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# BRACKET WIDTH OF CURRENT LIE ALGEBRAS

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Abstract. The length of an element $z$ of a Lie algebra $L$ is defined as the smallest number $s$ needed to represent $z$ as a sum of $s$ brackets. The bracket width of $L$ is defined as supremum of the lengths of its elements. Given a finite-dimensional simple Lie algebra $\mathfrak{g}$ over an algebraically closed field $\Bbbk$ of characteristic zero, we study the bracket width of current Lie algebras $L = \mathfrak{g} \otimes A$. We show that for an arbitrary $A$ the width is at most 2. For $A = \Bbbk[[t]]$ and $A = \Bbbk[t]$ we compute the width for algebras of types A and C.

## 1. Introduction

Given a Lie algebra $L$ over an infinite field $\Bbbk$, we define its bracket width as the supremum of lengths $\ell(z)$, where $z$ runs over the derived algebra $[L, L]$ and $\ell(z)$ is defined as the smallest number $n$ of Lie brackets $[x_i, y_i]$ needed to represent $z$ in the form $z = \sum_{i=1}^{n}[x_i, y_i]$.

There are many examples of Lie algebras of bracket width strictly bigger than one, see, e.g., [Rom16]. However, the width of any finite-dimensional complex *simple* Lie algebras is equal to one [Br63]. For finite-dimensional simple *real* Lie algebras the problem of existence of an algebra of width greater than one is still wide open, see [Ak15].

The first examples of simple Lie algebras of bracket width greater than one were found only recently in [DKR21, Theorem A] among complex *infinite-dimensional* algebras. Namely, they appeared among Lie algebras of vector fields $\mathrm{Vec}(C)$ on smooth affine curves $C$ with trivial tangent bundle, which are simple by [Jo86] and [Si96, Proposition 1]. More recently, it was proved in [MR23] that the bracket width of such Lie algebras is less than or equal to three, and if in addition $C$ is a plane curve with the unique place at infinity, the bracket width of $\mathrm{Vec}(C)$ equals two.

In the present paper, we study the bracket width of another class of infinite-dimensional Lie algebras, namely current Lie algebras.

Let $\Bbbk$ be an algebraically closed field of characteristic zero, $\mathfrak{g}$ be a finite-dimensional simple Lie $\Bbbk$-algebra, $A$ be a commutative associative $\Bbbk$-algebra with the identity. The current algebra corresponding to $\mathfrak{g}$ and $A$ is defined as the tensor product $\mathfrak{g} \otimes_{\Bbbk} A$ with the bracket

$$[x \otimes a, y \otimes b] := [x, y] \otimes ab.$$

With respect to this bracket $\mathfrak{g} \otimes_{\Bbbk} A$ is a Lie algebra.

Our first result provides an upper estimate for the bracket width of an arbitrary current algebra.

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**Theorem 1.** *The bracket width of $\mathfrak{g}\otimes_{\Bbbk} A$ is less than or equal to 2.*

The main object of our interest is the Lie algebra $\mathfrak{g}\otimes_{\Bbbk} A$ where $A=\Bbbk[[t]]$ is the algebra of formal power series. In this case we expect a more precise statement.

**Conjecture 2.** *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Then the bracket width of $\mathfrak{g}\otimes_{\Bbbk}\Bbbk[[t]]$ is equal to 2 if $\mathfrak{g}$ is of type $\mathrm{A}_n$ or $\mathrm{C}_n$ $(n\geq 2)$ and to 1 otherwise.*

Our results partially confirm this expectation.

**Theorem 3.**

(i) *The bracket width of $\mathfrak{sl}_2\otimes\Bbbk[[t]]$ is equal to 1.*

(ii) *If $\mathfrak{g}$ is of type $\mathrm{A}_n$ or $\mathrm{C}_n$ $(n\geq 2)$, the bracket width of $\mathfrak{g}\otimes\Bbbk[[t]]$ is equal to 2.*

Some arguments supporting the conjecture for the types other than $\mathrm{A}_n$ or $\mathrm{C}_n$ will be given later, in Section 3.

We deduce from (the proof of) Theorem 3 some results on other current algebras.

