [7, 10, 11, 23].

2020 *Mathematics Subject Classification.* 05A17; 14H70; 37K10.

Key words and phrases. Double ramification hierarchies; Quantum KdV; Partitions; Shifted symmetric functions.

To deal with the infinite sums in the definition (1.3) of these generating functions, we need to work with a completion $\mathcal{B}$ of $\widetilde{\mathcal{B}}$ (see [10, 16]), whose elements are (formal) power series in $\omega_{-1}, \omega_{-2}, \dots$ with coefficients which are polynomials in $\hbar, \omega_0, \omega_1, \omega_2, \dots$; the product of two such power series can always be brought into the form of another such power series by use of the commutation relations (1.1). Then, $g_k(\epsilon; z) \in \mathcal{B}[\epsilon][\![z^{\pm 1}]\!]$.

Remark 1.1. The variables $\epsilon, \hbar, z$ of the present paper coincide with $-\varepsilon^2, \mathrm{i}\hbar, \mathrm{e}^{\mathrm{i}x}$ in [10].

The results of [10] give interesting properties of these generating functions (generalizing to arbitrary cohomological field theories). In particular, in op. cit. it is shown that $g_k(\epsilon; z)$ can be computed by an explicit recursion in k (which we recall in Sec. 2.1 below). Moreover,

$$G_k(\epsilon) \;=\; \mathrm{Res}_z\, g_k(\epsilon; z) \frac{\mathrm{d}z}{z} \in \mathcal{B}[\epsilon] \qquad (k \geq 0) \tag{1.5}$$

where $\mathrm{Res}_z \sum_{n \in \mathbb{Z}} A_n\, z^n\, \mathrm{d}z = A_{-1}$ is the formal residue, commute pairwise, i.e.,

$$[G_k(\epsilon), G_\ell(\epsilon)] = 0 \qquad (k, \ell \geq 0). \tag{1.6}$$

The pairwise commuting elements $G_k(\epsilon)$ form the quantum double ramification hierarchy for the trivial cohomological theory.

Example 1.2. *When $k = 0$, only terms with $(g, n) = (0, 2)$, $(1, 1)$, or $(1, 0)$ contribute to the sum (1.3). These terms are readily computed (cf. [7, 10, 23]) and we have*

$$g_0(\epsilon; z) \;=\; \frac{1}{2} \sum_{a_1, a_2 \in \mathbb{Z}} {:}\omega_{a_1} \omega_{a_2}{:}\; z^{a_1 + a_2} \;+\; \frac{\epsilon}{24} \sum_{a \in \mathbb{Z}} a^2 \omega_a z^a \;-\; \frac{\hbar}{24}. \tag{1.7}$$

Similarly,

$$\begin{aligned} g_1(\epsilon; z) \;=\; \frac{1}{6} \sum_{a_1, a_2, a_3 \in \mathbb{Z}} {:}\omega_{a_1} \omega_{a_2} \omega_{a_3}{:}\; z^{a_1 + a_2 + a_3} \;+\; \frac{\epsilon}{24} \sum_{a_1, a_2 \in \mathbb{Z}} \frac{a_1^2 + a_2^2}{2}\, {:}\omega_{a_1} \omega_{a_2}{:}\; z^{a_1 + a_2} \\ +\; \frac{\epsilon^2}{1152} \sum_{a \in \mathbb{Z}} a^4 \omega_a\, z^a \;+\; \frac{\hbar}{24} \sum_{a \in \mathbb{Z}} (a^2 - 1) \omega_a\, z^a \;+\; \frac{\epsilon \hbar}{2880}. \end{aligned}$$

Remark 1.3 (Quantum KdV). In the classical limit, $\hbar = 0$ and the variables ω_a $(a \in \mathbb{Z})$ form a commutative algebra. Then, $G_k^{\mathrm{cl}}(\epsilon) = \lim_{\hbar \to 0} G_k(\epsilon)$ are the conserved quantities of the classical Korteweg–de Vries (KdV) hierarchy, cf. [7] (G_1^{cl} being the KdV Hamiltonian). We have $\{G_k^{\mathrm{cl}}(\epsilon), G_\ell^{\mathrm{cl}}(\epsilon)\} = 0$, where the Poisson bracket is defined by $\{\omega_a, \omega_b\} = \mathrm{i}\, a\, \delta_{a,-b}$, which is the *first* Poisson structure of the KdV hierarchy (in Fourier coordinates). Hence, the quantum double ramification hierarchy described above solves the quantization problem for the KdV hierarchy by explicitly providing an $\hbar$-deformation of the classical KdV conserved quantities which form a commuting family of quantum conserved charges, which we call the *quantum KdV hierarchy*. We direct the reader to [10, Section 2] and [15, Section 1] and references therein for more details.

It is worth noting that the classical KdV hierarchy also admits a *second* Poisson structure, whose quantization involves the Virasoro algebra and has been extensively studied in conformal field theory after the impetus of the seminal work by Bazhanov, Lukyanov, and Zamolodchikov [2].

1.2. **Spectral problem.** Consider the representation $\rho_c : \mathcal{B} \to \mathrm{End}\,\mathsf{B}$, where $\mathsf{B} = \mathbb{Q}[\boldsymbol{p}]$, $\boldsymbol{p} = (p_1, p_2, \dots)$, defined by

$$\rho_c(\hbar)\, \phi \;=\; \phi, \quad \rho_c(\omega_a)\, \phi \;=\; \begin{cases} p_a\, \phi, & \text{if } a > 0, \\ c\, \phi, & \text{if } a = 0, \\ -a \frac{\partial \phi}{\partial p_{-a}}, & \text{if } a < 0, \end{cases} \qquad (\phi \in \mathcal{B},\, a \in \mathbb{Z})\,. \tag{1.8}$$

depending on an arbitrary rational number c.

Remark 1.4. We could let $\hbar$ act by multiplication by an arbitrary nonzero constant. Without loss of generality, we set this constant to 1, as we can always reduce to this case by rescaling the variables p_k.

We obtain the family of commuting *quantum KdV Hamiltonian operators* on B

$$\widehat{G}_k(\epsilon, c) = \rho_c\big(G_k(\epsilon)\big) \qquad (k \geq 0), \tag{1.9}$$

polynomially depending on ϵ and c. It is readily checked that these operators preserve the grading $\mathsf{B} = \bigoplus_{n\geq 0} \mathsf{B}_n$ induced by assigning weight k to the variable p_k.

In this paper, we continue the study of the spectral problem for such a family of commuting operators on B. Rossi [33] (see also the work of Pogrebkov [31]) showed that $\widehat{G}_k(0;c)$ is quadratic in fermions under the boson-fermion correspondence, which we will recall in Section 3.3. Dubrovin then showed that, when $\epsilon = 0$, a basis of common eigenvectors is given by Schur functions. Namely, [15, Corollary 2.4]

$$\widehat{G}_k(0;c)\, s_\lambda \;=\; E_k^{[0]}(\lambda;c)\, s_\lambda \qquad (k \geq 0), \tag{1.10}$$

where the Schur functions $s_\lambda = s_\lambda(\boldsymbol{p})$ are labeled by partitions $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathscr{P}$ and are defined by (cf. [27])

$$s_\lambda(\boldsymbol{p}) \;=\; \det\big(h_{\lambda_i - i + j}(\boldsymbol{p})\big)_{i,j=1}^r, \quad \sum_{\ell\in\mathbb{Z}} h_\ell(\boldsymbol{p})\, y^\ell \;=\; \exp\Big(\sum\nolimits_{\ell\geq 1} \frac{p_\ell}{\ell} y^\ell\Big). \tag{1.11}$$

Moreover, Dubrovin showed that the eigenvalue in (1.10) is a *shifted symmetric* function on partitions, that is

$$E_k^{[0]}(\lambda;c) \;=\; \sum_{j=0}^{k+2} \frac{c^{k+2-j}}{(k+2-j)!}\, Q_j(\lambda), \tag{1.12}$$

where the functions $Q_k : \mathscr{P} \to \mathbb{Q}$ are given by $Q_0(\lambda) = 1$ and

$$Q_k(\lambda) \;=\; \frac{1}{(k-1)!} \sum_{i\geq 1} \Big[\big(\lambda_i - i + \tfrac{1}{2}\big)^{k-1} - \big(-i + \tfrac{1}{2}\big)^{k-1}\Big] \;+\; \beta_k \qquad (k \geq 1) \tag{1.13}$$

with

$$\sum_{k\geq 0} \beta_k\, y^k = \frac{y/2}{\sinh(y/2)}. \tag{1.14}$$

Equivalently, $\beta_k \;=\; \frac{1}{k!}\big(\frac{1}{2^{k-1}} - 1\big) B_k$, where B_k denotes the kth Bernoulli number.

Remark 1.5. The algebra of shifted symmetric functions is a deformation of the algebra of symmetric functions, for which the natural generators are the Q_k. A vector space basis is given by the central characters of the symmetric group. Shifted symmetric functions appear in asymptotic representation theory [22, 25, 32] and in enumerative geometry, e.g., in the Hurwitz/Gromov–Witten theory of curves [14, 17, 18, 19, 29, 30], and in computations of volumes and Siegel–Veech constants of the moduli space of flat surfaces [12, 13, 18]. For the latter, a remarkable relation to quasimodular forms [3, 36] is crucial, which will be recalled below (cf. Theorem 1.12).

Furthermore, it was shown in [34] that the commuting Hamiltonians $\widehat{G}_k$ admit a common basis of eigenvectors deforming the Schur polynomials. That is, for all partitions λ of size $|\lambda| = \sum_i \lambda_i$ there exist unique power series $r_\lambda(\boldsymbol{p};\epsilon) \in \mathsf{B}_{|\lambda|}[\![\epsilon]\!]$ satisfying

$$\widehat{G}_k(\epsilon, c)\, r_\lambda(\boldsymbol{p};\epsilon) \;=\; E_k(\lambda;\epsilon, c)\, r_\lambda(\boldsymbol{p};\epsilon) \tag{1.15}$$

such that

$$r_\lambda(\boldsymbol{p};\epsilon) = s_\lambda + \sum_{m\geq 1} \epsilon^m r_\lambda^{[m]}(\boldsymbol{p}), \tag{1.16}$$

$$E_k(\lambda;\epsilon,c) = E_k^{[0]}(\lambda;c) + \sum_{m\geq 1} \epsilon^m E_k^{[m]}(\lambda;c) \tag{1.17}$$

and

$$\langle s_\lambda, r_\lambda^{[m]}\rangle = 0 \qquad (m \geq 1), \tag{1.18}$$

where the standard scalar product on B is given by $\langle s_\lambda(\boldsymbol{p}), s_\mu(\boldsymbol{p})\rangle = \delta_{\lambda,\mu}$.

1.3. **Results.** Our first result gives expresses the eigenvalue $E_k^{[1]}(\lambda;c)$ as a shifted symmetric function in the partition λ.

Theorem 1.6. *We have*

$$E_k^{[1]}(\,\cdot\,;c) \;=\; \frac{1}{24}\sum_{\ell=0}^{k} \frac{c^\ell}{\ell!}\Big(2Q_2Q_{k-\ell} \,+\, (k-\ell)(k-\ell+3)Q_{k+3-\ell}\Big) \tag{1.19}$$

where the shifted symmetric function Q_k is defined by (1.13).

The proof is given in Section 4 and employs the boson-fermion correspondence. A challenging aspect of this proof is the fact that the linear part in ϵ of the quantum KdV Hamiltonian operators are quartic expressions in fermions, rather than simpler quadratic expression, cf. [15, 33].

We obtain the following application.

Corollary 1.7. *We have the following explicit evaluation of double Hodge intersection numbers on the moduli spaces of curves:*

$$\int_{\overline{\mathcal{M}}_{g,1}} \lambda_1\lambda_g\psi_1^{2g-3} \;=\; \frac{(-1)^g}{24}\Big(2g(2g-3)\beta_{2g} - \frac{1}{12}\beta_{2g-2}\Big) \qquad (g\geq 2), \tag{1.20}$$

where β_k is defined by (1.14).

For our second result, we focus on the eigenvectors in (1.15), also to first order in ϵ. We show that $\langle r_\lambda^{[1]}, s_\mu\rangle \neq 0$ if and only if the Hamming distance d, given by (5.1), of the Frobenius coordinates of two partitions λ and μ is exactly 2 (which is the minimal distance for two partitions of the same integers). If this is the case, λ can be obtained from μ, in exactly two ways, by removing a border strip γ_i and adding a border strip γ_i' ($i=1,2$, with $|\gamma_i| = |\gamma_i'|$, $\gamma_1\subset\gamma_2$, $\gamma_1'\subset\gamma_2'$). (More detailed explanations and proofs are given in Sections 3 and 5.)

