π_1 of Smooth Points of a Log Del Pezzo Surface is Finite : I

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Introduction

A normal projective surface S over C is called *a log del Pezzo surface* if S has at worst quotient singularities and $-K_S$ is ample, where K_S denotes the canonical divisor of S.

Recall that the divisor class group of a quotient singularity is always finite. Hence for any Weil divisor D on a log del Pezzo surface S, nD is a Cartier divisor for some integer $n \geq 1$.

The principal result of this paper is the following :

Main Theorem. *The fundamental group of the space of smooth points of a log del Pezzo surface is finite.*

In the case of a Gorenstein log del Pezzo surface, this result was proved in [13] by first classifying such surfaces. In this paper, we also give a very easy proof of the result in the case of Gorenstein log del Pezzo surfaces. So far, there are not many results about general log del Pezzo surfaces. Recently, V.A. Alekseev and V.V. Nikulin have classified all log del Pezzo surfaces of index ≤ 2 (i.e., where $2K_S$ is Cartier) (cf. [1]).

The index of a log del Pezzo surface S is defined to be the smallest positive integer n such that nK_S is a Cartier divisor. In [15], Nikulin has proved that the rank of the Picard group of a minimal resolution of S is bounded by a universal function of the index of S. From this also one can deduce Proposition 1.7 below.

M. Miyanishi has made the following :

Conjecture *Let S be a log del Pezzo surface of rank* 1. *Then there is a finite unramified covering of $S - SingS$ which contains a Zariski-open subset isomorphic to $C \times \mathbf{A}^1$, where C is a smooth curve.*

It follows easily from the Lemmas 1.2 and 2.2 of this paper that if Miyanishi's conjecture is true then the Main Theorem of this paper is true. The Main Theorem thus lends a partial support to Miyanishi's conjecture.

Due to the length of the proof of the Main Theorem, this paper is being written in two parts. We will now give some indication of key ideas used in the proof.

Following an important idea of Miyanishi and Tsunoda, in §3 we use a "minimal" (-1) curve C on the minimal resolution of singularities, $\widetilde{S}$, of S. Using the assumption that $-K_S$ is ample we analyse the intersection behavior of the exceptional divisor D with C. The proof splits into two main cases according as the linear system $|K_{\widetilde{S}} + C + D|$ is empty or non-empty. The bulk of the paper goes into handling the first case. The first case itself splits into the "2-component" case and the "3-component" case. The part **II** of this paper deals exclusively with the "2"-component" case. It should be remarked that we can prove much more precise results about the intersection behavior of C and D than given in §6, but the Main Theorem stated above has been our main goal in this paper and so we have given only those details which are crucial for the proof (cf. the remark after the proof of Theorem 6.14). Several sub-cases from the "3-component" case are reduced to the "2-component" case. We could have given a self-contained proof for the "3-component" case, but this would have made the proof even more technical. As a consequence, the proof of the Main Theorem (even in the "3-component" case) is completed only in the part **II** of this paper.

The main ingradients in the proof of the Main Theorem are the following :

1) Several results of the paper [18]. The lemmas 1.5, 1.6 from [18] are frequently used.

2) A reduction to the case when the Picard group of S is infinite cyclic.

3) A somewhat precise information about the configuration of singular points when Pic $S \simeq \mathbf{Z}$.

4) A version of the Lefschetz hyperplance section theorem for fundamental groups given in [16].

5) A version of Van- kampen's theorem for non-connected intersections

due to P. Wagreich.

There are easy examples of normal projective rational surfaces over $\mathbf{C}$ with quotient singularities (even double points) and with numerically effective anti-canonical divisor, such that the fundamental group of the space of smooth points is infinite. See §1.15. This shows that the condition about the ampleness of $-K_S$ in the Main Theorem cannot be dropped.

From the Main theorem, we see easily that any log del Pezzo surface S is a quotient of a log del Pezzo surface T modulo a finite group such that the space of smooth points of T is simply-connected (the group acting freely outside a finite set of points of T).

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§1. Some easy results

In this section we fix the following notations and terminology which will be used throughout the paper.

Let S be a log del Pezzo surface as defined in the introduction. Denote by $S^o := S-$ (Sing S) the smooth part of S. Let $f : \tilde{S} \to S$ be a minimal resolution of singularities and denote by $D := f^{-1}(\text{Sing } S)$ the exceptional divisor. A divisor H on $\tilde{S}$ is *numerically effective* (nef, for short) if and only if $H \cdot \Delta \geq 0$ for any curve Δ on $\tilde{S}$. A nef divisor H is *big* if $(H^2) > 0$. By a $(-n)$-*curve* on $\tilde{S}$ we mean a nonsingular rational curve of self intersection $-n$.

K_S : canonical divisor of S.

$f^*(H)$: total transform of H by f.

$f'(H)$: proper transform of H by a birational morphism f.

$H_1 \sim H_2$: linear equivalence.

$H_1 \equiv H_2$: numerical equivalence.

$\#H$: the number of irreducible components of *Supp H*.

The dual graphs of minimal resolutions of quotient singularities are classified in [2].

Write $D = \sum_{i=1}^{n} D_i$ where D_i is irreducible. The first part of Lemma 1.1 below follows from the definition of a quotient singularity. The second part is trivial and the third part follows from the ampleness of $-K_S$ (cf. [9]).

