Max-Planck-Institut für Mathematik
Bonn

Dynamical systems on some elliptic modular surfaces via operators on line arrangements

by

Lukas Kühne
Xavier Roulleau

Max-Planck-Institut für Mathematik
Preprint Series 2024 (3)

Date of submission: February 28, 2024

Dynamical systems on some elliptic modular surfaces via operators on line arrangements

by

Lukas Kühne
Xavier Roulleau

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Universität Bielefeld
Fakultät für Mathematik
33501 Bielefeld
Germany

Université d'Angers
CNRS, LAREMA
SFR Math-STIC
49000 Angers
France

MPIM 24-3

DYNAMICAL SYSTEMS ON SOME ELLIPTIC MODULAR SURFACES VIA OPERATORS ON LINE ARRANGEMENTS

LUKAS KÜHNE AND XAVIER ROULLEAU

Abstract. This paper further studies the matroid realization space of a specific deformation of the regular n-gon with its lines of symmetry. Recently, we obtained that these particular realization spaces are birational to the elliptic modular surfaces $\Xi_1(n)$ over the modular curve $X_1(n)$. Here, we focus on the peculiar cases when $n = 7, 8$ in more detail. We obtain concrete quartic surfaces in $\mathbb{P}^3$ equipped with a dominant rational self-map stemming from an operator on line arrangements, which yields K3 surfaces with a dynamical system that is semi-conjugated to the plane.

1. Introduction

A *line arrangement* $\mathcal{C} = \ell_1 + \cdots + \ell_k$ is a finite union of lines ℓ_j in the projective plane $\mathbb{P}^2$. Line arrangements are ubiquitous objects studied in various fields such as topology, algebra, algebraic geometry, see for instance [14, 18] for two surveys. In [11], the second author described a number of operators acting on line arrangements: if $\mathfrak{n}, \mathfrak{m}$ are sets of integers at least 2, the operator $\Lambda_{\mathfrak{m},\mathfrak{n}}$ associates to a line arrangement $\mathcal{C}$ the line arrangement $\Lambda_{\mathfrak{m},\mathfrak{n}}(\mathcal{C})$ which is the union of the lines that contain $n \in \mathfrak{n}$ points among the m-points of $\mathcal{C}$, for $m \in \mathfrak{m}$ (recall that an m-point of $\mathcal{C}$ is a point where exactly m lines of $\mathcal{C}$ meet). For example $\Lambda_{\{2\},\{3\}}(\mathcal{C})$ is the union of the lines that contain exactly three double points of $\mathcal{C}$ (the arrangement might be empty).

A labeled line arrangement $\mathcal{C} = (\ell_1, \dots, \ell_k)$ is a line arrangement for which one fixes the order of the lines. The configuration of a labeled line arrangement $\mathcal{C}$ is described by its associated *matroid* $M = M(\mathcal{C})$. Conversely, given a matroid M (a combinatorial object), one can look at line arrangements $\mathcal{C}$ for which $M(\mathcal{C}) = M$. When such a $\mathcal{C}$ exists, one says that $\mathcal{C}$ is a realization of M. Let us denote by $\mathcal{R} = \mathcal{R}(M)$ the moduli space of realizations of M: a point of $\mathcal{R}$ is the orbit under the action of the projective general linear group PGL_3 of a realization of M. The space of all realizations of M is denoted by $\mathfrak{U} = \mathfrak{U}(M)$ and there is a natural quotient map $\mathfrak{U} \to \mathcal{R}$.

In [7], we constructed a realizable matroid M_n for any $n \geq 7$ that is based on the regular n-gon. Interestingly, there exists an operator Λ among the ones we described above (for example if $n = 2k + 1$ is odd, then $\Lambda = \Lambda_{\{2\},\{k\}}$) which acts non-trivially on the realization space of M_n. Thus if $\mathcal{C}$ is a realization of M_n, then $\Lambda(\mathcal{C})$ is also a realization of M_n. We obtain in that way a dominant self-rational map λ on the realization space $\mathcal{R}_n = \mathcal{R}(M_n)$.

The main result of [7] establishes that the realization space $\mathcal{R}_n$ is an open dense sub-scheme of the *elliptic modular surface* $\Xi_1(n)$, a well-studied surface, see e.g. Shioda's paper [13]. Recall that this surface $\Xi_1(n)$ parametrizes (up to isomorphisms) triples (E, t, p) of an elliptic curve and points t, p on E such that t has order n. The modular curve $X_1(n)$ parametrizes (up to isomorphisms) pairs (E, t), where E, t are as above. The map $(E, t, p) \to (E, t)$ defines an elliptic fibration on $\Xi_1(n)$, with fiber over the point (E, t) isomorphic to E. For any integer m, there is a natural multiplication by m rational map of the elliptic surface $\Xi_1(n)$. We obtain in [7] that, through the identification of $\mathcal{R}_n$ as an open subscheme of $\Xi_1(n)$, the rational self map λ induced by Λ is the multiplication by -2 map acting on $\Xi_1(n)$, in particular λ has degree 4.

2000 *Mathematics Subject Classification.* Primary: 14N20, 14H50, 37D40.

The aim of the present paper is to study the peculiar cases when $n = 7, 8$ in more detail. In particular, we give another proof that the surface $\mathcal{R}_n$ is an open dense subscheme of $\Xi_1(n)$, and the degree of λ is 4 in these cases. From now on assume $n \in \{7, 8\}$; in those cases, we obtain (singular) models of $\Xi_1(n)$ as quartic surfaces in $\mathbb{P}^3$. There is a natural section $\mathcal{R}_n \to \mathfrak{U}_n = \mathfrak{U}(M_n)$ of the quotient map $\mathfrak{U}_n \to \mathcal{R}_n$, so that one may consider $\mathcal{R}_n$ as contained in $\mathfrak{U}_n$, and therefore one may consider a class as a realization of M_n. Using that fact, we are able to give explicit polynomials for the action $\lambda = \lambda(n)$ of $\Lambda = \Lambda(n)$ on $\mathcal{R}_n \subset \mathbb{P}^3$.

Recall that a dynamical system is a pair (X, λ) of a variety X and a dominant rational map $\lambda : X \to X$. Two dynamical systems (X, λ), (Y, μ) are called *semi-conjugated* if there exists a generically finite rational dominant map $\pi : X \to Y$ such that $\pi \circ \lambda = \mu \circ \pi$. A principle result of this article is the following.

Theorem 1. *For $n \in \{7, 8\}$, the dynamical system $(\mathcal{R}_n, \lambda)$ is semi-conjugated to $(\mathbb{P}^2, F)$ where $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is an explicitly described rational self map; the dominant rational map $\pi : \mathcal{R}_n \to \mathbb{P}^2$ such that $\pi \circ \lambda = F \circ \pi$ is a double cover of $\mathbb{P}^2$ branched along a sextic curve.*

The surfaces $\Xi_1(7)$, $\Xi_1(8)$ are K3 surfaces; to our knowledge these are the first examples of a degree > 1 dynamical system on a K3 surfaces that is semi-conjugated to the plane.

Let us describe the structure of this paper and some further results. In Section 2, we start by describing the line operators Λ and general results on matroids. In Subsection 2.3, we study under which conditions a K3 surface which is the double cover of the $\mathbb{P}^2$ may be semi-conjugated to $\mathbb{P}^2$. Subsequently, we study the case $n = 7$ in Section 3: we start by recalling the definition of the matroid M_7 and then show that $\Lambda_{\{2\},\{3\}}$ induces a rational self map $\lambda_{\{2\},\{3\}}$ on the quartic surface $\mathcal{R}_7 \subset \mathbb{P}^3$. We then compute the degree of $\lambda_{\{2\},\{3\}}$ and prove that $\mathcal{R}_7$ is an open subset of the elliptic modular surface $\Xi_1(7)$. The automorphism group of the matroid M_7 is the order 42 Frobenius group. There is a natural action of that group on the surface $\mathcal{R}_7$. We show that this action is faithful. The quotient surface $\mathcal{R}_7/\mathrm{Aut}(M_7)$ is the moduli space for unlabeled line arrangements coming from realizations of M_7: we obtain that this is a rational surface. In Subsection 3.6, we describe explicitly the semi-conjugacy of $\mathcal{R}_7$ (or equivalently $\Xi_1(7)$) with $\mathbb{P}^2$. The branch loci of the double cover $\Xi_1(7) \to \mathbb{P}^2$ is the union of a line and a singular quintic curve which we describe. Section 4 follows a similar pattern for the case $n = 8$. In that case, the branch loci of the double cover $\Xi_1(8) \to \mathbb{P}^2$ is union of a conic and a singular quartic curve. We moreover describe some 3-periodic line arrangements for Λ; their classes are fixed points for the action of λ on $\mathcal{R}_8$.

We remark that for $n = 9$, one may similarly obtain that $\mathcal{R}_9$ (contained as a sextic surface in $\mathbb{P}^3$) is birational to $\Xi_1(9)$. That elliptic surface is no longer a K3 surface and we could not find a semi-conjugacy with the plane.

Computations in this paper are based on Magma [2] and `OSCAR` [4]. In the arXiv ancillary file of this paper are some datas related to these computations.

Acknowledgments The authors are grateful to Bert van Geemen and to Keiji Oguiso for interesting discussions. LK is supported by the SFB-TRR 358 – 491392403 "Integral Structures in Geometry and Representation Theory" (DFG). XR is thankful to Max Planck Institute for Mathematics in Bonn for its hospitality and financial support. XR is also supported by the Centre Henri Lebesgue (ANR-11-LABX-0020-01).

2. Notations and definitions

Throughout this article we assume to be working over the field $\mathbb{C}$.

2.1. **Line arrangements and the operator $\Lambda_{\mathfrak{n},\mathfrak{m}}$.** A line arrangement $\mathcal{C} = \ell_1 + \cdots + \ell_n$ is a union of finitely many distinct lines in $\mathbb{P}^2$. A labeled line arrangement $\mathcal{C} = (\ell_1, \ldots, \ell_n)$ is a line arrangement for which one with a fixed order of the lines. We sometime put a superscript $^{\ell}$ (resp. ^{u})

when we want to emphasize that an arrangement or related objects has (resp. does not have) a labeling.