Theorem 1.8. *For all partitions λ*

$$r_\lambda^{[1]}(\boldsymbol{p}) \;=\; \frac{1}{12}\sum_{d(\lambda,\mu)=2} w_{\lambda,\mu}\,(-1)^{\mathrm{ht}(\gamma_1)+\mathrm{ht}(\gamma_1')}\left(\frac{|\gamma_1|}{|\gamma_2|} - \frac{|\gamma_2|}{|\gamma_1|}\right) s_\mu(\boldsymbol{p})\,. \tag{1.21}$$

Here $w_{\lambda,\mu} = 1$ if $\lambda_i > \mu_i$ for the smallest $i\geq 1$ such that $\lambda_i\neq\mu_i$, and $w_{\lambda,\mu} = -1$ otherwise. Moreover, $\mathrm{ht}(\gamma)$ and $|\gamma|$ are, respectively, the height and size of a border strip γ.

1.4. **Further comments and conjectures.** We denote by $\Lambda^* = \mathbb{Q}[Q_2,Q_3,\dots]$ the algebra of shifted symmetric functions on partitions. The expressions for $E_k^{[0]}(\lambda;c)$ in (1.12) and for $E_k^{[1]}(\lambda;c)$ in (1.19) prompt a conjecture, suggested to us by Don Zagier and already stated in [21]. Namely, we conjecture that $E_k^{[m]}(\lambda;c)$ belongs to $\Lambda^*[c]$. More precisely, by assigning weight k to Q_k and weight 1 to c the algebra $\Lambda^*[c] = \bigoplus_{n\geq 0}\Lambda^*[c]_n$ becomes a graded algebra, and we conjecture that $E_k^{[m]}(\lambda;c)$ is of homogeneous weight $k+2+m$ in $\Lambda^*[c]$ for all $k,m \geq 0$. Based on extensive numerical verification, which we report in Appendix A, we formulate the following refined version of that conjecture.

Conjecture 1.9. *For each integer $D \geq 0$ and each partition ν, there exists a polynomial $f_{D,\nu}$ with rational coefficients of degree $2D$ such that*

$$E_k(\,\cdot\,;\epsilon,0) \;=\; \sum_{\substack{\nu\in\mathscr{P}\\ D\geq 0}} \epsilon^{D+|\nu|} f_{D,\nu}(k)\, Q_{k+D+2-\ell(\nu)} \prod_i Q_{\nu_i+1}\,. \tag{1.22}$$

Remark 1.10. Note that by Lemma 2.3, we have $E_k(\lambda;\epsilon,c) = \sum_{\ell=0}^k \frac{c^\ell}{\ell!} E_{k-\ell}(\lambda;\epsilon,0)$, such that the restriction to $c=0$ in this conjecture does not imply any loss of generality.

Example 1.11. *We have*

$$f_{0,\emptyset}(k) = 1, \quad f_{1,\emptyset}(k) = \frac{1}{24}k(k+3) \quad \text{and} \quad f_{0,(1)}(k) = \frac{1}{12}, \tag{1.23}$$

and with these polynomials we recover Dubrovin's result (1.12), as well as Theorem 1.6.

See Appendix A for a table of such polynomials, which give a conjectural expression for $E_k^{[m]}$ up to $m \leq 5$. For example, we conjecture

$$E_k^{[2]}(\,\cdot\,;0) \;=\; \frac{1}{24^2}\biggl(\frac{1}{2}Q_3Q_{k+1} + \frac{1}{2}Q_2^2Q_k + \frac{1}{2}(k^2+2k-1)Q_2Q_{k+2} + \frac{1}{6}k(k+4)(3k^2+16k+17)Q_{k+4}\biggr). \tag{1.24}$$

We plan to provide an explicit construction for the polynomials $f_{D,\nu}$ in a future publication in collaboration with Don Zagier.

Supporting evidence for the above conjecture, in addition to Theorem 1.6 and Appendix A, is provided in our previous work [21] through a connection between KdV Hamiltonian operators and quasimodular forms which we briefly review now.

Let us first recall the following theorem of Bloch and Okounkov [3] (see also [36]) concerning the q-bracket of functions on partitions, given by

$$\langle f\rangle_q \;=\; \frac{\sum_{\lambda\in\mathscr{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda\in\mathscr{P}} q^{|\lambda|}} \in \mathbb{Q}[\![q]\!] \tag{1.25}$$

for any function $f:\mathscr{P}\to\mathbb{Q}$ on the set $\mathscr{P}$ of all partitions. Write $\widetilde{M}_k$ for the space of all weight k quasimodular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 1.12 ([3])**.** *For any $f\in\Lambda_k^*$, we have $\langle f\rangle_q \in \widetilde{M}_k$.*

Because of this theorem, the validity of Conjecture 1.9 would imply the quasimodularity of the q-brackets of the KdV eigenvalues. This has been independently verified in our previous work. We recall the relevant result as it will be useful later on. To this end, we extend the grading on the algebra of quasimodular forms to $\widetilde{M}[c,\epsilon]$ by assigning weight 1 to c and weight -1 to ϵ.

Theorem 1.13 ([21])**.** *For all $k\geq 0$, we have $\bigl\langle E_k(\,\cdot\,;\epsilon,c)\bigr\rangle_q \in \widetilde{M}[c,\epsilon]_{k+2}$.*

Remark 1.14. An intriguing aspect of this result is that the eigenvalues $E_k(\cdot;\epsilon,0)$ (taking $c=0$ for simplicity) interpolate between the generators Q_k of the algebra of shifted symmetric functions at $\epsilon=0$ and the generators S_k of the algebra of symmetric functions (namely, $S_k(\lambda) = -\frac{B_k}{2k} + \sum_i \lambda_i^{k-1}$) at $\epsilon=\infty$ (cf. [21, Appendix A]). Both these algebras of functions on partitions are *quasimodular algebras*, in the sense of [20], namely, q-brackets of homogeneous elements are quasimodular forms of pure weight. No other quasimodular algebras are known for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Then, Theorem 1.13 asserts that this ϵ-deformation respects quasimodularity.

Finally, given the commutativity (1.6), Conjecture 1.9 would imply the stronger result that for all $k_1, \ldots, k_r \geq 0$, we have

$$\Big\langle \prod_i E_{k_i}(\,\cdot\,;\epsilon,c) \Big\rangle_q \in \widetilde{M}[c,\epsilon]_{\sum_i (k_i+2)} \, . \tag{1.26}$$

Note that when $\epsilon = 0$, (1.26) coincides with the quasimodularity property first established by Okounkov and Pandharipande [30, Section 5] for the Gromov–Witten theory of an elliptic curve (see also [33]). Then, the conjectural relation (1.26) equation would imply that this quasimodularity property persists under suitable insertion of Hodge classes.

Outline of the rest of the paper. In Section 2 and 3 we recall some preliminary material on quantum double ramification hierarchies, partitions, and the boson-fermion correspondence, needed for proving our main theorems. Theorem 1.6 is proven in Section 4. We prove Theorem 1.8 in Section 5.2. Finally, Appendix A contains supplemental data to Conjecture 1.9.

Acknowledgments

We thank Konstantin Aleshkin, Alexandr Buryak, Giordano Cotti, Davide Masoero, Paolo Rossi, Di Yang, and Don Zagier for valuable discussions. A large part of this work was completed during our stay at the Max Planck Institute for Mathematics in Bonn, which we wish to thank for the excellent working conditions.

The first author was supported by the SFB/TRR 191 “Symplectic Structure in Geometry, Algebra and Dynamics”, funded by the DFG (Projektnummer 281071066 TRR 191). He also thanks the Grupo de Física Matemática (UIDB/00208/2020, UIDP/00208/2020, DOI: 10.54499/UIDB/00208/2020, 10.54499/UIDP/00208/2020) for their support during a one-week visit to Instituto Superior Técnico in Lisbon, to which he extends his gratitude for their hospitality.

The second author is supported by the FCT grant 2022.07810.CEECIND.

2. Quantum double ramification hierarchies

In this section, we recall several aspects of quantum double ramification hierarchies.

2.1. Recursion.

We recall the following result from [10, Theorem 3.5] (with small changes due to slightly different normalizations, for which we refer to Remark 1.1). To this end, it is convenient to define

$$g_{-1}(\epsilon;z) \;=\; \sum_{a\in\mathbb{Z}} \omega_a z^a. \tag{2.1}$$

Theorem 2.1 ([10]). *We have*

$$\left(k+2+\epsilon\frac{\partial}{\partial\epsilon}\right) z\partial_z\, g_{k+1}(\epsilon;z) \;=\; \frac{1}{\hbar}\big[g_k(\epsilon;z), G_1(\epsilon)\big] \qquad (k\geq -1) \tag{2.2}$$

and

$$G_1(\epsilon) \;=\; \frac{1}{6}\sum_{a_1,a_2\in\mathbb{Z}} {:}\omega_{a_1}\omega_{a_2}\omega_{-a_1-a_2}{:} + \frac{\epsilon}{24}\sum_{a\in\mathbb{Z}} a^2 \,{:}\omega_a\omega_{-a}{:} - \frac{\hbar}{24}\omega_0 + \frac{\epsilon\hbar}{2880}. \tag{2.3}$$

Remark 2.2. The formulation in [10, Theorem 3.5] is given in terms of differential polynomials (cf. Section 2.3) and it involves the operator $D = 2\epsilon\partial_\epsilon + 2\hbar\partial_\hbar + \sum_{k\geq 0} u_k \partial_{u_k}$, acting on polynomials in $\boldsymbol{u} = (u_0, u_1, \dots)$. We have made use of the homogeneity of $g_k(\epsilon; z)$ stemming from the dimensional constraint in (1.4) to rewrite the recursion as in (2.2), cf. [34, Equation (4.7)].

Formula (2.2), together with (2.1) allows us to recursively determine $g_k(\epsilon; z)$ up to a constant depending on $\epsilon, \hbar$, but not on the ω_a, for all $k \geq 0$, as explained in [10, Section 3.5]. Here, to compute the constant term in z of $g_k(\epsilon, z)$ one uses the fact that the coefficients (1.4) in (1.3) are polynomials in $a_1, \dots, a_n$ (see the next paragraph). Finally, the constants depending only on ϵ and $\hbar$ can be recovered by either the string equation [10, Lemma 3.7]

$$\frac{\partial g_{k+1}(\epsilon; z)}{\partial \omega_0} \;=\; g_k(\epsilon; z) \qquad (k \geq -1) \tag{2.4}$$

or by invoking the homogeneity expressed by Theorem 1.13.

It will be convenient to write

$$g_k(\epsilon; z) \;=\; \sum_{j=0}^{k+1} g_k^{[j]}(z)\left(\frac{\epsilon}{24}\right)^j, \quad G_k(\epsilon) \;=\; \sum_{j=0}^{k} G_k^{[j]}\left(\frac{\epsilon}{24}\right)^j, \qquad (k \geq 0) \tag{2.5}$$

(cf. [21, Appendix A]) such that we can express (2.2) as

$$\hbar\,(k+2+j)\,z\partial_z\, g_{k+1}^{[j]}(z) \;=\; \left[g_k^{[j]}(z), G_1^{[0]}\right] \;+\; \left[g_k^{[j-1]}(z), G_1^{[1]}\right] \qquad (k \geq 0,\; j \geq 0). \tag{2.6}$$

2.2. **Reduction to $c = 0$.** In what follows we set $c = 0$, cf. (1.8). This is without loss of generality, as we now show.

