A Pieri-type theorem for Lagrangian and odd Orthogonal Grassmannians

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0 Introduction

One of the main goals of the intersection theory on a smooth algebraic variety is to give a solution to the following two problems. The first problem is to find a possibly small family of subvarieties whose rational equivalence classes are additive generators of the Chow group of the variety. Then, the second question is to write down the multiplication table for the intersection product with respect to this set of generators. Usually, the first problem is easier than the second.

In the present paper we investigate the intersection rings of isotropic Grassmannians. More precisely, let V be a $2m$-dimensional (resp. $(2m+1)$- dimensional) complex vector space equipped with a nondegenerate symplectic or orthogonal form ϕ [1]. If $n \leq m$, then the set G of all n-dimensional subspaces of V that are isotropic with respect to ϕ, is an interesting and worth studying smooth algebraic variety. It is well known that $G = \mathrm{Sp}(2m)/P_n$ and $G' = \mathrm{SO}(2m+1)/P_n$ where P_n is the maximal parabolic subgroup corresponding to omitting the n-th simple root in the root system of type C_m (resp. B_m) (equal to $e_n - e_{n+1}$ if $n < m$, or $2e_m$ (resp. e_m) if $n = m$).

The solution to the first question above is a part of a general theory of cellular decompositions of the spaces of the form H/P where H is a reductive group and P is a parabolic subgroup of H (this goes back to Schubert, Bruhat, Ehresmann ...). The resulting additive basis of the intersection ring is formed by the generalized Schubert cycles.

Multiplicatively, this ring is generated by n Schubert cycles (called "special") which (up to a scalar) are Chern classes of the tautological isotropic bundle. Thus to describe multiplication in the intersection ring it suffices to express the intersection of an arbitrary Schubert cycle with a special Schubert cycle as a combination of other Schubert cycles. These "Pieri-type-formulas" (see Theorem 2.2 and Theorem 10.1) are the main subject of the present article.

The classical Schubert calculus for "usual" Grassmannians was invented in the end of the 19th century and the begining of the present one in the works of Schubert, Pieri and Giambelli. To the best of our knowledge, the analogous theory for the Lagrangian and Orthogonal Grassmannians (apart from some very particular cases of intersection formulas for the sets of linear spaces on a quadric) was not treated by classics of enumerative geometry. The technical difficulties (concerning especially the multiplicities) which are transparent in the present paper can serve for an explanation for a lack of a "symplectic and orthogonal" Schubert calculus in the classical literature. It is the theory

[1]These varieties are called respectively the Lagrangian and odd Orthogonal Grassmannians. In accordance with a tradition we will refer to these two cases as "type C" and "type B" respectively.

developed in [B–G–G] and [D2] in the seventies which allows one to deal with the Schubert Calculus for other classical groups.

The strategy used here follows the method invented in our previous paper [P–R1]. Let B be a Borel subgroup of H and T— the maximal torus of H contained in B. One has the Borel characteristic map $S_*(X(T)) \to A^*(H/B)$ ($X(T)$ is the group of characters of T), defined by sending a character χ to $c_1(L_\chi)$, where L_χ is a line bundle with transition functions determined by χ. Then we use a result (see [B–G–G] and [D2]) which asserts that the coefficients of the characteristic map in the basis of Schubert cycles are given by the "divided differences operators". For instance, in [D2], this fact is deduced from the geometry of Bott-Samelson schemes. This information allows us to reformulate problems from intersection theory into some questions of purely algebro-combinatorial nature.

The main technical task of this paper is to find an efficient way to calculate with the orthogonal and symplectic divided differences. We extensively use a Leibniz-type formula (and, especially its iterations) and find some optimal reduced words of simple reflections to work with. All of this relies heavily on a combinatorial technique of z- and v-ribbons which is invented and developed in the present paper. A rather detailed analysis allows us to determine "admissible" deformations of z- and v-ribbons and, as a consequence, possible shapes involved in our intersection formula (see Theorem 8.1).

Finally, appropriate divided differences and symmetrizations evaluated in elementary symmetric polynomials give precise multiplicities of Schubert cycles appearing in the formula.

More precisely, the paper is organized as follows.

Section 1 contains some general information about G (e.g. a description of its Chow ring $A^*(G)$. In particular we introduce some combinatorial objects called "shapes" which label the Schubert subvarieties of G. (A shape is a pair of strict partitions fulfilling certain conditions). We find this way of indexing of Schubert varieties the best suited to the purposes of intersection theory. (It generalizes the "$m = n$ - case" from [H–B] and [P–R1].)

Section 2 contains a formulation of our formula and some corollaries. The formulation uses properties of Ferrers' diagrams of partitions of shapes. However, it is much more subtle than the "$m = n$ - case" from loc. cit. A delicate point which appears is an interplay between the connected components of an "almost horizontal strip" added to the bottom part of a shape and the rows of the top one.

In Section 3 we collect facts on divided differences that we need. In particular we state a generalization of a Leibniz-type formula in terms of shapes, which is a base-point of our calculations.

In Section 4 we introduce the notion of a "mark" of a box in a shape and study its properties. Roughly speaking the mark encodes the way in which an element passes to its place in the barred permutation.

In Section 5 we collect several lemmas which are of constant use throughout this paper.

In Section 6 and 7, using properties of the marks of boxes in a shape we establish some necessary conditions for a summand in our generalized Leibniz-type formula, to be non-zero.

Section 8 is devoted to prove the key technical fact which says that there is at most one non-zero summand in the above generalized Leibniz-type formula. We give also an explicit algorithm for constructing this summand.

Finally, Section 9 contains the proof of our formula; in particular we calculate the multiplicities of summands involved, in terms of the original shape-data.

The Schubert varieties in $SO(2m)/P$, where P is a maximal parabolic subgroup are labelled by a poset which is different from the $\mathcal{P}_n$'s. We plan to describe a corresponding Pieri-type formula in a forthcoming paper.

A part of results of this paper was announced in [P–R2]. We refer to this note for a sketch of the proof of the main result of the present paper.

1 Notation, conventions and preliminaries

We start with some recollection of Lie theory in the symplectic case. A good reference for this material is [F–H]. The usual realization of the root system of type C_m is the set of vectors

$$\mathcal{R} = \{\pm e_i \pm e_j : 1 \leq i < j \leq m\} \cup \{\pm 2e_i : 1 \leq i \leq m\}$$

in the Euclidean space $\mathbb{R}^m = \bigoplus_{i=1}^{m} \mathbb{R}e_i$. Let W denote the group generated by the reflections s_β, $\beta \in \mathcal{R}$, where

$$s_\beta(x) = x - (x, \beta^\vee)\beta$$

and $\beta^\vee$ is the co-root $2(\beta,\beta)^{-1}\beta$ ($(\cdot\,,\,\cdot)$ denotes here the standard scalar product in $\mathbb{R}^m$). A set $\{\alpha_1, \ldots, \alpha_m\}$ where $\alpha_i = e_i - e_{i+1}$, $1 \leq i < m$ and $\alpha_m = 2e_m$ is the set of simple roots for $\mathcal{R}$. The group W (called the symplectic Weyl group) is generated by the simple reflections $\{s_i\}_{1\leq i \leq m}$, $s_i = s_{\alpha_i}$ and is isomorphic to the semidirect product $S_m \ltimes \mathbb{Z}_2^m$ where the symmetric group S_m acts on $\mathbb{Z}_2^m$ in the obvious way. We write a typical element of W as $w = (\tau, \varepsilon)$ where $\tau \in S_m$ and $\varepsilon \in \mathbb{Z}_2^m$; so that if $w' = (\tau', \varepsilon')$ is another element, their product in W is:

$$w \cdot w' = (\tau \circ \tau', \delta),$$

where "$\circ$" denotes the composition of permutations and $\delta_i = \varepsilon_{\tau'(i)} \cdot \varepsilon'_i$.

Now fix $n < m$, and consider W_n a subgroup of W generated by $\{s_i\}_{i \neq n}$. We have $W_n \simeq S_n \times (S_{m-n} \ltimes \mathbb{Z}_2^{m-n})$. Then $W^{(n)}$—the set of minimal length left coset representatives of W_n in W can be identified with the set of sequences of the form:

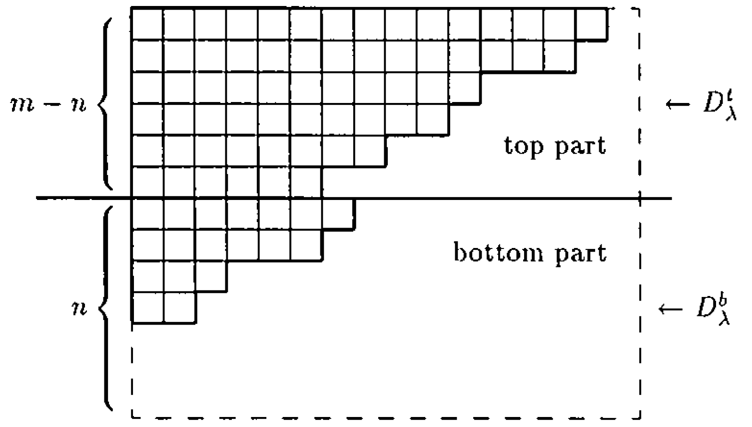
$$(y_1, \ldots, y_{n-k}; \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n})$$

in the standard "barred-permutation notation" (the bars indicate that $\varepsilon_i = -1$) where $y_1 < \ldots < y_{n-k}$; $z_k > \ldots > z_1$ and $v_1 < \ldots < v_{m-n}$ (see [H]). $W^{(n)}$ is a poset (with an order induced from the Bruhat order in W). For the purposes of this paper it will be convenient to use the following presentation of $W^{(n)}$.

Definition 1.1 A pair $\lambda = (\lambda^t // \lambda^b)$ of strict partitions λ^t and λ^b is called a *shape* if $\lambda^t \subset (m^{m-n})$, $\lambda^b \subset (m^n)$ and $\lambda^t_{m-n} \geq l(\lambda^b) + 1$.

Denote the set of shapes by $\mathcal{P}_n$. It will be useful to visualize shapes with the help of a set of boxes in the fourth quarter of the plane. Let D^t_λ and D^b_λ be the Ferrers diagrams of λ^t and λ^b (see [M]; also the other terminology related to the partitions, diagrams etc. is borrowed from loc. cit.). The diagram D_λ

of a shape $(\lambda^t//\lambda^b)$ is the juxtaposition of D^t_λ and D^b_λ with rows of successive lengths: $\lambda^t_1, \ldots, \lambda^t_{m-n}, \lambda^b_1, \ldots, \lambda^b_l, \quad l = l(\lambda^b)$:



Rows that are contained in the top (resp. bottom) part of D_λ will be called top (resp. bottom) rows. Note that the last condition of the definition of a shape means that D_λ must contain a triangular diagram of boxes with the row-lengths: $m - n + l(\lambda^b)$, $m - n + l(\lambda^b) - 1, \ldots, 2, 1$.

For an element $w = (y_1, \ldots, y_{n-k}; \overline{z}_k, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n})$ of $W^{(n)}$ we denote

$$d_r = d_r(w) := \mathrm{card}\{j : z_j < v_r,\ j = 1, \ldots, k\} \qquad r = 1, \ldots, m - n.$$

Lemma 1.2 *With the above notation, the assignment $w \mapsto \lambda = (\lambda^t//\lambda^b)$ given by*

$$\lambda^b_j = m + 1 - z_j \qquad j = 1, \ldots, k \tag{1}$$

$$\lambda^t_r = m + 1 - v_r + d_r \qquad r = 1, \ldots, m - n \tag{2}$$

gives a bijection between $W^{(n)}$ and $\mathcal{P}_n$.

Proof. Since the sequences (z_j) and (v_r) are increasing and (d_r) is nondecreasing, λ^b and λ^t are strict partitions contained in (m^n) and (m^{m-n}) respectively.

The inequality $\lambda^t_{m-n} \geq l(\lambda^b) + 1$ is equivalent to $v_{m-n} \leq m - (k - d_{m-n})$ because $k = l(\lambda^b)$. The latter inequality is clear since $k - d_{m-n} = \mathrm{card}\{z_j : z_j > v_{m-n},\ j = 1, \ldots, k\}$.

Conversely, suppose that a shape $\lambda = (\lambda^t//\lambda^b)$ is given. Using (1) we compute first z_j, $j = 1, \ldots, k = l(\lambda^b)$.

Since $\lambda^t_{m-n} \geq k + 1$, we have $\lambda^t_r \geq m - n - r + k + 1$ and consequently $p_r := \lambda^t_r - (m - n + k - r) > 0$, $r = 1, \ldots, m - n$. Then the recipe for (v_r) is as follows. v_{m-n} is the p_{m-n}-th element (counting from the right) in the sequence $(1,2,\ldots,m)$ with removed $\{z_j : j = 1,\ldots,k\}$. In general, v_r is the p_r-th element (counting from the right) in the sequence $(1,2,\ldots,m)$ with removed $\{z_j : j = 1,\ldots,k\}$ and $v_{m-n}, v_{m-n-1},\ldots,v_{r+1}$. Then, the cardinality of the elements in $(1,2,\ldots,m)$ appearing after v_r can be expressed as

$$\begin{aligned}&(p_r - 1) + \mathrm{card}\{v_{m-n},\ldots,v_{r+1}\} + \mathrm{card}\{j : z_j > v_r\}\\ &\quad = \ (\lambda^t_r - (m-n+k-r) - 1) + (m-n-r) + (k-d_r) = \lambda^t_r - d_r - 1.\end{aligned}$$

Equating this with $m - v_r$ we get $\lambda^t_r = m + 1 - v_r + d_r$, as desired. □

Denote by w_λ the element of W associated by the lemma with a shape $\lambda = (\lambda^t//\lambda^b)$. It follows from [H-B, Lemma 2.2] that

$$l(w_\lambda) = \sum_{i=n-k+1}^{n} (m+1-i) + \sum_{j=1}^{k} c_j + \sum_{h=1}^{n-k} (y_h - 1),$$

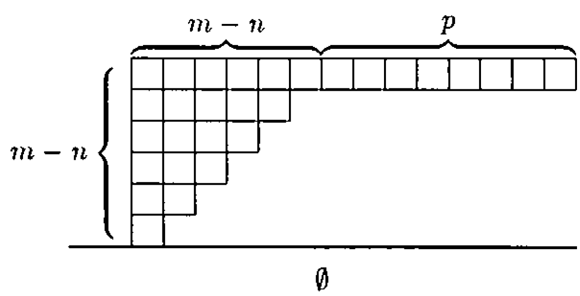
where $c_j = \mathrm{card}\{r : v_r > z_j \ r = 1,\ldots,m-n\}$. By using the equations $\sum_{j=1}^{k} c_j = \sum_{r=1}^{m-n} d_r$ and $\sum y_h + \sum z_j + \sum v_r = 1+2+\ldots+m$, one can rewrite this in the form

$$\begin{aligned} l(w_\lambda) &= \sum_{j=1}^{k}(m+1-z_j) + \sum_{r=1}^{m-k}(n+r-v_r+d_r)\\ &= |\lambda^b| + |\lambda^t| - \binom{m-n+1}{2}. \end{aligned} \tag{3}$$

Note a particular reduced decomposition of w_λ:

$$\begin{aligned} w_\lambda = \ & (s_{m-\lambda^b_k+1}\cdot s_{m-\lambda^b_k+2}\cdot\ldots\cdot s_{m-1}\cdot s_m)\cdot\ldots\\ &\cdot(s_{m-\lambda^b_1+1}\cdot s_{m-\lambda^b_1+2}\cdot\ldots\cdot s_{m-1}\cdot s_m)\cdot(s_{m-\lambda^t_{m-n}+1}\cdot\ldots\cdot s_{m-2}\cdot s_{m-1})\\ &\cdot(s_{m-\lambda^t_{m-n-1}+1}\cdot\ldots\cdot s_{m-3}\cdot s_{m-2})\cdot\ldots\cdot(s_{m-\lambda^t_1+1}\cdot\ldots\cdot s_{n-1}\cdot s_n). \end{aligned}$$

Example 1.3 $\lambda = ((m-n+p, m-n-1,\ldots,1)//\emptyset)$, $p = 1,\ldots,n < m$. Then $w_\lambda = s_{n-p+1}\cdot\ldots\cdot s_n = (1,\ldots,n-p,n-p+2,\ldots,n+1;\emptyset;n-p+1,n+2,\ldots,m)$ Note that the diagram of λ is



Let G denote the Grassmannian of n-dimensional isotropic subspaces in $\mathbb{C}^{2m}$ with respect to a nondegenerate symplectic form on $\mathbb{C}^{2m}$. It is known that $G = \mathrm{Sp}(2m)/P_n$ where P_n is the maximal parabolic corresponding to omitting the n-th simple root system of type C_m. Let B be the Borel subgroup contained in P_n (i.e. corresponding to the empty subset of the set of simple roots) and B^- its opposite. Then the homogeneous space $\mathrm{Sp}(2m)/B$ is identified with the flag variety F of (total) isotropic flags in $\mathbb{C}^{2m}$ (with respect to the same symplectic form). The canonical projection $p : \mathrm{Sp}(2m)/B \to \mathrm{Sp}(2m)/P_n$ induces the injection $p^* : A^*(G) \hookrightarrow A^*(F)$. Let $X_{w_\lambda} \in A^{|\lambda|}(F)$, $|\lambda| = |\lambda^t| + |\lambda^b| - \binom{m-n+1}{2}$, be the rational equivalence class of the closure of $B^- w_\lambda B/B$ is $\mathrm{Sp}(2m)/B$. Let $\sigma(\lambda) \in A^{|\lambda|}(G)$ denote the rational equivalence class of the closure of $B^- w_\lambda P_n/P_n$ in $\mathrm{Sp}(2m)/P_n$. Note that $p^*(\sigma(\lambda)) = X_{w_\lambda}$.

More precise information on $A^*(G)$ is contained in the following theorem and its proof.

