The Fourier transforms of weighted orbital integrals on semisimple groups of real rank one

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Introduction

Weighted orbital integrals are distributions on reductive groups over a local field or an adele ring, they arose in the Selberg-Arthur trace formula for non-compact quotients as a result of a truncation process. Unlike the usual orbital integrals they are not invariant under inner automorphisms of the group. Using weighted orbital integrals and their dual analogues, the weighted characters, J. Arthur has constructed invariant distributions I_M and proved an invariant trace formula, which is an identity beetween them. Under Fourier transform, invariant distributions on the group map to scalar-valued distributions on the unitary dual of the group, whereas non-invariant distributions yield operator-valued Fourier transforms. If one were able to calculate the Fourier transforms of the invariant distributions I_M, then the invariant trace formula could be written as an identity between distributions on the unitary dual just as in the classical case of $\mathrm{SL}(2,\mathbb{R})$. Since the definition of I_M is rather involved, this is apparently too much to hope for in general. In fact, many applications of the trace formula have been carried out without this precise information.

However, as we shall see in the present paper, the Fourier transforms of the distributions I_M can be explicitly calculated in the case when we consider a semisimple Lie group of real rank one. A good deal of work in this direction has been done by R. Herb et al. in [6] and G. Warner in [17], and we shall make use of it. As indicated in the introduction to [1], there is also unpublished work of J. Arthur on this question, although to my knowledge he did not obtain an explicit formula either. Our approach is to utilize the differential equations satisfied by weighted orbital integrals together with information about their asymptotic behaviour and their jumps at the singular set. This certainly restricts the method to the case of real groups. Recently, J. Arthur has shown in [4] with the help of his local trace formula that, for reductive groups over any local field of characteristic zero, the Fourier transforms of the I_M are distributions represented by smooth functions on a parameter space for the tempered dual.

We shall also express the distribution I_M as an integral transform (on certain subtori) of ordinary orbital integrals. Surprisingly, it turns out that the kernel of this integral transform is a rational expression in characters of the tori. It would be interesting to know whether this fact follows from a general principle. We remark that in the present case I_M is thereby explicitly calculated as a distribution on the group.

The paper is organized as follows. In the first section, we shall give the basic definitions and state our main result, a formula for the Fourier transform Ω_P of a distribution I_P, whose study is equivalent to that of I_M. For the time being, we shall restrict ourselves to regular orbits. The proof occupies sections 2–4. Namely, in section 2 we shall collect the properties of Ω_P, in particular, a differential equation it satisfies. In section 3, we shall find and investigate a particular solution $\tilde{\Omega}_P$ of this differential equation, and it will be shown in section 4 that $\Omega_P = \tilde{\Omega}_P$. The distribution I_P will be expressed as an integral transform of ordinary orbital integrals in section 5, and in section 6 the main result will be extended to any orbit for which I_M may be defined.

1 The main result

Suppose G is a connected semisimple Lie group of real rank one contained in its simply connected complexification $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. We denote by G' the set of regular semisimple elements. This means that $g \in G'$ iff the centralizer G_g of g is a Cartan subgroup of G. The Weyl discriminant of $g \in G$ is

$$D_G(g) = \det(\mathrm{Id} - \mathrm{Ad}(g))_{\mathfrak{g}/\mathfrak{g}_g}.$$

Given $g \in G'$ and a function f in the Schwartz space $\mathcal{C}(G)$ (see [9], p, 19), the orbital integral is defined as

$$J_G(g, f) = |D_G(g)|^{1/2} \int_{G/G_g} f(xgx^{-1})\, dx. \tag{1}$$

It depends on the choice of the Haar measures on G and G_g. One knows that the integral is absolutely convergent. The distribution $J_G(g)$ is invariant, which means that $J_G(g, f^y) = J_G(g, f)$ for $y \in G$, where $f^y(x) = f(yxy^{-1})$. Moreover, it is tempered, i. e., a continuous linear functional on $\mathcal{C}(G)$.

In order to define the weighted orbital integral, we need some preparation. Let M be a Levi subgroup of G, i. e., a Levi component of a parabolic subgroup P of G, and suppose that $M \neq G$. Then M is the centralizer in G of a unique maximal (i. e., one-dimensional) $\mathbb{R}$-split torus $A_R = \exp \mathfrak{a}_R$, and $M = M_I A_R$, where M_I is a compact subgroup. In fact, M is a Levi component of two different parabolic subgroups $P = MN$ and $\bar{P} = M\bar{N}$, whose unipotent radicals are N and $\bar{N}$. The roots of $\mathfrak{a}_R$ in $\mathfrak{n}$ determine a positive chamber (half-line) C_P in $\mathfrak{a}_R$, and we fix some $\lambda_P \in \mathfrak{a}_R^*$ with positive values on C_P to normalize the Haar measure on $A_R = \exp \mathfrak{a}_R$. Of course, $\lambda_{\bar{P}} = -\lambda_P$. If $m \in M \cap G'$, we shall assume that the Haar measure on the Cartan subgroup $A = G_m \subset M$ used in the definition of $J_G(m, f)$ is the product of the measure on A_R determined by λ_P and the measure on $A_I = A \cap M_I$ of total mass one.

Fix now a maximal compact subgroup K of G. We mention in passing that the symmetric space $K\backslash G$ is a hyperbolic space over either the real numbers, the complex numbers, the quaternions or the octonions. Due to the Iwasawa decomposition one can define maps H_P, $H_{\bar{P}} : G \to \mathfrak{a}_R$ by

$$H_P(kan) = \log a, \;\; H_{\bar{P}}(ka\bar{n}) = \log a \quad \text{for all } k \in K,\, a \in A_R,\, n \in N \text{ and } \bar{n} \in \bar{N}.$$

The function $v(x) = \lambda_P(H_P(x) - H_{\bar{P}}(x))$ is positive-valued and has the property $v(xm) = v(x)$ for $m \in M$. Given $m \in M \cap G'$ and $f \in \mathcal{C}(G)$, in our particular case the weighted orbital integral is defined as

$$J_M(m,f) = |D_G(m)|^{1/2} \int_{G/G_m} f(xmx^{-1})v(x)\,dx. \tag{2}$$

The integral is absolutely convergent and defines a noninvariant tempered distribution (see [1]). Note that it is independent of λ_P and proportional to the Haar measure on G.

While (weighted) orbital integrals appear on the geometric side of the trace formula, the corresponding objects on the spectral side are (weighted) characters. Let $\hat{G}_{\text{temp}}$ denote the set of equivalence classes of irreducible tempered (hence unitary) representations of G. For any $\pi \in \hat{G}_{\text{temp}}$, its character $\Theta_\pi(f) = \operatorname{Tr}(\pi(f))$ is an invariant tempered distribution, which is proportional to the Haar measure on G. (For the definition and traceability of $\pi(f)$ for $f \in \mathcal{C}(G)$, see [17], Lemma 11.1.) To stress the analogy with the geometric side, one sometimes writes $J_G(\pi,f) = \Theta_\pi(f)$. By Harish-Chandra's regularity theorem, Θ_π is represented by an analytic function on G', which is usually denoted by Θ_π, too. For any representation π, we shall denote the contragredient representation by $\check{\pi}$. If π is unitary, then $\Theta_{\check{\pi}} = \overline{\Theta_\pi}$.

Let us fix notations concerning the principal series of G. Let $\hat{M}$ denote the set of equivalence classes of irreducible (thus finite-dimensional) unitary representations of $M = M_I A_R$. Any $\sigma \in \hat{M}$ may be twisted by $\lambda \in \mathfrak{a}^*_{R,\mathbb{C}} = \mathfrak{a}^*_R \otimes \mathbb{C}$ according to $\sigma_\lambda(ma) = \sigma(ma)a^\lambda$, where $(\exp(H))^\lambda = e^{\lambda(H)}$ for $H \in \mathfrak{a}_R$. If we consider $\hat{M}_I$ as a subset of $\hat{M}$ in the obvious way, we get a bijection $\hat{M}_I \times i\mathfrak{a}^*_R \to \hat{M}$. The induced representation $\pi_{P,\sigma_\lambda} := \operatorname{Ind}_P^G(\delta_P^{1/2}\sigma_\lambda)$ acts on the space $\mathcal{H}_{P,\sigma_\lambda}$ of classes of measurable functions $\phi : G \to V_\sigma$ satisfying $\phi|_K \in L^2(K, V_\sigma)$ and $\phi(xmn) = \delta_P(m)^{-1/2}\sigma_\lambda(m)^{-1}\phi(x)$ for $m \in M$, $n \in N$ and a. e. $x \in G$. Here $\delta_P(m) = |\det \mathrm{Ad}_{\mathfrak{n}}(m)|$.

In order to define weighted characters, we need the intertwining operators $J_{\bar{P}|P}(\sigma_\lambda) : \mathcal{H}_{P,\sigma_\lambda} \to \mathcal{H}_{\bar{P},\sigma_\lambda}$ defined for $\sigma \in \hat{M}_I$ and $\operatorname{Re}\lambda$ positive on C_P by

$$\left(J_{\bar{P}|P}(\sigma_\lambda)\phi\right)(x) = \int_{\bar{N}} \phi(x\bar{n})\,d\bar{n}$$

and satisfying

$$J_{\bar{P}|P}(\sigma_\lambda)\pi_{P,\sigma_\lambda}(x) = \pi_{\bar{P},\sigma_\lambda}(x)J_{\bar{P}|P}(\sigma_\lambda).$$

Note that restricting functions to K defines an isomorphism $u_P(\sigma_\lambda) : \mathcal{H}_{P,\sigma_\lambda} \to \mathcal{H}_\sigma$, where the space $\mathcal{H}_\sigma$ is independent of λ. If we consider $J_{\bar{P}|P}(\sigma_\lambda)$ as an operator in $\mathcal{H}_\sigma$ via $u_P(\sigma_\lambda)$ and $u_{\bar{P}}(\sigma_\lambda)$, its restriction to K-finite vectors admits a meromorphic continuation to $\lambda \in \mathfrak{a}^*_{R,\mathbb{C}}$. If $\lambda \in i\mathfrak{a}^*_R$, $\lambda \neq 0$, then π_{P,σ_λ} is irreducible and $J_{\bar{P}|P}(\sigma_\lambda)$ is invertible. The Plancherel density $\mu(\sigma_\lambda)$ is defined by

$$\mu(\sigma_\lambda)J_{P|\bar{P}}(\sigma_\lambda)J_{\bar{P}|P}(\sigma_\lambda) = \mathrm{Id}. \tag{3}$$

It is known that $\pi_{P,\sigma}$ is reducible iff σ is $W(G, A_R)$-stable and $\mu(\sigma) \neq 0$ (see [14], Theorem 5).

If ϕ is a function of σ_λ meromorphic in λ we shall use the abbreviation

$$\partial_P \phi(\sigma) = \frac{d}{dz}\phi(\sigma_{z\lambda_P})\Big|_{z=0}.$$

Let $\sigma \in \hat{M}$ with $\mu(\sigma) \neq 0$ and $f \in \mathcal{C}(G)$. Then in our particular case the weighted character is defined as

$$J_P(\sigma, f) = -\operatorname{Tr}\left(\pi_{P,\sigma}(f) J_{\bar{P}|P}(\sigma)^{-1} \partial_P J_{\bar{P}|P}(\sigma)\right),$$

where the derivative is defined for the operator in $\mathcal{H}_\sigma$. $J_P(\sigma)$ is a noninvariant tempered distribution, which is proportional to the Haar measure on G and inversely proportional to λ_P. It does not depend on the Haar measure on $\bar{N}$ used to define $J_{\bar{P}|P}(\sigma)$.

In [3], J. Arthur actually defines a slightly different distribution $J_M(\sigma, f)$ using normalized intertwining operators $R_{\bar{P}|P}(\sigma) = r_{\bar{P}|P}(\sigma)^{-1} J_{\bar{P}|P}(\sigma)$ instead of $J_{\bar{P}|P}(\sigma)$, where the functions $r_{\bar{P}|P}(\sigma)$ satisfy the analogue of (3). However, the obvious relation

$$J_M(\sigma, f) = J_P(\sigma, f) + \frac{\partial_P r_{\bar{P}|P}(\sigma)}{r_{\bar{P}|P}(\sigma)} \Theta_{\pi_\sigma}(f)$$

allows one to pass back and forth. Here, we have denoted the equivalence class of $\pi_{P,\sigma}$ by π_σ. Unlike $J_M(\sigma)$, $J_P(\sigma)$ is sensitive to the permutation of P and $\bar{P}$:

$$J_{\bar{P}}(\sigma, f) = J_P(\sigma, f) - \frac{\partial_P \mu(\sigma)}{\mu(\sigma)} \Theta_{\pi_\sigma}(f). \tag{4}$$

Also, the distribution $J_M(\sigma)$ has the advantage of being defined for all $\sigma \in \hat{M}$, but it depends on the possible choices of $r_{\bar{P}|P}$. Nevertheless, for reasons which will become clear later we prefer to work with $J_P(\sigma)$.

If $f \in C_c^\infty(G)$, then $\pi_{P,\sigma_\lambda}(f)$ is an entire function in λ. In section 2 we shall see that, for $\sigma \in \hat{M}_I$,

$$\operatorname*{Res}_{z=0} J_P(\sigma_{z\lambda_P}, f) = n(\sigma)\Theta_{\pi_\sigma}(f), \tag{5}$$

where $2n(\sigma)$ is the order of the zero of $\mu(\sigma_\lambda)$ at $\lambda = 0$.

Now we are going to recall the notion of invariant Fourier transform. Remember that $\hat{G}_{\text{temp}}$ is the union of three disjoint sets: the discrete series $\hat{G}_{\text{dis}}$, the set of the irreducible components of all reducible $\pi_{P,\sigma}$ (two per each, which are called limits of discrete series), and the principal series, which is parametrized by the set $W(G, A_R)\backslash\hat{M}$ from which the σ's with reducible $\pi_{P,\sigma}$ have been removed. If $f \in \mathcal{C}(G)$, then $\pi \mapsto \Theta_\pi(f)$ is a function on $\hat{G}_{\text{temp}}$. In this way one gets a map ϕ_G from $\mathcal{C}(G)$ into the space of functions on $\hat{G}_{\text{temp}}$. It is the content of the trace Payley-Wiener theorem ([4], appendix) to describe the image $\mathcal{I}(G)$ of ϕ_G and the finest topology on it such that ϕ_G will be continuous.

Let I be an invariant distribution on $\mathcal{C}(G)$. If I is "supported on characters", i. e., if it vanishes on the kernel of ϕ_G (which is likely to be always true), it determines a distribution $\hat{I}$ on $\mathcal{I}(G)$ by $\hat{I}(\phi_G(f)) = I(f)$. One calls $\hat{I}$ the (invariant) Fourier transform of I. To calculate it, one has thus to exhibit $I(f)$ in terms of the characters $\Theta_\pi(f)$, $\pi \in \hat{G}_{\text{temp}}$.

E. g., the Fourier transform of the δ-distribution at 1_G is the Plancherel measure. The Fourier transform of the orbital integral $J_G(m)$ for $m \in M \cap G'$ is also well known ([15], p. 16):

$$J_G(m,f) = |D_M(m)|^{1/2} \sum_{\sigma \in \hat{M}_I} \frac{1}{2\pi i} \int_{i\mathfrak{a}_R^*} \Theta_{\delta_{-\lambda}}(m) \Theta_{\pi_{\sigma_\lambda}}(f)\, d\lambda,$$

where $i\mathfrak{a}_R^*$ is oriented by $i\lambda_P$ and $d\lambda$ is defined by $d(z\lambda_P) = dz$. Here Θ_σ is the character of the finite-dimensional representation σ, and D_M is defined by analogy to D_G. In the sequel it will be convenient to combine the sum over $\hat{M}_I$ and the integral over $i\mathfrak{a}_R^*$ into an integral over $\hat{M}$ as

$$J_G(m,f) = |D_M(m)|^{1/2} \frac{1}{2\pi i} \int_{\hat{M}} \Theta_{\check{\sigma}}(m) \Theta_{\pi_\sigma}(f)\, d\sigma \tag{6}$$

with the obvious interpretation. Sometimes we shall indicate the dependance on the ambient group by an upper index like $J_G = J_G^G$, $J_M = J_M^G$ etc. One may similarly define the orbital integral on M (the Haar measures on compact groups being normalized to total mass one) by

$$J_M^M(m,h) = |D_M(m)|^{1/2} \int_{M/M_m} h(m_1 m m_1^{-1})\, dm_1$$

for $m \in M'$ and $h \in \mathcal{C}(M)$. Its Fourier transform can be read off from the well-known identity

$$J_M^M(m,h) = |D_M(m)|^{1/2} \frac{1}{2\pi i} \int_{\hat{M}} \Theta_{\check{\sigma}}(m) \Theta_\sigma(h)\, d\sigma. \tag{7}$$

In [3], section 10, J. Arthur has defined invariant distributions $I_M(m)$, $m \in M \cap G'$. The general procedure simplifies considerably in our real rank one situation. Let ϕ_M be the map which assigns to each $f \in \mathcal{C}(G)$ the function $\sigma \mapsto J_M(\sigma, f)$ on $\hat{M}$. Due to th simple structure of M it is clear that $\mathcal{I}(\mathcal{M})$ is the Schwartz space on $\hat{M}$ in the obvious sense. It follows from [17], p. 99, that ϕ_M is a continuous map from $\mathcal{C}(G)$ to $\mathcal{I}(\mathcal{M})$. Denoting the invariant Fourier transform of $J_M^M(m)$ by $\hat{J}_M^M(m)$, one then defines the tempered distribution

$$I_M(m,f) = J_M(m,f) - \hat{J}_M^M(m, \phi_M(f)).$$

Note that $\phi_M(f)$ is proportional to the Haar measure on G, as is $J_M(m,f)$. Using (7), this can be written as

$$I_M(m,f) = J_M(m,f) - |D_M(m)|^{1/2}\frac{1}{2\pi i}\int_{\hat{M}} \Theta_{\check{\sigma}}(m) J_M(\sigma, f)\, d\sigma.$$

As noted earlier, instead of $J_M(\sigma)$ we use $J_P(\sigma)$, which is constructed from unnormalized intertwining operators. However, the latter have poles on the set $\hat{M} = \hat{M}_I \times i\mathfrak{a}_R^* \subset \hat{M}_I \times \mathfrak{a}_{R,\mathbf{C}}^*$. Let thus $D_{P,\sigma}$ be a contour in $\mathfrak{a}_{R,\mathbf{C}}^*$ which goes along $i\mathfrak{a}_R^*$ in the direction $i\lambda_P$ but evades zero to the side of λ_P if $J_{\bar{P}|P}(\sigma_\lambda)$ has a pole at $\lambda = 0$. Put $D_P = \{\sigma_\lambda : \sigma \in \hat{M}_I,\ \lambda \in D_{P,\sigma}\}$ with the inherited orientation. If $f \in C_c^\infty(G)$, then $J_P(\sigma_\lambda, f)$ is meromorphic in λ, and we may define the distribution

$$I_P(m,f) = J_M(m,f) - |D_M(m)|^{1/2}\frac{1}{2\pi i}\int_{D_P} \Theta_{\check{\sigma}}(m) J_P(\sigma, f)\, d\sigma$$

for $m \in M \cap G'$. The contour integral can be written as a Cauchy principal value plus half a residue:

$$I_P(m,f) = J_M(m,f) - |D_M(m)|^{1/2}\frac{1}{2\pi i}\,\mathrm{p.v.}\int_{\hat{M}} \Theta_{\check{\sigma}}(m) J_P(\sigma, f)\, d\sigma - |D_M(m)|^{1/2}\sum_{\sigma\in\hat{M}_I}\frac{n(\sigma)}{2}\,\Theta_{\check{\sigma}}(m)\Theta_{\pi_\sigma}(f), \tag{8}$$

where $n(\sigma)$ are the integrers from (5). The distributions $I_P(m)$ differ from $I_M(m)$ by a term which is already expressed in terms of characters:

$$I_P(m,f) = I_M(m,f) + |D_M(m)|^{1/2}\frac{1}{2\pi i}\,\mathrm{p.v.}\int_{\hat{M}} \Theta_{\check{\sigma}}(m)\,\frac{\partial_P r_{\bar{P}|P}(\sigma)}{r_{\bar{P}|P}(\sigma)}\,\Theta_{\pi_\sigma}(f)\, d\sigma - |D_M(m)|^{1/2}\sum_{\sigma\in\hat{M}_I}\frac{n(\sigma)}{2}\,\Theta_{\check{\sigma}}(m)\Theta_{\pi_\sigma}(f).$$

It is esy to check that the logarithmic derivative of $r_{\bar{P}|P}(\sigma)$ is slowly increasing on $\hat{M}$ (see [17], p. 99). Since Arthur's distributions $I_M(m)$ are tempered and invariant, so are the $I_P(m)$, and it suffices to calculate the invariant Fourier transform of the latter.

Let us fix the Cartan subgroup $A = G_m$ as above and study $I_P(a)$ for $a \in A' = A \cap G'$. Choose a half-system Σ of positive roots for $(\mathfrak{m}_\mathbf{C}, \mathfrak{a}_\mathbf{C})$. The union of Σ with the set of roots of $(\mathfrak{g}_\mathbf{C}, \mathfrak{a}_\mathbf{C})$ whose restrictions to C_P are positive-valued gives a half-system Σ_P of positive roots for $(\mathfrak{g}_\mathbf{C}, \mathfrak{a}_\mathbf{C})$. We write $H_\alpha \in \mathfrak{a}_\mathbf{C}$ for the coroot corresponding to $\alpha \in \pm\Sigma_P$, i. e., $\alpha'(H_\alpha) \in \mathbb{Z}$ for all $\alpha' \in \Sigma_P$, $\alpha(H_\alpha) = 2$. It is clear that $A' = \{a \in A : a^\alpha \neq 1 \text{ for } \alpha \in \Sigma_P\}$. Let $A'' = \{a \in A : a^\alpha \neq 1 \text{ for } \alpha \in \Sigma_P^+\}$, where $\Sigma_P^+ = \Sigma_P \setminus \Sigma$ is the set of positive complex or real roots. By [1], Lemma 1.1, the set Σ_P contains a real root (which we denote by β) iff $\operatorname{rk} G = \operatorname{rk} K$ iff (by classification) $K\backslash G$ is not an odd-dimensional real hyperbolic space. If β exists, we introduce $\gamma = \exp(\pi i H_\beta) \in G_{\mathbb{C}}$. It lies in the centre of M_I and satisfies $\gamma^2 = 1$. It is known (see [8], §24) that A_I and M_I are connected except for $\dim K\backslash G = 2$, i. e., except for G isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or a product of the latter with a compact group. If $G = \mathrm{SL}(2,\mathbb{R})$, then $A_I = M_I = \{1, \gamma\}$.