Lemma 1.1. (1) *There exists a* $\mathbf{Q}$*-coefficient divisor* $D^* = \sum_{i=1}^{n} \alpha_i D_i$ *such that* $0 \leq \alpha_i < 1$ *and*

$$f^*(K_S) \equiv K_{\tilde{S}} + D^*.$$

Moreover, $\alpha_j = 0$ *i financial support andf and only if the connected component of* D *containing* D_j *is contracted to a rational double point on* S.

(2) *Let* p *be the smallest positive integer such that* pD^* *is an integral divisor. Then* pK_S *is a Cartier divisor and*

$$f^*(pK_S) \sim p(K_{\tilde{S}} + D^*).$$

(3) $-(K_{\tilde{S}} + D^*)$ *is a nef and big divisor. Moreover,* $-(K_{\tilde{S}} + D^*) \cdot B = 0$ *if and only if the support of* B *is contained in* D.

(4) *Suppose that* B *is an irreducible curve on* $\tilde{S}$ *with negative self intersection. Then either* B *is a* (-1)*-curve or* $B \leq D$.

Proof. (4) Suppose that B is not contained in D. Then $B.K_{\tilde{S}} < 0$ by (3). Now it follows from the genus formula that B is a (-1)-curve.

Lemma 1.2. *Let* T *be a normal projective surface with a finite morphism* $\varphi : T \to S$ *which is unramified over* S^o. *Then* T *is a log del Pezzo surface.*

Proof. If $T^o = T-$ (Sing T), then clearly $K_{T^o} \sim \varphi^*(K_{S^o})$. This implies that $K_T \sim \varphi^*(K_S)$. Since $-K_S$ is ample, $-K_T$ is also ample. Since φ is unramified over S^o, the local fundamental group of T at any point is finite. Hence T is a log del Pezzo surface.

Lemma 1.3. *A log del Pezzo surface is rational.*

Proof. Let $f : \tilde{S} \to S$ be as in the beginning of this section. Then for suitable large integer p, $-p(K_{\tilde{S}} + D^*)$ is a Cartier divisor linearly equivalent to a nonzero effective divisor Δ. Hence $|pK_{\tilde{S}}| = \phi$. Now $\tilde{S}$ is a ruled surface or $\mathbf{P}^2$.

Suppose $\widetilde{S}$ is a ruled surface with a morphism $\varphi : \widetilde{S} \to B$ onto a smooth projective curve B such that a general fiber of φ is $\mathbf{P}^1$.

First we consider the case where one of the irreducible components of D maps surjectively onto B under φ. In this case $B \approx \mathbf{P}^1$ and $\widetilde{S}$ is rational.

Now we assume that D is contained in a union of fibers of φ. We have an induced $\mathbf{P}^1$-fibration $\varphi' : S \to B$. Clearly, rank $S \geq 2$. We borrow part of the argument from the proof of Lemma 2.1. We argue by a suitable induction on rank S.
By Kawamata's contraction theorem, there is a contraction $\sigma : S \to Y$. If Y is a surface then rank $Y <$ rank S, Y is a log del Pezzo surface. By induction, Y and hence S is rational.
Suppose Y is a non-singular curve. If a "horizontal" irreducible curve for φ' is contracted to a point by σ, then that "horizontal" curve is a rational curve and hence B is rational. Assume now that an irreducible curve C contained in a fiber of φ' is contracted by σ. As C generates an extremal ray, we see that every fiber of φ' is irreducible and φ' is the contraction map.
Suppose Y is not a rational curve. Using a branched covering $Z \to Y$ as in the proof of Lemma 2.1 with suitable ramification divisor on Y and Lemma 1.2, we see that $S_1 = \overline{S \times_Y Z}$ is a log del Pezzo surface with a $\mathbf{P}^1$-fibration $\varphi'' : S_1 \to Z$. All the fibers of φ'' are reduced. Hence we can now assume that φ' itself has all the fibers reduced. Now $\widetilde{S}$ is obtained from a minimal ruled fibration $\psi : X \to B$ by a composition of blowing ups. Using the fact that all the fibers of φ' are reduced, we see that the contraction $\widetilde{S} \to X$ can be so chosen that we have an induced morphism $S \to X$. But then $S = X$. We have $K_S^2 = 8(1-g)$, where g is the genus of B. From the ampleness of K_S we know that $K_S^2 > 0$. Hence $g = 0$ and B is rational.
This completes the proof of Lemma 1.2.

Lemma 1.4. *$H_1(S^\circ, \mathbf{Z})$ is finite.*

Proof. By Lemma 1.3, $\widetilde{S}$ is rational and hence $H_1(\widetilde{S}, \mathbf{Z}) = 0$. We consider the long exact cohomology sequence of the pair $(\widetilde{S}, D)$ with integral coefficients :

$$H^2(\widetilde{S}) \to H^2(D) \to H^3(\widetilde{S}, D) \to H^3(\widetilde{S}) \to (0).$$

The irreducible components of D give linearly independent homology classes in $H_2(\tilde{S})$ as the intersection matrix of D is negative definite. From this we see that the cokernel of the map $H^2(\tilde{S}) \to H^2(D)$ is finite. By Poincaré duality, $H^3(\tilde{S}) \cong H_1(\tilde{S}) = (0)$. Now the result follows.

Lemma 1.5. *$\overline{\kappa}(S^o) = -\infty$, where $\overline{\kappa}$ is the logarithmic Kodaira dimension as defined by S. Iitaka* (cf. [8]).

