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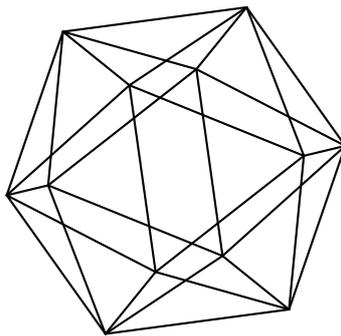
Boundary value problems for noncompact boundaries of Spin^c manifolds and spectral estimates

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BOUNDARY VALUE PROBLEMS FOR NONCOMPACT BOUNDARIES OF SPIN^c MANIFOLDS AND SPECTRAL ESTIMATES

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ABSTRACT. We study boundary value problems for the Dirac operator on Riemannian Spin^c manifolds of bounded geometry and with noncompact boundary. This generalizes a part of the theory of boundary value problems by C. Bär and W. Ballmann for complete manifolds with closed boundary. As an application, we derive the lower bound of Hijazi-Montiel-Zhang, involving the mean curvature of the boundary, for the spectrum of the Dirac operator on the noncompact boundary of a Spin^c manifold. The limiting case is then studied and examples are then given.

1. INTRODUCTION

In the last years, the spectrum of the Dirac operator on hypersurfaces of a Spin manifold has been intensively studied. Indeed, many extrinsic upper bounds have been obtained (see [2, 3, 1, 5, 8, 9, 27] and references therein) and more recently in [22, 23, 24, 26, 20, 21, 45], extrinsic lower bounds for the hypersurface Dirac operator are established. From these spectral estimates and their limiting cases, many topological and geometric informations on the hypersurface are derived.

In [22], O. Hijazi, S. Montiel and X. Zhang investigated the spectral properties of the Dirac operator on a compact manifold with boundary for the Atiyah-Patodi-Singer type boundary condition (or shortly APS-boundary condition) corresponding to the spectral resolution of the classical Dirac operator of the boundary hypersurface. They proved that, on the compact boundary $\Sigma = \partial M$ of a compact Riemannian Spin manifold (M^{n+1}, g) of nonnegative scalar curvature scal^M, the first nonnegative eigenvalue of the Dirac operator on the boundary satisfies

$$\lambda_1 \geq \frac{n}{2} \inf_{\Sigma} H, \tag{1}$$

where the mean curvature of the boundary H is calculated with respect to the inner normal and assumed to be nonnegative. Equality holds in (1) if and only if H is constant and every eigenspinor associated with the eigenvalue λ_1 is the restriction to Σ of a parallel spinor field

on M (and hence M is Ricci-flat). As application of the limiting case, they gave an elementary Spin proof of the famous Alexandrov theorem: *The only closed embedded hypersurface in $\mathbb{R}^{n+1}$ of constant mean curvature is the sphere of dimension n.*

Furthermore, Inequality (1) does not only give an extrinsic lower bound on the first nonnegative eigenvalue but can also be seen as an obstruction to positive scalar curvature of the interior given only in terms of a neighbourhood of the boundary. More precisely, let a neighbourhood of the boundary Σ be equipped with a metric of nonnegative scalar curvature and such that the boundary has nonnegative mean curvature. If the lowest positive eigenvalue of the Dirac operator on the boundary is smaller than $\frac{n}{2}\inf_\Sigma H$, then the metric cannot be extended to all of M such that the scalar curvature remains nonnegative.

Moreover, not just Spin geometry but especially Spin^c geometry became recently a field of active research with the advent of Seiberg-Witten theory whose applications to 4-dimensional geometry and topology are already notorious, see [28, 43, 39, 29, 30, 19]. From an intrinsic point of view, Spin, almost complex, complex, Kähler, Sasaki and some classes of CR manifolds have a canonical Spin^c structure carrying natural spinors like parallel or Killing spinor fields. Nowadays, and from the extrinsic point of view, it seems that it is more natural to work with Spin^c structures rather than Spin structures. Indeed, O. Hijazi, S. Montiel and F. Urbano [25] constructed on Kähler-Einstein manifolds with positive scalar curvature, Spin^c structures carrying Kählerian Killing spinors. The restriction of these spinors to minimal Lagrangian submanifolds provides topological and geometric restrictions on these submanifolds. In [35, 36] and via Spin^c spinors, hypersurfaces of some 3 and 4-dimensional manifolds are characterised. From these characterisations, an elementary proof for a Lawson type correspondence between constant mean curvature surfaces of 3-dimensional homogeneous manifolds with 4-dimensional isometry group is derived. Moreover, necessary and sufficient geometric conditions are given to immerse any 3-dimensional Sasaki manifold into the complex projective space or the complex hyperbolic space [37].

In this paper, we extend the lower bound (1) to the infimum of the nonnegative part of the Dirac spectrum of the noncompact boundary of a Riemannian Spin^c manifold. When shifting from the compact case to the noncompact case, many obstacles occur. Moreover, when shifting from the classical Spin geometry to Spin^c geometry, the situation is more general since the spectrum of the Dirac operator will not only depend on the geometry of the manifold but also on the geometry of the auxiliary line bundle associated with the fixed Spin^c structure.

When we consider a Riemannian Spin or Spin^c manifold with noncompact boundary, the main technical difference to the compact case is that we cannot restrict all our computations to smooth spinors. For compact manifolds, this is possible by using the spectral decomposition of L^2 by an eigenbasis. For complete manifolds, eigenspinors do not have to exist or even if they do, in general they do not form an orthonormal basis of L^2 since continuous spectrum can occur. Additionally, the proof of Inequality (1) in the closed case uses the existence of a solution of a boundary value problem defined under the APS-boundary condition. While for noncompact boundaries the idea of APS-boundary conditions can be transferred by using

the spectral theorem, it is not clear to us whether they still defines a nontrivial boundary condition, see Example 3.14. Moreover, even if they give a nontrivial boundary condition, there may occur problems due to continuous spectrum.

In order to circumvent all these problems, a large part of the paper is devoted to give a generalization of the theory of boundary value problems for noncompact boundaries, see Section 3. We stick to the part of the theory that gives existence of solutions of such boundary value problem, cf. Remark 3.13. For complete manifolds with closed boundary, the theory of boundary value problems is given in [7] by C. Bär and W. Ballmann. They did not only restrict to the classical Dirac operator but they generalized the traditional theory of elliptic boundary value problems to Dirac type operators. Additionally, they proved a decomposition theorem for the essential spectrum, a general version of Gromov and Lawson's relative index theorem and a generalization of the cobordism theorem.

In Section 3, we will classify boundary conditions for a Riemannian Spin^c manifold (M^{n+1}, g) with noncompact boundary $\Sigma := \partial M$ and of bounded geometry, see Definition 2.2. Indeed, we prove in Section 3 that the trace map or the restriction map $R : \varphi \mapsto \varphi|_\Sigma$ where φ is a compactly supported smooth spinor on M can be extended to a bounded operator

$$R : \operatorname{dom} D_{\max} \to H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma).$$

Here $\operatorname{dom} D_{\max}$ is the maximal domain of the Dirac operator on M, $\mathbb{S}_M|_\Sigma$ is the restriction of the Spin^c bundle $\mathbb{S}_M$ to Σ and for the definition of $H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$ see the Appendix. The map R is not surjective. But in Theorem 3.10, we show that the extension map – a right inverse to R – can be extented to a bounded linear operator from $R(\operatorname{dom} D_{\max})$ to $\operatorname{dom} D_{\max}$. This will allow to equip $R(\operatorname{dom} D_{\max})$ with a norm $\|.\|_R$ that turns it into a Banach space. With these ingredients, we can then classify the closed extensions of the Dirac operator D_{cc} acting on smooth compactly supported spinors on M: For every closed extension of the Dirac operator acting on smooth compactly supported spinors on M the set $B := R(\operatorname{dom} D) \subset H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$ is closed in $(R(\operatorname{dom} D_{\max}), \|.\|_R)$. Conversely, every closed subset $B \subset H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$ gives the domain dom, D_B of a closed extension. Such subsets B are called a boundary conditions.

Then, we generalize the existence result for boundary value problems to our noncompact setting. For this, we need the notion of *B-coercivity at infinity*, see Definition 3.1. This notion generalizes the notion of *coercivity at infinity* for closed boundaries as used in [7], where this assumption is also needed when characterising the Fredholmness of the Dirac operator. The *B-coercivity at infinity* condition will in general depend on the boundary condition B and under some additional assumptions, it coincides with the *coercivity at infinity* condition used in [7].

Proposition 1.1. *Let B be a boundary condition and the Dirac operator*

$$D_B : \operatorname{dom} D_B \subset L^2(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$$

be B-coercive at infinity. We consider $P_{B^\perp} := Id - P_B : R(\operatorname{dom} D_{\max}) \to R(\operatorname{dom} D_{\max})$, where P_B is the projection from $R(\operatorname{dom} D_{\max})$ to B. Then, for all $\psi \in L^2(M, \mathbb{S}_M)$ and

$\tilde{\rho} \in \operatorname{dom} D_{\max}$ *where* $\psi - D\tilde{\rho} \in (\ker(D_B)^*)^{\perp}$ *the boundary value problem*

$$\begin{cases} D\varphi = \psi & on \;\; M, \\ P_{B^{\perp}} R\varphi = P_{B^{\perp}} R\tilde{\rho} & on \;\; \Sigma, \end{cases}$$

has a unique solution $\varphi \in \operatorname{dom} D_{\max}$, *up to elements of the kernel* $\ker D_B$.

Proposition 1.1 will be one of the main ingredients to generalize Inequality (1) to our non-compact setting. As boundary condition B we will not take the APS-boundary condition as in the closed case but another one: $B_{\pm}$, cf. Section 4. For closed boundaries, the $B_{\pm}$ boundary condition was introduced in [24] to prove a conformal version of (1). Using Proposition 1.1 for the boundary condition $B_{\pm}$ and the Spin^c Reilly inequality on possibly open boundary domains, we obtain

Theorem 1.2. *Let* (M^{n+1}, g) *be a complete Riemannian* Spin^c *manifold with boundary* Σ *and* L *be the auxiliary line bundle associated to the* Spin^c-*structure. Assume that* (M, Σ) *and* L *are of bounded geometry, cf. Definition 2.2 and 2.3. Moreover, we assume that* Σ *has a nonnegative mean curvature* H *with respect to its the inner unit normal field of* Σ, *the Dirac operator* D *is* (B_+)- *or* (B_-)-*coercive at infinity and* $\operatorname{scal}^M + 2i\Omega\cdot$ *is a nonnegative operator where* $i\Omega$ *denotes the curvature 2-form of* L. *Then, the infimum* λ_1 *of the nonnegative part of the spectrum of the Dirac operator on* Σ *satisfies*

$$\lambda_1 \geq \frac{n}{2} \inf_{\Sigma} H. \tag{2}$$

If $\lambda_1 \geq 0$ *is an eigenvalue, equality holds if and only if* H *is constant and any eigenspinor corresponding to* λ_1 *is the restriction of a parallel* Spin^c *spinor* φ *on* M.

The paper is structured as follows: In Section 2, we give all the preliminaries as e.g. the Spin^c Dirac operator and the assumption on the bounded geometry. The theory of boundary values will be generalized to our noncompact setting in Section 3. The special boundary condition $B_{\pm}$ needed to proof the desired inequality is examined in Section 4. In Section 5, we study the coercivity condition for the Dirac operator. Then, we review the spinorial Reilly inequality in order to ready to proof the inequality in Section 7. In the remaining parts, we will compare the extrinsic Inequality (1) with the intrinsic Friedrich's inequality. Moreover, we will give some examples of the limiting case where the equality case cannot occur if we consider the Spin Dirac operator on these examples. Finally and in the appendix, we give a short proof of the extension of the trace map to $H_1(M, \mathbb{S}_M)$ and its right inverse $\mathcal{E}$.

2. Notations and preliminaries

In this section, we briefly review some basic facts about Spin^c geometry. Then, we give the necessary preliminaries on the Sobolev spaces on manifolds with boundary, the Trace Theorem and its implications, some basics of spectral theory and we recall the closed range theorem.

The Spin^c Dirac operator. Let (M^{n+1}, g) be an $(n+1)$-dimensional Riemannian Spin^c manifold with boundary. On such a manifold we have a Hermitian complex vector bundle $\mathbb{S}_M$ endowed with a natural scalar product $\langle ., . \rangle$ and with a connection ∇ wich parallelizes the metric. Moreover, the bundle $\mathbb{S}_M$, called the Spin^c bundle, is endowed with a Clifford multiplication denoted by "$\cdot$", $\cdot : TM \longrightarrow \operatorname{End}_{\mathbb{C}}(\mathbb{S}_M)$, such that at every point $x \in M$,

".″defines an irreducible representation of the corresponding Clifford algebra. Hence, the complex rank of $\mathbb{S}_M$ is $2^{[\frac{n+1}{2}]}$. Given a Spin^c structure on (M^{n+1}, g), one can prove that the determinant line bundle det $\mathbb{S}_M$, has a root of index $2^{[\frac{n+1}{2}]-1}$, see [15, Section 2.5]. We denote by L this root line bundle over M and call it the auxiliary line bundle associated with the Spin^c structure.

Locally, a Spin structure always exists. We denote by $\mathbb{S}'_M$ the possibly (globally) non-existent spinor bundle. Moreover, the square root of the auxiliary line bundle L always exists locally. But, $\mathbb{S}_M = \mathbb{S}'_M \otimes L^{\frac{1}{2}}$, see [15, Appendix D] and [35]. This essentially means that, while the spinor bundle and $L^{\frac{1}{2}}$ may not exist globally, their tensor product (the Spin^c bundle) is defined globally. Thus, the connection ∇ on $\mathbb{S}_M$ is the twisted connection of the one on the spinor bundle (coming from the Levi-Civita connection) and a fixed connection on L.

With these ingredients, we may define the Dirac operator D acting on the space of smooth sections of $\mathbb{S}_M$ – denoted by $\Gamma^\infty(M, \mathbb{S}_M)$ – by the composition of the metric connection and the Clifford multiplication. In local coordinates this reads

$$D = \sum_{j=1}^{n+1} e_j \cdot \nabla_{e_j}$$

where $\{e_j\}_{j=1,\cdots,n+1}$ is an orthonormal basis of TM. It is a first-order elliptic operator satisfying for all smooth spinors φ, ψ on M at least one of them being compactly supported

$$(D\psi, \varphi) - (\psi, D\varphi) = -\int_{\partial M} \langle \nu \cdot \psi|_{\partial M}, \varphi|_{\partial M}\rangle ds, \tag{3}$$

where $(.,.)$ is the L^2-scalar product given by $(\varphi, \psi) = \int_M \langle \varphi, \psi\rangle dv$, ∂M is the boundary of M, $|_{\partial M}$ denotes the restriction to the boundary, ν the inner unit normal vector of the immersion $\partial M \hookrightarrow M$ and dv (resp. ds) is the Riemannian volume form of M (resp. of ∂M). Hence, if $\partial M = \emptyset$, the Dirac operator is formally self-adjoint with respect to the L^2-scalar product.