Theorem 1.4 *$A^*(G)$ is isomorphic with:*

$$\mathrm{Sym}[x_1, \ldots, x_n]/(h_j(x_1^2, \ldots, x_n^2) : m - n + 1 \leq j \leq m),$$

where $\mathrm{Sym}[\ ]$ *means the ring of symmetric polynomials in the indicated variables, and $h_j(\ )$ denotes the j-th complete homogeneous polynomial in the indicated elements.*

Proof. Let $c : S.(X(T)) = \mathbb{Z}[x_1, \ldots, x_m] \longrightarrow A^*(Sp(2m, \mathbb{C})/B) = A^*(F)$ be the Borel characteristic map mentioned in Introduction. (Here, $x_1, \ldots, x_m$ are independent variables). By [D1] we know that c is surjective and $ker\ c$ is the ideal generated by symmetric polynomials in $x_1^2, \ldots, x_m^2$ without constant term. It follows from the comparison of the proof of [B–G–G, Theorem 5.5], with the proof of [D1, Proposition 5] that c induces a surjective map

$$c_G : \mathbb{Z}[x_1, \ldots, x_m]^{W_n} \longrightarrow A^*(G)$$

Consequently $ker\ c_G = ker\ c \cap \mathbb{Z}[x_1, \ldots, x_m]^{W_n}$. Recall that $W_n \simeq S_n \times (S_{m-n} \ltimes \mathbb{Z}_2^{m-n})$ and S_n acts on $x_1, \ldots, x_n$ via permutations, $S_{m-n} \ltimes \mathbb{Z}_2^{m-n}$ acts on $x_{n+1}, \ldots, x_m$ via barred permutations. Of course

$$\mathbb{Z}[x_1, \ldots, x_m]^{W_n} = \mathrm{Sym}[x_1, \ldots, x_n] \otimes \mathrm{Sym}[x_{n+1}^2, \ldots, x_m^2].$$

Denoting by $e_i(\ )$ the i-th elementary symmetric polynomial in the indicated elements, we have $ker\ c = (e_i(x_1^2, \ldots, x_m^2) : 1 \leq i \leq m)$. Since for every i

$$e_i(x_1^2, \ldots, x_m^2) = \sum_{k+l=i} e_k(x_1^2, \ldots, x_n^2) e_l(x_{n+1}^2, \ldots, x_m^2)$$

and in each summand, the first factor is S_n-invariant and the second is $S_{m-n} \ltimes \mathbb{Z}_2^{m-n}$-invariant, we infer $ker\, c_G = (e_i(x_1^2, \ldots, x_m^2) : 1 \leq i \leq m)$. Consequently

$$A^*(G) \simeq \mathrm{Sym}[x_1, \ldots, x_n] \otimes \mathrm{Sym}[x_{n+1}^2, \ldots, x_m^2]/(e_i(x_1^2, \ldots, x_m^2) : 1 \leq i \leq m).$$

We will need the following lemma.

Lemma 1.5 *Let $y_1, \ldots, y_m$ be independent variables. The following equality of ideals holds in* $\mathrm{Sym}[y_1, \ldots, y_n] \otimes \mathrm{Sym}[y_{n+1}, \ldots, y_m]$*:*
$(e_i(y_1, \ldots, y_m) : 1 \leq i \leq m) =$

$$(e_i(y_{n+1}, \ldots, y_m) + (-1)^i h_i(y_1, \ldots, y_n), 1 \leq i \leq m - n;$$
$$h_i(y_1, \ldots, y_n), \quad m - n + 1 \leq i \leq m)$$

Proof. Denote the former ideal by I and the latter by J. We first show that $J \subset I$. It follows from the relations:

$$e_i(y_{n+1}, \ldots, y_m) = \sum_j (-1)^j e_{i-j}(y_1, \ldots, y_m) h_j(y_1 \ldots, y_n)$$

that $e_i(y_{n+1}, \ldots, y_m) + (-1)^{i-1} h_i(y_1, \ldots, y_n) \in I$. Also it follows from the latter relations that

$$\begin{aligned} e_i(y_1, \ldots, y_m) &= \sum_{k+l=i} (-1)^l e_k(y_1, \ldots, y_n) e_l(y_{n+1}, \ldots, y_m) \\ &\equiv \sum_{k+l=i, l \leq m-n} e_k(y_1, \ldots, y_n) h_l(y_1, \ldots, y_n) \; mod\; I \end{aligned}$$

Hence for every i we get $\sum_{k+l=i, l \leq m-n} (-1)^l e_k(y_1, \ldots, y_n) h_l(y_1, \ldots, y_n) \in I$. Combining this with standard relations between the e's and h's we infer succesively $h_{m-n+1}(y_1, \ldots, y_n) \in I, \ldots, h_m(y_1, \ldots, y_n) \in I$.

Now we show $I \subset J$. We have for $1 \leq i \leq m - n$

$$\begin{aligned} e_i(y_1, \ldots, y_m) &= \sum_{k+l=i} e_k(y_1, \ldots, y_n) e_l(y_{n+1}, \ldots, y_m) \\ &\equiv \sum_{k+l=i} e_k(y_1, \ldots, y_n) (-1)^l h_l(y_1, \ldots, y_n) \; mod\, J. \end{aligned}$$

But the latter expression is tautologically zero because of standard relations between e's and the h's. Therefore $e_i(y_1, \ldots, y_m) \in J$. For $m - n \leq i \leq m$ we have

$$\begin{aligned} e_i(y_1, \ldots, y_m) &\equiv \sum_{k+l=i, l \leq m-n} e_k(y_1, \ldots, y_n)(-1)^l h_l(y_1, \ldots, y_n) \\ &\equiv \sum_{k+l=i} (-1)^l e_k(y_1, \ldots, y_n) h_l(y_1, \ldots, y_n) \; mod\, J \end{aligned}$$

because $h_j(y_1, \ldots, y_n \in J$ for $m-n+1 \leq j \leq m$. Since the latter expression is tautologically zero, $e_i(y_1, \ldots, y_m) \in J$. This proves the Lemma. $\square$

The end of the proof of Theorem 1.4:

Use the Lemma with $y_i = x_i^2$, $i = 1, \ldots, m$. The first set of relations in the second presentation J of the above ideal shows that the natural map:

$$\mathrm{Sym}[x_1, \ldots, x_n] \longrightarrow \mathrm{Sym}[x_1, \ldots, x_n] \otimes \mathrm{Sym}[x_{n+1}^2, \ldots, x_m^2]/(e_i(x_1^2, \ldots, x_m^2))$$

is surjective. Then the second set of relations generates the kernel of this map.

Thus the theorem has been proved. $\square$

Lemma 1.6 *Assume that $x_1, \ldots, x_n$ are algebraically independent over $\mathbb{Z}$. Then, for $m \geq n$, $h_{m-n+1}(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n)$ are algebraically independent over $\mathbb{Z}$.*

Proof. The assertion follows immediately from an isomorphism

$$\begin{aligned} & \mathrm{Sym}[x_1, \ldots, x_n] \otimes \mathrm{Sym}[x_{n+1}, \ldots, x_m]/(e_i(x_1, \ldots, x_m) : 1 \leq i \leq m) \\ \simeq \ & \mathrm{Sym}[x_1, \ldots, x_n]/(h_j(x_1, \ldots, x_n) : m-n+1 \leq j \leq m) \end{aligned}$$

induced by the identity on $\mathrm{Sym}[x_1, \ldots, x_n]$; $x_{n+1}, \ldots, x_m$ are independent variables transcendental over $x_1, \ldots, x_n$. (cf. Lemma 1.5). $\square$

Theorem 1.4 and Lemma 1.6 imply the following

Corollary 1.7 *Let $P(t) = \sum_i \mathrm{rk}_{\mathbb{Z}} A^i(G) T^i$ be the Poincaré series of G. Then*

$$P(t) = \frac{(1-T^{2m})(1-T^{2(m-1)})\ldots(1-T^{2(m-n+1)})}{(1-T^n)(1-T^{n-1})\ldots(1-T)}.$$

Moreover we have obviously

Corollary 1.8 *The elements $c(e_i(x_1, \ldots, x_n))$, $1 \leq i \leq n$, generate algebraically the ring $A^*(G)$ over $\mathbb{Z}$.*

2 Statement of the main result for type C

We assume here $n < m$; the case $m = n$ was treated in [H–B] (see also [P–R1]).

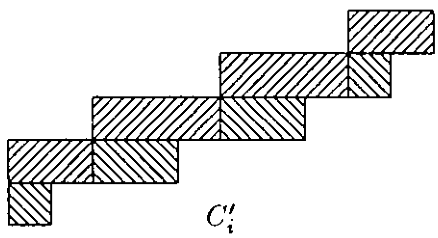
For $p = 1, \dots, n$ denote $\sigma_p := \sigma((m-n+p, m-n-1, \dots, 2, 1)//\emptyset)$. It turns out (see Lemma 3.2) that σ_p corresponds to $e_p(x_1, \dots, x_n)$ via the isomorphism from Theorem 1.4. By Corollary 1.8, $A^*(G) = \mathbb{Z}[\sigma_1, \dots, \sigma_n]$. The main theorem of this paper gives an explicit formula for intersecting a general Schubert cycle $\sigma(\lambda)$, $\lambda \in \mathcal{P}_n$, with σ_p. This Pieri-type formula is, however, more subtle than the other formulas of this type known to the authors. Let, for the moment, μ be a shape appearing nontrivially in the right hand side of $\sigma(\lambda) \cdot \sigma_p = \dots$. First "surprise" is that it can happen that $D^t_\mu \not\supset D^t_\lambda$. The second one is that $D^b_\mu \setminus D^b_\lambda$ can be a non-horizontal strip. Finally, the formula involves some nontrivial multiplicities (which are powers of 2).

To formulate the theorem we need several notions. Recall that we follow [M] for conventions and terminology concerning partitions, diagrams, shifted diagrams etc.

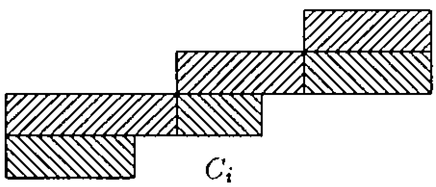
A skew diagram D (i.e. a difference of two Ferrers diagrams) is *connected* if each of the sets $\{i : \exists_j\ (i,j) \in D\}$ and $\{j : \exists_i\ (i,j) \in D\}$ is an interval in the set of positive integers. (In this work we will deal exclusively with skew diagrams which are differences of two *strict diagrams* i.e. diagrams of strict partitions.)

By an *almost horizontal strip* we mean a (possibly disconnected) skew diagram with at most two boxes in each column such that the set of the highest boxes in columns forms a horizontal strip, and the remaining boxes form a horizontal strip with pairwise disconnected rows.

Every almost horizontal strip has a decomposition $\bigcup C_i$ into connected components; we will denote by $C_i^{(1)}$ the set of highest boxes in columns and write $C_i^{(2)} := C_i \setminus C_i^{(1)}$. Pictorially

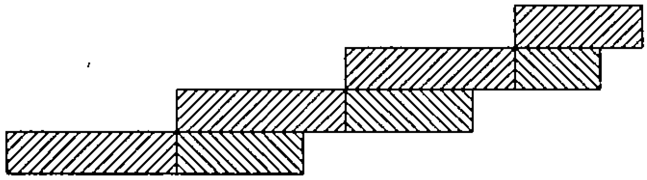


C_i'



C_i

(▨—visualizes boxes in $C_i^{(1)}$, ▧—boxes in $C_i^{(2)}$). Note that two boxes will appear neither in the leftmost nor in the rightmost column of the component in the situation of Definition 2.1 below. In other words a component of an almost horizontal strip which satisfies this definition can be depicted as



The set of boxes $C_i^{(2)}$ will be called the *excrescence* of C_i.

Now suppose that two shapes λ and μ are given. In what follows, by a *row* without further indications we will mean a row in the top part; by λ- (resp. μ-) part of a row we understand its restriction to D^t_λ resp. D^t_μ.

A row is called *exceptional* if its λ-part contains strictly its μ-part.

By a *component* we understand a shifted connected component of $D^b_\mu \setminus D^b_\lambda$. A component is called *extremal* if, before a shift, it meets the leftmost column.(Note that there exist at most one extremal component).

Since some combinatorics of shifted tableaux will emerge naturally, we will use the following conventions. We say that a box $\mathfrak{t} \in D^t_\lambda \cup D^t_\mu$ *lies over* a box $\mathfrak{b} \in D^b_\mu$ if $\mathfrak{t}$ and the shifted $\mathfrak{b}$ are in the same column. We will also say in this situation that $\mathfrak{b}$ *lies under* $\mathfrak{t}$. A subset T of $D^t_\lambda \cup D^t_\mu$ *lies over* a subset B of D^b_μ if every box from T lies over some box of B. We will also say that B *lies under* T if every box $\mathfrak{b}$ from B lies under some box $\mathfrak{t}$ from T. If T is contained in a row then we say that T *ends over* B if the rightmost box of T (called sometimes the *end* of T) lies over B.

Similarly for boxes $\mathfrak{b}_1$ and $\mathfrak{b}_2$ from D^b_μ, we will say that $\mathfrak{b}_1$ *lies over* (resp. *lies under*) $\mathfrak{b}_2$ if the column of the shifted $\mathfrak{b}_1$ is equal to the column of the shifted $\mathfrak{b}_2$ and the row number of $\mathfrak{b}_1$ is smaller (resp. bigger) than the row number of $\mathfrak{b}_2$

Finally, a box in $D^t_\mu \setminus D^t_\lambda$ will be called a $(\mu - \lambda)$-*box*.

Definition 2.1 $\mu = (\mu^t // \mu^b)$ is *compatible* with $\lambda = (\lambda^t // \lambda^b)$ if

1) $D^b_\mu \supset D^b_\lambda$ and $D^b_\mu \setminus D^b_\lambda$ is an almost horizontal strip and the extremal component is an (ordinary) horizontal strip.

   $D^t_\mu \setminus D^t_\lambda$ is a horizontal strip with pairwise disconnected rows.

2) The λ-part of at most one row ends over a component but none over the extremal one. If a row ends over a component we say that they are *related*. A component which is related to some row will be called *related*. Similarly a row which is related to some component will be called *related*.

3) Each exceptional row is related to a component, over which the μ-part of this row ends.

4) If a $(\mu - \lambda)$-box lies over a component then this component is neither extremal nor related and this box lies over the leftmost box of the component.

5) An excrescence can appear only in a related component under the λ-part of the related row; no box from the μ-part of the related row lies over the excrescence.

(In particular, an excrescence can appear only in a component related to an exceptional row.)

The main result of the present paper is:

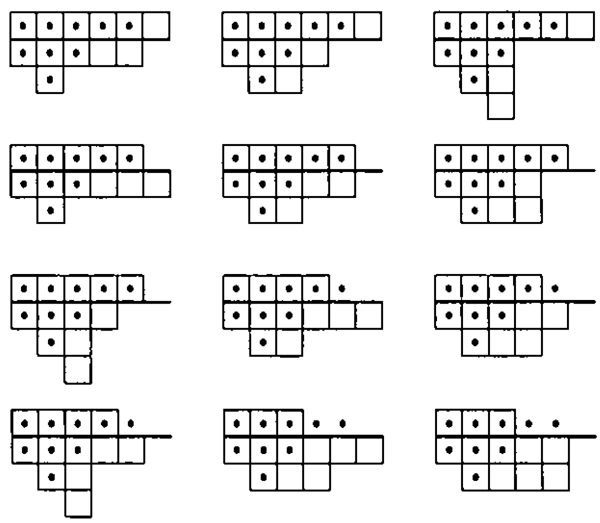
Theorem 2.2 *For every $\lambda \in \mathcal{P}_n$ and $p = 1, \ldots, n$*

$$\sigma(\lambda) \cdot \sigma_p = \sum 2^{e(\lambda,\mu)} \sigma(\mu),$$

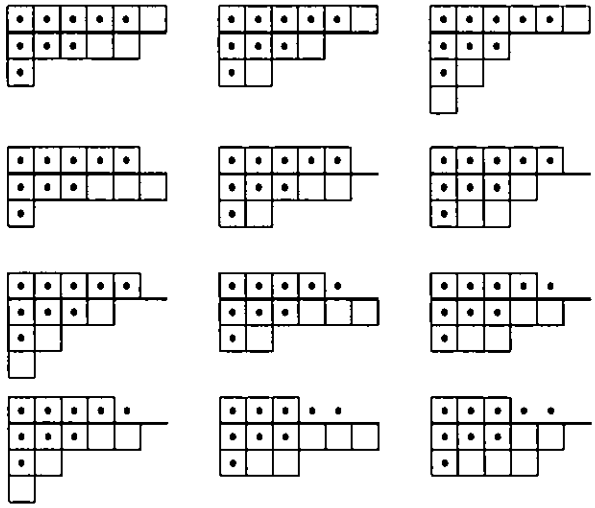
where the sum is over all μ compatible with λ, $|\mu| = |\lambda| + p$ and $e(\lambda, \mu)$ is the cardinality of the set of components that are not extremal, not related and have no $(\mu - \lambda)$-boxes over them.

Example 2.3 $m = 6$, $n = 5$

$$\begin{aligned}\sigma(5//3,1) \cdot \sigma_3 \;=\;& \sigma(6//5,1) + 2^2\sigma(6//4,2) + \sigma(6//3,2,1) + \\ & \sigma(5//6,1) + 2\sigma(5//5,2) + 2\sigma(5//4,3) + \\ & 2\sigma(5//4,2,1) + 2\sigma(4//6,2) + \sigma(4//5,3) + \\ & \sigma(4//5,2,1) + \sigma(3//6,3) + \sigma(3//5,4)\end{aligned}$$



The nonshifted version is:



We now consider some special cases of the theorem. We write for partitions I and K, $\sigma_{I//K} := \sigma(i_1 + m - n, \ldots, i_{m-n} + 1 // K)$, $\sigma_I := \sigma_{I//\emptyset}$.

Assume $l(I) \leq m - n - 1$. Then it follows from the theorem that

$$\sigma_I \cdot \sigma_p = \sum \sigma_J,$$

sum over $J \supset I$ such that $|J| = |I| + p$ and $J \setminus I$ is a horizontal strip. Arguing as in in [P, Section 6], we deduce from it and Pieri's and determinantal formulas for S-functions:

Corollary 2.4 *For every partition $I \subset (n^{m-n})$,*

$$\sigma_I = \mathrm{Det}[\sigma_{i_h - h + q}]_{1 \leq h, q \leq l(I)}.$$

In particular $\sigma_{(n^{m-n})} = (\sigma_n)^{m-n}$.

Assume now $l(I) = m - n$. Then the theorem gives

$$\sigma_I \cdot \sigma_p = \sum_{0 \leq h \leq i_{m-n}} \sum_{J_h} \sigma_{J_h // h},$$

where the sum is over $J_h \supset I$, $|J_h| = |I| + p - h$ and $J_h \setminus I$ is a horizontal strip.
Here is another special case. Let I be a partition with $i_1 = n$. Then

$$\sigma_{(n^{m-n})//I} \cdot \sigma_p = \sum_J 2^{f(I, nJ)} \sigma_{(n^{m-n})//nJ} + \sum_J 2^{f(I,(n+1)J)} \sigma_{(n^{m-n})//(n+1)J},$$

where the first sum is over (strict) partitions $J \supset (i_2, \ldots)$, $j_1 < n$, $|J| = |I| - n + p$; the second is over (strict) partitions $J \supset (i_2, \ldots)$, $j_1 < n + 1$, $|J| = |I| - n + p - 1$; and $nJ \setminus I$ (resp. $(n+1)J \setminus I$) is a horizontal strip. In the both sums $f(I, L)$ means the number of components of $L \setminus I$ not meeting the leftmost column, and—after shifting—not meeting the $m - n$ rightmost columns of the (basic) $m \times m$ square.