Proof. Suppose $|n(K_{\tilde{S}} + D)| \neq \phi$ for some $n \geq 1$. Since $-n(K_{\tilde{S}} + D^*)$ is a nef and big Cartier divisor for some integer $n >> 0$, the complete linear system $|n(K_{\tilde{S}} + D) - n(K_{\tilde{S}} + D^*)|$ has dimension ≥ 1. This contradicts the negative definiteness of the intersection matrix of D.

Remark 1.6. If the Picard group of S has rank one, then M. Miyanishi has proved the converse of Lemma 1.5 viz. in this case, if $\overline{\kappa}(S^o) = -\infty$, then S is a log del Pezzo surface (cf. [18, Remark 1.2]). This result is false if the rank of Pic S is bigger than one (cf. Example in §1.15).

The next result is a very useful step in the proof of the Main Theorem of this paper.

Proposition 1.7. *The algebraic fundamental group of S^o is finite.*

Proof. We have to show that S^o does not have finite unramified covers of arbitrarily large degrees.

So, suppose that $\cdots \to S_n \to S_{n-1} \to \cdots \to S_0 := S$ is a sequence of finite Galois covers of S unramified over S^o. Let q be any singular point of S. Then all the points in S_n lying over q are conjugate to each other under the Galois group of S_n over S. The local fundamental groups of S_n at these points are then mutually isomorphic and isomorphic to a subgroup of the local fundamental group of S at q.

From this observation we see easily that the maps $S_n \to S_{n-1}$ are unramified for large n. By Lemma 1.2 each S_n is a log del Pezzo surface and hence rational by Lemma 1.3. But any normal projective rational surface is simply-connected. Thus $S_n \to S_{n-1}$ is an isomorphism for large n. This proves the result.

In the remaining part of this section we collect together some known results which will be used crucially in the proof of the theorem.

The following result is proved in [11, Chapter 1, §2.1.2].

Let X be a smooth projective rational surface and $\Delta = \Delta_1 + \cdots \Delta_r$ be a reduced divisor with irreducible components Δ_i.

Let m be the number of connected components of $\mathrm{Supp}(\Delta)$.

Let $e(\Delta) = m - r + \Sigma_{i<j}\Delta_i \cdot \Delta_j$, which is clearly a nonnegative integer.

Lemma 1.8. *dim* $H^0(X, O(K_X + \Delta)) = \sum_{i=1}^{r} p_a(\Delta_i) + e(\Delta)$. *Further,* $H^0(X, O(K_X + \Delta)) = 0$ *if and only if Supp* Δ *is a normal crossing divisor of nonsingular rational curves such that each connected component of Supp* Δ *is a tree.*

The following two results are proved by Madhav Nori in [16].

Lemma 1.9. *Let X be a nonsingular quasi-projective surface with a surjective morphism $\varphi : X \to B$, where B is a nonsingular curve. Assume that the general fiber F of φ is connected and each scheme-theoretic fiber of φ contains a reduced irreducible component. Then the following sequence is exact :*

$$\pi_1(F) \to \pi_1(X) \to \pi_1(B) \to (1).$$

The next result is a very useful version of Lefschetz hyperplane section theorem.

Lemma 1.10. *Let X be a nonsingular projective surface and Δ any effective divisor on X such that the Iitaka D-dimension $\kappa(X, \Delta) \geq 2$. Let $R \subset X$ be any proper Zariski closed subset. Then for any open neighbourhood U of Δ, the homomorphism $\pi_1(U - R) \to \pi_1(X - R)$ is surjective.*

Using Lemma 1.10, we will now give an easy proof of the special case of the Main Theorem when S is Gorenstein. This was proved earlier in [13] by first classifying such surfaces.

Proposition 1.11. *Let S be a Gorenstein log del Pezzo surface. Then $\pi_1(S^o)$ is abelian and finite.*

Proof. By Lemma 1.5, it is enough to prove that $\pi_1(S^o)$ is abelian. By [4, Theorem 1, p.39], there is a nonsingular elliptic curve $A \in |-K_S|$. Since $-K_S$ is ample and $K_{\widetilde{S}} = f^*(K_S)$, the Iitaka D-dimension $\kappa(\widetilde{S}, -K_{\widetilde{S}}) = 2$. Also A is disjoint from *sing* S as S has only rational double points. Now by Lemma 1.10, we have a surjective map $\mathbf{Z} \times \mathbf{Z} = \pi_1(A) \to \pi_1(\widetilde{S} - D)$. Thus $\pi_1(S^o)$ is abelian.

Remark 1.12. The proof shows that if $|-K_S|$ contains a member A which is a rational cuspidal curve disjoint from *sing* S then $\pi_1(S^o) = (1)$ because $\pi_1(A) = (1)$.

The next result follows from the well-known result of Mumford giving the presentation of the fundamental group of the boundary of a nice tubular neighborhood of a tree of non-singular rational curves on a smooth complex surface. (cf. [14]).

Lemma 1.13 *Let X be a non-singular projective surface and Δ a connected normal crossing divisor on X with all the irreducible components non-singular rational curves. Assume one of the following two conditions :*

(1) The dual graph of Δ is linear and Δ supports a divisor with positive self-intersection.

(2) The dual graph of Δ has exactly one branch point and the three linear branches T_1, T_2, T_3 are such that :

(i) Δ supports a divisor with positive self-intersection

(ii) the intersection form on $T_1 + T_2 + T_3$ is negative definite and $1/d_1 + 1/d_2 + 1/d_3 > 1$, where d_i is the absolute value of the determinant of the intersection matrix of T_i.

If U is a "nice" tubular neighborhood of Δ in X, then $\pi_1(U - \Delta)$ is finite.