An important tool when examining the Dirac operator on Spin^c manifolds is the Schrödinger-Lichnerowicz formula:

$$D^2 = \nabla^*\nabla + \frac{1}{4}\text{scal}^M \text{Id}_{\Gamma(\mathbb{S}_M)} + \frac{i}{2}\Omega\cdot, \tag{4}$$

where ∇^* is the adjoint of ∇ with respect to the L^2-scalar product, $i\Omega$ is the curvature of the auxiliary line bundle L associated with a fixed connection (Ω is a real 2-form on M) and $\Omega\cdot$ is the extension of the Clifford multiplication to differential forms.

Example 2.1. (i) A Spin structure can be seen as a Spin^c structure with trivial auxiliary line bundle L and trivial connection (and so $i\Omega = 0$).
(ii) Every almost complex manifold $(M^{2m=n+1}, g, J)$ of complex dimension m has a canonical Spin^c structure. In fact, the complexified cotangent bundle $T^*M \otimes \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$ decomposes into the $\pm i$-eigenbundles of the complex linear extension of the complex structure J. Thus, the spinor bundle of the canonical Spin^c structure is given by

$$\mathbb{S}_M = \Lambda^{0,*}M = \oplus_{r=0}^m \Lambda^{0,r}M,$$

where $\Lambda^{0,r}M = \Lambda^r(\Lambda^{0,1}M)$ is the bundle of r-forms of type $(0,1)$. The auxiliary line bundle of this canonical Spin^c structure is given by $L = (K_M)^{-1} = \Lambda^m(\Lambda^{0,1}M)$, where K_M is the

canonical bundle of M [15, 32, 25, 35]. Let α be the Kähler form defined by the complex structure J, i.e. $\alpha(X,Y) = g(X, JY)$ for all vector fields $X, Y \in \Gamma(TM)$. The auxiliary line bundle $L = (K_M)^{-1}$ has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by $i\Omega = i\rho$, where ρ is the Ricci 2-form given by $\rho(X,Y) = \mathrm{Ric}(X, JY)$. Here Ric denotes the Ricci tensor of M. For any other Spin^c structure on M^{2m}, the spinorial bundle can be written as [15, 25]:

$$\mathbb{S}_M = \Lambda^{0,*}M \otimes \mathcal{L},$$

where $\mathcal{L}^2 = K_M \otimes L$ and L is the auxiliary bundle associated with this Spin^c structure. In this case, the 2-form α can be considered as an endomorphism of $\mathbb{S}_M$ via Clifford multiplication and we have the well-known orthogonal splitting $\mathbb{S}_M = \oplus_{r=0}^m \mathbb{S}_M^r$, where $\mathbb{S}_M^r$ denotes the eigensubbundle corresponding to the eigenvalue $i(m - 2r)$ of α, with complex rank $\binom{m}{k}$. The bundle $\mathbb{S}_M^r$ corresponds to $\Lambda^{0,r}M \otimes \mathcal{L}$. For the canonical Spin^c structure, the subbundle $\mathbb{S}_M^0$ is trivial. Hence and when M is a Kähler manifold, this Spin^c structure admits parallel spinors (constant functions) lying in $\mathbb{S}_M^0$ [32]. Of course, we can define another Spin^c structure for which the spinor bundle is given by $\Lambda^{*,0}M = \oplus_{r=0}^m \Lambda^r(T_{1,0}^*M)$ and the auxiliary line bundle by K_M. This Spin^c structure is called the anti-canonical Spin^c structure.

Any Spin^c structure on (M^{n+1}, g) induces a Spin^c structure on its boundary $\Sigma = \partial M$ and we have

$$\begin{cases} \mathbb{S}_M|_\Sigma & \simeq \mathbb{S}_\Sigma \quad \text{if } n \text{ is even,} \\ \mathbb{S}_M^+|_\Sigma & \simeq \mathbb{S}_\Sigma \quad \text{if } n \text{ is odd.} \end{cases}$$

We recall that if n is odd, the spinor bundle $\mathbb{S}_M$ splits into

$$\mathbb{S}_M = \mathbb{S}_M^+ \oplus \mathbb{S}_M^-,$$

by the action of the complex volume element. Moreover, Clifford multiplication with a vector field X tangent to Σ is given by

$$X \bullet \varphi = (X \cdot \nu \cdot \psi)|_\Sigma,$$

where $\psi \in \Gamma^\infty(M, \mathbb{S}_M)$ (or $\psi \in \Gamma^\infty(\mathbb{S}_M^+)$ if n is odd), φ is the restriction of ψ to Σ, '$\bullet$' is the Clifford multiplication on M. When n is odd we also get $\mathbb{S}_M^- \simeq \mathbb{S}_\Sigma$. In this case, the Clifford multiplication by a vector field X tangent to Σ is given by $X \bullet \varphi = -(X \cdot \nu \cdot \psi)|_\Sigma$ and hence we have $\mathbb{S}_M|_\Sigma \simeq \mathbb{S}_\Sigma \oplus \mathbb{S}_\Sigma$. Moreover, the corresponding auxiliary line bundle L^Σ on Σ is the restriction to Σ of the auxiliary line bundle L and $i\Omega^\Sigma = i\Omega|_\Sigma$. We denote by ∇^Σ the spinorial Levi-Civita connection on $\mathbb{S}_\Sigma$. For all smooth vector fields $X \in \Gamma^\infty(T\Sigma)$ and for every smooth spinor field $\psi \in \Gamma^\infty(M, \mathbb{S}_M)$, we consider $\varphi = \psi|_\Sigma$ and we have the following Spin^c Gauss formula [25, 35, 33]:

$$(\nabla_X \psi)|_\Sigma = \nabla_X^\Sigma \varphi + \frac{1}{2} II(X) \bullet \varphi, \tag{5}$$

where II denotes the Weingarten map with respect to ν. Moreover, let D and D^Σ be the Dirac operators on M and Σ. After denoting any smooth spinor and its restriction to Σ by the same symbol, we have on Σ (see [25, 33, 35]) that

$$\widetilde{D}\varphi = \frac{n}{2} H \varphi - \nu \cdot D\varphi - \nabla_\nu \varphi, \tag{6}$$

$$\widetilde{D}(\nu \cdot \varphi) = -\nu \cdot \widetilde{D}\varphi, \tag{7}$$

where $H = \frac{1}{n}\text{tr}(II)$ denotes the mean curvature and $\widetilde{D} = D^{\Sigma}$ if n is even and $\widetilde{D} = D^{\Sigma} \oplus (-D^{\Sigma})$ if n is odd. Note that $\sigma(\widetilde{D}) = \{\pm\lambda \mid \lambda \in \sigma(D^{\Sigma})\}$ where $\sigma(A)$ denotes the spectrum of an operator A.

Bounded geometry. In this paragraph, we recall the definition of manifolds of bounded geometry.

Definition 2.2. [38, Definition 2.2] Let (M^{n+1}, g) be a complete Riemannian manifold with boundary Σ. We say that (M, Σ) is of bounded geometry if the following is fulfilled

  (i) The curvature tensor of M and all its covariant derivatives are bounded.
  (ii) The injectivity radius of Σ is positive.
  (iii) There is a collar around Σ, i.e: There is $r_{\Sigma} > 0$ such that the geodesic collar

$$U_{\Sigma} = [0, r_{\Sigma}) \times \Sigma \to M, \ (t, x) \mapsto \exp_x(t\nu)$$

  is a diffeomorphism onto its image where ν is the inner unit normal field on Σ. We equip U_{Σ} with the induced metric and will identify U_{Σ} with its image.
  (iv) There exists $\varepsilon > 0$ such that the injectivity radius of each point $x \in M \setminus U_{\Sigma}$ is greater or equal than ε.
  (v) The mean curvature of Σ and all its covariant derivatives are bounded.

Definition 2.3. (cp. [40, A.1.1] together with [13, Theorem B]) Let E be a hermitian vector bundle over M where (M, Σ) is of bounded geometry. Then E is said to be of bounded geometry if its curvature and all its covariant derivatives are bounded.

Remark 2.4.   (1) Note that the above definition contains the usual definition of manifold of bounded geometry without boundary. Moreover, if (M, g) is of bounded geometry, then $(\Sigma, g|_{\Sigma})$ is also of bounded geometry [38, Corollary 2.24].
  (2) For the spinor bundle $\mathbb{S}'_M$ associated with a Spin structure, the bounded geometry follows automatically from the bounded geometry of M, [4, Section 3.1.3]. For a Spin^c manifold the situation is more general since the Spin^c bundle $\mathbb{S}_M$ does not only depend on the geometry of the underlying manifold but also on the geometry of the auxiliary line bundle L. But, $\mathbb{S}_M = \mathbb{S}'_M \otimes L^{\frac{1}{2}}$, where $\mathbb{S}'_M$ is the locally defined spinor bundle, $L^{\frac{1}{2}}$ is locally defined too and $\mathbb{S}_M$ is globally defined. Thus, the assumption that L is of bounded geometry assures that $\mathbb{S}_M$ is also of bounded geometry.

> **Assumption for the rest of the paper:** (M, Σ) and L are of bounded geometry.

Restriction of spinors to the boundary. We denote by $\Gamma_c^{\infty}(M, \mathbb{S}_M)$ all compactly supported smooth spinors on M. This allows boundary values if $\partial M \neq \emptyset$. The set of smooth spinors that are compactly supported in the interior of M is denoted by $\Gamma_{cc}^{\infty}(M, \mathbb{S}_M)$. We consider the restriction operator

$$\begin{aligned} R : \Gamma_c^{\infty}(M, \mathbb{S}_M) &\to \Gamma_c^{\infty}(M, \mathbb{S}_M) \\ \varphi &\mapsto \varphi|_{\Sigma}. \end{aligned}$$

By the Trace Theorem A.5, the operator R extends to a bounded operator

$$R : H_1(M, \mathbb{S}_M) \to H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_{\Sigma}).$$

For the definition of these spaces see below and the appendix. If it is clear from the context that $R\varphi$ is considered instead of φ, we will sometimes abbreviate by using φ only. Note that by Theorem A.6, the operator R has a bounded right inverse – an extension map $\mathcal{E}$.

For abbreviation we set $L^2 = L^2(M) = L^2(M, \mathbb{S}_M)$ and $L^2(\Sigma) = L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ and analogously for other function spaces. Moreover, $(.,.)$ shall always denote the L^2-scalar product on M and $(.,.)_\Sigma$ the one on Σ.

The Sobolev space H_1 on manifolds with boundary. We define the $H_1 = H_1(M, \mathbb{S}_M)$-norm on $\Gamma_c^\infty(M, \mathbb{S}_M)$ by

$$\|\varphi\|_{H_1(M, \mathbb{S}_M)}^2 = \|\varphi\|_{L^2(M, \mathbb{S}_M)}^2 + \|\nabla\varphi\|_{L^2(M, \mathbb{S}_M)}^2.$$

Finally, we define $H_1 = H_1(M, \mathbb{S}_M)$ as the closure of $\Gamma_c^\infty(M, \mathbb{S}_M)$ with respect to the H_1-norm defined above. Note that

$$H_1(M, \mathbb{S}_M) = \{\varphi \in L^2(M, \mathbb{S}_M) \mid \exists \tilde{\varphi} \in L^2(M, \mathbb{S}_M) \forall \psi \in \Gamma_{cc}^\infty(M, \mathbb{S}_M) : \ (\tilde{\varphi}, \psi) = (\varphi, \nabla\psi)\}.$$

Using the Lichnerowicz formula (4), (3), the Gauß theorem $(\nabla^*\nabla\varphi, \varphi) = \|\nabla\varphi\|_{L^2}^2 + \int_\Sigma \langle \nabla_\nu\varphi, \varphi\rangle ds$ and (6) we obtain another description of the H_1-norm: For all $\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M)$, we have

$$\|\varphi\|_{H_1}^2 = \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2 - \int_M \frac{\mathrm{scal}^M}{4}|\varphi|^2 dv - \int_M \frac{i}{2}\langle \Omega \cdot \varphi, \varphi\rangle dv + \int_\Sigma \langle \varphi|_\Sigma, D^W(\varphi|_\Sigma)\rangle ds, \tag{8}$$

where $D^W = \widetilde{D} - \frac{n}{2}H$ is the so-called Dirac-Witten operator. Note that due to the local expression of D and the Cauchy Schwarz inequality, we always have

$$\|D\varphi\|_{L^2}^2 \le \sum_{i=1}^{n+1} \int_M |\nabla_{e_i}\varphi|^2 dv \le (n+1)\|\nabla\varphi\|_{L^2}^2, \tag{9}$$

for all $\varphi \in H_1(M, \mathbb{S}_M)$.

Lemma 2.5. *For all $\varphi, \psi \in H_1(M, \mathbb{S}_M)$, Equalities (8) and (3) hold.*

Proof. The proof is a more or less straightforward usage of the Trace Theorem A.5 and the corresponding equalities on $\Gamma_c^\infty(M, \mathbb{S}_M)$. Indeed, let φ_i be a sequence in $\Gamma_c^\infty(M, \mathbb{S}_M)$ with $\varphi_i \to \varphi$ in $H_1(M, \mathbb{S}_M)$. The Trace Theorem A.5 gives $R\varphi_i \to R\varphi$ in $H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$ and, hence, $\widetilde{D}R\varphi_i \to \widetilde{D}R\varphi$ in $H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$, cf. Remark A.4.iii. Clearly, $\|\varphi_i\|_{H_1}^2 \to \|\varphi\|_{H_1}^2$ and with (9), this implies $\|\varphi_i\|_D^2 \to \|\varphi\|_D^2$. Moreover, the bounded geometry of (M, Σ) implies

$$\int_M \mathrm{scal}^M|\varphi_i|^2 dv \to \int_M \mathrm{scal}^M|\varphi|^2 dv, \qquad \int_\Sigma H|\varphi_i|^2 ds \to \int_\Sigma H|\varphi|^2 ds \ \text{ and}$$

$$\left|\int_M \langle \Omega \cdot \varphi_i, \varphi_i\rangle dv - \int_M \langle \Omega \cdot \varphi, \varphi\rangle dv\right| \le (\|\varphi_i - \varphi\|_{L^2}\|\varphi\|_{L^2} + \|\varphi_i\|_{L^2}\|\varphi_i - \varphi\|_{L^2})\sup_M |\Omega| \to 0.$$

Note that due to the bounded geometry of L, $\sup_M |\Omega|$ is finite. It remains to consider the term $\int_\Sigma \langle R\varphi, \widetilde{D}R\varphi\rangle ds$. First we note that due to the pairing in Lemma A.7, the Trace Theorem A.5 and $\widetilde{D} : H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma) \to H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$, this expression is finite for all $\varphi \in$