Lemma 2.3. *We have*

$$\widehat{G}_k(\epsilon, c) \;=\; \sum_{\ell=0}^{k} \frac{c^\ell}{\ell!}\, \widehat{G}_{k-\ell}(\epsilon, 0) \tag{2.7}$$

Proof. By (1.8), for all $X \in \mathcal{B}$ we have $\rho_c(X) \;=\; \rho_0\big(\mathrm{e}^{c\frac{\partial}{\partial\omega_0}} X\big)$. The proof follows by taking $X = G_k(\epsilon)$ and recalling (2.4). ☐

In view of this lemma, Theorem 1.6 is equivalent to the identity

$$E_k^{[1]}(\lambda; 0) \;=\; \frac{1}{24}\left(2\,Q_2(\lambda)\,Q_{k+1}(\lambda) \;+\; k(k+3)\,Q_{k+3}(\lambda)\right). \tag{2.8}$$

2.3. **Differential polynomials formalism.** Finally, even if not needed in what follows, it is worth mentioning that there is an equivalent description of the generating functions $g_k(\epsilon; z)$ in terms of differential polynomials. To explain it, we first recall that the integral of any tautological class against the double ramification cycle $\mathrm{DR}_g(-\sum_l a_l, a_1, \dots, a_n)$ is an even polynomial in the variables a_i of degree at most $2g$ with rational coefficients, cf. [10, Appendix B]. In particular, $I_{g,d;a_1,\dots,a_n}(y)$ is an even polynomial in the variables a_i. Therefore, introducing

$$u_k(z) \;=\; \sum_{a\in\mathbb{Z}} a^k\,\omega_a\, z^a \qquad (k \geq 0), \tag{2.9}$$

(agreeing that $0^0 = 1$) we can define polynomials $g_k^{poly}(\epsilon, \hbar; \boldsymbol{u})$ $(k \geq 0)$ in the variables $\boldsymbol{u} = (u_0, u_1, \dots)$ with rational coefficients by requiring

$$:\!g_k^{poly}\big(\epsilon, \hbar; \boldsymbol{u}(z)\big)\!: \;=\; g_k(\epsilon; z) \qquad (k \geq 0), \tag{2.10}$$

where $\boldsymbol{u}(z) = (u_0(z), u_1(z), \dots)$ and the normal order defined in (1.2) is extended by linearity. Note that $u_k(z) = (z\partial_z)^k u_0(z)$ and that $g_k^{poly}(\epsilon, \hbar; \boldsymbol{u})$ is even with respect to the grading defining by assigning weight k to u_k for $k \geq 0$ (and weight 0 to $\hbar$ and ϵ).

Example 2.4. *We have (cf. Example 1.2)*

$$g_0^{poly}(\epsilon,\hbar;\boldsymbol{u}) \;=\; \frac{1}{2}u_0^2+\frac{u_2}{24}-\frac{\hbar}{24}, \tag{2.11}$$

$$g_1^{poly}(\epsilon,\hbar;\boldsymbol{u}) \;=\; \frac{1}{6}u_0^3+\frac{\epsilon}{24}u_2u_0+\frac{\epsilon^2}{1152}u_4+\frac{\hbar}{24}(u_2-u_0)+\frac{\epsilon\hbar}{2880}. \tag{2.12}$$

3. Partitions and the boson-fermion correspondence

In this section, we recall some standard notions in the theory of partitions as well as the so-called infinite-wedge formalism and the boson-fermion correspondence.

3.1. **Partitions.** Denote by $\mathbb{F}=\mathbb{Z}+\frac{1}{2}$ the set of half-integers, and for a set X write $X^{\pm}=\{a\in X\mid \pm a>0\}$. As already mentioned, we denote by $\mathscr{P}$ the set of all partitions. For $\lambda\in\mathscr{P}$ we denote by $|\lambda|:=\sum_i\lambda_i$ the size of λ and by λ' the transposed partition. Each partition is represented by its Young diagram Y_λ, given by

$$Y_\lambda \;=\; \{(x,y)\in\mathbb{Z}^2 \mid 1\le y\le \lambda_x\}\,. \tag{3.1}$$

We define the *arm-length* $a(\xi)$, *leg-length* $b(\xi)$ and *hook-length* $h(\xi)$ of $\xi=(x,y)\in Y_\lambda$ by

$$a(\xi)\;=\;\lambda_x-y,\quad b(\xi)\;=\;\lambda'_y-x,\quad h(\xi)\;=\;a(\xi)+b(\xi)+1. \tag{3.2}$$

Moreover, given partitions λ,σ with $Y_\sigma\subset Y_\lambda$, we define a skew Young diagram λ/σ by removing the cells of Y_σ from the cells of Y_λ. We call λ/σ a *border strip*, if it is connected (through edges of boxes, not only through vertices) and if it does not contain a 2×2 block. The *height* $\mathrm{ht}(\gamma)$ of a border strip γ is the number of rows it touches minus 1. Moreover, the *size* $|\gamma|$ of a border strip γ is the number of boxes it consists of. Note there is a bijection between cells $\xi\in Y_\lambda$ and border strips γ contained in λ, such that $\mathrm{ht}(\gamma)=b(\xi)$ and $|\gamma|=h(\xi)$.

The *Frobenius coordinates* of λ are $(a_1,\dots,a_d\mid b_1,\dots,b_d)$, where $a_i=a(i,i)$, $b_i=b(i,i)$ and $d=\max\{i\ge 1\mid (i,i)\in Y_\lambda\}$. Clearly, the Frobenius coordinates of λ uniquely identify the partition λ. We also define the *modified Frobenius coordinates* of λ as

$$c_i=a_i+\frac{1}{2}\in\mathbb{F}^+,\quad c_i^*=-b_i-\frac{1}{2}\in\mathbb{F}^-\qquad(i=1,\dots,d). \tag{3.3}$$

We will often make use of the set C_λ of modified Frobenius coordinates, i.e.,

$$C_\lambda=\{c_i\mid i=1,\dots,d\}\cup\{c_i^*\mid i=1,\dots,d\}\,. \tag{3.4}$$

Observe that $\sum_{c\in C_\lambda}\mathrm{sgn}(c)\,c=|\lambda|$. More generally, the shifted symmetric functions (1.13) equal

$$Q_k(\lambda)=\beta_k+\frac{1}{(k-1)!}\sum_{c\in C_\lambda}\mathrm{sgn}(c)\,c^{k-1}. \tag{3.5}$$

This follows from assigning the sequence

$$S_\lambda=\big(\lambda_j-j+\tfrac{1}{2}\big)_{j=1}^{\infty} \tag{3.6}$$

to each $\lambda\in\mathscr{P}$ and observing that S_λ and C_λ are related by

$$\mathbb{F}^+\cap S_\lambda=C_\lambda^+,\qquad \mathbb{F}^-\setminus S_\lambda=C_\lambda^-. \tag{3.7}$$

Denote by $\mathcal{S}$ the set of all decreasing sequences $S=(s_i)_{i=1}^\infty$ of half-integers for which $s_{i+1}=s_i-1$ for sufficiently large i, and by $\mathcal{S}_0\subset\mathcal{S}$ the subset of those sequences S for which the finite sets $\mathbb{F}^+\cap S$ and $\mathbb{F}^-\setminus S$ have the same cardinality. Note that the map $\lambda\to S_\lambda$ gives a bijection between $\mathscr{P}$ and $\mathcal{S}_0$.

3.2. **Infinite wedge formalism.** Let us recall the formalism of the fermionic Fock space F; for more details, we refer to [1, 28, 30, 32]. Let F be the $\mathbb{Q}$-vector space consisting of finite linear combinations of formal expressions

$$v_S \;=\; \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots \tag{3.8}$$

associated to $S = (s_i)_{i=1}^{\infty} \in \mathcal{S}$. We endow F with the scalar product $\langle\ ,\ \rangle$ for which the basis $\{v_S \mid S \in \mathcal{S}\}$ is orthonormal.

For $k \in \mathbb{F}$ we consider the operator $\psi_k : \mathsf{F} \to \mathsf{F}$ defined by

$$\psi_k\, v_S \;=\; \underline{k} \wedge \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots \tag{3.9}$$

where it is understood that we can put this expression in the canonical form (3.8) by using the skew-symmetry of $\wedge$. Namely, the result is zero whenever k is already in S, otherwise, we anti-commute $\underline{k}$ to the right of the $\underline{s_i}$ such that $s_i > k$.

For $k \in \mathbb{F}$, let ψ_k^* be the adjoint of these operators; their action on the basis vectors v_S is

$$\psi_k^*\, v_S \;=\; \begin{cases} (-1)^{i-1} v_{S\setminus\{s_i\}}, & \text{if } k = s_i \in S, \\ 0, & \text{if } k \notin S. \end{cases} \tag{3.10}$$

The operators ψ_k, ψ_k^* satisfy the *canonical anticommutation relations*, namely:

$$\psi_k\psi_\ell \,+\, \psi_\ell\psi_k \;=\; 0, \quad \psi_k^*\psi_\ell^* \,+\, \psi_\ell^*\psi_k^* \;=\; 0, \quad \psi_k\psi_\ell^* \,+\, \psi_\ell^*\psi_k \;=\; \delta_{k\ell} \quad (k, \ell \in \mathbb{F}). \tag{3.11}$$

We shall be in particular interested in the *charge zero subspace* F_0, spanned by the v_S with $S \in \mathcal{S}_0$. We identify v_λ with v_{S_λ} under the bijection between $\mathscr{P}$ and $\mathcal{S}_0$. In particular, $v_\emptyset$ is associated with the set $\{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots\}$, and every vector v_λ can be written as

$$v_\lambda \;=\; \sigma_\lambda\, \psi_{c_1} \cdots \psi_{c_d} \psi_{c_d^*}^* \cdots \psi_{c_1^*}^*\, v_\emptyset, \qquad \sigma_\lambda \;=\; (-1)^{\sum_{i=1}^d b_i}, \tag{3.12}$$

where c_i and c_i^* are the modified Frobenius coordinates of λ, see (3.3), and b_i are (part of) the Frobenius coordinates of λ. We also have

$$\psi_{-a}\, v_\emptyset \;=\; 0, \quad \psi_a^*\, v_\emptyset \;=\; 0 \qquad (a \in \mathbb{F}^+). \tag{3.13}$$

Accordingly, for $a \in \mathbb{F}^+$, the operators ψ_{-a}, ψ_a^* are termed *annihilation operators* and ψ_a, ψ_{-a}^* are termed *creation operators*.

Next, we recall the *fermionic normal order* :—:, which is obtained by moving all annihilation operators to the right of creation operators (as if they commuted), inserting a minus sign each time we swap two operators. More explicitly, it is defined recursively by

$$\begin{aligned} {:}\psi_a\phi_1\cdots\phi_k{:} &= \begin{cases} \psi_a \;{:}\phi_1\cdots\phi_k{:}, & \text{if } a > 0, \\ (-1)^k \;{:}\phi_1\cdots\phi_k{:}\; \psi_a, & \text{if } a < 0, \end{cases} \\ {:}\psi_a^*\phi_1\cdots\phi_k{:} &= \begin{cases} (-1)^k \;{:}\phi_1\cdots\phi_k{:}\; \psi_a^*, & \text{if } a > 0, \\ \psi_a^* \;{:}\phi_1\cdots\phi_k{:}, & \text{if } a < 0, \end{cases} \end{aligned} \tag{3.14}$$

where $\phi_1, \ldots, \phi_k \in \{\psi_i \mid i \in \mathbb{F}\} \cup \{\psi_i^* \mid i \in \mathbb{F}\}$. Note in particular that

$${:}\phi_{\sigma(1)}\cdots\phi_{\sigma(k)}{:} = \operatorname{sgn}(\sigma)\;\, {:}\phi_1\cdots\phi_k{:}\,, \tag{3.15}$$

where ϕ_i are as above and σ is a permutation of $\{1, \ldots, k\}$. Finally, the formula (cf. [1, Section 2.4])

$$\phi_1 \;{:}\phi_2\ldots\phi_k{:} \;=\; {:}\phi_1\phi_2\ldots\phi_k{:} \;+\; \sum_{i=2}^{k} (-1)^i \langle v_\emptyset, \phi_1\phi_i v_\emptyset\rangle \;{:}\phi_1\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_k{:} \tag{3.16}$$

(where ϕ_i are as above) allows one to express normally ordered monomials in ψ_i, ψ_i^* as a linear combination of monomials in ψ_i, ψ_i^*, and vice versa, and will be useful below.

3.3. **Boson-fermion correspondence.** Our interest in the space F_0 is motivated by the *boson-fermion correspondence*, which we recall now. Let us introduce the following operators on F:

$$\alpha_n \;=\; \sum_{a\in\mathbb{F}} {:}\psi_a\psi^*_{a-n}{:} \qquad (n\in\mathbb{Z}). \tag{3.17}$$

Note that the subspace F_0 coincides with the kernel of α_0.

Theorem 3.1 (Boson-fermion correspondence)**.** *Let $\Phi:\mathsf{F}_0\to\mathsf{B}$ be the isomorphism of vector spaces defined by $\Phi\,v_\lambda = s_\lambda$. Then, for all $n\geq 1$ we have the following identities of operators on F_0:*

$$\Phi^{-1}\,p_n\,\Phi \;=\; \alpha_n, \quad \Phi^{-1}\,n\frac{\partial}{\partial p_n}\,\Phi \;=\; \alpha_{-n} \qquad (n\geq 1). \tag{3.18}$$

Remark 3.2. The map Φ is an isometry with respect to the scalar products defined above (for which the v_λ respectively the Schur polynomials s_λ are orthonormal).