In particular,

$$\sigma_{(n^{m-n})//(n,n-1,\ldots,1)} \cdot \sigma_p = \sigma_{(n^{m-n})//(n+1,n,\ldots,n-p+2,n-p,\ldots,1)}.$$

The latter equality is also a particular case of the following formula. Assume $I \supset (n, n-1, \ldots, 1)$. Then

$$\sigma_{(n^{m-n})//I} \cdot \sigma_p = \sum_J \sigma_{(n^{m-n})//J},$$

the sum over (strict) partitions $J \supset I$ such that $|J| = |I| + p$ and $J \setminus I$ is a horizontal *and* vertical strip.

We deduce from it and Pieri's and determinantal formulas for S-functions:

Corollary 2.5 *Let I be a partition contained in (n^n). Then for the conjugate partition J of I one has:*

$$\sigma_{(n^{m-n})//(n,n-1,\ldots,1)+I} = \sigma_{(n^{m-n})//(n,n-1,\ldots,1)} \cdot \mathrm{Det}[\sigma_{j_h-h+q}]_{1\le h,q\le n}.$$

It would be interesting to give an explicit formula expressing $\sigma_{I///J}$ as a polynomial in the σ_p's. For $n = m$ such a Giambelli-type formula was given in [P, Section 6] with the help of Schur's Q-polynomials.

3 Calculus of divided differences

Let $x = (x_1, \ldots, x_m)$ be a sequence of independent variables.

We have "symplectic divided differences":

$\partial_i : \mathbb{Z}[x] \to \mathbb{Z}[x]$ (of degree -1) $i = 1, \ldots, m$, defined by

$$\begin{aligned} \partial_i(f) &= (f - s_i f)/(x_i - x_{i+1}) \qquad i = 1, \ldots, m-1 \\ \partial_m(f) &= (f - s_m f)/2x_m \end{aligned}$$

(here, the s_i, $i < m$, acts on $\mathbb{Z}[x]$ by transposing x_i and x_{i+1} and s_m acts on $\mathbb{Z}[x]$ by sending x_m to $-x_m$, the other variables remaining invariant).

For every i the following Leibniz-type formula holds for $f, g \in \mathbb{Z}[x]$:

$$\partial_i(f \cdot g) = f \cdot (\partial_i g) + (\partial_i f) \cdot (s_i g) \tag{4}$$

We will need in the sequel the following formulas for generating functions. Let $\mathbf{a} = (a_m, a_{m-1}, \ldots, a_2, a_1) \in \{-1, 0, 1\}^m$. Define

$$E_{\mathbf{a}} = \prod_{i=1}^{m} (1 + a_i x_i).$$

For example for $\mathbf{a} = (0, \ldots, 0)$ we have $E_{\mathbf{a}} = 1$; if $\mathbf{a} = (\underbrace{0, \ldots, 0}_{m-n}, \underbrace{1, \ldots, 1}_{n})$ then $E_{\mathbf{a}} = \prod_{i=1}^{n} (1 + x_i) =: E$ say, is the generating function for the elementary symmetric polynomials in $x_1, \ldots, x_n$.

Lemma 3.1 a) $s_i(E_{\mathbf{a}}) = E_{\mathbf{a}'}$ *where*

$$\mathbf{a}' = \begin{cases} (a_m, \ldots, a_{i+2}, a_i, a_{i+1}, a_{i-1}, \ldots, a_1) & i < m \\ (-a_m, a_{m-1}, \ldots, a_1) & i = m \end{cases}$$

b) *For* $i = 1, 2, \ldots, m-1$

$$\partial_i(E_{\mathbf{a}}) = d \cdot E_{\mathbf{a}'} \qquad \textit{if } \; a_i = a_{i+1} + d \qquad (d = -2, -1, 0, 1, 2)$$

where $\mathbf{a}' = (a_m, \ldots, 0, 0, \ldots, a_1)$ *is the sequence* $\mathbf{a}$ *with* a_{i+1}, a_i *replaced by zeros. In particular if* Δ *is a composition of some* s- *and* ∂-*operations then for every* $\mathbf{a}$, $\Delta(E_{\mathbf{a}}) = (scalar) \cdot E_{\mathbf{a}'}$, *where* $\mathbf{a}'$ *is uniquely determined if this scalar is not zero.*

c) $\partial_m(E_{\mathbf{a}}) = a_m \cdot E_{(0, a_{m-1}, \ldots, a_1)}$. $\square$

Next, we recall that for any $w \in W$ and every reduced decomposition $w = s_{i_1} \cdot \ldots \cdot s_{i_l}$ one can define $\partial_w := \partial_{i_1} \circ \ldots \circ \partial_{i_l}$—an operator on $\mathbb{Z}[x]$ of degree $-\ l(w)$. In fact ∂_w does not depend on the reduced decomposition chosen. There exists a surjective ring homomorphism

$$c : \mathbb{Z}[x] \to A^*(F)$$

(called the characteristic map) whose value for a homogeneous $f \in \mathbb{Z}[x]$ is given by

$$c(f) = \sum_{l(w)=\deg f} \partial_w(f) X_w$$

(we refer to [B–G–G] and especially to [D2] for details concerning c).

Lemma 3.2 *For every $p = 1, \ldots, n$, denoting by $e_p(x_1, \ldots, x_n)$ the p-th elementary symmetric polynomial in $x_1, \ldots, x_n$, we have*

$$c(e_p(x_1, \ldots, x_n)) = X_{s_{n-p+1} \cdot \ldots \cdot s_{n-1} \cdot s_n} = \sigma_p \in A^p(G).$$

Proof. It is sufficient to show that $\partial_w(e_p(x_1, \ldots, x_n)) = 0$ unless $w = w_{(p)} := s_{n-p+1} \cdot \ldots \cdot s_{n-1} \cdot s_n$ and $\partial_{w_{(p)}}(e_p(x_1, \ldots, x_n)) = 1$. This follows easily by induction on n using the following properties of ∂_i:

$$\partial_i(e_q(x_1, \ldots, x_h)) \neq 0 \qquad \text{only if } \ h = i,$$

and

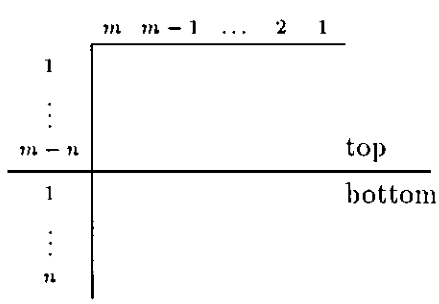
$$\partial_i(e_q(x_1, \ldots, x_i)) = e_{q-1}(x_1, \ldots, x_{i-1}). \ \square$$

Note that the lemma says that the p-th Chern class of the tautological isotropic n-bundle on G is equal to σ_p.

Let $f_\lambda \in \mathbb{Z}[x]$ be homogeneous such that $c(f_\lambda) = \sigma(\lambda)$. Then for $w \in W$, $l(w) = |\lambda|$, we have $\partial_w(f_\lambda) \neq 0$ iff $w = w_\lambda$ and $\partial_{w_\lambda}(f_\lambda) = 1$. Our goal is to find the coefficients m_μ in

$$c(f_\lambda \cdot e_p(x_1, \ldots, x_n)) = \sum m_\mu \sigma(\mu).$$

It is convenient to use the following coordinates for indexing boxes in shapes:



(i.e. the leftmost column now has the number m and we use a separate numbering of the top and the bottom rows). Note that the first coordinate of a given box will indicate the number of its row (it will be always clear whether it is a top or bottom row) and the second—the number of its column.

For a shape μ we denote by $\mathring{D}_\mu$ the set of boxes obtained by removing from D^t_μ the following set of boxes:

$$\begin{array}{llll} (1,m), & \ldots, & (1,n+2), & (1,n+1) \\ (2,m), & \ldots, & (2,n+2) & \\ & \vdots & & \\ (m-n,m) & & & \end{array} \tag{5}$$

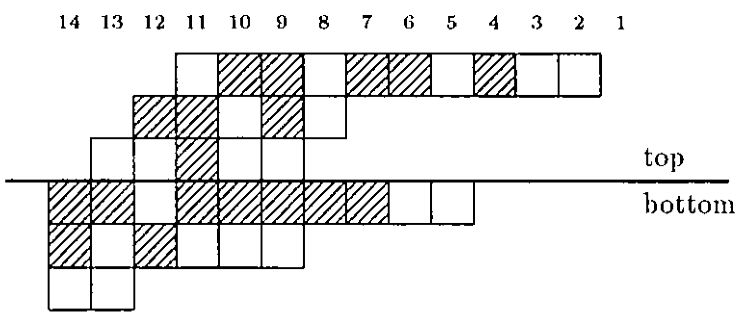
Consider now a subset D of $\mathring{D}_\mu$. The boxes in $\mathring{D}_\mu$ which belong to D will be called *D-boxes*; the boxes in $\mathring{D}_\mu \setminus D$ will be called $\sim$*D-boxes*. Denote by D^t (resp. D^b) the restriction of D to D^t_μ (resp. D^b_μ). Now we associate with D an operator ∂^D_μ , and a word r_D.

Definition 3.3 Read $\mathring{D}_\mu$ row by row left to right and from top to bottom. Every D-box (resp. $\sim$ D-box) in the i-th column gives us the s_i (resp. ∂_i). Then ∂^D_μ is the composition of the resulting s_i's and ∂_i's (the composition written from right to left).

Definition 3.4 Read $\mathring{D}_\mu$. Every D-box in the i-th column gives us the s_i. $\sim$ D-boxes give no contribution. Then r_D is the word obtained by writing the resulting s_i's from right to left.(In other words, one obtains r_D by erasing all the ∂_i's from ∂^D_μ.

Example 3.5 (i) Note that for $D = \mathring{D}_\lambda$, r_D is the reduced decomposition of w_λ given before Example 1.3.

(ii) $m = 14 \quad \mu = (12, 6, 5//10, 6, 2)$



(D-boxes are shaded here)

$$
\begin{aligned}
\partial_\mu^D &= \partial_{13}\circ\partial_{14}\circ\partial_9\circ\partial_{10}\circ\partial_{11}\circ\partial_{12}\circ\partial_{13}\circ\partial_{14}\circ\partial_5\circ\partial_6\circ s_7\circ s_8\circ s_9\circ s_{10}\\
&\circ s_{11}\circ\partial_{12}\circ s_{13}\circ s_{14}\circ\partial_9\circ\partial_{10}\circ s_{11}\circ\partial_{12}\circ\partial_{13}\circ\partial_8\circ s_9\circ\partial_{10}\\
&\circ s_{11}\circ s_{12}\circ\partial_2\circ\partial_3\circ s_3\circ s_4\circ\partial_5\circ s_6\circ s_7\circ\partial_8\circ s_9\circ s_9\circ s_{10}\circ\partial_{11},\\
r_D &= s_{12}\cdot s_{14}\cdot s_7\cdot s_8\cdot s_9\cdot s_{10}\cdot s_{11}\cdot s_{13}\cdot s_{14}\cdot s_{11}\cdot s_9\cdot s_{11}\cdot s_{12}\\
&\cdot s_4\cdot s_6\cdot s_7\cdot s_9\cdot s_{10}.
\end{aligned}
$$

Proposition 3.6 *In the above notation,*

$$m_\mu = \sum \partial_\mu^D(e_p(x_1,\ldots,x_n)),$$

where the sum is over all $D\subset \mathring{D}_\mu$ such that $r_D\in R(w_\lambda)$ [2]*.*

Proof. We know that $m_\mu = \partial_{w_\mu}(f_\lambda\cdot e_p(x_1,\ldots,x_n))$. Moreover we have $\partial_{w_\mu} = \partial_\mu^\emptyset$. The multiplicity $m_\mu = \partial_\mu^\emptyset(f_\lambda\cdot e_p(x_1,\ldots,x_n))$ can be computed by a consecutive applications of (4) used in this way: we apply only the ∂_i's (and the identity operators) to f_λ; and both the s_i's and ∂_i's to $e_p(x_1,\ldots,x_n)$. We get

$$m_\mu = \sum \partial_{r_D}(f_\lambda)\cdot\partial_\mu^D(e_p(x_1,\ldots,x_n)),$$

the sum over all $D\subset\mathring{D}_\mu$. The summand corresponding to such a D is not zero only if card $D\geq \deg f_\lambda$ and card$(D_\mu - D)\geq p$. But card $D_\mu = \deg f_\lambda + p$. Thus the above inequalites can be replaced by equalities. Consequently, it follows from the properties of f_λ that $\partial_{r_D}(f_\lambda)=0$ if $r_D\notin R(w_\lambda)$ and 1 if $r_D\in R(w_\lambda)$. The assertion follows. □

[2]For a given $w\in W$ we denote by $R(w)$ the set of its reduced decompositions.

4 z- and v-ribbons

We treat a reduced decomposition $w_\lambda = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_l}$ as a sequence of "simple transposition–operations" s_{i_h}, $h = 1, \ldots, l$, which produces w_λ from the identity permutation $(1, 2, \ldots, m)$:

$$(\ldots(((1,2,\ldots,m)\cdot s_{i_1})\cdot s_{i_2})\cdot\ldots)\cdot s_{i_l} = \\ (y_1,\ldots,y_{n-k};\overline{z}_k,\overline{z}_{k-1},\ldots,\overline{z}_1;v_1,\ldots,v_{m-n}).$$

These "simple transposition–operations" will be called "s_{i_h}-operations" $h = 1, \ldots, l$; or simply "$s_{i.}$-operations".

Assume that an element $w = (w_1, \ldots, w_m) \in W$ is written in the "barred permutation notation". Recall that s_i, $i < m$, acting from the right on W, interchanges the i-th and $(i+1)$-th component of w. The s_m supplies the m-th component of w with a bar, if this component is bar-free.

Proposition 4.1 *Either exactly one (bar-free) $z_.$ or exactly one $v_.$ is nontrivially involved in a "$s_{i.}$-operation". More precisely*

a) *If $i_h = m$, then the operation is:*

$$\ldots z_. \to \ldots \overline{z}_.$$

b) *If $i_h < m$, then the operation is:*

$$\ldots z_. x \ldots \to \ldots x z_. \ldots, \qquad (x \neq z_j, j = 1, \ldots, k)$$

or,

$$\ldots v_. x \ldots \to \ldots x v_. \ldots, \\ (x \neq z_j, j = 1, \ldots, k;\ x \neq v_r, r = 1, \ldots, m-n).$$

Proof. For every $j = 1, \ldots, k$, we define O_j to be the family of all "$s_{i.}$-operations" which move z_j forward, toward the m-th place or supply z_j with a bar.

For every $r = 1, \ldots, m-n$ we define O'_r to be the family of all "$s_{i.}$-operations" which move v_r forward. Obviously $\operatorname{card} O_j \geq m - z_j + 1$. Moreover, $\operatorname{card} O'_r \geq n + r - v_r + d_r$ where $d_r = \operatorname{card}\{j : z_j < v_r, j = 1, \ldots, k\}$. Indeed, this is a minimal number of simple transpositions which transform v_r from the v_r-place to the $(n+r)$-th place in w_λ. The summand "d_r" comes from the fact that every $z_\cdot$ goes under "$s_{i_\cdot}$-operations" to the m-th place; thus if $z_j < v_r$ then some "$s_{i_\cdot}$-operation" acts as

$$\ldots z_j v_r \ldots \longrightarrow \ldots v_r z_j \ldots .$$

It is clear that the sets O_j, $j = 1, \ldots, k$ and O'_r, $r = 1, \ldots, m-n$ are disjoint. Therefore the cardinality of $s_{i_\cdot}$-operations needed to pass from $(1, \ldots, m)$ to w_λ is at least equal to

$$\sum_j \operatorname{card} O_j + \sum_r \operatorname{card} O'_r \geq \sum_{j=1}^{k} (m + 1 - z_j) + \sum_{r=1}^{m-n} (n + r - v_r + d_r)$$

Comparing this with (1.3) we conclude that the sets O_j, $j = 1, \ldots, k$, O'_r, $r = 1, \ldots, m-n$, exhaust all the "$s_{i_\cdot}$-operations" (and, consequently, the above inequalities for $\operatorname{card} O_j$ and $\operatorname{card} O'_r$ can be replaced by equalities). This implies the assertion. □

Corollary 4.2 *No "$s_{i_\cdot}$-operation" as above can interchange $z_\cdot$ and z_*, $v_\cdot$ and v_*. A "$s_{i_\cdot}$-operation" can only cause the following changes:*

- *move $z_\cdot$ forward, toward the m-th place and v_* backward toward the first place.*

- *move $z_\cdot$ forward, toward the m-th place with moving no v_*.*

- *move $v_\cdot$ forward with moving no z_*.*

- *supply $z_\cdot$ (in the m-th place) with a bar.*

Now invoking definitions and notation from Section 3 we introduce the notion of a z- and v-mark of a D-box. Assume that $r_D \in R(w_\lambda)$. Suppose that a D-box appears in the i-th column ($i = m, m-1, \ldots, 2, 1$). We associate with it its mark $(a, b) \in \{0, 1, \ldots, k\} \times \{0, 1, \ldots, m-n\}$ where

1) If $i = m$, we put $b = 0$ and a is the subscript in $z_\cdot$ that is supplied with a bar by the operation $s_{i_\cdot}$,

2) If $i < m$, we denote by a (resp. by b) the subscript of $z_\cdot$ (resp. of v_*) that is moved by the operation $s_{i_\cdot}$. If no $z_\cdot$ (resp. v_*) is moved by the operation $s_{i_\cdot}$, then we put $a = 0$ (resp. $b = 0$).

We will call a (resp. b) the z- (resp. v-) *mark* of a D-box. A D-box with a mark (a, b), $a \geq 1$ will be called a *z-box*. A D-box with a mark (a, b), $b \geq 1$ will be called a *v-box*. A D-box with a mark $(0, b)$, $b \geq 1$ will be called a *pure v-box*.

Example 4.3 $m = 10$, $n = 7$, $\mu = (10, 8, 5//8, 7, 2, 1)$, $\lambda = (10, 8, 5//7, 4, 1)$, $w_\lambda = (2, 5, 6, 9; \overline{10}, \overline{7}, \overline{4}; 1, 3, 8)$. Two possible $D \subset \mathring{D}_\mu$ are depicted below together with marks of their D-boxes ("$\bigcirc$" visualizes here a $\sim D$-box)

$$
\begin{array}{cccccccccc}
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
 & & & (0,1) & (0,1) & (0,1) & (0,1) & (0,1) & (0,1) & (0,1) \\
 & & (0,3) & (0,3) & (0,3) & (0,3) & (0,3) & (0,3) & & \\
 & (0,8) & (0,8) & (0,8) & (0,8) & & & & & \\
\hline
(1,0) & (1,0) & (1,0) & (1,0) & (1,8) & (1,0) & (1,0) & \bigcirc & & \\
(2,0) & (2,0) & (2,0) & (2,8) & \bigcirc & \bigcirc & \bigcirc & & & \\
(3,0) & \bigcirc & & & & & & & & \\
\bigcirc & & & & & & & & &
\end{array}
$$

$$
\begin{array}{cccccccccc}
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
 & & & (0,1) & (0,1) & (0,1) & (0,1) & (0,1) & (0,1) & (0,1) \\
 & & (0,3) & (0,3) & (0,3) & (0,3) & \bigcirc & \bigcirc & & \\
 & (0,8) & (0,8) & (0,8) & (0,8) & & & & & \\
\hline
(1,0) & (1,0) & (1,0) & (1,0) & (1,8) & \bigcirc & (0,3) & (0,3) & & \\
(2,0) & (2,0) & (2,0) & (2,8) & \bigcirc & (1,0) & (1,0) & & & \\
\bigcirc & \bigcirc & & & & & & & & \\
(3,0) & & & & & & & & &
\end{array}
$$

Let us record some properties of z- and v-boxes.