$H_1(M, \mathbb{S}_M)$. Abbreviating $R\varphi$ by φ, we have

$$|(\widetilde{D}\varphi_i, \varphi_i)_\Sigma - (\widetilde{D}\varphi, \varphi)_\Sigma| \leq |(\widetilde{D}\varphi_i, \varphi - \varphi_i)_\Sigma| + |(\widetilde{D}\varphi - \widetilde{D}\varphi_i, \varphi)_\Sigma|$$

$$\leq \|\widetilde{D}\varphi_i\|_{H_{-\frac{1}{2}}} \|\varphi - \varphi_i\|_{H_{\frac{1}{2}}} + \|\widetilde{D}\varphi - \widetilde{D}\varphi_i\|_{H_{-\frac{1}{2}}} \|\varphi\|_{H_{\frac{1}{2}}},$$

which gives the convergence of the last term. This proves Equality (8) for all $\varphi \in H_1(M, \mathbb{S}_M)$. Now, let φ_i, ψ_j be sequences in $\Gamma_c^\infty(M, \mathbb{S}_M)$ with $\varphi_i \to \varphi$ and $\psi_j \to \psi$ in $H_1(M, \mathbb{S}_M)$. Then,

$$|(D\psi_j, \varphi_i) - (D\psi, \varphi)| \leq |(D\psi_j, \varphi_i) - (D\psi_j, \varphi)| + |(D\psi_j, \varphi) - (D\psi, \varphi)|$$

$$\leq \|D\psi_j\|_{L^2} \|\varphi_i - \varphi\|_{L^2} + \|D(\psi_j - \psi)\|_{L^2} \|\varphi\|_{L^2}.$$

Using (9) and that φ_i and ψ_j are uniformly bounded in H_1, we get for a certain constant $C > 0$ that

$$|(D\psi_j, \varphi_i) - (D\psi, \varphi)| \leq C\|\varphi_i - \varphi\|_{L^2} + C\|\psi_j - \psi\|_{H_1} \to 0.$$

Analogously, one obtains $(\psi_j, D\varphi_i) \to (\psi, D\varphi)$. Moreover, using again the Trace Theorem A.5, we get

$$\left| \int_\Sigma \langle \nu \cdot R\psi_j, R\varphi_i \rangle - \langle \nu \cdot R\psi_j, R\varphi \rangle ds \right| \leq \|R\psi_j\|_{L^2(\Sigma)} \|R(\varphi_i - \varphi)\|_{L^2(\Sigma)}$$

$$\leq C\|\psi_j\|_{H_1} \|\varphi_i - \varphi\|_{H_1} \to 0.$$

In the same way, $\left| \int_\Sigma \langle \nu \cdot R\psi_j, R\varphi \rangle - \langle \nu \cdot R\psi, R\varphi \rangle ds \right| \to 0$. Hence,

$$\left| \int_\Sigma \langle \nu \cdot R\psi_j, R\varphi_i \rangle - \langle \nu \cdot R\psi, R\varphi \rangle ds \right| \to 0.$$

$\square$

Spectral theory. In this paragraph, we shortly review the spectral theory of the Dirac operator $D : H_1(N, \mathbb{S}_N) \subset L^2(N, \mathbb{S}_N) \to L^2(N, \mathbb{S}_N)$ on a complete Riemannian Spin^c manifold N without boundary. Most of the follwing can be found in [6]. Then D is self-adjoint and the spectrum is real. A real number λ is an eigenvalue of D if there exists a nonzero spinor $\varphi \in H_1$ with $D\varphi = \lambda\varphi$. Then φ is called an eigenspinor to the eigenvalue λ. Standard local elliptic regularity theory gives that an eigenspinor is always smooth. The set of all eigenvalues is denoted by $\sigma_p(D^\Sigma)$ – the point spectrum. If N is closed, the Dirac operator has a pure point spectrum. But on open manifolds, the spectrum might have a continuous part. In general, the spectrum – denoted by $\sigma(D)$ – is composed of the point, the continuous and the residual spectrum. In case of a self-adjoint operator – as we have – there is no residual spectrum. Often another decomposition of the spectrum is used – the one into discrete spectrum $\sigma_d(D)$ and essential spectrum $\sigma_{ess}(D)$. A real number λ lies in the essential spectrum of D if there exists a sequence of smooth compactly supported spinors φ_i which are orthonormal with respect to the L^2-product and

$$\|(D - \lambda)\varphi_i\|_{L^2} \longrightarrow 0.$$

The essential spectrum contains amongst other elements all eigenvalues of infinite multiplicity. In contrast, the discrete spectrum $\sigma_d(D) := \sigma_p(D) \setminus \sigma_{ess}(D)$ consists of all eigenvalues of finite multiplicity.

Closed Range Theorem. Next, we want to recall briefly (a part of) the Closed Range Theorem for later use.

Theorem 2.6. [44, p.205] *Let $T : X \to Y$ be a closed linear operator between Banach spaces X, Y. Then the range $\operatorname{ran}(T)$ of T is closed in Y if and only if $\operatorname{ran}(T) = \ker(T^*)^\perp$ where T^* is the adjoint operator of T and $\ker(T^*)$ is the kernel of T^*.*

A linear operator $T : X \to Y$ between Banach spaces is called Fredholm if its kernel is finite dimensional and its image has finite codimension.

3. Boundary value problems

The general theory of boundary value problems for elliptic differential operators of order one on complete manifolds with closed boundary can be found in [7]. The aim of this section is to generalize a part of this theory to noncompact boundaries on manifolds of bounded geometry. We restrict to the part that gives existence of solutions of boundary value problems as in Theorem 3.15. The property needed to assure a solution to such a problem is the closedness of the range. For that we introduce a type of coercivity condition which in general can depend on the boundary values (that is not the case for closed boundaries). Moreover, we restrict to the classical Spin^c Dirac operator.

In the first part, we first give some generalities on domains of the Dirac operator and introduce a coercivity condition that implies closed range of the Dirac operator. Then, we extend the trace map R to the whole maximal domain of the Dirac operator and give some examples and properties of boundary conditions. In particular, we will introduce two boundary conditions $B_\pm$ which will be used to prove Theorem 1.2 in Section 7. At the end, we give an existence result for boundary value problems in our context.

General domains and closed range. Let D be the Dirac operator acting on $\Gamma_{cc}^\infty(M, \mathbb{S})$ on a manifold M with boundary Σ. If we want to emphasise that D acts on the domain $\Gamma_{cc}^\infty(M, \mathbb{S})$, we shortly write D_{cc}. We denote the graph norm of D by

$$\|\varphi\|_D^2 = \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2.$$

By $D_{\max} := (D_{cc})^*$ we denote the maximal extension of D. Here, A^* denotes the adjoint operator of A in the sense of functional analysis. Note that

$$\operatorname{dom} D_{\max} = \{\varphi \in L^2(M, \mathbb{S}_M) \mid \exists \tilde{\varphi} \in L^2(M, \mathbb{S}_M) \forall \psi \in \Gamma_{cc}^\infty(M, \mathbb{S}_M) : (\tilde{\varphi}, \psi) = (\varphi, D\psi)\}$$

and together with $\|.\|_D$, the space $\operatorname{dom} D_{\max}$ is a Hilbert space. Moreover, we denote by $D_{\min} := (D_{cc})^{**} = \overline{D_{cc}}^{\|.\|_D}$ the minimal extension of D. Here, $\overline{A}^{\|.\|_D}$ denotes the closure of the set A w.r.t. the graph norm. Any closed subset of $\operatorname{dom} D_{\max}$ between $\operatorname{dom} D_{\min}$ and $\operatorname{dom} D_{\max}$ gives the domain of a closed extension of $D : \Gamma_{cc}^\infty(M, \mathbb{S}_M) \to \Gamma_{cc}^\infty(M, \mathbb{S}_M)$. We generalize the notion of coercivity at infinity [7, Definition 8.2.] to our noncompact setting:

Definition 3.1. A closed linear operator $D : \operatorname{dom} D \subset L^2(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$ is said to be $(\operatorname{dom} D)$-coercive at infinity if there is a $c > 0$ such that

$$\forall \varphi \in \operatorname{dom} D \cap (\ker D)^\perp : \ \|D\varphi\|_{L^2} \geq c\|\varphi\|_{L^2}.$$

In Section 5, we will compare this coercivity condition with the originally one used in [7]. But first, we will see how this condition forces the range of the operator to be closed which is crucial in order to show existence of preimages of linear operator as we will do in Proposition 3.15.

Lemma 3.2. *If the closed linear operator* $D : \operatorname{dom} D \subset L^2(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$ *is* $(\operatorname{dom} D)$-*coercive at infinity, the range is closed.*

Proof. Let φ_i be a sequence in $\operatorname{dom} D$ with $D\varphi_i \to \psi$ in L^2. We have to show that ψ is in the image of D. W.l.o.g. we can assume that $\varphi_i \perp \ker D$. Then $(\operatorname{dom} D)$-coercivity at infinity gives that φ_i is bounded in L^2 and also in the graph norm of D. Thus, $\varphi_i \to \varphi$ weakly in $\|.\|_D$. Closedness of $\operatorname{dom} D$ then implies that $D\varphi = \psi$. $\qquad\square$

Extension of the trace map. The Trace Theorem A.5 extends the trace map

$$R : \Gamma_c^\infty(M, \mathbb{S}_M) \quad \to \quad \Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)$$
$$\varphi \quad \mapsto \quad \varphi|_\Sigma.$$

Here, we will extend R further to $\operatorname{dom} D_{\max}$. This will generalize the corresponding result [7, Theorem 6.7.ii] for closed boundaries to noncompact boundaries. Moreover, we give some auxiliary lemmata which are found in [7] for closed boundaries. Some of the proofs and the order of obtaining them will be a little bit different since we do not use (and cannot use, cf. Example 3.14.iv) the projection to the negative spectrum and we use an abstract extension map as given by Theorem A.6.

Lemma 3.3. *The space* $\Gamma_c^\infty(M, \mathbb{S}_M)$ *is dense in* $\operatorname{dom} D_{\max}$ *w.r.t. the graph norm.*

Proof. For a closed boundary, this is done in [7, Theorem 6.7.i]. We use a different proof here. Let $\varphi \in \operatorname{dom} D_{\max}$. Let K_i be a compact exhaustion of M that comes together with smooth cut-off functions $\eta_i : M \to [0, 1]$ such that $\eta_i = 1$ on K_i, $\eta_i = 0$ on K_{i+1} and $\max |d\eta_i| \leq \frac{2}{i}$. Then, $\varphi_i = \eta_i \varphi$ are compactly supported sections in $\operatorname{dom} D_{\max}$ fulfilling

$$\|\varphi_i - \varphi\|_D^2 \quad \leq \quad \|\varphi_i - \varphi\|_{L^2}^2 + \|D\varphi_i - D\varphi\|_{L^2}^2$$
$$\leq \quad \|(1-\eta_i)\varphi\|_{L^2}^2 + \left(\|(1-\eta_i)D\varphi\|_{L^2} + \frac{2}{i}\|\varphi\|_{L^2}\right)^2 \to 0.$$

Each φ_i has now compact support in K_{i+1}. Thus, there is a sequence $\varphi_{ij} \in \Gamma_c^\infty(M \setminus K_{i+1}, \mathbb{S}_M)$ with $\varphi_{ij} \to \varphi_i$ in the graph norm on $M \setminus K_{i+1}$. Choose $j = j(i)$ such that $\|\varphi_{ij} - \varphi\|_D \leq \|\varphi_{ij} - \varphi_i\|_D + \|\varphi_i - \varphi\|_D \leq i^{-1}$ and $i \leq j$. Then

$$\|\eta_j \varphi_{ij} - \varphi_{ij}\|_D^2 \quad \leq \quad \|(1-\eta_j)\varphi_{ij}\|_{L^2}^2 + (\|(1-\eta_j)D\varphi_{ij}\|\|_{L^2} + \|d\eta_j \cdot \varphi_{ij}\|_{L^2})^2$$
$$\leq \quad (\|(1-\eta_j)(\varphi_{ij} - \varphi_i)\|_{L^2} + \|(1-\eta_j)\eta_i\varphi\|_{L^2})^2 + \Big(\|(1-\eta_i)D(\varphi_{ij} - \varphi_i)\|_{L^2}$$
$$+\|(1-\eta_j)(\eta_i D\varphi + d\eta_i \cdot \varphi)\|_{L^2} + \frac{2}{j}\|\varphi_{ij} - \varphi_i\|_{L^2} + \frac{2}{j}\|\varphi\|_{L^2}\Big)^2 \to 0$$

for $i \to \infty$. Thus, we have a sequence $\hat{\varphi}_i := \eta_{j(i)}\varphi_{ij(i)} \in \Gamma_c^\infty(M, \mathbb{S}_M)$ such that $\hat{\varphi}_i \to \varphi$ in the graph norm as $i \to \infty$. $\qquad\square$

Theorem 3.4. *The trace map* $R : \Gamma_{cc}^\infty(M, \mathbb{S}_M) \to \Gamma_{cc}^\infty(\Sigma, \mathbb{S}_M|_\Sigma)$ *can be extended to a bounded operator*

$$R : \operatorname{dom} D_{\max} \to H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma).$$

Proof. Let $\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M)$ and $\psi \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$. Then by Theorem A.6, the spinor $\mathcal{E}\psi \in H_1(M, \mathbb{S}_M)$. Thus, we can use Lemma 2.5 and (9) to obtain

$$|(\nu \cdot \varphi|_\Sigma, \psi)_\Sigma| = |(D\varphi, \mathcal{E}\psi) - (\varphi, D\mathcal{E}\psi)| \leq \|D\varphi\|_{L^2}\|\mathcal{E}\psi\|_{L^2} + \|\varphi\|_{L^2}\|D\mathcal{E}\psi\|_{L^2}$$
$$\leq 2\|\varphi\|_D\|\mathcal{E}\psi\|_D \leq C\|\varphi\|_D\|\mathcal{E}\psi\|_{H_1} \leq C'\|\varphi\|_D\|\psi\|_{H_{\frac{1}{2}}(\Sigma)}.$$

Together with Lemma A.7, this implies

$$\|\varphi\|_{H_{-\frac{1}{2}}(\Sigma)} \leq C'\|\varphi\|_D.$$

Since $\Gamma_c^\infty(M, \mathbb{S}_M)$ is dense in $\operatorname{dom} D_{\max}$ w.r.t. the graph norm, cf. Lemma 3.3, the claim follows. $\qquad\square$

Remark 3.5. Note that R is not surjective here. For closed boundaries the image was specified in [7, Theorems 1.7 and 6.7.ii]. For noncompact boundaries we still don't have an explicit description of the image. Abstractly it will be considered in Lemma 3.11.

Lemma 3.6. *Equality* (3) *holds for all* $\varphi \in \operatorname{dom} D_{\max}$ *and* $\psi \in H_1(M, \mathbb{S}_M)$.