Let $\Lambda_A \subset i\mathfrak{a}^*$ be the set of A-integral weights, so that $\Lambda_A = \Lambda_{A_I} + i\mathfrak{a}_R^*$ with the obvious meaning. The expression a^λ makes sense for any $a \in A^0$ and $\lambda \in \Lambda_A$. Since $G_{\mathbb{C}}$ is simply connected by assumption, G is acceptable. Thus, $\rho_\Sigma = \frac{1}{2}\sum_{\alpha\in\Sigma}\alpha$ belongs to Λ_{A_I}, and

$$\Delta_\Sigma(\exp H) = \prod_{\alpha\in\Sigma}\left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right)$$

correctly defines a function Δ_Σ on A. One has $\Delta_\Sigma(a) = \varepsilon_\Sigma(a)|D_M(a)|^{1/2}$ for all $a \in A \cap M'$, where ε_Σ is locally constant and satisfies $\varepsilon_\Sigma(a)^4 = 1$. Recall the Weyl character formula

$$\Delta_\Sigma(a)\Theta_\sigma(a) = \sum_{w\in W(M,A)} \det(w) a^{w\lambda},$$

where $\lambda - \rho_\Sigma \in \Lambda_A$ is the Σ-highest weight of $\sigma \in \hat{M}$. Incidentally, if $\sigma \in \hat{M}_I$ (and thus $\lambda - \rho_\Sigma \in \Lambda_{A_I}$), then

$$n(\sigma) = \frac{1}{2}\#\left(\{\alpha \in \Sigma_P^+ \setminus \{\beta\} : \lambda(H_\alpha) = 0\}\right) + \begin{cases} 1, & \text{if } \beta \text{ exists and } \operatorname{sgn}\sigma = 1, \\ 0 & \text{otherwise} \end{cases}$$

(cf. [17], pp. 92–95), where $\operatorname{sgn}\sigma = \sigma(\gamma)$ if $\dim K\backslash G = 2$, and $\operatorname{sgn}\sigma = -\gamma^\lambda$ in all other cases. We shall need the character $\varepsilon_M(w) = (-1)^{\#(w\Sigma\cap-\Sigma)}$ of $W(G,A)$, which may not coincide with $\det(w)$ for $w \notin W(M,A)$ (see Lemma 1).

Now we are ready to state the main result, giving the Fourier transform of $I_P(a)$. Actually, we shall consider $I_{P,\Sigma}(a) = \varepsilon_\Sigma(a)I_P(a)$. Let us write $\tilde{\Delta}_\Sigma(a) = \Delta_\Sigma(a)|\det(\mathrm{Id} - \mathrm{Ad}(a))_{\mathfrak{g}/\mathfrak{m}}|^{1/2}$, then $\tilde{\Delta}_\Sigma(a) = \varepsilon_\Sigma(a)|D_G(a)|^{1/2}$ for $a \in A'$.

Theorem 1 *There exists a function $\Omega_{P,\Sigma}$ on $A' \times \hat{M}$ which is $W(G, A_R)$-invariant in the second argument and such, that*

$$I_{P,\Sigma}(a,f) = -\tilde{\Delta}_\Sigma(a)\sum_{\pi\in\hat{G}_{\mathrm{dis}}}\Theta_\pi(a)\Theta_\pi(f) + \frac{1}{2\pi i}\int_{\hat{M}}\Omega_{P,\Sigma}(a,\check{\sigma})\Theta_{\pi_\sigma}(f)\,d\sigma$$

for all $a \in A'$ and $f \in \mathcal{C}(G)$, the sum and the integral being absolutely convergent. Moreover, the function $\Omega_{P,\Sigma}$ extends real-analytically to $A'' \times \hat{M} \setminus A_I \times \hat{M}_I$, *and for $\sigma\big|_{A_R} \neq 1$ it equals*

$$\frac{1}{2}\sum_{w\in W(G,A)} \varepsilon_M(w) a^{w\lambda} \sum_{\alpha\in\Sigma_P^+} \lambda_P(H_\alpha) \times \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{a^{-n\alpha}}{n - w\lambda(H_\alpha)}, & \text{if } a^{\lambda_P} > 1, \text{ else} \\ \displaystyle\left(\sum_{n=0}^{\infty} \frac{a^{n\alpha}}{n + w\lambda(H_\alpha)} + \frac{\delta_{\alpha\beta}\pi \operatorname{sgn}\sigma}{\sin \pi w\lambda(H_\beta)}\right), \end{cases}$$

where $\lambda - \rho_\Sigma$ is the Σ-highest weight of σ, λ_P is extended to $\mathfrak{a}_\mathbb{C}$ in the obvious way, $\pi = 3.14\ldots$, and $\delta_{\alpha\beta}$ equals one if $\alpha = \beta$ and zero otherwise. If A is not connected, then the character a^λ of A^0 has to be so extended that $\gamma^\lambda = \sigma(\gamma)$. The series are absolutely convergent and may be written as Gaussian hypergeometric series.

The 1-form $d\sigma$ introduced in (6) is inversely proportional to λ_P, while Ω_P is proportional to λ_P, as it should be. Further, $\lambda(H_\beta)$ is purely imaginary, so there is actually the hyperbolic sine of a real number in the denominator.

The contribution of $\hat{G}_{\mathrm{dis}}$ to the Fourier transform of $I_P(a)$ (or of $J_M(a)$, what amounts to the same) has been calculated in [1] and interpreted in the form cited here in [2] (of course, in much greater generality). The reason for its simple form has been elucidated in [5].

It might seem that the function $\Omega_{P,\Sigma}$ had already been calculated in [17]. The point, however, is not only that the result is stated there as an infinite series of integrals which can hardly be computed explicitely, but that the proof relies (on p. 70) on the evenness of $\Omega_{P,\Sigma}$ in the variable $a_R \in A_R$, which is not satisfied. Even if this were true, the conclusion drawn from it might be false, as the function sinh on $(0,\infty)$ may very well be extended to an even function on $\mathbb{R}\setminus\{0\}$. Regardless of this mistake, we shall extract from [17] the existence and smoothness of $\Omega_{P,\Sigma}$ for $a^{\lambda_P} > 1$ and its limit as $a^{\lambda_P} \to +\infty$. The smoothness on A' alone has meanwhile been established for reductive groups of arbitrary rank over local fields of characteristic zero in [4]. In the case $G \cong \mathrm{SL}(2,\mathbb{R})$, $a^{\lambda_P} > 1$, our result has been obtained in [6] up to the calculation of an integral. In the present form it can be found in [12], Proposition 7.

2 Properties of the distributions and their Fourier transforms

We shall now review some properties of the distributions defined in section 2. First of all, the proof of Lemma 13.1 in [3] given for adelic groups can be translated to real groups, showing that $I_P(m,f)$ is independent of the choice of K. More explicitly, if one defines analogous distributions $J'_M(m,f)$ and $J'_P(\sigma,f)$ using a different maximal compact subgroup $K' = y^{-1}Ky$ of G, then

$$J'_M(m,f) = J_M(m,f^y) - v(y)J_G(m,f),$$
$$J'_P(\sigma,f) = J_P(\sigma,f^y) - v(y)\Theta_{\pi_\sigma}(f).$$

Inserting this into (8) and using (6) and the invariance of $I_M(m)$ proved in [3] one gets the assertion.

Fix a non-compact Cartan subgroup A of G as in Theorem 1. Since all such A's are conjugate, we may choose a maximal compact subgroup K of G such that A is stable under the corresponding Cartan involution θ. This places us in the situation of [1], where the distributions

$$T_f(a) = \tilde{\Delta}_\Sigma(a) \int_{G/A} f(xax^{-1}) v(x)\, dx$$

have been considered. Recall also that

$$F_f(a) = \tilde{\Delta}_\Sigma(a) \int_{G/A} f(xax^{-1})\, dx$$

is Harish-Chandra's invariant integral with respect to A, which provides a continuous map $\mathcal{C}(G) \to \mathcal{C}(A)$. If $a \in A'$, then $T_f(a) = \varepsilon_\Sigma(a) J_M(a, f)$ and $F_f(a) = \varepsilon_\Sigma(a) J_G(a, f)$. According to [1], Lemma 3.1, $T_f(a)$ is a tempered distribution for $a \in A \setminus A_I$ under the assumption that $\operatorname{rk} G = \operatorname{rk} K$. This assumption is, however, not used in the proof, which also applies if $a \in A''$. Indeed, if there is no real root, then $\mathfrak{n}$ is abelian, and the invertibility of $\mathrm{Id}_\mathfrak{n} - \mathrm{Ad}_\mathfrak{n}(\theta a)$ needed in the proof is satisfied for $a \in A''$. If $f \in C_c^\infty(G)$, then T_f is clearly smooth on A'', because it is then defined by an absolutely convergent integral. Moreover, it follows from [2], Lemma 8.1, that T_f is smooth on A' even for $f \in \mathcal{C}(G)$, and that the map $f \mapsto T_f\big|_\Omega$ extends to a continuous linear map from $\mathcal{C}(G)$ to $C^\infty(\Omega)$ for any relatively compact open subset Ω of A'. (One could replace A' by A'' here, but this is of no great importance.)

Next we come to the differential equations. We choose a non-degenerate $\mathrm{Ad}(G_\mathbf{C})$-invariant symmetric bilinear form $\langle\, .\,,\, .\,\rangle$ (e. g., the Killing form) on $\mathfrak{g}_\mathbf{C}$ which restricts to a negative definite form on $\mathfrak{k}$. Its restriction to $\mathfrak{a}_\mathbf{C}$ gives rise to a bijection $\mathfrak{a}_\mathbf{C} \to \mathfrak{a}_\mathbf{C}^*$ and a bilinear form on $\mathfrak{a}_\mathbf{C}^*$ which we denote in the same way. The latter is a $W(\mathfrak{g}_\mathbf{C}, \mathfrak{a}_\mathbf{C})$-invariant element of the symmetric algebra of $\mathfrak{a}_\mathbf{C}$ and defines a second-order differential operator D on A. Let ω be the element of the centre of the universal enveloping algebra of $\mathfrak{g}_\mathbf{C}$ which corresponds to D under the Harish-Chandra isomorphism. Then $\omega - \langle \rho_{\Sigma_P}, \rho_{\Sigma_P} \rangle$ is the Casimir element corresponding to $\langle\, .\,,\, .\,\rangle$. Generalizing Harish-Chandra's formula $F_{\omega f}(a) = DF_f(a)$, Theorem 5.1 of [1] asserts that, for $a \in A''$,

$$T_{\omega f}(a) = DT_f(a) - Q(a) F_f(a),$$

where

$$Q(a) = 2 \sum_{\alpha \in \Sigma_P^+} \frac{\langle \lambda_P, \alpha \rangle}{(a^\alpha - 1)(1 - a^{-\alpha})}.$$

If $\sigma \in \hat{M}$ has Σ-highest weight $\lambda - \rho_\Sigma$, then $\pi_{P,\sigma}(\omega f) = \langle \lambda, \lambda \rangle \pi_{P,\sigma}(f)$ and thus $J_P(\sigma, \omega f) = \langle \lambda, \lambda \rangle J_P(\sigma, f)$. Weyl's character formula implies $D(\Delta_\Sigma \Theta_\sigma) =$

$\langle\lambda,\lambda\rangle\Delta_\Sigma\Theta_\sigma$, and from (8) one now easily deduces that

$$I_{P,\Sigma}(a,\omega f) = DI_{P,\Sigma}(a,f) - Q(a)F_f(a). \tag{9}$$

Similar differential equations associated with all elements of the centre of the universal enveloping algebra are satisfied, but we shall not need them. No general closed formula for the corresponding Q's seems to be known.

Let us now consider the behaviour of our distributions under the action of $W(G,A)$ on $a \in A$. For later use, we prove

Lemma 1 *There is a direct product decomposition*

$$W(G,A) = W(M,A) \times \{1, w_0\}$$

with $w_0\Sigma = \Sigma$. *Moreover,*

$$\varepsilon_M(w) = \begin{cases} \det_I(w), & \textit{if } \beta \textit{ exists,} \\ \det(w) & \textit{otherwise,} \end{cases}$$

where $\det_I$ *denotes the determinant in* $\mathfrak{a}_I$.

Proof. There is a natural exact sequence

$$1 \to W(M,A) \to W(G,A) \to W(G,A_R) \to 1. \tag{10}$$

The exactness in the term $W(G,A_R)$ follows from the fact that all Cartan subgroups of M are conjugate. Since $W(M,A)$ acts transitively on the set of bases for the root system $\Sigma \cup -\Sigma$, one can find an element $w_0 \in W(G,A)$ with $w_0\Sigma = \Sigma$ which is mapped on the nontrivial element of $W(G,A_R)$. Thus, the sequence (10) splits. If there exists a real root $\beta \in \Sigma_P$, then w_0 is, of course, the simple reflection s_β. Otherwise, w_0 depends on Σ.

As the formula for $\varepsilon_M(w)$ is obvious for $w \in W(M,A)$, it remains to prove it for $w = w_0$. If w_0 is trivial on $\mathfrak{a}_I$, then it is a simple reflection in $\mathfrak{a}_I$, and there exists a real root. So we are in the first case, and there is nothing to prove. In the second case, w_0 is nontrivial on $\mathfrak{a}_I$. But an automorphism of order two of a root system of type D_n, $n \geq 2$, preserving a set of positive roots has determinant -1. Thus, $\det_I(w_0) = \det_R(w_0) = -1$, as asserted. (For $n = 1$, the assertion can be checked immediately.) □

Let $w \in W(G,A)$ and choose a representative $y \in K$ for w. Then we have $H_{wP}(xy^{-1}) = wH_P(x)$, hence $v(xy^{-1}) = v(x)$, and the obvious substitution yields

$$J_M(wa,f) = J_M(a,f). \tag{11}$$

The analogous property of $J_G(a,f)$ is well known. Now consider the map j : $\mathcal{H}_{P,\sigma} \to \mathcal{H}_{wP,w\sigma}$ or $\mathcal{H}_\sigma \to \mathcal{H}_{w\sigma}$ given by $(j\phi)(x) = \phi(xy)$. Of course, wP and $w\sigma$ depend only on the image of w in $W(G,A_R)$. We have

$$j\pi_{P,\sigma}(x) = \pi_{wP,w\sigma}(x)j, \qquad jJ_{\bar{P}|P}(\sigma) = J_{w\bar{P}|wP}(w\sigma)j.$$

Since $\partial_{wP}\phi(w\sigma) = \partial_P\phi(\sigma)$ and j commutes with $u_P(\sigma)$, we get

$$J_{wP}(w\sigma, f) = J_P(\sigma, f). \tag{12}$$

Combining now the definition (8) with (11), (12) and the identity $\Theta_{w\sigma}(wa) = \Theta_\sigma(a)$, we obtain

$$I_{wP}(wa, f) = I_P(a, f). \tag{13}$$

Let us tie up a loose end: the proof of equation (5). If a continuous representation π acts on a topological vector space, there is a representation π^* on the space of continuous antilinear functionals. If π is unitary, $\pi^* \cong \pi$ in a natural way. In our situation, $\pi_{P,\sigma^*} = \pi^*_{P,\sigma}$ and $J_{\bar{P}|P}(\sigma^*) = J_{P|\bar{P}}(\sigma)^*$ for $\sigma \in \hat{M}_I \otimes \mathfrak{a}^*_{R,\mathbb{C}}$. On the other hand, if $\sigma \in \hat{M}_I$,

$$\operatorname*{Res}_{z=0}\left(J_{\bar{P}|P}(\sigma_{z\lambda_P})^{-1}\frac{d}{dz}J_{\bar{P}|P}(\sigma_{z\lambda_P})\right) = -n_P(\sigma)\mathrm{Id}$$

for some $n_P(\sigma) \in \mathbb{C}$, since the left-hand side is an intertwining operator for $\pi_{P,\sigma}$. As $\sigma^*_{z\lambda_P} = \sigma_{\bar{z}\lambda_P}$, we now see that $n_P(\sigma) = n_{\bar{P}}(\sigma)$. Taking logarithmic derivatives and residues on both sides of (3), we obtain (5).

Now we shall recall one last property of $T_f(a)$ in the case $\operatorname{rk} G = \operatorname{rk} K$, namely, its behaviour as a approaches A_I. For this purpose, we consider

$$S_f(a) = T_f(a) + \lambda_P(H_\beta)\log|a^{\beta/2} - a^{-\beta/2}| \cdot F_f(a). \tag{14}$$

Note that $S_f(a)$ is independent of the choice of the Haar measure on A. This definition is at slight variance with [1], p. 568, where Arthur uses $\log|1-a^{-\beta}| = \log|a^{\beta/2} - a^{-\beta/2}| - \frac{1}{2}\beta(\log a_R)$, while it agrees with [2] and gives rise to the simple property

$$S_f(wa) = \varepsilon_M(w)S_f(a)$$

for $w \in W(G, A)$. In particular, with $w = s_\beta$ we see that $S_f(a)$ is even in a_R. Let $A''' = \{a \in A : a^\alpha \neq 1 \text{ for } \alpha \in \Sigma_P^+ \setminus \{\beta\}\}$. It has been proved in [1], p. 568, that for $a_I \in A_I''' = A_I \cap A'''$ the limit as $a_R \to 1$ of $S_f(a_I a_R)$ exists and is a tempered distribution. Moreover, the derivative of $S(a_I a_R)$ in a_R has a jump at $a_R = 1$, which is an invariant tempered distribution. We shall now give the details.

If T is a compact Cartan subgroup of G containing A_I, then there are unique root vectors $X_{\pm\beta} \in \mathfrak{g}_{\pm\beta}$ such that $[X_\beta, X_{-\beta}] = H_\beta$ and $X_\beta - X_{-\beta} \in \mathfrak{t}$. One special choice is $T \subset K$, in which case $X_{-\beta} = \theta X_\beta$, θ being the Cartan involution. Anyway, $\mathfrak{t} = \mathfrak{a}_I + \mathbb{R}(X_\beta - X_{-\beta})$, and we put $t_\theta = \exp\theta(X_\beta - X_{-\beta})$ for $\theta \in \mathbb{R}$. It is known that $t_\pi = \gamma$. Let $y = \exp\frac{\pi i}{4}(X_\beta + X_{-\beta}) \in G_{\mathbb{C}}$. Then $\mathrm{Ad}(y)$ leaves $\mathfrak{a}_I$ pointwise fixed and maps $i(X_\beta - X_{-\beta})$ on H_β, thus $\mathrm{Ad}(y)$ is a Cayley transform $\mathfrak{t}_{\mathbb{C}} \to \mathfrak{a}_{\mathbb{C}}$. If we put $\alpha^y(X) = \alpha(\mathrm{Ad}(y)X)$ for $\alpha \in \pm\Sigma_P$, $X \in \mathfrak{t}_{\mathbb{C}}$, we get a system Σ_P^y of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and the corresponding function

$$\Delta_{\Sigma_P^y}(\exp X) = \prod_{\alpha\in\Sigma_P^y}\left(e^{\alpha(X)/2} - e^{-\alpha(X)/2}\right),$$

is correctly defined on the connected group T. Put

$$F_f^T(t) = \Delta_{\Sigma_P^\nu}(t) \int_{G/T} f(xtx^{-1})\,dx$$

for $f \in \mathcal{C}(G)$ and $t \in T \cap G'$. If $a_I \in A_I'''$ and $f \in C_c^\infty(G)$, then the limits

$$\partial_\beta^\pm S_f(a_I) = \lim_{t\to 0\pm} \frac{d}{dt} S_f(a_I \exp tH_\beta),$$

$$\partial_{\beta^\nu} F_f^T(a_I) = \lim_{\theta\to 0} \frac{d}{d\theta} F_f^T(a_I t_\theta)$$

exist, where t and θ are real variables. By [2], Lemmas 8.1 and 8.3, this is even true for $f \in \mathcal{C}(G)$. Actually, $\partial_\beta^+ S_f(a_I) + \partial_\beta^- S_f(a_I) = 0$ because $S_f(a)$ is even in a_R. Lemma 4.1, Theorem 6.4 and Lemma 7.1 of [1] together yield

$$\partial_\beta^+ S_f(a_I) - \partial_\beta^- S_f(a_I) = -2i\partial_{\beta^\nu} F_f^T(a_I) \tag{15}$$

for $a_I \in A_I'''$ and $f \in C_c^\infty(G)$. This is a special case of [2], Theorem 6.1, which extends to $f \in \mathcal{C}(G)$ by Corollary 8.4 of the same paper. Actually, the assumptions of the latter result reqire, in addition, that $\Delta_\Sigma(a_I) \neq 0$, but this is not important, because otherwise (15) reduces to $0 = 0$. Consult [2], p. 260, for a correction of the normalizing constants used in [1].

Our next task is to exploit the results of Warner in [17]. Since the results of the previous section are known for $\dim K\backslash G = 2$ by Proposition 7 of [12], we may exclude this case from now on, as Warner does. He uses the techniques of wave packets. Given a unitary double representation τ of K on a finite dimensional Hilbert space V_τ such that each irreducible subrepresentation occurs with multiplicity one, let $L(\tau)$ be the finite-dimensional space of functions $\psi : M_I \to V_\tau$ such that $\psi(m_1 m m_2) = \tau(m_1)\psi(m)\tau(m_2)$. Recall that $L(\tau)$ is the direct sum of all $L(\tau,\sigma) = L(\tau) \cap C_\sigma(M_I) \otimes V_\tau$, where $C_\sigma(M_I)$ is the space spanned by the matrix coefficients of $\sigma \in \hat{M}_I$. Using the Eisenstein integral associated to P and $\psi \in L(\tau,\sigma)$, and a Schwartz function $\alpha \in \mathcal{C}(i\mathfrak{a}_R^*)$, Harish-Chandra has built the wave packet $\phi_\alpha \in \mathcal{C}(G) \otimes V_\tau$. By Proposition 8.8 of [17], if $a \in A$ with $a^{\lambda_P} > 1$, the weighted orbital integral of ϕ_α can be evaluated as

$$T_{\phi_\alpha}(a) = \frac{1}{i} \int_{i\mathfrak{a}_R^*} \alpha(\lambda) P_\tau H(a,\lambda)\,d\lambda + R_\alpha(a), \tag{16}$$

where

$$P_\tau v = \int_K \tau(k) v \tau(k^{-1})\,dk$$

and $H : \{a \in A : a^{\lambda_P} > 1\} \times i\mathfrak{a}_R^* \to V_\tau$ is a smooth function of polynomial growth in λ such, that $H(a,\lambda) \to 0$ for $a^{\lambda_P} \to +\infty$, uniformly on compacta in (a_I, λ). Note that our λ is Warner's $i\nu\lambda^*$. He gives a formula for $R_\alpha(a)$ in terms of Harish-Chandra's c-functions in Lemma 8.8.