We will need the following generalization of the Van- Kampen theorem proved by P. Wagreich (cf. [17,Prop. 2.1]).

Lemma 1.14 *Suppose A is a connected simplicial complex with connected subcomplexes A_0, A_1 such that $A = A_0 \cup A_1, A_0 \cap A_1 = B_0 \cup B_1$ where, B_i is a connected subcomplex of A_j for all i, j and $B_0 \cap B_1 = \phi$. Let $\varphi_{ij} : \pi_1(B_i) \to \pi_1(A_j)$ be the map induced by the inclusion. Then $\pi_1(A)$ is* *isomorphic to* $\pi_1(A_0) * \pi_1(A_1) * \mathbf{Z}(u)/G$ *where,* $\mathbf{Z}(u)$ *denotes the free group with one generator* u *and* G *is the normal subgroup generated by the relations* :

$$\varphi_{0,0}(b) = \varphi_{0,1}(b)$$

for all $b \in \pi_1(B_0)$

$$\varphi_{1,0}(b) = u^{-1}\varphi_{1,1}(b)u$$

for all $b \in \pi_1(B_1)$.

§1.15. An example

Let σ be an involution which acts diagonally and non-trivially on a product $\mathbf{P}^1 \times E$, where E is an elliptic curve and let T be the quotient of this product modulo σ. Then T has only singularities of type A_1. The quotient morphism g is unramified over the smooth part of T. It follows that the fundamental group of $T - SingT$ is infinite. Clearly, $g^*(K_T) \simeq K_{\mathbf{P}^1 \times E}$. Hence we see by projection formula that $-K_T$ is numerically effective but not big. It is easy to see that T is a rational surface and *Pic* T has rank > 1.

§2. Reduction to the rank one case

In this section, using Kawamata's contraction theorem, we will show that it is enough to prove the main theorem when Pic $S \cong \mathbf{Z}$. (Note that since S is simply-connected, Pic S is isomorphic to $\mathbf{Z}$ if the rank of Pic S is one.)

Suppose rank Pic $S \geq 2$. Since K_S is not nef, there is a contraction $\varphi : S \to Y$ of an extremal ray by [10, Theorem 3.2.1]. (Note that a two-dimensional quotient singularity is nothing but a log-terminal singularity (cf. [9].) We have two cases :

Case 1. Y is a surface.

In this case φ is birational and the exceptional divisor of φ is an irreducible curve Δ (cf. [10, Prop. 5.1.6]).

Lemma 2.1. *Y is a log del Pezzo surface.*

Proof. From [9] we know that S has at worst log-terminal singularities. The proof of the Contraction Theorem shows that Y also has log-terminal singularities, hence quotient singularities.

Clearly, $\varphi_*(K_S) = K_Y$. We can write

$$K_S = \varphi^*(K_Y) + a\Delta \, for\, some\, a \in \mathbf{Q}.$$

We have $K_S \cdot \Delta = a\Delta \cdot \Delta < 0$ and $\Delta \cdot \Delta < 0$ hence $a > 0$. Let $Z \in \overline{NE}(Y) - \{0\}$. By projection formula,

$$(-K_Y) \cdot Z = -(\varphi^* K_Y) \cdot \varphi^*(Z) = -K_S \cdot \varphi^*(Z) + a\Delta \cdot \varphi^*(Z) = -K_S \cdot \varphi^*(Z) > 0$$

by the ampleness of $-K_S$. Now by Kleiman's criterion of ampleness, $-K_Y$ is ample.

This proves that Y is a log del Pezzo surface.

Now $S - \Delta \cong Y - \varphi(\Delta)$, hence $Y^o -$ {one smooth point} is a Zariski-open subset of S^o. This implies that we have a surjection $\pi_1(Y^o) \to \pi_1(S^o)$. On the other hand, rank Pic $Y <$ rank Pic S.

Case 2. Y is a smooth projective curve.

In this case by Lemma 1.3, $Y \cong \mathbf{P}^1$. We claim that a general fiber of φ is isomorphic to $\mathbf{P}^1$. For, if F is a general fiber of φ, then $-K_S \cdot F > 0$ and $F \cdot F = 0$. Now by adjunction formula we see that $F \cong \mathbf{P}^1$.

By restriction, we get a surjective morphism $S^0 \to \mathbf{P}^1$ whose general fibers are $\mathbf{P}^1$. For any scheme-theoretic fiber F of $S^o \to \mathbf{P}^1$, the *g.c.d.* of the multiplicities of the irreducible components of F is called *the multiplicity of* F.

Let $F_1, F_2, \cdots, F_r$ be all the multiple fibers of $S^o \to \mathbf{P}^1$ with multiplicities $m_1, m_2, \cdots, m_r$ bigger than one.

Suppose first $r \geq 3$. Then by the solution of Fenchel's conjecture due to R. Fox, there is a finite Galois morphism $B \to Y$ such that for any point in B lying over $\varphi(F_i)$, the ramification index is m_i (cf. [5]). By usual arguments, the normalization T^o of $S^o \times_Y B$ in its function field is an etale covering of S^o. The normalization of S in the function field of T^o is therefore a log del Pezzo surface T by Lemma 1.2. By Lemma 1.3, T is rational and hence $B \cong \mathbf{P}^1$. The morphism $T^o \to B$ has no fibers of multiplicity > 1 and has

$\mathbf{P}^1$ as a general fiber. Then the proof of Lemma 1.9 shows that we have a surjection $\pi_1(F) \to \pi_1(T^o)$ for a general fiber F of $T^o \to B$.

