COMBINATORIAL HOMOTOPY

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We describe the "CW-tower of categories" which approximates the homotopy category $\underline{\underline{CW}}/\simeq$ of connected CW-complexes with basepoints. The CW-tower is an important new tool for the homotopy classification problems:

(1) Classify the homotopy types of finite dimensional CW-complexes by algebraic data!

(2) Compute the homotopy classes of maps between finite dimensional CW-complexes!

(3) Compute the group of homotopy equivalences of a finite dimensional CW-complex!

A fundamental example for the solution of problem (1) is J.H.C. Whitehead's classification of 1-connected 4-dimensional polyhedra [2]. Below we show that this result is a nice consequence of the properties of the CW-tower. More generally we obtained solutions of problem (1) (in particular, for 4-dimensional CW-complexes and for 1-connected 5-dimensional CW-complexes) which will appear elsewhere. These results as well are derived from the CW-tower. The starting point of this paper is J.H.C. Whitehead's "Combinatorial Homotopy". In fact, many of Whitehead's results in [2],[3],[4] are consequences of the CW-tower. In my book "Algebraic Homotopy" [1] various further properties of the CW-tower are discussed, including all proofs.

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§ 1 The cellular chain complex of the universal covering

Let $\underline{\underline{CW}} = \underline{\underline{CW}}_0^*$ be the following category. Objects are CW-complexes X with trivial 0-skeleton $X^0 = *$, where $*$ is the basepoint of X, and morphisms are cellular maps $f : X \longrightarrow Y$. The skeleta of X are denoted by X^n and the map f is cellular if $f(X^n) \subset Y^n$. In particular the map f is basepoint preserving. Let $I = [0,1]$ be the unit interval which is a CW-complex with 0-skeleton $I^0 = \{0,1\}$. Whence also the reduced cyclinder

(1.1) $\quad I_*X = (I \times X)/(I \times *)$

is a CW-complex in $\underline{\underline{CW}}$. Let $i_t : X \longrightarrow I_*X$ be given by $i_t(x) = (t,x)$ for $t \in I$, $x \in X$. We call a map $H : I_*X \longrightarrow Y$ a <u>homotopy</u> $H : H_0 \simeq H_1$ with $H_t = Hi_t$. We say that $H : H_0 \overset{0}{\simeq} H_1$ is a <u>0-homotopy</u> if H_t is cellular for all $t \in I$ and we call $H : H_0 \overset{1}{\simeq} H_1$ a <u>1-homotopy</u> if H is a cellular map. This yields the quotient functors

(1.2) $\quad \underline{\underline{CW}} \longrightarrow \underline{\underline{CW}}/\overset{0}{\simeq} \longrightarrow \underline{\underline{CW}}/\overset{1}{\simeq} = \underline{\underline{CW}}/\simeq$

for the corresponding homotopy categories. We now describe the <u>chain functor</u>

(1.3) $\quad \hat{C}_* : \underline{\underline{CW}}/\overset{0}{\simeq} \longrightarrow \underline{\underline{Chain}}_{\mathbb{Z}}^{\wedge}$

which carries X to the cellular chain complex of the universal covering $p : \hat{X} \longrightarrow X$. For each X we fix a basepoint $* \in \hat{X}$ with $p(*) = *$. The covering space $\hat{X}$ is a CW-complex with n-skeleton $\hat{X}^n = p^{-1}(X^n)$. Moreover, the fundamental group $\pi = \pi_1(X)$ acts cellularly from the right on $\hat{X}$ by covering transformations (denoted by $x \longmapsto x^{\alpha}$, $\alpha \in \pi$; $x \in \hat{X}$). A map $f : Y \longrightarrow X$ in $\underline{\underline{CW}}$ induces a unique basepoint preserving covering maps $\hat{f} : \hat{Y} \longrightarrow \hat{X}$ (with $p\hat{f} = fp$) which is φ-equivariant, $\varphi = \pi_1(f)$, and which is cellular. Let $\hat{C}_*X$ with

(1.4) $\quad \hat{C}_nX = H_n(\hat{X}^n, \hat{X}^{n-1})$

be the cellular chain complex of $\hat{X}$. This is a chain complex of free right π-modules and $\hat{f}$ induces a φ-equivariant chain map $\hat{C}_*(f) = \hat{f}_*$. This leads to the definition of the following category .

(1.5) <u>Definition</u>: Objects in the <u>category</u> $\underline{\underline{Chain}}_{\mathbb{Z}}^{\wedge}$ are pairs (π, C) where π is a group and where $C = (C_n, d_n; n \in \mathbb{Z})$ is a chain complex of right π-modules. Maps $(\varphi, F) : (\pi', C') \longrightarrow (\pi, C)$ are φ-equivariant chain maps $F : C' \longrightarrow C$ where $\varphi : \pi' \longrightarrow \pi$ is a homomorphism. Two such chain maps are homotopic $(\varphi,F) \simeq (\Psi,G)$, if $\varphi = \Psi$ and if there exists a φ-equivariant map $\alpha : C' \longrightarrow C$ of degree $+1$ with $d\alpha + \alpha d = -F + G$. The chain map (φ,F) is a weak equivalence if φ is an isomorphism and if F induces an isomorphism in homology.

Since we have basepoints $* \in \hat{X}$ we know that

$$(1.6) \qquad \hat{C}_0 X = \mathbb{Z}[\pi]$$

is the group ring of π. Moreover, $\hat{f}$ induces

$$(1.7) \qquad \varphi_{\#} : \mathbb{Z}[\pi'] \longrightarrow \mathbb{Z}[\pi]$$

in degree 0 with $\varphi_{\#}[\beta] = [\varphi\beta]$. Here $[\alpha] \in \mathbb{Z}[\pi]$ denotes the generator given by $\alpha \in \pi$. The isomorphism (1.6) carries the 0-cell $*$ to the unit $[0]$ of the group ring.