Proposition 4.4 a) *z-boxes with a fixed z-mark and pure v-boxes with a fixed v-mark form a connected set of boxes in each row.*

b) *(Separateness) In a fixed row, any two sets of D-boxes are disconnected (i.e. there is at least one $\sim D$-box between them) provided:*

— *they are equipped with two different z-marks ≥ 1,*

— *they are pure v-boxes and equipped with two different v-marks,*

— *one of them consists of z-boxes with a fixed mark and the second—of pure v-boxes with a fixed mark.*

c) *The set of z-boxes with a mark (j, b) where $j \geq 1$ is fixed and b can vary, is contained entirely in the bottom part and is of the form*

$$(t_m, m), (t_{m-1}, m-1), \ldots, (t_{z_j}, z_j),$$

where $t_m \leq t_{m-1} \leq \ldots \leq t_{z_j}$.

d) *The z-marks of boxes in a fixed column (strictly) increase from top to bottom.*

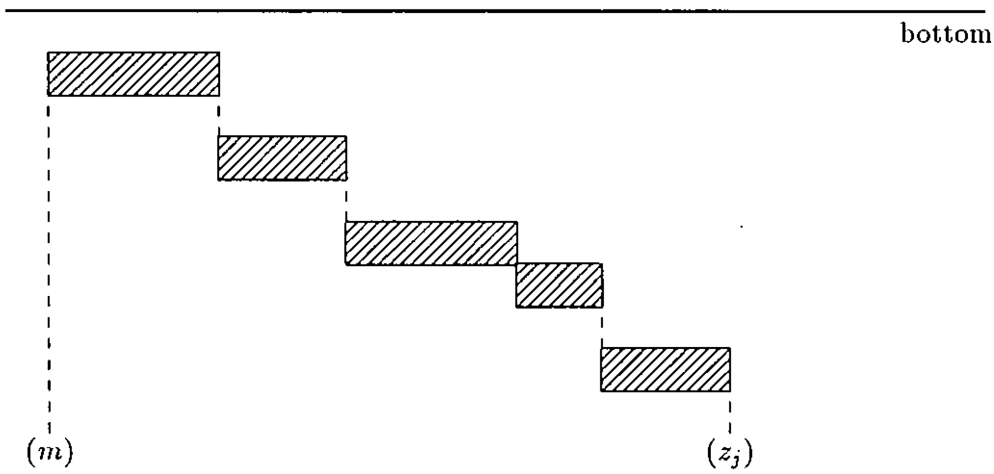
e) *In D^t only pure v-boxes appear. Their v-marks in a fixed column (strictly) increase from top to bottom of D^t.*

Proof. The assertions a) and b) are obvious. As for c), the proof of the second assertion is exactly the same as the one of Lemma 4.3c) in [P–R1]; then the first assertion follows because the leftmost box in every z-ribbon must appear in the m-th column. Finally, the proof of d) and e) is the same as the one of Lemma 4.3d) in [P–R1]. □

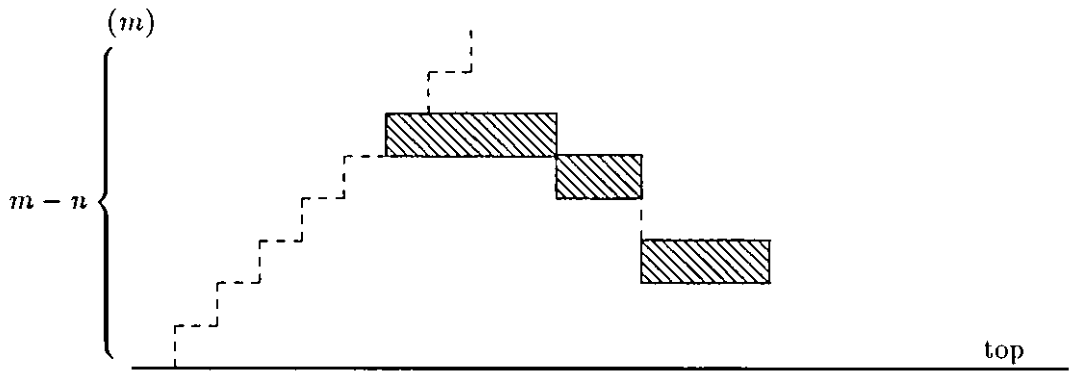
Definition 4.5 A set of z-boxes with a mark (j, b) where $j \geq 1$ is fixed and b can vary will be called a *z-ribbon* of z-mark j.

A set of v-boxes with a mark (a, r) where $r \geq 1$ is fixed and a can vary will be called a *v-ribbon* of v-mark r.

Remark 4.6 (i) A z-ribbon of mark j can be depicted as:



(ii) In a similar way one can visualize the top part of a v-ribbon:



The property depicted here is analogous to Proposition 4.4(c). Its proof is mutatis mutandis the same as the one of Lemma 4.3c) in [P–R1]. (Recall that all top v-boxes are pure.)

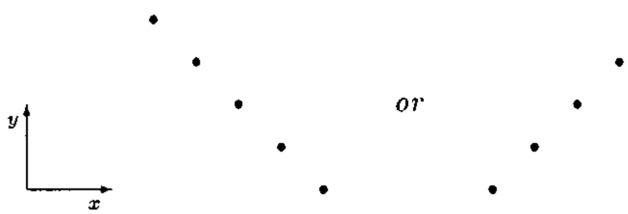
Note that Proposition 4.4 admits the following obvious complement concerning v-boxes in the bottom part of D_μ.

Proposition 4.7 *Read the bottom part of v-ribbon of mark r. Then the graph of the function:*

x = the number of a box in the bottom part of the ribbon	$\mapsto$	*y = the column number of the box*

has properties:

1° *It is a sum of sets of points of the form (which we will call a decreasing and increasing part of the graph, respectively)*



of cardinality ≥ 1. (Note that a set consisting of a single point only can be both an increasing or decreasing part of the graph).

2° *No two decreasing (resp. increasing) parts of the graph can appear successively.*

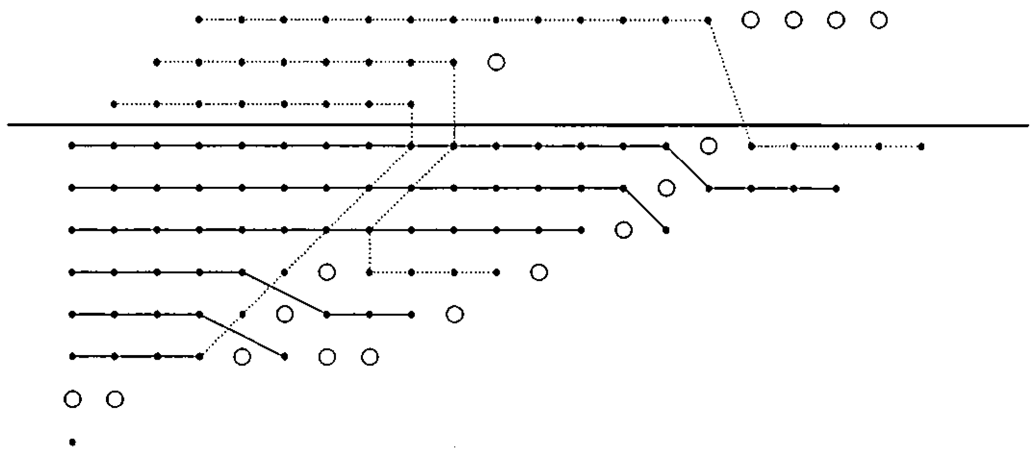
3° *The end and the beginning of two successive parts have the same y-coordinate.*

(Under this identification the function $y(x)$ is decreasing over pure v-boxes and increasing over non pure v-boxes).

Example 4.8

$$\begin{aligned}\mu &= (23, 11, 9//21, 19, 15, 12, 10, 8, 2, 1)\\ \lambda &= (21, 14, 9//19, 14, 13, 9, 6, 4, 1).\end{aligned}$$

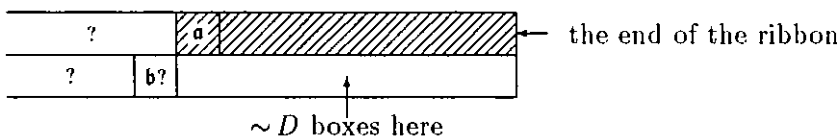
In the picture below "$\circ$" visualizes $\sim D$-boxes, z-ribbons are depicted with "___" and v-ribbons—with "..........".



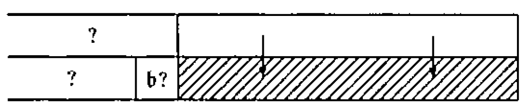
We end this Section with some information on deforming of z-ribbons, which is the key technical tool of the present paper.

Fix $j = 1, \ldots, k$ and consider the z-ribbon of mark j. Choose a box in the last row of the ribbon.

Lemma 4.9 *Assume that the following configuration of D-boxes appears ($\mathfrak{a}$ can be in the m-th column, $\mathfrak{b}$—if exists—is a $\sim$D-box).*

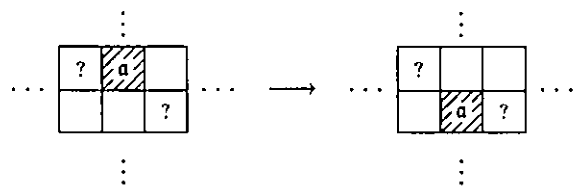


Such an $\mathfrak{a}$ will be called a breaking box. *Then the change of the configuration to:*



gives us a D' such that $r_{D'} \in R(w_\lambda)$ and the z-ribbon with mark j in D' is obtained through the same deformation from the z-ribbon in D.

Proof. Arguing by induction on the distance from $\mathfrak{a}$ to the end of the ribbon, it is sufficient to prove the assertion when $\mathfrak{a}$ is the last box of the ribbon. The assertion for the change:



(unshaded boxes are not in D) follows from the relations $s_i s_j = s_j s_i$, $|i-j| > 1$, applied to all D-boxes appearing in reading between the initial and final place of the deformed box. □

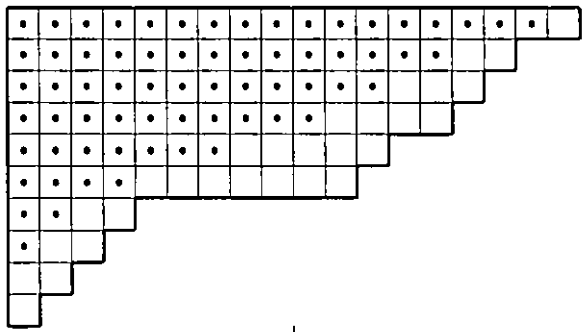
Now suppose that $D^b_\lambda \subset D^b_\mu$. We shall now describe a process of deforming rows of D^b_λ inside of D^b_μ. The description is by descending induction on the number of a row, starting from the last row.

1) If $(l(\lambda^b), m)$ is a breaking box we apply the deformation from the lemma so many times as it is possible. In other words we push down in a maximal way the last row of D^b_λ in D^b_μ.

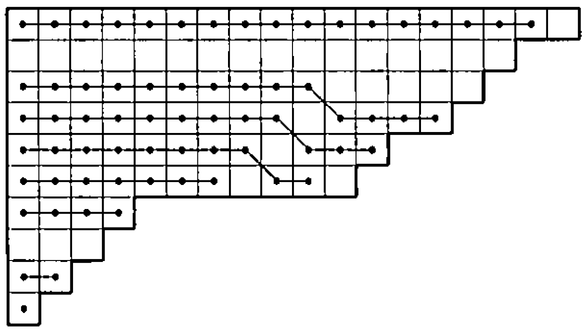
2) Pick a row in D^b_λ and suppose that lower rows in D^b_λ have been already deformed in D^b_μ. Let $\mathfrak{a}$ be the first breaking box in the row, appearing in reading. Apply the deformation from the lemma so many times as it is possible. Assume that $\mathfrak{b}$ is the first breaking box in the deformed part of the row; apply to $\mathfrak{b}$ the deformation from the lemma so many times as it is possible. Then repeat the same with the first breaking box of the resulting part of the row etc.

It is clear that this procedure defines a certain subset D^b in D^b_μ such that $r_{D^b \cup D^t_\lambda} \in R(w_\lambda)$, together with its decomposition into z-ribbons. We will call this subset the *maximal deformation* of D^b_λ in D^b_μ and denote $(D^b)^{\lambda,\mu}$ (this notation differs from the one in [P–R1]).

Example 4.10 The maximal deformation for
$\mu^b = (18, 16, 15, 14, 12, 11, 4, 3, 2, 1)$
$\lambda^b = (17, 14, 12, 10, 7, 4, 2, 1)$.
(Boxes in D^b_λ and their deformations are marked with dots. Deformed z-ribbons are depicted with "___".)

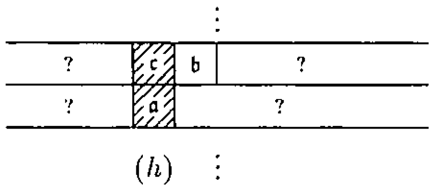


↓



5 Some omnibus lemmas

Lemma 5.1 *Assume that $r_D \in R(w_\lambda)$. Then, the following configuration of D-boxes cannot appear:*



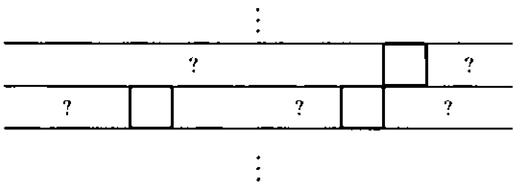
where $\mathfrak{a}$ *is a pure v-box,* $\mathfrak{b}$*—$\sim$D-box,* $\mathfrak{c}$*—D-box.*

Proof. The s-operator of $\mathfrak{a}$ changes $(\dots, v_\cdot, x, \dots) \to (\dots, x, v_\cdot, \dots)$, where $x \neq z_\cdot, v_*$ (the pair $(v_\cdot, x)$ occupies the h-th and $(h+1)$-th places). The existence of $\mathfrak{b}$ implies that the s-operator of $\mathfrak{c}$ changes the position of x from the h-th to the $(h+1)$-th place. This contradicts Proposition 4.1. $\square$

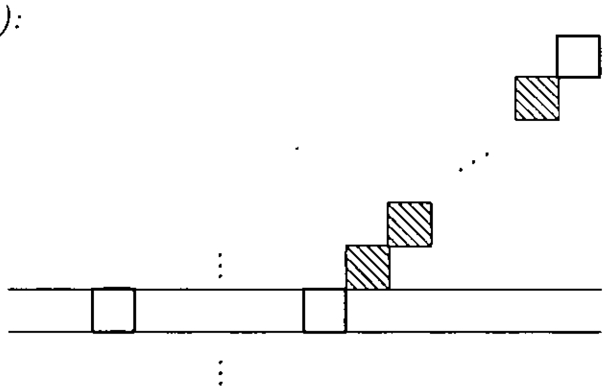
The next two lemmas can be proved by a direct calculation.

Lemma 5.2 *The following configuration of $\sim$D-boxes in D^b_μ implies $\partial^D_\mu(E) = 0$:*

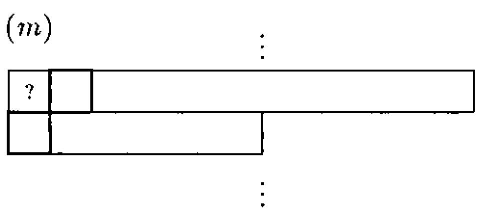
(i)



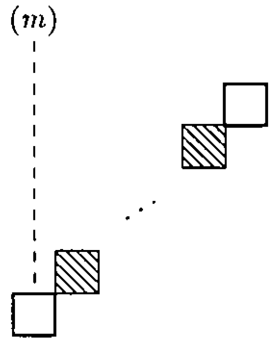
and more generally (shaded boxes are D-boxes and marked unshaded boxes are $\sim$D boxes):



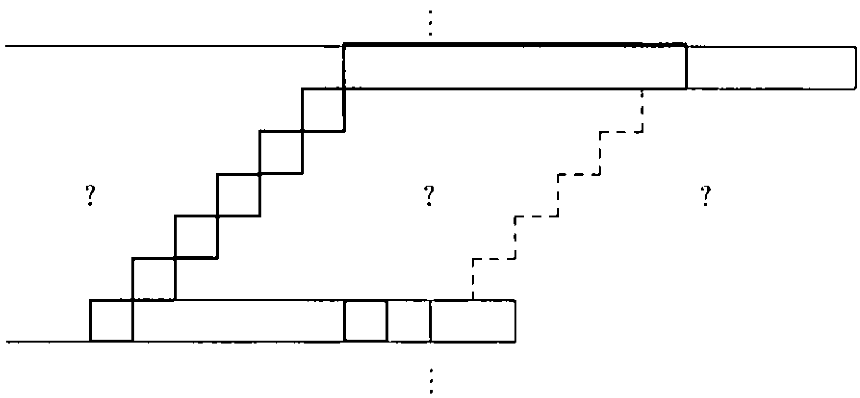
(ii)



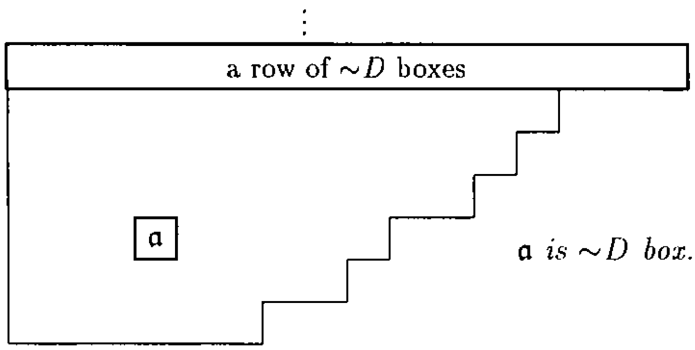
and more generally



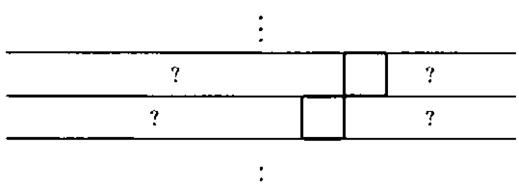
(iii)