Hence T^o is simply connected and hence $\pi_1(S^o)$ is finite.

Suppose $r \leq 2$.

If $r = 2$, let F_1, F_2 be the multiple fibers with multiplicities m_1, m_2 and $d = g.c.d.(m_1, m_2)$. we consider the cyclic d–fold covering $B \to Y$ ramified precisely over $\varphi(F_1)$ and $\varphi(F_2)$ with ramification index d. Then we work with $S^o \times_Y B$ exactly as above and complete the proof.

The case $r = 1$ is also easy.

This completes the proof of the Main Theorem when there is a Kawamata contraction of fiber type.

For future use, we state the following result whose proof is completely similar to the proof in Case 2 above.

Lemma 2.2. *Let Y be a log del Pezzo surface with a morphism $\varphi : Y \to \mathbf{P}^1$. Assume that a general fiber of φ is isomorphic to either $\mathbf{C}$ or $\mathbf{C}^*$. Then $\pi_1(Y - SingY)$ is finite.*

Combining the arguments in Cases 1 and 2, by a repeated application of contractions of extremal rays we reduce the proof of the main theorem to the case when Pic $S \cong \mathbf{Z}$.

§3. Some analysis of the rank one case

In this section we give a somewhat detailed description of rank one log del Pezzo surfaces.

So let S be a log del Pezzo surface of rank one. We use the notation introduced in the beginning of §1. Let p be the smallest positive integer such that pD^* is an integral divisor. Then for every curve B on $\tilde{S}$ not contained in D, $-(K_{\tilde{S}} + D^*) \cdot B \in \frac{1}{p}\mathbf{N} = \{n/p | n \in \mathbf{N}\}$ (cf. Lemma 1.1). From this we obtain the following :

Definition and Lemma 3.1. *(1) There exists an irreducible curve C on $\tilde{S}$ such that $-(K_{\tilde{S}} + D^*) \cdot C$ attains the smallest positive value. Such a curve satisfies $C^2 \geq -1$ (cf. Lemma 1.1,(4)).*
(2) A curve C as in (1) above is called minimal.

For the time being, we fix the curve C of Lemma 3.1. We shall treat the two cases $|K_{\widetilde{S}} + C + D| \neq \phi, = \phi$ separately.

§3.1. The case $|K_{\widetilde{S}} + C + D| \neq \phi$

In this subsection, we always assume $|K_{\widetilde{S}} + C + D| \neq \phi$.

Lemma 3.2 (cf. [18, Lemma 2.1]). *Let C be as in Lemma 3.1. Suppose $|C + D + K_{\widetilde{S}}| \neq \phi$. Then there exists a unique decomposition $D = D' + D''$ such that :*

(1) $K_{\widetilde{S}} + C + D" \sim 0$,

(2) *D' is disjoint from $C \cup D''$ and consists of (-2)-curves; hence D' is contracted to rational double points on S.*

Remark 3.3. Write $\widetilde{C} := f^*f_*(C) = C + G$. As Pic $S \cong \mathbf{Z}$, $C + G$ is a nef and big divisor and G is an effective divisor with support contained in D''. In particular, the Iitaka D-dimension $\kappa(\widetilde{S}, C + D'') = 2$.

Remark 3.4. We can divide the case $|C + D + K_{\widetilde{S}}| \neq \phi$ into the following subcases :

Case (I-1) $D" = 0$. Then S is a log del Pezzo surface with only rational double points. By Proposition 1.11, $\pi_1(S^o)$ is finite abelian.

In the following subcases, assume that $D'' \neq 0$. Now from $K_{\widetilde{S}}+C+D'' \sim 0$ and from Lemma 1.8 we see that each irreducible component of $C \cup D''$ is isomorphic to $\mathbf{P}^1$, e.g. $K_{\widetilde{S}} + C \sim -D''$ implies that $|K_{\widetilde{S}} + C| = \phi$, etc.

Case (I-2) $D'' \neq 0$ and $C + D$ is a divisor with only simple normal crossings. By Lemmas 1.8 and 3.2, there is a loop Δ of nonsingular rational curves contained in $C + D''$ and we have $|K_{\widetilde{S}} + \Delta| \neq \phi$. Now $K_{\widetilde{S}} + C + D'' \sim 0$ implies that $\Delta = C + D''$ and $C + D''$ is a simple loop of nonsingular rational curves, i.e., each irreducible curve in $C + D''$ meets exactly two other irreducible components of $C + D''$.

Case (I-3) $D'' \neq 0$ and $(C^2) \geq 0$. This case can be reduced to the case(I-2) above by replacing C with a new irreducible curve linearly equivalent to C. Indeed, by the Riemann-Roch theorem, the Serre duality and the genus formula, we have :

$$dim|C| = h^1(\tilde{S}, \mathcal{O}(C)) + \frac{1}{2}(C, C - K_{\tilde{S}}) \geq 1.$$

Then $|C|$ has no base points (as $C \cong \mathbf{P}^1$). Choose $C' \in |C|$ such that $C' + D$ has only simple normal crossings. Then $-(K_{\tilde{S}} + D^*) \cdot C'$ attains the smallest positive value and $|K_{\tilde{S}} + C' + D| \neq \phi$.

Case (I-4) $D'' \neq 0, (C^2) \leq -1$ and $C + D$ is not a divisor with only simple normal crossings. Then C is a (-1)-curve by Lemma 3.1 and the arguments as in Case (I-2) shows that one of the following two cases occurs.