We say that a chain complex (π,C) is n-realizable if there exists a CW-complex $X = X^n$ in CW with $\hat{C}_* X \cong (\pi, C^n)$ in $\text{Chain}^{\wedge}_{\mathbb{Z}}$. Here C^n denotes the n-skeleton of C given by C_i, $i \leq n$.

(1.8) Definition: Let chain be the following subcategory of $\text{Chain}^{\wedge}_{\mathbb{Z}}$. Objects (π,C) are chain complexes which are 2-realizable and for which C_n, $n \in \mathbb{Z}$, is a free π-module with $C_0 = \mathbb{Z}[\pi]$. Maps in chain are chain maps (φ,F) which coincide with $\varphi_{\#}$ in degree 0. Moreover, two such maps are homotopic, $(\varphi,F) \simeq (\Psi,G)$, if there exists a homotopy α as in (1.5) with $\alpha(C'_0) = 0$.

$\hat{C}_*$ in (1.3) induces the commutative diagram of functors

$$(1.9) \qquad \begin{array}{ccc} \text{CW}/\overset{0}{\simeq} & \xrightarrow{\hat{C}_*} & \text{chain} \\ \downarrow p & & \downarrow p \\ \text{CW}/\overset{1}{\simeq} & \xrightarrow{\hat{C}_*} & \text{chain}/\simeq \end{array}$$

where p is a quotient functor. The following theorems are classical results of J.H.C. Whitehead.

(1.10) Theorem: *A map f in CW is a homotopy equivalence in CW/$\simeq$ if and only if $\hat{C}_* f$ is a weak equivalence.*

Let CW^n and chain^n be the full subcategories of CW and chain respectively consisting of objects of dimension $\leq n$.

(1.11) Theorem: *The functor*

$$\hat{C}_* : \underline{\underline{CW}}^2/\simeq \longrightarrow \underline{\underline{chain}}^2/\simeq$$

is an equivalence of categories and the functor

$$\hat{C}_* : \underline{\underline{CW}}^3/\simeq \longrightarrow \underline{\underline{chain}}^3/\simeq$$

is full. Moreover, each object in $\underline{\underline{chain}}$ *is 4-realizable.*

We derive from (1.10) and (1.11)

(1.12) Corollary: *Homotopy types of 3-dimensional CW-complexes in* $\underline{\underline{CW}}/\simeq$ *are 1-1 corresponded to homotopy types in the category* $\underline{\underline{chain}}^3/\simeq$.

The corollary shows that the chain complex $\hat{C}_*X$ of a 3-dimensional CW-complex $X = X^3$ determines the homotopy type of X. This is, however, not true for 4-dimensional CW-complexes.

§ 2 Homotopy systems

Let $n \geq 2$. A homotopy system of order $(n+1)$ is a triple

(2.1) (C, f_{n+1}, X^n)

where X^n is an object in $\underline{\underline{CW}}^n$ and where $(\pi_1 X^n, C)$ is a chain-complex which coincides with $\hat{C}_* X^n$ in degree $\leq n$. Moreover, f_{n+1} is a homorphism of right π-modules for which the following diagram commutes

$$(1) \qquad \begin{array}{ccc} C_{n+1} & \xrightarrow{f_{n+1}} & \pi_n(X^n) \\ d \downarrow & & \downarrow j \\ C_n & \xleftarrow[\cong]{h_n} & \pi_n(X^n, X^{n-1}) \end{array}$$

Here d is the boundary in C and

$$(2) \qquad h_n : \pi_n(X^n, X^{n-1}) \xrightarrow[\cong]{p_*^{-1}} \pi_n(\hat{X}^n, \hat{X}^{n-1}) \xrightarrow[\cong]{h} H_n(\hat{X}^n, \hat{X}^{n-1})$$

is given by the Hurewicz isomorphism h. In addition f_{n+1} satisfies the <u>cocycle condition</u>

(3) $$f_{n+1} d(C_{n+2}) = 0 \quad .$$

A <u>map</u> between homotopy systems of order $(n+1)$ is a pair (ξ,η),

(4) $$(\xi,\eta) : (C, f_{n+1}, X^n) \longrightarrow (C', g_{n+1}, Y^n) \quad ,$$

with the following properties. The map $\eta : X^n \longrightarrow Y^n$ is a morphism in $\underline{\underline{CW}}/\overset{0}{\simeq}$ and $\xi : C \longrightarrow C'$ is a $\pi_1(\eta)$-equivariant chain map which coincides with $\hat{C}_*(\eta)$ in degree $\leq n$ and for which the following diagram commutes:

(5) $$\begin{array}{ccc} C_{n+1} & \xrightarrow{\xi_{n+1}} & C'_{n+1} \\ \downarrow f_{n+1} & & \downarrow g_{n+1} \\ \pi_n X^n & \xrightarrow[\eta_*]{} & \pi_n Y^n \end{array}$$

Let $\underline{\underline{H}}^c_{n+1}$ be the <u>category of homotopy systems of order $(n+1)$</u> and of such maps. Clearly composition is defined by $(\xi,\eta)(\bar{\xi},\bar{\eta}) = (\xi\bar{\xi},\eta\bar{\eta})$. We have obvious functors, $n \geq 2$,

(2.2) $$\underline{\underline{CW}} \xrightarrow[r_{n+1}]{} \underline{\underline{H}}^c_{n+1} \xrightarrow[\lambda]{} \underline{\underline{H}}^c_n \xrightarrow[C]{} \underline{\underline{chain}}$$

with $C\lambda r_{n+1} = \hat{C}_*$ and $\lambda r_{n+1} = r_n$. Clearly r_{n+1} carries the CW-complex X to $r_{n+1}X = (C, f_{n+1}, X^n)$ where $C = \hat{C}_* X$ and where f_{n+1} is the composition

(1) $$f_{n+1} : C_{n+1} \cong \pi_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial} \pi_n(X^n) \quad .$$

Here we use the isomorphism h_{n+1} in (2.1)(2). Moreover, C in (2.2) is the forgetful functor which carries (C, f_n, X^{n-1}) to $(\pi_1 X^n, C)$.