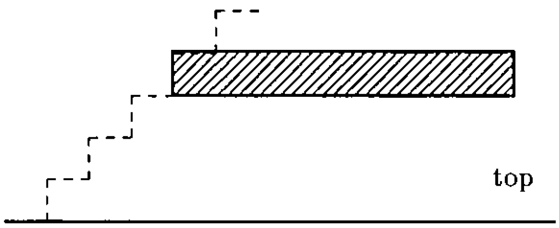
(iv)



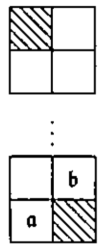
Lemma 5.3 *The following configuration of* $\sim D$*-boxes in* D^t_μ *implies* $\partial^D_\mu(E) = 0$*:*



Lemma 5.4 *The top segment of each v-ribbon in $D^\iota \subset \mathring{D}_\mu$ such that $\partial_\mu^D(E) \neq 0$, is of the form (cf. Remark 4.5):*

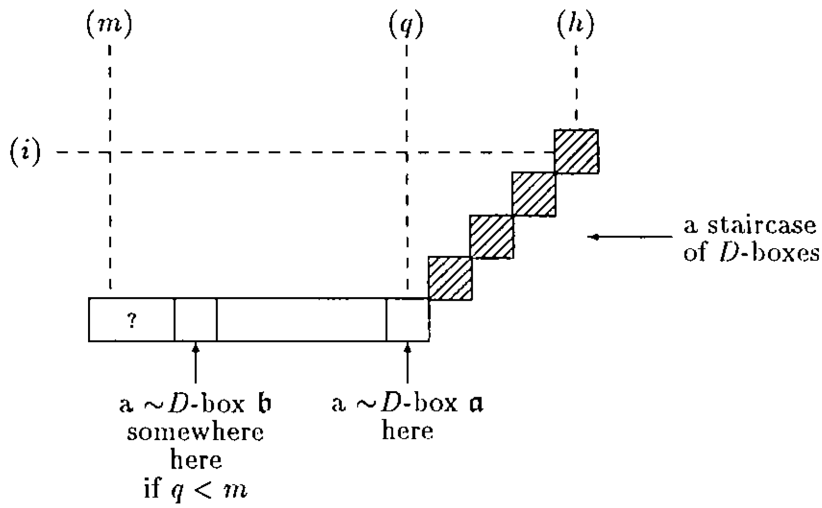


Proof. Suppose, conversely that the top segment of a v-ribbon contains a configuration:



Since obviously $\mathfrak{a}$ and $\mathfrak{b}$ are $\sim D$-boxes, we get a contradiction with Lemma 5.3. The assertion follows. □

Lemma 5.5 *Let $D \subset D_\mu$. Assume that for some i the following configuration of D and $\sim D$-boxes appears in the bottom part of D:*



Let Δ be the "part" of the operator ∂_μ^D formed by the composition of s- and ∂-operators of all D-boxes above the i-th row. Assume $\Delta(E) = (scalar) \cdot E_{\mathbf{a}}$ where $a_h = 0$. Then $\partial_\mu^D(E) = 0$.

This situation occurs—for example—if

- $i = 1$; a row of D^t ends in the h-th column.
- $i = 1$; a $\sim D$-box appears in D^t_μ in the h-th column.
- $i > 1$ and the $(i-1)$-th row consists of $\sim D$-boxes only.

Proof. The s-operators of marked D-boxes from the staircase change successively the position of 0 in the sequence $\mathbf{a}$ from the h-th place to the q-th place. Then the ∂-operator of $\mathfrak{a}$ causes $\partial^D_\mu(E) = 0$ in virtue of the existence of $\mathfrak{b}$. □

Lemma 5.6 *Assume $D \subset D_\mu$ for a shape μ, such that D^t is a strict diagram and $D^t_\mu \setminus D^t$ is a horizontal strip with pairwise disconnected rows. Let Δ be the "part" of ∂^D_μ formed by the composition of s- and ∂-operators of all D^t-boxes. Then $\Delta(E) = 1 \cdot E_{\mathbf{a}}$ where the sequence $\mathbf{a}$ is built as follows:*

$$a_h = \begin{cases} 0 & \textit{if } h \textit{ is the column number of the end of a row of } D^t \\ 0 & \textit{if } h \textit{ is the column number of a box in } D_\mu \setminus D \\ 1 & \textit{in the opposite case.} \end{cases}$$

The proof is a straightforward calculation.

6 Admissible deformations of z-ribbons

Assume that $\lambda, \mu \in \mathcal{P}_n$, $n = 1, \ldots, m$, and $D \subset \mathring{D}_\mu$.

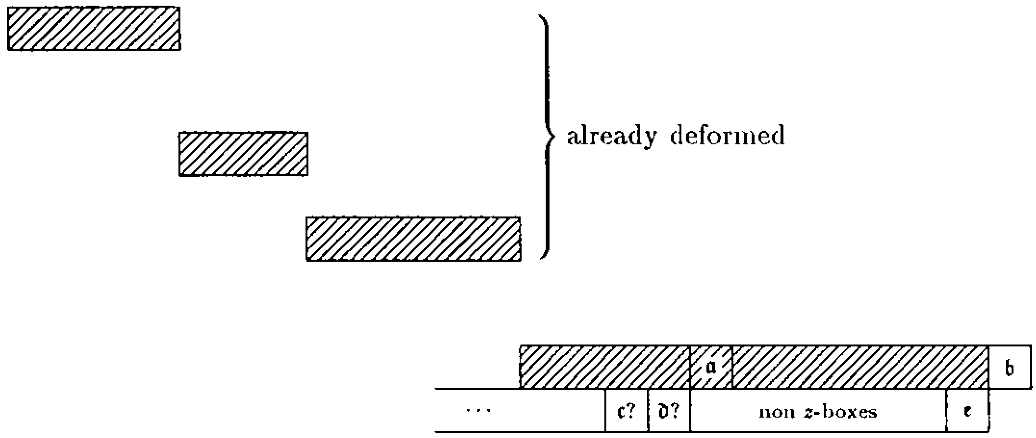
Proposition 6.1 *The conditions*

$$r_D \in R(w_\lambda) \qquad \text{and} \qquad \partial_\mu^D(E) \neq 0 \tag{6}$$

hold only if the set of z-boxes of D is equal to the maximal deformation of D_λ^b in D_μ^b.

A proof will be divided into several steps. We claim first that $D_\lambda^b \subset D_\mu^b$. Indeed D_λ^b is obtained from the set of z-boxes of D by the operation reverse to the maximal deformation (see Section 4). It is now clear from the form of a z-ribbon (Proposition 4.4(c)) and the way of deforming, that this forces $D_\lambda^b \subset D_\mu^b$.

We will now show that the maximal deformation of D_λ^b in D_μ^b is necessary in order to avoid the vanishing $\partial_\mu^D(E) = 0$ (for short: "vanishing"). Fix a row in D_λ^b and assume that the assertion has been proved for lower rows (i.e. they have been maximally deformed to avoid the vanishing). Assume that the maximal deformations have been performed in all preceding breaking boxes of the row and fix the next free breaking box $\mathfrak{a}$. Our situation is depicted as follows:



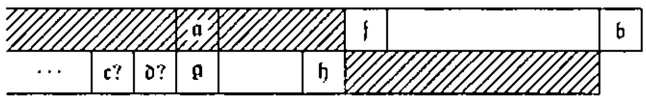
(Note that a box $\mathfrak{b}$ must exist because μ^b is strict.) We will show that to avoid the vanishing, the maximal deformation of $\mathfrak{a}$ together with its right-hand part is necessary. If we do no changes at all, then after adding pure v-boxes we have

- $\mathfrak{b}$ is a $\sim D$-box: this follows from the separateness property (Proposition 4.4(b))

- $\mathfrak{e}$ is a $\sim D$-box: this follows from Lemma 5.1.
- in the row of $\mathfrak{e}$ there exists a $\sim D$-box left to $\mathfrak{e}$ (the only exception is that $\mathfrak{e}$ lies in the m-th column which can happen iff $\mathfrak{a}$ is in the m-th column; but this leads to a contradiction by Lemma 5.2(ii)). Indeed, if $\mathfrak{a}$ lies in the column with number $< m$, then regardless $\mathfrak{c}$ exists or not, a non z-box $\mathfrak{d}$ must be a $\sim D$-box (use the separateness property or the fact that no pure v-box appears in the m-th column).

But then, the configuration of boxes $\{\mathfrak{d}, \mathfrak{e}, \mathfrak{b}\}$ contradicts Lemma 5.2(i).

Now suppose that we can escape the vanishing by pushing down a final segment of the row starting after $\mathfrak{a}$. Pictorially, the effect of such a deformation is:

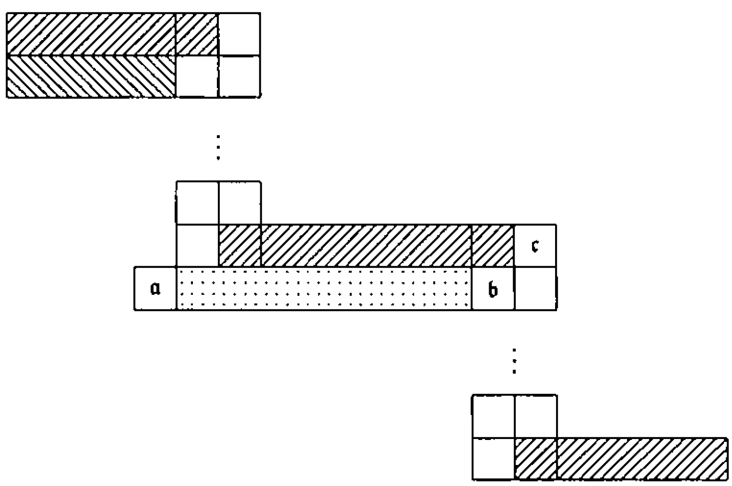


Here, $\mathfrak{f}$ is not a pure v-box by the separateness property, and $\mathfrak{h}$ is not a pure v-box by Lemma 5.1. If $\mathfrak{a}$ is in the m-th column, then $\mathfrak{g}$ is not a pure v-box and the configuration of $\sim D$-boxes $\{\mathfrak{g}, \mathfrak{h}, \mathfrak{f}\}$ contradicts Lemma 5.2(i) or (ii). If $\mathfrak{a}$ is not in the m-th column then $\mathfrak{d}$ is not a pure v-box and the configuration of $\sim D$-boxes $\{\mathfrak{d}, \mathfrak{h}, \mathfrak{f}\}$ contradicts Lemma 5.2(i).

The proposition has been proved. □

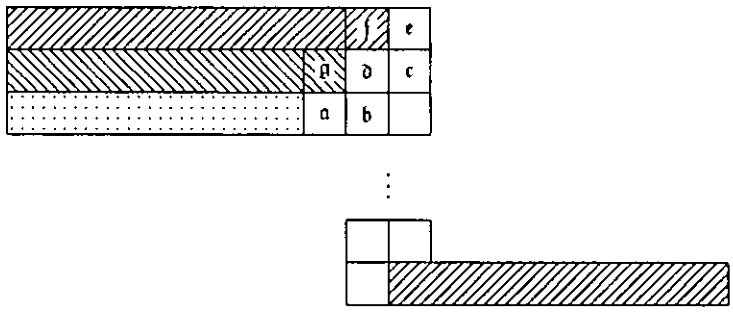
Proposition 6.2 *In $D \subset \mathring{D}_\mu$ satisfying (6), the z-boxes with the same z-mark can appear in at most two succesive rows.*

Proof. Assume first that some z-boxes with the same mark appear in three different rows. We visualize the situation as follows (boxes which are not depicted are irrelevant):



The existence of segments of z-boxes depicted here as and follows from the construction of maximal deformation that is necessary by Proposition 6.1. It follows from Proposition 4.4(d) that the z-marks of - and -boxes are equal (being both bigger by 1 than the z-mark of -boxes). Thus, by Proposition 4.4(d), $\mathfrak{a}$ is a $\sim D$-box. Since, by the separateness property $\mathfrak{b}$ and $\mathfrak{c}$ are also $\sim D$-boxes, the configuration $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ contradicts Lemma 5.2(i).

Now suppose that some z-boxes with the same z-mark appear in two rows that are not successive. We visualize the situation as follows (boxes which are not depicted are irrelevant):

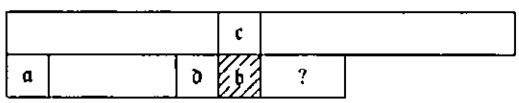


The existence of (nonempty) segments of z-boxes depicted here as and follows from the construction of maximal deformation that is necessary by Proposition 6.1. The non z-boxes $\mathfrak{a}$,$\mathfrak{d}$,$\mathfrak{e}$ are $\sim D$-boxes by the separateness property. If $\mathfrak{c}$ (resp. $\mathfrak{b}$) is a D-box then its s-operator moves forward the same element as the s-operator of $\mathfrak{f}$ (resp. $\mathfrak{g}$). Hence $\mathfrak{c}$ and $\mathfrak{b}$ are not pure v-boxes. Moreover $\mathfrak{c}$ is not a z-box by Proposition 4.4(d), and $\mathfrak{b}$ is not a z-box by Proposition 4.4(c). Then the configuration of $\sim D$-boxes $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ contradicts Lemma 5.2 (i).

The proposition has been proved. $\square$

Lemma 6.3 *In $D \subset \overset{\circ}{D}_\mu$ satisfying (6), no row with a $\sim D$-box in the m-th column contains a z-box.*

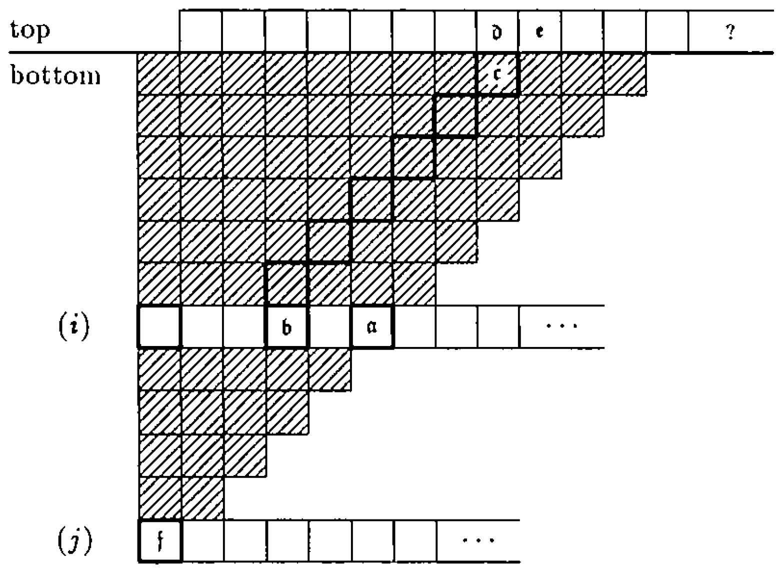
Proof. Assume conversely that there exists a row containing a z-box and a $\sim D$-box $\mathfrak{a}$ in the m-th column. This cannot be the first row of the bottom part, because the z-ribbon containing a z-box must contain a box in the m-th column above this z-box. Pick a leftmost z-box $\mathfrak{b}$ in the row. Pictorially,



Then $\mathfrak{c}$ must be a $\sim D$-box (in the opposite case the s-operator of $\mathfrak{c}$ will move the z., that is pushed forward by the s-operator of $\mathfrak{b}$, backward—which is not possible (see Section 4). But the configuration of $\sim D$-boxes $\{\mathfrak{a}, \mathfrak{d}, \mathfrak{c}\}$ contradicts Lemma 5.2(i) or (ii). The assertion has been proved. □

Proposition 6.4 *In $D \subset \overset{\circ}{D}_\mu$ satisfying (6), there are no two rows with $\sim D$-boxes in the m-th column. Consequently $l(\mu^b) \leq l(\lambda^b) + 1$.*

Proof. Suppose, conversely, that there exist two rows with $\sim D$-boxes in the m-th column. Pick up a pair of such rows with the smallest row numbers $i < j$. By the lemma both the i-th and j-th rows do not contain z-boxes. Thus by the construction of maximal deformation that is necessary by Proposition 6.1, the $(i-1)$-th bottom row must contain an initial segment of a z-ribbon (of z-mark $(i-1)$) of length $\geq \lambda_i + 1$, similarly the $(i-2)$-th row must contain an initial segment of a z-ribbon (of z-mark $(i-2)$) of length $\geq \lambda_i + 2, \ldots,$ finally the first row must contain an initial segment of a z-ribbon (of z-mark 1) of length $\geq \lambda_i + i - 1$. A similar argument, if $j > i+1$, shows that the rows with numbers $j-1, j-2, \ldots, i+1$ must contain similar initial segments of z-ribbons of length $\geq \lambda_j + 1, \geq \lambda_j + 2, \ldots, \geq \lambda_j + (j-i) - 1$ respectively. Let $\mathfrak{a}$ denote the box $(i, m-(j-i))$. The following picture (where only those z-boxes which are relevant for our purposes, are shaded) will be helpful to end the proof:



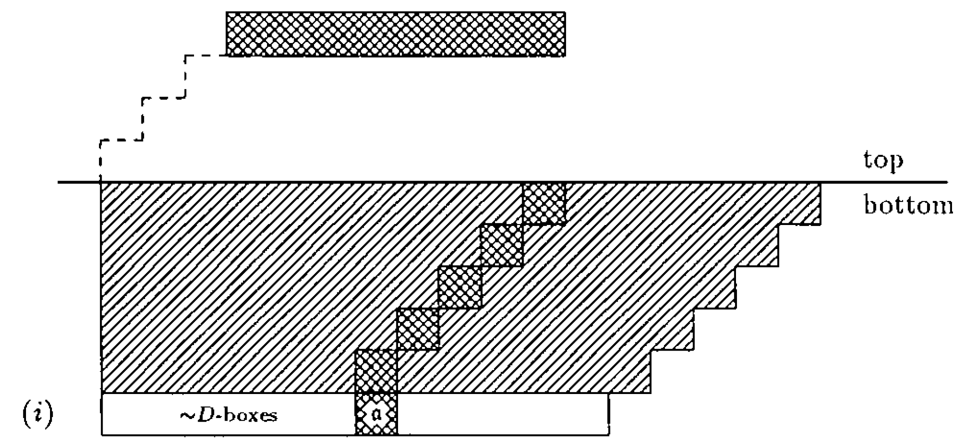
(shaded boxes belong to D)

If $\mathfrak{a}$ is a $\sim D$-box, then Lemma 5.5 shows that the ∂-operator of $\mathfrak{f}$ involved in ∂^D_μ causes the vanishing. If $\mathfrak{a}$ is a D-box (i.e. a pure v-box) then we pick the leftmost pure v-box $\mathfrak{b}$ in the row of $\mathfrak{a}$. The bottom part of the v-ribbon of $\mathfrak{b}$ above the i-th row consists of non pure v-boxes and is a "staircase" which ends with a box $\mathfrak{c}$, say (see Proposition 4.7). Note that $\mathfrak{d}$ and $\mathfrak{e}$ belong to D_μ because of the condition $\mu^t_{m-n} \geq l(\mu^b) + 1$. If $\mathfrak{d}$ is a $\sim D$-box then we get immediately the vanishing by Lemmas 5.6 and 5.5. If $\mathfrak{d} \in D$ then $\mathfrak{d}$ is a pure v-box (with the same mark as the v-mark of $\mathfrak{c}$). By the property of v-ribbons from Lemma 5.4, $\mathfrak{d}$ must be the rightmost box of the initial (horizontal) segment of the v-ribbon of $\mathfrak{c}$. Then, obviously $\mathfrak{e}$ is a $\sim D$ box and applying Lemmas 5.6 and 5.5 we get the vanishing.