Case (I-4a) D'' is an irreducible curve such that $C \cdot D'' = 2$ and $C \cap D''$ is a single point. By Remark 3.3, the intersection matrix of $C + D''$ has one positive eigenvalue and hence $(D'')^2 = -2$ or -3.

Case (I-4b) D'' consists of two irreducible components D''_1, D''_2 such that $C \cdot D''_1 = C \cdot D''_2 = 1$ and $C \cap D''_1 \cap D''_2$ consists of a single point. By the same reasoning as in Case (I-4a), we have $((D''_1)^2, (D''_2)^2) = (-2, -2), (-2, -3), (-2, -4)$ after interchanging the subscripts of D''_i, if necessary.

§3.2. The Case $|K_{\tilde{S}} + C + D| = \phi$

In this section we always assume that $|K_{\tilde{S}} + C + D| = \phi$. First of all, by Lemma 1.8, we have the following :

Lemma 3.5. *$C + D$ has only simple normal crossings, consists of non-singular rational curves and has a disjoint union of trees as the dual graph.*

We need the following results from [18].

Proposition 3.6 (cf. the proof of [18, Lemma 2.2]). *Let C be as in Lemma 3.1. Suppose $|C + D + K_{\tilde{S}}| = \phi$. Then either C is a (-1)-curve or S*

is the Hirzebruch surface with the minimal section contracted. In the latter case, S^o is simply connected.

From now on till the end of the present section, we assume always that C is a (-1)-curve.

Lemma 3.7 (cf. [18, Lemma 4.1]). *Let $D_1, \cdots, D_r$ exhaust all irreducible components of D with $(C, D_i) > 0$. Suppose $(D_1^2) \geq \cdots \geq (D_r^2)$. Then $\{-(D_1^2), \cdots, -(D_r^2)\}$ is one of the following :*

$$\{2^a, n\}(n \geq 2), \{2^a, 3, 3\}, \{2^a, 3, 4\}, \{2^a, 3, 5\}$$

where 2^a signifies that 2 is repeated a-times.

Lemma 3.8 (1) *Suppose C meets exactly one irreducible component D_0 of D. Then $(D_0^2) = -2$.*

(2) *C meets at least one component of D.*

Proof. (1) Suppose $(D_0^2) \leq -3$. Then $C + D$ is contractible to quotient singularities. This leads to $1+\#(D) = \rho(\tilde{S}) \geq 1+\#(C+D)$, a contradiction.

(2) can be similarly verified.

Lemma 3.9 (cf. [18, Lemma 4.4]). *Suppose C meets exactly two irreducible components D_0, D_1 of D. Then $(D_i^2) = -2$ for $i = 0$ or 1.*

Lemma 3.10 (cf. [18, Lemma 4.3]). *Assume that one of the following cases takes place :*

(1) C meets only one irreducible component D_0 of D.

(2) C meets exactly two irreducible components D_0, D_1 of D with $(D_1^2) \leq -3$.

Let $\sigma : \tilde{S} \to \tilde{T}$ be the blowing-down of the (-1)-curve C, let $E = \sigma(D_0)$ and let $B = \sigma(D - D_0)$. Then there exists a log del Pezzo surface T of Picard number one and there exists a birational morphism $g : \tilde{T} \to T$ such that g is a minimal resolution and $B = g^{-1}(Sing T)$.

Remark 3.11. Let $D_1, \cdots, D_r$ be all irreducible components of D with $(C, D_i) > 0$ (hence $(C, D_i) = 1$ by Lemma 3.5). Suppose $(D_1^2) \geq \cdots \geq (D_r^2)$.

By virtue of Lemmas 3.5, 3.7, 3.8 and 3.9, in the case where C is a (-1)-curve, we can divide into the following cases :

Case (II-1) $r \geq 2$ and $(D_1^2) = (D_2^2) = -2$.

Case (II-2) $r = 1$ and $(D_1^2) = -2$.

Case (II-3) $r = 3$ and $\{(D_1^2), (D_2^2), (D_3^2)\} = \{-2, -3, -3\}, \{-2, -3, -4\}$ or $\{-2, -3, -5\}$.

Case (II-4) $r = 2$ and $(D_1^2) = -2, (D_2^2) \leq -3$.

We shall consider these cases separately in §5, §6 and part **II.**
As remarked in the Introduction, Cases (II-3), (II-4) are "3-component case" and "2-component case" respectively.

In §6 and **II** we shall be tacitly using the following useful result very often for estimating the coefficients of irreducible components in D^*. (For proof, cf. [18, Lemma 1.7]). Write $D = \Sigma_{i=1}^n D_i$.

Lemma 3.12 *Let $\{B_1, \cdot\cdot, B_r\}(1 \leq r \leq n)$ be a subset of $\{D_1, \cdot\cdot D_n\}$, say $B_i = D_i (1 \leq i \leq r)$. Assign formally the numbers B_i^2 to B_i so that $D_i^2 \leq B_i^2 \leq -2$ and $B_i \cdot K_{\widetilde{S}} := -2 - B_i^2$. Write $D^* = \Sigma_{i=1}^n alpha_i D_i$. Define rational numbers $b_1, \cdots, b_r$ by the conditions*

$$B_j \cdot (K_{\widetilde{S}} + \Sigma_{i=1}^r b_i B_i) = 0 \ (j = 1, \cdots, r),$$

where $B_i \cdot B_j := D_i \cdot D_j$ if $i \neq j$.