Proof. The proof is done as the one of Lemma 2.5 starting with $\psi_j, \varphi_i \in \Gamma_c^\infty(M, \mathbb{S}_M)$ where $\psi_j \to \psi$ in H_1 and $\varphi_i \to \varphi$ in the graph norm of D and using the (extended) Trace Theorem 3.4. The only difference is seen in the estimate of the boundary integrals which now read e.g.

$$\left| \int_\Sigma \langle \nu \cdot R\psi_j, R\varphi_i - R\varphi \rangle ds \right| \leq \|R\psi_j\|_{H_{\frac{1}{2}}(\Sigma)}\|R(\varphi_i - \varphi)\|_{H_{-\frac{1}{2}}(\Sigma)} \leq C\|\psi_j\|_{H_1}\|\varphi_i - \varphi\|_D \to 0$$

where the last inequality uses both versions of the Trace Theorem A.5 and 3.4. $\qquad\square$

The next Lemma gives a full description of $\operatorname{dom} D_{\min}$:

Lemma 3.7. *The* H_1*-norm and the graph norm* $\|.\|_D$ *are equivalent on*

$$\{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi = 0\}.$$

In particular,

$$\operatorname{dom} D_{\min} = \overline{\Gamma_{cc}^\infty(M, \mathbb{S}_M)}^{\|.\|_D} = \overline{\Gamma_{cc}^\infty(M, \mathbb{S}_M)}^{\|.\|_{H_1}} \quad = \quad \{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi = 0\}$$
$$= \quad \{\varphi \in H_1(M, \mathbb{S}_M) \mid R\varphi = 0\}.$$

Proof. Firstly we show the equivalence on $\{\psi \in \Gamma_c^\infty(M, \mathbb{S}_M) \mid R\psi = 0\}$: Let φ be in this set. Then by (8) we have

$$\|\varphi\|_{H_1}^2 = \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2 - \int_M \frac{\operatorname{scal}^M}{4}|\varphi|^2 dv - \int_M \frac{i}{2}\langle \Omega \cdot \varphi, \varphi \rangle dv \leq C\|\varphi\|_D^2,$$

where we used that M and L are of bounded geometry and, hence, $|\operatorname{scal}^M|$ and $|\Omega|$ are uniformly bounded on all of M. The reverse inequality was seen in (9). From the definition of $\operatorname{dom} D_{\min}$ and the equivalence of the norms from above, we already have $\operatorname{dom} D_{\min} = \overline{\Gamma_{cc}^\infty}^{\|.\|_D} = \overline{\Gamma_{cc}^\infty}^{\|.\|_{H_1}}$. From the Trace Theorem 3.4, we get

$$\overline{\Gamma_{cc}^\infty}^{\|.\|_D} \subset \{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi = 0\}.$$

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Next we want to show that $D : \{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi = 0\} \to L^2(M, \mathbb{S}_M)$ already equals $D_{\min}$. First we note that by the Trace Theorem 3.4, D is a closed extension of D_{cc}. It suffices to show that $D^* = D_{\max}$. By definition, we have

$$\operatorname{dom} D^* = \{\vartheta \in L^2(M, \mathbb{S}_M) \mid \exists \chi \in L^2(M, \mathbb{S}_M) \, \forall \psi \in \operatorname{dom} D_{\max}, R\psi = 0 : (\vartheta, D\psi) = (\chi, \psi)\}.$$

Let $\vartheta \in \operatorname{dom} D_{\max}$. By Lemma 3.3, there exists a sequence $\vartheta_i \in \Gamma_c^\infty(M, \mathbb{S}_M)$ with $\vartheta_i \to \vartheta$ in the graph norm. Hence, for all $\psi \in \operatorname{dom} D_{\max}$ with $R\psi = 0$ we have $(\vartheta, D\psi) = \lim_{i \to \infty} (\vartheta_i, D\psi)$. Then by Lemma 3.6 and $R\psi = 0$, we obtain

$$(\vartheta, D\psi) = \lim_{i \to \infty} (D\vartheta_i, \psi) = (D\vartheta, \psi),$$

which implies that $\vartheta \in \operatorname{dom} D^*$. Together with

$$\operatorname{dom} D_{\min} = \overline{\Gamma_{cc}^\infty}^{\|\cdot\|_D} \subset \{\varphi \in H_1(M, \mathbb{S}_M) \mid R\varphi = 0\} \subset \{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi = 0\} = \operatorname{dom} D_{\min},$$

the rest of the Lemma follows. $\qquad\square$

Now we can describe H_1 in terms of its image under the trace map.

Lemma 3.8. *We have $H_1(M, \mathbb{S}_M) = \{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)\}$.*

Proof. The inclusion '$\subset$' is clear from the Trace Theorem A.5. It remains to prove '$\supset$': Let $\varphi \in \operatorname{dom} D_{\max}$. Then with Theorem A.6, $R\varphi \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$ implies that $\psi := \mathcal{E}R\varphi \in H_1(M, \mathbb{S}_M)$. Thus, $\varphi - \psi \in \operatorname{dom} D_{\max}$ and $R(\varphi - \psi) = 0$. But due to Lemma 3.7, $\varphi - \psi \in H_1(M, \mathbb{S}_M)$ and, hence, $\varphi \in H_1(M, \mathbb{S}_M)$. $\qquad\square$

Let $\tilde{\nu}$ be an extension of the unit inner normal vector field ν on Σ to a unit smooth vector field on U_Σ such that $|\nabla \tilde{\nu}|$ is uniformly bounded on U_Σ. Such a $\tilde{\nu}$ always exists since (M, Σ) is of bounded geometry. Note that $\nu \cdot R\varphi = R(\tilde{\nu} \cdot \varphi)$ for any φ with support in U_Σ.

Lemma 3.9. *On the set $\mathcal{U} := \{\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M) \mid \operatorname{supp} \varphi \subset U_\Sigma\}$, we have $\|\tilde{\nu} \cdot \varphi\|_D \sim \|\varphi\|_D$ and $\|\tilde{\nu} \cdot \varphi\|_{H_1} \sim \|\varphi\|_{H_1}$ where $\sim$ denotes the equivalence of the norms. Moreover, there is a constant $C > 0$ such that for all $\varphi \in \mathcal{U}$, we have*

$$\|\mathcal{E}R\varphi\|_D \leq C\|\varphi\|_D.$$

Proof. Since $\tilde{\nu}$ is a unit vector field, we get immediately $\|\tilde{\nu} \cdot \varphi\|_{L^2} = \|\varphi\|_{L^2}$. Moreover, $D(\tilde{\nu} \cdot \varphi) = \tilde{\nu} \cdot D\varphi + \nabla \tilde{\nu} \cdot \varphi$ and the uniform bound on $\nabla \tilde{\nu}$ gives

$$\|D(\tilde{\nu} \cdot \varphi)\|_{L^2} \leq \|D\varphi\|_{L^2} + C\|\varphi\|_{L^2}.$$

Hence, $\|\tilde{\nu} \cdot \varphi\|_D \leq C'\|\varphi\|_D$. Using this inequality for $\varphi = \tilde{\nu} \cdot \psi$ we obtain the corresponding converse inequality and, thus, the first claimed equivalence. Analogously one obtains the equivalence for the H_1-norms. By Inequality (9), Theorems A.5 and A.6 we get for certain constants C_i that for all $\varphi \in \mathcal{U}$

$$\|\mathcal{E}R\varphi\|_D^2 \leq C_1\|\mathcal{E}R\varphi\|_{H_1}^2 \leq C_2\|R\varphi\|_{H_{\frac{1}{2}}}^2 \leq C_3\|\varphi\|_{H_1}^2. \tag{10}$$

Moreover, by (3) and the norm equivalences from above

$$\int_\Sigma |R\varphi|^2 ds = -\int_\Sigma \langle \nu \cdot R(\tilde{\nu} \cdot \varphi), R\varphi \rangle ds = (D(\tilde{\nu} \cdot \varphi), \varphi) - (D\varphi, \tilde{\nu} \cdot \varphi)$$

$$\leq \|D(\tilde{\nu} \cdot \varphi)\|_{L^2}\|\varphi\|_{L^2} + \|D\varphi\|_{L^2}\|\tilde{\nu} \cdot \varphi\|_{L^2} \leq C\|\varphi\|_D^2. \tag{11}$$

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Let us first consider those $\varphi \in \mathcal{U}$ with $\int_\Sigma \langle R\varphi, \tilde{D}R\varphi \rangle ds \leq 0$: Then by (10), (11), (8) and the bounded geometry of (M, Σ) and L, we have

$$\|\mathcal{E}R\varphi\|_D^2 \leq C_3 \|\varphi\|_{H_1}^2$$

$$= C_3 \left(\|\varphi\|_D^2 - \int_M \frac{1}{4} \langle \mathrm{scal}^M \varphi + 2i\Omega \cdot \varphi, \varphi \rangle dv + \int_\Sigma \left\langle R\varphi, \left(\tilde{D} - \frac{nH}{2} \right) R\varphi \right\rangle ds \right)$$

$$\leq C_3 \|\varphi\|_D^2 + C_4 \|\varphi\|_{L^2}^2 + C_5 \int_\Sigma |R\varphi|^2 ds \leq C_6 \|\varphi\|_D^2.$$

It remains to consider those $\varphi \in \mathcal{U}$ with $\int_\Sigma \langle R\varphi, \tilde{D}R\varphi \rangle ds > 0$: Then by (7),

$$\int_\Sigma \langle R(\tilde{\nu} \cdot \varphi), \tilde{D}R(\tilde{\nu} \cdot \varphi) \rangle ds = \int_\Sigma \langle \nu \cdot R\varphi, -\nu \cdot \tilde{D}R\varphi \rangle ds = -\int_\Sigma \langle R\varphi, \tilde{D}R\varphi \rangle ds < 0.$$

Hence, together with the norm equivalences, we obtain similar as above that

$$
\begin{aligned}
\|\mathcal{E}R\varphi\|_D^2 &\leq C_3 \|\varphi\|_{H_1}^2 \leq C_7 \|\tilde{\nu} \cdot \varphi\|_D^2 \\
&= C_7 \Big(\|\varphi\|_D^2 - \int_M \frac{1}{4} \langle (\mathrm{scal}^M + 2i\Omega \cdot) \tilde{\nu} \cdot \varphi, \tilde{\nu} \cdot \varphi \rangle dv \\
&\quad + \int_\Sigma \left\langle R(\tilde{\nu} \cdot \varphi), \left(\tilde{D} - \frac{nH}{2} \right) R(\tilde{\nu} \cdot \varphi) \right\rangle ds \Big) \\
&\leq C_7 \|\varphi\|_D^2 + C_8 \|\varphi\|_{L^2}^2 + C_9 \int_\Sigma |R\varphi|^2 ds \leq C_{10} \|\varphi\|_D^2.
\end{aligned}
$$

Setting $C^2 := \max\{C_6, C_{10}\}$ we obtain the claim. $\square$

Theorem 3.10. *There is a constant $C > 0$ such that for all $\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M)$*

$$\|\mathcal{E}R\varphi\|_D \leq C\|\varphi\|_D.$$

In particular, the map $\mathcal{E}R : \Gamma_c^\infty(M, \mathbb{S}_M) \to \Gamma_c^\infty(M, \mathbb{S}_M)$ extends to a bounded linear map from $\mathrm{dom}\, D_{\max}$ to itself.

Proof. Let $\eta : M \to [0, 1]$ be a smooth cut-off function that is 1 on $\Sigma \times [0, \frac{r_\partial}{2}]$ and 0 outside U_Σ. Moreover, let η be such that $d\eta$ is uniformly bounded. Let $\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M)$. Then, $\eta\varphi$ is a compactly supported smooth spinor with support in U_Σ. Moreover, $R(\eta\varphi) = R\varphi$. Hence by Lemma 3.9, we have

$$
\begin{aligned}
\|\mathcal{E}R\varphi\|_D &= \|\mathcal{E}R(\eta\varphi)\|_D \leq C\|\eta\varphi\|_D \leq C\|\varphi\|_{L^2} + C\|\eta D\varphi + d\eta \cdot \varphi\|_{L^2} \\
&\leq C\|\varphi\|_{L^2} + C\|D\varphi\|_{L^2} + C \sup |d\eta| \|\varphi\|_{L^2} \leq C'\|\varphi\|_D.
\end{aligned}
$$

$\square$

The last theorem can be seen as a workaround against a lack of an explicit description of $R(\mathrm{dom}\, D_{\max})$. For closed boundaries, it is a consequence of [7, Lemmas 6.1 and 6.2]. For us, it is enough to equip $R(\mathrm{dom}\, D_{\max})$ with a norm. On $R(\mathrm{dom}\, D_{\max})$, we set

$$\|\psi\|_R := \|\mathcal{E}R\varphi\|_D,$$

where $R\varphi = \psi$. By Theorem 3.10, this is well defined.

Lemma 3.11. *The space $(R(\mathrm{dom}\, D_{\max}), \|.\|_R)$ is a Banach space.*

Proof. From the definition of $\|.\|_R$ and the fact that $\|.\|_D$ is a norm, we get immediately that $\|.\|_R$ is a norm on $R(\mathrm{dom}\, D_{\max})$. It remains to show completeness: For that we consider a Cauchy sequence ψ_i in $R(\mathrm{dom}\, D_{\max})$. Then, there is a sequence $\varphi_i \in \mathrm{dom}\, D_{\max}$ with $R\varphi_i = \psi_i$. With the definition of the R-norm, we get that $\mathcal{E}R\varphi_i$ is a Cauchy sequence in the Banach space $(\mathrm{dom}\, D_{\max}, \|.\|_D)$ and, hence, there is a $\varphi \in \mathrm{dom}\, D_{\max}$ with $\mathcal{E}R\varphi_i \to \varphi$ w.r.t. the graph norm. By Theorem 3.10, we get

$$\|\mathcal{E}R(\varphi_i - \varphi)\|_D \leq C\|\mathcal{E}R\varphi_i - \varphi\|_D \to 0.$$

Thus, $\mathcal{E}R\varphi = \varphi$ and $\|\psi_i - R\varphi\|_R = \|\mathcal{E}(R\varphi_i - R\varphi)\|_D \to 0$. Hence, $\psi_i \to \psi$ in the R-norm. $\square$

Boundary conditions. In this part, we show that each closed extension of D_{cc} can be realized by a closed subset of $R(\mathrm{dom}\, D_{\max})$. We give some examples and comment on a difference to the case of closed boundaries.

Lemma 3.12. *Let D be a closed extension of D_{cc} with $B := R(\mathrm{dom}\, D) \subset H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$. Then, its domain $\mathrm{dom}\, D$ equals $\mathrm{dom}\, D_B := \{\varphi \in \mathrm{dom}\, D_{\max} \mid R\varphi \in B\}$ and B is a closed subset of $(R(\mathrm{dom}\, D_{\max}), \|.\|_R)$. Conversely, for every closed subset $B \subset R(\mathrm{dom}\, D_{\max})$ the operator $D_B : \mathrm{dom}\, D_B \to L^2(M, \mathbb{S}_M)$ is a closed extension of D_{cc}.*

Due to this Lemma, a closed subset B of $R(\mathrm{dom}\, D_{\max})$ is called boundary condition.