**Corollary 4.** *Let $\mathfrak{g}=\mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$ $(n\geq 2)$. Then for $A=\Bbbk[t]$ the width of $\mathfrak{g}\otimes_{\Bbbk} A$ is equal to 2.*

This statement can be generalized to a wider class of rings $A$ as follows.

**Corollary 5.** *Let $\mathfrak{g}=\mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$ $(n\geq 2)$. Let $A$ be a ring containing an ideal $\mathfrak{a}$ such that the quotient $\bar{A}=A/\mathfrak{a}$ is a two-dimensional $\Bbbk$-algebra. Then the width of $\mathfrak{g}\otimes_{\Bbbk} A$ is equal to 2.*

## 2. Proofs

We begin with the following general statement on finite-dimensional simple Lie algebras [BN11, Theorem 26].

**Proposition 6.** *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra defined over an arbitrary infinite field of characteristic not 2 or 3. Then there exist $w_1, w_2\in\mathfrak{g}$ such that*

$$\mathfrak{g}=[w_1,\mathfrak{g}]+[w_2,\mathfrak{g}].$$

This immediately implies Theorem 1.

*Proof of Theorem 1.* Consider a linear basis of $A$, $A=\langle 1=a_0,a_1,a_2,\dots\rangle$. We have

$$\mathfrak{g}\otimes A=\mathfrak{g}\otimes 1\oplus\mathfrak{g}\otimes a_1\oplus\mathfrak{g}\otimes a_2\oplus\dots$$

Any element $z$ of $\mathfrak{g}\otimes A$ can be written in the form $z=\sum_{i=0}^{k} z_i\otimes\alpha_i a_i$ with $z_i\in\mathfrak{g}$, $\alpha_i\in\Bbbk$. By Proposition 6, for every $z_i$ there exist $x_i,y_i\in\mathfrak{g}$ such that

$$z_i=[w_1,x_i]+[w_2,y_i].$$

Thus, we have:

$$z=\sum_{i=0}^{k} z_i\otimes\alpha_i a_i=\left[w_1,\sum_{i=0}^{k} x_i\otimes\alpha_i a_i\right]+\left[w_2,\sum_{i=0}^{k} y_i\otimes\alpha_i a_i\right].$$

This completes the proof. ☐

*Proof of Theorem 3.* Our first step consists in reformulating the property of $L:=\mathfrak{g}\otimes\Bbbk[[t]]$ to be of bracket width one as some condition on $\mathfrak{g}$.

**Proposition 7.**

(i) *The bracket width of $L$ is equal to $1$ if and only if $\mathfrak{g}$ satisfies the following condition $(*)$: every nonzero element $c \in \mathfrak{g}$ can be represented as a bracket of elements without common centralizer, i.e. there exist $a, b \in \mathfrak{g}$ such that $c = [a, b]$ and*

$$C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b) = (0). \tag{1}$$

(ii) *Assume that $A$ satisfies the conditions of Corollary 5. Then condition $(*)$ is necessary for $\mathfrak{g} \otimes A$ to be of bracket width $1$.*

The following simple lemma is needed for the proof of Proposition 7.

**Lemma 8.** *Condition (1) is equivalent to the following one:*

$$\operatorname{im}(\operatorname{ad} a) + \operatorname{im}(\operatorname{ad} b) = \mathfrak{g}. \tag{2}$$

*Proof of Lemma 8.* Let $(,)$ denote the Killing form on $\mathfrak{g}$, and let $V \subset \mathfrak{g}$ denote the orthogonal complement to $\operatorname{im}(\operatorname{ad} a) + \operatorname{im}(\operatorname{ad} b)$.

Suppose that condition (1) holds and prove (2). Assume to the contrary that (2) does not hold, i.e. $V \neq (0)$. Let $d$ be a nonzero element of $V$. Then for any $e \in \mathfrak{g}$ we have $([e, a], d) = ([e, b], d) = 0$. As the Killing form is invariant, this gives $(e, [a, d]) = (e, [b, d]) = 0$. Since $e$ is an arbitrary element of $\mathfrak{g}$ and the Killing form is non-degenerate, we have $[a, d] = [b, d] = 0$, i.e. $d$ centralizes both $a$ and $b$, contradiction.