For an integer $k \geq 2$, a k-point of the line arrangement $\mathcal{C}$ is a point where exactly k lines of $\mathcal{C}$ meet. As in [11], for a subset $\mathfrak{n}$ of integers at least 2, let us denote by $\mathcal{P}_{\mathfrak{n}}(\mathcal{C})$ the set of k-points of $\mathcal{C}$ for all $k \in \mathfrak{n}$. We denote by $t_k = t_k(\mathcal{C}) = |\mathcal{P}_{\{k\}}(\mathcal{C})|$ the number of k-points of $\mathcal{C}$. For a finite set of point $\mathcal{P}$ in $\mathbb{P}^2$ and $\mathfrak{n}$ as above, we denote by $\mathcal{L}_{\mathfrak{n}}(\mathcal{P})$ the set of lines which contain exactly n points in $\mathcal{P}$ for some $n \in \mathfrak{n}$.

For subsets $\mathfrak{n}, \mathfrak{m}$ of integers at least 2, let us denote by $\Lambda_{\mathfrak{n},\mathfrak{m}}(\mathcal{C}) = \mathcal{L}_{\mathfrak{m}} \circ \mathcal{P}_{\mathfrak{n}}(\mathcal{C})$ the line arrangement that contains all lines of $\mathbb{P}^2$ containing exactly m points of $\mathcal{P}_{\mathfrak{n}}(\mathcal{C})$ for $m \in \mathfrak{m}$. For example $\Lambda_{\{2\},\{3,4\}}(\mathcal{C})$ is the union of the lines that contain three or four double points of $\mathcal{C}$. The arrangement could be the empty arrangement if no such lines exists.

2.2. **Matroids and the period map of the moduli of a matroid.** A matroid is a fundamental and actively studied object in combinatorics. Matroids generalize linear dependency in vector spaces as well as forests in graphs. See e.g. [10] for a comprehensive treatment of matroids. We just briefly mention a few concepts about matroids that are relevant for this article.

A *matroid* is a pair $M = (E, \mathcal{B})$, where E is a finite *ground set* of elements called atoms and $\mathcal{B}$ is a nonempty collection of subsets of E, called *bases*, satisfying an exchange property reminiscent from linear algebra.

The prime examples of matroids arise by choosing a finite set of vectors E in a vector space and declaring the maximal linearly independent subsets of E as bases. In our case we obtain matroids through line arrangements: If $\mathcal{C} = (\ell_1, \ldots, \ell_m)$ is a labeled line arrangement, the subsets $\{i, j, k\} \subseteq \{1, \ldots, m\}$ such that the lines ℓ_i, ℓ_j, ℓ_k meet in three distinct points are the bases of a matroid $M(\mathcal{C})$ over the set $\{1, \ldots, m\}$. We say that $M(\mathcal{C})$ is the matroid associated to $\mathcal{C}$.

We denote by $\mathrm{Aut}(M)$ the *automorphism group* of the matroid M, i.e., the set of isomorphisms from M to M.

A *realization* (over some field) of a matroid $M = (E, \mathcal{B})$ is a converse operation to the association $\mathcal{C} \to M(\mathcal{C})$: it is a $3 \times m$-matrix with non-zero columns $C_1, \ldots, C_m$, which are considered up to a multiplication by a scalar (thus as point in the projective plane) such that a subset $\{i_1, i_2, i_3\}$ of E of size 3 is a basis if and only if the 3×3 minor $|C_{i_1}, C_{i_2}, C_{i_3}|$ is nonzero. We denote by ℓ_i the line with normal vector the point $C_i \in \mathbb{P}^2$.

If $\mathcal{C} = (\ell_1, \ldots, \ell_m)$ is a realization of M and $\gamma \in PGL_3$, then $(\gamma\ell_1, \ldots, \gamma\ell_m)$ is another realization of M; we denote by $[\mathcal{C}]$ the orbit of $\mathcal{C}$ under that action of PGL_3. The *moduli space* $\mathcal{R}(M)$ of realizations of M parametrizes the orbits $[\mathcal{C}]$ of realizations. A more detailed introduction to these moduli spaces together with a description of a software package in `OSCAR` that can compute these spaces is given in [4].

In this article, we always assume that each subset of three elements of the the first four atoms is a basis (otherwise, we replace M by a matroid isomorphic to it). Then in the moduli space $\mathcal{R}(M)$, one can always map the first four vectors of $\mathcal{C} \in [\mathcal{C}]$ to the canonical basis, so that each element $[\mathcal{C}]$ of $\mathcal{R}(M)$ has a canonical representative, which we will identify with $[\mathcal{C}]$.

A useful tool for the computations related to the moduli space $\mathcal{R} = \mathcal{R}(M)$ of realizations of a matroid M is what we call the period map: Let us denote by $\mathfrak{U} = \mathfrak{U}(M)$ the scheme of all realizations of M in $\mathbb{P}^2$. By analogy with similar objects, we call the quotient map

$$\mathfrak{q} : \mathfrak{U}(M) \to \mathcal{R}(M)$$

the period map; a point c of $\mathcal{R} = \mathcal{R}(M)$ is the class $c = [\mathcal{C}]$ of a realization $\mathcal{C}$. Once a basis is fixed, each class c has a unique representative $\mathcal{C}_0$ and we can (and we will) identify c with that representative.

It often occurs that $\mathcal{R}$ is embedded in a space $\mathbb{S} = \mathbb{S}(y_1, \dots, y_k)$ (affine or projective) of small dimension, like $\mathbb{P}^3$. The coordinates of the normal vectors $n^{(j)} = (n_1^{(j)} : n_2^{(j)} : n_3^{(j)})$ of $\mathcal{C}_0$ are then polynomials $n_1^{(j)} = P_1^{(j)}(y), \dots, n_3^{(j)} = P_3^{(j)}(y)$ in the coordinates $y_1, \dots, y_k$ of $\mathcal{R}$ in $\mathbb{S}$.

One often arrives at the natural question on computing the point $y = (y_1, \dots, y_k)$ in $\mathcal{R}$ from the knowledge of the normal vectors n. In other words, we need an explicit form of the period map $\mathfrak{q}$ as a map from $\mathfrak{U}$ to the scheme $\mathcal{R}$ embedded in the space $\mathbb{S}$. The answer to that problem are polynomials (or rational functions) $Q_1, \dots, Q_k$ in the coordinates of the normal vectors $n^{(1)}, \dots, n^{(m)}$ etc.; here m is the number of lines in an arrangement.

2.3. **Degree two K3 surfaces semi-conjugated to the plane.** Let $C_1 : Q_1 = 0$ be a sextic curve with at most ADE singularities, so that the desingularization X^s of the associated double cover

$$X = \{y^2 = Q_1(z_1, z_2, z_3)\} \hookrightarrow \mathbb{P}(3, 1, 1, 1)$$

is a K3 surface. Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be a rational self-map defined by coprime homogeneous polynomials (F_1, F_2, F_3) of degree m. Suppose that $F^*C_1 = C_1 + 2D$, for an effective divisor D; in algebraic terms, that means that we assume that

$$Q_1(F_1, F_2, F_3) = Q_1 \cdot R^2,$$

for some polynomial R. Then the following relation holds

$$(y\, R(z))^2 = Q_1(z)\, R(z)^2 = Q_1(F_1(z), F_2(z), F_3(z)),$$

where $z = (z_1 : z_2 : z_3) \in \mathbb{P}^2$. Hence, the rational map

$$\tilde{F} : (y; z) \dashrightarrow (y\, R(z); F_1(z) : F_2(z) : F_3(z))$$

is a rational self-map acting on the K3 surface X^s. Let $\pi : X \to \mathbb{P}^2$ be the double cover map. The following diagram

$$\begin{array}{ccc} X & \overset{\tilde{F}}{\dashrightarrow} & X \\ \pi \downarrow & & \pi \downarrow \\ \mathbb{P}^2 & \overset{F}{\dashrightarrow} & \mathbb{P}^2 \end{array} \tag{2.1}$$

is commutative and, by analogy with other dynamical systems, we say that the dynamical system $(X, \tilde{F})$ is *semi-conjugated* to $(\mathbb{P}^2, F)$.

Example 2. Let C be an irreducible curve of degree 6 with 10 nodes. A Coble surface Y is the blow-up of $\mathbb{P}^2$ at the 10 nodal singularities of C. The group of birational transformations G preserving C is infinite, it is generated by Bertini involutions centered at the nodal points of C. When C is generic, the group G lifts to Y and the elements of G become automorphisms of Y. The automorphism group $G \subset \mathrm{Aut}(Y)$ preserves the pull-back C' of C, thus taking the double cover of Y branched over C', one gets a smooth K3 surface X and the group G is in fact the automorphism group of X (see e.g. [3]). The surface X is also the minimal desingularization of the double cover branched over C and the diagram (2.1) is commutative.

3. The heptagon

3.1. $\Lambda_{\{2\},\{3\}}$ **is a rational self-map on** $\mathcal{R}_7$ **and** $\mathfrak{U}_7$.

3.1.1. *Definition of the matroid M_7.* The matroid M_7 has 14 atoms $1,\dots,7,1',\dots,7'$ and the bases are the triples $\{a,b,c\}$ with $\{a,b\}\subset\{1,\dots,7\}$ and $c\in\{1',\dots,7'\}$ such that $a+b\neq 2c \bmod 7$. A sketch of M_7 is described in Figure 3.1, where the atoms $i\in\{1,\dots,7\}$ and $j\in\{1',\dots,7'\}$ correspond to the lines ℓ_i and ℓ'_j, resp., and three lines form a bases if they do not meet in one point. Note that the central singularity of arrangement in Figure 3.1 is not part of the matroid and therefore removed.

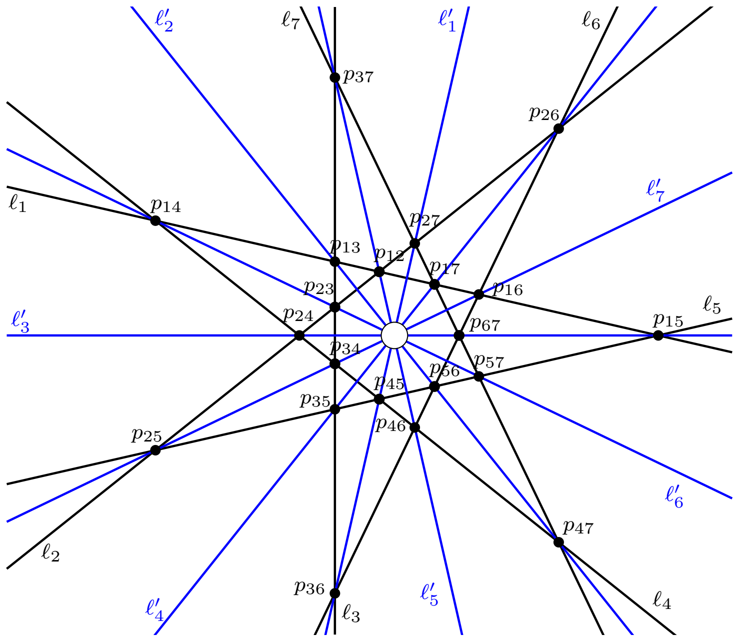

FIGURE 3.1. The matroid M_7 whose construction is based on the regular heptagon.