For the proof, we refer to [1, 24, 28].

4. First order correction to the eigenvalues

4.1. **Quadratic and quartic fermionic operators.** Let us introduce the following operators on F_0, where $a,b,c,d\in\mathbb{F}$:

$$\Xi_{ab} \;={:}\psi_a\psi^*_b{:}\,, \qquad \Xi_{abcd} \;={:}\psi_a\psi^*_b\psi_c\psi^*_d{:}\;. \tag{4.1}$$

Moreover, consider the following variations of the Kronecker delta function:

$$\eta_{abuv} \;=\; \delta^-_{av}\delta^+_{bu}-\delta^+_{av}\delta^-_{bu}\,, \qquad \delta^\pm_{lm} \;=\; \begin{cases}1 & \text{if } l=m \text{ and } \pm l>0,\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

Note that

$$\langle v_\emptyset,\psi_l\psi^*_m v_\emptyset\rangle = \delta^-_{lm}, \qquad \langle v_\emptyset,\psi^*_m\psi_l v_\emptyset\rangle = \delta^+_{lm}\,. \tag{4.3}$$

Lemma 4.1. *We have*

$$[\Xi_{ab},\Xi_{uv}] \;=\; \delta_{bu}\Xi_{av}-\delta_{av}\Xi_{ub}+\eta_{abuv}\,, \tag{4.4}$$

$$\begin{aligned}[\Xi_{abcd},\Xi_{uv}] \;=\;& \delta_{du}\Xi_{abcv}-\delta_{cv}\Xi_{abud}+\delta_{bu}\Xi_{avcd}-\delta_{av}\Xi_{ubcd}\\ &+\eta_{cduv}\Xi_{ab}+\eta_{abuv}\Xi_{cd}-\eta_{cbuv}\Xi_{ad}-\eta_{aduv}\Xi_{cb}\,.\end{aligned} \tag{4.5}$$

In particular, when $u=v$, the above formulae simplify to

$$[\Xi_{ab},\Xi_{uu}] \;=\; (\delta_{bu}-\delta_{au})\,\Xi_{ab}, \tag{4.6}$$

$$[\Xi_{abcd},\Xi_{uu}] \;=\; (\delta_{du}-\delta_{cu}+\delta_{bu}-\delta_{au})\,\Xi_{abcd}\,. \tag{4.7}$$

Proof. By (3.16) we get

$$\psi^*_b\;{:}\psi_u\psi^*_v{:} = {:}\psi^*_b\psi_u\psi^*_v{:} + \langle v_\emptyset,\psi^*_b\psi_u v_\emptyset\rangle\psi^*_v \;= {:}\psi^*_b\psi_u\psi^*_v{:} + \delta^+_{bu}\psi^*_v \tag{4.8}$$

and, again by an application of the same formula, we get

$$\begin{aligned}\psi_a\;{:}\psi^*_b\psi_u\psi^*_v{:} =\;& {:}\psi_a\psi^*_b\psi_u\psi^*_v{:} + \langle v_\emptyset,\psi_a\psi^*_b v_\emptyset\rangle\;{:}\psi_u\psi^*_v{:} + \langle v_\emptyset,\psi_a\psi^*_v v_\emptyset\rangle\;{:}\psi^*_b\psi_u{:}\\ =\;& {:}\psi_a\psi^*_b\psi_u\psi^*_v{:} + \delta^-_{ab}\;{:}\psi_u\psi^*_v{:} - \delta^-_{av}\;{:}\psi_u\psi^*_b{:}\end{aligned} \tag{4.9}$$

Noting that $\Xi_{ab} = \psi_a\psi^*_b - \delta^-_{ab}$ and combining these two identities we conclude that

$$\Xi_{ab}\Xi_{uv} \;=\; \Xi_{abuv} \;-\; \delta^-_{av}\Xi_{ub} \;+\; \delta^+_{bu}\Xi_{av} \;+\; \delta^-_{av}\delta^+_{bu}. \tag{4.10}$$

(This is essentially a special case of Wick's theorem.) Subtract from this the same relation after the substitutions $a\leftrightarrow u$ and $b\leftrightarrow v$ to obtain (4.4). Here, we use $\Xi_{abuv}=\Xi_{uvab}$, cf. (3.15). Next, using (4.10), we have

$$[\Xi_{abcd},\Xi_{uv}] = [\Xi_{ab}\Xi_{cd},\Xi_{uv}] + \delta^-_{ad}[\Xi_{cb},\Xi_{uv}] - \delta^-_{bc}[\Xi_{ad},\Xi_{uv}]$$

$$= \Xi_{ab}[\Xi_{cd}, \Xi_{uv}] + [\Xi_{ab}, \Xi_{uv}]\Xi_{cd} + \delta^-_{ad}[\Xi_{cb}, \Xi_{uv}] - \delta^-_{bc}[\Xi_{ad}, \Xi_{uv}] \,. \tag{4.11}$$

After some algebraic simplifications and using (4.4) and (4.10), we obtain (4.5). Finally, (4.6) and (4.7) follow from (4.4) and (4.5), respectively, by the identity $\eta_{abuu} = 0$. ☐

4.2. **Fermionic expressions for KdV Hamiltonian operators.** Let us denote $\widehat{g}^{[j]}_k(z) = \rho_0\big(g^{[j]}_k(z)\big) \in (\mathrm{End}\,\mathsf{B})[\![z^{\pm 1}]\!]$, cf. (1.8) and (2.5). Recall that Φ denotes the isomorphism in the boson-fermion correspondence, cf. Theorem 3.1. The following result was proved in [33].

Proposition 4.2 ([33, Theorem 5.3])**.** *We have*

$$\Phi^{-1}\, \widehat{g}^{[0]}_k(z)\, \Phi \;=\; \beta_k \;+\; \frac{1}{(k+1)!} \sum_{a,b\in\mathbb{F}} \left(\frac{a+b}{2}\right)^{k+1} \Xi_{ab}\, z^{b-a} \qquad (k \ge 0), \tag{4.12}$$

where the numbers β_k are defined by (1.14).

Next, we establish a formula for $\Phi^{-1}\, \widehat{g}^{[1]}_k(z)\, \Phi$ which involves quartic fermionic operators.

Proposition 4.3. *For all $k \ge 0$, we have*

$$\Phi^{-1}\, \widehat{g}^{[1]}_k(z)\, \Phi \;=\; \sum_{a,b,c,d\in\mathbb{F}} A_k(a,b,c,d)\, \Xi_{abcd}\, z^{d-c+b-a} \;+\; \sum_{a,b\in\mathbb{F}} B_k(a,b)\, \Xi_{ab}\, z^{b-a} \;+\; \gamma_k\,, \tag{4.13}$$

where $\gamma_k \in \mathbb{Q}$ and A_k and B_k are polynomials of degree $k+1$ and $k+2$ respectively, which satisfy the following intitial values

$$A_{-1}(a,b,c,d) \;=\; 0\,, \qquad B_{-1}(a,b) \;=\; 0 \tag{4.14}$$

and recursions $(k \ge 0)$

$$(k+2)\,(d-c+b-a)\, A_k(a,b,c,d) \;=\; \frac{d^2-c^2+b^2-a^2}{2} A_{k-1}(a,b,c,d) \;+\; P_k(a,b,c,d) \tag{4.15}$$

$$(k+2)\,(b-a)\, B_k(a,b) \;=\; \frac{b^2-a^2}{2} B_{k-1}(a,b) \;+\; R_k(a,b) \;+\; T_k(a,b) \tag{4.16}$$

where

$$P_k(a,b,c,d) \;=\; \frac{1}{k!\,2^k}\Big((a-b)^2\big((-a+b+c+d)^k - (a-b+c+d)^k\big) + (c-d)^2\big((a+b-c+d)^k - (a+b+c-d)^k\big)\Big) \tag{4.17}$$

$$R_k(a,b) \;=\; [\zeta^k]\, \mathrm{e}^{\zeta(a-b)/2}\big((b-\partial_\zeta)^2 - (a-b)^2\big)\frac{1-\mathrm{e}^{-(a-b)\zeta}}{\sinh(\zeta/2)} \tag{4.18}$$

$$T_k(a,b) \;=\; \frac{2}{3\,k!}\,\big(b(b^2-\tfrac14) - a(a^2-\tfrac14)\big)\left(\frac{a+b}{2}\right)^k. \tag{4.19}$$

In (4.18), the notation $[\zeta^k]f(\zeta)$ stands for the coefficient of ζ^k in the power series $f(\zeta) \in \mathbb{Q}[\![\zeta]\!]$.

Remark 4.4. The numbers γ_k are not computed in this proposition. It will be shown below, in the proof of Theorem 1.6, that $\gamma_k = 2\beta_2\beta_{k+1} + k(k+3)\beta_{k+3}$, cf. (1.14).

Before giving the proof of this proposition, let us define

$$\widehat{G}^{[j]}_k \;=\; \rho_0(G^{[j]}_k), \tag{4.20}$$

cf. (1.8) and (2.5), and let us state and prove a lemma.

Lemma 4.5. *We have*

$$\Phi^{-1}\,\widehat{G}_1^{[0]}\,\Phi \;=\; \sum_{a\in\mathbb{F}} \frac{a^2}{2}\,\Xi_{aa}\,, \tag{4.21}$$

$$\Phi^{-1}\,\widehat{G}_1^{[1]}\,\Phi \;=\; \sum_{\substack{a,b,c,d\in\mathbb{F},\\ a+c=b+d}} (a-b)^2\,\Xi_{abcd} \,+\, \sum_{a\in\mathbb{F}} \frac{2}{3}a\left(a^2-\frac{1}{4}\right)\Xi_{aa} \,+\, \frac{1}{120}\,. \tag{4.22}$$

Proof. The first identity is an immediate consequence of (4.12), so we only have to prove the second one. By (2.3), we have

$$G_1^{[1]} \;=\; \sum_{a\in\mathbb{Z}} a^2\; {:}\omega_a\omega_{-a}{:}\; + \frac{\hbar}{120}\,. \tag{4.23}$$

Hence, $\widehat{G}_1^{[1]} = 2\sum_{n\in\mathbb{Z}_{\geq 1}} n^3 p_n \frac{\partial}{\partial p_n} \,+\, \frac{1}{120}$, and so (according to Theorem 3.1)

$$\Phi^{-1}\,\widehat{G}_1^{[1]}\,\Phi \;-\; \frac{1}{120} \;=\; 2\sum_{n\geq 1} n^2\,\alpha_n\,\alpha_{-n} \;=\; 2\sum_{n\geq 1} n^2 \sum_{a,b\in\mathbb{F}} \Xi_{a,a-n}\,\Xi_{b,b+n}\,. \tag{4.24}$$

By an application of (4.10), we rewrite this expression as

$$\begin{aligned} &2\sum_{n\geq 1} n^2 \sum_{a,b\in\mathbb{F}} \Big(\Xi_{a,a-n,b,b+n} - \delta^-_{a,b+n}\Xi_{b,a-n} + \delta^+_{a-n,b}\Xi_{a,b+n} + \delta^-_{a,b+n}\delta^+_{a-n,b}\Big) \\ &\quad = 2\sum_{n\geq 1} n^2 \left(\sum_{a,b\in\mathbb{F}} \Xi_{a,a-n,b,b+n}\right) + 2\sum_{a\in\mathbb{F}} \Xi_{aa}\left(\delta_{a>0}\sum_{n=1}^{a-\frac{1}{2}} n^2 - \delta_{a<0}\sum_{n=1}^{-a-\frac{1}{2}} n^2\right). \end{aligned} \tag{4.25}$$

We finally use the identity $\Xi_{abcd} = \Xi_{cdab}$ in the first term and

$$\delta_{a>0}\sum_{n=1}^{a-\frac{1}{2}} n^2 - \delta_{a<0}\sum_{n=1}^{-a-\frac{1}{2}} n^2 \;=\; \tfrac{1}{3}a(a^2-\tfrac{1}{4}) \qquad (a\in\mathbb{F}) \tag{4.26}$$

in the second term to rewrite this expression as in (4.22). ☐

Proof of Proposition 4.3. Denote $h_k(z)$ the right-hand side in (4.13). As recalled in Section 2.1, the recursion (2.2) uniquely determines the $g_k(\epsilon;z)$ up to an additive constant. Therefore, we only need to show that the sequence $h_k(z)$ satisfies (2.6), namely, that for all $k\geq 0$ we have

$$(k+3)\,z\partial_z\,h_{k+1}(z) \;=\; \big[h_k(z),\Phi^{-1}\widehat{G}_1^{[0]}\Phi\big] \;+\; \big[\Phi^{-1}\widehat{g}_k^{[0]}(z)\Phi,\Phi^{-1}\widehat{G}_1^{[1]}\Phi\big]. \tag{4.27}$$