Proof. Let D be a closed extension of D_{cc} with domain $\mathrm{dom}\, D$ and $B := R(\mathrm{dom}\, D)$. Clearly, $\mathrm{dom}\, D \subset \mathrm{dom}\, D_B$. We have to show that also the converse is true: Let $\varphi \in \mathrm{dom}\, D_B$. Then, there exists $\psi \in \mathrm{dom}\, D$ with $R\varphi = R\psi$. By Lemma 3.7, $\varphi - \psi \in \mathrm{dom}\, D_{\min} \subset \mathrm{dom}\, D$ and, hence, $\varphi \in \mathrm{dom}\, D$. This implies that $\mathrm{dom}\, D = \mathrm{dom}\, D_B$. Next, we show that B is closed. For that let $\rho_i \in B$ such that $\rho_i \to \rho \in R(\mathrm{dom}\, D_{\max})$ w.r.t. $\|.\|_R$. By definition of the R-norm, we get $\mathcal{E}(\rho_i - \rho) \to 0$ in the graph norm. Together with $\mathcal{E}\rho_i \in \mathrm{dom}\, D_B = \mathrm{dom}\, D$ and the closedness of D, we get $\mathcal{E}\rho \in \mathrm{dom}\, D$ and, thus, $\rho \in B$. Hence, $B \subset R(\mathrm{dom}\, D_{\max})$ is closed. Let now $B \subset R(\mathrm{dom}\, D_{\max})$ be a closed subset. Then, $\mathrm{dom}\, D_{\min} \subset \mathrm{dom}\, D_B \subset \mathrm{dom}\, D_{\max}$. Let $\varphi_i \in \mathrm{dom}\, D_B \to \varphi \in \mathrm{dom}\, D_{\max}$ in the graph norm. Then, Theorem 3.10 gives

$$\|R\varphi_i - R\varphi\|_R = \|\mathcal{E}R(\varphi_i - \varphi)\|_D \leq C\|\varphi_i - \varphi\|_D \to 0.$$

By the closedness of B, this implies $R\varphi \in B$ and, hence, $\varphi \in \mathrm{dom}\, D_B$. Thus, D_B is closed. $\square$

Remark 3.13. The definition of $\mathrm{dom}\, D_B$ in [7, Section 7] uses the space $H_1^D := \overline{\Gamma_c^\infty(M, \mathbb{S}_M)}^{\|.\|_{H_1^D}}$ instead of H_1 where the H_1^D-norm is given by

$$\|\varphi\|_{H_1^D}^2 = \|\chi\varphi\|_{H_1}^2 + \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2.$$

Here χ denotes an appropriate cut-off function such that $\chi\varphi$ only lives on a small collar of the boundary. Since we work with the classical Dirac operator on Spin^c manifolds and assume (M, Σ) and L being of bounded geometry, both norms coincide. C. Bär and W. Ballmann consider a more general situation where it suffices that M is only complete but not necessarily of bounded geometry. Then the H_1^D-norm is needed. We could also switch to this more general setup when dropping the condition (i) and (iii) in the Definition 2.2 while still assuming that $(\Sigma, g|_\Sigma)$ is of bounded geometry and that the curvature tensor and its derivatives are bounded on U_Σ. For that situation, we would also obtain Theorem 1.2. But in order to simplify notation we stick to the bounded geometry of (M, Σ).

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Example 3.14.   (1) **Minimal and maximal extension.** $B = 0$ gives the minimal extension $D_{B=0} = D_{\min}$, cf. Lemma 3.7. The maximal extension is obtained with $B = R(\operatorname{dom} D_{\max})$.

(2) $D_{B=H_{\frac{1}{2}}} : H_1(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$ is an extension of D_{cc} but not closed (if the boundary is nonempty): Since $\Gamma_c^\infty(M, \mathbb{S}_M) \subset H_1$ and $\Gamma_c^\infty(M, \mathbb{S}_M)$ dense in $\operatorname{dom} D_{\max}$, the closure of $D_{B=H_{\frac{1}{2}}}$ is $D_{\max}$.

(3) [24, Section 6] Let $P_\pm : L^2(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma)$, $\varphi \mapsto \frac{1}{2}(\varphi \pm \nu \cdot \varphi)$ and

$$D_\pm : \operatorname{dom} D_\pm := \{\varphi \in \operatorname{dom} D_{\max} \mid P_\pm R \varphi = 0\} \to L^2(M, \mathbb{S}_M).$$

In Section 4, we will show that $D_\pm$ is a closed extension and that $D_\pm = D_{B_\pm}$ where

$$B_\pm = \{\varphi \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma) \mid P_\pm \varphi = 0\}.$$

Each φ decomposes uniquely into $\varphi = P_+ \varphi + P_- \varphi$ and if $\varphi \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$, $P_\pm \varphi \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$, too. This assures that the $B_\pm$'s are honestly larger than the trivial boundary condition $B = \{0\}$. More properties of this boundary condition can be found in Section 4.

(4) **APS boundary conditions.** An obvious way to generalize the APS boundary conditions for a closed boundary to our situation is given by the following: Let $\{E_I\}_{I \subset \mathbb{R}}$ be the family of projector-valued measures belonging to the self-adjoint operator

$$\widetilde{D} : H_1(\Sigma, \mathbb{S}_M|_\Sigma) \subset L^2(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma).$$

Define for $a \in \mathbb{R}$

$$\pi_{\geq a \ (\leq a)} : L^2(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma), \ \varphi \mapsto E_{[a,\infty)}\varphi \ (E_{(-\infty,a]}\varphi)$$

and the spaces

$$\Gamma^{\mathrm{APS}}_{\geq a \ (\leq a)} = \{\varphi \in L^2(\Sigma, \mathbb{S}_M|_\Sigma) \mid \varphi = \pi_{\geq a \ (\leq a)}\varphi\}.$$

We set $B^{\mathrm{APS}}_{\geq a \ (\leq a)} = R(\operatorname{dom} D_{\max}) \cap \Gamma^{\mathrm{APS}}_{\geq a(\leq a)}$. Up to now, we have still the following question: When is $\pi_{\geq a(\leq a)}H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma) \subset H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$ as for closed manifolds or can it even happen that $B^{\mathrm{APS}}_{\geq a \ (\leq a)} = \{0\}$? In case of a closed boundary Σ (or more generally in case that $L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ has an orthonormal basis of eigenspinors) $\pi_{\geq a(\leq a)}H_{\frac{1}{2}} \subset H_{\frac{1}{2}}$ is trivially fulfilled. Then, $B^{\mathrm{APS}}_{\geq a \ (\leq a)}$ really define nontrivial boundary conditions with $B^{\mathrm{APS}}_{\geq a} \cup B^{\mathrm{APS}}_{\leq a} = H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$. Even if this is still true in general, $B^{\mathrm{APS}}_{\geq a} \cup B^{\mathrm{APS}}_{\leq a}$ is a disjoint union only if $a \notin \sigma(\tilde{D})$. One could make this disjoint by using $B^{\mathrm{APS}}_{\geq a} \cup B^{\mathrm{APS}}_{< a}$ but then $B^{\mathrm{APS}}_{< a}$ may not be closed.

Boundary value problems. Now we can just collect all preliminary work to obtain

Proposition 3.15. *Let B be a boundary condition and the Dirac operator*

$$D_B : \operatorname{dom} D_B \subset L^2(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$$

be B-coercive at infinity. Let $P_{B^\perp} := \operatorname{Id} - P_B : R(\operatorname{dom} D_{\max}) \to R(\operatorname{dom} D_{\max})$ where P_B is the projection from $R(\operatorname{dom} D_{\max})$ to B. Then, for all $\psi \in L^2(M, \mathbb{S}_M)$ and $\tilde{\rho} \in \operatorname{dom} D_{\max}$

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where $\psi - D\tilde{\rho} \in (\ker(D_B)^*)^\perp$, the boundary value problem

$$\begin{cases} D\varphi = \psi & \text{on } M, \\ P_{B^\perp} R\varphi = P_{B^\perp} R\tilde{\rho} & \text{on } \Sigma \end{cases}$$

has a unique solution $\varphi \in \operatorname{dom} D_{\max}$ up to elements of the kernel $\ker D_B$.

Proof. Since D is B-coercive at infinity, its range is closed by Lemma 3.2. Thus, due to the Closed Range Theorem 2.6, the spinor $\psi - D\tilde{\rho} \in \operatorname{ran} D_B$. Hence, there exists $\hat{\varphi} \in \operatorname{dom} D_B$ with $D\hat{\varphi} = \psi - D\tilde{\rho}$. Setting $\varphi = \hat{\varphi} + \tilde{\rho}$, we get $\varphi \in \operatorname{dom} D_{\max}$, $D\varphi = \psi$ and $P_{B^\perp}\varphi = P_{B^\perp}\hat{\varphi} + P_{B^\perp} R\tilde{\rho} = P_{B^\perp} R\tilde{\rho}$. $\qquad\square$

Corollary 3.16. *Let B be a boundary condition such that $B \subset H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$. We assume that the Dirac operator $D : \operatorname{dom} D_B \subset L^2(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$ is B-coercive at infinity and we consider $P_{B^\perp} := Id - P_B$ be as above. Then, for all $\psi \in L^2(M, \mathbb{S}_M)$ and $\rho \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$ where for all $\chi \in \ker(D_B)^*$*

$$(\psi, \chi) + (\nu \cdot \rho, R\chi) = 0 \tag{12}$$

the boundary value problem

$$\begin{cases} D\varphi = \psi & \text{on } M, \\ P_{B^\perp} R\varphi = P_{B^\perp}\rho & \text{on } \Sigma \end{cases}$$

has a unique solution $\varphi \in H_1(M, \mathbb{S}_M)$ up to elements of the kernel $\ker D_B$.

Proof. By Lemma 3.8, $B \subset H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$ implies $\operatorname{dom} D_B \subset H_1(M, \mathbb{S}_M)$. We set $\tilde{\rho} = \mathcal{E}\rho$. By the Trace Theorem A.5, $\tilde{\rho} \in H_1(M, \mathbb{S}_M)$. Moreover, by Lemma 3.6 the integrability condition (12) implies this $\psi - D\tilde{\rho} \in (\ker(D_B)^*)^\perp$. Hence, as in Proposition 3.15, we get $\hat{\varphi} \in \operatorname{dom} D_B \subset H_1(M, \mathbb{S}_M)$ and the solution $\varphi = \hat{\varphi} + \tilde{\rho} \in H_1(M, \mathbb{S}_M)$. $\qquad\square$

Remark 3.17. In order to give a full generalization of the theory given in [7] it would be interesting to examine the following questions:
- Consider general boundary condition, in particular we would like to identify the image of the extended trace map in Theorem 3.4.
- Give a generalization of the definition for elliptic boundary conditions for noncompact boundaries (of bounded geometry) and study them.
- Consider more generally complete Dirac-type operators as in [7].

4. ON THE BOUNDARY CONDITION $B_\pm$

In this section, we briefly recall and give some basic facts on $P_\pm$. Some of them can be found in [24, Section 6]. Moreover, we prove the claims of Example 3.14.(3).

Lemma 4.1. *Let $P_\pm : L^2(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ be the map $\varphi \mapsto \frac{1}{2}(\varphi \pm i\nu \cdot \varphi)$ and consider $B_\pm := \{\varphi \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma) \mid P_\pm\varphi = 0\}$. Then, the following hold*

   (i) *$P_\pm$ are self-adjoint projections, orthogonal to each other and $\nu P_\pm = P_\pm \nu = \mp i P_\pm$.*

   (ii) *For all $s \in \mathbb{R}$ $P_\pm(\varphi) = \frac{1}{2}(\varphi \pm i\nu \cdot \varphi)$ gives an operator from $H_s(\Sigma, \mathbb{S}_M|_\Sigma)$ to itself such that for all $\varphi \in H_s(\Sigma, \mathbb{S}_M|_\Sigma)$ and $\psi \in H_{-s}(\Sigma, \mathbb{S}_M|_\Sigma)$ we have $(P_+\varphi, P_-\psi)_\Sigma = 0$ and $(P_\pm\varphi, \psi)_\Sigma = (\varphi, P_\pm\psi)_\Sigma$.*

   (iii) *$\widetilde{D} P_\pm = P_\mp \widetilde{D}$.*

(iv) $D_\pm$ is a closed extension of D_{cc}.
(v) $D_\pm = D_{B_\pm}$.
(vi) $(D_{B_\pm})^* = D_{B_\mp}$.
(vii) $\ker D_{B_\pm} = 0$.

Proof. Assertions (i) and (ii) follow directly by simple calculation and (iii) follows directly from (7). For (iv) we have by definition of $D_\pm$ (see Example 3.14.iii) that $D_\pm = D_{\tilde{B}_\pm}$ where $\tilde{B}_\pm = \{\varphi \in R(\mathrm{dom}\, D_{\max}) \mid P_\pm \varphi = 0\}$. In order to show the closedness of $D_\pm$ we want to apply Lemma 3.12. For that, we have to have w that $\tilde{B}_\pm$ is closed in $R(\mathrm{dom}D_{\max})$: Let $\varphi_i \in \tilde{B}_\pm$ with $\varphi_i \to \varphi$ in $R(\mathrm{dom}\, D_{\max})$. Then,

$$\|P_\pm \varphi\|_R = \|P_\pm(\varphi - \varphi_i)\|_R \le \|\varphi - \varphi_i\|_R \to 0.$$

Hence, $P_\pm \varphi = 0$ and $\varphi \in \tilde{B}_\pm$. For (v), we have clearly that $\mathrm{dom}\, D_{B_\pm} \subset \mathrm{dom}\, D_\pm$. It remains to show that any $\varphi \in \mathrm{dom}\, D_\pm$ is already in $H_1(M, \mathbb{S}_M)$. By Lemma 3.3, there is a sequence $\varphi_i \in \Gamma_c^\infty(M, \mathbb{S}_M)$ with $\varphi_i \to \varphi$ in the graph norm. Consider $\mathcal{E}P_\pm R\varphi_i$. By Theorem 3.10 and using η and $\tilde{\nu}$ from Lemma 3.9, we have

$$\|\mathcal{E}P_\pm R\varphi_i\|_D = \|\mathcal{E}P_\pm R\eta(\varphi_i - \varphi)\|_D = \frac{1}{2}\|\mathcal{E}R\eta((\varphi_i - \varphi) \pm i\tilde{\nu} \cdot (\varphi_i - \varphi))\|_D$$

$$\le \frac{1}{2}(\|\eta(\varphi_i - \varphi)\|_D + \|\tilde{\nu} \cdot \eta(\varphi_i - \varphi)\|_D) \le C\|\varphi_i - \varphi\|_D \to 0.$$

Hence, $\psi_i := \varphi_i - \mathcal{E}P_\pm R\varphi_i \in \mathrm{dom}\, D_\pm$ and $\psi_i \to \varphi$ in the graph norm. This implies that $\mathrm{dom}\, D_{B_\pm}$ is dense in $\mathrm{dom}\, D_\pm$. Moreover, note that with (iii) we have

$$\int_\Sigma \langle R\psi_i, \tilde{D}R\psi_i\rangle ds = \int_\Sigma \langle P_\mp R\psi_i, \tilde{D}P_\mp R\psi_i\rangle ds = \int_\Sigma \langle P_\mp R\psi_i, P_\pm \tilde{D}R\psi_i\rangle ds = 0.$$

Hence, together with the Lichnerowicz formula in Lemma 2.5, the bounded geometry and an estimate as in (11), we get

$$\|\psi_i - \psi_j\|_{H_1}^2 = \|\psi_i - \psi_j\|_D^2 - \frac{1}{4}\int_M \langle(\mathrm{scal}^M + 2i\Omega\cdot)(\psi_i - \psi_j), (\psi_i - \psi_j)\rangle dv - \frac{n}{2}\int_\Sigma H|\psi_i - \psi_j|^2 ds$$

$$\le C\|\psi_i - \psi_j\|_D^2.$$

Thus, ψ_i is even a Cauchy sequence in H_1 which implies that φ is already in $H_1(M, \mathbb{S}_M)$. For (vi), the domain of the adjoint is defined by

$$\mathrm{dom}\,(D_+)^* = \{\vartheta \in L^2(M, \mathbb{S}_M) \mid \exists \chi \in L^2(M, \mathbb{S}_M)\, \forall \psi \in \mathrm{dom}\, D_+ : (\chi, \psi) = (\eta, D\psi)\}.$$