Conversely, suppose that condition (2) holds and prove (1). Assume to the contrary that $C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b) \neq (0)$. Let $d \neq 0$ centralize both $a$ and $b$. Then the same argument as above shows that $d \in V$, contradiction. □

*Proof of Proposition 7.* (i) Suppose that $\mathfrak{g}$ satisfies condition $(*)$ and show that the bracket width of $L = \mathfrak{g} \otimes \Bbbk[[t]]$ is equal to 1. Let

$$z = z_0 + z_1 \otimes t + z_2 \otimes t^2 + \dots, \quad z_i \in \mathfrak{g},$$

be an arbitrary element of $L$. We want to represent it as $z = [x, y]$ where

$$x = x_0 + x_1 \otimes t + x_2 \otimes t^2 + \dots, \quad y = y_0 + y_1 \otimes t + y_2 \otimes t^2 + \dots, \quad x_i, y_i \in \mathfrak{g},$$

which gives the equation

$$\sum_{k=0}^{\infty} \sum_{i+j=k} [x_i, y_j] \otimes t^k = \sum_{k=0}^{\infty} z_k \otimes t^k,$$

which, in turn, yields the system of equations

$$\begin{aligned} [x_0, y_0] &= z_0 \\ [x_0, y_1] + [x_1, y_0] &= z_1 \\ &\dots \\ [x_0, y_k] + [x_k, y_0] &= z_k - \sum_{i=1}^{k-1} [x_i, y_{k-i}] \\ &\dots \end{aligned} \tag{3}$$

Without loss of generality we can assume $z_0 \neq 0$. Condition $(*)$ together with Lemma 8 allows one to find $x_0$, $y_0$ and then $x_1$, $y_1$. By induction, we find all other $x_k$ and $y_k$.

Conversely, assuming that the bracket width of $L$ equals 1, looking at the zeroth and first equations of the above system and applying Lemma 8 once again, we conclude that condition $(*)$ holds in $\mathfrak{g}$.

(ii) Suppose that $A$ satisfies the conditions of Corollary 5 and that the width of $\mathfrak{g}\otimes A$ is equal to 1. We have to show that condition $(*)$ holds. We argue as in the necessity part of the proof of (i). Namely, let $\{1,\bar{t}\}$ be a linear basis of $\bar{A}=A/\mathfrak{a}$, and fix a preimage $t$ of $\bar{t}$. Let $z=z_0\otimes 1+z_1\otimes t$ be an element of $\mathfrak{g}\otimes A$ with $z_0\neq 0$. Any such $z$ can be represented in the form $z=[x,y]$ with

$$x=x_0+x_1\otimes t+\sum_{i\geq 2}x_i\otimes a_i,\quad y=y_0+y_1\otimes t+\sum_{i\geq 2}y_i\otimes b_i$$

with $a_i,b_i\in\mathfrak{a}$. We then arrive at the system consisting of the first two equations in (3). By Lemma 8, condition $(*)$ holds in $\mathfrak{g}$. □

We now continue the proof of Theorem 3 using the criterion obtained in Proposition 7.

*Proof of Theorem 3* (i). This case is easy because any element $c$ of $\mathfrak{g}=\mathfrak{sl}_2$ is either nilpotent or semisimple.

First assume that $c$ is nilpotent. We can use the natural representation of $\mathfrak{g}$ and write $c=e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Take $a=h/2=\operatorname{diag}(1/2,-1/2)$, $b=c$. We obtain $[a,b]=c$, $C_{\mathfrak{g}}(a)=\operatorname{span}(a)$, $C_{\mathfrak{g}}(b)=\operatorname{span}(b)$, so that $C_{\mathfrak{g}}(a)\cap C_{\mathfrak{g}}(b)=(0)$.

Let now $c$ be semisimple, write $c=h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Take $a=e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $b=\begin{pmatrix}0&0\\1&0\end{pmatrix}$. We obtain $[a,b]=c$, $C_{\mathfrak{g}}(a)=\operatorname{span}(a)$, $C_{\mathfrak{g}}(b)=\operatorname{span}(b)$, so that $C_{\mathfrak{g}}(a)\cap C_{\mathfrak{g}}(b)=(0)$, as above. Condition $(*)$ is satisfied, hence the bracket width of $L$ is equal to 1, as claimed.