Let $\mathcal{A}_1$ be a line arrangement realizing the matroid M_7. We write $\mathcal{A}_1=\mathcal{C}_0\cup\mathcal{C}_1$ where $\mathcal{C}_0$ are the first seven lines and $\mathcal{C}_1$ are the seven last ones. By the combinatorics of the matroid M_7, the property $\mathcal{C}_1=\Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$ holds, and – that will be important for us – the image of $\mathcal{C}_0$ by the operator $\Lambda_{\{2\},\{3\}}$ has a natural labeling: for any $j\in\{1,\dots,7\}$, the six line arrangement

$$H_j=\sum_{k\in\{1,\dots,7\},\,k\neq j}\ell_k \tag{3.1}$$

is such that the line arrangement $\Lambda_{\{2\},\{3\}}(H_j)$ is a unique line ℓ'_j, moreover:

$$\mathcal{C}_1=\ell'_1+\cdots+\ell'_7.$$

Since $\mathcal{C}_1=\Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$, we will often speak of $\mathcal{C}_0$ as a realization of M_7 instead of $\mathcal{C}_0\cup\mathcal{C}_1$ to shorten our notations.

The singularities of $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) are 21 double points. The 21 singularities on $\mathcal{C}_0$ become the triple points on $\mathcal{C}_0\cup\mathcal{C}_1$, moreover $t_2(\mathcal{C}_0\cup\mathcal{C}_1)=28$.

3.1.2. *Equation of the quartic surface Z_7 and realization space of M_7.* Consider Z_7, the quartic surface in $\mathbb{P}^3$ given by the equation

$$y_1^2y_2^2+y_1^2y_2y_3-y_1y_2^2y_3-y_1y_2y_3^2-y_1^2y_2y_4-y_1y_2^2y_4+y_1y_2y_3y_4-y_2y_3^2y_4+y_1y_2y_4^2+y_3^2y_4^2=0. \tag{3.2}$$

The eight singularities of Z_7 are of type $4A_1+A_2+3A_3$, at the points respectively

$$s_1=(0:0:0:1), s_2=(1:0:0:1), s_3=(0:0:1:0), s_4=(1:0:1:0),$$
$$s_5=(0:1:0:0), s_6=(0:1:0:1), s_7=(1:-1:1:0), s_8=(1:0:0:0).$$

The minimal desingularization of Z_7 is a K3 surface which we denote by Z_7^s. Let x_1, x_2, x_3 be the coordinates on the affine chart $y_4 \neq 0$. For a generic point $x = (x_1, x_2, x_3)$ on the surface Z_7 in the chart $y_4 \neq 0$, let us define the labeled arrangement of seven lines $\mathcal{C}_0 = \mathcal{C}_0(x)$ with normal vectors the points $p_1, \dots, p_7$ respectively defined by

$$\begin{array}{c}(1:0:0),(0:1:0),(0:0:1),(-1:1:1)\\ (-x_1x_2^2 - x_1x_2x_3 + x_1x_2 - x_2x_3 + x_3 \,:\, x_1x_2 + x_1x_3 - x_1 \,:\, x_2 - 1)\\ (-x_1x_2^2 - x_1x_2x_3 + x_1x_2 - x_2x_3 + x_3 \,:\, x_1x_2 + x_1x_3 - x_1 + x_2^2\\ +x_2x_3 - 2x_2 - x_3 + 1 \,:\, x_2^2 + x_2x_3 - x_2 - x_3)\\ (-x_1x_2^2 - x_1x_2x_3 + x_1x_2 + x_3^2 \,:\, x_1x_2 + x_1x_3 - x_1 - x_2x_3\\ -x_3^2 + x_3 \,:\, x_2^2 + x_2x_3 - x_2 - x_3).\end{array} \tag{3.3}$$

Let us also define the lines arrangement $\mathcal{C}_1 = \mathcal{C}_1(x)$ with normal vectors

$$\begin{array}{c}(-x_1x_2^2 - x_1x_2x_3 + x_1x_2 + x_3^2 \,:\, x_1x_2^2 + 2x_1x_2x_3 - x_1x_2 + x_1x_3^2\\ -x_1x_3 - x_2^2x_3 - 2x_2x_3^2 + x_2x_3 - x_3^3 + x_3^2 \,:\, x_2^2 + x_2x_3 - x_2 - x_3),\\ (-x_1x_2 - x_1x_3 + x_1 \,:\, x_1x_2 + x_1x_3 - x_1 \,:\, x_2 - 1), (-x_2 : 1 : 0),\\ (-x_1x_2^3 - 2x_1x_2^2x_3 + x_1x_2^2 - x_1x_2x_3^2 + x_1x_2x_3 - x_2^2x_3 - x_2x_3^2 + x_2x_3\\ +x_3^2 \,:\, x_1x_2 + x_1x_3 - x_1 + x_2^2 + x_2x_3 - 2x_2 - x_3 + 1 \,:\, x_2^2 + x_2x_3 - x_2 - x_3),\\ (-x_1x_2^2 - x_1x_2x_3 + x_1x_2 - x_2x_3 + x_3 \,:\, 0 \,:\, x_2^2 + x_2x_3 - x_2 - x_3),\\ (-x_2^2 - x_2x_3 + x_2 + x_3 \,:\, x_1x_2 + x_1x_3 - x_1 - x_2x_3 - x_3^2 + x_3 \,:\, x_2^2 + x_2x_3\\ -x_2 - x_3), (0:1:1).\end{array} \tag{3.4}$$

A computation in `OSCAR` yields the following concrete description of the moduli space $\mathcal{R}_7 = \mathcal{R}(M_7)$.

Proposition 3. *The moduli space $\mathcal{R}_7$ is an open sub-scheme of Z_7: for $x \in \mathcal{R}_7$, the line arrangement $\mathcal{A} = \mathcal{C}_0(x) \cup \mathcal{C}_1(x)$ is a realization of M_7, and conversely any realization of M_7 is projectively equivalent to a unique such line arrangement.*

The complement of $\mathcal{R}_7$ in Z_7 is the union of 20 irreducible curves described in Section 3.2.

From the definition of the matroid M_7, if $\mathcal{A} = \mathcal{C}_0 \cup \mathcal{C}_1$ is a realization of M_7, one has $\Lambda_{\{2\},\{3\}}(\mathcal{C}_0) = \mathcal{C}_1$, but the following result on $\mathcal{C}_2 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$ is unexpected:

Theorem 4. *Let $\mathcal{A}_0 = \mathcal{C}_0 \cup \mathcal{C}_1$ be a generic realization of M_7 and define $\mathcal{C}_2 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$. The labeled line arrangement $\mathcal{A}_1 = \mathcal{C}_1 \cup \mathcal{C}_2$ is again a realization of M_7. The operator $\Lambda_{\{2\},\{3\}}$ induces a rational self-map on the schemes $\mathfrak{U}_7$ of all realizations of M_7 and its moduli space $\mathcal{R}_7$.*

We denote by $\lambda_{\{2\},\{3\}} : Z_7 \dashrightarrow Z_7$ the rational self-map on Z_7 induced by $\Lambda_{\{2\},\{3\}}$. Since $\mathcal{C}_1$ may be recovered from the relation $\Lambda_{\{2\},\{3\}}(\mathcal{C}_0) = \mathcal{C}_1$, we will often identify $\mathcal{C}_0$ and $\mathcal{A}_0$ and by abuse, speak of $\mathcal{C}_0$ as a realization of M_7.

Proof. Up to projective automorphism, one can suppose that the line arrangement $\mathcal{A}_0$ is of the form $\mathcal{A}_0 = \mathcal{C}_0(x) \cup \mathcal{C}_1(x)$ for x generic in Z_7: concretely, we use $x = (x_1, x_2, x_3)$, where $x_1, x_2, x_3 \in \mathbb{C}(Z_7)$ are considered as rational functions. A direct computation (with Magma) then shows that $\mathcal{C}_2 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$ is a line arrangement of seven lines. It has a canonical labeling as described in the previous Subsection and we then check that the matroid associated to $\mathcal{C}_1 \cup \mathcal{C}_2$ is equal to M_7, so that $\mathcal{C}_1 \cup \mathcal{C}_2$ is a realization of M_7. Using the period map, one computes $\lambda_{\{2\},\{3\}}$ and obtain that it is a dominant rational map. The reader can find the polynomials defining $\lambda_{\{2\},\{3\}}$ in an ancillary file of this paper on arXiv; it can be also retrieved from the polynomials given in Section 3.6. That describes action of $\Lambda_{\{2\},\{3\}}$ on the space of realization $\mathfrak{U}_7$ and on the moduli space $\mathcal{R}_7$. □

3.2. **The open surface $\mathcal{R}_7$ inside Z_7.** The scheme $Z_7 \setminus \mathcal{R}_7$ is the union of the following curves:

- The 12 lines

$L_1 : y_2 = y_3 = 0,$ $L_2 : y_1 = y_3 = 0,$ $L_3 : y_2 = y_4 = 0,$
$L_4 : y_1 - y_3 = y_4 = 0,$ $L_5 : y_1 = y_4 = 0,$ $L_6 : y_2 - y_4 = y_3 = 0,$
$L_7 : y_1 - y_3 - y_4 = y_2 + y_3 = 0,$ $L_8 : y_1 - y_3 = y_2 + y_3 = 0,$ $L_9 : y_2 + y_3 = y_4 = 0,$
$L_{10} : y_1 - y_4 = y_3 = 0,$ $L_{11} : y_1 - y_3 = y_2 - y_4 = 0,$ $L_{12} : y_1 = y_2 - y_4 = 0.$

These lines are also the lines contained in the quartic surface Z_7 that contain at least two double points of Z_7.
- The conic C_o defined by $y_1y_3 - y_3^2 - y_1y_4 = y_2 + y_3 - y_4 = 0$.
- Seven curves $E_1, \dots, E_7$ of geometric genus one. For example, one of these curves is given by
$$y_1^2 - 2y_1y_3 + y_3^2 - y_1y_4 = y_2^2 + y_2y_3 + y_1y_4 - y_3y_4 - y_4^2 = 0.$$

The j-invariant of the normalizations of the curves E_i is to $-5^6/28$. The elliptic curve with this j-invariant is known as the modular curve $X_1(14)$ parametrizing pairs (E, t) where E is an elliptic curve and t is an order 14 torsion element of E. For a generic point p on the curves $E_1, \dots, E_7$, the line arrangement $\mathcal{C}_0(p)$ with normal vectors as in (3.3) is well-defined. The line arrangement $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$ has seven lines, but its singularities are $t_2 = 6, t_3 = 5$, and one has $\Lambda_{\{2\},\{3\}}(\mathcal{C}_1) = \emptyset$. Moreover, the singularities of $\mathcal{C}_0 \cup \mathcal{C}_1$ are $t_2 = 13, t_3 = 26$.