Since, $\Gamma_{cc}^\infty(M, \mathbb{S}_M) \subset \mathrm{dom}\, D_+$, we get $\mathrm{dom}\,(D_+)^* \subset \mathrm{dom}\, D_{\max}$. Thus,

$$\mathrm{dom}\,(D_+)^* = \{\vartheta \in \mathrm{dom}\, D_{\max} \mid \forall \psi \in \mathrm{dom}\, D_+ : (D\vartheta, \psi) = (\vartheta, D\psi)\}.$$

Due to Lemma 3.6, the definition of $\mathrm{dom}\, D_+$ and (i), we get

$$\mathrm{dom}\,(D_+)^* = \left\{\vartheta \in \mathrm{dom}\, D_{\max} \mid \forall \psi \in H_1(M, \mathbb{S}_M) : \int_\Sigma \langle R\vartheta, \nu \cdot P_- R\psi\rangle ds = 0\right\}.$$

By (i) and (ii), we have

$$\int_\Sigma \langle R\vartheta, \nu \cdot P_- R\psi\rangle ds = -i\int_\Sigma \langle R\vartheta, P_- R\psi\rangle ds = -i\int_\Sigma \langle R_- R\vartheta, P_- R\psi\rangle ds$$

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and $P_- R\vartheta \in H_{-\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$. Hence, together with Lemma A.7,

$$\operatorname{dom}(D_+)^* = \left\{ \vartheta \in \operatorname{dom} D_{\max} \mid \forall \hat\psi \in H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma) : \int_\Sigma \langle P_- R\vartheta, \hat\psi \rangle ds = 0 \right\}$$

$$= \{ \vartheta \in \operatorname{dom} D_{\max} \mid P_- R\vartheta = 0 \} = \operatorname{dom} D_-.$$

The assertion (vii) is proven as in the closed case [24, Corollary]: Let $\varphi \in \ker D_\pm$, i.e. $D\varphi = 0$ on M and $P_\pm R\varphi = 0$ on Σ. Using this, (3), Lemma 3.6 and (i), we compute

$$0 = \int_M \langle D\varphi, i\varphi \rangle dv - \int_M \langle \varphi, iD\varphi \rangle dv = \int_\Sigma \langle \nu \cdot R\varphi, iR\varphi \rangle ds$$

$$= \int_\Sigma \langle \nu \cdot P_\mp R\varphi, iP_\mp R\varphi \rangle ds = \mp \int_\Sigma |R\varphi|^2 ds.$$

Hence, $R\varphi = 0$ and $\varphi \in \operatorname{dom} D_{\min}$, cf. Lemma 3.7. But due to the unique continuation property of the Dirac operator, $D_{\min}\varphi = 0$ implies $\varphi = 0$. $\qquad\square$

5. Examples and the coercivity condition

In Definition 3.1, we defined when an operator D_B is $(\operatorname{dom} D_B)$-coercive at infinity. When working with B, we will also use the short version – B-coercive at infinity. In this passage, we will compare this notion with the one of coercivity at infinity given in [7, Definition 8.2] as cited below and give some examples.

Definition 5.1. [7, Definition 8.2] $D : \operatorname{dom} D_{\max} \subset L^2(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$ is coercive at infinity if there is a compact subset $K \subset M$ and a constant $c > 0$ such that

$$\|D\varphi\|_{L^2} \geq c\|\varphi\|_{L^2},$$

for all $\varphi \in \Gamma_c^\infty(M \setminus K, \mathbb{S}_M)$.

By Lemma [7, 8.4], D is coercive at infinity for a closed boundary Σ if and only if there is a compact subset $K \subset M$ and a constant $c > 0$ such that for all $\varphi \in \Gamma_{cc}^\infty(M \setminus K, \mathbb{S}_M)$ we have $\|D\varphi\|_{L^2} \geq c\|\varphi\|_{L^2}$. For noncompact boundaries, just the 'only if'-direction survives since in contrast to closed boundaries there is no compact K such that $\Gamma_c^\infty(M \setminus K, \mathbb{S}_M) \subset \Gamma_{cc}^\infty(M, \mathbb{S}_M)$.

Before we compare those different coercivity conditions we give some examples:

Example 5.2.   (i) By the unique continuation property, the kernel of $D_{\min}$ is trivial. Thus, together with Lemma 3.7, we have that D is $(B = 0)$–coercive at infinity if and only if there is a constant $c > 0$ such that for all $\varphi \in \Gamma_{cc}^\infty(M, \mathbb{S}_M)$

$$\|D\varphi\|_{L^2} \geq c\|\varphi\|_{L^2}.$$

For closed boundaries, this implies coercivity at infinity by Lemma [7, 8.4] which was cited above. We will see that for closed boundaries also the converse is true, cf. Corollary 5.6.

(ii) By Lemma 4.1, $\ker D_{B_\pm} = \{0\}$. Thus, D is $B_\pm$-coercive at infinity if and only if there is a constant $c > 0$ such that

$$\|D\psi\|_{L^2} \geq c\|\psi\|_{L^2}$$

for all $\psi \in H_1(M, \mathbb{S}_M)$ with $P_\pm R\psi = 0$. In particular, this implies $(B = 0)$-coercivity at infinity. More generally, if $B_1 \subset B_2$ and $\ker D_{B_1} = \ker D_{B_2}$, then B_2-coercivity at infinity implies B_1-coercivity at infinity.

Lemma 5.3. *Let D be coercive at infinity and B be a boundary condition. Assume that* $\operatorname{dom} D_B \cap (\ker D_B)^\perp \subset H_1(M, \mathbb{S}_M)$ *and that the H_1-norm and the graph norm are equivalent on* $\operatorname{dom} D_B \cap (\ker D_B)^\perp$. *Then, D is B-coercive at infinity.*

Proof. Since D is coercive at infinity, there is a compact subset $K \subset M$ and a constant $c > 0$ such that $\|D\varphi\|_{L^2} \geq c\|\varphi\|_{L^2}$ for all $\varphi \in \Gamma_c^\infty(M \setminus K, \mathbb{S}_M)$. Assume that D is not B-coercive at infinity. Then, there is a sequence $\varphi_i \in \operatorname{dom} D_B \cap (\ker D_B)^\perp$ with $\|\varphi_i\|_{L^2} = 1$ and $\|D\varphi_i\|_{L^2} \to 0$. By equivalence of the norms, φ_i is also bounded in H_1. This implies $\varphi_i \to \varphi$ weakly in H_1 and, thus, locally strongly in L^2. Moreover, $D\varphi = 0$. Together with $\varphi_i \perp \ker D_B$, this implies $\varphi = 0$. Thus, for each compact subset $K' \subset M$ we have $\int_{K'} |\varphi_i|^2 dv \to 0$ as $i \to \infty$. Let $\eta : M \to [0,1]$ be a cut-off function and K' be a compact subset such that $K \subset K' \subset M$ and $\eta = 0$ on K, $\eta = 1$ on $M \setminus K'$ and $|d\eta| \leq a$ for a constant $a > 0$ big enough. Then, $\operatorname{supp}(\eta\varphi_i) \subset M \setminus K$, $\|D(\eta\varphi_i)\|_{L^2} \leq a\|\varphi_i\|_{L^2(K')} + \|D\varphi_i\|_{L^2} \to 0$ and

$$1 \geq \|\eta\varphi_i\|_{L^2} \geq \|\varphi\|_{L^2} - \|(1-\eta)\varphi_i\|_{L^2} \geq 1 - \|\varphi_i\|_{L^2(K')} \to 1.$$

By Lemma 3.3, we can choose a sequence $(\varphi_{ij})_j \subset \Gamma_c^\infty(M, \mathbb{S}_M)$ with $\varphi_{ij} \to \varphi_i$ in the graph norm as $j \to \infty$. Then, $\eta\varphi_{ij} \to \eta\varphi_i$ in the graph norm and $\operatorname{supp}\eta\varphi_{ij} \in M \setminus K$. Thus, we can find $j = j(i)$ such that $\|D(\eta\varphi_{ij(i)})\|_{L^2} \to 0$ and $\|\eta\varphi_{ij(i)}\|_{L^2} \to 1$ as $i \to \infty$. But this contradicts the assumption that D is coercive at infinity. $\square$

Corollary 5.4. *If D is coercive at infinity, then D is $(B = 0)$-coercive at infinity and $(B_\pm)$-coercive at infinity.*

Proof. By Lemma 3.7, the H_1-norm and the graph norm are equivalent on $\operatorname{dom} D_{\min} = \operatorname{dom} D_{B=0}$. The unique continuation property of D gives $\ker D_{B=0} = \{0\}$. Thus, the above Lemma gives immediately the $(B = 0)$-coercivity at infinity. Now we consider $D_\pm$. By Lemma 4.1, $\ker D_\pm = \{0\}$. Thus, in order to apply Lemma 5.3, it is enough to show that the H_1-norm and the graph norm are equivalent on $\operatorname{dom} D_{B_\pm}$: Let $\varphi \in \operatorname{dom} D_{B_\pm}$. Then, $\varphi \in H_1(M, \mathbb{S}_M)$ with $P_\pm R\varphi = 0$. By (8), we have

$$\|\varphi\|_{H_1}^2 = \|\varphi\|_D^2 - \int_M \frac{\operatorname{scal}^M}{4}|\varphi|^2 dv - \int_M \frac{i}{2}\langle \Omega \cdot \varphi, \varphi\rangle dv + \int_\Sigma \langle R\varphi, \widetilde{D}R\varphi\rangle ds - \frac{n}{2}\int_\Sigma H|R\varphi|^2 ds.$$

From $P_\pm R\varphi = 0$ and Lemma 4.1.(iii) we see that

$$\int_\Sigma \langle R\varphi, \widetilde{D}R\varphi\rangle ds = \int_\Sigma \langle P_\mp R\varphi, \widetilde{D}P_\mp R\varphi\rangle ds = \int_\Sigma \langle P_\mp R\varphi, P_\pm\widetilde{D}R\varphi\rangle ds = 0.$$

Together with the bounded geometry and the estimate (11), we obtain for a certain constant $c > 0$ that

$$\|\varphi\|_{H_1}^2 \leq \|\varphi\|_D^2 + c\|\varphi\|_D^2 \leq (c+1)\|\varphi\|_D^2.$$

The corresponding inequality for the converse direction holds for all $\varphi \in H_1(M, \mathbb{S}_M)$, cf. (9). $\square$

Next we give some (very restrictive) conditions that are sufficient to prove that B-coercivity at infinity implies coercivity at infinity. Those additional assumptions are needed to make sure that the φ_i appearing in Definition 5.1 are in $\operatorname{dom} D_B$.

Lemma 5.5. *Let B be a boundary condition with $B \subset H_{\frac{1}{2}}(\Sigma, \mathbb{S}_M|_\Sigma)$. Assume that there exists a compact subset $K' \subset M$ with $\Gamma_c^\infty(M \setminus K', \mathbb{S}_M) \subset \mathrm{dom}\, D_B$. If $D : \mathrm{dom}(D_B) \subset L^2(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ has a finite dimensional kernel and D is B-coercive at infinity, then D is coercive at infinity.*

Proof. Assume that D is not coercive at infinity. Then, for all compact subsets $K \subset M$ there exists a sequence $\varphi_i \in \Gamma_c^\infty(M \setminus K, \mathbb{S})$ with $\|\varphi_i\|_{L^2} = 1$ and $\|D\varphi_i\|_{L^2} \to 0$. We choose K such that $K' \subset K$. Then, all those $\varphi_i \in \mathrm{dom}\, D_B$. Thus, $\varphi_i \to \varphi \in \mathrm{dom}\, D_B$ weakly in the graph norm of D, $\varphi \in \ker D_B$ and $\varphi = 0$ on K. We decompose $\varphi_i = \varphi_i^k + \varphi_i^\perp$ where $\varphi_i^k \in \ker D_B$ and $\varphi_i^\perp \in (\ker D_B)^\perp$. Then $\|D\varphi_i^\perp\|_{L^2} \to 0$. Moreover, we assume that the kernel is finite dimensional, i.e. $\varphi_i^k = \sum_{j=1}^l a_{ij}\psi_j$ where the ψ_j's form an orthonormal basis of $\ker D_B$. Thus, $\|\varphi_i^k\|_{L^2}^2 = \sum_{j=1}^l |a_{ij}|^2$. If $\|\varphi_i^\perp\|_{L^2} \to 0$, then $\varphi_i^\perp \to 0$ in the graph norm. But $\|\varphi_i\|_{L^2} = 1$. This implies that there is at least one $j \in \{1, \ldots, l\}$ with $|a_{ij}|$ is bounded away from zero for almost all i, i.e. φ cannot be zero everywhere. Since φ is zero on K, this is a contradiction to the unique continuation principle. Thus, there exists $c > 0$ with $\|\varphi_i^\perp\|_{L^2} > c$ and D is not B-coercive at infinity. $\qquad\square$

Note that the assumption on the existence of K' is very restrictive. If the boundary is closed, it is automatically satisfied and we get the corollary below. If the boundary is noncompact, for a general $\mathrm{dom}\, D$ e.g. for the minimal domain of D, it is not true. For $\mathrm{dom}\, D = H_1$, this is true since we can choose $K' = \varnothing$, but then the kernel might be infinite dimensional.

Corollary 5.6. *Let the boundary Σ be closed. If B is an elliptic boundary condition as defined in [7, Definition 7.5], B-coercivity at infinity implies coercivity at infinity. In particular, D is $(B = 0)$-coercive at infinity if and only if it is coercive at infinity.*

Proof. If the boundary is closed and B is elliptic, D_B has a finite kernel [7, Theorem 8.5]. The rest of the assumption in Lemma 5.5 is trivially fulfilled which gives the first claim . The rest follows with Corollary 5.4. $\qquad\square$

For closed boundaries, assuming uniformly positive scalar curvature at infinity is a sufficient condition to have that D is coercive at infinity, see [7, Example 8.3]. For noncompact boundaries, we obtain the following

Lemma 5.7.   (i) *If $\frac{1}{2}\mathrm{scal}^M + i\Omega\cdot$ is a positive operator, the Dirac operator D is $\mathrm{dom}\, D_{\min}$-coercive at infinity.*
  (ii) *If $\frac{1}{2}\mathrm{scal}^M + i\Omega\cdot$ is a positive operator and $H \geq 0$, the Dirac operator D is $B_\pm$-coercive at infinity.*

Proof. Let $c > 0$ such that $\frac{1}{2}\mathrm{scal}^M + i\Omega\cdot \geq 2c$. The Lichnerowicz formula (8) and Lemma 2.5 give

$$
\begin{aligned}
\|D\varphi\|_{L^2}^2 &= \|\nabla\varphi\|_{L^2}^2 + \int_M \frac{\mathrm{scal}^M}{4}|\varphi|^2 dv + \int_M \frac{i}{2} <\Omega\cdot\varphi, \varphi> dv - \int_\Sigma \langle R\varphi, \tilde{D}(R\varphi)\rangle ds \\
&\quad + \frac{n}{2}\int_\Sigma H|R\varphi|^2 ds \geq c\|\varphi\|_{L^2}^2 - \int_\Sigma \langle R\varphi, \tilde{D}(R\varphi)\rangle ds + \frac{n}{2}\int_\Sigma H|R\varphi|^2 ds,
\end{aligned}
$$

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for all $\varphi \in H_1(M, \mathbb{S}_M)$. Then (i) follows directly with Lemma 3.7. For (ii), let now $H \geq 0$ and $R\varphi \in B_\pm$. Then, together with Lemma 4.1, it implies

$$\|D\varphi\|_{L^2}^2 \geq c\|\varphi\|_{L^2}^2 - \int_\Sigma \langle R\varphi, \tilde{D}(R\varphi)\rangle ds = c\|\varphi\|_{L^2}^2 - \int_\Sigma \langle P_\pm R\varphi, \tilde{D}(P_\pm R\varphi)\rangle$$

$$= c\|\varphi\|_{L^2}^2 - \int_\Sigma \langle P_\pm R\varphi, P_\mp \tilde{D}(R\varphi)\rangle = c\|\varphi\|_{L^2}^2.$$

$\square$

6. Spin^c Reilly inequality on possibly open boundary domains

In this section, we shortly review the spinorial Reilly inequality. This inequality together with those boundary value problems discussed in Section 3 will be the main ingredient in the proof of Theorem 1.2.