Let Z_n be the set of n-cells in X. Then we know that

(2.3) $$C_n = \bigoplus_{Z_n} \mathbb{Z}[\pi]$$

is the free π-module generated by Z_n. The map f_n carries the generator $x \in Z_n$ to $f_n(x) \in \pi_{n-1}(X^{n-1})$. Here $f_n(x)$ corresponds to the attaching map of the cell x. Whence we can choose a map

$$(2.4) \qquad f : \bigvee_{Z_n} S^{n-1} \longrightarrow X^{n-1}$$

(determined up to homotopy by f_n above) such that the mapping cone C_f is homotopy equivalent to X^n (under X^{n-1}). This gives us the coaction

$$(2.5) \qquad \mu : X^n \longrightarrow X^n \vee \bigvee_{Z_n} S^n$$

which induces the action

$$(2.6) \qquad [X^n,Y] \times E(X^n,Y) \xrightarrow{+} [X^n,Y] .$$

The set $[U,V]$ denotes the set of homotopy classes $U \longrightarrow V$ in $\underline{\underline{CW}}/\simeq$ and $E(X^n,Y)$ is the abelian group

$$(1) \qquad E(X^n,Y) = [\bigvee_{Z_n} S^n, Y]$$

$$(2) \qquad\qquad = \mathrm{Hom}_\varphi(C_n, \pi_n Y) .$$

Here $\varphi : \pi_1 X \longrightarrow \pi_1 Y$ is any homorphism of groups and Hom_φ denotes the abelian group of φ-equivariant homorphisms; the isomorphism (2) is given by (2.3). The action (2.6) is defined by $F + \alpha = \mu^*(F,\alpha)$. We now are ready to define the homotopy relation, $\simeq$, in the category $\underline{\underline{H}}^c_{n+1}$.

(2.7) <u>Definition</u>: Let

$$(\xi,\eta),\ (\xi',\eta') : (C, f_{n+1}, X^n) \longrightarrow (C', g_{n+1}, Y^n)$$

be maps in $\underline{\underline{H}}^c_{n+1}$. We set $(\xi,\eta) \simeq (\xi',\eta')$ if $\pi_1(\eta) = \pi_1(\eta') = \varphi$ and if there exist φ-equivariant homomorphisms $\alpha_{j+1} : C_j \longrightarrow C'_{j+1}$, $j \geq n$, such that

(a) $\qquad \{\eta\} + g_{n+1}\,\alpha_{n+1} = \{\eta'\}$ and

(b) $\qquad \xi'_k - \xi_k = \alpha_k d + d\alpha_{k+1},\ k \geq n+1.$

The action $+$ in (a) is defined in (2.6) above; $\{\eta\}$ denotes the homotopy class of η in $[X^n, Y^n]$. We call $\alpha : (\xi, \eta) \simeq (\xi', \eta')$ a <u>homotopy</u> in $\underline{\underline{H}}^c_{n+1}$.

One can check that this homotopy relation is a natural equivalence relation and that the functors in (2.2) induce functors between homotopy categories $(n \geq 2)$

$$(2.8) \qquad \underline{\underline{CW}}/\simeq \xrightarrow[r_{n+1}]{} \underline{\underline{H}}^c_{n+1}/\simeq \xrightarrow[\lambda]{} H^c_n/\simeq \xrightarrow[C]{} \underline{\underline{chain}}/\simeq$$

(2.9) <u>Theorem</u>: *The functor*

$$C : \underline{\underline{H}}^c_3 \longrightarrow \underline{\underline{chain}}$$

is full and the functor

$$C : \underline{\underline{H}}^c_3/\simeq \longrightarrow \underline{\underline{chain}}/\simeq$$

is an equivalence of categories.

A similar result was obtained by J.H.C. Whitehead for the category of "crossed chain complexes" which he called "homotopy systems". Theorem (2.9) is a consequence of (VI. 5.13), (VI. 4.9) in [1]

§ 3 <u>The obstruction</u>

We use the groups Γ_n which were introduced by J.H.C. Whitehead. Let X be a CW-complex in $\underline{\underline{CW}}$ and let

$$(3.1) \qquad \Gamma_n(X) = \mathrm{image}(\pi_n(X^{n-1}) \longrightarrow \pi_n(X^n))$$

be the group given by the inclusion $X^{n-1} \subset X^n$ of skeleta. Clearly $\Gamma_2 X = 0$ since $\pi_2(X^1) = 0$. Moreover, for $n \geq 3$ the group $\Gamma_n(X)$ is a $\pi_1(X)$-module which is embedded in the "certain exact sequence" of $\pi_1(X)$-modules

$$(3.2) \qquad \longrightarrow H_{n+1}\hat{X} \xrightarrow{b} \Gamma_n X \xrightarrow{j} \pi_n X \xrightarrow{h_n} H_n \hat{X} \longrightarrow \quad .$$