The left-hand side of (4.27) is

$$\begin{aligned} (k+3)\Bigg(&\sum_{a,b,c,d\in\mathbb{F}} (d-c+b-a)\,A_{k+1}(a,b,c,d)\,\Xi_{abcd}\,z^{d-c+b-a} \\ &\qquad + \sum_{a,b\in\mathbb{F}} (b-a)\,B_{k+1}(a,b)\,\Xi_{ab}\,z^{b-a}\Bigg). \end{aligned} \tag{4.28}$$

The first term on the right-hand side of (4.27) is, using Lemma 4.5,

$$\begin{aligned} &\big[h_k(z),\Phi^{-1}\widehat{G}_1^{[0]}\Phi\big] \\ &= \sum_{\ell,a,b,c,d\in\mathbb{F}} z^{d-c+b-a}\frac{\ell^2}{2}A_k(a,b,c,d)[\Xi_{abcd},\Xi_{\ell\ell}] \,+\, \sum_{\ell,a,b\in\mathbb{F}} z^{b-a}\frac{\ell^2}{2}B_k(a,b)[\Xi_{ab},\Xi_{\ell\ell}] \end{aligned}$$

$$= \sum_{a,b,c,d\in\mathbb{F}} z^{d-c+b-a}\frac{d^2-c^2+b^2-a^2}{2}A_k(a,b,c,d)\Xi_{abcd} + \sum_{a,b\in\mathbb{F}} z^{b-a}\frac{b^2-a^2}{2}B_k(a,b)\Xi_{ab}\,. \tag{4.29}$$

The second term on the right-hand side of (4.27) is, once again using Lemma 4.5,

$$\begin{aligned}\left[\Phi^{-1}\widehat{g}_k^{[0]}(z)\Phi, \Phi^{-1}\widehat{G}_1^{[1]}\Phi\right] &= \frac{1}{(k+1)!}\sum_{\substack{a,b,c,d,u,v\in\mathbb{F},\\ a+c=b+d}}(a-b)^2\left(\frac{u+v}{2}\right)^{k+1}[\Xi_{uv},\Xi_{abcd}]\,z^{v-u}\\ &+\frac{1}{(k+1)!}\sum_{a,u,v\in\mathbb{F}}\frac{2}{3}a\left(a^2-\frac{1}{4}\right)\left(\frac{u+v}{2}\right)^{k+1}[\Xi_{uv},\Xi_{aa}]\,z^{v-u}.\end{aligned} \tag{4.30}$$

It follows from Lemma 4.1 that this expression contains terms which are either quartic or quadratic in the ψ_i, ψ_i^*, which we now consider separately.

The quartic terms come from the first line only and they are

$$\begin{aligned}&\frac{-1}{(k+1)!}\sum_{\substack{a,b,c,d,u,v\in\mathbb{F},\\ a+c=b+d}}(a-b)^2\left(\frac{u+v}{2}\right)^{k+1}(\delta_{du}\Xi_{abcv}-\delta_{cv}\Xi_{abud}+\delta_{bu}\Xi_{avcd}-\delta_{av}\Xi_{ubcd})\,z^{v-u}\\ &\quad= -\frac{1}{(k+1)!}\sum_{a,b,c,v\in\mathbb{F}}(a-b)^2\left(\frac{a-b+c+v}{2}\right)^{k+1}\Xi_{abcv}\,z^{v-c+b-a}\\ &\qquad+\frac{1}{(k+1)!}\sum_{a,b,d,u\in\mathbb{F}}(a-b)^2\left(\frac{u+b+d-a}{2}\right)^{k+1}\Xi_{abud}\,z^{d+b-a-u}\\ &\qquad-\frac{1}{(k+1)!}\sum_{a,c,d,v\in\mathbb{F}}(c-d)^2\left(\frac{a+c-d+v}{2}\right)^{k+1}\Xi_{avcd}\,z^{v+d-c-a}\\ &\qquad+\frac{1}{(k+1)!}\sum_{b,c,d,u\in\mathbb{F}}(c-d)^2\left(\frac{u+b+d-c}{2}\right)^{k+1}\Xi_{ubcd}\,z^{d-c+b-u}\\ &\quad= \sum_{a,b,c,d\in\mathbb{F}}P_{k+1}(a,b,c,d)\,\Xi_{abcd}\,z^{d-c+b-a}\,,\end{aligned} \tag{4.31}$$

with $P_k(a,b,c,d)$ as in (4.17). This shows that the coefficients of $\Xi_{abcd}z^{a-b+c-d}$ in both sides of (4.27) match, provided (4.15) holds.

The quadratic terms instead come from both lines on the right-hand side of (4.30). Those coming from the first line are

$$\frac{-1}{(k+1)!}\sum_{\substack{a,b,c,d,u,v\in\mathbb{F},\\ a+c=b+d}}(a-b)^2\left(\frac{u+v}{2}\right)^{k+1}(\eta_{cduv}\Xi_{ab}+\eta_{abuv}\Xi_{cd}-\eta_{cbuv}\Xi_{ad}-\eta_{aduv}\Xi_{cb})\,z^{v-u}\,. \tag{4.32}$$

By renaming summation indexes in the second and fourth terms by the transformations $a\leftrightarrow c$ and $b\leftrightarrow d$, we see that this expression equals

$$\begin{aligned}&-\frac{2}{(k+1)!}\sum_{\substack{a,b,c,d,u,v\in\mathbb{F},\\ a+c=b+d}}(a-b)^2\left(\frac{u+v}{2}\right)^{k+1}+\left(\eta_{cduv}\Xi_{ab}-\eta_{cbuv}\Xi_{ad}\right)z^{v-u}\\ &\quad= -\frac{2}{(k+1)!}\sum_{\substack{a,b,c,d,u,v\in\mathbb{F},\\ a+c=b+d}}\left((a-b)^2-(a-d)^2\right)\left(\frac{u+v}{2}\right)^{k+1}+\eta_{cduv}\Xi_{ab}\,z^{v-u}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{(k+1)!}\Bigg(\sum_{\substack{a,b,c\in\mathbb{F},\\ c<0,\ c+a-b>0}} \big((a-b)^2-(b-c)^2\big)\left(\frac{a+2c-b}{2}\right)^{k+1} + \Xi_{ab}\, z^{b-a} \\
&\qquad - \sum_{\substack{a,b,c\in\mathbb{F},\\ c>0,\ c+a-b<0}} \big((a-b)^2-(b-c)^2\big)\left(\frac{a+2c-b}{2}\right)^{k+1} + \Xi_{ab}\, z^{b-a}\Bigg) \\
&= \sum_{a,b\in\mathbb{F}} \widehat{R}_{k+1}(a,b)\,\Xi_{ab}\, z^{b-a}, \qquad (4.33)
\end{aligned}
$$

where $\widehat{R}_k(a,b)$ is defined as

$$
\widehat{R}_k(a,b) = \frac{2}{k!}\Bigg(\sum_{\substack{c\in\mathbb{F}^-,\\ c+a-b>0}} \big((b-c)^2-(a-b)^2\big)\big(\tfrac{a+2c-b}{2}\big)^k - \sum_{\substack{c\in\mathbb{F}^+,\\ c+a-b<0}} \big((b-c)^2-(a-b)^2\big)\big(\tfrac{a+2c-b}{2}\big)^k\Bigg). \qquad (4.34)
$$

Let us show that $\widehat{R}_k(a,b) = R_k(a,b)$ as defined in (4.18). Indeed, we rewrite (4.34)

$$
\begin{aligned}
\widehat{R}_k(a,b) &= 2[\zeta^k]\, \mathrm{e}^{\zeta(a-b)/2}\big((b-\partial_\zeta)^2-(a-b)^2\big)\Bigg[\sum_{\substack{c\in\mathbb{F}^-,\\ c+a-b>0}} \mathrm{e}^{c\,\zeta} - \sum_{\substack{c\in\mathbb{F}^+,\\ c+a-b<0}} \mathrm{e}^{c\,\zeta}\Bigg] \\
&= [\zeta^k]\mathrm{e}^{\zeta(a-b)/2}\big((b-\partial_\zeta)^2-(a-b)^2\big)\frac{1-\mathrm{e}^{-(a-b)\zeta}}{\sinh(\zeta/2)} = R_k(a,b)\,. \qquad (4.35)
\end{aligned}
$$

The quadratic terms coming from the second line of (4.30) are computed by (4.6) as

$$
\begin{aligned}
&\frac{1}{(k+1)!}\sum_{u,v\in\mathbb{F}} \frac{2}{3}\left(v\left(v^2-\frac{1}{4}\right) - u\left(u^2-\frac{1}{4}\right)\right)\left(\frac{u+v}{2}\right)^{k+1} \Xi_{uv}\, z^{v-u} \\
&\qquad = \sum_{a,b\in\mathbb{F}} T_{k+1}(a,b)\,\Xi_{ab}\, z^{b-a}\,. \qquad (4.36)
\end{aligned}
$$

Therefore, also the terms of the form $\Xi_{ab} z^{b-a}$ in (4.27) coincide, provided (4.16) holds. This shows (4.27). The initial conditions (4.14) follow from (2.1). ☐

Corollary 4.6. *We have*

$$
\Phi^{-1}\,\widehat{G}_k^{[1]}\,\Phi = \sum_{a,b,c\in\mathbb{F}} A_k(a,b,c,a+c-b)\,\Xi_{a,b,c,a+c-b} + \sum_{a\in\mathbb{F}} B_k(a,a)\,\Xi_{aa} + \gamma_k\,, \qquad (4.37)
$$

where the polynomials A_k and B_k are defined in Proposition 4.3.

4.3. **Proof of Theorem 1.6.** We are now ready for the proof of Theorem 1.6. We first have the following simple lemma.

Lemma 4.7. *We have*

$$
E_k^{[1]}(\lambda;0) = \frac{1}{24}\langle s_\lambda, \widehat{G}_k^{[1]} s_\lambda\rangle = \frac{1}{24}\langle v_\lambda, \Phi^{-1}\widehat{G}_k^{[1]}\Phi v_\lambda\rangle\,. \qquad (4.38)
$$

Proof. The second equality follows from Theorem 3.1. The first one follows by considering terms of order $O(\epsilon)$ in (1.15) (setting $c=0$). ☐

According to Lemma 4.7 and Corollary 4.6, we have

$$
24\,E_k^{[1]}(\lambda;0) = \sum_{a,b,c\in\mathbb{F}} A_k(a,b,c,a+c-b)\langle v_\lambda, \Xi_{a,b,c,a+c-b} v_\lambda\rangle + \sum_{a\in\mathbb{F}} B_k(a,a)\langle v_\lambda, \Xi_{aa} v_\lambda\rangle + \gamma_k\,. \qquad (4.39)
$$

All ingredients needed to simplify this expression are given by the following proposition. Recall the constant terms β_k of the shifted symmetric functions Q_k were defined by (1.14).

Proposition 4.8. *Let $a,b,c,d \in \mathbb{F}$ and $k \geq 0$. The following are true:*

(i) $\langle v_\lambda, \Xi_{aa} v_\lambda\rangle \;=\; \sum_{c\in C_\lambda} \operatorname{sgn}(c)\,\delta_{a,c}$.

(ii) $B_k(a,a) \;=\; \dfrac{k(k+3)}{(k+2)!}a^{k+2} \;-\; \dfrac{1}{12\,k!}a^k \;+\; 2\beta_{k+1}a \;-\; 2(k+1)\beta_{k+2},$.

(iii) *The scalar product $\langle v_\lambda, \Xi_{abcd} v_\lambda\rangle$ is nonzero only if ($a=b$ and $c=d$), or ($a=d$ and $b=c$).*

(iv) $\langle v_\lambda, \Xi_{abba} v_\lambda\rangle \;=\; \sum_{c,c'\in C_\lambda} \operatorname{sgn}(c\,c')\,\delta_{a,c}\,\delta_{b,c'} \qquad (a \neq b)$.

(v) $A_k(a,a,b,b) = 0$ *and* $A_k(a,b,b,a) = \dfrac{(a-b)(b^k-a^k)}{k!}$.