Then $\alpha_i \geq b_i \geq 0 \, (i = 1, \cdots, r)$. Taking $r = 1$, we obtain $\alpha_i \geq 1 + 2/D_i^2$.

§4. The proof of the Main theorem when $|K_{\widetilde{S}} + C + D| \neq \phi$

In this section we prove the Main Theorem stated in the introduction under assumption that $|K_{\widetilde{S}} + C + D| \neq \phi$.

By the discussion in Remark 3.4, we need only to consider the cases (I-2), (I-4a) and (I-4b).

First we dispose of the cases (I-4a) and (I-4b).

Consider the case (I-4a). By two blowing-ups we get a smooth projective surface X with a morphism $g : X \to \widetilde{S}$ such that the total transform B of

$C \cup D$" is a divisor with simple normal crossings and has four irreducible components with a (-1)-curve B_0 meeting the three other components B_1, B_2, B_3. Further, $B_1^2 = -2, B_2^2 = -3, B_3^2 = -4$ or -5.

Let U be a small nice "tubular" neighbourhood of $B_0 \cup B_1 \cup B_2 \cup B_3$ in X. Then Mumford's result in [14] shows that $\pi_1(\partial U)$ has the following presentation :

$$< e_0, e_1, e_2, e_3 | e_1^2 = e_2^3 = e_3^\ell = e_0, e_1 e_2 e_3 = e_0 >,$$

where $\ell = 4$ or 5.

It is well known that $\pi_1(\partial U)$ is then a finite group. On the other hand, the intersection matrix of $B_0 + B_1 + B_2 + B_3$ has one positive eigenvalue. Hence by Lemma 1.10, we have a surjection

$$\pi_1(U - B_0 \cup B_1 \cup B_2 \cup B_3) \to \pi_1(X - g^{-1}(C \cup D)).$$

We have also a surjection $\pi_1(\tilde{S} - C \cup D) \to \pi_1(\tilde{S} - D)$. Now it follows from $\pi_1(X - g^{-1}(C \cup D)) \cong \pi_1(\tilde{S} - C \cup D)$ that $\pi_1(\tilde{S} - D)$ is finite.

The proof for the case (I-4b) is completely similar.

Now we consider the case (I-2). Then $C + D''$ has simple normal crossings, the dual graph of $C + D''$ is a simple loop of smooth rational curves and $D' = D - D''$ is disjoint from $C + D''$.

Let U_0 be a small nice tubular neighbourhood of D'' in $\tilde{S}$ and U_1 that of C in $\tilde{S}$.

We can write $(U_0 - D) \cap (U_1 - D)$ as a disjoint union $N_0 \cup N_1$, where each N_i is homeomorphic to $\Delta^* \times \Delta$, where

$$\Delta = \{z \in \mathbf{C} | |z| < 1\} \; \Delta^* = \Delta - \{0\}.$$

By Mumford's presentation for $\pi_1(\partial U_0)$, we see immediately that "the" loop γ_1 in N_0 around D'' generates $\pi_1(\partial U_0)$. Similarly, the loop γ_2 in N_1 around D'' generates $\pi_1(\partial U_0)$. We can assume that γ_1 is a small loop in C around one point in $C \cap D''$ and γ_2 a small loop in C around the other point in $C \cap D''$. In $\pi_1(C - D)$, we have $\gamma_1 \cdot \gamma_2 = 1$. Further, $\pi_1(U_1 - D) \cong \mathbf{Z}$ generated by γ_1.

Now we use Lemma 1.14.

We apply this to the space $A = (U_0 - D) \cup (U_1 - D)$ with $A_0 = U_0 - D, A_1 = U_1 - D$. Since D" is contracted to a quotient singularity on S, $\pi_1(U_0 - D)$ is a cyclic finite group. Then $\pi_1(U_0 \cup U_1 - D)$ has the presentation :

$$\pi_1(U_0 - D) * \pi_1(U_1 - D) * \mathbf{Z}(u)$$

with relations

$$\gamma_1 = g_0, g_0^n = u^{-1}\gamma_1^{-1}u$$

where g_0 is the generator of $\pi_1(U_0 - D)$ coming from γ_1 and g_0^n the generator of $\pi_1(U_0 - D)$ coming from γ_2.

It follows that we have an exact sequence

$$(1) \to \pi_1(U_0 - D) \to \pi_1(U_0 \cup U_1 - D) \to \mathbf{Z} \to (1).$$

The intersection matrix of $C + D$ has one positive eigenvalue. Hence by Lemma 1.10, we have a surjection

$$\pi_1(U_0 \cup U_1 - D) \to \pi_1(S^o).$$

Let K be the kernel of this homomorphism. Then we get an isomorphism

$$\pi_1(U_0 \cup U_1 - D)/(\pi_1(U_0 - D) \cdot K) \cong \mathbf{Z}/(a)$$

for some $a \geq 0$, i.e.,

$$\pi_1(S^o)/(\pi_1(U_0 - D) \cdot K/K) \cong \mathbf{Z}/(a)$$

The group $(\pi_1(U_0 - D) \cdot K/K)$ is clearly finite.

Now by Proposition 1.7, $\pi_1(S^o)$ does not have normal subgroups of arbitrarily large indices. It follows that $\pi_1(S^o)$ is finite.

§5. The proof of the main theorem in the case (II-1) and (II-2)

We consider the case (II-1) or (II-2) in Remark 3.11. We shall employ the notation there. First of all, we have the following Theorem 5.1 which is the consequence of §4 , Theorem 5.2 below and Lemma 2.2.