Here $h_n = h(p_*)^{-1}$ is defined by the Hurewicz homomorphism, see (2.1)(2). The map j is induced by the inclusion $X^n \subset X$ and b is the "secondary boundary" for which the diagram

$$(3.3)\qquad \begin{array}{ccc} H_{n+1}\hat{X} & \xrightarrow{\quad b \quad} & \Gamma_n X \\ \uparrow & & \cap \\ H_{n+1}\hat{X}^{n+1} \subset C_{n+1} & \xrightarrow[f_{n+1}]{} & \pi_n X^n \end{array}$$

commutes, see (2.2)(1). The sequence (3.2) is natural for maps in $\underline{\underline{CW}}/\simeq$. We point out that for an object $X = (C, f_n, X^{n-1})$ in $\underline{\underline{H}}^c_n$ the group $\Gamma_n(X)$ in (3.1) as well is defined by choosing X^n as in (2.4). Let $\varphi : \pi_1(X) \longrightarrow \pi'$ be a homomorphism and let Γ be a right π'-module, then we obtain the cohomology groups

$$(3.4)\qquad \hat{H}^m(X, \varphi^*\Gamma) = H^m \mathrm{Hom}_\varphi(C, \Gamma)$$

where $C = \hat{C}_* X$ and where $\mathrm{Hom}_\varphi(C,\Gamma)$ is the cochains complex of φ-equivariant homomorphisms $C_n \longrightarrow \Gamma$, $n \in \mathbf{Z}$. The cohomology groups (3.4) are defined as well if X is an object in $\underline{\underline{H}}^c_n$.

(3.5) Proposition: Let X, Y be objects in $\underline{\underline{H}}^c_{n+1}$ and let $(\xi, \eta) : \lambda X \longrightarrow \lambda Y$ be a map in $\underline{\underline{H}}^c_n$ with $\varphi = \pi_1(\eta)$. Then an element

$$O_{X,Y}(\xi,\eta) \in \hat{H}^{n+1}(X, \varphi^*\Gamma_n Y)$$

is defined such that $O_{X,Y}(\xi,\eta) = 0$ if and only if there exists a map $(\xi, \bar{\eta}) : X \longrightarrow Y$ in $\underline{\underline{H}}^c_{n+1}$ with $\lambda(\xi,\bar{\eta}) = (\xi,\eta)$.

We define the obstruction $O_{X,Y}(\xi,\eta)$ in (3.5) as follows. Since (ξ,η) is a map in $\underline{\underline{H}}^c_n$ we can choose a map $F : X^n \longrightarrow Y^n$ in $\underline{\underline{CW}}/\overset{0}{\simeq}$ which extends η and for which $\hat{C}_* F$ coincides with ξ in degree $\leq n$. The diagram

$$(1)\qquad \begin{array}{ccc} C_{n+1} & \xrightarrow[\xi_{n+1}]{} & C'_{n+1} \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle g_{n+1}} \\ \pi_n X^n & \xrightarrow[F_*]{} & \pi_n Y^n \end{array}$$

needs not to be commutative. The difference

(2) $$O(F) = -g_{n+1}\xi_{n+1} + F_* f_{n+1} ,$$

maps C_{n+1} to $\Gamma_n Y \subset \pi_n Y^n$ and this difference is a cocycle in $\mathrm{Hom}_\varphi(C_{n+1}, \Gamma_n Y)$. The obstruction

(3) $$O_{X,Y}(\xi,\eta) = \{O(F)\}$$

is the cohomology class represented by the cocycle $O(F)$. This class does not depend on the choice of F in (1).

(3.6) <u>Proposition</u>: Let $(\xi,\eta) \simeq (\xi',\eta')$ be a homotopy in $\underline{\underline{H}}_n^c$. Then the obstructions

$$O_{X,Y}(\xi,\eta) = O_{X,Y}(\xi',\eta')$$

coincide.

(3.7) <u>Proposition</u>: The obstruction has the "derivation property", that is

$$O_{X,Z}((\overline{\xi},\overline{\eta})(\xi,\eta)) = \overline{\xi}^* O_{X,Y}(\xi,\eta) + \eta_* O_{Y,Z}(\overline{\xi},\overline{\eta})$$

where $(\xi,\eta) : \lambda X \longrightarrow \lambda Y$, $(\overline{\xi},\overline{\lambda}) : \lambda Y \longrightarrow \lambda Z$.

Moreover the obstruction yields an action on the set of realizations. To this end we define

(3.8) <u>Definition</u>: Let $\lambda : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ be a functor and let B be an object in $\underline{\underline{B}}$. We consider all pairs (A,b) where A is an object in $\underline{\underline{A}}$ and where $b : \lambda A \cong B$ is an isomorphism in $\underline{\underline{B}}$. We define an equivalence relation $\sim$ on such pairs by

$$(A,b) \sim (A',b') \iff \begin{cases} \exists\ g : A' \cong A & \text{in } \underline{\underline{A}} \\ \text{with } \lambda(g) = b^{-1}b' \end{cases} .$$

Let $\mathrm{Real}_\lambda(B)$ be the set of all such equivalence classes. We denote by $\{A,b\}$ the equivalence class of (A,b), often A is sufficient notation for $\{A,b\}$. We call $\{A,b\}$ a <u>realization</u> of B.

(3.9) Proposition: Let B be an object in $\underline{\underline{H}}_n^c/\simeq$ and let $\lambda : \underline{\underline{H}}_{n+1}^c/\simeq \longrightarrow \underline{\underline{H}}_n^c/\simeq$ be the functor in (2.8). Then the group $\hat{H}^{n+1}(B,\Gamma_n B)$ acts transitively and effectively on the set $\mathrm{Real}_\lambda(B)$ provided this set is non empty.