Theorem 6.1. Spin^c Reilly inequality. *For all $\psi \in H_1(M, \mathbb{S}_M)$, we have*

$$\int_\Sigma \left(\langle \tilde{D}\psi, \psi\rangle - \frac{n}{2}H|\psi|^2\right)ds \geq \int_M \left(\frac{1}{4}\mathrm{scal}^M|\psi|^2 + \frac{1}{2}\langle i\Omega \cdot \psi, \psi\rangle - \frac{n}{n+1}|D\psi|^2\right)dv, \quad (13)$$

where dv (resp. ds) is the Riemannian volume form of M (resp. Σ). Moreover, equality occurs if and only if the spinor field ψ is a twistor-spinor, i.e. if and only if $P\psi = 0$, where P is the twistor operator acting on $\mathbb{S}_M$ locally given by $P_X\psi = \nabla_X\psi + \frac{1}{n+1}X \cdot D\psi$ for all $X \in \Gamma(TM)$.

Proof. The inequality is proved for $\psi \in \Gamma_c^\infty(M, \mathbb{S}_M)$ analogously as in the compact Spin case [22, (17)]. For the convenience of the reader, we will shortly recall it here. For all $\psi \in H_1(M, \mathbb{S}_M)$ the claim follows using the Trace Theorem A.5 in the same way as in Lemma 2.5: We define 1-forms α and β on M by $\alpha(X) = \langle X \cdot D\psi, \psi\rangle$ and $\beta(X) = \langle \nabla_X\psi, \psi\rangle$ for all $X \in \Gamma^\infty(TM)$. Then α and β satisfy

$$\delta\alpha = \langle D^2\psi, \psi\rangle - |D\psi|^2, \quad \delta\beta = -\langle \nabla^*\nabla\psi, \psi\rangle + |\nabla\psi|^2.$$

Applying the divergence theorem with (4) and (6), we get

$$\int_\Sigma \left(\langle \tilde{D}\psi, \psi\rangle - \frac{n}{2}H|\psi|^2\right)ds = \int_M \left(|\nabla\psi|^2 - |D\psi|^2 + \frac{1}{4}\mathrm{scal}^M|\psi|^2 + \frac{i}{2}\langle\Omega \cdot \psi, \psi\rangle\right)dv. \quad (14)$$

On the other hand, for any spinor field ψ we have

$$|\nabla\psi|^2 = |P\psi|^2 + \frac{1}{n+1}|D\psi|^2. \quad (15)$$

Combining the identities (15), and (14) and $|P\psi|^2 \geq 0$, the result follows. Equality holds if and only if $|P\psi|^2 = 0$, i.e. the spinor ψ is a twistor spinor. $\square$

7. A lower bound for the first nonnegative eigenvalue of the Dirac operator on the boundary

In this section, we prove our main theorem. For that we won't follow the original proof given in [22] due to our problems concerning the APS-boundary conditions as remarked at the end of Example 3.14.iv. But we will use $B_\pm$ as given in Example 3.14.iii.

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Proof of Theorem 1.2. Since Σ is complete, $\tilde{D} : H_1(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ is self-adjoint and, hence, λ_1 is an eigenvalue or in the essential spectrum of $\tilde{D}$. In both cases, there is a sequence $\varphi_i \in H_1(\Sigma, \mathbb{S}_M|_\Sigma)$ with $\|\varphi_i\|_{L^2(\Sigma)} = 1$ and $\|(\tilde{D} - \lambda_1)\varphi_i\|_{L^2(\Sigma)} \to 0$. Then, $\varphi_i \to \varphi$ weakly in $L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ (In case that $\varphi \neq 0$, φ is an eigenspinor of $\tilde{D}$ to the eigenvalue λ_1 otherwise λ_1 is in the essential spectrum of $\tilde{D}$). We assumed that D is B_--coercive at infinity (everything which follows is also true when assuming B_+-coercivity at infinity when switching the signs). Then by Lemma 3.2, the range of D_{B_-} is closed. Moreover, from Lemma 4.1 we have $\ker(D_{B_-})^* = \ker D_{B_+} = \{0\}$. Thus, due to Proposition 3.15 for each i there exists a unique $\Psi_i \in H_1(M, \mathbb{S}_M)$ with $D\Psi_i = 0$ and $P_+ R\Psi_i = P_+\varphi_i$. Using Theorem 6.1 and $\mathrm{scal}^M + 2i\Omega \geq 0$, we obtain

$$0 \leq \int_\Sigma \left(\langle \tilde{D}R\Psi_i, R\Psi_i \rangle - \frac{n}{2}H|R\Psi_i|^2 \right) ds.$$

Moreover,

$$(\tilde{D}(P_+R\Psi_i + P_-R\Psi_i), P_+R\Psi_i + P_-R\Psi_i) = (\tilde{D}P_+R\Psi_i, P_-R\Psi_i) + (\tilde{D}P_-R\Psi_i, P_+R\Psi_i)$$
$$= (\tilde{D}P_+R\Psi_i, P_-R\Psi_i) + (P_-R\Psi_i, \tilde{D}RP_+\Psi_i),$$

where we used that $\tilde{D}$ is self-adjoint on $H_1(\Sigma, \mathbb{S}_M|_\Sigma)$ and Lemma 4.1. Hence, summarizing we get that

$$\frac{n}{2}\int_\Sigma H|R\Psi_i|^2 ds \leq 2\Re \int_\Sigma \langle \tilde{D}P_+R\Psi_i, P_-R\Psi_i \rangle ds = 2\Re \int_\Sigma \langle P_-\tilde{D}\varphi_i, P_-R\Psi_i \rangle ds$$
$$\leq 2\Re \int_\Sigma \langle P_-(\tilde{D} - \lambda_1)\varphi_i, P_-R\Psi_i \rangle ds + 2\Re\lambda_1 \int_\Sigma \langle P_-\varphi_i, P_-R\Psi_i \rangle ds.$$

Using $2\Re \int_\Sigma \langle P_-\varphi_i, P_-R\Psi_i \rangle ds \leq \|P_-\varphi_i\|^2_{L^2(\Sigma)} + \|P_-R\Psi_i\|^2_{L^2(\Sigma)}$ and $\lambda_1 \geq 0$, we obtain

$$\frac{n}{2} \inf_\Sigma H \|R\Psi_i\|^2_{L^2(\Sigma)} \leq 2\|(\tilde{D} - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|_{L^2} + \lambda_1(\|P_-\varphi_i\|^2_{L^2(\Sigma)} + \|P_-R\Psi_i\|^2_{L^2(\Sigma)}).$$

Moreover, $(\tilde{D}P_\pm\varphi_i, P_\mp\varphi_i) = (P_\mp(\tilde{D} - \lambda_1)\varphi_i, P_\mp\varphi_i) + \lambda_1\|P_\mp\varphi_i\|^2_{L^2}$. Since $\tilde{D}$ is self-adjoint, $\Re(\tilde{D}P_+\varphi_i.P_-\varphi_i) = \Re(\tilde{D}P_-\varphi_i.P_+\varphi)$. Together with $|(P_\mp(\tilde{D} - \lambda_1)\varphi_i, P_\mp\varphi_i)| \to 0$ as $i \to \infty$, we have that $\lim_{i\to\infty}\|P_-\varphi_i\|_{L^2} = \lim_{i\to\infty}\|P_+\varphi_i\|_{L^2} = \frac{1}{2}$ for $\lambda_1 > 0$. Hence, for certain ε_i with $\varepsilon_i \to 0$ as $i \to \infty$

$$\frac{n}{2} \inf_\Sigma H \|R\Psi_i\|^2_{L^2(\Sigma)} \leq 2\|(\tilde{D} - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|_{L^2} + \lambda_1(\|P_+\varphi_i\|^2_{L^2(\Sigma)} + \varepsilon_i + \|P_-R\Psi_i\|^2_{L^2(\Sigma)})$$
$$\leq 2\|(\tilde{D} - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|_{L^2} + \lambda_1(\|P_+R\Psi_i\|^2_{L^2(\Sigma)} + \varepsilon_i + \|P_-R\Psi_i\|^2_{L^2(\Sigma)})$$
$$\leq 2\|(\tilde{D} - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|_{L^2} + \lambda_1(\|R\Psi_i\|^2_{L^2(\Sigma)} + \varepsilon_i).$$

Hence,

$$\frac{n}{2} \inf_\Sigma H \leq 2\|(\tilde{D} - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|^{-1}_{L^2} + \lambda_1(1 + \varepsilon_i\|R\Psi_i\|^{-2}_{L^2}).$$

With $\|R\Psi_i\|_{L^2} \geq \|P_+R\Psi_i\|_{L^2} = \|P_+\varphi_i\|_{L^2} \to \frac{1}{2}$, we finally get

$$\frac{n}{2} \inf_\Sigma H \leq \lambda_1.$$

Next we collect all conditions that have to be fulfilled to obtain the equality $\frac{n}{2}\inf_\Sigma H = \lambda_1$:

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(1) From the spinorial Reilly Inequality (13), $\int_M |P\Psi_i|^2 dv \to 0$ which implies together with $D\Psi_i = 0$ that $\int_M |\nabla \Psi_i|^2 dv \to 0$.

(2) $\int_M \mathrm{scal}^M |\Psi_i|^2 + 2i\langle \Omega \cdot \Psi_i, \Psi_i \rangle dv \to 0$

(3) $\|\varphi_i - R\Psi_i\|_{L^2(\Sigma)} \to 0$

(4) $\int_\Sigma (H - \inf_\Sigma H)|R\Psi_i|^2 ds \to 0$.

In case that λ_1 is an eigenvalue of $\widetilde{D}$ with eigenspinor φ, one can choose $\varphi_i = \varphi$ for all i. Then $\Psi_i =: \Psi$ for all i and those equality conditions reduce to $\varphi = R\Psi$, Ψ is a parallel spinor on M, H is constant and $\int_M \mathrm{scal}^M |\Psi|^2 + 2i\langle \Omega \cdot \Psi, \Psi \rangle dv = 0$. $\qquad\square$

8. Dirac spectrum of product manifolds

As an example we give special cases of product manifolds where the Inequality (2) follows directly from the Inequality (1) for the closed case. This also serves as an example that in (2) equality can also occur for noncompact manifolds. That is different to Friedrich's inequality where equality already implies that the manifold is closed, see the next Section.

Assume that Σ' is a closed embedded hypersurface in a Riemannian Spin^c manifold (N^{n+1}, h) with mean curvature H'. If we assume that H' is nonnegative and $\frac{1}{2}\mathrm{scal}^N + i\Omega^N\cdot$ is nonnegative, then the first positive eigenvalue λ_1 of the Dirac operator $(\widetilde{D})'$ on Σ' satisfies [35, 37]

$$\lambda'_1 \geq \frac{n}{2}\inf_{\Sigma'} H'. \tag{16}$$

Equality holds if and only if H' is constant and every eigenspinor corresponding to λ'_1 is the restriction to Σ' of a parallel spinor on N.

We recall that since the hypersurface Σ' is closed, the Dirac operator $(\widetilde{D})'$ has a discrete spectrum

$$\cdots \leq \lambda'_{-l} \leq \cdots \leq \lambda'_{-1} \leq 0 \leq \lambda'_1 \cdots \leq \lambda'_l \cdots.$$

But from (7), we have $\lambda'_{-l} = -\lambda'_l$ for all $l \in \mathbb{Z}$. Then, the spectrum of $(\widetilde{D})'$ is symmetric. Let $(M = N \times M', g = h + g')$ be the Riemannian product of (N, h) and (M', g') where (M', g') is any complete Riemannian Spin^c k-manifold without boundary. Then, $\Sigma = \Sigma' \times M'$ is a complete embedded hypersurface of M with mean curvature $H((\xi, x) \in \Sigma' \times M') = \frac{n}{n+k}H'(\xi)$. We recall that if $\lambda'_1 \geq 0$ is the first nonnegative eigenvalue of $(\widetilde{D})'$ defined on Σ', then we have

$$\lambda_1^2 = (\lambda'_1)^2 + \mu_1^2$$

where μ_1^2 is the infimum of the spectrum of $(D^{M'})^2$ and λ_1^2 is the infimum of the spectrum of $\widetilde{D}^2$ on Σ (See [11] for $M' = \mathbb{R}^k$, [12, Chapter 3] for M' compact. The general result is seen analogously using the spectral theorem for self-adjoint operators). Note that due to the symmetry of the spectrum of $\widetilde{D}$, λ_1 is already the first nonnegative eigenvalue of $\widetilde{D}$. Together with (16) we have,

$$\lambda_1^2 = (\lambda'_1)^2 + \mu_1^2 \geq \left(\frac{n}{2}\right)^2 \inf_{\Sigma'}(H')^2 = \left(\frac{n+k}{2}\right)^2 \inf_\Sigma H^2.$$

Equality holds if and only if it holds in (16) and $\mu_1 = 0$. Moreover, $\mu_1 = 0$ is an eigenvalue if and only if λ_1 is an eigenvalue, cf. [12, Section 3.2]. Summarizing we get the following proposition:

Proposition 8.1. *Let Σ' be a compact embedded hypersurface in a Riemannian $\mathrm{Spin^c}$ manifold (N^{n+1}, h). Let (M', g') be a complete Riemannian $\mathrm{Spin^c}$ k-manifold. Moreover, let $(M = N \times M', g = h + g')$. Then, $\Sigma := \Sigma' \times M'$ is a complete embedded hypersurface of M. Assume that the mean curvature H of Σ is nonnegative and that N satisfies $\mathrm{scal}^N + 2i\Omega^N \geq 0$. Then, the infimum λ_1 of the nonnegative part of the spectrum of $\widetilde{D}$ on Σ satisfies*

$$\lambda_1 \geq \frac{n+k}{2} \inf_{\Sigma} H. \tag{17}$$

Equality holds in (17) if and only if equality holds in (16) and $0 \in \sigma(D^{M'})$. In this case, λ_1 is an eigenvalue of $\widetilde{D}$ if and only if 0 is an eigenvalue of $D^{M'}$.