Section 9 of [17] is an attempt to determine $P_\tau H$ as

$$P_\tau H = C(P,\sigma)\cdot P_\tau\psi(1)\cdot\Omega, \tag{17}$$

where $C(P,\sigma)$ is a certain constant and Ω is a scalar function not depending on ψ, but only on σ. Since this argument contains a mistake, we cannot use the form of $P_\tau H$ given in (17). We only know that, by construction, H depends linearly on $\psi \in L(\tau,\sigma)$. Nevertheless, it will be convenient to apply sections 10–13 of [17] in order to interpret (16) in terms of Fourier transform.

Choose a finite subset $F \subset \hat{K}$ and denote by $\chi_F \in C^\infty(K)$ the associated idempotent. Put

$$\mathcal{C}(G;F) = \{f \in \mathcal{C}(G) : \chi_F *_K f *_K \chi_F = f\}.$$

Recall the decomposition $\mathcal{C}(G) = \mathcal{C}_{\rm dis}(G) \oplus \mathcal{C}_{\rm con}(G)$, where $\mathcal{C}_{\rm dis}(G)$ is the space of cusp forms and $\mathcal{C}_{\rm con}(G)$ is the space spanned by wave packets. The associated projections $f \mapsto f_{\rm dis}$, $f \mapsto f_{\rm con}$ leave $\mathcal{C}(G;F)$ invariant, thus $\mathcal{C}(G;F) = \mathcal{C}_{\rm dis}(G;F) \oplus \mathcal{C}_{\rm con}(G;F)$. If $f \in \mathcal{C}(G;F)$, then $\pi_{P,\sigma_\lambda}(f)$ actually acts in a finite-dimensional subspace $\mathcal{H}_{\sigma,F}$ of $\mathcal{H}_\sigma$, the projector p_F on the latter being given by

$$p_F\phi(k) = \int_K \chi_F(k_1)\phi(k_1^{-1}k)\,dk_1.$$

Note that $\mathcal{H}_{\sigma,F} = \{0\}$ unless σ occurs in $\delta\big|_{M_I}$ for some $\delta \in F$.

Now let τ be the double representation of K on $C^\infty(K\times K)$ given by

$$(\tau(k_1')v\tau(k_2'))(k_1,k_2) = v(k_1k_1', k_2'k_2).$$

Then $V_F = \{v \in C^\infty(K\times K) : \tau(\chi_F)v\tau(\chi_F) = v\}$ is stable under τ and finite-dimensional. Denote $\tau_F = \tau\big|_{V_F}$. Recall that the wave packet is bilinear in $\psi \in L(\tau,\sigma)$ and $\alpha \in \mathcal{C}(i\mathfrak{a}_R^*)$, hence may be defined on $L(\tau,\sigma)\otimes\mathcal{C}(i\mathfrak{a}_R^*)$. Using Harish-Chandra's bijection $T \mapsto \psi_T$ from $\operatorname{End}(\mathcal{H}_{\sigma,F})$ to $L(\tau_F,\sigma)$, one may build a wave packet out of $\psi_{\pi_{P,\sigma_\lambda}(\check{f})}$, where $\check{f}(x) = f(x^{-1})$, $\lambda \in i\mathfrak{a}_R^*$. Summing up these wave packets, suitably normalized, over all $\sigma \in \hat{M}_I$ and applying the map $V_F \to \mathbb{C}$, $v \mapsto v(1,1)$, one recovers $f_{\rm con}$ (cf. [17], p. 86). If we insert this sum of wave packets into (16), we obtain

$$T_{f_{\rm con}}(a) = \sum_{\sigma\in\hat{M}_I}\frac{1}{2\pi i}\int_{i\mathfrak{a}_R^*} h_F(a,\sigma_\lambda)\pi_{P,\sigma_\lambda}(\check{f})\,d\lambda + R_f(a), \tag{18}$$

where $h_F(a,\sigma_\lambda)$ is defined by the commutative diagram of linear maps

$$\begin{array}{ccccc}
\operatorname{End}(\mathcal{H}_{\sigma,F}) & \xrightarrow{h_F(a,\sigma_\lambda)} & \mathbb{C} & & P_{\tau_F}v(1,1) \\
\psi_\cdot \Big\downarrow \wr & & \Big\uparrow & & \Big\uparrow \\
L(\tau_F,\sigma) & \xrightarrow{H(a,\lambda)} & V_F & & v
\end{array}$$

Actually, we have absorbed into $h_F(a,\sigma_\lambda)$ certain normalizing constants.

The term $R_f(a)$ has been explicated in Theorem 12.8 of [17] as $|\lambda^*|$ times

$$\Delta_\Sigma(a)\sum_{\sigma\in\hat{M}_I}\Theta_\sigma(a)\frac{1}{2\pi i}\,\text{p.v.}\int_{i\mathfrak{a}_R^*}a^\lambda\,\text{Tr}\left(\partial_P J_{P|\bar{P}}(\sigma_\lambda)\cdot J_{P|\bar{P}}(\sigma_\lambda)^{-1}\pi_{P,\sigma_\lambda}(\check{f})\right)\,d\lambda$$
$$-\frac{1}{2}\Delta_\Sigma(a)\sum_{\sigma\in\hat{M}_I}\Theta_\sigma(a)\,\text{Tr}\left(\operatorname*{Res}_{z=0}(\partial_P J_{P|\bar{P}}(\sigma_{z\lambda_P})\cdot J_{P|\bar{P}}(\sigma_{z\lambda_P})^{-1})\pi_{P,\sigma}(\check{f})\right),$$

if written in our notation. Indeed, for $f\in C_c^\infty(G)$, both terms may be combined into a sum of contour integrals over $-D_\sigma$ with upward orientation similar to (8). Since a contour integral of a (logarithmic) derivative is homogeneous of degree zero under linear substitutions of the complex variable, we have simply replaced Warner's $i\nu\lambda^*$ by our $\lambda = z\lambda_P$ and his contour by our one. Note that from p. 88 of [17] on, $d\nu$ stands for the measure $\frac{|\lambda^*|}{2\pi}d\nu$.

In the definition of $T_f(a)$, Warner uses λ^* to define the weighting function $v(x)$, and he normalizes the Haar measure on A_R with the help of the Killing form. Since we use λ_P for both purposes, we may omit the prefacing factor $|\lambda^*|$. We would like to have an expression in terms of f instead of $\check{f}$. Given a linear operator B in a Hilbert space H, let $\check{B}$ denote the transposed operator acting in the dual space of H. If B is of trace class, so is $\check{B}$, and $\text{Tr}\,B=\text{Tr}\,\check{B}$. If π is a representation of G in H, then $\check{\pi}(f)=\pi(\check{f})\check{}$. In our situation, $\check{\pi}_{P,\sigma}\cong\pi_{P,\check{\sigma}}$ and $J_{P|\bar{P}}(\sigma)\check{}=J_{\bar{P}|P}(\check{\sigma})$. Therefore,

$$\text{Tr}\left(\partial_P J_{P|\bar{P}}(\sigma)\cdot J_{P|\bar{P}}(\sigma)^{-1}\pi_{P,\sigma}(\check{f})\right)=\text{Tr}\left(\pi_{P,\check{\sigma}}(f)J_{\bar{P}|P}(\check{\sigma})^{-1}\partial_P J_{\bar{P}|P}(\check{\sigma})\right).$$

Inserting this and substituting σ_λ for $(\sigma_\lambda)\check{}=\check{\sigma}_{-\lambda}$, we obtain a sum of contour integrals over D_σ with downward orientation. Thus $R_f(a)$ equals

$$-\Delta_\Sigma(a)\sum_{\sigma\in\hat{M}_I}\Theta_{\check{\sigma}}(a)\frac{1}{2\pi i}\,\text{p.v.}\int_{i\mathfrak{a}_R^*}a^{-\lambda}\,\text{Tr}\left(\pi_{P,\sigma_\lambda}(f)J_{\bar{P}|P}(\sigma_\lambda)^{-1}\partial_P J_{\bar{P}|P}(\sigma_\lambda)\right)\,d\lambda$$
$$-\frac{1}{2}\Delta_\Sigma(a)\sum_{\sigma\in\hat{M}_I}\Theta_{\check{\sigma}}(a)\,\text{Tr}\left(\pi_{P,\sigma}(f)\operatorname*{Res}_{z=0}(J_{\bar{P}|P}(\sigma_{z\lambda_P})^{-1}\partial_P J_{\bar{P}|P}(\sigma_{z\lambda_P}))\right).$$

Due to (5),

$$R_f(a)=\Delta_\Sigma(a)\frac{1}{2\pi i}\,\text{p.v.}\int_{\hat{M}}\Theta_{\check{\sigma}}(a)J_P(\sigma,f)\,d\sigma+\Delta_\Sigma(a)\sum_{\sigma\in\hat{M}_I}\frac{n(\sigma)}{2}\Theta_{\check{\sigma}}(a)\Theta_{\pi_\sigma}(f).$$

If we multiply (8) through with $\varepsilon_\Sigma(a)$, we get

$$T_{f_{\text{con}}}(a)=I_{P,\Sigma}(a,f_{\text{con}})+R_f(a),$$

and a glance at (18) shows that

$$I_{P,\Sigma}(a, f_{\rm con}) = \sum_{\sigma\in\hat{M}_I} \frac{1}{2\pi i}\int_{i\mathfrak{a}_R^*} h_F(a,\sigma_\lambda)\pi_{P,\sigma_\lambda}(\check{f})\,d\lambda.$$

It is clear that $h_F(a,\sigma_\lambda)$ uniquely determines $\tilde{h}_F(a,(\sigma_\lambda)^{\check{}}) \in \mathrm{End}(\mathcal{H}_{\check{\sigma},F})$ such that

$$h_F(a,\sigma_\lambda)(B) = \mathrm{Tr}\left(\tilde{h}_F(a,(\sigma_\lambda)^{\check{}})\check{B}\right)$$

for all $B \in \mathrm{End}(\mathcal{H}_{\sigma,F})$. Thus, in the simplified notation of (6),

$$I_{P,\Sigma}(a, f_{\rm con}) = \frac{1}{2\pi i}\int_{\hat{M}} \mathrm{Tr}(\tilde{h}_F(a,\sigma)\pi_{P,\sigma}(f))\,d\sigma \tag{19}$$

for all $f \in \mathcal{C}(G;F)$ and $a \in A'$ with $a^{\lambda_P} > 1$. Note that we have defined $I_{P,\Sigma}(a)$ for $a \in A'$ only. Of course, it can be smoothly extended to A''. However, we shall define it on the singular set $A \setminus A'$ in another, more significant way in section 6.

Let Φ be a smooth function on $\hat{M}$ whose values on $\{\sigma_\lambda : \lambda \in i\mathfrak{a}_R^*\}$ are in $\mathrm{End}(\mathcal{H}_{\sigma,F})$ for each $\sigma \in \hat{M}_I$. We call Φ Weyl-invariant if

$$jJ_{\bar{P}|P}(\sigma)\Phi(\sigma) = \Phi(w_0\sigma)jJ_{\bar{P}|P}(\sigma)$$

for all $\sigma \in \hat{M}$, where j is as in the proof of (12). Inserting for f varying wave packets, one sees that $\pi_{P,\sigma}(f)$ runs through all Weyl-invariant Schwartz functions on $\hat{M}$ in the sense described above. Thus, replacing $\tilde{h}_F$ by its $W(G,A_R)$-average if necessary, we may assume that that it is Weyl-invariant in the second variable, and with this property $\tilde{h}_F$ is uniquely determined by (19). In particular, if $F \subset F' \subset \hat{K}$, then

$$\tilde{h}_F(a,\sigma_\lambda) = p_F\tilde{h}_{F'}(a,\sigma_\lambda)\big|_{\mathcal{H}_{\sigma,F}},$$

hence the $\tilde{h}_F$'s fit togehter and form a homomorphism $\tilde{h}(a,\sigma_\lambda)$ from the K-finite subspace of $\mathcal{H}_\sigma$ into $\mathcal{H}_\sigma$.

We already know that $I_P(a)$ is an invariant distribution, thus $I_P(a, f*g) = I_P(a, g*f)$ for all $f,\ g \in \mathcal{C}(G)$. If f and g run through all K-finite elements of $\mathcal{C}(G)$, then (19) implies that $\tilde{h}(a,\sigma)$ commutes with all $\pi_{P,\sigma}(g)$, and from the irreducibility of $\pi_{P,\sigma}$ for generic σ we see that

$$\tilde{h}(a,\sigma) = \Omega_{P,\Sigma}(a,\check{\sigma})\mathrm{Id}$$

for some scalar function $\Omega_{P,\Sigma}$. Thereby we have recovered (17), and one may check that $\Omega_{P,\Sigma} = \Omega$. However, we shall henceforth use the language of invariant Fourier transform instead of working with wave packets. Let us restate what we have extracted from [17].

Proposition 2 *There exists a smooth function $\Omega_{P,\Sigma}$ on $\{a \in A : a^{\lambda_P} > 1\} \times \hat{M}$ of polynomial growth in the second variable and such, that*

$$I_{P,\Sigma}(a,f) = -\Delta_\Sigma(a) \sum_{\pi \in \hat{G}_{\mathrm{dis}}} \Theta_{\check{\pi}}(a)\Theta_\pi(f) + \frac{1}{2\pi i}\int_{\hat{M}} \Omega_{P,\Sigma}(a,\check{\sigma})\Theta_{\pi_\sigma}(f)\,d\sigma$$

for all $f \in \mathcal{C}(G)$ and $a \in A'$ with $a^{\lambda_P} > 1$. Moreover, $\Omega_{P,\Sigma}(a, w\sigma) = \Omega_{P,\Sigma}(a,\sigma)$ for $w \in W(G, A_R)$, and $\Omega_{P,\Sigma}(a,\sigma) \to 0$ for $a^{\lambda_P} \to +\infty$ uniformly on compacta in (a_I, σ).

Note that the contribution of f_{dis} has been determined in [1], [2]. The K-finiteness assumption on f has been removed in section 13 of [17].

We have indicated the dependence of $\Omega_{P,\Sigma}$ on P in the notation not without reason. Equations (4) and (8) imply

$$\begin{aligned} I_{P,\Sigma}(a,f) &= I_{\bar{P},\Sigma}(a,f) + \Delta_\Sigma(a)\frac{1}{2\pi i}\,\mathrm{p.v.}\int_{\hat{M}} \Theta_{\check{\sigma}}(a)\frac{\partial_P \mu(\check{\sigma})}{\mu(\check{\sigma})}\Theta_{\pi_\sigma}(f)\,d\sigma \\ &= I_{\bar{P},\Sigma}(a,f) + \Delta_\Sigma(a)\frac{1}{4\pi i}\int_{\hat{M}} (\Theta_{\check{\sigma}}(a) - \Theta_{w_0\check{\sigma}}(a))\frac{\partial_P \mu(\check{\sigma})}{\mu(\check{\sigma})}\Theta_{\pi_\sigma}(f)\,d\sigma. \end{aligned}$$

Here we have used that $\mu(\check{\sigma}) = \mu(\sigma) = \mu(w_0\sigma)$ and $\partial_{\bar{P}} = -\partial_P$. The logarithmic derivative of μ has a simple pole at $\sigma \in \hat{M}_I$ if $\mu(\sigma) = 0$. But, in this case, $w_0\sigma \cong \sigma$ (see [10], p. 195), and the difference of characters makes the last integrand smooth on $\hat{M}$. We may now extend Proposition 2 to all $a \in A'$, $a \notin A_I$, by putting

$$\Omega_{P,\Sigma}(a,\sigma) = \Omega_{\bar{P},\Sigma}(a,\sigma) + \frac{1}{2}\Delta_\Sigma(a)(\Theta_\sigma(a) - \Theta_{w_0\sigma}(a))\frac{\partial_P\mu(\sigma)}{\mu(\sigma)} \qquad (20)$$

for $a^{\lambda_P} < 1$. Of course, $\Omega_{P,\Sigma}$ remains w_0-invariant in σ.

The notation $\Omega_{P,\Sigma}(a,\sigma)$ has its final meaning only for $a \in A'$. It will be redefined on the singular set in section 6.

We come now to the last part of this section. If f is in the K-finite subspace of $\mathcal{C}(G)$, then $\Theta_{\pi_\sigma}(f)$ is a w_0-invariant Schwartz function of $\sigma \in \hat{M}$ supported on finitely many connected components of $\hat{M} = \hat{M}_I \times i\mathfrak{a}_R^*$. Again using wave packets, one proves that this map $f \mapsto \Theta_{\pi_\sigma}(f)$ is surjective. (This is the easiest part of the trace Payley-Wiener theorem of [4].) Thus, Proposition 2 allows us to translate the properties of $I_P(a)$ collected before into properties of $\Omega_{P,\Sigma}$. E. g., (13) implies

$$\Omega_{wP,\Sigma}(wa,\sigma) = \varepsilon_M(w)\Omega_{P,\Sigma}(a,\sigma) \qquad (21)$$

for $w \in W(G,A)$. In order to derive the differential equation for $\Omega_{P,\Sigma}$, we rewrite (6) as

$$F_f(a) = \Delta_\Sigma(a)\frac{1}{4\pi i}\int_{\hat{M}} (\Theta_{\check{\sigma}}(a) + \Theta_{w_0\check{\sigma}}(a))\Theta_{\pi_\sigma}(f)\,d\sigma.$$

As we do not insist that $\operatorname{rk} G = \operatorname{rk} K$, the slight simplification provided by Lemma 9.1 of [17] is not available. Inserting the last equation and the formula from Proposition 2 into (9) and varying $f \in \mathcal{C}_{\text{con}}(G)$, we obtain in view of $\Theta_{\pi_\sigma}(\omega f) = \langle\lambda, \lambda\rangle \Theta_{\pi_\sigma}(f)$ the differential equation

$$D\Omega_{P,\Sigma}(a, \sigma) = \langle\lambda, \lambda\rangle \Omega_{P,\Sigma}(a, \sigma) + \frac{1}{2} Q(a) \Delta_\Sigma(a)(\Theta_\sigma(a) + \Theta_{w_0\sigma}(a)), \tag{22}$$

where $\lambda - \rho_\Sigma$ is the Σ-highest weight of σ.

Finally, we study the behaviour of $\Omega_{P,\Sigma}$ as a approaches the singular set A_I. By Proposition 2, $\Omega_{P,\Sigma}$ extends to a smooth function on A'' with values in the space of w_0-invariant distributions on $\hat{M}$ which are tempered on each connected component.

This yields nothing new in the case $\operatorname{rk} G = \operatorname{rk} K$, to which we now turn. In view of the definition (14), let us put

$$\Psi_{P,\Sigma}(a, \sigma) = \Omega_{P,\Sigma}(a, \sigma) + \frac{1}{2} \lambda_P(H_\beta) \log |a^{\beta/2} - a^{-\beta/2}| \cdot \Delta_\Sigma(a)(\Theta_\sigma(a) + \Theta_{w_0\sigma}(a)). \tag{23}$$

Just as we constructed the invariant distribution I_P from J_M in (8), we may construct an invariant distribution from S_f, for which the analogue of Lemma 2 will be true with $\Omega_{P,\Sigma}$ replaced by $\Psi_{P,\Sigma}$. The last terms in (8) are smooth, hence the behaviour of S_f at the singular set will be inherited by the corresponding invariant distribution. In particular,

$$\lim_{a_R \to 1} \left(\Psi_{P,\Sigma}(a_I a_R, \sigma) - \Psi_{P,\Sigma}(a_I a_R^{-1}, \sigma)\right) = 0 \tag{24}$$

for $a_I \in A_I'''$ as a distribution on $\hat{M}$. We shall now translate (15) into a condition on $\Psi_{P,\Sigma}$. The Fourier transform of F_f^T has been calculated in [15]. If $f \in \mathcal{C}_{\text{con}}(G)$ then, in our notation,

$$F_f^T(t) = \frac{1}{\#(W(M, A))} \int_{\hat{M}} \sum_{w \in W(G,T)} \det(w) \Phi_\Sigma(wt, \check{\sigma}) \Theta_{\pi_\sigma}(f) \, d\sigma, \tag{25}$$

where the function Φ_Σ is given by

$$\Phi_\Sigma(a_I t_\theta, \sigma_\lambda) = \frac{\lambda_P(H_\beta)}{4} \Delta_\Sigma(a_I) \Theta_\sigma(a_I) \frac{\sin((\theta - \pi \operatorname{sgn} \theta)\lambda(H_\beta)) - \sin \theta \lambda(H_\beta)}{\sin \pi \lambda(H_\beta)}$$

for $a_I \in A_I$, $0 < |\theta| < \pi$, $\sigma \in \hat{M}_I$ and $\lambda \in i\mathfrak{a}_R^*$. Note that our $\lambda(H_\beta)$ equals $i\nu$ in the notation of [15]. Inserting (25) and the analogue of Lemma 2 for S_f into the identity (15), we deduce that

$$\partial_\beta^+ \Psi_{P,\Sigma}(a_I, \sigma) - \partial_\beta^- \Psi_{P,\Sigma}(a_I, \sigma) = \tfrac{4\pi}{\#(W(M,A))} \sum_{w \in W(G,T)} \det(w) \partial_{w\beta^\vee} \Phi_\Sigma(wa_I, \sigma),$$

for $a_I \in A_I'''$, $\sigma \in \hat{M}$. Here we have put $\partial_{w\beta^y} f(t) = \lim_{\theta\to 0} \frac{d}{d\theta} f(t \cdot wt_\theta)$. The limits $\partial_\beta^\pm \Psi_{P,\Sigma}$ are understood in the sense of distributions on $\hat{M}$.

It is clear that $yW(\mathfrak{g}_\mathbf{C}, \mathfrak{t}_\mathbf{C})y^{-1} = W(\mathfrak{g}_\mathbf{C}, \mathfrak{a}_\mathbf{C})$ and that $W(M,A) \cong W(M_I, A_I)$ via restriction to $\mathfrak{a}_I$.

Lemma 2 *Let $W(G,T,A_I) = \{w \in W(G,T) : wA_I = A_I\}$. Then the map $w \mapsto w|_{\mathfrak{a}_I}$ is an isomorphism $W(G,T,A_I) \to W(M_I,A_I)$.*

Proof. First we show that the image is contained in $W(M_I, A_I)$. Given $w \in W(G,T,A_I)$, $ywy^{-1}\Sigma$ is a system of positive roots for $(\mathfrak{m}_\mathbf{C}, \mathfrak{a}_\mathbf{C})$, thus $ywy^{-1}\Sigma = w_1\Sigma$ for some $w_1 \in W(M,A)$. Now $s_\beta^k w_1^{-1} ywy^{-1}$ must stabilize Σ_P for some $k \in \{0,1\}$, hence $ywy^{-1} = w_1 s_\beta^{-k}$, and $w|_{\mathfrak{a}_I} = w_1|_{\mathfrak{a}_I}$, because $\mathrm{Ad}(y)$ is trivial on $\mathfrak{a}_I$.