The image of the curves $C_o, E_1, \dots, E_7$ under the map $\lambda_{\{2\},\{3\}}$ are lines L_k; when defined, the image of the lines L_k are lines $L_{k'}$ or points.

3.3. **The degree of $\lambda_{\{2\},\{3\}}$.** Recall that $\lambda_{\{2\},\{3\}} : Z \dashrightarrow Z$ denotes the action of the operator $\Lambda_{\{2\},\{3\}}$ on the K3 surface Z_7. One has:

Theorem 5. *The operator $\lambda_{\{2\},\{3\}}$ acts on Z_7 as a degree 4 rational self-map.*

In order to prove Theorem 5, let us describe the period map:
Let $\ell_1, ..., \ell_7$ be the lines of $\mathcal{C}_0$ with normal vectors as in Equation (3.3). Let us denote by $p_{i,j}$ the intersection point of the lines ℓ_i and ℓ_j. The point $p_{5,7}$ is $(1 : x_2 : x_3)$, so that one may recover x_2, x_3 from the knowledge of that point. Also the point $p_{1,7}$ is

$$(0 : -x_1{x_2}^2 - x_1x_2x_3 + x_1x_2 + {x_2}^2 - x_2 : x_2x_3 + {x_3}^2 - x_3), \tag{3.5}$$

this is linear in x_1, so that from the knowledge of $p_{5,7}$ and $p_{1,7}$, one may recover the point $(x_1, x_2, x_3) \in Z_7$.

Proof of Theorem 5. Let $A \in PGL_3(\mathbb{C})$ be the projective transformation that sends the first four lines of $\mathcal{C}_1$ to the four lines having the the same normal vectors as the one of $\mathcal{C}_0$. Let $\mathcal{C}_1' = (\ell_1', \dots, \ell_7')$ be the image of $\mathcal{C}_1$ by A. Using the period map, one can determine the points $p_{5,7}'$ and $p_{1,7}'$ and we obtain a point $x' = (x_1', x_2', x_3')$ (in the function field of Z_7). The line arrangements $\mathcal{C}_0(x_1', x_2', x_3')$ and $\mathcal{C}_1'$ are equal, and the action of $\Lambda_{\{2\},\{3\}}$ on Z_7 is through the map
$$\lambda_{\{2\},\{3\}} : (x_1, x_2, x_3) \to (x_1', x_2', x_3').$$
The rational self-map $\lambda_{\{2\},\{3\}} : Z_7 \dashrightarrow Z_7$ is studied in Section 3.6.

Let us compute the degree of $\lambda_{\{2\},\{3\}}$; we apply the method from [17]. Let $f(x_1, x_2, x_3)$ be the equation of the quartic Z_7 in the chart $U_4 : y_4 \neq 0$. The space of global non-vanishing differential 2-forms is generated by a form ω, which one can choose so that on an open set of U_4 one has:
$$\omega = \frac{dx_2 \wedge dx_3}{\partial f / \partial x_1}.$$
The rational self-map $\lambda_{\{2\},\{3\}}$ preserves U_4, and by a direct computation one obtains that
$$\lambda_{\{2\},\{3\}}^* \omega = -2\omega.$$

The above expression shows that when applying $\lambda_{\{2\},\{3\}}$, the volume form $\omega\bar{\omega}$ is multiplied by 4, which gives the degree of $\lambda_{\{2\},\{3\}}$. □

3.4. **Action of** $\mathrm{Aut}(M_7)$ **on the K3 surface** Z_7^s**.** The automorphism group of M_7 is generated by the order 7 and 6 permutations

$$\sigma_1 = (1,7,4,3,6,5,2)(8,14,11,10,13,12,9) \text{ and } \sigma_2 = (1,3,5,6,7,2)(8,10,12,13,14,9).$$

This group is isomorphic to the the Frobenius group $F_7 = \mathbb{Z}/6\mathbb{Z} \rtimes \mathbb{Z}/7\mathbb{Z}$. These automorphisms act on the K3 surface Z.

Proposition 6. *The action of* $\mathrm{Aut}(M_7)$ *on* Z_7 *is faithful.*

Proof. As in the proof of Theorem 5, let $\mathcal{C}_0 = \mathcal{C}_0(x_1, x_2, x_3)$ be the generic line arrangement in $\mathcal{R}_7$, where $x_1, x_2, x_3 \in \mathbb{C}(Z_7)$ are considered as rational functions.

For $\sigma \in \mathrm{Aut}(M_7)$, let $\mathcal{C}_0^\sigma$ be the image of $\mathcal{C}_0$ under the action of σ (that is just the permutation of the lines under σ). We apply the period map to the line arrangement $\mathcal{C}_0^\sigma$, where $\mathcal{C}_0 = \mathcal{C}_0(x)$. Using the period map, we obtain the point $\sigma(x) = (x_1', x_2', x_3')$ which is a zero of the equation of Z_7 and such that $\mathcal{C}_0(\sigma(x))$ is projectively equivalent to $\mathcal{C}_0^\sigma$.

When $\sigma = \sigma_1$, the automorphism σ_1 acts on Z_7 through the map in $\mathbb{P}^3$ given by the ring homomorphism which to (y_1, y_2, y_3, y_4) associates

$$\begin{gathered}(y_1y_2^2y_3 + y_1y_2y_3^2 - y_2^2y_3^2 - y_2y_3^3 - y_1y_2y_3y_4 - y_2^2y_3y_4 + y_2y_3y_4^2 + y_3^2y_4^2,\\ y_1y_2^3 + y_1y_2^2y_3 + y_2^2y_3^2 + y_2y_3^3 - 2y_1y_2^2y_4 - y_1y_2y_3y_4 - 2y_2y_3^2y_4 - y_3^3y_4 + y_1y_2y_4^2 + y_3^2y_4^2,\\ y_1y_2^2y_3 + y_1y_2y_3^2 - y_2^2y_3^2 - y_2y_3^3 - y_1y_2y_3y_4 + y_2y_3^2y_4,\\ y_2^3y_3 + 2y_2^2y_3^2 + y_2y_3^3 - 2y_2^2y_3y_4 - 3y_2y_3^2y_4 - y_3^3y_4 + y_2y_3y_4^2 + y_3^2y_4^2).\end{gathered}$$

For σ_2, we obtain that it acts on the surface Z_7 through the map which to (y_1, y_2, y_3, y_4) associates

$$\begin{gathered}(-y_2^2y_3 - y_2y_3^2 + y_2y_3y_4, -y_1y_2y_3 + y_2y_3^2 + y_2y_3y_4 - y_3y_4^2,\\ y_1y_2^2 + y_1y_2y_3 - y_2^2y_3 - y_2y_3^2 - y_1y_2y_4 + y_2y_3y_4, y_2y_3y_4 - y_3y_4^2);\end{gathered}$$

thas map is a birational transformation of $\mathbb{P}^3$. In order to check that the action of $\mathrm{Aut}(M_7)$ is faithful on Z_7, it is then enough to check that the orbit of one point (for example the point $(-6 : -25/8 : 5 : 1)$ in Z_7) has 42 elements, which is a direct computation.

The fixed points under the order seven element σ_1 are the singularities s_5, s_7, s_8; (there is a unique conjugacy class of elements of order 7 in F_7).

The fixed points locus of σ_2 and the order 3 automorphism σ_2^2 acting on Z_7 are:

(i) The A_3 singularity $(0 : 1 : 0 : 1)$,
(ii) The four points $p = (r^2+1 : r^2-r+2 : r : 1)$ where r is any complex root of $X^4 - X^3 + 3X^2 - X + 1$. These points are in $Z_7 \setminus \mathcal{R}_7$; they are periodic of period 2 for the rational self-map $\lambda_{\{2\},\{3\}}$, moreover the (unlabeled) line arrangements $\mathcal{C}_0(p), \mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0), \mathcal{C}_2 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$ have $7, 10$ and 37 lines respectively. It seems likely that the number of lines of the sequence $\mathcal{C}_{n+1} = \Lambda_{\{2\},\{3\}}(\mathcal{C}_n)$ goes to infinity.
(iii) The points $(w+1 : -w : w : 1)$ where $w^2+w+1 = 0$, which are fixed by the rational self-map $\lambda_{\{2\},\{3\}}$; these two points are in $Z_7 \setminus \mathcal{R}_7$.