Example 8.2. (1) $M' = K \times S$ where K is a complete Kähler manifold endowed with the canonical $\mathrm{Spin^c}$ structure and S is any Spin complete manifold with a parallel spinor. Then M' is a complete $\mathrm{Spin^c}$ manifold with a parallel spinor [32] and hence $0 \in \sigma(D^{M'})$. Thus, equality holds in (17) if and only if equality holds in (16). Examples of compact embedding hypersurfaces satisfying the equality case in (16) are given in [22, 23, 24].

(2) Let M' be $\mathbb{R}^p$ or $\mathbb{H}^p$ and N a closed $\mathrm{Spin^c}$ manifold of dimension $n+1$. Let $M = N \times M'$ be of positive scalar curvature. We have $\sigma_{ess}(D^{M'}) = \sigma(D^{M'}) = (-\infty, \infty)$ and, hence, any embedding $\Sigma \hookrightarrow N$ that gives equality in (16) gives equality in (17).

9. The intrinsic Friedrich lower bound

In this part, we shortly want to compare the extrinsic Inequality (2) with the intrinsic Friedrich's inequality: Let Σ^n be a complete Riemannian $\mathrm{Spin^c}$ manifold. If scal^Σ denotes the scalar curvature of the metric on Σ, we have the intrinsic Friedrich $\mathrm{Spin^c}$ inequality, see [14] for the closed Spin case, [16] for the complete Spin case and [35, 34] for the $\mathrm{Spin^c}$ case:

$$(\lambda_1^+)^2 \geq \frac{n}{4(n-1)} \inf_{\Sigma}(\mathrm{scal}^\Sigma - c_n |\Omega^\Sigma|) \tag{18}$$

where λ_1^+ denotes the infimum of the nonnegative part of the spectrum of D^Σ.

Let now Σ be additionally the boundary of a Riemannian $\mathrm{Spin^c}$ manifold M^{n+1} of bounded geometry such that the induced metric and $\mathrm{Spin^c}$-connection on Σ are the original ones. A consequence from the Gauss formula for the embedding, see [10], is that

$$\mathrm{scal}^\Sigma = \mathrm{scal}^M - 2\mathrm{Ric}^M(\nu, \nu) + n^2 H^2 - |II|^2, \tag{19}$$

From that, it is clear that in general we cannot hope getting a relation between scal^Σ and H allowing us to compare the Friedrich inequality (18) and the inequality

$$\lambda_1 \geq \frac{n}{2} \inf_{\Sigma} H, \tag{20}$$

where λ_1 is the infimum of the nonnegative spectrum of $\tilde{D}$. Note that always, $\lambda_1^+ \geq \lambda_1$ and in case the spectrum of D^Σ is symmetric they already coincide. But in special cases, we can compare them as in the closed case [22]:

Proposition 9.1. *Let (M, Σ) and L be of bounded geometry. Let Σ be an embedded complete hypersurface on a Riemannian* Spin^c *manifold M. Let the Einstein tensor* $\mathrm{Ric}^M - \frac{\mathrm{scal}^M}{2}g$ *of M be positive semidefinite. Then the extrinsic lower bound* (20) *for the infimum of the nonnegative spectrum of the Dirac operator $\widetilde{D}$ of Σ is sharper than the Friedrich inequality* (18). *The two lower bounds coincide if and only if the embedding is totally umbilical and the restricted* Spin^c *structure has a flat line bundle.*

Proof. Since the Einstein tensor $\mathrm{Ric}^M - \frac{\mathrm{scal}^M}{2}g$ is positive semidefinite, we get by (19) that

$$\mathrm{scal}^\Sigma - c_n|\Omega^\Sigma| \le n^2 H^2 - |II|^2 - c_n|\Omega^\Sigma|.$$

Using the Cauchy-Schwarz inequality on II and $|\Omega^\Sigma| \ge 0$, we get

$$\mathrm{scal}^\Sigma - c_n|\Omega^\Sigma| \le n(n-1)H^2,$$

and the result follows. Moreover, the two lower bounds coincide if and only if $\Omega^\Sigma = 0$, $\lambda_1 = \lambda_1^+$ and $II(X) = H\,X$ for all $X \in \Gamma(T\Sigma)$. $\qquad\square$

Example 9.2. Recall an example from the compact case, cf. [22, p. 11]: When the ambient space is the Euclidean space with a closed hypersurface Σ. If $\mathrm{scal}^\Sigma > 0$, then H w.r.t. the normal pointing into the interior of Σ is positive too (see [31, Lemma 1]), but it is possible for an embedded closed hypersurface to have everywhere positive mean curvature and somewhere negative scalar curvature (for example, consider the compact revolution tori T^2 in $\mathbb{R}^3$). So there are situations in which only inequality (2) will be significant.

From this example one can easily construct noncompact ones, e.g. $T^2 \times \mathbb{R}^n$ embedded in $\mathbb{R}^3 \times \mathbb{R}^n$. From Proposition 8.1, it is clear that the lower bound (17) still holds but the Friedrich inequality is still trivial.

Appendix A. Trace theorem

The goal is to sketch briefly the Trace and Extension Theorem on manifolds of bounded geometry and review the basic definitions that are involved. For more details on the definition of bounded geometry on manifolds with boundary see [38]. For the equivalence of all those different definitions of Sobolev-norms involved here and the corresponding theorems for submanifolds (not necessarily hypersurfaces) see [18]. For the convenience of the reader, we start by briefly recalling the definition of fractional Sobolev spaces for functions on $\mathbb{R}^n$ with values in a trivial hermitian $\mathbb{C}^r$-bundle:

Definition A.1. [41, Definition 3.1] Let $s \in \mathbb{R}$. The $H_s := H_s^2$-norm of a compactly supported $f : \mathbb{R}^n \to \mathbb{C}^r$ is defined as

$$\|f\|_{H_s(\mathbb{R}^n, \mathbb{C}^r)}^2 := \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 (1 + |\xi|)^s d\xi$$

where $\hat{f}(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(\xi)d\xi$ denotes the Fourier transform of f. The space $H_s(\mathbb{R}^n, \mathbb{C}^r)$ is then defined as the completion of $\Gamma_c^\infty(\mathbb{R}^n, \mathbb{C}^r)$, the space of smooth compactly supported functions on $\mathbb{R}^n$ with values in $\mathbb{C}^r$, with respect to the H_s-norm.

From now on, let M be a Riemannian manifold possibly with boundary and of bounded geometry, as in Definition 2.2. Moreover, let E be a hermitian vector bundle over M. We assume that E is also of bounded geometry, see Definition 2.3. Before we use the above

definition to define Sobolev spaces for sections of E, we define when we call a trivialization of E synchronous:

Definition A.2. [38, Definition 2.3] Let (M^n, Σ) be of bounded geometry, see Definition 2.2 and the notions defined therein. Let $r = \min\{r_\Sigma, r_M, r_\partial\}$ where r_Σ is the injectivity radius of Σ and r_M the one of M. Let $p_\alpha^\Sigma \in \Sigma$ and $p_\beta \in M$ be points such that

- the metric balls $B_r^\Sigma(p_\alpha^\Sigma)$ in Σ (i.e. w.r.t. the metric $g|_\Sigma$) give a locally finite cover of Σ
- the metric balls $B_r(p_\beta)$ in M cover $M \setminus U_\Sigma$ and are locally finite on all of M.

Let $(V_\gamma)_\gamma$ be a locally finite covering of M where each V_γ is of the form $B_r(p_\beta)$ or $U_{p_\alpha^\Sigma}^\Sigma = F(B_r^{m-k} \times B_r^\Sigma(p_\alpha^\Sigma))$. By construction the covering $(V_\gamma)_\gamma$ is locally finite. Coordinates on V_γ are chosen to be geodesic normal coordinates around p_β in case $V_\gamma = B_r(p_\beta)$. Otherwise coordinates are given by Fermi coordinates

$$\kappa_\alpha : V_{p_\alpha^\Sigma}^\Sigma := [0, r_\Sigma) \times B_r(0) \subset \mathbb{R}^n \to U_{p_\alpha^\Sigma}^\Sigma, \quad (t, x) \mapsto \exp_{\exp_{p_\alpha^\Sigma}^\Sigma(x)}(t\nu)$$

where ν is the inner normal field of Σ and $\exp^\Sigma$ is the exponential map on Σ w.r.t. the induced metric. We call such coordinates $(V_\gamma, \kappa_\gamma)_\gamma$ Fermi coordinates for (M, Σ). If $V_\gamma = B_r(p_\gamma)$, E is trivialized via parallel transport along radial geodesic and identify E with $\mathbb{C}^r$. Otherwise, E is trivialized via parallel transport along radial geodesic of the boundary and along the normal direction. The obtained trivialization is denoted by $(\xi_\gamma)_\gamma$.

In case of manifolds without boundary, the Definition of ξ_γ in A.2 is the usual definition of synchronous trivialization as found in [4, Section 3.1.3]. Note that by construction the restriction of a synchronous trivialization of E over a manifold M to its boundary Σ gives a synchronous trivialization of $E|_\Sigma$.

Definition A.3. [18] Let $s \in \mathbb{R}$. Let $(V_\alpha)_\alpha$ be a covering of M together with a synchronous trivialization ξ_α of E as defined above. Moreover, let the covering be locally finite and h_α a partition of unity subordinated to V_α. Then

$$\|\varphi\|_{H_s(M,E)} := \sum_\alpha \|\xi_{\alpha*}(h_\alpha \varphi)\|_{H_s(\mathbb{R}_+^n, \mathbb{C}^r)}.$$

Remark A.4. (i) For $s \in \mathbb{N}$ the definition of $H_s(M, E)$ from above coincides with the usual definition given by

$$\|\varphi\|_{H_s(M,E)} := \sum_{i=0}^s \|\underbrace{\nabla^E \cdots \nabla^E}_{i \text{ times}} \varphi\|_{L_2(M,E)},$$

cp. [38], [18].
(ii) [17, Lemma 26] Let $f = (f_1, \ldots, f_r) : \mathbb{R}^n \to \mathbb{C}^r$. Then the norms $\|f\|_{H_s(\mathbb{R}^n, \mathbb{C}^r)}$ and $\sum_{i=1}^r \|f_i\|_{H_s(\mathbb{R}^n, \mathbb{C})}$ are equivalent.
(iii) For $s \leq t$ we have $\|\varphi\|_{H_s(M,E)} \leq \|\varphi\|_{H_t(M,E)}$. That is seen for $M = \mathbb{R}_+^n$ immediately using $(1 + |\xi|)^s \leq (1 + |\xi|)^t$. For general M, one just lifts this result by using a partition of unity and a synchronous trivialization.
(iv) Let $D^\Sigma : \Gamma_c^\infty(\Sigma, \mathbb{S}_\Sigma) \to \Gamma_c^\infty(\Sigma, \mathbb{S}_\Sigma)$ be a Dirac operator on Σ. For any $s \in \mathbb{R}$, there is closed extension of D^Σ from $H_s(\Sigma, \mathbb{S}_\Sigma) \to H_{s-1}(\Sigma, \mathbb{S}_\Sigma)$.

Theorem A.5. *Let M^n be a Riemannian manifolds with boundary Σ. Assume that (M, Σ) is of bounded geometry and that E is an hermitian vector bundle over M that is also of bounded geometry. Then, for all $s \in \mathbb{R}$ with $s > \frac{1}{2}$ the operator $R : \Gamma_c^\infty(M, E) \to \Gamma_c^\infty(\Sigma, E|_\Sigma)$ with $\varphi \mapsto \varphi|_\Sigma$ extends to a bounded linear operator from $H_s(M, E)$ to $H_{s-\frac{1}{2}}(\Sigma, E|_\Sigma)$.*

Proof. First we note that in case of real valued Sobolev spaces on $M = \mathbb{R}^n_+$ with $\Sigma := \partial M = \mathbb{R}^{n-1}$ the trace theorem is the one in the original formulation and can be found in [41, Theorem I.3.4]. As a next step we consider Sobolev spaces on $M = \mathbb{R}^n_+$ with values in a trivial $\mathbb{C}^r$-bundle E over M. Thus, $\varphi = (\varphi_1, \ldots, \varphi_r)$ and the Trace Theorem holds for each component separately. Using Remark A.4.ii the trace theorem for the trivial $\mathbb{C}^r$ follows immediately. Let now M be arbitrary. We choose a covering V_α together with a synchronous trivialization ξ_α of E and a subordinated partition of unity h_α. Moreover, we choose the covering such that each point in M is covered by at most L charts (for a big enough L). The restrictions $V_\alpha \cap \Sigma$ cover Σ. Let $\varphi \in H_s(M, E)$. Then, for all α we have $\xi_{\alpha*}(h_\alpha \varphi) \in H_s(\mathbb{R}^n_+, \mathbb{C}^r)$. Thus, there exists a $C > 0$ with $\|R(\xi_{\alpha*}(h_\alpha \varphi))\|_{H_{s-1}(\mathbb{R}^{n-1}, \mathbb{C}^r)} \leq C\|\xi_{\alpha*}(h_\alpha \varphi)\|_{H_{s-\frac{1}{2}}(\mathbb{R}^n_+, \mathbb{C}^r)}$. With $R(\xi_{\alpha*}(h_\alpha \varphi)) = \xi_{\alpha*}(h_\alpha R\varphi)$ we get after summing up $\|R\varphi\|_{H_{s-1}(\Sigma, E|_\Sigma)} \leq C\|\varphi\|_{H_{s-\frac{1}{2}}(M, E)}$ since ξ_α is still a synchronous trivialization for $E|_\Sigma$. $\square$

Theorem A.6. *Let the assumptions of Theorem A.5 be satisfied. Then, for all $s > \frac{1}{2}$ there is a bounded right inverse $\mathcal{E} : H_{s-\frac{1}{2}}(\Sigma, E_\Sigma) \to H_s(M, E)$ of the trace map $R : H_s(M, E) \to H_{s-\frac{1}{2}}(\Sigma, E_\Sigma)$. In particular, $\mathcal{E}(\Gamma_c^\infty(\Sigma, E_\Sigma)) \subset \Gamma_c^\infty(M, E_M))$*

Proof. This is proven analogously as the Trace Theorem using the original Euclidean version $\mathcal{E} : H_{s-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H_s(\mathbb{R}^n)$ which can be found in [42, 4.4.1(4)]. The last inclusion follows in immediately from $\mathcal{E}(\Gamma_c^\infty(\mathbb{R}^{n-1})) \subset \Gamma_c^\infty(\mathbb{R}^n)$. $\square$

Lemma A.7. *The L^2-product $(\varphi, \psi) = \int_\Sigma \langle \varphi, \psi \rangle dv$ for $\varphi, \psi \in \Gamma_c^\infty(\Sigma, E|_\Sigma)$ extends to a perfect pairing $H_s(\Sigma, E|_\Sigma) \times H_{-s}(\Sigma, E|_\Sigma) \to \mathbb{C}$ for all $s \in \mathbb{R}$.*

Proof. This is also proven in the same way as above – by lifting the corresponding result from the Euclidean case [41, Section I.3]. $\square$

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