Next we check injectivity. If w is as above with $w|_{\mathfrak{a}_I} = \mathrm{Id}$, then $w = s_{\beta^y}^k$ with $k \in \{0,1\}$. But β^y is a noncompact root, as one easily sees by a calculation in the centralizer $G(\beta)$ of A_I, thus $w = 1$. (Note that the complexified Lie algebra of $G(\beta)$ is $\mathfrak{g}_\mathbf{C}(\beta) = \mathfrak{a}_\mathbf{C} + \mathfrak{g}_{\mathbf{C},\beta} + \mathfrak{g}_{\mathbf{C},-\beta}$.)

Since $W(M_I, A_I)$ is generated by simple reflections, it remains to show that $s_\delta|_{\mathfrak{a}_I} = s_{\delta^y}|_{\mathfrak{a}_I}$ is in the image for any $\delta \in \Sigma$. If δ is strongly orthogonal to β (i. e., if $\alpha = \beta + \delta \notin \Sigma_P$), then $\mathfrak{g}_{\mathbf{C},\delta}$ commutes with $X_{\pm\beta}$, hence with y. So $\mathfrak{g}_{\mathbf{C},\delta^y} = \mathfrak{g}_{\mathbf{C},\delta}$, and δ^y is a compact root. Otherwise, $\Sigma_P = \{\alpha, \bar{\alpha}, \beta, \delta\}$, and a calculation in $\mathrm{Sp}(1,1)$ shows that, although δ^y is a noncompact root, the roots α and $\bar{\alpha}$ are compact. Clearly, $s_{\alpha^y} s_{\bar{\alpha}^y}$ leaves $\mathfrak{a}_I$ invariant, and $s_{\alpha^y} s_{\bar{\alpha}^y}|_{\mathfrak{a}_I} = s_{\delta^y}|_{\mathfrak{a}_I}$. As $W(G,T) = W(K,T)$ is generated by the simple reflections in the compact roots, we are done. □

Since $\Phi_\Sigma(wt,\sigma) = \det(w)\Phi_\Sigma(t,\sigma)$ for $w \in W(G,T,A_I)$, the Lemma allows us to write

$$\partial_\beta^+ \Psi_{P,\Sigma}(a_I,\sigma) - \partial_\beta^- \Psi_{P,\Sigma}(a_I,\sigma) = 4\pi \sum_{w \in W(G,T,A_I)\backslash W(G,T)} \det(w) \partial_{w\beta^y} \Phi_\Sigma(wa_I,\sigma). \tag{26}$$

The term for $w = 1$ has a simple expression:

$$\partial_{\beta^y} \Phi_\Sigma(a_I, \sigma_\lambda) = \frac{\lambda_P(H_\beta)}{4} \Delta_\Sigma(a_I) \Theta_\sigma(a_I) \lambda(H_\beta) \frac{\cos \pi\lambda(H_\beta) - \operatorname{sgn}\sigma}{\sin \pi\lambda(H_\beta)}.$$

3 A particular solution of the differential equation

In order to determine $\Omega_{P,\Sigma}$, we shall first find some solution $\tilde{\Omega}_{P,\Sigma}$ of the inhomogeneous differential equation (22), which leaves us with the associated homogeneous equation. We shall check that $\tilde{\Omega}_{P,\Sigma}$ has basically all properties established in the previous section for $\Omega_{P,\Sigma}$, which will finally allow us to conclude that both functions are equal.

First of all, we want to explicate the last term of (22). If $\sigma \in \hat{M}$ has Σ-highest weight $\lambda - \rho_\Sigma$, then $w_0\sigma$ has Σ-highest weight $w_0\lambda - \rho_\Sigma$. Weyl's character formula and Lemma 1 now imply

$$\Delta_\Sigma(a)(\Theta_\sigma(a) + \Theta_{w_0\sigma}(a)) = \sum_{w\in W(G,A)} \varepsilon_M(w) a^{w\lambda}. \tag{27}$$

It is therefore natural to look for a solution of the differential equation

$$D\phi(a,\lambda) = \langle\lambda,\lambda\rangle\phi(a,\lambda) + Q(a)a^\lambda$$

first, where $\lambda \in \Lambda_A$. The expansion

$$Q(a) = 2\sum_{\alpha\in\Sigma_P^+} \langle\lambda_P,\alpha\rangle \frac{a^{-\alpha}}{(1-a^{-\alpha})^2} = 2\sum_{\alpha\in\Sigma_P^+} \langle\lambda_P,\alpha\rangle \sum_{n=1}^\infty n a^{-n\alpha},$$

which is convergent for $a^{\lambda_P} > 1$, suggests to look for ϕ in the form

$$\phi(a,\lambda) = a^\lambda \sum_{\alpha\in\Sigma_P^+} \langle\lambda_P,\alpha\rangle \sum_{n=1}^\infty c_n(\alpha,\lambda) a^{-n\alpha}.$$

Inserting this and remembering that $Da^\lambda = \langle\lambda,\lambda\rangle a^\lambda$, we get

$$\sum_{\alpha\in\Sigma_P^+} \langle\lambda_P,\alpha\rangle \sum_{n=1}^\infty c_n(\alpha,\lambda)(\langle\lambda - n\alpha, \lambda - n\alpha\rangle - \langle\lambda,\lambda\rangle) a^{\lambda-n\alpha}$$
$$= 2\sum_{\alpha\in\Sigma_P^+} \langle\lambda_P,\alpha\rangle \sum_{n=1}^\infty n a^{\lambda-n\alpha}.$$

As the characters of A are linearely independent, this implies that

$$c_n(\alpha,\lambda) = \frac{2}{\langle n\alpha - 2\lambda, \alpha\rangle}.$$

The resulting function ϕ is independent of the choice of the form $\langle\,.\,,\,.\,\rangle$. Using the fact that $2\langle\lambda,\alpha\rangle = \langle\alpha,\alpha\rangle\lambda(H_\alpha)$, ϕ can be written as

$$\phi(a,\lambda) = a^\lambda \sum_{\alpha\in\Sigma_P^+} \lambda_P(H_\alpha) \sum_{n=1}^\infty \frac{a^{-n\alpha}}{n - \lambda(H_\alpha)},$$

which is absolutely convergent for $a^{\lambda_P} > 1$ provided $\lambda(H_\alpha)$ is not a positive integer for any $\alpha \in \Sigma_P^+$. If we exclude, for the time being, these values of $w\lambda$, the function

$$\tilde{\Omega}_P(a,\lambda) = \frac{1}{2}\sum_{w\in W(G,A)} \varepsilon_M(w) a^{w\lambda} \sum_{\alpha\in\Sigma_P^+} \lambda_P(H_\alpha) \sum_{n=1}^\infty \frac{a^{-n\alpha}}{n - w\lambda(H_\alpha)} \tag{28}$$

is a solution of (22) for $a^{\lambda_P} > 1$. Note that Σ_P^+ is independent of the choice of Σ. If $\sigma \in \hat{M}$ with Σ-highest weight $\lambda - \rho_\Sigma$, then $\tilde{\Omega}_{P,\Sigma}(a,\sigma) := \tilde{\Omega}_P(a,\lambda)$ is our candidate for $\Omega_{P,\Sigma}$.

We can apply (28) to $\tilde{\Omega}_{w'P}(w'a,\lambda)$ for any $w' \in W(G,A)$. Using $\Sigma_{w'P}^+ = w'\Sigma_P^+$, we may substitute $\alpha = w'\alpha'$. The obvious substitution in the sum over w now yields

$$\tilde{\Omega}_{w'P}(w'a,\lambda) = \varepsilon_M(w')\tilde{\Omega}_P(a,\lambda), \tag{29}$$

which should be compared with (21). On the other hand, $\Sigma_{\bar{P}}^+ = -\Sigma_P^+$, thus

$$\tilde{\Omega}_{\bar{P}}(a,\lambda) = \frac{1}{2}\sum_{w\in W(G,A)} \varepsilon_M(w)a^{w\lambda} \sum_{\alpha\in\Sigma_P^+} \lambda_P(H_\alpha)\sum_{n=1}^{\infty}\frac{a^{n\alpha}}{n+w\lambda(H_\alpha)}$$

for $a^{\lambda_P} < 1$. In view of (20) it is natural to extend $\tilde{\Omega}_{P,\Sigma}$ to all $a \in A$, $a \notin A_I$, by putting

$$\tilde{\Omega}_{P,\Sigma}(a,\sigma) = \tilde{\Omega}_{\bar{P},\Sigma}(a,\sigma) + \frac{1}{2}\Delta_\Sigma(a)(\Theta_\sigma(a) - \Theta_{w_0\sigma}(a))\frac{\partial_P\mu(\sigma)}{\mu(\sigma)} \tag{30}$$

for $a^{\lambda_P} < 1$. Since $\tilde{\Omega}_{P,\Sigma} - \tilde{\Omega}_{\bar{P},\Sigma} = \Omega_{P,\Sigma} - \Omega_{\bar{P},\Sigma}$, equations (22) and (29) are satisfied by the function so extended. Let us calculate it explicitly. By [8], [10] or [15],

$$\mu(\sigma) = \text{const} \prod_{\alpha\in\Sigma_P^+} \lambda(H_\alpha) \times \begin{cases} \tan(\pi\lambda(H_\beta)/2), & \text{if } \beta \text{ exists and } \operatorname{sgn}\sigma = 1, \\ \cot(\pi\lambda(H_\beta)/2), & \text{if } \beta \text{ exists and } \operatorname{sgn}\sigma = -1, \\ 1 & \text{otherwise,} \end{cases}$$

where σ and λ are connected as before. Thus,

$$\frac{\partial_P\mu(\sigma)}{\mu(\sigma)} = \sum_{\alpha\in\Sigma_P^+} \lambda_P(H_\alpha)\left(\frac{1}{\lambda(H_\alpha)} + \frac{\delta_{\alpha\beta}\pi\operatorname{sgn}\sigma}{\sin\pi\lambda(H_\beta)}\right).$$

Similarly to how we got (27), we see that

$$\Delta_\Sigma(a)(\Theta_\sigma(a) - \Theta_{w_0\sigma}(a)) = \sum_{w\in W(G,A)} \det_R(w)\varepsilon_M(w)a^{w\lambda},$$

$\det_R$ denoting the determinant in $\mathfrak{a}_R$. Multiplying the last two formulas and substituting $\det_R(w)w\alpha$ for α in the (now inner) sum over Σ_P^+, we finally get that $\tilde{\Omega}_P(a,\lambda,\operatorname{sgn}\sigma) = \tilde{\Omega}_{P,\Sigma}(a,\sigma)$ equals

$$\frac{1}{2}\sum_{w\in W(G,A)} \varepsilon_M(w)a^{w\lambda}\sum_{\alpha\in\Sigma_P^+}\lambda_P(H_\alpha)\left(\sum_{n=0}^{\infty}\frac{a^{n\alpha}}{n+w\lambda(H_\alpha)} + \frac{\delta_{\alpha\beta}\pi\operatorname{sgn}\sigma}{\sin\pi w\lambda(H_\beta)}\right) \tag{31}$$

for $a^{\lambda_P} < 1$ and $\lambda \in \Lambda_A$ with $\lambda(H_\alpha) \notin \mathbb{Z}$ for any $\alpha \in \Sigma_P^+$. Thus, we have obtained a function $\tilde{\Omega}_P(a, \lambda, \varepsilon)$ which is defined for $\varepsilon = \pm 1$ and λ not necessarily Σ-dominant and which satisfies

$$\tilde{\Omega}_P(a, w\lambda, \varepsilon) = \varepsilon_M(w)\tilde{\Omega}_P(a, \lambda, \varepsilon) \tag{32}$$

for $w \in W(G, A)$. If $\dim K\backslash G > 2$, then $\operatorname{sgn}\sigma = -\gamma^\lambda$, and we need only the simplified form $\tilde{\Omega}_P(a, \lambda) = \tilde{\Omega}_P(a, \lambda, -\gamma^\lambda)$.

Lemma 3 *The function $\tilde{\Omega}_P$ extends to a real-analytic function on $A'' \times \Lambda_A \setminus A_I \times \Lambda_{A_I}$.*

Proof for $\operatorname{rk} G = \operatorname{rk} K$. We rewrite the formula (28) for $\tilde{\Omega}_P(a, \lambda)$ as

$$\frac{1}{2} \sum_{w \in W(M,A)} \varepsilon_M(w) \sum_{\alpha \in \Sigma_P^+} \lambda_P(H_\alpha) \sum_{n=1}^{\infty} \left(\frac{a^{w\lambda - n\alpha}}{n - w\lambda(H_\alpha)} + \frac{a^{ww_0\lambda - n\alpha}}{n - ww_0\lambda(H_\alpha)} \right).$$

Singularities cannot occur in the terms belonging to the real root β, because $w\lambda(H_\beta) \in i\mathbb{R}$. Thus, we fix a complex root $\alpha \in \Sigma_P^+$ and write $\lambda = \lambda_I + \lambda_R$ in the obvious meaning. Then $w\lambda_I(H_\alpha) \in \mathbb{R}$ and $w\lambda_R(H_\alpha) \in i\mathbb{R}$. (It will be clear from the last assertion of Lemma 4 below that $w\lambda_I(H_\alpha) \in \frac{1}{2}\mathbb{Z}$.) Consequently, the n-th term in the above sum is smooth unless $w\lambda_I(H_\alpha) = n$, in which case it becomes

$$\frac{a^{w\lambda - n\alpha}}{-w\lambda_R(H_\alpha)} + \frac{a^{ww_0\lambda - n\alpha}}{-ww_0\lambda_R(H_\alpha)} = \frac{a_R^{-w\lambda_R} - a_R^{w\lambda_R}}{w\lambda_R(H_\alpha)} a_I^{w\lambda_I} a^{-n\alpha},$$

because $w_0 = s_\beta$ acts trivially on $\mathfrak{a}_I$. This is an analytic function on $\{a \in A : a^{\lambda_P} > 1\} \times i\mathfrak{a}_R^*$, and so is therefore $\tilde{\Omega}_P$. In connection with (20) we remarked that

$$\Delta_\Sigma(a)(\Theta_\sigma(a) - \Theta_{w_0\sigma}(a)) \frac{\partial_P \mu(\sigma)}{\mu(\sigma)}$$

is also analytic. This proves the lemma in the present case, since $A'' = A \setminus A_I = \{a \in A : a^{\lambda_P} \neq 1\}$. □

The proof for the case $\operatorname{rk} G \neq \operatorname{rk} K$ will be given in short. For this purpose, we have to look more closely at the structure of G. Given $\alpha \in \Sigma_P^+$, put $C(\alpha) = \{a \in A : a^\alpha = 1\}$. The complexified Lie algebra $\mathfrak{c}_\mathbb{C}(\alpha)$ of this group is the common kernel of α and $\bar{\alpha}$ in $\mathfrak{a}_\mathbb{C}$, and we have $C(\alpha) = C(\bar{\alpha}) \subset A_I$ with equality iff $\alpha = \beta$. Of course, $A'' = A \setminus \bigcup_{\alpha \in \Sigma_P^+} C(\alpha)$. Let $G(\alpha)$ denote the centralizer of $C(\alpha)$ in G. Then A is a Cartan subgroup of $G(\alpha)$, and if β exists, then $G(\beta)$ is contained in all the other $G(\alpha)$.

Lemma 4 *The group $G(\alpha)$ is connected and reductive with centre $C(\alpha)$. The derived subgroup $G(\alpha)^1$ of $G(\alpha)$ satisfies the same assumptions as G, its absolute rank is one if $\alpha = \beta$ and two otherwise. Moreover, $\gamma = \exp \pi i(H_\alpha - H_{\bar{\alpha}})$ for $\alpha \neq \beta$.*

Proof. Let $G_{\mathbb{C}}(\alpha)$ be the centralizer of $C(\alpha)$ in $G_{\mathbb{C}}$. Its Lie algebra is

$$\mathfrak{g}_{\mathbb{C}}(\alpha) = \mathfrak{a}_{\mathbb{C}} + \sum_{\delta\in\pm\Sigma_P(\alpha)} \mathfrak{g}_{\mathbb{C},\delta},$$

where $\Sigma_P(\alpha)$ is the intersection of Σ_P with the subspace spanned by α and $\bar{\alpha}$. Hence $\mathfrak{g}_{\mathbb{C}}(\alpha)$ is reductive, and its derived subalgebra has rank equal to $\dim(\mathbb{C}\alpha+\mathbb{C}\bar{\alpha})$. The algebraic group $G_{\mathbb{C}}(\alpha)$ is defined over $\mathbb{R}$ and may thus serve as a complexification of $G(\alpha)$. The real rank of $G(\alpha)$ is one, because $A \subset G(\alpha) \subset G$. The centre of $G(\alpha)$ is contained in $C(\alpha)$ as it must centralize $\mathfrak{g}_{\mathbb{C},\alpha}$, so it equals $C(\alpha)$.

We assert that $G(\alpha) = G(\alpha)^0 A$. (For the case $\alpha = \beta$, see [1], Lemma 1.3.) Thus, let $x \in G(a)$. Since any two noncompact Cartan subalgebras of $\mathfrak{g}(\alpha)$ are $G(\alpha)^0$-conjugate, we have $x = ym$, where $y \in G(\alpha)^0$ and $m \in G(\alpha)$ normalizes A. Moreover, there exists the analogue $w_0(\alpha)$ of w_0 in $W(G(\alpha)^0, A)$. Thus we may assume that m centralizes A_R, which means that $m \in M \cap G(\alpha) = M(\alpha)$. Observe that $\mathrm{Ad}_{\mathfrak{a}_{\mathbb{C}}}(m)$ fixes both $\mathfrak{a}_{R,\mathbb{C}}$ and $\mathfrak{c}_{\mathbb{C}}(\alpha)$, which span a hyperplane, so it is trivial or a simple reflection $s \in W(M,A)$. In the latter case, there exists an imaginary root $\delta \in \Sigma_P$ with $s_\delta = s$. Obviously, δ is a root of $M(\alpha)$, a group compact mod A, and s_δ has a representative in $M(\alpha)^0$. In each case we may thus assume that m centralizes $\mathfrak{a}_{\mathbb{C}}$, which proves our assertion. If G contains no simple factor isomorphic to $\mathrm{SL}(2,\mathbb{R})$, then A is connected, but in the remaining case we have $A = A^0 \cup \gamma A^0$, $\gamma \in G(\beta)^0$. Thus we see that $G(\alpha)$ is connected, and so is therefore $G(\alpha)^1$.

Now denote by L the kernel of the map $\exp : \mathfrak{a}_{\mathbb{C}} \to A_{\mathbb{C}}$ and by $L(\alpha)$ its intersection with the subspace spanned by H_α and $H_{\bar{\alpha}}$. Since $G_{\mathbb{C}}$ is simply connected, L is generated by the root system $R = \{2\pi i H_\delta : \delta \in \pm\Sigma_P\}$. In order to prove that $G_{\mathbb{C}}(\alpha)^1$ is simply connected, we have to show that $L(\alpha)$ is spanned by the root system $R(\alpha) = \{2\pi i H_\delta : \delta \in \pm\Sigma_P(\alpha)\}$. For this purpose, choose a real linear functional λ_0 on $\mathfrak{a}_I + i\mathfrak{a}_R$ such, that $\ker\lambda_0 \cap R = R(\alpha)$, and introduce a linear order on $\mathfrak{a}_I + i\mathfrak{a}_R$ which satisfies $\lambda_0(H) > 0 \Rightarrow H > 0$. Then the basis of R determined by this order must contain a basis of $R(\alpha)$, and our assertion follows.

Let us prove the equality $\gamma = \exp \pi i(H_\alpha - H_{\bar{\alpha}})$. If α and $\bar{\alpha}$ are not orthogonal, then, by classification, $G(\alpha)^1 \cong \mathrm{SU}(2,1)$, $\beta = \alpha + \bar{\alpha}$, and $H_\beta = H_\alpha + H_{\bar{\alpha}}$. On the other hand, if α and $\bar{\alpha}$ are orthogonal, then $G(\alpha)^1 \cong \mathrm{Sp}(1,1)$, $\beta = (\alpha+\bar{\alpha})/2$, and again $H_\beta = H_\alpha + H_{\bar{\alpha}}$. Thus, we have to prove that $\exp \pi i(H_\alpha + H_{\bar{\alpha}}) = \exp \pi i(H_\alpha - H_{\bar{\alpha}})$, which is equivalent to the obvious equality $\exp 2\pi i H_{\bar{\alpha}} = 1$. □

If the absolute rank of G is greater than one, we shall denote the contribution of the complex roots to Ω_P by Ω_P^0. Thus,

$$\Omega_P^0(a,\lambda) = \frac{1}{2} \sum_{w\in W(G,A)} \varepsilon_M(w) a^{w\lambda} \sum_{\alpha\in\Sigma_P^+\setminus\{\beta\}} \lambda_P(H_\alpha) \sum_{n=\{ {1 \atop 0}}^{\infty} \frac{a^{\mp n\alpha}}{n \mp w\lambda(H_\alpha)}. \quad (33)$$

In the rank one case, we simply put $\Omega_P^0 = \Omega_P$. It is clear that our whole discussion would remain true if we considered instead of G the almost direct product of G with some compact connected abelian group C. In particular, we may replace G, A, M and Σ by $G(\alpha) = G(\alpha)^1 C(\alpha)$, A, $M(\alpha) = M \cap G(\alpha)$ and $\Sigma(\alpha) = \Sigma \cap (\mathbb{R}\alpha + \mathbb{R}\bar{\alpha})$. We can then define functions $\Omega^0_{P(\alpha)}$ on $(A \backslash A_I) \times \hat{M}$. By classification, $\Sigma(\alpha) \neq \emptyset$ iff $G(\alpha)^1$ is isomorphic to $\mathrm{Sp}(1,1)$.

Lemma 5 *We have*

$$\Omega_P(a, \lambda, \varepsilon) = \sum_{\{\alpha, \bar{\alpha}\} \subset \Sigma_P^+} \sum_{w \in W(M(\alpha), A) \backslash W(M, A)} \varepsilon_M(w) \Omega^0_{P(\alpha)}(a, w\lambda, \varepsilon).$$

Proof. Lemma 1 applies to $G(\alpha)^1$. Note that the group $W(G(\alpha), A)$ consists of all $w \in W(G, A)$ which leave $C(\alpha)$ pointwise fixed. Thus the formula for ε_M applies to $G(\alpha)$, too, and we conclude that $\varepsilon_{M(\alpha)}(w) = \varepsilon_M(w)$ for $w \in W(G(\alpha), A)$. In particular, the analogue of (32) reads

$$\Omega^0_{P(\alpha)}(a, w\lambda, \varepsilon) = \varepsilon_M(w) \Omega^0_{P(\alpha)}(a, \lambda, \varepsilon)$$

for $w \in W(G(\alpha), A)$, which shows that the right-hand side of the asserted identity is well defined.