*Proof of Theorem 3* (ii). We have to prove that for $\mathfrak{g}=\mathfrak{sl}_n$ $(n\geq 3)$ and $\mathfrak{g}=\mathfrak{sp}_{2n}$ $(n\geq 2)$ the bracket width of $L$ is greater than 1. Together with the upper estimate from Theorem 1 this will imply that the width is equal to 2.

We thus have to prove that $\mathfrak{g}$ does not satisfy condition $(*)$. This means that we have to exhibit an element $c$ such that any $a,b$ with $[a,b]=c$ have a nonzero common centralizer. We shall choose $c$ to be a rank 1 matrix in the natural representation of $\mathfrak{g}$. This will allow us to apply the following general lemma from linear algebra.

**Lemma 9.** [Gu79] *Let $A,B$ be square matrices such that $\operatorname{rk}(AB-BA)\leq 1$. Then one can simultaneously conjugate $A$ and $B$ to upper triangular form.*

**Remark 10.** See [EG02, Lemma 12.7] for an alternative proof of Guralnick's lemma (attributed to Rudakov).

We now go over to a more general set-up, using the notion of *almost commuting scheme* of $\mathfrak{g}$, see [GG06], [Lo21]. First, let us define it for $\mathfrak{g}=\mathfrak{sl}_n$.

Let $R$ denote the vector space $\mathfrak{sl}_n^{\oplus 2}\oplus\mathbb{C}^n\oplus(\mathbb{C}^n)^*$. The subscheme $M_n\subset R$ is defined as

$$\{(x,y,i,j)\in\mathfrak{sl}_n^{\oplus 2}\oplus\mathbb{C}^n\oplus(\mathbb{C}^n)^*\mid [x,y]+ij=0\}\tag{4}$$

and is called the almost commuting scheme of $\mathfrak{sl}_n$.

In a similar way, for $\mathfrak{g}=\mathfrak{sp}_{2n}$ we consider its natural representation $\mathbb{C}^{2n}$, identify $S^2(\mathbb{C}^{2n})$ with $\mathfrak{sp}_{2n}$ and thus view $i^2\in S^2(\mathbb{C}^{2n})$ as an element of $\mathfrak{sp}_{2n}$. The almost commuting scheme

$X_n$ of $\mathfrak{sp}_{2n}$ is then defined similarly to (4):

$$X_n := \{(x, y, i)\} \in \mathfrak{sp}_{2n}^{\oplus 2} \oplus \mathbb{C}^{2n} \mid [x, y] + i^2 = 0\}. \tag{5}$$

Note that both varieties carry a natural action of $G = \mathrm{SL}_n$ or $\mathrm{Sp}_{2n}$, respectively. Say, $G = \mathrm{SL}_n$ acts on $M_n$ by the formula

$$g(x, y, i, j) = (gxg^{-1}, gyg^{-1}, gi, jg^{-1}).$$

Note that such an action on $M_n$ is well-defined since $(gxg^{-1}, gyg^{-1}, gi, jg^{-1}) \in M_n$ whenever $(x, y, i, j) \in M_n$ as the following computations show:

$$[gxg^{-1}, gyg^{-1}] + gijg^{-1} = g[x, y]g^{-1} + gijg^{-1} = g(-ij)g^{-1} + gijg^{-1} = 0.$$

We have the following generalization of Guralnick's lemma, see [EG02, Lemma 12.7] and [Lo21, Lemma 2.1].

**Lemma 11.** *Let $(x, y, i, j) \in M_n$ (resp. $(x, y, i) \in X_n$). Then there is a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ that contains both $x$ and $y$.* □

We continue the proof of Theorem 3(ii). In our new notation, we have to prove that given any $(a, b, i, j) \in M_n$ (resp. $(a, b, i) \in X_n$), the elements $a$ and $b$ have nonzero common centralizer in $\mathfrak{g}$.