For $X_0 \in \mathrm{Real}_\lambda(B)$ and $\alpha \in \hat{H}^{n+1}(B,\Gamma_n B)$ we denote the action in (3.7) by $X_0 + \alpha$. In fact, we have $X = X_0 + \alpha$ if and only if

$$(3.10) \qquad O_{X,X_0}(1_B) = \alpha \quad .$$

Here 1_B is the identity of B and $X, X_0 \in \mathrm{Real}_\lambda(B)$.

$$(3.11) \qquad O_{X+\alpha,Y+\beta}(\xi,\eta) - O_{X,Y}(\xi,\eta) = \eta_*\alpha - \xi^*\beta \quad .$$

This shows that this difference corresponds to an "inner derivation". These properties of the obstruction O lead to the notion of a "linear covering of categories", compare (IV, §4) in [1].

§ 4 The action

Let X,Y be CW-complexes in $\underline{\underline{CW}}$ or let X,Y be objects in $\underline{\underline{H}}_m^c$ and let $\varphi : \pi_1 X \longrightarrow \pi_1 Y$ be a homomorphism. We denote by $[X,Y]^n$ the set of all morphisms $X_0 \longrightarrow Y_0$ in $\underline{\underline{H}}_n^c/\simeq$ where X_0 and Y_0 are the images of X and Y respectively in the category $\underline{\underline{H}}_n^c$, $n \leq m$. Here we use the functors in (2.2). Moreover,

$$(4.1) \qquad [X,Y]^n_\varphi \subset [X,Y]^n$$

denotes the subset of all morphisms which induce φ on fundamental groups. This subset can be empty. The functor λ yields the function

$$(4.2) \qquad \lambda : [X,Y]^{n+1}_\varphi \longrightarrow [X,Y]^n_\varphi \; .$$

(4.3) Proposition: There is an action (denoted by +) of the group $\hat{H}^n(X,\varphi^*\Gamma_n Y)$ on the set $[X,Y]^{n+1}_\varphi$ such that the following "exactness property" is satisfied. For $f,g \in [X,Y]^{n+1}_\varphi$ we have $\lambda f = \lambda g$ if and only if there exists $\alpha \in \hat{H}^n(X,\varphi^*\Gamma_n Y)$ with $g = f + \alpha$.

We define the action as follows. Let $(\xi,\eta) : X \longrightarrow Y$ be a map in $\underline{\underline{H}}^{c}_{n+1}$ which induces $\varphi = \pi_1(\eta)$. Moreover, let $\{\alpha\} \in \hat{H}^n(X,\varphi^*\Gamma_n Y)$ be a class represented by the cocycle

(1) $\qquad \alpha \in \mathrm{Hom}_{\varphi}(C_n,\Gamma_n(Y))$.

Then we obtain by $i : \Gamma_n(Y) \subset \pi_n Y^n$ the composition $i\alpha$ such that $\eta + i\alpha = \mu^*(\eta,i\alpha)$ is defined by μ in (2.5). Here $\eta + i\alpha$ is a well defined map in $\underline{\underline{CW}}/\overset{0}{\simeq}$. We now set

(2) $\qquad \{(\xi,\eta)\} + \{\alpha\} = \{(\xi,\eta + i\alpha)\}$.

Here $\{(\xi,\eta)\}$ denotes the homotopy class of (ξ,η) in $\underline{\underline{H}}^{c}_{n+1}/\simeq$. The action (2) is well defined and it has the properties in (4.3).

(4.4) Remark: The isotropy groups of the action (4.3) can be computed by use of a spectral sequence compare (VI. 5.16) in [1].

(4.5) Proposition: The action (4.3) satisfies the following "linear distributivity law". For $g \in [X,Y]^{n+1}_{\varphi}$, $f \in [Y,Z]^{n+1}_{\psi}$ we have the formula

$$(f + \alpha)(g + \beta) = fg + (f_*\beta + g^*\alpha) \quad .$$

These properties of the action (4.3) lead to the notion of a "linear extension of categories", compare (IV. 3.2) in [1].

§ 5 The CW-tower of categories

For a category $\underline{\underline{K}}$ let $F(\underline{\underline{K}})$ be the "category of factorizations" in $\underline{\underline{K}}$. Objects in $F(\underline{\underline{K}})$ are the morphisms in $\underline{\underline{K}}$ and morphisms $(\alpha,\beta) : f \longrightarrow g$ in are the commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ f\uparrow & & \uparrow g \\ B & \xleftarrow[\beta]{} & B' \end{array}$$

in $\underline{\underline{K}}$. Hence $\alpha f \beta = g$ is a factorization of g. Composition is defined by $(\alpha',\beta')(\alpha,\beta) = (\alpha'\alpha,\beta\beta')$. We call a functor from $F(\underline{\underline{K}})$ to the category $\underline{\underline{Ab}}$ of abelian groups a natural system on $\underline{K}$. For example we have the natural system

(5.1) $$H^m\Gamma_n : F(\underline{\underline{H}}^c_n/\simeq) \longrightarrow \underline{\underline{Ab}}$$

which carries the object $f : X \longrightarrow Y$ to the abelian group

$$H^m\Gamma_n(f) = \hat{H}^m(X, \varphi^*\Gamma_n Y)$$

where $\varphi = \pi_1(f)$. We say that

(5.2) $$H^n\Gamma_n \xrightarrow{+} H^c_{n+1}/\simeq \xrightarrow{\lambda} H^c_n/\simeq \xrightarrow{O} H^{n+1}\Gamma_n$$

is an exact sequence for the functor λ in (2.8) since an obstruction operator O and an action $+$ with all the properties in § 3 and § 4 are given. Whence we have a collection of exact sequences $(n \geq 3)$ which form the following diagram.