We now combine the terms for each $\{\alpha, \bar{\alpha}\}$ in Ω_P and split the sum over $W(G, A)$. The exterior sum runs over the set

$$\begin{aligned} W(G(\alpha), A) \backslash W(G, A) &= W(G(\alpha), A) \backslash W(G(\alpha), A) W(M, A) \\ &\cong W(G(\alpha), A) \cap W(M, A) \backslash W(M, A) \\ &= W(M(\alpha), A) \backslash W(M, A). \end{aligned}$$ □

Let us consider the series

$$b(s, z) = \sum_{n=1}^{\infty} \frac{z^n}{n + s},$$

which is absolutely convergent for s, $z \in \mathbb{C}$ with $s \notin \{-1, -2, \ldots\}$, $|z| < 1$, and conditionally convergent for $|z| = 1$, $z \neq 1$. This is a hypergeometric series, namely,

$$b(s, z) = \frac{z}{s+1} F(s+1, 1; s+2; z) = \frac{1}{s}(F(s, 1; s+1; z) - 1),$$

and has in similar form been considered in [12], section 4. We have

$$b(s, z) = z \int_0^1 t^s (1 - zt)^{-1} \, dt \tag{34}$$

for $\mathrm{Re}\, s > -1$, which provides the analytic continuation of $b(s, z)$ to all $z \notin [1, \infty)$ for the given values of s. (We might continue $b(s, z)$ across the slit $(1, \infty)$

by deforming the contour of integration as to evade the point z^{-1}. This would yield a multi-valued function with logarithmic branching point at $z = 1$.) After that we may analytically continue to all $s \notin \{-1, -2, \dots\}$ using the identity

$$b(s-1, z) = z\left(b(s, z) + s^{-1}\right). \tag{35}$$

From this and the definition of b it is clear that $\lim_{|s|\to\infty} b(s, z) = 0$. We shall say more about the asymptotic as $|\operatorname{Im} s| \to \infty$ in Lemma 7.

Let us denote the restriction of Ω_P^0 to the set $\{a \in A : a^{\pm\lambda_P} > 1\}$ by $\Omega_P^{\pm}$. The series occurring in $\Omega_P^{\pm}$ may be expressed as

$$\sum_{n=1}^{\infty} \frac{a^{-n\alpha}}{n - w\lambda(H_\alpha)} = b(-w\lambda(H_\alpha), a^{-\alpha}),$$
$$\sum_{n=0}^{\infty} \frac{a^{n\alpha}}{n + w\lambda(H_\alpha)} = a^{-\alpha} b(w\lambda(H_\alpha) - 1, a^{\alpha}).$$

If $a_I \in A_I'''$, these series are conditionally convergent. If now $\operatorname{rk} G = \operatorname{rk} K$, then $w_0\alpha = s_\beta\alpha = -\bar{\alpha}$, and a substitution yields

$$\Omega_P^+(a_I, \lambda) - \Omega_P^-(a_I, \lambda) = -\frac{1}{2} \sum_{w\in W(G,A)} \varepsilon_M(w) a_I^{w\lambda} \sum_{\alpha\in\Sigma_P^+\setminus\{\beta\}} \lambda_P(H_\alpha) \frac{1}{w\lambda(H_\alpha)}$$
$$= \frac{1}{2} \sum_{w\in W(M,A)} \varepsilon_M(w) a_I^{w\lambda} \sum_{\alpha\in\Sigma_P^+\setminus\{\beta\}} \lambda_P(H_\alpha) \left(\frac{1}{w\lambda(H_{\bar{\alpha}})} - \frac{1}{w\lambda(H_\alpha)}\right) = 0, \tag{36}$$

because $\bar{\alpha}$ runs through $\Sigma_P^+ \setminus \{\beta\}$ if α does.

Using the properties of $b(s, z)$, we may continue $\Omega_P^{\pm}$ as a real-analytic function across $A_I''' = A_I \setminus \bigcup_{\alpha\neq\beta} C(\alpha)$ to the set $\{a \in A : a^{\mp\alpha} \notin [1, \infty)\, \forall\alpha \in \Sigma_P^+ \setminus \{\beta\}\}$. The extended function will still satisfy the differential equation (22). As in [12], Lemma 22, one shows that

$$z^{-1} b(s-1, z) - b(-s, z^{-1}) = \int_0^\infty t^{s-1}(1 - zt)^{-1}\, dt$$
$$= (-z)^{-s} \int_{z\mathbf{R}_-} t^{s-1}(1+t)^{-1}\, dt = (-z)^{-s} B(s, 1-s) = \frac{\pi(-z)^{-s}}{\sin \pi s}$$

for $z \notin [0, \infty)$ and $0 < \operatorname{Re} s < 1$, where we use the branch of the complex power on $\mathbb{C} \setminus (-\infty, 0]$ satisfying $1^s = 1$. This equality extends meromorphically to all $s \notin \mathbb{Z}$ and allows us to deduce that

$$\Omega_P^+(a, \lambda) - \Omega_P^-(a, \lambda) = \frac{1}{2} \sum_{w\in W(G,A)} \varepsilon_M(w) a^{w\lambda}$$
$$\times \sum_{\alpha\in\Sigma_P^+\setminus\{\beta\}} \lambda_P(H_\alpha)\big(b(-w\lambda(H_\alpha), a^{-\alpha}) - a^{-\alpha} b(w\lambda(H_\alpha) - 1, a^{\alpha})\big)$$

$$= -\frac{1}{2} \sum_{w \in W(G,A)} \varepsilon_M(w) a^{w\lambda} \sum_{\alpha \in \Sigma_P^+ \setminus \{\beta\}} \lambda_P(H_\alpha) \frac{\pi(-a^{-\alpha})^{w\lambda(H_\alpha)}}{\sin \pi w\lambda(H_\alpha)}. \qquad (37)$$

Proof of Lemma 9 for $\operatorname{rk} G \neq \operatorname{rk} K$. In view of Lemma 5, it suffices to consider $G = \mathrm{SL}(2, \mathbb{C})$, in which case $\Sigma = \emptyset$, $\Sigma_P = \{\alpha, \bar\alpha\}$, and

$$\Omega_P^+(a,\lambda) = \frac{\lambda_P(H_\alpha)}{2}\big(a^\lambda b(-\lambda(H_\alpha), a^{-\alpha}) + a^\lambda b(-\lambda(H_{\bar\alpha}), a^{-\bar\alpha}) + a^{-\lambda} b(\lambda(H_\alpha), a^{-\alpha}) + a^{-\lambda} b(\lambda(H_{\bar\alpha}), a^{-\bar\alpha}\big).$$

The first and the last term have singularities for $\lambda_I(H_\alpha) = n \in \{1, 2, \ldots\}$. But $b(s,z) = \frac{z^n}{n+s} +$ an analytic function near $s = -n$, and so the singular parts of the aforementioned terms combine to the analytic function of $(a_R, \lambda_R) \neq (1,0)$

$$\frac{a^{-\lambda - n\bar\alpha} - a^{\lambda - n\alpha}}{\lambda_R(H_\alpha)} = \frac{a_R^{-\lambda_R} - a_R^{\lambda_R}}{\lambda_R(H_\alpha)} a_R^{-n\alpha}.$$

Here we have used that λ_I and $n\alpha$ have the same value n on $(H_\alpha - H_{\bar\alpha})/2$, thus coincide on $\mathfrak{a}_I$. A similar assertion is true for the middle terms, hence Ω_P^+ is analytic for $a^{-\alpha} \notin [1,\infty)$ and $(a_R, \lambda_R) \neq (1,0)$. In the same way we see that Ω_P^- is analytic for $a^\alpha \notin [1,\infty)$ and $(a_R, \lambda_R) \neq (1,0)$.

If $\gamma = \exp \pi i(H_\alpha - H_{\bar\alpha})$ (in fact, $\gamma = -\mathrm{Id} \in \mathrm{SL}(2,\mathbb{C})$), then $\gamma^\alpha = \gamma^{\bar\alpha} = 1$, $\gamma^\lambda = e^{\pi i \lambda(H_\alpha - H_{\bar\alpha})}$, thus $\sin \pi\lambda(H_\alpha) = \gamma^\lambda \sin \pi\lambda(H_{\bar\alpha})$, and equation (37) specializes to

$$\Omega_P^+(a,\lambda) - \Omega_P^-(a,\lambda) = \frac{\pi\lambda_P(H_\alpha)}{2\sin \pi\lambda(H_\alpha)}\big(f(a,\lambda) - f(a^{-1},\lambda)\big),$$

where

$$f(a,\lambda) = a^{-\lambda}(-a^\alpha)^{\lambda(H_\alpha)} - \gamma^\lambda a^\lambda (-a^{-\bar\alpha})^{\lambda(H_{\bar\alpha})}$$

for $a^\alpha \notin [0,\infty)$. We assert that $f(a,\lambda) \equiv 0$. If $s_\alpha \in W(\mathfrak{g}_\mathbb{C}, \mathfrak{a}_\mathbb{C})$ is the simple reflection in α, then $s_\alpha s_{\bar\alpha} = -\mathrm{Id}$, thus $s_{\bar\alpha}\lambda = -s_\alpha\lambda$ or, explicitly, $\lambda - \lambda(H_{\bar\alpha})\bar\alpha = -\lambda + \lambda(H_\alpha)\alpha$. Thus, $f(a,\lambda) = a_R^{s_\alpha\lambda} f(a_I, \lambda)$. The quotient of the two terms in $f(a_I,\lambda)$ equals

$$g(a_I,\lambda) = \gamma^\lambda a_I^{-2\lambda}(-a_I^\alpha)^{\lambda(H_\alpha - H_{\bar\alpha})}.$$

Observe that $\lambda(H_\alpha - H_{\bar\alpha})\alpha$ and 2λ have the same restriction to $\mathfrak{a}_I$, hence g is locally constant on $A_I \setminus C(\alpha)$. But

$$(-a_I^\alpha)^{\lambda(H_\alpha - H_{\bar\alpha})} \to e^{\pm\pi i(H_\alpha - H_{\bar\alpha})} = \gamma^\lambda$$

for $a_I \to 1$, hence $g(a_I,\lambda) \to 1$, $f(a,\lambda) \equiv 0$ and $\Omega_P^+ = \Omega_P^-$. □

Finally, we suppose that $\operatorname{rk} G = \operatorname{rk} K$ and study the behaviour of $\tilde\Omega_P$ near the singular set A_I. In analogy to (23), we put

$$\tilde\Psi_{P,\Sigma}(a,\sigma) = \tilde\Omega_{P,\Sigma}(a,\sigma) + \frac{1}{2}\lambda_P(H_\beta)\log|a^{\beta/2} - a^{-\beta/2}| \cdot \Delta_\Sigma(a)(\Theta_\sigma(a) + \Theta_{w_0\sigma}(a)).$$

If $\sigma \in \hat{M}$ has Σ-highest weight $\lambda - \rho_\Sigma$, we put $\tilde{\Psi}_P(a, \lambda, \operatorname{sgn}\sigma) = \tilde{\Psi}_{P,\Sigma}(a, \sigma)$, which extends to an analytic function on $A'' \times \Lambda_A \setminus A_I \times \Lambda_{A_I}$ satisfying (32).

Lemma 6 *If $G \cong \mathrm{SL}(2, \mathbb{R})$, then*

$$\lim_{a \to 1} \tilde{\Psi}_P(a, \lambda, \varepsilon) = \lambda_P(H_\beta)\left(\psi(1) - \frac{\psi(1 + \lambda(H_\beta)) + \psi(1 - \lambda(H_\beta))}{2}\right),$$

$$\partial_\beta^+ \tilde{\Psi}_P(1, \lambda, \varepsilon) - \partial_\beta^- \tilde{\Psi}_P(1, \lambda, \varepsilon) = \pi \lambda_P(H_\beta)\lambda(H_\beta)\frac{\cos \pi\lambda(H_\beta) - \varepsilon}{\sin \pi\lambda(H_\beta)},$$

where $\lambda \in i\mathfrak{a}^*$, $\varepsilon = \pm 1$, and ψ denotes the logarithmic derivative of the gamma function.

Proof. Let us fix λ_P with $\lambda_P(H_\beta) = 1$ and use the variables $t \in \mathbb{R}$, $s \in i\mathbb{R}$ defined by $a^\beta = e^{2t}$, $\lambda(H_\beta) = s$. Then $a^\lambda = e^{st}$,

$$\tilde{\Psi}_P(t, s, \varepsilon) = \tfrac{1}{2}\left(e^{st} b(-s, e^{-2t}) + e^{-st} b(s, e^{-2t}) + (e^{st} + e^{-st})\log(e^t - e^{-t})\right)$$

for $t > 0$ and

$$\tilde{\Psi}_P(t, s, \varepsilon) - \tilde{\Psi}_P(-t, s, \varepsilon) = \frac{1}{2}(e^{st} - e^{-st})\left(\frac{1}{s} + \frac{\pi\varepsilon}{\sin \pi s}\right).$$

Substituting e^{2t-u} for the variable of integration in (34), we get, for $\operatorname{Re} s > -1$ and $t > 0$,

$$\begin{aligned} b(s, e^{-2t}) + \log(e^t - e^{-t}) &= b(s, e^{-2t}) - b(0, e^{-2t}) + t \\ &= \int_{2t}^{\infty} \frac{e^{2st - (s+1)u} - e^{-u}}{1 - e^{-u}}\, du + t \\ &\to \psi(1) - \psi(s + 1) \end{aligned}$$

as $t \to 0+$. Since, for $s \neq 0$, $\tilde{\Psi}_P(t, s, \varepsilon) - \tilde{\Psi}_P(-t, s, \varepsilon) \to 0$ as $t \to 0$, we have proved the first assertion.

In view of the equality

$$\frac{d}{dt}\left(e^{-2st} b(s, e^{-2t})\right) = -2\frac{e^{-2(s+1)t}}{1 - e^{-2t}}$$

we have, for $t > 0$,

$$\frac{d}{dt}\left(e^{-st}(b(s, e^{-2t}) - b(0, e^{-2t}))\right) = se^{-st}\left(b(s, e^{-2t}) + b(0, e^{-2t})\right).$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\tilde{\Psi}_P(t, s, \varepsilon) &= \frac{s}{2}\left(e^{-st} b(s, e^{-2t}) - e^{st} b(-s, e^{-2t})\right) \\ &\qquad -s \sinh st \log(e^t - e^{-t}) + \cosh st \\ &\to \frac{s}{2}\left(\psi(1 - s) - \psi(1 + s)\right) + 1 = \frac{\pi s}{2}\cot \pi s + \frac{1}{2} \end{aligned}$$

as $t \to 0+$. If $t < 0$, then

$$\frac{d}{dt}\tilde{\Psi}_P(t,s,\varepsilon) = \frac{d}{dt}\tilde{\Psi}_P(-t,s,\varepsilon) + \frac{s}{2}(e^{st}+e^{-st})\left(\frac{1}{s}+\frac{\pi\varepsilon}{\sin\pi s}\right)$$
$$\to -\frac{\pi s}{2}\cot\pi s + \frac{1}{2} + \frac{\pi s\varepsilon}{\sin\pi s}$$

as $t \to 0-$. Altogether,

$$\lim_{t\to 0+}\frac{d}{dt}\tilde{\Psi}_P(t,s,\varepsilon) - \lim_{t\to 0-}\frac{d}{dt}\tilde{\Psi}_P(t,s,\varepsilon) = \pi s\frac{\cos\pi s - \varepsilon}{\sin\pi s}.$$

□

In the general case, $G(\beta)^1 \cong \mathrm{SL}(2,\mathbb{R})$. Put $P(\beta)^1 = P \cap G(\beta)^1$. From the definition (28) and (31) of $\tilde{\Omega}_P$ we see that

$$\tilde{\Omega}_{P,\Sigma}(a,\sigma_\lambda) = \Omega^0_{P,\Sigma}(a,\sigma_\lambda) + \Delta_\Sigma(a_I)\Theta_\sigma(a_I)\tilde{\Omega}_{P(\beta)^1}(a_R,\lambda,\operatorname{sgn}\sigma)$$

for $a \in A''$, $\sigma \in \hat{M}_I$, $\lambda \in i\mathfrak{a}_R^*$. Thus,

$$\tilde{\Psi}_{P,\Sigma}(a,\sigma_\lambda) = \Omega^0_{P,\Sigma}(a,\sigma_\lambda) + \Delta_\Sigma(a_I)\Theta_\sigma(a_I)\tilde{\Psi}_{P(\beta)^1}(a_R,\lambda,\operatorname{sgn}\sigma),$$

and Lemma 6 together with (36) implies

$$\lim_{a_R\to 1}(\tilde{\Psi}_{P,\Sigma}(a_Ia_R,\sigma_\lambda) - \tilde{\Psi}_{P,\Sigma}(a_Ia_R^{-1},\sigma_\lambda)) = 0, \tag{38}$$

$$\partial^+_\beta\tilde{\Psi}_{P,\Sigma}(a_I,\sigma_\lambda) - \partial^-_\beta\tilde{\Psi}_{P,\Sigma}(a_I,\sigma_\lambda) = \partial^+_\beta\Omega^0_{P,\Sigma}(a_I,\sigma_\lambda) - \partial^-_\beta\Omega^0_{P,\Sigma}(a_I,\sigma_\lambda)$$
$$+ \pi\lambda_P(H_\beta)\Delta_\Sigma(a_I)\Theta_\sigma(a_I)\lambda(H_\beta)\frac{\cos\pi\lambda(H_\beta) - \operatorname{sgn}\sigma}{\sin\pi\lambda(H_\beta)} \tag{39}$$

in the space of w_0-invariant distributions on $\hat{M}$ which are tempered on each connected component, provided $a_I \in A_I'''$.

4 Completion of the proof of Theorem 1

We are now in a position to determine the function $\Omega_{P,\Sigma}$ appearing in Proposition 2. In order to prove Theorem 1, we have to show that this function coincides with $\tilde{\Omega}_{P,\Sigma}$, which was defined by equations (28) and (31).

Thus, let

$$\Upsilon_\Sigma = \Omega_{P,\Sigma} - \tilde{\Omega}_{P,\Sigma},$$

a smooth function on $(A \setminus A_I) \times \hat{M}$ by Lemma 3. This notation is justified as $\Omega_{P,\Sigma} - \Omega_{\bar{P},\Sigma} = \tilde{\Omega}_{P,\Sigma} - \tilde{\Omega}_{\bar{P},\Sigma}$ by (20) and (30). Let us again suppose $\dim K\backslash G > 2$ and write $\Upsilon_\Sigma(a,\sigma) = \Upsilon(a,\lambda)$, where $\lambda - \rho_\Sigma$ is the Σ-highest weight of σ. Since $\Upsilon_{w\Sigma} = \varepsilon_M(w)\Upsilon_\Sigma$, we see by varying Σ that $\Upsilon(a,\lambda)$ is defined for all $\lambda \in \Lambda_A$ and satisfies $\Upsilon(a,w\lambda) = \varepsilon_M(w)\Upsilon(a,\lambda)$ for $w \in W(G,A)$. By Proposition 2,

$\Omega_{P,\Sigma}(a,\sigma)$ is of polynomial growth in σ, and $\Omega_{P,\Sigma}(a,\sigma) \to 0$ as $a^{\lambda_P} \to +\infty$, uniformly on compacta in (a_I, σ). The same may be checked immediately for $\tilde{\Omega}_{P,\Sigma}$, hence it is true for Υ, too. From (21) and (29) we get

$$\Upsilon(wa, \lambda) = \varepsilon_M(w)\Upsilon(a, \lambda) \tag{40}$$

for $w \in W(G, A)$. Therefore, $\Upsilon(a, \lambda) \to 0$ for $a^{-\lambda_P} \to +\infty$ as well. The differential equation (22), which is also satisfied by $\Omega_{P,\Sigma}$ by construction, implies that Υ satisfies the homogeneous differential equation

$$D\Upsilon(a, \lambda) = \langle\lambda, \lambda\rangle\Upsilon(a, \lambda) \tag{41}$$

for $a \in A \setminus A_I$. If we were able to show that Υ extends to a solution on all of A, it would be easy to deduce that $\Upsilon \equiv 0$.

However, it is difficult to see this directly. If, e. g., $\operatorname{rk} G = \operatorname{rk} K$, then (24) and (38) imply that $\Upsilon(a_I a_R, \lambda)$ extends to a continuous function of a_R with values in the space of skew $W(G, A)$-invariant distributions on Λ_A which are tempered on each connected component, while (26) and (39) imply that

$$\begin{aligned}\partial_\beta^+ \Upsilon(a_I, \lambda) - \partial_\beta^- \Upsilon(a_I, \lambda) = 4\pi \sum_{\substack{w \in W(G,T,A_I)\backslash W(G,T) \\ w \neq 1}} \det(w)\partial_{w\beta^\vee}\Phi(wa_I, \lambda) \\ -(\partial_\beta^+ \Omega_P^0(a_I, \lambda) - \partial_\beta^- \Omega_P^0(a_I, \lambda)),\end{aligned} \tag{42}$$

provided $a_I \in A_I'''$. Here, $\Phi_\Sigma(t, \sigma) = \Phi(t, \lambda)$ in the obvious sense. At the moment we only know that the limit on the left-hand side exists in the aforementioned space of distributions on Λ_A. Note that the contribution from the trivial coset $W(G, T, A_I)$ has canceled that from the real root β. But it would be a tedious work to deduce from the explicit formulas that the whole expression vanishes.