The fixed-point locus of the involution σ_2^3 acting on Z_7 is the union of the line L_7 and a curve E_j, which is in $Z_7 \setminus \mathcal{R}_7$ (see Section 3.2). There is a unique conjugacy class of involutions in F_7, so that similarly, any involution from $\mathrm{Aut}(M_7)$ fixes a curve and a line. □

There is an open set in the quotient surface $Z_7/\mathrm{Aut}(M_7)$ which parametrizes unlabeled line arrangements $\mathcal{C}_0^o$ associated to $\mathcal{C}_0$ in $\mathcal{R}_7$. One has:

Corollary 7. *The surface* $Z_7/\mathrm{Aut}(M_7)$ *is rational.*

Proof. Since an involution of $\mathrm{Aut}(M_7)$ fixes a one dimensional curve, it is non-symplectic (see [5]), thus $Z_7/\mathrm{Aut}(M_7)$ is rational. □

For a labeled line arrangement $\mathcal{C}_0 = (\ell_1, \dots, \ell_k) \in \mathfrak{U}_7$ and $j \in \{1, \dots, 7\}$, let us denote by $H_j(\mathcal{C}_0)$ the line arrangement $H_j = \sum_{k \neq j} \ell_k$. The labeled line arrangement $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$ is

$$\mathcal{C}_1 = (\Lambda_{\{2\},\{3\}}(H_1), \dots, \Lambda_{\{2\},\{3\}}(H_7)).$$

An element $\sigma \in \mathrm{Aut}(M_7)$ permutes the lines of $\mathcal{C}_0$: it can also be seen as a permutation of $\{1, \dots, 7\}$. The $\sigma(j)^{th}$ line of $\sigma.\mathcal{C}_1$ is $\Lambda_{\{2\},\{3\}}(H_{\sigma(j)}(\mathcal{C}_0))$. Since

$$H_{\sigma(j)}(\mathcal{C}_0) = \sum_{k \neq j} \ell_{\sigma(k)} = H_j(\sigma.\mathcal{C}_0),$$

the $\sigma(j)^{th}$ line of $\sigma.\mathcal{C}_1$ is $\Lambda_{\{2\},\{3\}}(H_j(\sigma.\mathcal{C}_0))$. Thus

$$\sigma.\Lambda_{\{2\},\{3\}}(\mathcal{C}_0) = (\Lambda_{\{2\},\{3\}}(H_{\sigma(1)}), \dots, \Lambda_{\{2\},\{3\}}(H_{\sigma(7)})) = \Lambda_{\{2\},\{3\}}(\sigma.\mathcal{C}_0)$$

and we obtain that:

Proposition 8. *The action of* $\mathrm{Aut}(M_7)$ *commutes with the action of* $\Lambda_{\{2\},\{3\}}$, *that is for all* $\sigma \in \mathrm{Aut}(M_7)$ *it holds that*

$$\Lambda_{\{2\},\{3\}} \circ \sigma = \sigma \circ \Lambda_{\{2\},\{3\}}.$$

Remark 9. The group $\mathrm{Aut}(M_7)$ acts faithfully on the surface $Z_7 \subset \mathbb{P}^3$, but does not extend canonically to a well-defined action on the ambient space $\mathbb{P}^3$. For example, the action of σ_2 we computed is the restriction of an order 6 birational map $\tilde{\sigma_2}$ of $\mathbb{P}^3$; in particular $\tilde{\sigma_2}^3$ is a birational involution of $\mathbb{P}^3$ defined by degree 5 coprime polynomials. If instead one starts with $\sigma_2^3 = (1,6)(2,5)(3,7)(8,13)(9,12)(10,14)$ and computes the action of $\widetilde{\sigma_2^3}$ on Z_7 as we did above for σ_1 and σ_2, one obtains that, surprisingly, the defining coprime polynomials of the rational map $\widetilde{\sigma_2^3} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ have degree 4, although the maps $\widetilde{\sigma_2^3}$ and $\tilde{\sigma_2}^3$ have the same effect on Z_7. Moreover although $(\widetilde{\sigma_2^3})^2$ is the identity on the surface Z_7, it is not the identity on $\mathbb{P}^3$ (it is defined by degree 6 coprime polynomials). Moreover, one can compute that the rational map $(\widetilde{\sigma_2^3})^4$ is defined by degree 21 coprime polynomials.

3.5. **Fibration preserved by $\lambda_{\{2\},\{3\}}$ and the elliptic modular surface $\Xi_1(7)$.** The line L_6 : $y_2 - y_4 = y_3 = 0$ is contained in the surface Z_7. Let $\gamma : Z_7 \to \mathbb{P}^1$ be the elliptic fibration induced by the projection from that line. One obtains a smooth cubic affine model A in $\mathbb{A}^2_{\mathbb{Q}(t)} = \mathbb{A}^2_{\mathbb{Q}(t)}(x,y)$ of that elliptic fibration by substituting $(x, 1+ty, y, 1)$ in the equation of Z_7. A computation shows that $Z_7^s \to \mathbb{P}^1$ is (isomorphic to) the elliptic surface Y associated to the elliptic curve $E_{/\mathbb{Q}(t)}$ with Weierstrass model

$$E : y^2 = x^3 + \tfrac{(t^4-2t^3+3t^2+6t+1)}{(t+1)^2}x^2 + \tfrac{8t^3(t^2-t-1)}{(t+1)^3}x + 16\tfrac{t^6}{(t+1)^4}.$$

The map between A and Y sends $(0,0)$ to the zero section. The elliptic fibration $Y \to \mathbb{P}^1$ has singular fibers $3I_7 + 3I_1$ at the points

$$\infty, 0, -1, t^3 - 5t^2 - 8t - 1 = 0,$$

respectively.

We recall that the curve $X_1(7)$ parametrizes (up to isomorphisms) the pairs (E,p) where E is an elliptic curve and p is a torsion point of order 7 on E. A Weierstrass model E' of the elliptic modular surface $\Xi_1(7)$ over the curve $X_1(7) \simeq \mathbb{P}^1$ is computed in [15]. The j-invariant maps $j_E(t), j_{E'}(t) \in \mathbb{Q}$ of E and E' are related by the equality $j_{E'}(t) = j_E(-\frac{1}{t})$, which shows that E is isomorphic to E' and Z_7^s is isomorphic to the elliptic modular surface $X_1(7)$.

The Mordell–Weil group of E is isomorphic to $\mathbb{Z}/7\mathbb{Z}$; it is generated by the point

$$p_t = (0 : 4t^3 : (t+1)^2) \in E.$$

We thus obtained the first part of the following theorem:

Theorem 10. *a) The K3 surface Z_7^s is isomorphic to the modular elliptic surface $\Xi_1(7)$.*
b) The rational self-map $\lambda_{\{2\},\{3\}}$ preserves the elliptic fibration $\gamma : Z_7 \to \mathbb{P}^1$ and acts on the base curve $\mathbb{P}^1$ through the order 3 map $t \to -1/(t+1)$. There exists an automorphism σ_0 coming from $\mathrm{Aut}(M_7)$ such that $\sigma_0\lambda_{\{2\},\{3\}}$ preserves the fibration γ and acts on E as the multiplication by 2 map.

The last property implies that the operator $\Lambda_{\{2\},\{3\}}$ preserves the moduli interpretation of $X_1(7)$.

Proof. Using the period map and the function field of A, one computes that the action of $\lambda_{\{2\},\{3\}}$ on the base $\mathbb{P}^1$ of the fibration $A \to \mathbb{P}^1$ is through the map $t \to -1/(t+1)$.

An automorphism $\sigma \in \mathrm{Aut}(M_7)$ acts on the surface $Z_7 \cap \{y_4 \neq 0\}$ and on the affine model A. Using the period map and again the generic point of A, one computes the action of the rational self-maps $\sigma\lambda_{\{2\},\{3\}}$ ($\sigma \in \mathrm{Aut}(M_7)$) on A. For 14 of these maps, the action on the base curve $\mathbb{P}^1$ is trivial. This is the case for example for

$$\sigma_0 = (1,2,4)(3,6,7)(8,9,11)(10,13,14).$$

The map $\mu = \sigma_0\lambda_{\{2\},\{3\}}$ also acts on E, one can thus compute its action on the generic point of E. Knowing that action, we are now able to compute the pull-back of a non-zero holomorphic one-form ω by μ, which is: $\mu^*\omega = 2\omega$. Using the seven torsion points, one computes that μ fixes the origin, thus $\mu = [2]$. □

Among the 12 lines contained in Z_7, in the complement of $\mathcal{R}_7$, eight are contained in the singular fibers of the fibration γ, and 4 are sections.

Using the pull-back to Z_7^s of the lines contained in Z_7 and the (-2)-curves of the desingularization, one may compute the Néron–Severi lattice of Z_7^s, and obtain that it has discriminant -7 and rank 20. The modular elliptic surface $\Xi_1(7)$ is well-known and studied; it is known as the unique K3 surface with Néron–Severi lattice of rank 20 and discriminant -7: in obtain in that way another proof that Z^s is isomorphic to $\Xi_1(7)$. The inequivalent fibrations of $\Xi_1(7)$ have been classified (see [8]). Another remarkable fact is that $\Xi_1(7)$ is a ball-quotient surface: there exists a co-compact lattice Γ in the automorphism group of the unit ball $\mathbb{B}_2$ such that $\Xi_1(7) \simeq \mathbb{B}_2/\Gamma$ [9]. The automorphism group of $\Xi_1(7)$ is studied in [16].

3.6. **The K3 surface Z_7^s is semi-conjugated to the plane.** The rational self-map $\lambda_{\{2\},\{3\}}$ acting on the quartic $Z_7 \hookrightarrow \mathbb{P}^3(y_1,\dots,y_4)$ is defined by

$$\lambda_{\{2\},\{3\}} = (P_1 : \cdots : P_4),$$

where $P_1,\dots,P_4$ are four homogeneous degree 11 polynomials computed via the period map. These polynomials are given in the ancillary file in the arXiv version of this paper; they also may be obtained from the polynomials Q_1, Q_2, Q_3 and R below. A remarkable fact about the polynomials $P_1,\dots,P_4$ is that

$$\deg_{y_1}(P_1) = 1,\ \deg_{y_1}(P_2) = \deg_{y_1}(P_3) = \deg_{y_1}(P_4) = 0, \tag{3.6}$$

where $\deg_{y_1}$ denote the degree relative to the variable y_1.