Proof. (i) Let us first show that

$$\Xi_{aa} v_\lambda \;=\; \left[\sum_{c\in C_\lambda} \operatorname{sgn}(c)\,\delta_{ac}\right] v_\lambda\,. \tag{4.40}$$

We start with the special case

$$\Xi_{aa} \;=\, :\!\psi_a\psi_a^*\!: \,=\begin{cases}\psi_a\psi_a^*, & \text{if } a>0,\\ -\psi_a^*\psi_a, & \text{if } a<0\end{cases} \tag{4.41}$$

of (3.14). Therefore, when $a\in\mathbb{F}^+$, $\Xi_{aa}v_\lambda = \sigma_\lambda\psi_a\psi_a^*\psi_{c_1}\cdots\psi_{c_d}\psi^*_{c_d^*}\cdots\psi^*_{c_1^*}v_\emptyset$, cf. (3.12). Using the commutation relations (3.11) and that $\psi_a^* v_\emptyset = 0$, we infer that this expression is nonzero only if $a = c_i$ for some i, in which case $\Xi_{aa}v_\lambda = v_\lambda$. Similarly, when $a\in\mathbb{F}^-$, $\Xi_{aa}v_\lambda = -\sigma_\lambda\psi_a^*\psi_a\psi_{c_1}\cdots\psi_{c_d}\psi^*_{c_d^*}\cdots\psi^*_{c_1^*}v_\emptyset$, and since $\psi_a v_\emptyset = 0$ this expression is nonzero only if $a = c_i^*$ for some i, in which case $\Xi_{aa}v_\lambda = -v_\lambda$. Equation (4.40) follows and (i) is an immediate consequence.

(ii) Because of (4.16), the polynomials B_k satisfy

$$(k+3)B_{k+1}(a,a) \;=\; a\,B_k(a,a) + \widetilde{R}_{k+1}(a) + \widetilde{T}_{k+1}(a)\,, \tag{4.42}$$

where

$$\begin{aligned}\widetilde{R}_k(a) \;&=\; \lim_{b\to a}\frac{R_k(a,b)}{b-a}\\ &=\; [\zeta^k]\left((a-\partial_\zeta)^2\frac{\zeta}{\sinh(\zeta/2)}\right)\\ &=\; -2\beta_k a^2 \;+\; 4(k+1)\beta_{k+1}a \;-\; 2(k+1)(k+2)\beta_{k+2}\,.\end{aligned} \tag{4.43}$$

$$\widetilde{T}_k(a) \;=\; \lim_{b\to a}\frac{T_k(a,b)}{b-a} \;=\; \frac{2}{k!}a^k\left(a^2-\tfrac{1}{12}\right). \tag{4.44}$$

(Here we use (1.14).) It is readily checked that the right-hand side of (ii) satisfies the same recursion and initial data as in (4.42), thus proving (ii).

(iii) According to (4.10), we have

$$\langle v_\lambda, \Xi_{abcd}v_\lambda\rangle \;=\; \langle v_\lambda, \Xi_{ab}\Xi_{cd}v_\lambda\rangle + \delta^-_{ad}\langle v_\lambda, \Xi_{cb}v_\lambda\rangle - \delta^+_{bc}\langle v_\lambda, \Xi_{ad}v_\lambda\rangle - \delta^-_{ad}\delta^+_{bc}\,. \tag{4.45}$$

Since $\langle v_\lambda, \Xi_{pq}v_\lambda\rangle$ can be nonzero only if $p=q$, the last three terms on the right-hand side can be nonzero if and only if $a=d$ and $b=c$. Since $(\Xi_{ab})^* = \Xi_{ba}$, the first term on the right-hand side is $\langle \Xi_{ba}v_\lambda, \Xi_{cd}v_\lambda\rangle$, which can be nonzero only if $a=d$ and $b=c$ or if $a=b$ and $c=d$.

(iv) If $a\neq b$, it can be checked from (3.14) that $\Xi_{abba} = \Xi_{aa}\Xi_{bb}$ and (iv) follows from (4.40).

(v) follows directly from (4.15). ☐

We are ready to prove our first main result.

Proof of Theorem 1.6. We start from (4.39) and use Proposition 4.8 to obtain the following expression for $24\,E_k^{[1]}(\lambda;0)$:

$$\sum_{c,c'\in C_\lambda}\frac{c^kc'+c\,c'^k-c^{k+1}-c'^{k+1}}{k!}\mathrm{sgn}(c\,c')+$$
$$+\sum_{c\in C_\lambda}\left(\frac{k(k+3)}{(k+2)!}c^{k+2}-\frac{1}{12\,k!}c^k+2\beta_{k+1}c-2(k+1)\beta_{k+2}\right)\mathrm{sgn}(c)+\gamma_k\,. \tag{4.46}$$

Observe, using (3.5) and $\sum_{c\in C_\lambda}\mathrm{sgn}(c)=0$, that the first line is $2(Q_2-\beta_2)(Q_{k+1}-\beta_{k+1})$ and that the second line is

$$k(k+3)(Q_{k+3}-\beta_{k+3})+2\beta_2(Q_{k+1}-\beta_{k+1})+2\beta_{k+1}(Q_2-\beta_2)+\gamma_k\,. \tag{4.47}$$

Hence,

$$24\,E_k^{[1]}(\lambda;0)=2Q_2Q_{k+1}+k(k+3)Q_{k+3}+\gamma_k-2\beta_2\beta_{k+1}-k(k+3)\beta_{k+3}$$

Using Theorems 1.12 and 1.13, we deduce that the constant term $\gamma_k-2\beta_2\beta_{k+1}-k(k+3)\beta_{k+3}$ must vanish, which concludes the proof (and yields the explicit value for γ_k anticipated in Remark 4.4). This proves (2.8), and hence Theorem 1.6. ☐

4.4. Proof of Corollary 1.7.

Proof. Recalling (1.3) and (1.4), the coefficient of ϵ in the constant term of $G_k(\epsilon)$ is

$$I_{g,k;\emptyset}=\int_{\mathrm{DR}_g(0)}\psi_1^k\lambda_1=(-1)^g\int_{\overline{\mathcal{M}}_{g,1}}\lambda_1\lambda_g\psi^k, \tag{4.48}$$

where we use the fact that $\mathrm{DR}_g(0)$ is Poincaré dual to $(-1)^g\lambda_g$ [23, Section 0.5.3]. This intersection number is nonzero only if $3g-2=1+g+k$, i.e., only if $k=2g-3$. On the other hand, according to Theorem 1.6, the same term is computed as the value of (2.8) at $\lambda=\emptyset$, for which we finally use the identity $Q_k(\emptyset)=\beta_k$, cf. (1.13). ☐

5. First order correction to the eigenvector

5.1. A metric on partitions. Given $\lambda,\mu\in\mathscr{P}$, we define the *Hamming distance* between λ and μ as

$$d(\lambda,\mu)=\frac{1}{2}\big|C_\lambda\,\Delta\,C_\mu\big| \tag{5.1}$$

where $X\,\Delta\,Y=(X\cup Y)\setminus(X\cap Y)$ is the symmetric difference of sets X,Y.

Lemma 5.1.
(i) *The Hamming distance d is an integer-valued metric on $\mathscr{P}$.*
(ii) *We have $d(\lambda,\mu)=\frac{1}{2}\big|S_\lambda\,\Delta\,S_\mu\big|$, where S_λ is given by (3.6).*
(iii) *If $|\lambda|=|\mu|$, then $d(\lambda,\mu)\neq 1$.*
(iv) *For all $k\geq 0$, if $d(\lambda,\mu)=2k$, then $|S_\lambda\setminus S_\mu|=2k=|S_\mu\setminus S_\lambda|$.*

Proof. It is well-known that the symmetric difference defines a metric. Hence, for (i) it remains to show that d only takes integer values. Since $C_\lambda\,\Delta\,C_\mu=(C_\lambda\setminus C_\mu)\sqcup(C_\mu\setminus C_\lambda)$ and $\sum_{c\in C_\lambda}\mathrm{sgn}(c)=0$ we obtain that

$$\sum_{c\in C_\lambda\setminus C_\mu}\mathrm{sgn}(c)=\sum_{c\in C_\mu\setminus C_\lambda}\mathrm{sgn}(c). \tag{5.2}$$

In particular, $|C_\lambda\setminus C_\mu|$ and $|C_\mu\setminus C_\lambda|$ have the same parity, proving that d is integer-valued.

For (ii), recall from (3.7) that $C^+_\lambda = S^+_\lambda$ and $C^-_\lambda = \mathbb{F}^- \setminus S^-_\lambda$. Hence,

$$C_\lambda \,\Delta\, C_\mu = (C^+_\lambda \,\Delta\, C^+_\mu) \cup (C^-_\lambda \,\Delta\, C^-_\mu) = (S^+_\lambda \,\Delta\, S^+_\mu) \cup (S^-_\lambda \,\Delta\, S^-_\mu) = S_\lambda \,\Delta\, S_\mu, \tag{5.3}$$

where we use that the symmetric difference is invariant under taking complements, i.e.,

$$(Z \setminus X) \,\Delta\, (Z \setminus Y) = X \Delta Y \qquad \text{(for all sets } X, Y \subseteq Z).$$

Next, for (iii) recall that $|\lambda| = \sum_{c\in C_\lambda} |c|$ for all $\lambda \in \mathscr{P}$. Hence, if $|\lambda| = |\mu|$ we have

$$\sum_{c\in C_\lambda \setminus C_\mu} |c| = \sum_{c\in C_\mu \setminus C_\lambda} |c|. \tag{5.4}$$

If $d(\lambda, \mu) = 1$, i.e., $|C_\mu \,\Delta\, C_\lambda| = 2$, this implies that $|C_\lambda \setminus C_\mu| = 1 = |C_\mu \setminus C_\lambda|$, and hence because of (5.2) that $C_\lambda \setminus C_\mu = C_\mu \setminus C_\lambda$, contradiction.

Finally, for any pair of partitions λ, μ let us denote $b^\pm_{\lambda\mu} = |S^\pm_\lambda \setminus S^\pm_\mu|$. If $d(\lambda, \mu) = 2k$ we must have

$$b^+_{\lambda\mu} + b^+_{\mu\lambda} + b^-_{\lambda\mu} + b^-_{\mu\lambda} = 4k \tag{5.5}$$

We have $b^+_{\lambda\mu} = |C^+_\lambda \setminus C^+_\mu|$ and $b^-_{\lambda\mu} = |C^-_\mu \setminus C^-_\lambda|$ (note the order of λ and μ). Hence, condition (5.2) implies

$$b^+_{\lambda\mu} - b^-_{\mu\lambda} = b^+_{\mu\lambda} - b^-_{\lambda\mu}. \tag{5.6}$$

Therefore, combining the last identity with (5.5) we get $b^+_{\lambda\mu} + b^-_{\lambda\mu} = 2k = b^+_{\mu\lambda} + b^-_{\mu\lambda}$, and (iv) is proved. □

Let us introduce the following *neighborhood* of λ:

$$\mathcal{U}(\lambda) \;=\; \big\{\mu \in \mathscr{P} \;\big|\; |\mu| = |\lambda| \text{ and } d(\lambda, \mu) = 2\big\}. \tag{5.7}$$

According to the last point in the previous lemma, for $\mu \in \mathcal{U}(\lambda)$, there exist $1 \le a < b$ and $1 \le a' < b'$ such that

$$S_\lambda \setminus S_\mu = \{\lambda_a - a + \tfrac12,\, \lambda_b - b + \tfrac12\} \qquad S_\mu \setminus S_\lambda = \{\mu_{a'} - a' + \tfrac12,\, \mu_{b'} - b' + \tfrac12\}. \tag{5.8}$$

Note that, since $|\lambda| = |\mu|$, for all K large enough

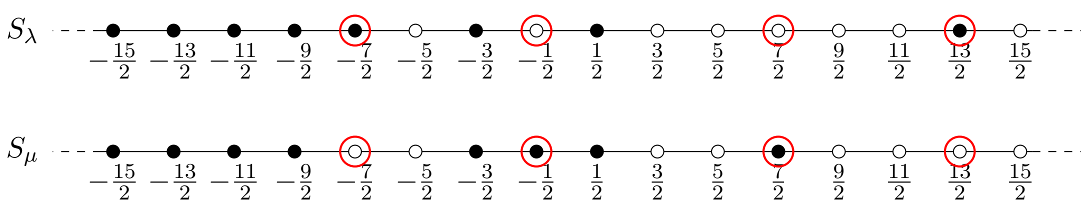
$$\sum_{i=1}^{K} (\lambda_i - i) = \sum_{i=1}^{K} (\mu_i - i) \tag{5.9}$$

(in fact, $K \ge \ell(\lambda), \ell(\mu)$ suffices) and so

$$\lambda_a - a + \lambda_b - b = \mu_{a'} - a' + \mu_{b'} - b'. \tag{5.10}$$

Before proceeding, it is convenient to illustrate our arguments with one example.