Instead we only observe some properties of these functions. Taking some $X = \theta(X_\beta - X_{-\beta})$, we have $i\theta\lambda(H_\beta) = -\lambda^\vee(X)$, which allows us to rewrite $\Phi(t, \lambda)$. As $t_\theta^{\beta^\vee} = e^{-2i\theta}$, the function $\Phi(t, \lambda)$ is defined for $t^{\beta^\vee} \neq 1$, and

$$\Phi(t \exp X, \lambda) = \sum_{w \in \widetilde{W}(G,T,A_I)} c_w(t, \lambda) e^{w\lambda^\vee(X)}$$

for all X in a neighbourhood of zero in $\mathfrak{t}$ depending on t. Here the $c_w(t, \lambda)$ are analytic functions in $\lambda \neq 0$, and $\widetilde{W}(G, T, A_I) = W(G, T, A_I) \times \{1, s_{\beta^\vee}\}$. Choosing now $t \in T$ such that $t^\delta \neq 1$ for all $\delta \in \Sigma^\vee \setminus \{\beta^\vee\}$, we get

$$4\pi \sum_{\substack{w \in W(G,T,A_I)\backslash W(G,T) \\ w \neq 1}} \det(w)\Phi(w(t \exp X), \lambda) = \sum_{\substack{w \in \widetilde{W}(G,T) \\ wA_I \neq A_I}} c_w(t, \lambda) e^{w\lambda^\vee(X)}$$

for X in a neighbourhood of zero in $\mathfrak{t}$, where $\widetilde{W}(G, T) = W(G, T) \cdot \{1, s_{\beta^\vee}\}$. As we have excluded the coset $W(G, T, A_I)$, $t^{\beta^\vee} = 1$ is allowed, so we can apply

∂_{β^y} at some point $a_I \exp H \in A_I$, where $a_I^\delta \neq 1$ for all $\delta \in \Sigma^y \setminus \{\beta^y\}$ and $H \in \mathfrak{a}_I$ is small.

Similarly, if we choose $a \in A$ with $a^\alpha \notin (0, \infty)$ for all $\alpha \in \Sigma_P^+ \setminus \{\beta\}$, we can write (37) in the form

$$\Omega_P^+(a \exp H) - \Omega_P^-(a \exp H) = \sum_{w \in W(G,A)} \sum_{\alpha \in \Sigma_P^+ \setminus \{\beta\}} d_{w,\alpha}(a, \lambda) e^{s_\alpha w\lambda(H)}$$

for all H in a neighbourhood of zero in $\mathfrak{a}$. Inserting these expressions in (42), we can choose $a_I \in A_I$ such that

$$\begin{aligned} &\partial_\beta^+ \Upsilon(a_I \exp H, \lambda) - \partial_\beta^- \Upsilon(a_I \exp H, \lambda) \\ &= \sum_{\substack{w \in \widetilde{W}(G,T) \\ wA_I \neq A_I}} \tilde{c}_w(a_I, \lambda) e^{w\lambda^y(H)} - \sum_{w \in W(G,A)} \sum_{\alpha \in \Sigma_P^+ \setminus \{\beta\}} \tilde{d}_{w,\alpha}(a_I, \lambda) e^{s_\alpha w\lambda(H)} \quad (43) \end{aligned}$$

for all H in a neighbourhood U_{a_I} of zero in $\mathfrak{a}_I$.

Let us return to the general case (without the equirank assumption) and expand $\Upsilon(a, \lambda)$ into a Fourier series with respect to a_I. Each term in the Fourier series has the form $g(a_R) a_I^\mu$ with $g \in C^\infty(A_R \setminus \{1\})$, $\mu \in \Lambda = \Lambda_{A_I}$, it must satisfy (41) and $g(a_R) \to 0$ as $a_R^{\pm\lambda_P} \to \infty$. If $\langle \mu, \mu \rangle \neq \langle \lambda, \lambda \rangle$, then $g(a_R)$ is a linear combination of the functions a_R^ν, where $\nu \in \mathfrak{a}_{R,\mathbf{C}}^*$ is any of the two solutions of $\langle \mu, \mu \rangle + \langle \nu, \nu \rangle = \langle \lambda, \lambda \rangle$. The condition on the limit forces $\nu \in \mathfrak{a}_R^*$ and leaves only one solution on each component of $A_R \setminus \{1\}$. Thus, if $\langle \lambda, \lambda \rangle \neq \langle \mu, \mu \rangle$ for all $\mu \in \Lambda$, then

$$\Upsilon(a, \lambda) = \sum_{\mu \in \Lambda(\lambda)} c_\mu^\pm(\lambda) a^{\mu \pm \nu_{\mu,\lambda}}$$

for $a^{\pm\lambda_P} > 1$, where $\Lambda(\lambda) = \{\mu \in \Lambda \,:\, \langle \mu, \mu \rangle < \langle \lambda, \lambda \rangle\}$, a finite set, and $\nu_{\mu,\lambda} \in \mathfrak{a}_R^*$ is the solution of $\langle \mu, \mu \rangle + \langle \nu, \nu \rangle = \langle \lambda, \lambda \rangle$ negative on C_P.

We know that Υ extends to a smooth function on A'' with values in the space of distributions on Λ_A. If $\operatorname{rk} G \neq \operatorname{rk} K$, then A_I'' is open and dense in A_I, and $\Upsilon(a_I a_R, \lambda)$ is smooth in a_R iff $c_\mu^\pm(\lambda) = 0$ for all $\lambda \in \Lambda(\mu)$. Thus, $\Upsilon(a, \lambda) = 0$ in this case.

If $\operatorname{rk} G = \operatorname{rk} K$, then (40) implies $c_\mu^+ = c_\mu^-$, and

$$\partial_\beta^+ \Upsilon(a_I, \lambda) - \partial_\beta^- \Upsilon(a_I, \lambda) = 2 \sum_{\mu \in \Lambda(\lambda)} c_\mu^+(\lambda) \nu_{\mu,\lambda}(H_\beta) a_I^\mu. \quad (44)$$

We have seen in (43) that this expression, evaluated at $a_I \exp H$ for some $a_I \in A_I$ and all $H \in U_{a_I}$, equals a linear combination of characters $e^{w\lambda(H)}$ of $\mathfrak{a}_I$, where w runs through a subset of $W(\mathfrak{g}_\mathbf{C}, \mathfrak{a}_\mathbf{C})$ with $w\mathfrak{a}_I \neq \mathfrak{a}_I$, i. e., $w\mathfrak{a}_R \neq \mathfrak{a}_R$. Let $B \subset \Lambda_A = \Lambda \times i\mathfrak{a}_R^*$ be the set of all λ for which $\langle \lambda, \lambda \rangle \neq \langle \mu, \mu \rangle$ for all $\mu \in \Lambda$, and $w\lambda \notin \Lambda \times \mathfrak{a}_{R,\mathbf{C}}^*$ for all $w \in W(\mathfrak{g}_\mathbf{C}, \mathfrak{a}_\mathbf{C})$ with $w\mathfrak{a}_R \neq \mathfrak{a}_R$. This is an open dense subset of Λ_A. For each $\lambda \in B$, the characters $e^{w\lambda(H)}$ of $\mathfrak{a}_I$ occurring in

(43) are different from those occurring in (44). Since $U_{\mathfrak{a}_I}$ generates the group $\mathfrak{a}_I$ and the characters of $\mathfrak{a}_I$ are linearly independent, their restrictions to $U_{\mathfrak{a}_I}$ are linearly independent, too. Thus, $c_w^+(\lambda) = 0$ for all $\lambda \in B$, $\mu \in \Lambda(\lambda)$. This means $\Upsilon(a,\lambda) = 0$ for $\lambda \in B$, hence for all λ by continuity.

5 Connection with orbital integrals

In this section we are going to express the invariant distributions I_P in terms of the orbital integrals I_G. In view of the identity (6), this amounts to calculating the inverse Fourier transform on A of the distribution Ω_P.

So far we have parametrized representations $\sigma \in \hat{M}$ by their Σ-highest weight $\lambda - \rho_\Sigma \in \Lambda_A$. This is not enough in the case $\dim K\backslash G = 2$, where A is not connected. As we shall now have to take this case into account, we shall sometimes replace λ by the character $\chi \in \hat{A}$ defined by $\chi(a) = \sigma(a)$ if $\dim K\backslash G = 2$, and $\chi(a) = a^\lambda$ otherwise. Thus we write, for $a \in A'$, $\chi \in \hat{A}$ and $\varepsilon = \pm 1$,

$$\Omega_P(a,\chi,\varepsilon) = \frac{1}{2} \sum_{w \in W(G,A)} \varepsilon_M(w) w\chi(a) \sum_{\alpha \in \Sigma_P^+} \phi_\alpha(a, w\chi, \varepsilon), \tag{45}$$

where $\phi_\alpha(a,\chi,\varepsilon)$ equals

$$\phi_\alpha(a,\lambda,\varepsilon) = \begin{cases} \lambda_P(H_\alpha) b(-\lambda(H_\alpha), a^{-\alpha}) & \text{if } a^{\lambda_P} > 1, \\ \lambda_P(H_\alpha)\left(b(\lambda(H_\alpha), a^\alpha) + \dfrac{1}{\lambda(H_\alpha)} + \dfrac{\delta_{\alpha\beta}\pi\varepsilon}{\sin \pi\lambda(H_\beta)}\right) & \text{if } a^{\lambda_P} < 1. \end{cases}$$

provided $\lambda \in \Lambda_A$ is the differential of χ. Recall that if χ is the Σ-parameter of $\sigma \in \hat{M}$ in the aforementioned sense, then

$$\Omega_{P,\Sigma}(a,\sigma) = \Omega_P(a,\chi,\operatorname{sgn}\sigma) = \begin{cases} \Omega_P(a,\chi,\chi(\gamma)) & \text{if } \dim K\backslash G = 2, \\ \Omega_P(a,\chi,-\chi(\gamma)) & \text{otherwise.} \end{cases}$$

For certain reasons, we introduce a notation for the inverse Fourier transform of Ω_P^0 (see equation (33)) rather than Ω_P, i. e., we put formally

$$K_P^\pm(a,a') = \frac{1}{2\pi i} \int_{\hat{A}} \Omega_P^0(a,\bar{\chi},\pm\chi(\gamma))\chi(a')\,d\chi, \tag{46}$$

where the measure on $\widehat{A^0} \cong \Lambda_A$ has been fixed earlier and extends in the obvious way in the case $\dim K\backslash G = 2$, in which $A = A^0 \times \{1,\gamma\}$. This measure is inversely proportional to λ_P, hence K_P is independent of any normalization of measures. We shall reduce the calculation to the case $\operatorname{rk} G \le 2$, where the distribution $K_P^\pm$ will turn out to be a function. First of all, we have to calculate the inverse Fourier transform of $b(s,z)$.

Lemma 7 *Let $m \in \mathbb{Z}$, $\nu \in \mathbb{R}$, $z \in \mathbb{C}$, $|z| < 1$.*

(i) $b(\frac{m+i\nu}{2}, z) = \frac{2z}{(1-z)i\nu} + O(|\nu|^{-2})$ *as* $\nu \to \infty$.

(ii) If $m \geq -1$, *then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} b\Big(\frac{m+i\nu}{2}, z\Big)e^{i\nu t}d\nu = \begin{cases} \dfrac{2ze^{-mt}}{e^{2t}-z} & \text{if } t > 0, \\ 0 & \text{if } t < 0 \end{cases}$$

as an improper integral.

(iii) If $m \leq -1$, *then the Cauchy principal value (at* $\nu = 0$*) of the same integral equals*

$$\begin{Bmatrix} \dfrac{e^{2t}+z}{e^{2t}-z} z^{-m/2} & \text{for even } m \\ \dfrac{2e^t}{e^{2t}-z} z^{-(m-1)/2} & \text{for odd } m \end{Bmatrix} - \begin{cases} 0 & \text{if } t > 0, \\ \dfrac{2ze^{-mt}}{e^{2t}-z} & \text{if } t < 0. \end{cases}$$

(iv)
$$\frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{(m+i\nu)t}}{m+i\nu}\,d\nu = \operatorname{sgn} m + \operatorname{sgn} t,$$

where the principal value is necessary for $m = 0$ *only.*

(v) If $\varepsilon = \pm 1$, $t \neq 0$, *then*

$$\frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\left(\frac{1}{i\nu} + \frac{\pi\varepsilon}{\sin \pi i\nu}\right)e^{i\nu t}d\nu = \varepsilon\frac{e^{\varepsilon|t|}-1}{e^t - e^{-t}}.$$

In (ii)-(v), the principal value of the integral for $t = 0$ *is half-way between the one-sided limits* $t \to 0\pm$.

Note that the right-hand side of (iii) is regular at $e^{2t} = z$.

Proof. Substituting e^{-2t} for the variable of integration in (34), we get

$$b\Big(\frac{m+i\nu}{2}, z\Big) = \int_0^{\infty} \frac{2ze^{-mt}}{e^{2t}-z}e^{-i\nu t}dt$$

for $|z| < 1$, $m \geq -1$. This is the Fourier transform of the function $f(t)$ given by the right-hand side of (ii). Let $g(t) = f(t) - \frac{2z}{1-z}e^{-t}$ for $t > 0$ and $g(t) = 0$ for $t \leq 0$. As the distributional derivative of g is in $L^1(\mathbb{R})$ and has bounded total variation, its Fourier transform is $O(|\nu|^{-1})$ as $\nu \to \infty$ (see [7], §3). Thus, the Fourier transform of g itself is $O(|\nu|^{-2})$ as $\nu \to \infty$. This proves (i) for $m \geq -1$, and the general case follows from (35) in view of

$$z\left(\frac{2z}{(1-z)i\nu} + \frac{2}{m+i\nu}\right) = \frac{2z}{(1-z)i\nu}\left(1 + \frac{m(z-1)}{m+i\nu}\right).$$

Assertion (ii) follows from the inversion formula for the Fourier transform, the convergence of the integral is clear from (i).

Using residue calculus and the formula

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\nu t}{\nu}\,d\nu=\operatorname{sgn}t$$

(improper integral), one easily proves (iv). If we denote, for any $m\in\mathbb{Z}$, by $h_m(t)$ the left-hand side of (ii) taken as principal value, then (iv) and equation (35) imply

$$h_{m-2}(t)=z(h_m(t)+(\operatorname{sgn}m+\operatorname{sgn}t)e^{-mt}).$$

This allows to prove (iii) by induction. To prove (v), observe that, for $m\in\{0,1\}$ and $\operatorname{Re}s>0$,

$$\begin{aligned}\frac{1}{s}+\frac{\pi(-1)^m}{\sin\pi s}&=\frac{(-1)^m}{2}\left(\psi\Big(\frac{1+s}{2}\Big)-\psi\Big(\frac{1-s}{2}\Big)-\psi\Big(m+\frac{s}{2}\Big)+\psi\Big(m-\frac{s}{2}\Big)\right)\\&=(-1)^m\int_0^{\infty}\frac{e^{2mt}-e^{-t}}{1-e^{-2t}}(e^{st}-e^{-st})\,dt\\&=(-1)^m\int_{-\infty}^{\infty}\frac{e^{(1-2m)t}-1}{e^t-e^{-t}}e^{-st}dt,\end{aligned}$$

whereby the left-hand side (with $s=i\nu$) can be considered as distributional Fourier transform. □

Lemma 8 *Let $t,\ \theta\in\mathbb{R},\ t\neq 0,\ z\in\mathbb{C}$. If $|z|<1$, then*

$$\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty}b\Big(\frac{m+i\nu}{2},z\Big)e^{im\theta+i\nu t}d\nu=\frac{z}{e^{2i\theta}-z}\cdot\frac{e^{t-i\theta}+1}{e^{t-i\theta}-1}.$$

If $|z|>1$, then the value of

$$\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\left(b\Big(-\frac{m+i\nu}{2},z^{-1}\Big)-\frac{2}{m+i\nu}\right)e^{im\theta+i\nu t}d\nu$$

is given by the same formula. The integrals are conditionally convergent at infinity, the principal value is taken at $\nu=0$ (if necessary).

Proof. For the first assertion, one simply has to apply the summation formula for geometric progressions to formulae (ii) and (iii) of Lemma 7, which yields

$$\frac{z}{e^{2t}-z}\left(\frac{2}{1-e^{-t+i\theta}}+\frac{2e^{t+i\theta}+e^{2t}+z}{e^{2i\theta}-z}\right).$$

This can be simplified to the expression stated in the Lemma. In the same way one obtains

$$\sum_{m=-\infty}^{\infty} \frac{1}{\pi} \,\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{e^{im\theta+i\nu t}}{m+i\nu}\,d\nu = \frac{e^{t-i\theta}+1}{e^{t-i\theta}-1}$$

from (iv), which allows to deduce the second assertion from the first one. □

We shall now calculate $K_P^{\pm}$, as defined by (46), for groups G with $\mathrm{rk}\, G \leq 2$. By classification, G is then isomorphic to one of the groups $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{SU}(2,1)$ or $\mathrm{Sp}(1,1)$. Here, we realize $\mathrm{U}(2,1)$ as the automorphism group of the hermitian form $z_1\bar{z}_3 + z_2\bar{z}_2 + z_3\bar{z}_1$, and $\mathrm{Sp}(1,1)$ as the automorphism group of the quaternionic hermitian form $q_1\bar{q}_2 + q_2\bar{q}_1$. This gives us the possibility, in each case, to let A be the subgroup of diagonal matrices and P the subgroup of upper triangular matrices in G. In order to make use of the preceding Lemmas, we resort to the following explicit parametrizations:

If $G = \mathrm{SL}(2,\mathbb{R})$, let $a_t = \mathrm{diag}(e^t, e^{-t})$,

if $G = \mathrm{SL}(2,\mathbb{C})$, let $a_z = \mathrm{diag}(e^z, e^{-z})$,

if $G = \mathrm{SU}(2,1)$, let $a_z = \mathrm{diag}(e^z, e^{\bar{z}-z}, e^{-\bar{z}})$,

if $G = \mathrm{Sp}(1,1)$, let $a_z = \mathrm{diag}(e^z, e^{-\bar{z}})$,

where $t \in \mathbb{R}$, $z \in \mathbb{C}$.

Proposition 3 *Let t, $t' \in \mathbb{R}$, z, $z' \in \mathbb{C}$, such that $t \neq 0$, $\mathrm{Re}\, z \neq 0$, $t \neq \pm t'$, $\mathrm{Re}\, z \neq \pm \mathrm{Re}\, z'$.*
If $G = \mathrm{SL}(2,\mathbb{R})$, then

$$K_P^{\pm}(a_t, a_{t'}) = K_P^{\pm}(\gamma a_t, \gamma a_{t'}) = \begin{cases} \dfrac{1}{e^{|t|+|t'|}-1} & \text{if } |t'| > |t|, \\ 0 & \text{if } |t'| < |t|, t > 0, \\ \mp 1 & \text{if } |t'| < |t|, t < 0, \end{cases}$$

$$K_P^{\pm}(a_t, \gamma a_{t'}) = K_P^{\pm}(\gamma a_t, a_{t'}) = \begin{cases} 0 & \text{if } t > 0, \\ \pm\dfrac{1}{2}\left(\tanh\dfrac{t+t'}{2} + \tanh\dfrac{t-t'}{2}\right) & \text{if } t < 0. \end{cases}$$

If $G = \mathrm{SL}(2,\mathbb{C})$, then

$$K_P(a_z, a_{z'}) = -\,\mathrm{Re}\,\frac{1}{\left(e^{z-z'}-1\right)\left(e^{\bar{z}+\bar{z}'}-1\right)}.$$

If $G = \mathrm{SU}(2,1)$, then

$$K_P(a_z, a_{z'}) = -\frac{1}{2}\,\mathrm{Re}\left(\frac{1}{e^{\bar{z}-z'+\bar{z}'}-1}\left(\frac{e^z+e^{z'}}{e^z-e^{z'}} + \frac{e^z+e^{-\bar{z}'}}{e^z-e^{-\bar{z}'}}\right)\right).$$

If $G = \mathrm{Sp}(1,1)$,

$$K_P(a_z, a_{z'}) = -\frac{1}{2}\,\mathrm{Re}\left(\frac{1}{e^{z+\bar{z}-z'+\overline{z'}}-1}\left(\frac{e^z+e^{z'}}{e^z-e^{z'}}+\frac{e^z+e^{-\overline{z'}}}{e^z-e^{-\overline{z'}}}\right)\right)$$
$$+\frac{1}{2}\,\mathrm{Re}\left(\frac{1}{e^{z+\bar{z}+z'-\overline{z'}}-1}\left(\frac{e^z+e^{\overline{z'}}}{e^z-e^{\overline{z'}}}+\frac{e^z+e^{-z'}}{e^z-e^{-z'}}\right)\right).$$

The integral in (46) is conditionally convergent at infinity on each connected component of $\hat{A}$.

Obviously, the distribution (depending on a) given by the integrable function $K_P^{\pm}(a, a')$ is then the distributional inverse Fourier transform on A of $\Omega_P^0(a, \chi, \pm\chi(\gamma))$.

Proof. Let us insert (45) into (46). We shall see in short that one can integrate, in the sense of Cauchy, the individual terms, i. e.,

$$k_\alpha^{\pm}(a, a') = \frac{1}{2\pi i}\,\mathrm{p.v.}\int_{\hat{A}} \phi_\alpha(a, \bar{\chi}, \pm\chi(\gamma))\overline{\chi(a)}\chi(a')\,d\chi.$$

Since $\overline{\lambda(H_\alpha)} = \bar{\lambda}(H_{\bar{\alpha}}) = -\lambda(H_{\bar{\alpha}})$ and $\overline{a^\alpha} = a^{\bar{\alpha}}$ for $\lambda \in \Lambda_A$, $\alpha \in \Sigma_P^+ \setminus \{\beta\}$ and $a \in A$, we have $\overline{\phi_\alpha(a, \chi)} = \phi_{\bar{\alpha}}(a, \bar{\chi})$, hence $\overline{k_\alpha(a, a')} = k_{\bar{\alpha}}(a, a')$. We may thus write

$$K_P^{\pm}(a, a') = \begin{cases} \frac{1}{2}(k_\beta^{\pm}(a, a') + k_\beta^{\pm}(a, a'^{-1})) & \text{if } G = \mathrm{SL}(2,\mathbb{R}), \\ \displaystyle\sum_{w \in W(G,A)} \varepsilon_M(w)\,\mathrm{Re}\,k_a(a, wa') & \text{otherwise.} \end{cases}$$

The parametrization a_t resp. a_z identifies A^0 with $\mathbb{R}$ resp. $\mathbb{C}/2\pi i\mathbb{Z}$, hence any $\lambda \in \Lambda_A$ defines a character of A^0 of the form $a_{t+i\theta}^{\lambda} = e^{im\theta+i\nu t}$ with $m \in \mathbb{Z}$, $\nu \in \mathbb{R}$ (where $m = \theta = 0$ for $\mathrm{SL}(2,\mathbb{R})$) and will be denoted by $\lambda = \lambda_{m+i\nu}$. We choose the Haar measure $da_t = dt$ on A_R, thereby fixing λ_P. The dual measure on Λ_A will then be $d\lambda_{m+i\nu} = i d\nu$. Now we turn to the various cases of G.