(5.3)
$$\begin{array}{ccccc}
 & & \underline{\underline{CW}}/\simeq & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
H^n\Gamma_n & \xrightarrow{+} & \underline{\underline{H}}^c_{n+1}/\simeq & & \\
 & & \downarrow\lambda & & \\
 & & \underline{\underline{H}}^c_n/\simeq & \xrightarrow{O} & H^{n+1}\Gamma_n \\
 & & \downarrow & & \\
 & & \vdots & & \\
H^3\Gamma_3 & \xrightarrow{+} & \underline{\underline{H}}^c_4/\simeq & & \\
 & & \downarrow\lambda & & \\
 & & \underline{\underline{H}}^c_3/\simeq & \xrightarrow{O} & H^4\Gamma_3 \\
 & \sim & \downarrow C & & \\
 & & \underline{chain}/\simeq & &
\end{array}$$

We call such a diagram a "tower of categories", in particular, this diagram is the CW-tower of categories which approximates the homotopy category of finite dimensional CW-complexes in $\underline{\underline{CW}}/\simeq$.

Let X,Y be CW-complexes in $\underline{\underline{CW}}$ and let $\varphi : \pi_1 X \longrightarrow \pi_1 Y$ be a homomorphism. As in (4.1) we have the subset

$$(5.4) \qquad [X,Y]_\varphi \subset [X,Y]$$

of all maps $\{\eta\} : X \longrightarrow Y$ in $\underline{\underline{CW}}/\simeq$ with $\pi_1(\eta) = \varphi$. Similarly, let

$$(5.5) \qquad [C,C']_\varphi \subset [C,C']$$

be the set of all $\{\xi\} : C \longrightarrow C'$ in $\underline{\text{chain}}/\simeq$ which induce φ. Then the CW-tower yields the following diagram of exact sequences of sets.

(5.6)

$$\begin{array}{ccccc}
 & & [X,Y]_\varphi & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
\hat{H}^n(X,\varphi^*\Gamma_n Y) & \xrightarrow{+} & [X,Y]^{n+1}_\varphi & & \\
 & & \downarrow \lambda & & \\
 & & [X,Y]^n_\varphi & \xrightarrow{O} & \hat{H}^{n+1}(X,\varphi^*\Gamma_n Y) \\
 & & \downarrow & & \\
 & & \vdots & & \\
H^3(X,\varphi^*\Gamma_3 Y) & \xrightarrow{+} & [X,Y]^4_\varphi & & \\
 & & \downarrow \lambda & & \\
 & & [X,Y]^3_\varphi & \xrightarrow{O} & H^4(X,\varphi^*\Gamma_3 Y) \\
 & & \approx \downarrow C & & \\
 & & [\hat{C}_* X,\hat{C}_* Y]_\varphi & &
\end{array}$$

We have $\text{kernel}(O) = \text{image}(\lambda)$ and we have $\lambda(f) = \lambda(g)$ if and only if there exists α with $g = f + \alpha$. Moreover, the definition of $\underline{\underline{H}}^c_n$ yields the

(5.7) <u>Proposition</u>: Let $X = X^N$, then

$$r_n : [X,Y]_\varphi \longrightarrow [X,Y]^n_\varphi$$

is bijective for $n = N+1$ and is surjective for $n = N$.

Next we derive from the CW-tower a structure theorem for the group of homotopy equivalence. For a CW-complex X in $\underline{\underline{CW}}$ let

(5.8) $\quad \mathrm{Aut}(X)^* \subset [X,X]$

be the <u>group of homotopy equivalences</u> of X in $\underline{\underline{CW}}/\simeq$. Moreover, let $\mathrm{Aut}(C)$ be the group of homotopy equivalences of C in the category $\underline{\underline{chain}}/\simeq$ and let

(5.9) $\quad E_n(X) \subset [X,X]^n \quad , \quad n \geq 3,$

be the group of equivalences of $r_n X$ in $\underline{\underline{H}}_n^c/\simeq$. Then the CW-tower yields the following tower of groups where the arrows O denote derivations and where all the other arrows are homomorphisms between groups.

(5.10)

$$
\begin{array}{ccccc}
 & & \mathrm{Aut}(X)^* & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
\hat{H}^n(X,\Gamma_n X) & \xrightarrow{1^+} & E_{n+1}(X) & & \\
 & & \downarrow \lambda & & \\
 & & E_n(X) & \xrightarrow{O} & \bigcup_{\varphi \in A} \hat{H}^{n+1}(X,\varphi^*\Gamma_n X) \\
 & & \downarrow & & \\
 & & \vdots & & \\
\hat{H}^3(X,\Gamma_3 X) & \xrightarrow{1^+} & E_4(X) & & \\
 & & \downarrow \lambda & & \\
 & & E_3(X) & \xrightarrow{O} & \bigcup_{\varphi \in A} \hat{H}^4(X,\varphi^*\Gamma_3 X) \\
 & & \approx \downarrow C & & \\
 & & \mathrm{Aut}(\hat{C}_* X) & & \\
 & & \downarrow & & \\
 & & \mathrm{Aut}(\pi_1 X) = A & &
\end{array}
$$

Here we define the obstruction O by the obstruction O in (5.6) and we define 1^+ by the action in (5.6), namely $1^+(\alpha) = 1 + \alpha$ where 1 is the identity. The linear distributivity law in (4.5) shows that 1^+ is a homomorphism of groups. Moreover we have the exactness image(1^+) = kernel(λ) and image(λ) = kernel(O). The isomorphism C is given by (2.9). As in (5.8) we get

(5.11) <u>Proposition</u>: Let $X = X^N$, then

$$r_n : \mathrm{Aut}(X)^* \longrightarrow E_n(X)$$

is an isomorphism for $n = N+1$ and is an epimorphism for $n = N$.