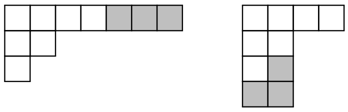
Example 5.2. *Let $\lambda = (7, 2, 1)$ and $\mu = (4, 2, 2, 2)$, which satisfy $|\lambda| = |\mu| = 10$ and $d(\lambda, \mu) = 2$. Consider the sequences S_λ and S_μ depicted as Maya diagram* [28, 32]:



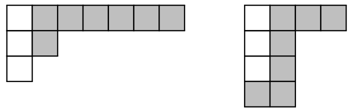
Here, the black dots correspond to elements in S_λ and S_μ, and we indicated in red the four places at which S_λ and S_μ differ; note that, in this case, the indices in (5.10) *are given by $a = 1$, $b = 4$, $a' = 1$, $b' = 3$.*

Next, recall that in Maya diagrams, removing a border strip is represented by swapping a black dot with any white dot to its left. Therefore we see that there are exactly two ways to remove a border strip from λ and one from μ such that the resulting partitions coincide. Namely, we either swap the white and black dots at $\frac72$ and $\frac{13}{2}$ in S_λ and at $-\frac72$ and $-\frac12$

in S_μ*, or we swap white and black dots at* $\frac12$ *and* $\frac{13}{2}$ *in* S_λ *and at* $-\frac72$ *and* $\frac72$ *in* S_μ*. These border strips are visualized in the Young diagrams as follows*



and



The general situation is completely parallel, as shown in the following lemma.

Lemma 5.3. *Let* $\mu \in \mathcal{U}(\lambda)$*. There exist exactly two border strips* γ_1, γ_2 *in* λ *and two border strips* γ_1', γ_2' *in* μ *such that* $\lambda \setminus \gamma_i = \mu \setminus \gamma_i'$ $(i = 1, 2)$*. These border strips satisfy*

(i) $\gamma_1 \subset \gamma_2\,,\ \gamma_1' \subset \gamma_2'\,,$

(ii) $|\gamma_1| = |\gamma_1'| = |\lambda_a - a - \mu_{a'} + a'|,\ |\gamma_2| = |\gamma_2'| = |\lambda_a - a - \mu_{b'} + b'|$

(iii) *If* $\lambda_a - a > \mu_{a'} - a'$*, then*

$$\mathrm{ht}(\gamma_1) = a' - a,\ \mathrm{ht}(\gamma_1') = b - b',\ \mathrm{ht}(\gamma_2) = b' - a - 1,\ \mathrm{ht}(\gamma_2') = b - a' \tag{5.11}$$

and if $\lambda_a - a < \mu_{a'} - a'$*, then*

$$\mathrm{ht}(\gamma_1) = b' - b,\ \mathrm{ht}(\gamma_1') = a - a',\ \mathrm{ht}(\gamma_2) = b' - a,\ \mathrm{ht}(\gamma_2') = b - a' - 1. \tag{5.12}$$

Proof. We can assume without loss of generality that $\lambda_a - a > \mu_{a'} - a'$ (otherwise we just interchange the roles of λ and μ). Since a and a' are minimal such that $\lambda_a - a$ and $\mu_{a'} - a'$ are not elements of S_μ and S_λ respectively, this implies $a' \geq a$. Using (5.10) one concludes $a \leq a' < b' \leq b$. There are exactly two ways to remove border strips from λ and μ such that the resulting partitions coincide: we either swap white and black dots at $\mu_{a'} - a' + \frac12$ and $\lambda_a - a + \frac12$ in S_λ and those at $\lambda_b - b + \frac12$ and $\mu_{b'} - b' + \frac12$ in S_μ (corresponding to border strips γ_1, γ_1' respectively), or we swap white and black dots at $\mu_{b'} - b' + \frac12$ and $\lambda_a - a + \frac12$ in S_λ and those at $\lambda_b - b + \frac12$ and $\mu_{a'} - a' + \frac12$ in S_μ (corresponding to border strips γ_2, γ_2' respectively). The properties (i) and (ii) are clear by construction. The last one follows from the fact that the height of a border strip in a partition ν is read off the Maya diagram as follows: the border strip itself is a pair of white and black dots, the white one to the left of the right one, and the height is the number of black dots (i.e., of elements of S_ν) in between them (see, e.g., [32]). ☐

We now compute the scalar products relevant to the proof of Theorem 1.8.

Lemma 5.4. *Let* $\lambda, \mu \in \mathscr{P}$ *with* $\lambda \neq \mu$*.*

(i) *If* $\mu \notin \mathcal{U}(\lambda)$*, then* $\langle s_\mu, \widehat{G}_k^{[1]} s_\lambda \rangle = 0$ *for all* $k \geq 0$*.*

(ii) *If* $\mu \in \mathcal{U}(\lambda)$*, then (with the notations of the previous lemma)*

$$\begin{aligned}\langle s_\mu, \widehat{G}_1^{[1]} s_\lambda \rangle &= 2\,(-1)^{\mathrm{ht}(\gamma_1)+\mathrm{ht}(\gamma_1')}|\gamma_1|^2 + (-1)^{\mathrm{ht}(\gamma_2)+\mathrm{ht}(\gamma_2')}|\gamma_2|^2 \\ &= 2\,(-1)^{a+a'+b+b'}\left((\lambda_a - a - \mu_{a'} + a')^2 - (\lambda_a - a - \mu_{b'} + b')^2\right).\end{aligned} \tag{5.13}$$

Remark 5.5. Note that the last identity involves twice the quantity $\sum_{m\geq1} m^3 \left\langle s_\mu, p_m \frac{\partial s_\lambda}{\partial p_m}\right\rangle$. It is straightforward to adapt the proof below to show that, more generally, we have

$$\begin{aligned}\sum_{m\geq1} m^{n+1}\left\langle s_\mu, p_m \frac{\partial s_\lambda}{\partial p_m}\right\rangle &= (-1)^{\mathrm{ht}(\gamma_1)+\mathrm{ht}(\gamma_1')}|\gamma_1|^n + (-1)^{\mathrm{ht}(\gamma_2)+\mathrm{ht}(\gamma_2')}|\gamma_2|^n \\ &= (-1)^{a+a'+b+b'}\left((\lambda_a - a - \mu_{a'} + a')^n - (\lambda_a - a - \mu_{b'} + b')^n\right)\end{aligned} \tag{5.14}$$

for any integer $n \geq 0$ and partitions λ, μ satisfying $\mu \in \mathcal{U}(\lambda)$.

Proof. Recalling Corollary 4.6 we obtain that, when $\lambda \neq \mu$, $\langle s_\mu, \widehat{G}_k^{[1]} s_\lambda \rangle$ is a linear combination of $\langle v_\mu, \Xi_{a,b,c,a+c-b} v_\lambda \rangle$ and $\langle v_\mu, \Xi_{aa} v_\lambda \rangle$, for $a, b, c \in \mathbb{F}$. That the latter terms do not contribute is clear from the fact that Ξ_{aa} is diagonal on v_λ, cf. (4.40). Moreover, $\Xi_{abcd} v_\lambda$

is a linear combination of v_μ such that at most four modified Frobenius coordinates of μ are different from those of λ, i.e., such that $d(\mu,\lambda) \le 2$. Since $|\lambda| = |\mu|$ and $\lambda \neq \mu$ also these other terms do not contribute, cf. point (iii) in Lemma 5.1). It remains to prove the last assertion. To this end, we use (4.24) to write

$$\langle s_\mu, \widehat{G}_1^{[1]} s_\lambda\rangle \;=\; 2\sum_{n\ge1} n^2 \langle v_\lambda, \alpha_n\alpha_{-n} v_\mu\rangle \qquad (\lambda\neq\mu) \tag{5.15}$$

where α_n are the operators defined in (3.17). It is well known (see, for example, [32]) that

$$\alpha_{-n} v_\lambda \;=\; \sum_{\gamma\in \mathrm{BS}(\lambda,n)} (-1)^{\mathrm{ht}(\gamma)} v_{\lambda\setminus\gamma} \qquad (n\ge 1) \tag{5.16}$$

where $\mathrm{BS}(\lambda,n)$ is the set of border strips of size n in a partition λ. Since the operators α_n and α_{-n} are mutually adjoint, cf. [32], we get

$$\langle s_\mu, \widehat{G}_1^{[1]} s_\lambda\rangle \;=\; 2\sum_{n\ge1} n^2 \langle \alpha_{-n}v_\mu, \alpha_{-n} v_\lambda\rangle \;=\; 2\sum_{n\ge1} n^2 \sum_{\substack{\gamma\in\mathrm{BS}(\lambda,n)\\ \gamma'\in\mathrm{BS}(\mu,n)}} (-1)^{\mathrm{ht}(\gamma)+\mathrm{ht}(\gamma')}\,\delta_{\lambda\setminus\gamma,\mu\setminus\gamma'}\,. \tag{5.17}$$

Finally, we can use Lemma 5.3 to complete the proof. □

5.2. **Proof of Theorem 1.8.** With the notations of (1.15) and (1.16), we define coefficients $c(\lambda,\mu)$, depending on pairs of partitions λ,μ, by

$$r_\lambda^{[1]}(\boldsymbol{p}) \;=\; \frac{1}{24}\sum_{\mu\in\mathscr{P}} c(\lambda,\mu)\, s_\mu(\boldsymbol{p})\,. \tag{5.18}$$

By (1.18), $c(\lambda,\lambda)=0$ for all $\lambda\in\mathscr{P}$. Obviously, since $r_\lambda(\epsilon)\in \mathsf{B}_{|\lambda|}[\![\epsilon]\!]$, one has $c(\lambda,\mu)=0$ unless $|\lambda|=|\mu|$.

Lemma 5.6. *For all $\lambda,\mu\in\mathscr{P}$ we have*

$$c(\lambda,\mu) \;=\; \begin{cases} \dfrac{\langle s_\mu, \widehat{G}_1^{[1]} s_\lambda\rangle}{Q_3(\lambda)-Q_3(\mu)} & \text{if } Q_3(\lambda)\neq Q_3(\mu),\\ 0 & \text{otherwise.}\end{cases} \tag{5.19}$$

Proof. Considering the linear terms in ϵ in (1.15) (after setting $c=0$) yields

$$\widehat{G}_k^{[1]} s_\lambda \;+\; \sum_{\nu\in\mathscr{P}} c(\lambda,\nu)\,\widehat{G}_k^{[0]} s_\nu \;=\; 24\,E^{[1]}(\lambda)\,s_\lambda \;+\; Q_k(\lambda)\sum_{\nu\in\mathscr{P}} c(\lambda,\nu)\,s_\nu \tag{5.20}$$

where we use (1.12), (4.20) (with $c=0$) and (5.18). Using (1.10) we have $G_k^{[0]} s_\nu = Q_k(\nu) s_\nu$ and so taking the scalar product of (5.20) with s_μ we obtain

$$\langle s_\mu, \widehat{G}_k^{[1]} s_\lambda\rangle \;+\; Q_k(\mu)\,c(\lambda,\mu) \;=\; Q_k(\lambda)\,c(\lambda,\mu). \tag{5.21}$$

Setting $k=3$, this shows the part of the statement relative to the case $Q_3(\lambda)\neq Q_3(\mu)$. Moreover, the same equation shows that, if instead $Q_3(\lambda)=Q_3(\mu)$ and $\lambda\neq\mu$, then $\langle s_\mu, \widehat{G}_1^{[1]} s_\lambda\rangle = 0$. Lemma 5.4 then implies that $\mu\notin\mathcal{U}(\lambda)$ and, therefore, that $\langle s_\mu, \widehat{G}_k^{[1]} s_\lambda\rangle = 0$ for all $k\ge 0$. Finally, there must exist a $k>3$ such that $Q_k(\lambda)\neq Q_k(\mu)$. Hence, (5.21) implies that $c(\lambda,\mu)=0$ in this case. □

The previous lemmas imply that Theorem 1.8 is equivalent to the following statement.