If $G = \mathrm{SL}(2,\mathbb{R})$, then $\Sigma_P = \{\beta\}$, $a_t^\beta = e^{2t}$, and $H_\beta = \mathrm{diag}(1, -1) \in \mathfrak{sl}(2,\mathbb{R})$. We have $\lambda_{i\nu}(H_\beta) = i\nu$, $\lambda_P(H_\beta) = 1$. Remember that $\gamma = -\mathrm{Id}$ and that the Haar measure on $A_I = \{1, \gamma\}$ has total mass one. Taking into account that $\phi_\alpha(\gamma a, \chi, \varepsilon) = \phi_\alpha(a, \chi, \varepsilon)$, we obtain

$$k_\beta^{\pm}(\gamma^m a_t, \gamma^{m'} a_{t'}) = \sum_{\varepsilon=\pm 1} \frac{1}{2\pi}\,\mathrm{p.v.}\int_{-\infty}^{\infty} \phi_\beta(a_t, \lambda_{-i\nu}, \pm\varepsilon)\varepsilon^{m'-m} e^{i\nu(t'-t)}\,d\nu.$$

Using Lemma 7, we see that if $t > 0$, then

$$k_\beta^{\pm}(a_t, a_{t'}) = \begin{cases} \dfrac{2}{e^{t+t'}-1} & \text{if } t' > t, \\ 0 & \text{if } t' < t, \end{cases}$$

$$k_\beta^{\pm}(a_t, -a_{t'}) = 0,$$

whereas if $t < 0$, then

$$k_\beta^\pm(a_t, a_{t'}) = \pm\,\mathrm{sgn}(t-t') + \begin{cases} \dfrac{2}{e^{-t-t'}-1} & \text{if } t' < t, \\ 0 & \text{if } t' > t, \end{cases}$$

$$k_\beta^\pm(a_t, -a_{t'}) = \pm \tanh \frac{t-t'}{2}.$$

If $G = \mathrm{SL}(2,\mathbb{C})$, then $\Sigma_P = \{\alpha, \bar\alpha\}$, $a_z^\alpha = e^{2z}$, $a_z^{\bar\alpha} = e^{2\bar z}$, $W(G,A) = \{1, w_0\}$ and $w_0 a_z = a_{-z}$. In the co-ordinate z the vector H_α acts as $\frac{\partial}{\partial z}$. If $\lambda \in \Lambda_A$, then $\lambda(H_\alpha) = \frac{\partial}{\partial z} a_z^\lambda\big|_{z=0}$, hence $\lambda_{m+i\nu}(H_\alpha) = \frac{m+i\nu}{2}$, and $\lambda_P(H_\alpha) = \frac{1}{2}$. With the help of Lemma 8 we get

$$\begin{aligned} k_\alpha(a_{t+i\theta}, a_{t'+i\theta'}) &= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\,\mathrm{p.v.}\int_{-\infty}^{\infty} \phi_\alpha(a_{t+i\theta}, \lambda_{-m-i\nu}) e^{im(\theta'-\theta)+i\nu(t'-t)} d\nu \\ &= \frac{1}{2\big(e^{2(t+i\theta')}-1\big)} \cdot \frac{1+e^{(t-i\theta)-(t'-i\theta')}}{1-e^{(t-i\theta)-(t'-i\theta')}}. \end{aligned}$$

The assertion now follows in view of

$$K_P(a_z, a_{z'}) = \mathrm{Re}\big(k_\alpha(a_z, a_{z'}) + \overline{k_\alpha(a_z, a_{-z'})}\big).$$

If $G = \mathrm{SU}(2,1)$, then $\Sigma_P = \{\alpha, \bar\alpha, \beta = \alpha + \bar\alpha\}$, $a_{t+i\theta}^\alpha = e^{t+3i\theta}$, $a_{t+i\theta}^{\bar\alpha} = e^{t-3i\theta}$, $H_\alpha = \mathrm{diag}(1,-1,0)$, $H_{\bar\alpha} = \mathrm{diag}(0,1,-1) \in \mathfrak{g}_\mathbb{C} = \mathrm{sl}(3,\mathbb{C})$, and $W(G,A) = \{1, s_\beta\}$. Since $a_z = \exp(zH_\alpha + \bar z H_{\bar\alpha})$, we have $\lambda(H_\alpha) = \frac{\partial}{\partial z} a_z^\lambda\big|_{z=0}$, hence $\lambda_{m+i\nu}(H_\alpha) = \frac{m+i\nu}{2}$. Moreover, $\lambda_P = \frac{\beta}{2}$, $\lambda_P(H_\alpha) = \frac{1}{2}$. Using Lemma 8, we obtain

$$k_\alpha(a_{t+i\theta}, a_{t'+i\theta'}) = \frac{1}{2\big(e^{t+i\theta+2i\theta'}-1\big)} \cdot \frac{e^{t'-i\theta'} + e^{t-i\theta}}{e^{t'-i\theta'} - e^{t-i\theta}}.$$

If $G = \mathrm{Sp}(1,1)$, then $\Sigma_P = \{\alpha, \bar\alpha, \beta = \frac{\alpha+\bar\alpha}{2}, \delta = \frac{\alpha-\bar\alpha}{2}\}$, $a_z^\alpha = e^{2z}$, $a_z^{\bar\alpha} = e^{2\bar z}$, and $W(G,A) = \{1, s_\beta\} \times \{1, s_\delta\}$. Since $a_{t+i\theta}^{m\delta+i\nu\beta} = e^{2im\theta+2i\nu t}$, we have $\lambda_{m+i\nu} = \frac{1}{2}(m\delta + i\nu\beta)$, $\lambda_{m+i\nu}(H_\alpha) = \frac{m+i\nu}{2}$. Moreover, $\lambda_P = \frac{\beta}{2}$, $\lambda_P(H_\alpha) = \frac{1}{2}$. With the help of Lemma 8 we see that $k_\alpha(a_{t+i\theta}, a_{t'+i\theta'})$ is given by the same expression as in the case of $\mathrm{SL}(2,\mathbb{C})$.

In each case, the integrals appearing in K_P turned out to be of the type considered in Lemma 7, and since $t \neq \pm t'$, $\mathrm{Re}\, z \neq \pm\, \mathrm{Re}\, z'$, they are conditionally convergent at infinity. We know from Lemma 3 that the singularities at $\nu = 0$ of the various terms cancel, therefore no principal value is necessary. □

We come now to the main result of this section. In the notation of Lemma 4, let $A(\alpha) = A \cap G(\alpha)^1$ and $P(\alpha)^1 = P(\alpha) \cap G(\alpha)^1$, thus $A = C(\alpha)A(\alpha)$, $P(\alpha) = C(\alpha)P(\alpha)^1$. Remember that $F_f(a) = \varepsilon_\Sigma(a) I_G(a,f)$ for $a \in A'$.

Theorem 4 *If $f \in \mathcal{C}_{\text{con}}(G)$, $a \in A'$, then*

$$I_{P,\Sigma}(a,f) = \sum_{\{\alpha,\bar\alpha\}\subset\Sigma_P^+} \int_{W(M(\alpha),A)\backslash\widehat{A(\alpha)}} K^{\pm}_{P(\alpha)^1}(a_\alpha, a')F_f(c_\alpha a')\,da',$$

where we take, for each $\{\alpha,\bar\alpha\} \subset \Sigma_P^+$, a decomposition $a = c_\alpha a_\alpha$ with $c_\alpha \in C(\alpha)$, $a_\alpha \in A(\alpha)$. The function $K^{\pm}_{P(\alpha)^1}$ is given by Proposition 3 as applied to $G(\alpha)^1$. One has to take $K^{+}_{P(\beta)^1}$ if $\dim K\backslash G = 2$ and $K^{-}_{P(\beta)^1}$ otherwise.

Proof. Given $f \in \mathcal{C}(G)$ and $\chi \in \hat{A}$, put

$$\theta_\chi(f) = \int_A \chi(a)F_f(a)\,da.$$

Then $\theta_{w\chi}(f) = \varepsilon_M(w)\theta_\chi(f)$ for $w \in W(G,A)$, and if χ is the Σ-parameter of $\sigma \in \hat{M}$ as explained in the beginning of this section, then $\Theta_{\pi_\sigma}(f) = (-1)^{\#(\Sigma)}\theta_\chi(f)$. It is, of course, this equality which leads to (6).

We stick to our conventions concerning normalization of measures, which also apply if we replace G by $G(\alpha)^1$. Then the compact group $A/A(\alpha)$ has volume one, and

$$\theta_\chi(f) = \int_{A/A(\alpha)} \chi(a) \int_{A(\alpha)} \chi(a')F_f(aa')\,da'.$$

Fourier inversion on $A/A(\alpha)$ yields, for each $\chi' \in \widehat{A(\alpha)}$ and $a \in A$,

$$\sum_{\substack{\chi\in\hat{A} \\ \chi|_{A(\alpha)}=\chi'}} \overline{\chi(a)}\theta_\chi(f) = \int_{A(\alpha)} \chi'(a')F_f(aa')\,da'. \tag{47}$$

If χ is the Σ-parameter of σ, then $\bar\chi$ is the $(-\Sigma)$-parameter of $\check\sigma$, and $\Omega_{P,\Sigma}(a,\check\sigma) = (-1)^{\#(\Sigma)}\Omega_P(a,\bar\chi,\pm\chi(\gamma))$, where the sign depends on G as described in the Theorem. By Lemma 5 we now get, for $a \in A'$,

$$\begin{aligned}
\int_{\hat{M}} \Omega_{P,\Sigma}(a,\check\sigma)\Theta_{\pi_\sigma}(f)\,d\sigma &= \int_{W(M,A)\backslash\hat{A}} \Omega_P(a,\bar\chi,\pm\chi(\gamma))\theta_\chi(f)\,d\chi \\
&= \sum_{\{\alpha,\bar\alpha\}\subset\Sigma_P^+} \int_{W(M(\alpha),A)\backslash\hat{A}} \Omega^0_{P(\alpha)}(a,\bar\chi,\pm\chi(\gamma))\theta_\chi(f)\,d\chi \\
&= \sum_{\{\alpha,\bar\alpha\}\subset\Sigma_P^+} \int_{W(M(\alpha),A)\backslash\widehat{A(\alpha)}} \sum_{\substack{\chi\in\hat{A} \\ \chi|_{A(\alpha)}=\chi'}} \Omega^0_{P(\alpha)}(a,\bar\chi,\pm\chi(\gamma))\theta_\chi(f)\,d\chi'.
\end{aligned}$$

Now we insert (45) and consider the sum over χ for fixed χ':

$$\sum_{\substack{\chi\in\hat{A} \\ \chi|_{A(\alpha)}=\chi'}} \varepsilon_M(w)\overline{w\chi(a)}\phi_\alpha(a,\overline{w\chi},\pm\chi(\gamma))\theta_\chi(f)$$

$$= \phi_\alpha(a,\overline{w\chi'},\pm\chi'(\gamma)) \sum_{\substack{\chi\in\hat{A} \\ \chi|_{A(\alpha)}=\chi'}} \overline{w\chi(a)}\theta_{w\chi}(f)$$

$$= \phi_\alpha(a,\overline{w\chi'},\pm\chi'(\gamma)) \int_{A(\alpha)} w\chi'(a')F_f(aa')\,da'$$

$$= \phi_\alpha(a_\alpha,\overline{w\chi'},\pm\chi'(\gamma))\overline{w\chi'(a_\alpha)} \int_{A(\alpha)} w\chi'(a')F_f(c_\alpha a')\,da',$$

where we have used (47) and the fact that $\phi_\alpha(a,\lambda,\varepsilon)$ depends only on a mod $C(\alpha)$ and $\lambda|_{\mathfrak{a}(\alpha)}$. Substituting wa' for a' and interpreting the result in terms of $G(\alpha)^1$, we see that the whole formula takes the form

$$\sum_{\{\alpha,\bar{\alpha}\}\subset\Sigma_P^+} \int_{W(M(\alpha),A)\backslash\widehat{A(\alpha)}} \Omega^0_{P(\alpha)^1}(a_\alpha,\overline{\chi'},\pm\chi'(\gamma)) \int_{A(\alpha)} \chi'(a')F_f(c_\alpha a')\,da'\,d\chi'.$$

Each term can be written as $\#(W(M(\alpha),A)\backslash\widehat{A(\alpha)})^{-1}$ times an integral over $\widehat{A(\alpha)}$. By distributional Fourier inversion on the group $A(\alpha)$,

$$\frac{1}{2\pi i}\int_{\hat{M}} \Omega_P(a,\check{\sigma})\Theta_{\pi_\sigma}(f)\,d\sigma$$

is given by the right-hand side of the asserted formula. In view of Theorem 1, this is what we had to prove. □

6 Passage to the limit

So far, the weighted orbital integral $J_M(m,f)$ was only defined for $m\in M\cap G'$. In [4], J. Arthur has generalized the definition to any $m\in M$. We shall now recall this definition for our special situation of real rank one groups and extend Theorem 1 to the limiting cases.

The extension proceeds in two stages. First of all, the definitions (1) and (2) make still sense for $f\in C_c^\infty(G)$ and those $m\in M$ for which $G_m^0\subset M$, because the integrals are compactly supported and $v(xy)=v(x)$ for $y\in G_m$. Indeed, y normalizes G_m^0 and its unique maximal $\mathbb{R}$-split torus A_R. Thus y represents some $w\in W(G,A_R)$, and the M-invariance of v reduces our assertion to the case $y\in K$, which has been dealt with in connection with (11). Also, there is no problem in defining the invariant distribution $I_P(m,f)$ by equation (8). However, $|D_G|^{1/2}$ is only upper semicontinuous, and if one passed to the limit in a naive way, one would simply get zero in view of (13) and (21). Incidentally, one may prove that $G_m^0 \subset M$ iff $G_m \subset M$. Indeed, in the preceding argument, $y \notin M \Longrightarrow m \in M_I$, which suffices if a real root exists. Otherwise one has to use the assumption that G is contained in its simply connected complexification, which is essential here, as the example $\operatorname{diag}(i, -i) \in \mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$ shows. Since the condition $G_m^0 \subset M$ is the one which generalizes, we shall not go into detail.

Let us again fix a Cartan subgroup A of M. Now $a \in A$ satisfies $G_a^0 \subset M$ iff $a \in A''$. As before, we choose a half-system Σ of positive roots for $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$, which enters implicitly in $F_f(a)$ and $T_f(a)$ as defined in section 2. Given $a \in A$, let $\Sigma_a = \{\alpha \in \Sigma : a^\alpha = 1\}$ and define $\varepsilon_\Sigma(a)$ by

$$\prod_{\alpha \in \Sigma \setminus \Sigma_a} \left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right) = \varepsilon_\Sigma(\exp H) |D_M(\exp H)|^{1/2}.$$

This is consistent with our earlier notation for $a \in A'$, and $\varepsilon_\Sigma(a)^4 = 1$ as before. We write $J_{G,\Sigma}(a, f) = \varepsilon_\Sigma(a) J_G(a, f)$, $J_{M,\Sigma}(a, f) = \varepsilon_\Sigma(a) J_M(a, f)$ and $I_{P,\Sigma}(a, f) = \varepsilon_\Sigma(a) I_P(a, f)$ for $a \in A''$.

Following Harish-Chandra, we introduce the element

$$\Pi_a = \prod_{\alpha \in \Sigma_a} H_\alpha$$

of the symmetric algebra $S(\mathfrak{a}_{I,\mathbb{C}}) \subset S(\mathfrak{a}_{\mathbb{C}})$, which depends implicitly on Σ. Of course, $\Pi_a = 1$ for $a \in A'$, and we shall write $\Pi = \Pi_1$. Elements of $S(\mathfrak{a}_{\mathbb{C}})$ may be considered as polynomial functions on $\mathfrak{a}_{\mathbb{C}}^*$. Let us denote, for each $p \in S(\mathfrak{a}_{\mathbb{C}})$, by D_p the corresponding invariant differential operator on A. Then

$$\lim_{a' \to a} D_{\Pi_a} \Delta_\Sigma(a') = C_{M_a} \varepsilon_\Sigma(a) |D_M(a)|^{1/2}.$$

The positive integer C_{M_a} depends only on the group M_a, and one has

$$C_M = \#(W(M, A)) \Pi(\rho_\Sigma).$$

Now Harish-Chandra's formula

$$\lim_{a' \to a} D_{\Pi_a} F_f(a') = C_{M_a} J_{G,\Sigma}(a, f)$$

(see [9], p. 33) follows as well as

$$\lim_{a' \to a} D_{\Pi_a} T_f(a') = C_{M_a} J_{M,\Sigma}(a, f).$$

Moreover, from (8) we get

$$\lim_{a' \to a} D_{\Pi_a} I_{P,\Sigma}(a', f) = C_{M_a} I_{P,\Sigma}(a, f), \tag{48}$$

where a' has to remain within A'.

Theorem 5 *If $a \in A''$ and $f \in C_c^\infty(G)$, then*

$$I_{P,\Sigma}(a,f) = -\varepsilon_\Sigma(a) \sum_{\pi \in \hat{G}_{\rm dis}} |D_G(a)|^{1/2}\Theta_{\check{\pi}}(a)\Theta_\pi(f) + \frac{1}{2\pi i}\int_{\hat{M}} \Omega_{P,\Sigma}(a,\check{\sigma})\Theta_{\pi_\sigma}(f)\,d\sigma,$$

provided we define $\Omega_{P,\Sigma}$ as follows. If $\lambda - \rho_\Sigma$ is the Σ-highest weight of $\sigma \in \hat{M}$ with $\sigma\big|_{A_R} \neq 1$, then

$$\Omega_{P,\Sigma}(a,\sigma) = \frac{1}{2}C_{M_*}^{-1} \sum_{w \in W(G,A)} \varepsilon_M(w)a^{w\lambda} \sum_{\alpha \in \Sigma_P^+} \Pi_a(s_\alpha w\lambda)\phi_\alpha(a, w\lambda, \operatorname{sgn}\sigma),$$

where $\phi_\alpha(a,\lambda,\varepsilon)$ is given as in (45). Moreover, $I_P(a)$ is a tempered distribution for $a \in A''$, and $\Omega_{P,\Sigma}(a,\sigma)$ depends real-analytically on $\sigma \in \hat{M}$.

I refrain from writing $\Omega_P(a,\lambda)$ here because this function of λ depends on Σ (through Π). Observing that, for $w \in W(G_a, A)$, one has $\phi_{w\alpha}(a,w\lambda) = \phi_\alpha(a,\lambda)$, $\Pi_a(s_{w\alpha}w\lambda) = \Pi_a(ws_\alpha\lambda) = \varepsilon_{M_a}(w)\Pi_a(s_\alpha\lambda)$ and $\varepsilon_{M_a}(w) = \varepsilon_M(w)$ by Lemma 1, one can replace the sum over $W(G,A)$ by $\#(W(G_a,A))$ times a sum over $W(G_a,A)\backslash W(G,A)$.

For the proof we need a lemma. We denote by $S_r(\mathfrak{a}_\mathbb{C})$ the subspace of elements of degree at most r. Let X be an indeterminate.

Lemma 9 *If $\alpha \in \Sigma_P^+$, $p \in S_r(\mathfrak{a}_{I,\mathbb{C}})$, then there exists $q(X) \in S_{r-1}(\mathfrak{a}_\mathbb{C}) \otimes \mathbb{C}(X)$ such, that*

$$D_p(a^\lambda\phi_\alpha(a,\lambda,\varepsilon)) = a^\lambda(p(s_\alpha\lambda)\phi_\alpha(a,\lambda,\varepsilon) + q(a^\alpha,\lambda))$$

for all $a \in A \setminus A_I$, $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $\varepsilon = \pm 1$, where we consider q as a meromorphic function on $\mathbb{C} \times \mathfrak{a}_\mathbb{C}^*$. Moreover, $(1-X)^r q(X)$ is a polynomial in X.

Proof. To start with, let us exclude the case $\alpha = \beta$, $a^{\lambda_P} = 1$. Then we have, for $H \in \mathfrak{a}_\mathbb{C}$ and $a^{\pm\lambda_P} > 1$,

$$\begin{aligned} D_H\phi_\alpha(a,\lambda) &= \lambda_P(H_\alpha) \sum_{n=\left\{\substack{1\\0}\right.}^{\infty} \frac{\mp n\alpha(H)}{n \mp \lambda(H_\alpha)} a^{\mp n\alpha} \\ &= \alpha(H)\left(\frac{\lambda_P(H_\alpha)}{1-a^\alpha} - \lambda(H_\alpha)\phi_\alpha(a,\lambda)\right), \end{aligned}$$

and therefore

$$\begin{aligned} D_H(a^\lambda\phi_\alpha(a,\lambda)) &= a^\lambda(\lambda(H) + D_H)\phi_\alpha(a,\lambda) \\ &= a^\lambda\left(\lambda(s_\alpha H)\phi_\alpha(a,\lambda) + \frac{\lambda_P(H_\alpha)\alpha(H)}{1-a^\alpha}\right). \end{aligned}$$

In the exceptional case we excluded, there is an additional term

$$D_H\left(a^\lambda \frac{\pi\varepsilon}{\sin\pi\lambda(H_\beta)}\right) = a^\lambda \lambda(H)\frac{\pi\varepsilon}{\sin\pi\lambda(H_\beta)}.$$

If we restrict H to $\mathfrak{a}_{I,\mathbf{C}}$, then $H = s_\beta H$, and the lemma follows for $p = H$, since $s_\alpha\lambda(H) = \lambda(s_\alpha H)$.