The kernel of 1^+ can be computed by a spectral sequence, compare (4.4). There are many applications of the CW-tower for the homotopy classification problems. The following result is immediate.

(5.12) <u>Theorem</u>: *Let* X, Y *be objects in* $\underline{\underline{CW}}$ *and let* X *be finite dimensional. Assume that*

$$(*) \qquad \hat{H}^p(X, \varphi^*\Gamma_n Y) = 0$$

for $p = n, n+1$, $n \geq 3$, $\varphi \in \mathrm{Hom}(\pi_1 X, \pi_1 Y)$. *Then the functor* $\hat{C}_*$ *yields the bijection of sets*

$$\hat{C}_* : [X,Y] \xrightarrow{\approx} [\hat{C}_* X, \hat{C}_* Y] \ .$$

Assume that (*) *holds for* $X = Y$ *then*

$$\hat{C}_* : \mathrm{Aut}(X)^* \xrightarrow{\cong} \mathrm{Aut}(\hat{C}_* X)$$

is an isomorphism of groups.

This theorem is a special case of (VI. 6.15) in [1].

§ 6 <u>On the classification of 4-dimensional homotopy types</u>

We first recall that for X in $\underline{\underline{CW}}$ we have a natural isomorphism

$$(6.1) \qquad \Gamma_3(X) = \Gamma(\pi_2 X) = \Gamma(H_2 \hat{X})$$

of $\pi_1(X)$-modules. Here Γ is the quadratic functor of J.H.C. Whitehead [3]. We derive from the exact sequence (3.2) the natural isomorphisms

(6.2) $$H_5(K(A,2)) \overset{b}{\cong} \Gamma(A) \overset{j}{\cong} \pi_3(M(A,2))$$

where $K(A,2)$ and $M(A,2)$ denote the Eilenberg-Mac Lane space and the Moore-space respectively of the abelian group A. We now consider the functor

(6.3) $$C\lambda = \hat{C}_* : \underline{\underline{CW}}^4/\simeq \longrightarrow \underline{\underline{chain}}^4/\simeq .$$

By (1.11), (2.9), and (3.9) we know

(6.4) Theorem: *For* C *in* $\underline{\underline{chain}}^4$ *the set* $\mathrm{Real}_{C\lambda}(C)$ *is non empty and the group* $\hat{H}^4(C,\Gamma(H_2C))$ *acts transitively and effectively on this set*

Moreover, we derive from (3.5), (3.6) and (2.9) the result

(6.5) Theorem: *Let* X, Y *be CW-complexes in* $\underline{\underline{CW}}^4$ *and let* $\{\varphi,\xi\} : \hat{C}_*X \to \hat{C}_*Y$ *be a map in* $\underline{\underline{chain}}^4/\simeq$*. Then there exists a map* $\{F\} : X \longrightarrow Y$ *in* $\underline{\underline{CW}}/\simeq$ *with* $\hat{C}_*\{F\} = \{\varphi,\xi\}$ *if and only if the obstruction*

$$O_{X,Y}\{\varphi,\xi\} \in \hat{H}^4(X,\varphi^*\Gamma(H_2\hat{Y}))$$

vanishes.

In my talk in Louvain la Neuve I described algebraic models of 4-dimensional CW-complexes which allow a formula for the obstruction in (6.5). This, in fact, yields the homotopy classification of 4-dimensional CW-complexes. We now consider for simplicity 1-connected 4-dimensional CW-complexes. In this case we obtain by (6.4) and (6.5) a classical result of J.H.C. Whitehead [2], see (6.9).

(6.6) Definition: Let Γ-$\underline{\underline{sequence}}^4$ be the following category. Objects are the exact sequences

$$S = (H_4 \longrightarrow \Gamma(H_2) \longrightarrow \pi_3 \longrightarrow H_3 \longrightarrow 0)$$

of abelian groups where H_4 is free abelian. A morphism $f : S \longrightarrow S'$ is a triple $f = (f_4,f_3,f_2)$ of homorphisms $f_i : H_i \longrightarrow H_i'$ for which there exists a homorphism φ such that the diagram

$$\begin{array}{ccccccccc}
H_4 & \longrightarrow & \Gamma(H_2) & \longrightarrow & \pi_3 & \longrightarrow & H_3 & \longrightarrow & 0 \\
\downarrow f_4 & & \downarrow \Gamma(f_2) & & \downarrow \varphi & & \downarrow f_3 & & \\
H'_4 & \longrightarrow & \Gamma(H'_2) & \longrightarrow & \pi'_3 & \longrightarrow & H'_3 & \longrightarrow & 0
\end{array}$$

commutes.

Let $\underline{\underline{rCW}}^4$ be the full subcategory of $\underline{\underline{CW}}^4$ consisting of CW-complexes X with $X^1 = *$. By (3.2) and (6.1) we obtain the functor

(6.7) $\Gamma S : \underline{\underline{rCW}}^4/\simeq \longrightarrow \Gamma\text{-}\underline{\underline{\text{sequence}}}^4$

which carries the CW-complex X to the exact sequence

(6.8) $H_4X \xrightarrow{b_4^X} \Gamma(H_2X) \longrightarrow \pi_3X \longrightarrow H_3X \longrightarrow 0$

where H_iX is the homology of $X = \hat{X}$. This sequence is natural in X and whence the functor ΓS is well defined.

Following J.H.C. Whitehead we define the following conditions on a functor $p : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$.

(a) Sufficiency: For objects A,A' in $\underline{\underline{A}}$ a morphism $\alpha : A \longrightarrow A'$ is an equivalence if and only if $p\alpha : pA \longrightarrow pA'$ is an equivalence in $\underline{\underline{B}}$.