Let us define the polynomials $\tilde{P}_k = P_{k+1}(0, z_1, z_2, z_3)$ for $k \in \{1,2,3\}$ (where z_1, z_2, z_3 are the three coordinates on the plane $\mathbb{P}^2 : y_1 = 0$). The polynomials $\tilde{P}_k$, $k \in \{1,2,3\}$ define a rational self-map $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$; the base locus of the linear system generated by $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ is the quintic curve B defined by

$$Q = z_1^3 z_2^2 + 2z_1^2 z_2^3 + z_1 z_2^4 + 2z_1^3 z_2 z_3 + 4z_1^2 z_2^2 z_3 + 2z_1 z_2^3 z_3 + z_1^3 z_3^2 \\ -4z_1^2 z_2 z_3^2 - 9z_1 z_2^2 z_3^2 - 4z_2^3 z_3^2 - 2z_1^2 z_3^3 + 2z_1 z_2 z_3^3 + 4z_2^2 z_3^3 + z_1 z_3^4.$$

That curve is irreducible, has geometric genus 1 and its normalization has j-invariant $-5^6/28$. By removing the base locus B, one obtains that the rational self-map F is defined by the following degree 6 polynomials

$$\begin{aligned} Q_1 =& z_1 Q, \\ Q_2 =& - z_1^5 z_2 - 3z_1^4 z_2^2 - 3z_1^3 z_2^3 - z_1^2 z_2^4 + z_1^4 z_2 z_3 + 2z_1^3 z_2^2 z_3 + z_1^2 z_2^3 z_3 \\ &+ z_1^3 z_2 z_3^2 + 2z_1^2 z_2^2 z_3^2 + z_1 z_2^3 z_3^2 - z_1^2 z_2 z_3^3 + z_2^3 z_3^3 - z_2^2 z_3^4, \\ Q_3 =& 2z_1^4 z_2 z_3 + 4z_1^3 z_2^2 z_3 + 2z_1^2 z_2^3 z_3 + z_1^4 z_3^2 - 4z_1^3 z_2 z_3^2 - 8z_1^2 z_2^2 z_3^2 - 3z_1 z_2^3 z_3^2 \\ &- 2z_1^3 z_3^3 + 2z_1^2 z_2 z_3^3 + 4z_1 z_2^2 z_3^3 + z_2^3 z_3^3 + z_1^2 z_3^4, \end{aligned}$$

and the indeterminacy locus of $F = (Q_1 : Q_2 : Q_3)$ are the 8 points

$$q_1 = (0:0:1), q_2 = (1:0:1), q_3 = (0:1:0), q_4 = (-1:1:0), q_5 = (1:0:0), \; q_r = (-r^2 + 2r : r : 1)$$

where r is any root of $X^3 - 4X^2 + 3X + 1$ (the field $\mathbb{Q}(r)$ is the degree 3 real subfield of $\mathbb{Q}(\zeta_7)$).

Let us define the projection map $\pi_1 : Z_7 \to \mathbb{P}^2(z_1, z_2, z_3)$ from the point $s_8 : y_2 = y_3 = y_4 = 0$ contained in surface Z_7. This point is an A_3 singularity on Z_7, in particular it has multiplicity 2, thus the map π_1 from the quartic to the plane has degree 2. One has:

Lemma 11. *The branch loci of π_1 is the union of the quintic curve $B = \{Q = 0\}$ and the line $L : z_1 = 0$.*

Proof. The ramification locus of π_1 is the discriminant of the equation of Z_7 (given in (3.2)) with respect to the variable y_1. The image of the ramification curve by π_1 is the curve $B + L$. □

The curve B has singularities of type A_4, A_4, A_2 at the points q_2, q_4, q_5, respectively. The union $L + B$ has singularities of type $A_1, A_3, A_4, A_3, A_4, A_2$ at the points $q_0 = (0:1:1), q_1, q_2, q_3, q_4, q_5$, respectively.

A direct computation shows that

$$Q_1(Q_1, Q_2, Q_3) = Q_1 R^2$$

for $R = \frac{1}{8} z_2^2 (z_1 - z_3)^2 R_4 R_7$, where

$$\begin{aligned} R_4 =& z_1^4 + 2z_1^3 z_2 + z_1^2 z_2^2 - z_1^2 z_3^2 - z_1 z_2 z_3^2 - z_2 z_3^3, \\ R_7 =& z_1^6 z_2 + 4z_1^5 z_2^2 + 6z_1^4 z_2^3 + 4z_1^3 z_2^4 + z_1^2 z_2^5 + z_1^6 z_3 - 7z_1^4 z_2^2 z_3 - 11z_1^3 z_2^3 z_3 - 6z_1^2 z_2^4 z_3 - z_1 z_2^5 z_3 \\ &- z_1^5 z_3^2 + 3z_1^3 z_2^2 z_3^2 + 2z_1^2 z_2^3 z_3^2 + 3z_1^2 z_2^2 z_3^3 + 5z_1 z_2^3 z_3^3 + 2z_2^4 z_3^3 - 2z_1 z_2^2 z_3^4 - 2z_2^3 z_3^4 - z_1 z_2 z_3^5. \end{aligned}$$

The images of the curves $z_2 = 0$, $z_1 - z_3 = 0$ and $R_4 = 0$ by the rational self-map $F = (Q_1 : Q_2 : Q_3)$ of $\mathbb{P}^2$ are the indeterminacy points q_2, q_4, q_2, respectively. The image of the curve $R_7 = 0$ under the map F is the quintic curve B. The image of the quintic curve B under the map F is the line $L : z_1 = 0$. The rational map F preserves L and the action of F on L is through the map $(z_2 : z_3) \to (z_2 - z_3 : z_3)$.

From the above description and Subsection 2.3, the surface Z_7^s is the minimal desingularization of the double cover

$$X : \{y^2 = Q_1(z_1, z_2, z_3\} \hookrightarrow \mathbb{P}(3, 1, 1, 1)$$

branched over $L + B$. The birational map between X and Z_7 is given by the equalities $y_{i+1} = z_i$ for $i \in \{1, 2, 3\}$ and

$$y_1 = \frac{1}{2}(y + z_2^2 z_3 + z_2 z_3^2 + z_2^2 z_4 - z_2 z_3 z_4 - z_2 z_4^2)/(z_2^2 + z_2 z_3 - z_2 z_4).$$

We continue to denote by $\lambda_{\{2\},\{3\}}$ the rational self-map

$$(y; z) \to (yR(z); F(z)).$$

Applying the results of Section 2.3, we obtain that:

Theorem 12. *The dynamical system* $(Z_7^s, \lambda_{\{2\},\{3\}})$ *is semi-conjugated to* $(\mathbb{P}^2, F)$.

Remark 13. a) The degrees of the coprime polynomials defining the rational maps F, F^2, F^3 are $6, 21, 82$, respectively. // b) It would be interesting to construct rational self-maps on some other degree two K3 surfaces.

4. The Octagon and the operator $\Lambda_{\{2\},\{3,4\}}$

4.1. **The matroid M_8 constructed from the regular octagon.** Consider the 16 lines in Figure 4.1: the black lines $\ell_1, \dots, \ell_8$ are the 8 lines of the regular octagon $\mathcal{C}_1$ and the blue lines $\ell'_1, \dots, \ell'_8$ are the 8 lines symmetries of $\mathcal{C}_0$. The image of $\mathcal{C}_0 = \ell_1 + \dots + \ell_8$ by the operator $\Lambda_{\{2\},\{3,4\}}$ is the line arrangement $\mathcal{C}_1 = \ell'_1 + \dots + \ell'_8$.

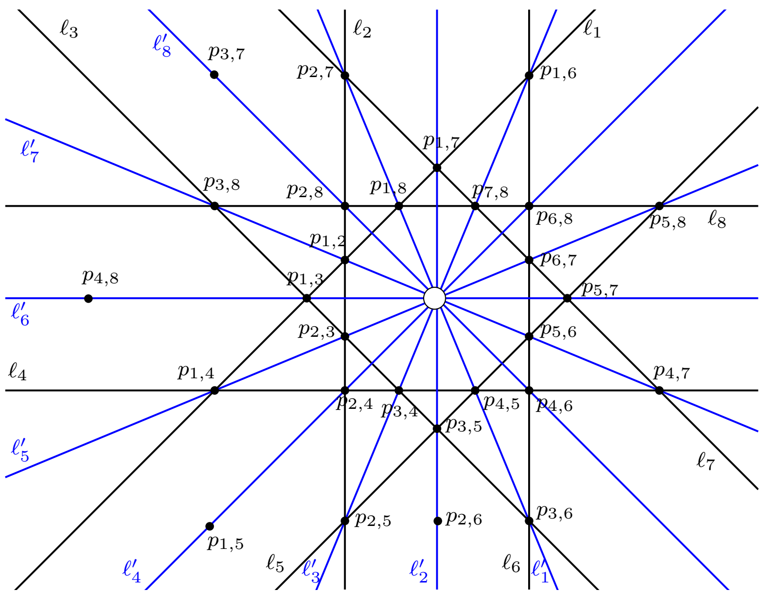

Figure 4.1. The regular Octagon and its axes of symmetries.

The 8 lines ℓ_i, ℓ_j of $\mathcal{C}_0 = (\ell_1, \dots \ell_8)$ meet in 28 double points denoted by $p_{i,j}$ (some points are at infinity). The lines $\ell'_1, \dots, \ell'_8$ are the lines containing the points in sets $S_1, \dots, S_8$ which are respectively

$$\begin{array}{c}\{p_{1,8}, p_{2,7}, p_{3,6}, p_{4,5}\}, \{p_{1,7}, p_{2,6}, p_{3,5}\}, \{p_{1,6}, p_{2,5}, p_{3,4}, p_{7,8}\}, \{p_{1,5}, p_{2,4}, p_{6,8}\}, \\ \{p_{1,4}, p_{2,3}, p_{5,8}, p_{6,7}\}, \{p_{1,3}, p_{4,8}, p_{5,7}\}, \{p_{1,2}, p_{3,8}, p_{4,7}, p_{5,6}\}, \{p_{2,8}, p_{3,7}, p_{4,6}\}.\end{array} \tag{4.1}$$

These sets S_k, $k = 1, \dots, 8$ form a partition of the 28 double points of $\mathcal{C}_0$; these 28 points are the triple points of $\mathcal{C}_0 \cup \mathcal{C}_1$. One has the relation $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0) = \mathcal{C}_1$ (as unlabeled line arrangements).

Let M_8 be the matroid associated to the incidences between the 16 labeled lines $\ell_1, \dots, \ell_8, \ell'_1, \dots, \ell'_8$ and the 28 triple points: it is obtained from the matroid associated to the labeled line arrangement

$\mathcal{C}_0 \cup \mathcal{C}_1$, but we discard all non-bases coming from the central point, so that M_8 has 16 atoms and only 28 non-bases. We denote by $\mathcal{R}_8$ the moduli space of realizations of M_8 (over $\mathbb{C}$).

Remark 14. A priori, there is no canonical choice for the labelings of the lines $\ell'_1, \dots, \ell'_8$ in the unlabeled line arrangement $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)$. The choice we made in Equation (4.1) will be justified later, see Remark 19.