Let us now prove the lemma for general p by induction on r. For $r = 0$ there is nothing to prove. We assume that the assertion is true for $p \in S_r(\mathfrak{a}_{I,\mathbf{C}})$ and let $H \in \mathfrak{a}_{I,\mathbf{C}}$. Then

$$\begin{aligned} D_{Hp}(a^\lambda\phi_\alpha(a,\lambda,\varepsilon)) &= p(s_\alpha\lambda)D_H(a^\lambda\phi_\alpha(a,\lambda,\varepsilon)) + D_H(a^\lambda q(a^\alpha,\lambda)) \\ &= a^\lambda\Bigg(p(s_\alpha\lambda)s_\alpha\lambda(H)\phi_\alpha(a,\lambda,\varepsilon) + \frac{\lambda_P(H_\alpha)\alpha(H)}{1-a^\alpha}p(s_\alpha\lambda) \\ &\qquad + \lambda(H)q(a^\alpha,\lambda) + \alpha(H)a^\alpha q'(a^\alpha,\lambda)\Bigg), \end{aligned}$$

where $q'(X) = \frac{d}{dX}q(X)$. Now $(1-X)^{r+1}q'(X)$ is a polynomial, and the last three terms have degree at most r in λ. This proves the assertion for $Hp \in S_{r+1}(\mathfrak{a}_{I,\mathbf{C}})$, and the general case follows by linearity. □

Proof of Theorem 5. In view of formula (48), we only have to apply D_{Π_a} and pass to the limit $a' \to a$ in Theorem 1. Concerning the contribution from $\hat{G}_{\text{dis}}$, recall thet $\tilde{\Delta}_\Sigma\Theta_\pi$ is explicitly given in [9], p. 96, as a linear combination of characters on each connected component of A''. Being skew $W(M,A)$-invariant, this function may be written as a linear combination of terms $\Delta_\Sigma\Theta_\sigma$ for appropriate $\sigma \in \hat{M}$. Therefore Θ_π is smooth on A'', and

$$\lim_{a'\to a} D_{\Pi_a}(\tilde{\Delta}_\Sigma(a')\Theta_\pi(a')) = C_{M_a}\varepsilon_\Sigma(a)|D_G(a)|^{1/2}\Theta_\pi(a).$$

It remains to show that the expression

$$C_{M_a}^{-1}\lim_{a'\to a} D_{\Pi_a}\Omega_P(a',\lambda,\operatorname{sgn}\sigma)$$

(which clearly defines a real-analytic function of $\lambda \in \Lambda_A$ in view of Lemma 3) is given by the formula stated in the Theorem. The preceding lemma tells us that both expressions differ by $\frac{1}{2}C_{M_a}^{-1}$ times

$$\begin{aligned} \lim_{a'\to a} \sum_{w\in W(G,A)} \varepsilon_M(w)a'^{w\lambda} \sum_{\alpha\in\Sigma_P^+} q_\alpha(a'^\alpha, w\lambda) \\ = \sum_{\alpha\in\Sigma_P^+} \lim_{a'\to a} D_{q_\alpha(a'^\alpha)}\big(\Delta_\Sigma(a')(\Theta_\sigma(a') + \Theta_{w_0\sigma}(a'))\big). \end{aligned}$$

Let us consider a monomial $H_1\ldots H_k$ entering in $q_\alpha(a^\alpha)$, where $k < \#(\Sigma_a)$ by the preceding lemma. If we apply $D_{H_1\ldots H_k}$ according to the Leibniz rule to

$\Delta_\Sigma(a')\Theta_\sigma(a')$, which is the product of $\prod_{\delta\in\Sigma_a}(a'^\delta - 1)$ with a smooth function, then we get a sum, in each term of which at least one factor $(a'^\delta - 1)$ remains undifferentiated. Thus, in the limit $a \to a'$ we get zero. $\square$

We turn now to the definition of $J_M(m, f)$ for $m \in M$ with $G_m^0 \not\subset M$. In this case, $m \in M_I$, and $J_M(ma_R, f)$ blows up as $a_R \to 1$ in $A_R \setminus \{1\}$. Let β_m be the only reduced root of $(\mathfrak{g}_m, \mathfrak{a}_R)$ positive on the chamber C_P, and define the coroot $H_{\beta_m} \in \mathfrak{a}_R$ by $\beta_m(H_{\beta_m}) = 2$. It follows from the calculations in [1], section 4, that the limit

$$J_M(m, f) = \lim_{a_R\to 1}\left(J_M(ma_R, f) + \lambda_P(H_{\beta_m})\log|a_R^{\beta_m/2} - a_R^{-\beta_m/2}| \cdot J_G(ma_R, f)\right)$$

exists for $f \in C_c^\infty(G)$ and equals

$$J_M(m, f) = |D_G(m)|^{1/2}\int_K\int_{N/N_m}\int_{N_m} f(kn'mnn'^{-1}k^{-1})\delta_m(n)\,dn\,dn'\,dk,$$

where K is such that its Cartan involution θ stabilizes A_R, the Haar measure on N is so normalized that

$$\int_G f(x)\,dx = \int_K\int_N\int_{A_R} f(kna)\,da\,dn\,dk,$$

and the function δ_m on N_m can be described as follows. If $X \in \mathfrak{g}_{m,\beta_m}$, $Y \in \mathfrak{g}_{m,2\beta_m}$ and $X \neq 0$, then

$$\delta_m(\exp(X+Y)) = \lambda_P(H_{\beta_m})\log\frac{\sqrt{2}|X|}{r_m},$$

where $|X|^2 = -\langle X, \theta X\rangle$ and $r_m = |H_{\beta_m}|$. Although the function $\Delta^*(a)$ used in [1] vanishes whenever $\Sigma_a \neq \emptyset$, it can easily be replaced by $|D_G(a)|^{1/2}$ without impairing the argument. Note that P and $\bar{P}$ are interchangeable in the definition of J_M.

The function $\tilde{\jmath}_m(X) = \frac{\sqrt{2}}{r_m}|X|$ appearing in the definition of δ_m can be characterized as follows. It is the square root of an $M_{I,m}$-invariant positive definite quadratic form on $\mathfrak{g}_{m,\beta_m}$, and $\tilde{\jmath}(X) = 1$ iff $(X, H_{\beta_m}, -\theta X)$ is a Lie triple (cf. [13]). Indeed, if $X \in \mathfrak{g}_{m,\beta_m}$, then $\theta X \in \mathfrak{g}_{m,-\beta_m}$, and $[X, \theta X] \in \mathfrak{m}_m$. As $\theta[X, \theta X] = -[X, \theta X]$, we see that $[X, \theta X]$ is a multiple of H_{β_m}. Our assertion now follows from the equality

$$\langle [X, -\theta X], H_{\beta_m}\rangle = \langle X, [H_{\beta_m}, \theta X]\rangle = 2|X|^2.$$

Of course, both $\tilde{\jmath}_m$ and the measure on N depend on the choice of K.

Our definition of $J_M(m, f)$ for $G_m^0 \not\subset M$, which generalizes (14), slightly differs from the general one given in [4], where J. Arthur uses a^{β_m} instead of $a^{\beta_m/2}$ in order to cover the p-adic case. In this connection, cf. his remark on

p. 289 bottom. In [17], section 15, G. Warner has given a different expression for $J_M(m,f)$, which obviously converges for $f \in \mathcal{C}(G)$ and shows that $J_M(m,f)$ is a tempered distribution.

Next we look at the definition (8) and insert ma_R. As the last two terms are continuous in a_R, we may put

$$I_P(m,f) = \lim_{a_R \to 1} \left(I_P(ma_R, f) + \lambda_P(H_{\beta_m}) \log\left|a_R^{\beta_m/2} - a_R^{-\beta_m/2}\right| \cdot J_G(ma_R, f) \right).$$

If we specialize $m = a_I \in A_I$, we may define $I_{P,\Sigma}(a_I, f) = \varepsilon_\Sigma(a_I) I_P(a_I, f)$. Of course, we are now going to pass to the limit in Theorem 5. For each $\pi \in \hat{G}_{\mathrm{dis}}$, the function $a_R \mapsto |D_G(a_I a_R)|^{1/2} \Theta_\pi(a_I a_R)$ is even and equals a linear combination of characters on each connected component of A_R''. Thus we may extend it to $a_R = 0$ by continuity.

Theorem 6 *If $a_I \in A_I$ and $f \in \mathcal{C}(G)$, then $I_{P,\Sigma}(a_I, f)$ is still given by the formula from Theorem 5, provided we define $\Omega_{P,\Sigma}(a_I, \sigma)$ as follows. If $\lambda - \rho_\Sigma$ is the Σ-highest weight of $\sigma \in \hat{M}$ with $\sigma\big|_{A_R} \neq 1$, then*

$$\Omega_{P,\Sigma}(a_I, \sigma) = \frac{1}{2} C_{M_{a_I}}^{-1} \sum_{w \in W(G,A)} \varepsilon_M(w) a_I^{w\lambda} \sum_{\alpha \in \Sigma_P^+} \Pi_{a_I}(s_\alpha w\lambda) \phi_\alpha(a_I, w\lambda),$$

where $\phi_\alpha(a_I, \lambda)$ is defined as before if $a_I^\alpha \neq 1$ (the series now being only conditionally convergent), while

$$\phi_\alpha(a_I, \lambda) = \lambda_P(H_\alpha)\big(\psi(1) - \psi(1 - \lambda(H_\alpha)) - \log(\alpha(H_{\beta_{a_I}})/2)\big)$$

if $a_I^\alpha = 1$. Here, ψ denotes the logarithmic derivative of the gamma function. Moreover, $\Omega_{P,\Sigma}(a_I, \sigma)$ depends real-analytically on $\sigma \in \hat{M}$.

Remember that $-\psi(1)$ is the Euler-Mascheroni constant. The remark made after Theorem 5 applies here as well.

Proof. Since $\log\left|a_R^{\beta_m/2} - a_R^{-\beta_m/2}\right| = \log\left|1 - a_R^{-\beta_m}\right| + \frac{1}{2}\beta_m(\log a_R)$, we can write, with the abbreviations $a = a_I a_R$, $\beta' = \beta_{a_I}$,

$$I_{P,\Sigma}(a_I, f) = \lim_{a_R \to 1} \left(I_{P,\Sigma}(a, f) + \lambda_P(H_{\beta'}) \log\left|1 - a_R^{-\beta'}\right| \cdot J_{G,\Sigma}(a, f) \right).$$

This formula remains valid for any $f \in \mathcal{C}(G)$ (cf. [17], Lemma 15.2). Thus, our assertion would be true if we defined $\Omega_{P,\Sigma}(a_I, \sigma)$ to be the limit as $a_R \to 1$ of $\Omega_{P,\Sigma}(a, \sigma)$ plus

$$\lambda_P(H_{\beta'}) \log\left|1 - a_R^{-\beta'}\right| \cdot \frac{1}{2} C_{M_a}^{-1} \lim_{a' \to a} D_{\Pi_a}\big(\Delta_\Sigma(a')(\Theta_\sigma(a') + \Theta_{w_0\sigma}(a'))\big)$$

in the space of w_0-invariant tempered distributions on $\hat{M}$. If we assume that $a_R \to 1$ while $a_R^{\lambda_P} > 1$, then this limit equals

$$\frac{1}{2} C_{M_{a_I}}^{-1} \sum_{w \in W(G,A)} \varepsilon_M(w) a_I^{w\lambda}$$
$$\times \lim_{a_R \to 1} \left(\sum_{\alpha \in \Sigma_P^+} \Pi_{a_I}(s_\alpha w\lambda) \phi_\alpha(a, w\lambda) + \lambda_P(H_{\beta'}) \Pi_{a_I}(w\lambda) \log\left(1 - a_R^{-\beta'}\right) \right).$$

If $a_I^\alpha \neq 1$, then $\lim_{a_R \to 1} \phi_\alpha(a, \lambda) = \phi_\alpha(a_I, \lambda)$. It remains to consider the contribution from $\Sigma_{P,a_I}^+ = \{\alpha \in \Sigma_P^+ : a_I^\alpha = 1\}$.

Note that $\log(1-z) = -b(0,z)$. Equation (34) implies

$$\lim_{z \to 1} (b(s,z) - b(0,z)) = \int_0^1 \frac{t^s - 1}{1 - t} \, dt$$

for $\operatorname{Re} s > -1$. The substitution $t = e^{-u}$ shows that this equals $\psi(1) - \psi(s+1)$, and the equality extends by analyticity to all $s \notin \{-1, -2, \ldots\}$. Applying this formula to $\phi_\alpha(a, \lambda) = \lambda_P(H_\alpha) b(-\lambda(H_\alpha), a^{-\alpha})$, we see that

$$\lim_{a_R \to 1} \left(\sum_{\alpha \in \Sigma_{P,a_I}^+} \Pi_{a_I}(s_\alpha \lambda) \phi_\alpha(a, \lambda) + \lambda_P(H_{\beta'}) \Pi_{a_I}(\lambda) \log\left(1 - a_R^{-\beta'}\right) \right)$$
$$= \sum_{\alpha \in \Sigma_{P,a_I}^+} \lambda_P(H_\alpha) \Pi_{a_I}(s_\alpha \lambda) \left(\psi(1) - \psi(1 - \lambda(H_\alpha))\right)$$
$$+ \lim_{a_R \to 1} \left(\lambda_P(H_{\beta'}) \Pi_{a_I}(\lambda) \log(1 - a_R^{-\beta'}) - \sum_{\alpha \in \Sigma_{P,a_I}^+} \lambda_P(H_\alpha) \Pi_{a_I}(s_\alpha \lambda) \log(1 - a_R^{-\alpha}) \right)$$
$$= \sum_{\alpha \in \Sigma_{P,a_I}^+} \Pi_{a_I}(s_\alpha \lambda) \phi_\alpha(a_I, \lambda) + p_{a_I}(\lambda) \lim_{a_R \to 1} \log\left(1 - a_R^{-\beta'}\right),$$

where $\phi_\alpha(a_I, \lambda)$ is as defined in the Theorem and

$$p_{a_I}(\lambda) = \lambda_P\left(H_{\beta_{a_I}}\right) \Pi_{a_I}(\lambda) - \sum_{\alpha \in \Sigma_{P,a_I}^+} \lambda_P(H_\alpha) \Pi_{a_I}(s_\alpha \lambda).$$

The existence of $\lim_{a_R \to 1} \Omega_{P,\Sigma}(a, \sigma)$ in the space of tempered distributions implies the curious fact that

$$\sum_{w \in W(G,A)} \varepsilon_M(w) a_I^{w\lambda} p_{a_I}(w\lambda) = 0.$$

This proves our formula and shows that the limit exists pointwise on $\hat{M} \setminus \hat{M}_I$.

We come now to the smoothness of $\Omega_{P,\Sigma}(a_I,\sigma)$. First of all, we observe that, for $\lambda_I \in \mathfrak{a}^*_{I,\mathbb{C}}$ and $H \in \mathfrak{a}_{I,\mathbb{C}}$, we have $(s_\alpha\lambda_I - s_{\bar\alpha}\lambda_I)(H) = \lambda_I(H_\alpha + H_{\bar\alpha})\alpha(H) = 0$. Thus, $\Pi_{\mathfrak{a}_I}(s_\alpha\lambda_I) = \Pi_{\mathfrak{a}_I}(s_{\bar\alpha}\lambda_I)$ for all $\lambda_I \in \Lambda_{A_I}$. If $a_I^\alpha \neq 1$, then, as we have seen in the proof of Lemma 3,

$$\sum_{w\in W(G(\alpha),A)} \varepsilon_M(w) a_I^{w\lambda}\left(\phi_\alpha(a_I,w\lambda) + \phi_{\bar\alpha}(a_I,w\lambda)\right)$$

is real-analytic in $\lambda \in \Lambda_A$. Since $\phi_\alpha(a_I,\lambda)$ has at most simple poles on the subset $\Lambda_{A_I} \subset \Lambda_A$, it follows from the preceding remark that

$$\sum_{w\in W(G(\alpha),A)} \varepsilon_M(w) a_I^{w\lambda}\Big(\Pi_{\mathfrak{a}_I}(s_\alpha\lambda)\phi_\alpha(a_I,w\lambda) + \Pi_{\mathfrak{a}_I}(s_{\bar\alpha}\lambda)\phi_{\bar\alpha}(a_I,w\lambda)\Big)$$

is real-analytic, too.

As for the roots α with $a_I^\alpha = 1$, we recall that $\psi(s) = \frac{1}{n-s} +$ a holomorphic function for $\operatorname{Re} s$ near $n \in \{-1,-2,\ldots\}$. Therefore, if $\lambda \in \Lambda_A$ with fixed $\lambda_I(H_\alpha) = n$,

$$\begin{aligned}\Pi_{\mathfrak{a}_I}(s_\alpha\lambda)\psi(1+\lambda(H_\alpha)) &+ \Pi_{\mathfrak{a}_I}(s_{\bar\alpha}\lambda)\psi(1+\lambda(H_{\bar\alpha}))\\ &= \frac{\Pi_{\mathfrak{a}_I}(s_{\bar\alpha}\lambda) - \Pi_{\mathfrak{a}_I}(s_\alpha\lambda)}{\lambda_R(H_\alpha)} + \text{ a real-analytic function,}\end{aligned}$$

and the above observation shows that this whole expression is analytic. In analogy with Lemma 5 it is easy to see that $\Omega_{P,\Sigma}(a_I,\sigma)$ is a linear combination of functions of the two types just considered. This implies our assertion. □

Remark. Obviously, $p_{\mathfrak{a}_I}(\lambda)$ depends only on the restriction of λ to $\mathfrak{a}_{\mathfrak{a}_I}$. If we consider $G_{\mathfrak{a}_I}$ in the role of G, we see that $p_{\mathfrak{a}_I} = 0$. It would be interesting to have a direct proof for this fact, i. e., for the identity

$$\sum_{\alpha\in\Sigma_P^+} \lambda_P(H_\alpha)s_\alpha\Pi = \lambda_P(H_{\beta_1})\Pi.$$

Corollary. If ζ lies in the centre of G, then $\Omega_{P,\Sigma}(\zeta,\sigma) = \zeta^\lambda\Omega_{P,\Sigma}(1,\sigma)$, and

$$\begin{aligned}\Omega_{P,\Sigma}(1,\sigma) = \lambda_P(H_{\beta_1})d(\sigma)\psi(1) &- \sum_{\alpha\in\Sigma_P^+}\lambda_P(H_\alpha)\frac{\Pi(s_\alpha\lambda)}{\Pi(\rho_\Sigma)}\\ &\times\left(\frac{\psi(1+\lambda(H_\alpha)) + \psi(1-\lambda(H_\alpha))}{2} + \log\big(\alpha(H_{\beta_1})/2\big)\right),\end{aligned}$$

where $d(\sigma)$ denotes the degree of σ and the other notations are as in Theorem 6. This follows from Theorem 6 if one remembers that $\Pi(s_\alpha w_0\lambda) = \Pi(s_{-w_0\alpha}\lambda)$ and $w_0\lambda(H_\alpha) = -\lambda(H_{-w_0\alpha})$.

Finally, let us have a look at the case when f belongs to the space $\mathcal{C}(G//K)$ of K-biinvariant functions in $\mathcal{C}(G)$. If now $\pi \in \hat{G}_{\text{temp}}$, then $\pi(f) = 0$ unless

$\pi \cong \pi_{P,1_\lambda}$, where $1 = 1_{M_I}$ is the trivial representation of M_I, and $\lambda \in i\mathfrak{a}_R^*$. Let $c(\lambda)$ be the restriction of the intertwining operator $J_{\bar{P}|P}(1_\lambda)$ to the one-dimensional subspace of K-invariant vectors in $\mathcal{H}_1$, and let $\Theta_\lambda(f) = \Theta_{\pi_{1_\lambda}}(f)$ denote the spherical transform of $f \in \mathcal{C}(G//K)$. Then the definition of $J_P(\sigma, f)$ simplifies to

$$J_P(1_\lambda, f) = -\Theta_\lambda(f)\frac{\partial_P c(\lambda)}{c(\lambda)}.$$

Using definition (8) and Theorem 1, the weighted orbital integral $J_{M,\Sigma}(a, f) = \varepsilon_\Sigma(a)J_M(a, f)$ for $a \in A'$ can now be expressed in terms of $\Theta_\lambda(f)$, namely,

$$J_{M,\Sigma}(a, f) = \frac{1}{2\pi i}\int_{i\mathfrak{a}_R^*} \Omega_{M,\Sigma}(a, \lambda)\Theta_\lambda(f)\,d\lambda + \frac{1}{2}\Delta_\Sigma(a)\Theta_0(f), \tag{49}$$

where

$$\Omega_{M,\Sigma}(a, \lambda) = \Omega_P(a, \lambda + \rho_\Sigma) - \frac{1}{2}\Delta_\Sigma(a)\left(a^{-\lambda}\frac{\partial_P c(\lambda)}{c(\lambda)} + a^{\lambda}\frac{\partial_P c(-\lambda)}{c(-\lambda)}\right).$$

By forming the even part in λ we have canceled the simple pole of the logarithmic derivative of $c(\lambda)$ at $\lambda = 0$. For the function $c(\lambda)$, one has the explicit formula

$$c(\lambda) = c_0\frac{2^{-\lambda(H_{\beta_1})/2}\Gamma(\frac{1}{2}\lambda(H_{\beta_1}))}{\Gamma(\frac{1}{4}(m(\beta_1) + 2 + \lambda(H_{\beta_1})))\Gamma(\frac{1}{4}(\lambda + \rho_P)(H_{\beta_1}))}$$

(see, e. g., [11], Ch. IV, Theorem 6.4), which also shows that $n(1_{M_I}) = 1$. Here, $m(\beta_1)$ is the multiplicity of the only reduced root β_1 of $(\mathfrak{g}, \mathfrak{a}_R)$ positive on C_P, $\rho_P(H) = \frac{1}{2}\operatorname{Tr}\operatorname{ad}_{\mathfrak{n}}(H)$ for $H \in \mathfrak{a}_R$, and the constant c_0 depends on the normalization of the measure on $\bar{N}$. Thus,

$$\begin{aligned}\frac{\partial_P c(\lambda)}{c(\lambda)} = \frac{1}{4}\lambda_P(H_{\beta_1})\Big(&2\psi(\tfrac{1}{2}\lambda(H_{\beta_1}))\\ &-\psi(\tfrac{1}{4}(m(\beta_1) + 2 + \lambda(H_{\beta_1}))) - \psi(\tfrac{1}{4}(\lambda + \rho_P)(H_{\beta_1})) - 2\log 2\Big).\end{aligned}$$

It is an easy matter to extend (49) to the limiting cases, using Theorems 5 and 6. E. g.,

$$J_M(1, f) = \frac{1}{2\pi i}\int_{i\mathfrak{a}_R^*} \Omega_M(1, \lambda)\Theta_\lambda(f)\,d\lambda + \frac{1}{2}\Theta_0(f),$$

where

$$\Omega_M(1, \lambda) = \Omega_{P,\Sigma}(1, 1_\lambda) - \frac{1}{2}\left(\frac{\partial_P c(\lambda)}{c(\lambda)} + \frac{\partial_P c(-\lambda)}{c(-\lambda)}\right).$$

Of course, $\Omega_{P,\Sigma}(1, 1_\lambda)$ is given here by the corollary of Theorem 6. For this special case, the function $\Omega_M(1, \lambda)$ has been calculated earlier with the help of the inversion formula for the Abel integral transform. Namely, it follows from [16], Theorem 9.3, that $\Omega_M(1,\lambda)$ equals

$$\frac{1}{2}\lambda_P(H_{\beta_1})\left(\psi\left(\tfrac{m(\beta_1)}{2}\right)+\psi(1)-\psi(1+\tfrac{1}{2}\lambda(H_{\beta_1}))-\psi(1-\tfrac{1}{2}\lambda(H_{\beta_1}))\right).$$

Unfortunately, our formulas imply a more complicated expression for $\Omega_M(1,\lambda)$. I have checked the equality of both expressions for $\mathrm{rk}\, G \leq 2$ using the duplication formula $2\psi(2s) = \psi(s) + \psi(s+1) + 2\log 2$, but I did not find an argument for variable rank. Since both formulas are proved, this is, of course, only a cross-check.

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