First suppose that both $a$ and $b$ are nilpotent. Then they are both contained in the nilradical $\mathfrak{n}$ of $\mathfrak{b}$. Since $\mathfrak{n}$ is nilpotent, its centre is nontrivial, and any its element is a common centralizer of $a$ and $b$.

So assume that at least one of $a$ and $b$ is not nilpotent and consider the orbit $O = G(a, b, i, j)$ (resp. $G(a, b, i)$). For the sake of brevity, in both cases we denote it by $O_{a,b}$.

In the sequel, we shall use $Q_\mathfrak{g}$ as a common notation for $M_n$ and $X_n$. Lemma 11 implies the following description of closed orbits in $Q_\mathfrak{g}$.

**Lemma 12.** *The orbit $O_{a,b}$ is closed if and only if $a$ and $b$ are commuting semisimple elements.*

*Proof.* Case A: By Guralnick's Lemma, it is sufficient to consider the case of upper triangular matrices. But then the orbit of a one-parameter group of matrices (the one-parameter torus corresponding to the coweight $2\rho^\vee$)

$$\{T_t := \mathrm{diag}(t^{n-1}, t^{n-3}, \ldots, t^{-n+1}) \mid t \in \Bbbk^*\}$$

is a quasi-affine subvariety

$$\{(T_t a T_t^{-1}, T_t b T_t^{-1}, T_t i, j T_t^{-1}) \mid t \in \Bbbk^*\} \subset Q_\mathfrak{g}$$

which contains $(a_s, b_s, 0, 0)$ in its closure. This proves the statement for $\mathfrak{g} = \mathfrak{sl}_n$.

Case C: A similar argument works in the case of $\mathfrak{g} = \mathfrak{sp}_{2n}$ (see also [Lo21, Corollary 2.2]). □

By Lemmas 11 and 12, we may assume that the closure of $O_{a,b}$ contains the closed orbit $O_{a_s,b_s}$ where $a_s$ and $b_s$ are commuting diagonal matrices.

Following [Lo21, Section 2.2], denote by $\mathfrak{l}$ the common centralizer in $\mathfrak{g}$ of the (commuting) elements $a_s$ and $b_s$, it is a Levi subalgebra of $\mathfrak{g}$. Denote by $L$ the corresponding Levi subgroup of $G$.

We are going to apply Luna's slice theorem [Lu73]. We will use the exposition of the slice method from lecture notes by Kraft [Kr15]. So let $X = Q_\mathfrak{g}$, $O = O_{a_s,b_s}$, then the almost commuting scheme $Q_{\mathfrak{l}}$ of $\mathfrak{l}$ is the required étale slice $S$ (see [Lo21, Lemma 2.4]), so that in an étale neighbourhood of $O$ we have an excellent morphism

$$\varphi\colon G \times^L S \to X \tag{6}$$

taking the image $[g, s] \in G \times^L S$ of the pair $(g, s) \in G \times S$ to $gs$; in particular, $\varphi$ is étale, and its image is affine and open in $X$, see [Kr15, Theorem 4.3.2].

Since $L$ is a *reductive* group of the form $\prod_{i=1}^k \mathrm{GL}_{n_i} \times \mathrm{Sp}_{2n_0}$ where each of the first $k$ factors necessarily has a nontrivial centre, $S = Q_{\mathfrak{l}}$ is of the form

$$\mathbb{C}^{2k} \times \prod_{i=1}^{k} M_{n_i} \times X_{n_0}, \tag{7}$$

where the $n_i$ correspond to the partition $\operatorname{rk} L = n_0 + n_1 + \cdots + n_k$ with $n_0 \geq 0$, $n_i > 0$ $(i = 1, \ldots, k)$, and $\mathbb{C}^{2k}$ is identified with $\mathfrak{z}(\mathfrak{l})^{\oplus 2}$, see [Lo21, 2.2].

The presence of the nontrivial $\mathfrak{z}(\mathfrak{l})$ is of critical importance: it guarantees the existence of a pair $(z, z') \in \mathfrak{z}(\mathfrak{l})^{\oplus 2}$ with nonzero components each of those centralizes both $x_s$ and $y_s$ (of course, the simplest choice is $z = x_s$, $z' = y_s$).