The proposition has been proved. □

Proposition 6.5 *For $D \subset \overset{\circ}{D}_\mu$ satisfying (6), if $(i,m) \notin D^b$ then the i-th bottom row consists entirely of $\sim D$ boxes. Moreover, each bottom row with number $> i$ consists entirely of D-boxes. In particular, $\mu_{h+1} = \lambda_h$ for $h \geq i$.*

Proof. By Lemma 6.3 the only D-box which can possibly appear in the i-th bottom row is a pure v-box. Suppose that such a box appears and assume that $\mathfrak{a}$ is the leftmost pure v-box in the i-th row. By the construction of maximal deformation that is necessary by Proposition 6.1, arguing as in the proof of Proposition 6.4, we infer that the part of the v-ribbon of $\mathfrak{a}$ above the i-th bottom row looks like (▨ denotes here z-boxes, ⊠—v-boxes);



(Note that the top segment of a v-ribbon is determined by Lemma 5.4). Now, Lemma 5.5 applies and gives the vanishing. The resulting contradiction proves the first assertion. The second assertion then follows from Lemma 5.2(iv). □

Note that from the above results of this section we infer:

Corollary 6.6 *If (8) is satisfied then, under the maximal deformation, every z-box can be moved down by one row at most.*

Proposition 6.7 *The conditions (8) are satisfied only if $D^b_\mu \setminus D^b_\lambda$ is an almost horizontal strip and its component meeting the leftmost column is an (ordinary) horizontal strip.*

Proof. The latter assertion is a consequence of Proposition 6.5. To prove the former we pick up a component of $D^b_\mu \setminus D^b_\lambda$ not meeting the leftmost column. Suppose first that there are 3 boxes of the component in one column. Pick up the leftmost column with this property and the highest triple of boxes $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ in it. Pictorially

$$\begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \mathfrak{d} & \mathfrak{a} \\ \cline{2-4} \multicolumn{1}{c|}{} & \mathfrak{e} & \mathfrak{i} & \mathfrak{b} \\ \hline \mathfrak{g} & \mathfrak{h} & \mathfrak{f} & \mathfrak{c} \\ \hline \end{array}$$

The box $\mathfrak{d}$ belongs to D^b_λ. Then $\mathfrak{e} \in D^b_\lambda$; otherwise $\mathfrak{d}$ can be deformed to the place of $\mathfrak{f}$ in contradiction with Corollary 6.6 (note that $\mathfrak{i}, \mathfrak{f}, \mathfrak{h} \notin D^b_\lambda$ because λ is strict). If $\mathfrak{g} \in D^b_\lambda$ then neither $\mathfrak{e}$ nor $\mathfrak{d}$ can be moved down. So we have z-boxes: $\mathfrak{d}$, $\mathfrak{e}$, $\mathfrak{g}$ and non z-boxes: $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{i}$, $\mathfrak{f}$, $\mathfrak{h}$. The boxes $\mathfrak{a}$, $\mathfrak{i}$, $\mathfrak{h}$ are $\sim D$-boxes by the separateness property. Then the configuration: a z-box $\mathfrak{d}$, a $\sim D$-box $\mathfrak{a}$, a $\sim D$-box $\mathfrak{i}$ implies that $\mathfrak{b}$ is a $\sim D$-box. Also, the configuration: a z-box $\mathfrak{e}$, a $\sim D$-box $\mathfrak{i}$, a $\sim D$-box $\mathfrak{h}$ implies that $\mathfrak{f}$ is a $\sim D$-box. The configuration of $\sim D$-boxes $\{\mathfrak{b}, \mathfrak{f}, \mathfrak{h}\}$ contradicts Lemma 5.2(i). If $\mathfrak{g} \notin D^b_\lambda$ then $\mathfrak{e}$ (together with the suitable part of its row in D^b_λ) is moved down exactly by one row (Corollary 6.6). Then $\mathfrak{d}$ (together with the suitable part of its row in D^b_λ) is moved down exactly by one row. So after the deformation, the former place of $\mathfrak{i}$ now is occupied by $\mathfrak{d}$ and the former place of $\mathfrak{h}$ now is occupied by $\mathfrak{e}$. Nothing arrives to the places of $\mathfrak{b}$ and $\mathfrak{f}$. By the separateness property $\mathfrak{b}$ and $\mathfrak{f}$ are $\sim D$-boxes. Since in the $\mathfrak{f}$-left part of the row of $\mathfrak{f}$ there exists a $\sim D$-box $\mathfrak{j}$ (the left-hand side neighbour of the breaking box pushed down), the configuration $\{\mathfrak{j}, \mathfrak{b}, \mathfrak{f}\}$ contradicts Lemma 5.2 (i).

Now divide the boxes of the component of $D^b_\mu \setminus D^b_\lambda$ into the horizontal strip of the highest boxes in columns and the set of remaining boxes. Decompose this latter into different rows. Assume that there exists a pair of such rows which are not disconnected. Let $\mathfrak{a}$ be the rightmost box of the lower row and $\mathfrak{b}$ the leftmost box of the higher one: ($\mathfrak{c}, \mathfrak{d} \notin D^b_\lambda$ as they are $\sim D$-boxes over $\mathfrak{a}$ and $\mathfrak{b}$, $\mathfrak{g} \notin D^b_\lambda$ because λ is strict)

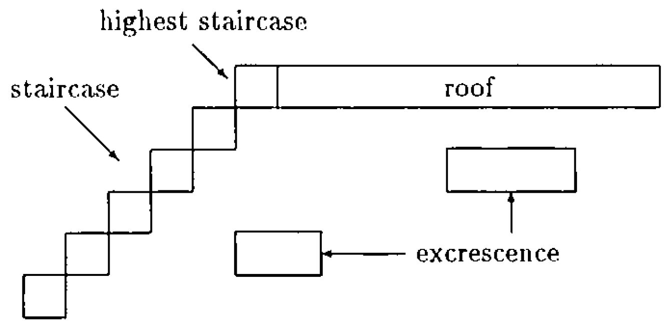
$$\begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \mathfrak{e} & \mathfrak{d} \\ \cline{2-4} \multicolumn{1}{c|}{} & \mathfrak{f} & \mathfrak{c} & \mathfrak{b} \\ \hline \mathfrak{h} & \mathfrak{g} & \mathfrak{a} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

The box $\mathfrak{e}$ belongs to D_λ by the previous step of the proof. Then $\mathfrak{f} \in D_\lambda$; otherwise $\mathfrak{e}$ can be deformed to the place of $\mathfrak{a}$ in contradiction with Corollary 6.6. Hence $\mathfrak{g} \notin D^b_\lambda$ and we have two possibilities for $\mathfrak{h}$. Suppose first that $\mathfrak{h} \in D_\lambda$. Then neither $\mathfrak{f}$ nor $\mathfrak{e}$ can be moved down. So we have z-boxes: $\mathfrak{e}$, $\mathfrak{f}$, $\mathfrak{h}$ and non z-boxes: $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{g}$. The boxes $\mathfrak{d}$, $\mathfrak{c}$, $\mathfrak{g}$ are $\sim D$-boxes by the separateness property. The configuration: a z-box $\mathfrak{e}$, a $\sim D$-box $\mathfrak{d}$, a $\sim D$-box $\mathfrak{c}$ implies that $\mathfrak{b}$ is a $\sim D$-box. Similarly, the configuration: a z-box $\mathfrak{f}$, a $\sim D$-box $\mathfrak{c}$, a $\sim D$-box $\mathfrak{g}$ implies that $\mathfrak{a}$ is a $\sim D$-box. Then the configuration of $\sim D$-boxes $\{\mathfrak{g}, \mathfrak{a}, \mathfrak{b}\}$ contradicts Lemma 5.2(i)

Now suppose $\mathfrak{h} \notin D^b_\lambda$. Then $\mathfrak{f}$ (together with a suitable part of its row in D^b_λ) is moved down exactly by one row. Also, $\mathfrak{e}$ (together with a suitable part of its row in D^b_λ) is moved down exactly by one row. So after the deformation the former place of $\mathfrak{g}$ now is occupied by $\mathfrak{f}$ and the former place of $\mathfrak{c}$ now is occupied by $\mathfrak{e}$. Nothing arrives to the places of $\mathfrak{a}$ and $\mathfrak{b}$. By the separateness property $\mathfrak{a}$ and $\mathfrak{b}$ are $\sim D$-boxes. Since in the $\mathfrak{a}$-left part of the row of $\mathfrak{a}$ there exists a $\sim D$-box $\mathfrak{i}$ (the left-hand side neighbour of the breaking box pushed down), the configuration $\{\mathfrak{i}, \mathfrak{a}, \mathfrak{b}\}$ contradicts Lemma 5.2(i)

The proposition has been proved. □

We end this Section with some definitions needed in the sequel. Suppose that $D^b_\mu \setminus D^b_\lambda$ is like in Proposition 6.7. Decompose $D^b_\mu \setminus D^b_\lambda$ into its connected components. Then use the maximal deformation procedure of Section 4. While the extremal component deforms to a single row meeting the leftmost column, a "typical" nonextremal component C_i looks like after the maximal deformation:



(7)

Here the roof and the staircase are the result of the deformation of $C_i^{(1)}$ (we use the notation before Definition 2.1). More precisely, the *roof* of a deformed (nonextremal) component is the segment of a row consisting of the highest boxes of the deformed component, without the leftmost box; the *staircase* of such a component is its deformation without the roof. The *excrescence* is the result of the deformation of $C_i^{(2)}$ which, in fact, remains unchanged under the deformation. The latter remark justifies a use here of the same name as the one in Definition 2.1. We will use freely the names from the picture. The existence of excrescences is the main difference between the case $n = m$ (where they do not appear) and the case $n < m$.

Let us record some properties of excrescences which are rather immediate consequences of the maximal deformation procedure.

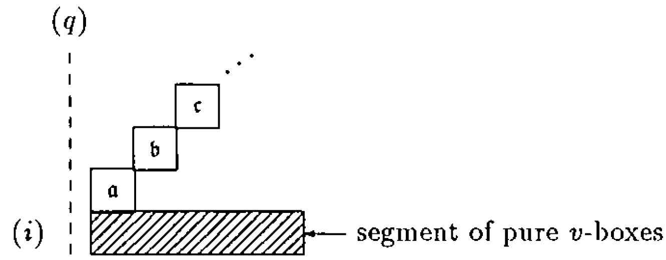
Lemma 6.8 *An excrescence can appear only under the roof of a deformed component and there is no pair of boxes of excrescences lying one over the other. (We follow the terminology before Definition 2.1). Moreover, a segment of a row between the staircase box and an excrescence box must contain a z-box.*

This means that an excrescence cannot be "too big". We will see in Section 9 that excrescences can only appear in components satisfying some rather special properties.

7 Admissible positions of pure v-boxes

Lemma 7.1 *Let $D \subset \mathring{D}_\mu$. Assume that for some i a segment of pure v-boxes appears in the i-th bottom row of D_μ. Let Δ be the "part" of ∂_μ^D formed by the composition of s- and ∂-operators of all boxes in D_μ above the i-th row. If $\Delta(E) = (scalar) \cdot E_{\mathbf{a}}$, then $a_q = 0$ where $q-1$ is the column number of the leftmost box of the segment.*

Proof. If $i = 1$, the assertion follows from Lemma 5.6. Now assume that the highest segment of pure v-boxes appears in some later row. Consider a "staircase" of boxes starting just over the leftmost box of the segment:



If $\mathfrak{a}$ is a $\sim D$-box then the ∂-operator of $\mathfrak{a}$ gives $a_q = 0$. If $\mathfrak{a} \in D$ then we have three possibilities for $\mathfrak{b}$:

- $\mathfrak{b}$ is a $\sim D$-box; then the ∂-operator of $\mathfrak{b}$ together with the s-operator of $\mathfrak{a}$ give $a_q = 0$.
- $\mathfrak{b} \notin D_\mu$; this means that $\mathfrak{a}$ lies in the first bottom row and the top part of the v-ribbon containing the segment ends exactly over the pure v-box $\mathfrak{a}$ (which is the first box in the bottom part of the ribbon). Applying Lemma 5.6 we get $a_{q-1} = 0$; then the s-operator of $\mathfrak{a}$ gives us $a_q = 0$.
- $\mathfrak{b} \in D$; then we have analogous three possibilities for $\mathfrak{c}$. Continuing this way we prove the assertion for the highest segment of pure v-boxes.

For lower segments of pure v-boxes the reasoning is similar. Let Δ' be the "part" of ∂_μ^D formed by the composition of s- and ∂-operators of all boxes of D_μ above the row containing the preceding segment of pure v-boxes. If $\Delta'(E) = (scalar) \cdot E_{\mathbf{a}}$ then (by induction) we can assume that $a_{q'} = 0$ where $q' - 1$ is the column number of the leftmost box of this segment.

We can now apply—word by word—the previous reasoning to our segment of pure v-boxes in question; the role of the top part of the v-ribbon is now played by the preceding segment of pure v-boxes. $\square$

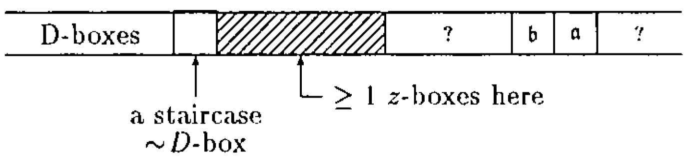
Corollary 7.2 *Let $D \subset \mathring{D}_\mu$. Assume that for some i a segment of pure v-boxes appears in the i-th bottom row of D_μ. If the column number of the leftmost box in the segment is $m-1$, or, it is $< m-1$ and two $\sim$D-boxes appear left to the segment (in the i-th row), then $\partial_\mu^D(E) = 0$.*

Proof. Combine Lemma 7.1 and Lemma 5.5. □

Let $\lambda, \mu \in \mathcal{P}_n$, $n = 1, \ldots, m$, and $D \subset \mathring{D}_\mu$. From now on we assume $r_D \in R(w_\lambda)$ and $\partial_\mu^D(E) \neq 0$. In what follows by d.c. we will mean a deformed component of $D_\mu^b \setminus D_\lambda^b$ in D_μ^b through the maximal deformation process (see the end of Section 6).

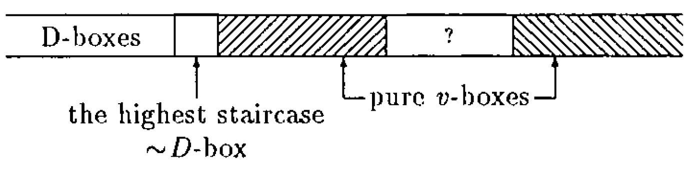
Proposition 7.3 *Pure v-boxes can appear only in the roof of a d.c., and form a segment starting from the leftmost box of the roof. In particular, no two pure v-boxes with different marks can appear in the same roof.*

Proof. By the separateness property, it is clear that no pure v-box can appear in the staircase of a d.c.. Suppose that a pure v-box appears in some excrescence of a d.c.. Pick such a leftmost box $\mathfrak{a}$. Then the row of the excrescence looks like (see Lemma 6.8)



The existence of a $\sim D$-box $\mathfrak{b}$ follows from the separateness property for segments of z- and pure v-boxes. Applying Corollary 7.2 we get the vanishing—contradiction.

It is clear from Corollary 7.2 again, that the leftmost box of the leftmost segment of pure v-boxes in the roof must be the same as the leftmost box of the roof. To show the last assertion, suppose conversely that pure v-boxes with different mark also appear in the roof. Pictorially

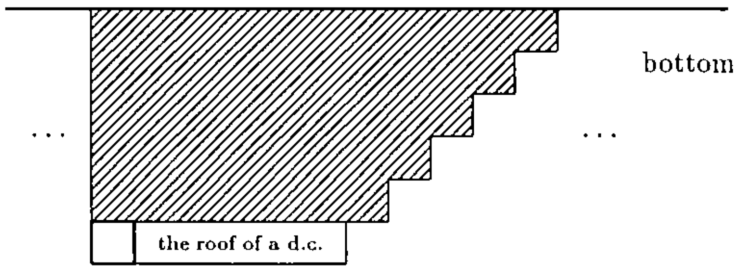


Since the marks of ▨- and ▧-boxes are different, by the separateness property there exists a $\sim D$-box in $\boxed{\ \ ?\ \ }$. Again, Corollary 7.2 now applied to the ▧-ribbon implies the vanishing—contradiction.

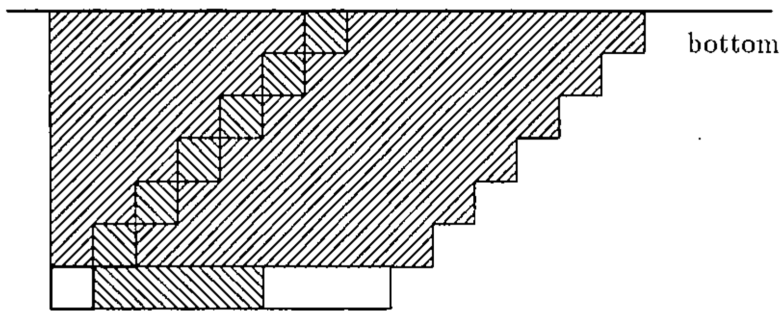
The proposition has been proved. □

Proposition 7.4 *No two different roofs can contain pure v-boxes of the same v-ribbon.*

Proof. It follows from the construction of maximal deformation which is necessary by Proposition 6.1, that the bottom part of D_μ over the roof of a d.c. looks like (compare the proof of Proposition 6.4 for details):



where shaded boxes are D-boxes. Therefore, the bottom part of the v-ribbon containing a segment of pure v-boxes from the roof of a d.c. above this roof looks like (see Proposition 4.7):



It is now clear that this v-ribbon cannot meet the roof of another d.c.. □

We end this Section with the following easy fact.

Lemma 7.5 *The lengths of rows in D^t form a decreasing sequence (i.e. D^t is a strict diagram).*

Proof. Note first that such a formulation makes sense by Lemma 5.4. In w_λ we have $v_1 < \ldots < v_{m-n}$. Since in the process of moving the v_r's, which is coded by boxes of D^t, there is no change $\ldots v.v_* \ldots \to \ldots v_* v. \ldots$, the order of the v_r's before performing "s_i-operations" of the boxes of D^t is the same. But this is a restatement of the assertion. □

Thus the proof of the proposition gives as a by-product the following:

Corollary 7.6 *The marks of segments of pure v-boxes in the roofs of consecutive deformed components, increase from top to bottom.*

8 When are $r_D \in R(w_\lambda)$ and $\partial_\mu^D(E) \neq 0$?

In this Section we will prove the following theorem.