4.2. **The moduli space $\mathcal{R}_8$ of M_8.** A direct computation in `OSCAR` shows that the moduli space $\mathcal{R}_8$ is two-dimensional, and an open sub-set of the quartic surface Z_8 in $\mathbb{P}^3$ with the equation

$$y_1y_2^2y_3 - y_1^2y_2y_4 + y_1y_2^2y_4 + y_1^2y_3y_4 - 2y_1y_2y_3y_4 - y_1y_3^2y_4 + y_1y_3y_4^2 - y_2y_3y_4^2 + y_3^2y_4^2 = 0.$$

The surface Z_8 has singularities $A_2, A_2, A_3, A_4, A_3, A_1$ at the respective points

$$(1:0:0:0):(0:1:0:0):(0:0:1:0):(0:0:0:1):(1:1:1:1):(1:0:1:0).$$

Its minimal desingularization Z_8^s is a K3 surface. The realization $\mathcal{A}(x)$ corresponding to a generic point $x = (x_1, x_2, x_3)$ of Z_8 in the affine chart $\mathbb{A}^3 = \{y_4 \neq 0\}$ is the union $\mathcal{A}(x) = \mathcal{C}_0(x) \cup \mathcal{C}_1(x)$, where $\mathcal{C}_0(x)$ is the line arrangement with eight lines with normal vectors the four vectors of the canonical basis and the following four vectors

$$\begin{gathered}(x_1 - x_2 : x_1^2 - x_1x_2 - x_1x_3 + x_1 - x_2 + x_3 : x_1 - x_2x_3 - x_2 + x_3),\\ (x_1x_2 - x_1x_3 - x_2 + x_3 : x_1x_2^2 - x_1x_2 - x_1x_3 + x_1 - x_2 + x_3 : x_1x_2x_3 - 2x_1x_3 + x_1 - x_2 + x_3)\\ (x_1 - 1 : x_1x_2 - x_2 : x_1 - x_2), (1 : x_1 : x_3).\end{gathered}$$

Moreover, $\mathcal{C}_1(x)$ is the line arrangement with normal vectors

$$\begin{gathered}(x_1x_2 - x_1x_3 - x_2 + x_3 : x_1x_2^2 - x_1x_2 - x_1x_3 + x_1 - x_2 + x_3 : x_1x_2 - x_1x_3 - x_2^2 + x_2x_3),\\ (x_1^2x_2 - x_1^2x_3 - x_1x_2^2 + x_1x_2x_3 - x_1x_2 + x_1x_3 + x_2^2 - x_2x_3 : x_1^3x_2 - x_1^3x_3 - x_1^2x_2^2 + x_1^2x_3^2\\ +2x_1x_2x_3 - x_1x_2 - 2x_1x_3^2 + x_1x_3 + x_2^2 - 2x_2x_3 + x_3^2 : x_1^2x_2x_3 - x_1^2x_2 - x_1^2\\ x_3 + x_1^2 + x_1x_2^2 - 2x_1x_2 - x_1x_3^2 + 2x_1x_3 + x_2^2 - 2x_2x_3 + x_3^2),\\ (x_1 - x_2 : x_1 - x_2 : x_1 - x_2x_3 - x_2 + x_3), (x_3 : x_1x_2 : x_3), (0 : 1 : 1),\\ (x_1 - 1 : 0 : x_1 - x_3), (1 : x_1 : 0), (1 : x_2 : x_3).\end{gathered}$$

From the definition of the matroid M_8, if $\mathcal{A}_0 = \mathcal{C}_0 \cup \mathcal{C}_1$ is a realization of M_8 and $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) denotes its first (resp. last) eight lines then, $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0) = \mathcal{C}_1$ as unlabeled line arrangements. The following operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$ gives a labeling to $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)$:

Definition 15. The operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$ associates to a labeled line arrangement L_8 of 8 lines $\ell_1, \dots, \ell_8$, the labeled line arrangement $\ell'_1, \dots, \ell'_8$ where ℓ'_j is the set of lines containing all the points in S_k defined in (4.1) (ℓ'_j is a line or the empty set).

For a generic arrangement L_8 of eight lines, one has $\Lambda^{\ell}_{\{2\},\{3,4\}}(L_8) = \emptyset$. The operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$ is constructed so that if $\mathcal{A}_0$ is any realization of M_8 and $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) denotes its first (resp. last) eight lines then $\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_0) = \mathcal{C}_1$ as labeled line arrangements (and of course $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0) = \mathcal{C}_1$ if one forgets the labels).

4.3. **The operator $\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$ acts as a rational self-map on $\mathcal{R}_8$.** A priori the line arrangement $\Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$ could be empty, however:

Theorem 16. *Suppose that $\mathcal{A}_0 = \mathcal{C}_0 \cup \mathcal{C}_1$ is generic among the realizations of M_8. Then the labeled line arrangement $\mathcal{C}_2 = \Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$ has 8 lines and $\mathcal{A}_1 = \mathcal{C}_1 \cup \mathcal{C}_2$ is a realization of $\mathcal{R}_8$.*

Proof. Using the function field of $\mathcal{R}_8$, we realize the generic element of $\mathcal{R}_8$ using the formulas for $\mathcal{A}(x)$. Then we compute $\mathcal{C}_2 = \Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$ and obtain eight lines. Finally we check that $\mathcal{C}_1 \cup \mathcal{C}_2$ defines the same matroid as $\mathcal{A}_0$. ∎

The operator $\Lambda^{\ell}_{\{2\},\{3,4\}}$ acts on realizations of M_8, sending $\mathcal{A}_0 = \mathcal{C}_0 \cup \mathcal{C}_1$ to $\mathcal{A}_1 = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_2 = \Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$. It therefore acts on the moduli space Z_8: we denote by

$$\lambda_{\{2\},\{3,4\}} : Z_8 \dashrightarrow Z_8$$

that action. In order to obtain the explicit polynomials defining $\lambda_{\{2\},\{3,4\}}$, we remark that one may recover the coordinates x_1, x_2, x_3 of the line arrangement $\mathcal{A}_0(x)$ from the two last normal vectors $(1 : x_1 : 0), (1 : x_2 : x_3)$ of $\mathcal{C}_1(x)$. Then one computes the unique line arrangement $\tilde{\mathcal{C}_1} \cup \tilde{\mathcal{C}_2}$ projectively equivalent to $\mathcal{A}_1(x) = \mathcal{C}_1(x) \cup \mathcal{C}_2(x)$ such that the first four normal vectors are the canonical basis. The image of x by $\lambda_{\{2\},\{3,4\}}$ is the point $x' = (x'_1, x'_2, x'_3)$ such that the two last normal vectors of $\tilde{\mathcal{C}_2}$ are $(1 : x'_1 : 0), (1 : x'_2 : x'_3)$ (and $\tilde{\mathcal{C}_1} \cup \tilde{\mathcal{C}_2} = \mathcal{A}_0(x')$). Taking the homogenization to $\mathbb{P}^3$, one obtains that the map $\lambda_{\{2\},\{3,4\}}$ is defined by the four degree 10 coprime polynomials $P_1, \dots, P_4$ given in the ancillary file of the arXiv version of this paper. The base points of $\lambda_{\{2\},\{3,4\}}$ are

$$(-\sqrt{2} - 1 : \sqrt{2} + 2 : 2\sqrt{2} + 3 : 1), (\sqrt{2} - 1 : -\sqrt{2} + 2 : -2\sqrt{2} + 3 : 1),$$
$$(i : 0 : 1 : 1), (-i : 0 : 1 : 1), (1 : 1 : 0 : 1), (0 : 1 : 1 : 0), (0 : 1 : 0 : 1).$$

The line arrangements $\mathcal{C}_0 \cup \mathcal{C}_1$ associated to the first two points are the regular octagon and its lines of symmetries. The line arrangements $\mathcal{C}_0$ associated to the third and fourth points are such that $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)$ is the Ceva line arrangement with 12 lines; it contains $\mathcal{C}_0$.

Using the explicit polynomials $P_1, \dots, P_4$, we obtain that:

Proposition 17. *The degree of the rational self-map $\lambda_{\{2\},\{3,4\}}$ on Z_8^s is* 4.

Proof. We again apply the method from [17]. Let $f(x_1, x_2, x_3)$ be the equation of the quartic Z_8 in the chart $U_4 : y_4 \neq 0$. The space of global non-vanishing differential 2-forms is generated by a form ω, which one can choose so that on an open set of U_4 one has: $\omega = \frac{dx_2 \wedge dx_3}{\partial f / \partial x_1}$. The rational self-map $\lambda_{\{2\},\{3,4\}}$ preserves U_4. A direct computation gives that $\lambda^*_{\{2\},\{3,4\}}\omega = -2\omega$. The pull-back by $\lambda_{\{2\},\{3,4\}}$ of the volume form $\omega\bar{\omega}$ is therefore $4\omega\bar{\omega}$, thus the degree of $\lambda_{\{2\},\{3,4\}}$ is 4. □

4.4. **The dynamical system $(Z_8, \lambda_{\{2\},\{3,4\}})$ is semi-conjugated to the plane.** The four polynomials $P_1, \dots, P_4$ such that $\lambda_{\{2\},\{3,4\}} = \lambda_{\{2\},\{3,4\}} = (P_1 : \cdots : P_4)$ verify $\deg_{y_1}(P_1) = 1$ and $\deg_{y_k}(P_1) = 0$ for $k \geq 2$. Let $\pi : Z_8 \dashrightarrow \mathbb{P}^2$ be the double cover obtained by projecting from the double point $(1 : 0 : 0 : 0)$ of Z_8.

Lemma 18. *The branch curve B of π is the union of the conic $C = \{z_1^2 - z_2 z_3 = 0\}$ and the quartic curve*

$$Q = \{z_1^2 z_2^2 + 2z_1^2 z_2 z_3 - 4z_1 z_2^2 z_3 - z_2^3 z_3 + z_1^2 z_3^2 - 4z_1 z_2 z_3^2 + 6z_2^2 z_3^2 - z_2 z_3^3\}.$$

Proof. The ramification locus of π_1 is the discriminant of the equation of Z_8 with respect to the variable y_1. The image of the ramification curve by π_1 is the curve B. □

The quartic Q has geometric genus 0 and is singular at the points $(1 : 0 : 0), (1 : 1 : 1)$ with singularities A_3 and A_1. The curve $B = C + Q$ is singular at the points

$$(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)$$

with singularities A_3, A_5, A_5, D_4, respectively.