Thus for any element of

$$Q_{\mathfrak{l}} \subset \oplus_{i=1}^k (\mathfrak{gl}_{n_i}^{\oplus 2} \oplus \mathbb{C}^{2n_i} \oplus (\mathbb{C}^*)^{2n_i}) \oplus \mathfrak{sp}_{2n_0}^{\oplus 2} \oplus \mathbb{C}^{2n_0}$$

of the form

$$(x_{n_1}, y_{n_1}, i_{n_1}, j_{n_1}, \ldots, x_{n_k}, y_{n_k}, i_{n_k}, j_{n_k}, x_{n_0}, y_{n_0}, i_{n_0})$$

the elements $x = (x_{n_1}, \ldots, x_{n_k}, x_{n_0})$, $y = (y_{n_1}, \ldots, y_{n_k}, y_{n_0}) \in \oplus_{i=1}^k \mathfrak{gl}_{n_i} \oplus \mathfrak{sp}_{2n_0}$ have a nonzero common centralizer.

Given any finite-dimensional simple Lie algebra $\mathfrak{g}$, denote by $F_{\mathfrak{g}}$ the set of pairs $(x, y) \in \mathfrak{g}^{\oplus 2}$ such that $x$ and $y$ have a nonzero common centralizer, and let $U_{\mathfrak{g}} := \mathfrak{g}^{\oplus 2} \setminus F_{\mathfrak{g}}$ denote its complement.

The following lemma is a variation on a theme of Arzhantsev [Ar24, Section 5].

**Lemma 13.** *The set $U_{\mathfrak{g}}$ is open and Zariski dense in $\mathfrak{g}^{\oplus 2}$.*

*Proof.* First fix a pair $(a, b) \in \mathfrak{g} \oplus \mathfrak{g}$ and define a linear map $T_{a,b}\colon \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ by

$$T_{a,b}(x) = ([a, x], [b, x]).$$

Let now $V$ denote the vector space of linear maps $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$. Define $\psi\colon \mathfrak{g} \oplus \mathfrak{g} \to V$ by $\psi(a, b) = T_{a,b}$, it is a linear map. Let $W \subset V$ denote the set of maps of maximal rank, it is open in $V$. Consider the preimage $\psi^{-1}(W)$. Note that

$$\ker T_{a,b} = C_{\mathfrak{g}}(a) \cap C_{\mathfrak{g}}(b).$$

Hence we have $\psi^{-1}(W) = U_{\mathfrak{g}}$ because if $a$ and $b$ have a non-zero common centralizer, then $\ker T_{a,b} \neq 0$ and therefore the rank of $T_{a,b}$ is strictly less than $\dim \mathfrak{g}$. Thus $U_{\mathfrak{g}}$ is open in $\mathfrak{g} \oplus \mathfrak{g}$ as the preimage of an open set. It remains to note that $U_{\mathfrak{g}}$ is non-empty. Indeed (see [Ar24, Remark 3]), any simple finite-dimensional Lie algebra $\mathfrak{g}$ is two-generated and centreless, so that any pair of generators $(a, b)$ belongs to $U_{\mathfrak{g}}$. The lemma is proven. ☐

Let $F'_{\mathfrak{g}} := F_{\mathfrak{g}} \oplus \mathbb{C}^{2n}$, embed it into $\mathfrak{g}^{\oplus 2} \oplus \mathbb{C}^{2n}$ and define $F''_{\mathfrak{g}} := F'_{\mathfrak{g}} \cap Q_{\mathfrak{g}}$.

By Lemma 13, $U_{\mathfrak{g}}$ is open and Zariski dense in $\mathfrak{g}^{\oplus 2}$. Hence $U'_{\mathfrak{g}} := U_{\mathfrak{g}} \oplus \mathbb{C}^{2n}$, embedded into $\mathfrak{g}^{\oplus 2} \oplus \mathbb{C}^{2n}$, is also open. Therefore $F'_{\mathfrak{g}}$ is closed in $\mathfrak{g}^{\oplus 2} \oplus \mathbb{C}^{2n}$, and thus $F''_{\mathfrak{g}}$ is closed in $Q_{\mathfrak{g}}$.