Theorem 8.1 *Assume that $\lambda \in \mathcal{P}_n$, $n = 1, \ldots, m$. Then for every $\mu \in \mathcal{P}_n$ there exists at most one $D \subset \mathring{D}_\mu$ such that*

$$r_D \in R(w_\lambda) \qquad \textit{and} \qquad \partial_\mu^D(E) \neq 0 \tag{8}$$

Let $D \subset \mathring{D}_\mu$. Assume, from now on, that D satisfies (8).

By Section 6 we know that the connected components of $D_\mu^b \setminus D^b$ are determined uniquely and the columns of their appearance form pairwise disjoint sets a.s. [3]

For a given $D \subset \mathring{D}_\mu$, $\overline{D}$ will denote the set D with the added set (5).

We will say for a given component that a top row of $\overline{D}$ is *associated* with the component if the row ends in the leftmost column of the shifted component.

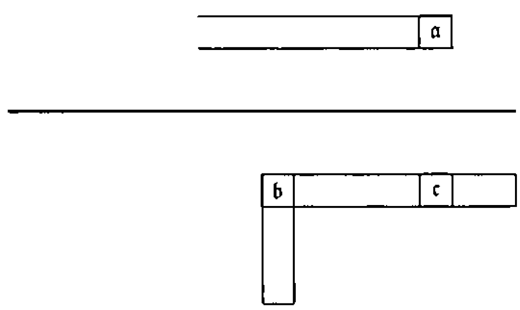
Assume for the moment that this is the r-th top row. Then by properties of v-ribbons (Section 7) pure v-boxes with mark r can appear in the bottom part only in the roof of precisely one (associated) component, where they form a segment starting in the leftmost box of the roof. Moreover the following equality holds

$$\lambda_r^t = (\text{length of the } r\text{-th row in } \overline{D}^t) + (\text{length of the segment}) \tag{9}$$

We record the following property of D satisfying (8). (From now on we follow the terminology before Definition 2.1)

Lemma 8.2 *No top row of D satisfying (8) can end over the roof of a deformed component.*

Proof. Suppose, conversely, that there exists a top row which ends over the roof of a deformed component. Pictorially



(a.s.)

[3]The abbreviation "a.s." means here and in the sequel "after shifting"(of the bottom part of the diagram)

Then the configuration of $\sim D$-boxes $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ causes the vanishing because of Lemmas 5.5 and 5.6—contradiction. $\square$

Lemma 8.3 *If for some $D \subset \overset{\circ}{D}_\mu$ satisfying* (8) *a certain component is associated with the r-th top row, say, then the same is true for any $D' \subset \overset{\circ}{D}_\mu$ satisfying* (8).

Proof. Consider the r-th top row in $\overline{D}'$. If it is not associated with the component, then using the previous lemma we conclude that it ends in the column which is, either

1) left to the leftmost column of the shifted component, or

2) right to the rightmost column of the shifted component.

Case 1) cannot occur because otherwise λ_r^t would be bigger than it actually is by (9).

In case 2), by properties of v-ribbons (Section 7) an eventual bottom segment of pure v-boxes with a mark r should occupy—a.s.—columns left to the leftmost column of the component. In virtue of (9) and remarks preceding (9), this would imply that λ_r^t is smaller than it actually is—contradiction. The lemma is proved. $\square$

Proof of Theorem 8.1

It is sufficient to show that $\overline{D}^t$ is determined uniquely by conditions (8). By the lemma we know the rows in $\overline{D}^t$ that are associated, are determined uniquely. In virtue of remarks preceding the lemma and by (9), we know that the lengths of the remaining i.e. non-associated rows are the same as in D_λ^t. Since, by Lemma 7.5, $\overline{D}^t$ is a strict diagram, we conclude that the decreasing sequence of the row lengths in $\overline{D}^t$ is uniquely determined. But this means that $\overline{D}^t$ is uniquely determined.

The theorem has been proved. $\square$

As a corollary we get an explicit recipe to construct D satisfying (8) for given λ, μ (for which such a D exists).

Recipe 8.4 *Assume that λ and μ are two shapes satisfying the conditions:*

1) $D_\mu^b \supset D_\lambda^b$ and $D_\mu^b \setminus D_\lambda^b$ is an almost horizontal strip whose extremal component is a horizontal strip.

2) At most one row from D_λ^t ends over a component but none over the extremal component.

The recipe says:

(1) Perform the maximal deformation of D^b_λ in D^b_μ to get $(D^b)^{\lambda,\mu} \subset D^b_\mu$, say, as described in Section 4. The connected components of $D^b_\mu \setminus D^b_\lambda$ deform to a family of subsets of $D^b_\mu \setminus (D^b)^{\lambda,\mu}$ (called deformed components) of the form (7). Different deformed components occur in pairwise disjoint sets of columns a.s..

(2) Shift the bottom part from now on. For every component pick a row which ends over the component. Subtract the segment of the row which lies over the roof of the deformed component and push it down to the roof.

Recall that the necessity of the condition 1) above to obtain D fulfilling (8) follows from Proposition 6.7. Similarly, the necessity of the condition 2) above follows from Lemma 8.2 combined with Proposition 7.3; the assertion about the extremal component being a consequence of Lemma 8.2 combined with Proposition 6.5. Moreover, note that pushing down of segments of top rows can be performed in an arbitrary order, as this takes place in disjoint sets of columns.

Lemma 8.5 *The conditions (8) for D from the recipe hold only if*

(i) $D^t_\mu \setminus D^t_\lambda$ is a horizontal strip with pairwise disconnected rows.

(ii) No $(\mu - \lambda)$-box lies over the staircase of a related component.

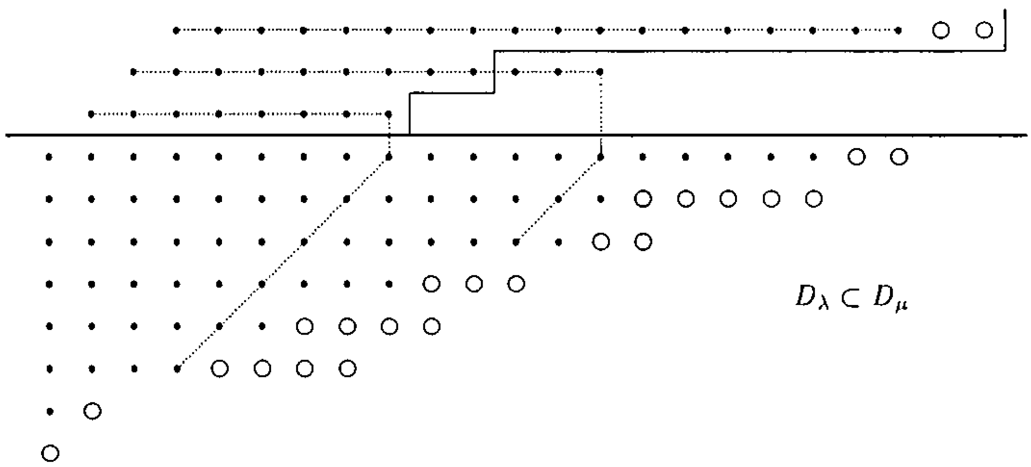
Proof.
(i) Regardless of the deforming or not deforming of some boxes in D^t_λ, the assertion follows from Lemma 5.3.
(ii) An eventual box $\mathfrak{a}$ lying over the staircase of a related component, can appear only in the row situated just below the related one. Then the box $\mathfrak{b}$ in the related row and in the column next right to the column of $\mathfrak{a}$ is a $\sim D$-box (because D^t_μ is a strict diagram). Since the configuration $\mathfrak{a}$,$\mathfrak{b}$ contradicts Lemma 5.3, the box $\mathfrak{a}$ cannot exist. $\square$

Observe that the conditions (i) and (ii) from the lemma imply that $D^t_\mu \setminus D^t$ is a horizontal strip with pairwise disconnected rows.

Example 8.6

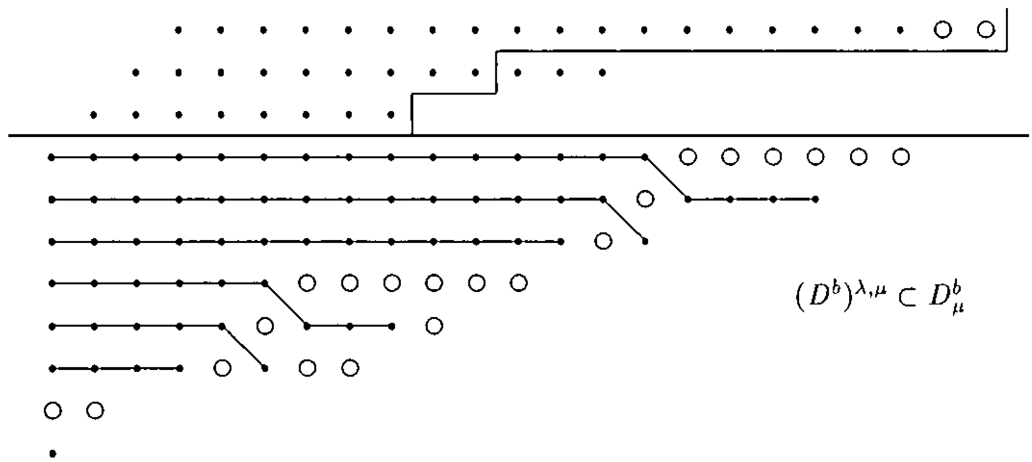
$$\begin{aligned} \mu &= (23, 11, 9//21, 19, 15, 12, 10, 8, 2, 1), \\ \lambda &= (21, 14, 9//19, 14, 13, 9, 6, 4, 1). \end{aligned}$$

In the picture below "○" visualizes $\sim D$-boxes, z-ribbons are depicted with "___" and v-ribbons—with "……".



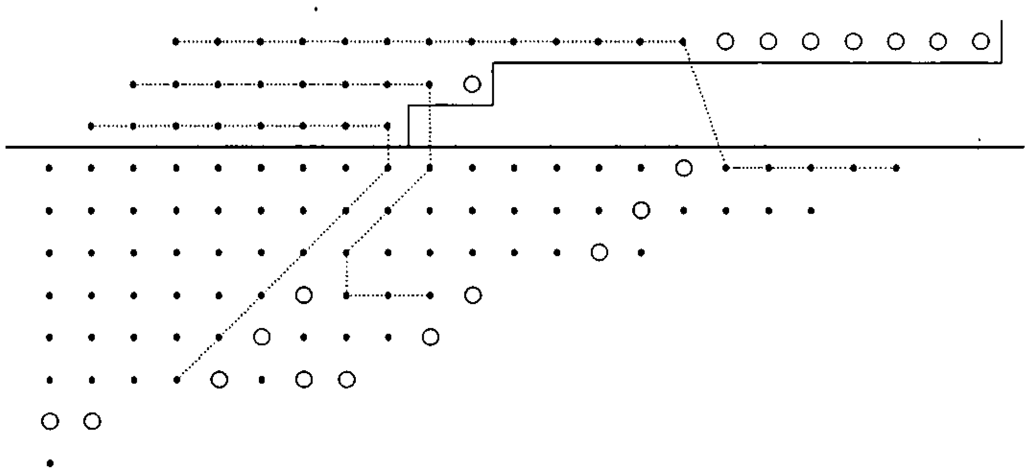
$D_\lambda \subset D_\mu$

1) After the first step of the recipe,we obtain:

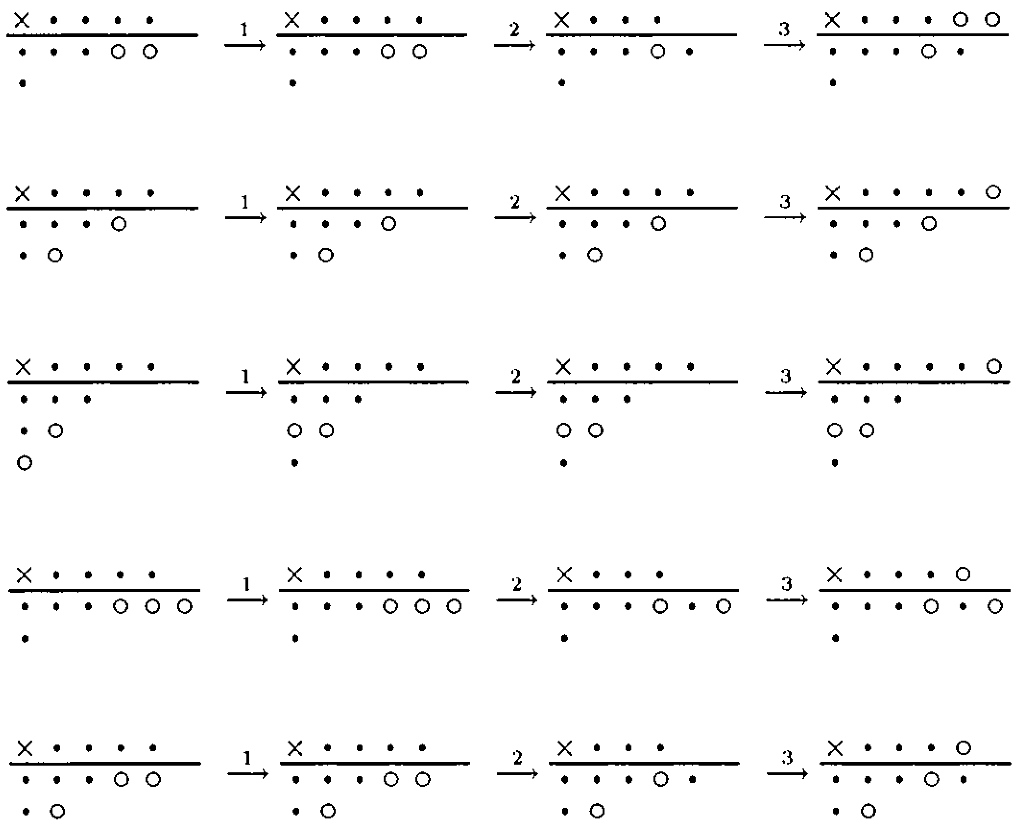


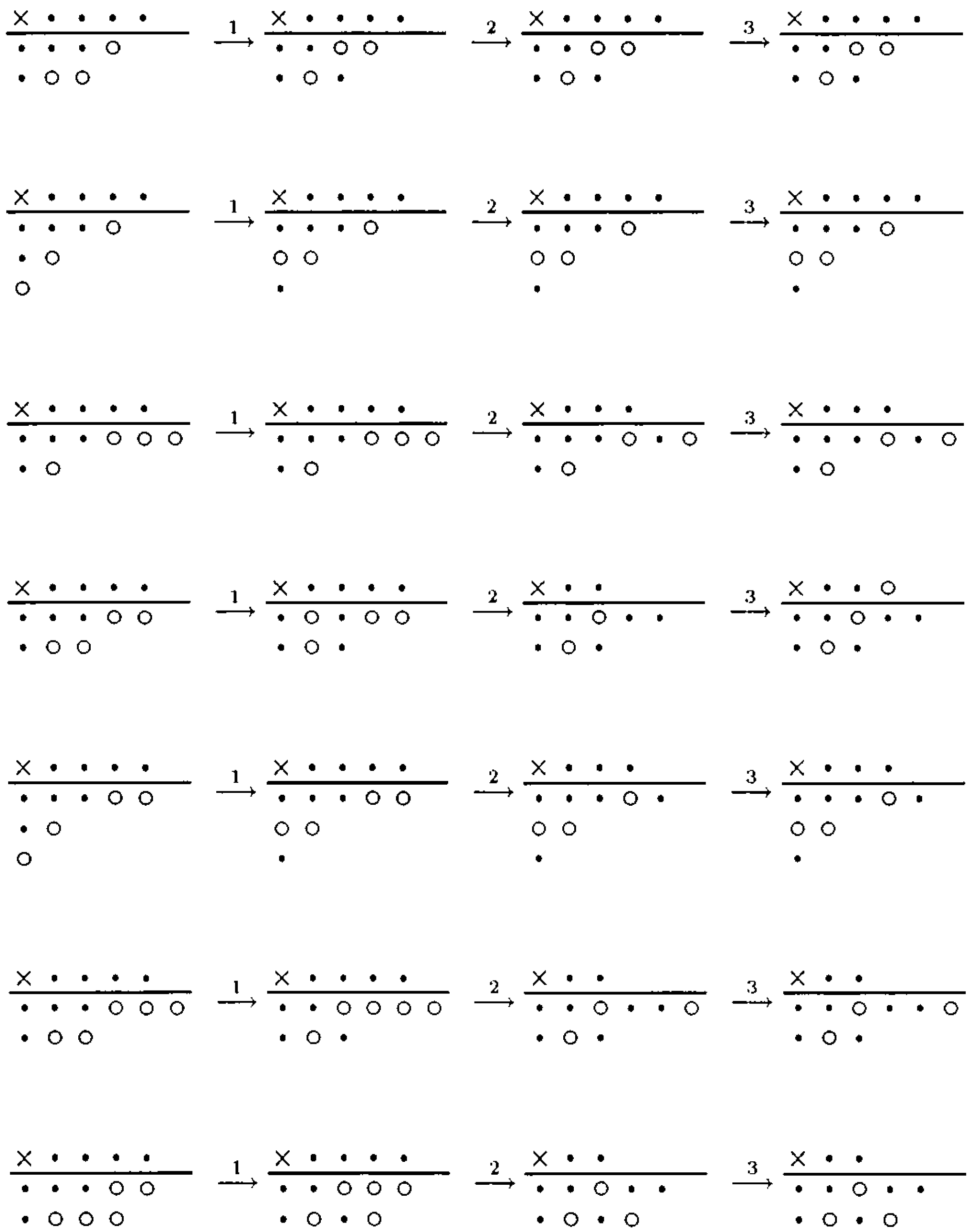
$(D^b)^{\lambda,\mu} \subset D^b_\mu$

2) Finally we get the following set depicted with "•":



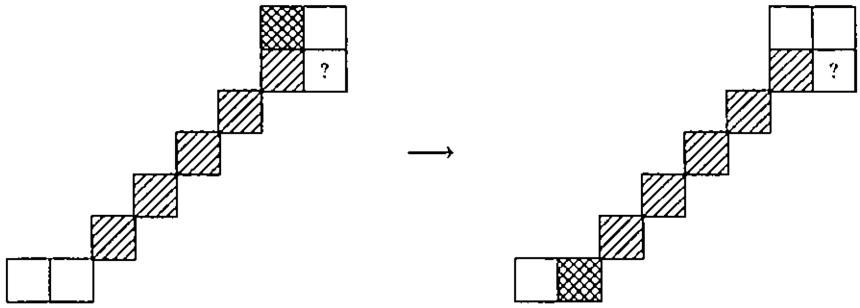
Example 8.7 We show here, how the summands in Example 2.3 are obtained using the algorithm described in this Section.