(b) Realizability: The functor p is full and for each object B on $\underline{\underline{B}}$ there is an object A in $\underline{\underline{A}}$ together with an equivalence $pA \xrightarrow{\sim} B$ in $\underline{\underline{B}}$.

We say that p is a detecting functor if p satisfies both the sufficiency and the realizability conditions.

(6.9) Theorem: *The functor* ΓS *in (6.7) is a detecting functor.*

(6.10) Corollary: *Homotopy types of 1-connected 4-dimensional CW-complexes are 1-1 corresponded to isomorphism classes of exact sequences in the category* $\Gamma\text{-}\underline{\underline{\text{sequence}}}^4$ *above.*

(6.11) Proof of theorem (6.8): Let r chain^4 be the full subcategory of chain^4 consisting of objects (π,C) with $\pi = 0$ and $C_1 = 0$. Whence objects in r chain^4 are given by chain complexes of free abelian groups

(1) $$C = (C_4 \longrightarrow C_3 \longrightarrow C_2)$$

with $C_i=0$ for $i>4$, $i<2$. Let C,C' be two such chain complexes with homology groups $H_* = H_*C$, $H'_* = H_*C$. Then we have the short exact sequence

(2) $$\bigoplus_{i=2,3} \mathrm{Ext}(H_i,H'_{i+1}) \overset{i}{\rightarrowtail} [C,C'] \twoheadrightarrow \mathrm{Hom}(H_*,H'_*)$$

where $[C,C']$ is the set of homotopy classes of chain maps. Moreover, we have for an abelian group Γ the short exact sequence

(3) $$\mathrm{Ext}(H_3,\Gamma) \overset{\Delta}{\rightarrowtail} H^4(C,\Gamma) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(H_4,\Gamma).$$

For $\Gamma = \Gamma(H_2C)$ we have the function

(4) $$b_4 : \mathrm{Real}_{C\lambda}(C) \longrightarrow \mathrm{Hom}(H_4,\Gamma)$$

which carries the realization X to the secondary boundary b_4^X, see (6.8). From (3.3) we derive

(5) $$b_4(X+\alpha) = b_4(X) + \mu(\alpha)$$

where we use the action in (6.4) and where we use μ in (3). Since μ is surjective, this implies that b_4 in (4) is surjective. For $b \in \mathrm{Hom}(H_4,\Gamma)$ we thus have the function

(6) $$\pi : b_4^{-1}(b) \longrightarrow \mathrm{Ext}(H_3,\mathrm{cok}(b))$$

which carries X with $b_4(X) = b$ to the extension $\{\pi_3X\}$ given by the short exact sequence

(7) $$\mathrm{cok}(b) \rightarrowtail \pi_3X \twoheadrightarrow H_3$$

in (6.8). The function π satisfies

(8) $\pi(X+\Delta(\beta)) = \pi(X) + q_*\beta$

where $\beta \in \mathrm{Ext}(H_3,\Gamma)$. Again we use (6.4) and (3) and $q : \Gamma \longrightarrow \mathrm{cok}(b)$ is the quotient map. Since q is surjective, also q_* in (8) is surjective and whence π in (6) is surjective. Surjectivity of (6) and (4) shows that the functor ΓS satisfies the realizability condition for objects. We now show that the functor ΓS is full. Let $f : S = \Gamma S(X) \longrightarrow \Gamma S(Y) = S'$ be a map in Γ-sequence[4] and let $C = C_*X$, $C' = C_*Y$ be the cellular chain complexes. By (2) we know that there is a chain map $\xi : C \longrightarrow C'$ which induces f in homology. For the obstruction (6.5) and for μ in (3) one readily gets

(9) $\mu O_{X,Y}(\zeta) = b_4' f_4 - \Gamma(f_2) b_4 = 0$.

This element is trivial since f makes the diagram in (6.6) commutative. Whence by (3) the element

(10) $\Delta^{-1} O_{X,Y}(\xi) \in \mathrm{Ext}(H_3,\Gamma H_2')$

is defined. For $q' : \Gamma H_2' \longrightarrow \mathrm{cok}(b_4')$ we get

(11) $q_*' \Delta^{-1} O_{X,Y}(\xi) = f_3^*\{\pi_3 X\} - \Gamma(f_2)_*\{\pi_3 Y\} = 0$

where we use the elements given by (7). The element (11) again is trivial since the diagram in (6.6) commutes. Finally we obtain for $\alpha \in \mathrm{Ext}(H_3,H_4')$ the formula

(12) $O_{X,Y}(\xi+\alpha) = O_{X,Y}(\xi) + \Delta(b_{4*}'\alpha)$.

Now the sequence

(13) $\mathrm{Ext}(H_3,H_4') \xrightarrow{b_{4*}'} \mathrm{Ext}(H_3,\Gamma H_2') \xrightarrow{q_*'} \mathrm{Ext}(H_3,\mathrm{cok}\ b_4')$

is exact. Whence by (11) we can choose α with

(14) $(b_4')_*\alpha = \Delta^{-1} O_{X,Y}(\xi)$.

Therefore (12) shows

(15) $O_{X,Y}(\xi - i\alpha) = 0$

and thus by (6.5) there exists a realization $F : X \longrightarrow Y$ with $\{\hat{C}_* F\} = \{\xi - i\alpha\}$. Here $\xi - i\alpha$ induces f in homology, whence we get $\Gamma S(F) = f$. This completes the proof that ΓS satisfies the realizability condition. By the Whitehead theorem (1.10) the functor ΓS satisfies the sufficiency condition.

In (IX. § 4) of my book [1] we show that the "same proof" as above yields as well the classification of certain R-local spaces ($R \subset \mathbb{Q}$) as well as the classification of certain chain algebras.

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