Let us define the polynomials $Q_k = P_{k+1}(0, z_1, z_2, z_3)$ $(k = 1, 2, 3)$ and the rational self-map $\mu : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $\mu = (Q_1 : Q_2 : Q_3)$. One has $\mu^*(B) = B + 2D$ for a degree 27 curve D. Using Subsection 2.3, the double cover of $\mathbb{P}^2$ branched over $B = C + Q$ is birational to the surface Z_8 and $(Z_8, \lambda_{\{2\},\{3,4\}})$ is semi-conjugated to $(\mathbb{P}^2, \mu)$.

The indetermination points of μ are the 9 points

$$(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1), (1 : 0 : 1), (1 : 1 : 0),$$
$$(0 : 1 : 1), (-\sqrt{2} + 2 : -2\sqrt{2} + 3 : 1), (\sqrt{2} + 2 : 2\sqrt{2} + 3 : 1).$$

The image by μ of Q is the conic C; the rational map μ restricts to the identity on C.

Remark 19. The choice for the labelings of the lines in the unlabeled line arrangement $\Lambda_{\{2\},\{3,4\}}(\mathcal{C}_0)$ was made so that the defining polynomials of the rational self-map $\lambda_{\{2\},\{3,4\}}$ are of low degree. Moreover, for the other choices we tried, the degrees of the polynomials defining the analog of $\lambda_{\{2\},\{3,4\}}$ with respect to any variables y_i were never $1,0,0,0$, so that it was not possible to understand that rational self-map $\lambda_{\{2\},\{3,4\}}$ as a semi-conjugacy with the plane.

4.5. **The K3 surface Z_8 and the modular surface $\Xi_1(8)$.** One has:

Proposition 20. *The K3 surface Z_8^s is the unique K3 surface with discriminant -8 and Picard number* 20.

Proof. The eight lines with equations

$$(y_1 = y_3 = 0), (y_1 = y_4 = 0), (y_2 = y_3 = 0), (y_2 = y_4 = 0), (y_3 = y_4 = 0),$$
$$(y_1 - y_4 = y_2 - y_4 = 0), (y_1 - y_3 = y_2 - y_4 = 0), (y_2 - y_4 = y_3 - y_4 = 0)$$

are contained in the surface Z_8. Using Magma, one can compute that their strict transforms on Z_8^s together with the 15 (-2)-curves coming from the resolutions of the singularities of Z_8, generate a rank 20 lattice with discriminant -8. There is no K3 surface with Picard number 20 and discriminant -2 and there is a unique K3 surface with Picard number 20 and discriminant -8 (see e.g. [12]) which yields the conclusion. □

Proposition 21. *The surface Z_8 is (isomorphic to) the elliptic modular surface $\Xi_1(8)$ above the modular curve $X_1(8)$.*

Proof. The projection map from the line $y_2 - y_4 = y_3 - y_4 = 0$ induces a fibration $Z_8 \to \mathbb{P}^1$. By evaluating the Equation of Z_8 at $(X, 1 + t(Y-1), Y, 1)$, one gets the cubic affine model

$$(t-1)X^2 - t^2XY^2 + XY + (t-1)^2X + (t-1)Y = 0$$

of the generic fiber, where t is the parameter of $\mathbb{P}^1$. One computes that the Weierstrass model of it is the elliptic curve

$$E : y^2 = x^3 + (4t^4 - 8t^3 + 4t^2 + 1)/t^4x^2 + 8(t-1)^2/t^6x + 16(t-1)^4/t^8.$$

The associated elliptic surface is a smooth model of the K3 surface Z_8: it is isomorphic to Z_8^s. One computes that the singular fibers of the fibration are $2I_8 + I_4 + I_2 + 2I_1$, at the points $1, 0, \infty, 1/2, t^2 - t - 1/4 = 0$, respectively.

By [15, Section 2.3.3], the equation of a Weierstrass model of the elliptic surface $\Xi_1(8)$ above the modular curve $X_1(8)$ is

$$E' : \ \eta^2 = \xi^3 + (2 - s^2)\xi^2 + \xi,$$

where $s = 2t^2/(t^2-1)$. To check that $\Xi_1(8)$ is isomorphic to Z_8^s, one just has to compare the two j-invariants $j(E)(t) \in \mathbb{Q}(t)$ and $j(E')(t) \in \mathbb{Q}(t)$. We compute that $j(E)(\frac{1}{2}(1-\frac{1}{t})) = j(E')(t)$, therefore E is isomorphic to E', and $\Xi_1(8) \simeq Z_8^s$. □

4.6. **Action of $\mathrm{Aut}(M_8)$.** The automorphism group of M_8 is generated by the involutions

$$s_1 = (2,4)(3,7)(6,8)(9,11)(10,14)(13,15), s_2 = (2,6)(4,8)(9,13)(11,15),$$
$$s_3 = (1,2)(3,8)(4,7)(5,6)(9,13)(10,12)(14,16).$$

The group $\mathrm{Aut}(M_8)$ is the semi-direct product $\mathbb{Z}/8\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. One computes that it acts faithfully on the K3 surface Z_8. The map s_2 (acting on Z_8) is given in the ancillary file of the arXiv version of this paper. It is a birational involution of $\mathbb{P}^3$.

The group of elements σ commuting with the action of $\lambda_{\{2\},\{3,4\}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. The involution $s = (1,5)(2,6)(3,7)(4,8)$ is the unique automorphism of $\mathrm{Aut}(M_8)$ such that $\lambda_{\{2\},\{3,4\}} \circ s = \lambda_{\{2\},\{3,4\}}$.

4.7. **Periodic line arrangements.** Let us prove:

Proposition 22. *There exists a curve C_3 of geometric genus 5 in Z_8 such that each point of C_3 is fixed by $\lambda_{\{2\},\{3,4\}}$ and for a generic point x of C_3, the line arrangement $\mathcal{C}_0(x)$ in $\mathbb{P}^2$ is periodic of period 3 for the action of $\Lambda^{\ell}_{\{2\},\{3,4\}}$.*

Proof. Let K be the finite field $K = \mathbb{F}_{1013}$. Consider the point $x = (794 : 582 : 116 : 1) \in \mathbb{P}^3(K)$. It is a fix-point of the endomorphism $\lambda_{\{2\},\{3,4\}}$ and the associated line arrangement $\mathcal{C}_0(x)$ is periodic of period 3 for $\Lambda^{\ell}_{\{2\},\{3,4\}}$. The line arrangement $L_{24} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ has 24 lines and the singularities $t_2 = 24$, $t_3 = 84$. One computes that the matroid N_{24} associated to L_{24} has an irreducible one dimensional moduli space $\mathcal{R}(N_{24})$ over $\mathbb{C}$. The geometric genus of the compactification of $\mathcal{R}(N_{24})$ is 5. The matroid associated to the first 16 lines $\mathcal{C}_0 \cup \mathcal{C}_1$ is the equal to the matroid M_8, thus there exists a natural map $\mathcal{R}(N_{24}) \to \mathcal{R}_8$, which to a realization $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ of N_{24} associates $\mathcal{C}_0 \cup \mathcal{C}_1$, that map is one-to-one onto its image C_3 since one may recover $\mathcal{C}_2$ (and therefore $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$) as $\mathcal{C}_2 = \Lambda^{\ell}_{\{2\},\{3,4\}}(\mathcal{C}_1)$. The curve C_3 is fixed by $\lambda_{\{2\},\{3,4\}}$. □

References

[1] Bertin M.J., Lecacheux O., Elliptic Fibrations on the Modular Surface Associated to $\Gamma_1(8)$, Fields Inst. Commun., 67 Springer, New York, 2013, 153–199.

[2] Bosma W., Cannon J., Playoust C., The Magma algebra system. I. The user language, J. Symbolic Comput. 24, 1997, 3–4, 235–265.

[3] Cantat S,. Dolgachev I., Rational surfaces with a large group of automorphisms, J. Amer. Math. Soc. 25 (2012), no. 3, 863–905.

[4] Corey D., Kühne L., Schröter B., Matroids in OSCAR, arXiv:2311.08792.

[5] Huybrechts D., Lectures on K3 surfaces. Cambridge Studies in Advanced Mathematics, 158. Cambridge University Press, Cambridge, 2016. xi+485 pp.

[6] Kawaguchi S., Silverman J.H., On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties. J. Reine Angew. Math. 713, 21–48 (2016)

[7] Kühne L, Roulleau X., Regular polygons, line operators, and elliptic modular surfaces as realization spaces of matroids, arXiv:2312.03470.

[8] Lecacheux O., Weierstrass equations for all elliptic fibrations on the modular K3 surface associated to $\Gamma_1(7)$, Rocky Mountain J. Math.45 (2015), no.5, 1481–1509.

[9] Naruki I., On a K3 surface which is a ball quotient, Max Planck Institute for Mathematics Preprint series 1985

[10] Oxley J., Matroid Theory, second ed., Oxford Graduate Texts in Mathematics, vol. 21, 2011. xiv+684pp.

[11] Roulleau X., On some operators acting on line arrangements and their dynamics, arXiv:2306.01053.

[12] Schütt M., K3 surfaces with Picard rank 20, Algebra Number Theory 4 (2010), no. 3, 335–356.

[13] Shioda T., Elliptic modular surfaces, J. Math. Soc. Japan 24 (1972), 20–59.

[14] Suciu A. I., Hyperplane arrangements and Milnor fibrations, Annales de la Faculté des sciences de Toulouse : Mathḿatiques, Serie 6, Volume 23 (2014) no. 2, pp. 417-481.

[15] Top J., Yui N., Explicit equations of some elliptic modular surfaces, Rocky Mountain J. Math.37 (2007), no.2, 663–687.

[16] Ujikawa M., The automorphism group of the singular K3 surface of discriminant 7, Comment. Math. Univ. St. Pauli 62 (2013), no.1, 11–29.

[17] Voisin C., Intrinsic pseudo-volume forms and K-correspondences, The Fano Conference, 761–792. Univ. di Torino, Turin, 2004.

[18] Yoshinaga M., Freeness of hyperplane arrangements and related topics, Annales de la Faculté des sciences de Toulouse : Mathématiques, Serie 6, Volume 23 (2014) no. 2, pp. 483-512.

Lukas Kühne, Universität Bielefeld, Fakultät für Mathematik, Bielefeld, Germany
Email address: lkuehne@math.uni-bielefeld.de

Xavier Roulleau, Université d'Angers, CNRS, LAREMA, SFR Math-STIC, F-49000 Angers, France
Email address: xavier.roulleau@univ-angers.fr