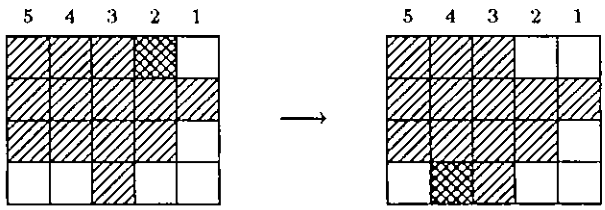


We start with D_λ with added an almost horizontal strip to D_λ^b; then we perform a maximal deformation of the bottom part ($\xrightarrow{1}$); then we perform the deformation from Recipe 2) ($\xrightarrow{2}$); finally we add additional μ-boxes to the top part, if possible ($\xrightarrow{3}$).)

Remark 8.8 A pushing down of a segment of a top row to the roof of a deformed component can be presented as a composition of the following operations:



(unshaded boxes are not in D). This operation can be justified with the help of Coxeter relations. The following example allows one to understand the general case. The change



corresponds to the sequence of equalities:

$$
\begin{aligned}
& s_5 s_4 s_3 \underline{s_2} s_5 s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_3 && \text{(commutativity relations)} \\
&= s_5 s_4 s_3 s_5 s_4 (\underline{s_2} s_3 s_2) s_1 s_5 s_4 s_3 s_2 s_3 && \text{(longer relations)} \\
&= s_5 s_4 s_3 s_5 s_4 (s_3 s_2 \underline{s_3}) s_1 s_5 s_4 s_3 s_2 s_3 && \text{(commutativity relations)} \\
&= s_5 s_4 s_3 s_5 s_4 s_3 s_2 s_1 s_5 (\underline{s_3} s_4 s_3) s_2 s_3 && \text{(longer relations)} \\
&= s_5 s_4 s_3 s_5 s_4 s_3 s_2 s_1 s_5 (s_4 s_3 \underline{s_4}) s_2 s_3 && \text{(commutativity relations)} \\
&= s_5 s_4 s_3 s_5 s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2 \underline{s_4} s_3
\end{aligned}
$$

9 Computation of multiplicities

In this Section we follow the terminology before or in Definition 2.1. Let us fix two shapes λ and μ satisfying the following properties:

1) $D^b_\mu \supset D^b_\lambda$ and $D^b_\mu \setminus D^b_\lambda$ is an almost horizontal strip such that the extremal component is a horizontal strip.

2) $D^t_\mu \setminus D^t_\lambda$ is a horizontal strip with pairwise disconnected rows.

3) The λ-part of at most one row from D^t_λ ends over a component but none over the extremal one.

4) No $(\mu - \lambda)$-box lies over the staircase of a related component.

For such λ and μ we define a subset $D^{\lambda,\mu} \subset \overset{\circ}{D}_\mu$ to be the result of applying Recipe 8.4 to the D_λ and D_μ. Let us write $D = D^{\lambda,\mu}$ for brevity. We know by Sections 6-8 that the above conditions are necessary for the conditions: $\partial^D_\mu(E) \neq 0$ and $r_D \in R(w_\lambda)$ to be satisfied. Note the following consequences of Recipe 8.4(2)

i) $D \subset D_\mu$ satisfies the hypothesis of Lemma 5.6 (see the remark after Lemma 8.5).

ii) The end of no row in D^t lies either over an extremal component or over the roof of a nonextremal one (in particular the end of no row in D^t lies over an excrescence).

Lemma 9.1 *Let Δ be the part of ∂^D_μ formed by the composition of s- and ∂-operators of boxes in D^t. Then $\Delta(E) = E_{\mathbf{a}}$ with $a_m = a_{m-1} = \ldots = a_{m-(q-1)} = 1$ where $q = l(\mu^b)$ if no extremal component appears and, in the opposite case, q is the row number of the deformed extremal component.*

Proof. Assume first that $l(\mu^b) = l(\lambda^b) = k$. Then $\lambda^b_k \geq 1$, $\lambda^b_{k-1} \geq 2$, $\ldots$ $\lambda^b_1 \geq k$, $\lambda^t_{m-n} \geq k+1$, $\ldots$. We see that no box from the segment of the $k+1$ leftmost boxes in the $(m-n)$-th row of D^t_λ is moved down by Recipe 8.4(2). The assertions follows from Lemma 5.6.

We now assume that $l(\mu^b) = l(\lambda^b) + 1$. Assume that the deformed extremal component lies in the q-th row where $1 \leq q \leq l(\mu^b)$. Since the boxes from D^b_μ that lie above this component belong to D^b_λ (because of the construction of the maximal deformation from Section 4), we get $\lambda^t_{m-n} \geq q$ (by the definition of a shape). We claim that $\lambda^t_{m-n} \geq q+1$. Indeed, otherwise the λ-part of the

$(m-n)$-th row in the top part would end over the extremal component, which is impossible. The assertion now follows from Lemma 5.6.

The Lemma has been proved. $\square$

For a given box $\mathfrak{a} \in D_\mu \setminus D$ define $\Delta_\mathfrak{a}$ to be the " part" of ∂_μ^D formed by the s- and ∂-operators of boxes read before $\mathfrak{a}$. Define $\mathbf{a}$ by the equation $\Delta_\mathfrak{a}(E) = (scalar) \cdot E_\mathbf{a}$ with the help of Lemma 3.1. Note that this definition of $\mathbf{a}$ makes sense if the lefthand side is nonzero. So, whenever we will have $\mathfrak{a}$ associated with $\mathfrak{a}$ via the above equation, we will tacitly assume $\Delta_\mathfrak{a}(E) \neq 0$. Moreover, let h be the column number of $\mathfrak{a}$.

Corollary 9.2 **i)** *If $h < m$ then one has either $a_{h+1} = 0$ or $a_{h+1} = -1$. The both corresponding cases hold iff there is a $\sim D$-box in the $\mathfrak{a}$-left part of the row of $\mathfrak{a}$ (resp. there is not).*

ii) *If $h = m$ then $a_m = 1$.*

Let $h < m$. We say that a box $\mathfrak{a} \in D_\mu \setminus D$ is *bad* if $a_{h+1} = a_h = 0$. We say that a box $\mathfrak{a}$ is *essential* if $a_{h+1} = -1$, $a_h = 1$. (It follows from Lemma 5.6 that both bad and essential boxes lie in D_μ^b).

Proposition 9.3 *The multiplicity m_μ (see Section 3) is not zero if no bad boxes appear.*

Proof. For $\mathfrak{a} \in D_\mu \setminus D$ define an integer $m_\mathfrak{a}$ by the equality $\partial_h(E_\mathbf{a}) = m_\mathfrak{a} \cdot E_{\mathbf{a}'}$ from Lemma 3.1(b). Then $\partial_\mu^D(E) = \prod_\mathfrak{a} m_\mathfrak{a}$, the product taken over $\mathfrak{a} \in D_\mu \setminus D$ (by Lemma 5.6 it suffices to take this product over $\mathfrak{a} \in D_\mu^b \setminus D$). Therefore $\partial_\mu^D(E) \neq 0$ iff $m_\mathfrak{a} \neq 0$ for every $\mathfrak{a} \in D_\mu^b \setminus D$. Thus we have reduced to showing $m_\mathfrak{a} = 0$ iff $\mathfrak{a} \in D_\mu^b \setminus D$ is bad. It follows from the corollary that $m_\mathfrak{a} = 0$ only if $h < m$. Then by Lemma 3.1 one has $m_\mathfrak{a} = 0$ iff $a_{h+1} = a_h (= -1, 0\, or\, 1)$. The third possibility is ruled out by Corollary 9.2(i).We claim that $a_h \neq -1$. Indeed, $a_h = -1$ only if h is the column number of the end of some row in in D_μ^b above $\mathfrak{a}$; since D_μ^b is strict, this is not possible. The proposition has been proved. $\square$

Proposition 9.4 *If there are no bad boxes then $m_\mu = 2^e$ where e is the number of essential boxes.*

Proof. In the above notation we have $m_\mu = \prod_\mathfrak{a} m_\mathfrak{a}$, the product taken over $\mathfrak{a} \in D_\mu^b \setminus D$. It suffices to show that $|m_\mathfrak{a}| > 1$ iff $\mathfrak{a}$ is essential and if this happens then $m_\mathfrak{a} = 2$. It follows from Corollary 9.2(i) that $|m_\mathfrak{a}| > 1$ only if $h < m$. Then Lemma 3.1 and Corollary 9.2(i) imply $|m_\mathfrak{a}| > 1$ iff $a_{h+1} = -1, a_h = 1$ (i.e. $\mathfrak{a}$ is essential). Moreover in that case $m_\mathfrak{a} = 2$. $\square$

We will give a diagrammatic answer to the following question:

When a box can be bad?

By definition $\mathfrak{a}$ is bad iff $h < m$ and $a_{h+1} = a_h = 0$. We get by Corollary 9.2(i) that $a_{h+1} = 0$ iff there is a $\sim D$-box in the $\mathfrak{a}$-left part of the row of $\mathfrak{a}$. It follows from the form of a deformed component (see the end of Section 6) that a bad box can appear either as a non leftmost box of the deformed extremal component or in the roof or in the excrescence of the deformation of a nonextremal one. Let h' be the column number of $\mathfrak{a}$ after shift. Let $\Delta(E) = E_{\mathbf{a}'}$. By Lemma 5.6 we have $a'_{h'} = 0$ (resp. $a'_{h'} = 1$) iff a $\sim D$-box in D^t_μ lies over $\mathfrak{a}$ (resp. no $\sim D$-box in D^t_μ lies over $\mathfrak{a}$). If $\mathfrak{a}$ appears either in the deformed extremal component or in the roof of a nonextremal one then $a_h = a'_{h'}$ ($a'_{h'}$ is moved down to the h-th place in $\mathbf{a}$ by successive transpositions of adjacent places). For $\mathfrak{a}$ lying in the excrescence of a component with $a'_{h'} = 1$ one has $a_h = 0$ iff there is a $\sim D$-box which lies over $\mathfrak{a}$ and is situated in the roof of the component. This exhausts all the possibilities.

We can summarize this discussion in:

Lemma 9.5 *A box $\mathfrak{a}$ is bad iff there exist a $\sim D$-box in the $\mathfrak{a}$-left part of $\mathfrak{a}$ and there is a $\sim D$-box from D_μ which lies over $\mathfrak{a}$ (Recall that we are in the situation when the property (ii) before Lemma 9.1 holds.)*

The above discussion gives the following criterion for the absence of bad boxes.

Proposition 9.6 *The set of bad boxes is empty iff the following conditions are satisfied:*

1) *No $\sim D$-box lies over the deformed extremal component.*

2) *No $\sim D$-box (in D^t_μ) lies over a $\sim D$-box situated in the roof of a deformed component.*

3) *Over each box of the excrescence of a component there exists a D-box in the roof of the component. No $\sim D$-box in D^t_μ lies over the excrescence.*

Now we pass to a diagrammatic answer to the following question:

When a box $\mathfrak{a}$ can be essential?

Here we assume $h < m$. We know from Corollary 9.2(i) that $a_{h+1} = -1$ iff all $\mathfrak{a}$-left boxes in the roof of $\mathfrak{a}$ are D-boxes. Thus an essential box cannot appear in the deformed extremal component, and it can lie only in the staircase of the deformation of a nonextremal one. The condition $a_h = 1$ holds only if $\mathfrak{a}$ is situated in the highest staircase. Using Lemma 5.6 and arguing like in the above analysis of bad boxes we see that the highest staircase can support an essential box $\mathfrak{a}$ iff neither a $\sim D$-box (in D^t_μ) nor the end of of a row in D^t lies over $\mathfrak{a}$. We can summarize this in:

Lemma 9.7 *A box* $\mathfrak{a}$ *is essential iff all* $\mathfrak{a}$*-left boxes of a of the row of* $\mathfrak{a}$ *are* D*-boxes and neither a* $\sim D$*-box nor the end of a row in* D^t *lies over* $\mathfrak{a}$.

We also get:

Proposition 9.8 *The number of essential boxes is the cardinality of the set of nonextremal components whose staircase lies under neither a* $\sim D$*-box nor the end of a row in* D^t.

We will now translate the content of Proposition 9.6 to our initial shape data.

We will first show that for given two shapes λ and μ satisfying the condition 1)—4) from the begining of this Section plus an obvious condition: $D^{\lambda,\mu} \subset D_\mu$, the absence of bad boxes is equivalent to the compatibility of μ with λ (see Definition 2.1). Suppose first that 2.1(1) and 2.1(2) hold, no $\mu \setminus \lambda$-box lies over the staircase of a related component and bad boxes do not exist. Then

- Both assertions of 2.1(3) follow from $D^{\lambda,\mu} \subset D_\mu$ and Recipe 8.4(2)
- The assertion of 2.1(4) saying that no $(\mu - \lambda)$-box lies over an extremal component is a consequence of 9.6(1) combined with the fact that the extremal component is not related (2.1(2)). The assertion of 2.1(4) saying that no $(\mu - \lambda)$-box lies over the roof of a component follows from 9.6(2) combined with Recipe 8.4(2) regardless the component is related or not.
- 2.1(5) follows from 9.6(3) combined with Recipe 8.4(2).

Conversely, we now prove that if μ is compatible with λ then bad boxes do not exist. Indeed:

- 9.6(1) holds because the extremal component is not related and no $(\mu-\lambda)$-box lies over it.
- An eventual $\sim D$-box (in D^t_μ) lying over a $\sim D$-box situated in the roof of a deformed component must be a $(\mu - \lambda)$-box regardless the component is related or not. Hence the assertion of 9.6(2) follows immediately from 2.1(4).
- 9.6(3) follows from 2.1(5) in virtue of Recipe 8.4(2).

Now assume that μ is compatible with λ.
The next proposition enumerates essential boxes.

Proposition 9.9 *The number of essential boxes is equal to the number of componets that are not extremal, not related and no* $(\mu - \lambda)$*-boxes lie over them.*

Proof. Using Proposition 9.8 we must show that that for extremal component C the following conditions are equivalent:

1) Neither a $\sim D$-box nor the end of a row in D^t lies over the staircase of C.

2) C is not related and no $(\mu - \lambda)$-box appears over C.

1) $\Rightarrow$ 2) Since the end of no row in D^t lies over the highest staircase then by Recipe 8.4(2), C is not related. Then the absence of $\sim D$-boxes over the staircase implies that no $(\mu - \lambda)$-box lies over this staircase. The latter statement is equivalent to saying that no $(\mu - \lambda)$-box appears over C (by 2.1(4)).

2)$\Rightarrow$1) If C is not related then the λ-part of every row ends either left or right to C. This implies, by Recipe 8.4(2) that the staircase of C does not lie under the end of a row in D^t. Finally, an eventual $\sim D$-box lying over the staircase must be a $(\mu - \lambda)$-box and its appearance would contradict 2).

The proposition has been proved. □

We have translated informations about the existence and absence of a subset $D \subset D_\mu$ such that $\partial_\mu^D(E) \neq 0$ and $r_D \in R(w_\lambda)$ into initial shape-data modulo the following fact. Note that the set of columns of the shifted extremal component is, in general, different from the set of columns of the first deformed and then shifted extremal component. Hence to complete the proof of Theorem 2.2 we only need:

Lemma 9.10 *(i) The end of the λ-part of a top row lies over the extremal component iff it lies over the deformation of this component.*
(ii) A $(\mu - \lambda)$-box lies over the extremal component iff it lies over the deformation of this component.

Proof. Both assertions follow easily from the fact that the λ-part of every top row ends right to the $l(\lambda^b)$-th column, in the numbering from left to right (cf. Definition 1.1). □

This finishes the proof of Theorem 2.2.

Remark 9.11 The method of bad and essential boxes used here, gives a simplification of calculations in [P–R1, Section 6].

10 The main result for type B

Let now G' be the Grassmannian of isotropic n-subspaces of a $(2m+1)$-dimensional complex vector space equipped with an orthogonal nondegenerate form. Then the Schubert subvarietes of G' are labelled by the same poset $\mathcal{P}_n$ as above. The divided differences ∂_i, $i < m$ are the same as in the symplectic case but $\partial_m(f) = (f - s_m f)/x_m$. We have $c(e_p(x_1, \ldots, x_n)) = \sigma_p$, $p = 1, \ldots, n < m$ (this is in contrast to the "$m = n$ - case" where $c(e_p) = 2\sigma_p$). With these divided differences, Lemma 3.1 a), b) goes through without changes, but $\partial_m(E_{\mathbf{a}}) = 2a_m \cdot E_{(0,a_{m-1},\ldots,a_1)}$.

Consequently an analog of Theorem 2.2 for type B reads as follows:

Theorem 10.1 *For every $\lambda \in \mathcal{P}_n$ and $p = 1, \ldots, n$, one has in $A^*(G')$,*

$$\sigma(\lambda) \cdot \sigma_p = \sum 2^{e'(\lambda,\mu)} \sigma(\mu),$$

where the sum is over all μ compatible with λ, $|\mu| = |\lambda| + p$ and $e'(\lambda, \mu)$ is the cardinality of the set of components that are not related and have no $(\mu - \lambda)$-boxes over them. (Note that $A^*(G')$ as a ring admits the same description as $A^*(G)$ in Theorem 1.4; however, the σ_p's generate it algebraically only after tensoring by $\mathbb{Z}[1/2]$).

Remark 10.2 In a recent paper [S], the author gives a "triple intersection theorem" for G'. The result gives a necessary condition for a nontrivial intersection of two arbitrary Schubert cycles and a "special" Schubert cycle.(Note that unlike to the present paper, where the special Schubert cycles are (up to a scalar) the Chern classes of the tautological subbundle on G'', the "special" cycles in [S] are (up to a scalar) the Chern classes of the tautological quotient bundle on G'.) However, the main theorem of [S] gives only a partial insight in the intersection theory on G'. Firstly, the condition given is not sufficient for the nontrivial intersection which obstructs a deduction of a Pieri-type formula from it (Recall that one possible approach to the classical Schubert calculus, used for instance by Hodge and Pedoe, derives the Pieri formula from an appropriate triple intersection theorem). Moreover, the approach used gives no information about the multiplicities occuring in the intersection of Schubert cycles.

Note added in proof. After the first version of this paper was written, S.Kumar has informed us that a result similar to our Proposition 3.6 was given quite independently in [K–K, Proposition 4.31]. Note, however, that [K–K] gives no sufficient condition for the intersection multiplicities, denoted by $p^u_{v,w}$ in loc.cit., to be nonzero, in terms of v, w and u (of course we speak here about the case $l(u) = l(v) + l(w)$). Moreover, [K–K] gives no expression for $p^u_{v,w}$ as a cardinality of an explicitly given set (again when $l(u) = l(v) + l(w)$). As a matter of fact, the algebro-combinatorial methods invented and developed in the present paper are a result of our attempt to solve these two problems mentioned above which were not treated in [K–K].

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