Theorem 5.7. *For all partitions λ*

$$r_\lambda^{[1]} \;=\; \frac{1}{12}\sum_{\mu\in\mathcal{U}(\lambda)} (-1)^{a+a'+b+b'}\left(\frac{\lambda_a-a-\mu_{a'}+a'}{\lambda_a-a-\mu_{b'}+b'} - \frac{\lambda_a-a-\mu_{b'}+b'}{\lambda_a-a-\mu_{a'}+a'}\right) s_\mu.$$

Proof. With the notation in (5.8), we note that for $\mu \in \mathcal{U}(\lambda)$ we have

$$\begin{aligned} Q_3(\lambda) - Q_3(\mu) &= \frac{1}{2}\Big(\big(\lambda_a - a + \tfrac{1}{2}\big)^2 + \big(\lambda_b - b + \tfrac{1}{2}\big)^2 - \big(\mu_{a'} - a' + \tfrac{1}{2}\big)^2 - \big(\mu_{b'} - b' + \tfrac{1}{2}\big)^2\Big) \\ &= \big(\lambda_a - a - \mu_{a'} + a'\big)\big(\lambda_a - a - \mu_{b'} + b'\big)\,, \end{aligned} \tag{5.22}$$

as it follows from (3.5) and (5.10). (Note that this quantity also equals $w_{\lambda\mu}\,|\gamma_1|\,|\gamma_2|$, in the notation of Theorem 1.8.) Then, from Lemma 5.4 and Lemma 5.6 and elementary algebraic simplification using (5.10), we obtain the statement of the theorem. ☐

Appendix A. Tables supporting Conjecture 1.9

Polynomials obtained by matching Conjecture 1.9 with the eigenvalues computed in SageMath for $k = 0, \ldots, 10$. Note that we did not restrict the degree of $f_{D,\nu}$ to be $2D$, but only verified this bound in each of the examples.

ν	D	$f_{D,\nu}(k)$
$\emptyset$	0	1
$\emptyset$	1	$\frac{1}{24}k(k+3)$
$\emptyset$	2	$\frac{1}{3456}k(k+4)(3k^2+16k+17)$
$\emptyset$	3	$\frac{1}{1244160}k(k+5)(15k^4+210k^3+896k^2+1405k+50)$
$\emptyset$	4	$\frac{1}{597196800}k(k+6)(75k^6+1950k^5+17570k^4+68042k^3+89913k^2 -100568k-277382)$
$\emptyset$	5	$\frac{1}{300987187200}k(k+7)(315k^8+13020k^7+199920k^6+1433012k^5+4539665k^4 +142128k^3-39506516k^2-99890840k-59164224)$
(1)	0	$\frac{1}{12}$
(1)	1	$\frac{1}{288}\left(k^2+2k-1\right)$
(1)	2	$\frac{1}{207360}(k+1)\left(15k^3+95k^2-148k-616\right)$
(1)	3	$\frac{1}{14929920}\left(15k^6+240k^5+467k^4-5229k^3-24476k^2-33159k-7482\right)$
(1)	4	$\frac{1}{50164531200}\big(525k^8+14700k^7+91910k^6-448826k^5-5919635k^4 -17900130k^3-11934720k^2+25552736k+32402640\big)$
(2)	0	$\frac{1}{288}$
(2)	1	$\frac{1}{103680}\left(15k^2+323k+278\right)$
(2)	2	$\frac{1}{4976640}\left(15k^4+696k^3+3170k^2+3631k-30\right)$
(2)	3	$\frac{1}{2508226560}\big(105k^6+7833k^5+85309k^4+27485k^3-1345564k^2-3262016k -1993680\big)$
(1^2)	0	$\frac{1}{288}$
(1^2)	1	$\frac{1}{6912}k(k+1)$
(1^2)	2	$\frac{1}{4976640}\left(15k^4+80k^3-421k^2-486k-60\right)$
(1^2)	3	$\frac{1}{358318080}\left(15k^6+195k^5-832k^4-9599k^3-24029k^2-22834k-4056\right)$

ν	D	$f_{D,\nu}(k)$
(3)	0	$-\frac{1}{6912}$
(3)	1	$-\frac{1}{165888}k(k+1)$
(3)	2	$\frac{1}{119439360}\left(-15k^4-80k^3+421k^2+486k+60\right)$
$(2,1)$	0	$\frac{1}{3456}$
$(2,1)$	1	$\frac{1}{1244160}\left(15k^2+308k+30\right)$
$(2,1)$	2	$\frac{1}{59719680}\left(15k^4+666k^3+1690k^2+2271k+1052\right)$
(1^3)	0	$\frac{1}{10368}$
(1^3)	1	$\frac{1}{248832}\left(k^2+3\right)$
(1^3)	2	$\frac{1}{179159040}\left(15k^4+50k^3-699k^2+994k+60\right)$
(4)	0	$-\frac{967}{829440}$
(4)	1	$\frac{1}{418037760}\left(-20307k^2-290329k-229408\right)$
$(3,1)$	0	$-\frac{109}{622080}$
$(3,1)$	1	$\frac{1}{29859840}\left(-218k^2+1247k-1488\right)$
(2^2)	0	$\frac{1}{165888}$
(2^2)	1	$\frac{1}{59719680}\left(15k^2+601k+1232\right)$
$(2,1^2)$	0	$\frac{1}{82944}$
$(2,1^2)$	1	$\frac{1}{29859840}\left(15k^2+293k-188\right)$
(1^4)	0	$\frac{1}{497664}$
(1^4)	1	$\frac{1}{11943936}\left(k^2-k+8\right)$
(5)	0	$\frac{253}{2903040}$
$(4,1)$	0	$\frac{967}{238878720}$
$(3,2)$	0	$-\frac{109}{8599633920}$
$(3,1^2)$	0	$\frac{109}{206391214080}$
$(2^2,1)$	0	$\frac{1}{660451885056}$
$(2,1^3)$	0	$-\frac{1}{23776267862016}$
(1^5)	0	$\frac{1}{5706304286883840}$

References

[1] A. Alexandrov, A. Zabrodin. Free fermions and tau-functions. *J. Geom. Phys.* **67** (2013), 37–80.
[2] V. Bazhanov, S. Lukyanov, A. Zamolodchikov. Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz. *Comm. Math. Phys.* **177** (1996), no. 2, 381–398.
[3] S. Bloch, A. Okounkov. The character of the infinite wedge representation. *Adv. Math.* **149** (2000), no. 1, 1–60.

[4] X. Blot. The quantum Witten–Kontsevich series and one-part double Hurwitz numbers. *Geom. Topol.* **26** (2022), no. 4, 1669–1743.

[5] X. Blot, A. Buryak. Quantum intersection numbers and the Gromov–Witten invariants of $\mathbb{CP}^1$. Preprint arXiv:2402.16464.

[6] X. Blot, D. Lewanski, S. Shadrin. On the strong DR/DZ conjecture. Preprint arXiv:2405.12334.

[7] A. Buryak. Double ramification cycles and integrable hierarchies. *Comm. Math. Phys.* **336** (2015), no. 3, 1085–1107.

[8] A. Buryak, B. Dubrovin, J. Guéré, P. Rossi. Integrable systems of double ramification type. *Int. Math. Res. Not. IMRN* (2020), no. 24, 10381–10446.

[9] A. Buryak, J. Guéré, P. Rossi. DR/DZ equivalence conjecture and tautological relations. *Geom. Topol.* **23** (2019), no. 7, 3537–3600.

[10] A. Buryak, P. Rossi. Double ramification cycles and quantum integrable systems. *Lett. Math. Phys.* **106** (2016), no. 3, 289–317.

[11] A. Buryak, S. Shadrin, L. Spitz, D. Zvonkine. Integrals of ψ-classes over double ramification cycles. *Amer. J. Math.* **137** (2015), no. 3, 699–737.

[12] D. Chen, M. Möller, D. Zagier. Quasimodularity and large genus limits of Siegel–Veech constants. *J. Amer. Math. Soc.* **31** (2018), no. 4, 1059–1163.

[13] D. Chen, M. Möller, A. Sauvaget, D. Zagier. Masur–Veech volumes and intersection theory on moduli spaces of Abelian differentials. *Invent. Math.* **222** (2020), no. 1, 283–373.

[14] R. Dijkgraaf. Mirror symmetry and elliptic curves. *The moduli space of curves (Texel Island, 1994)*, 149–163, *Progr. Math.*, 129, Birkhäuser Boston, Boston, MA, 1995.

[15] B. Dubrovin. Symplectic field theory of a disk, quantum integrable systems, and Schur polynomials. *Ann. Henri Poincaré* **17** (2016), no. 7, 1595–1613.

[16] Y. Eliashberg, A. Givental, H. Hofer. Introduction to symplectic field theory. *Geom. Funct. Anal.* 2000, Special Volume, Part II, 560–673.

[17] P. Engel. Hurwitz theory of elliptic orbifolds, I. *Geom. Topol.* **25** (2021), no. 1, 229–274.

[18] A. Eskin, A. Okounkov. Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials. *Invent. Math.* **145** (2001), no. 1, 59–103.

[19] M. A. Hahn, J.-W. M. van Ittersum, F. Leid. Triply mixed coverings of arbitrary base curves: quasimodularity, quantum curves and a mysterious topological recursion. *Ann. Inst. Henri Poincaré D* **9** (2022), no. 2, 239–296.

[20] J.-W. M. van Ittersum. A symmetric Bloch–Okounkov theorem. *Res. Math. Sci.* **8** (2021), no. 2, Paper No. 19, 42 pp.

[21] J.-W. M. van Ittersum, G. Ruzza. Quantum KdV hierarchy and quasimodular forms. *Comm. Number Theory Phys.* (2024), to appear.

[22] V. Ivanov, G. Olshanski. Kerov's central limit theorem for the Plancherel measure on Young diagrams. *Symmetric functions 2001: surveys of developments and perspectives*, 93–151, vol. 74 of *NATO Sci. Ser. II Math. Phys. Chem.*, Kluwer Acad. Publ., Dordrecht, 2002.

[23] F. Janda, R. Pandharipande, A. Pixton, D. Zvonkine. Double ramification cycles on the moduli spaces of curves. *Publ. Math. Inst. Hautes Études Sci.* **125** (2017), 221–266.

[24] V. G. Kac, A. Raina, N. Rozhkovskaya. Bombay lectures on highest weight representations of infinite dimensional Lie algebras. Second edition. Adv. Ser. Math. Phys., 29 *World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ,* 2013. xii+237 pp.

[25] S. Kerov, G. Olshanski. Polynomial functions on the set of Young diagrams. *C. R. Acad. Sci., Paris, Sér. I*, **319** (1994), no. 2, 121–126.

[26] M. Kontsevich, Yu. Manin. Gromov–Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.* 164 (1994), no. 3, 525–562.

[27] I. G. Macdonald. Symmetric functions and Hall polynomials. Oxford Classic Texts in the Physical Sciences. *The Clarendon Press, Oxford University Press, New York,* 2015.

[28] T. Miwa, M. Jimbo, E. Date. Solitons. Differential equations, symmetries and infinite-dimensional algebras. Translated from the 1993 Japanese original by Miles Reid. Cambridge Tracts in Math., 135. *Cambridge University Press, Cambridge,* 2000.

[29] H. Ochiai. Counting functions for branched covers of elliptic curves and quasi-modular forms. Representation theory of vertex operator algebras and related topics (Japanese) (Kyoto, 2000), No. 1218 (2001), 153–167, 2001.

[30] A. Okounkov, R. Pandharipande. Gromov–Witten theory, Hurwitz theory, and completed cycles. *Ann. of Math. (2)* **163** (2006), no. 2, 517–560.

[31] A. Pobregkov. The boson-fermion correspondence and quantum integrable and dispersionless models. *Uspekhi Mat. Nauk* **58** (2003), no. 5, 163–196; translation in *Russian Math. Surveys* **58** (2003), no. 5, 1003–1037.

[32] R. Ríos-Zertuche. An introduction to the half-infinite wedge. *Mexican mathematicians abroad: recent contributions*, 197–237, Contemp. Math., 657, Aportaciones Mat., *Amer. Math. Soc., Providence, RI*, 2016.
[33] P. Rossi. Gromov–Witten invariants of target curves via symplectic field theory. *J. Geom. Phys.* **58** (2008), no. 8, 931–941.
[34] G. Ruzza, D. Yang. On the spectral problem of the quantum KdV hierarchy. *J. Phys. A* **54** (2021), no. 37, Paper No. 374001, 27 pp.
[35] E. Witten. Two-dimensional gravity and intersection theory on moduli space. *Surveys in differential geometry (Cambridge, MA, 1990),* 243–310.
[36] D. Zagier. Partitions, quasimodular forms, and the Bloch–Okounkov theorem. *Ramanujan J.* **41** (2016), no. 1-3, 345–368.

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
Current address: Department of Mathematics and Computer Science, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
Email address: j.w.ittersum@uni-koeln.de

Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, Campo Grande Edifício C6, 1749-016, Lisboa, Portugal
Grupo de Física Matemática, Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
Email address: gruzza@fc.ul.pt