We wish to prove that $F''_{\mathfrak{g}} = C_{\mathfrak{g}}$. This will establish the statement (ii) of the theorem.

Assume to the contrary that there exists a quadruple $(x, y, i, j) \in M_n$ (resp. a triple $(x, y, i) \in X_n$) such that $(x, y) \notin F_{\mathfrak{g}}$. Consider the morphism $\varphi$ defined in (6). As said, its image is open. On the other hand, for all elements $(x, y, i, j)$ (resp. $(x, y, i)$) lying in this image we have $(x, y) \in F''_{\mathfrak{g}}$ because the corresponding property to have a nonzero common centralizer holds in $S = Q_{\mathfrak{l}}$, as said above. Since the image of $\varphi$ is open, the closure of $F''_{\mathfrak{g}}$ is $X = Q_{\mathfrak{g}}$, contradiction. This proves the statement. □

Corollary 5 now follows from Theorem 1, the proof of Theorem 3(ii) and Proposition 7(ii). Since $A = \Bbbk[t]$ satisfies the conditions of Corollary 5 with $\mathfrak{a} = t^2\Bbbk[t]$, Corollary 4 follows as well.

## 3. Concluding remarks

We finish with some remarks on what was not done and what should (and hopefully will) be done in the near future.

• The first tempting goal is to settle the remaining cases of Conjecture 2. By Theorem 1, the width of $\mathfrak{g} \otimes \Bbbk[[t]]$ is at most 2. To prove that it is equal to 2, we have to exhibit an element of $c \in \mathfrak{g}$ such that for every representation $c = [a, b]$ the elements $a$ and $b$ have a nonzero common centralizer, as in the proof of Theorem 3(ii). There we took an element $c$ of the minimal nonzero nilpotent orbit $\mathbb{O}_{\min}$ (it is well known that there exists a unique such orbit [CM93, Theorem 4.3.3 and Remark 4.3.4]) and used simultaneous triangularization of $a$ and $b$. However, this method breaks down for all types other than $\mathrm{A}_n$ and $\mathrm{C}_n$ as shown by Losev in [Lo21, Remark 2.3]: in simple algebras of all those types there are elements $c = [a, b] \in \mathbb{O}_{\min}$ such that $a$ and $b$ do not lie in a common Borel subalgebra. This gives a certain evidence that these algebras are of width 1.

• It would be interesting to look at other current algebras. Say, by Theorem 1 it is known that the bracket width of $\mathfrak{sl}_2 \otimes \Bbbk[t]$ is less than or equal to 2. Although we know that $\mathfrak{sl}_2 \otimes \Bbbk[[t]]$ has bracket width 1, we still do not know the bracket width of $\mathfrak{sl}_2 \otimes \Bbbk[t]$.

• In a similar vein, it would be interesting to compute the width of the loop algebras $\mathfrak{g} \otimes \Bbbk[t, t^{-1}]$.

• Our final remark concerns the parallel results on the width of the finite-dimensional Lie $R$-algebras $\mathfrak{gl}_n(R)$ obtained for various rings $R$ in a slightly different context. Namely, it is known that the width of such a Lie algebra is at most 2. This was first proved by Amitsur and Rowen [AR94] for division rings $R$ and then generalized to arbitrary commutative rings [Ros97] (and even to non-commutative rings [Me06]). This looks like an almost full analogue of our Theorem 1, modulo the transition from $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$, which may be a non-trivial task, see [St18] (in the latter paper it is also shown that the width of $\mathfrak{sl}_n(R)$ is equal to 1 if $R$ is a principal ideal domain).

However, none of these results implies the other: the bracket width of the *infinite-dimensional* Lie $\Bbbk$-algebra $\mathfrak{sl}_n \otimes_{\Bbbk} R$ is *a priori* unrelated to the bracket width of the *finite-dimensional* $R$-algebra $\mathfrak{sl}_n(R)$. In light of the existing parallels, it would be interesting to compute the bracket width of the finite-dimensional simple Lie $R$-algebras $\mathfrak{g}(R)$.

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