Theorem 5.1. *Assume the case (II-1) of Remark 3.11 takes place. Then $\pi_1(S^o)$ is a finite group.*

Theorem 5.2 (cf. [18, Theorem 5.1]). *Assume the case (II-1) of Remark 3.11 takes place. Then one of the following cases occurs :*

(1) *S^o is affine-ruled.*

(2) There is an irreducible curve C' such that $-C' \cdot (K_{\tilde{S}} + D^*) = -C \cdot (K_{\tilde{S}} + D^*)$ while $|C' + D + K_{\tilde{S}}| \neq \phi$.

(3) *$C + D$ has the configuration given in [18, Picture 10]. In particular, there exists a $\mathbf{P}^1$-fibration $\varphi = \Phi_{|2C+D_1+D_2|} : \tilde{S} \to \mathbf{P}^1$ and there are two irreducible components D_3, D_4 of D such that $D - D_3 - D_4$ are contained in fibers and D_3 and D_4 are cross-sections. Hence the restriction morphism $\varphi_{|S^o} : S^o \to \mathbf{P}^1$ is a $\mathbf{C}^*$-fibration.*

Next we consider the case(II-2) of Remark 3.11. We employ the following notations. Let Δ be the connected component of D containing D_1. Then either Δ is a linear chain, or a fork with a central component R and three twigs T_i's, i.e., $\Delta = R + T_1 + T_2 + T_3$.

Remark 5.3. Denote by $d_i = d(T_i)$ the absolute value of the determinant of the intersection matrix of T_i. Suppose $d_1 \leq d_2 \leq d_3$. Then $\{d_1, d_2, d_3\}$ is one of the following : $\{2,2,n\}, \{2,3,3\}, \{2,3,4\}, \{2,3,5\}$. In particular, $\sum \frac{1}{d_i} > 1$.

Now we shall prove the following Theorem 5.4.

Theorem 5.4. *Assume that the case(II-2) of Remark 3.11 takes place. Then $\pi_1(S^o)$ is a finite group.*

Proof. Let Δ be the connected component of D such that C meets the irreducible component D_o of Δ. Let U be a small tubular neighborhood of $C \cup \Delta$ in $\tilde{S}$. A small loop γ around D_o can be taken to be in C around the point $C \cap D_o$. As S has rank 1, $C + \Delta$ supports an effective divisor with strictly positive self-intersection. By Lemma 1.10 we have a surjection $\pi_1(U - \Delta) \to \pi_1(S^o)$. We can write U as a union $U_1 \cup U_2$, where U_1 is a small tubular neighborhood of Δ and U_2 that of C. Then $U - \Delta = (U_1 - \Delta) \cup (U_2 - D_o)$ and $(U_1 - \Delta) \cap (U_2 - D_o)$ is homeomorphic to $B^* \times B$, where $B = \{z \in \mathbf{C} | |z| < 1\}$ and $B^* = B - \{0\}$. Since $U_2 - D_o$ is a disc bundle over $\mathbf{A}^1$, we see by Van-Kampen theorem that $\pi_1(U - \Delta)$ is a homomorphic image of $\pi_1(U_1 - \Delta)$.

As Δ contracts to a quotient singularity, $\pi_1(U_1 - \Delta)$ is finite and hence so is $\pi_1(S^o)$.

§6. The proof of the main theorem in the case (II-3)

In the present section, we consider the case (II-3) in Remark 3.11. So, the (-1)-curve C meets exactly three irreducible components D_1, D_2, D_3 of D and $D_1^2 = -2, D_2^2 = -3, D_3^2 = -3, -4, -5$. Let Δ_i $(i = 1,2,3)$ be the connected component of D containing D_i. Since $|K + C + D| = \phi$ in our case, $C + \Delta_1 + \Delta_2 + \Delta_3$ is a tree of $\mathbf{P}^1$'s (cf. Lemma 3.5).

We shall prove the following Theorem 6.1 which is a consequence of Lemma 6.5, Theorems 6.12, 6.14 and 6.15.

Theorem 6.1. *Suppose that the case(II-3) in Remark 3.11 occurs. Then either $\pi_1(S^o)$ is finite or there is a minimal (-1)-curve E on $\tilde{S}$ such that Case (II-4) in Remark 3.11, with C replaced by E, takes place.*

First of all, we quote the following lemma from [18 , Lemma 2.3]).

Lemma 6.2. *Suppose the case(II-3) occurs. Then either $G(:= K_{\tilde{S}} + 2C + D_1 + D_2 + D_3) \sim 0$, or there exists a (-1)-curve Γ such that $G \sim \Gamma$ and $\Gamma \cap (C + D_1 + D_2 + D_2) = \phi$.*

Lemma 6.3. *Suppose $K_{\tilde{S}} + 2C + D_1 + D_2 + D_3 \sim 0$. Then D_i is an isolated irreducible component of D for $i = 1,2$ and 3.*

Proof. Suppose to the contrary that D_i is not an isolated irreducible component of D for some i. Then D_i meets an irreducible component B_i of $D - D_i$. This leads to $0 = B_i.(K_{\tilde{S}} + 2C + D_1 + D_2 + D_3) \geq B_i.D_i > 0$, a contradiction.

Remark 6.4. In fact, the converse of Lemma 6.3 is also true. Namely, assume that D_i is isolated for $i = 1,2$ and 3. Then $G(:= K_{\tilde{S}} + 2C + D_1 + D_2 + D_3) \sim 0$.

Lemma 6.5. *Suppose that for $i = 1,2$ and 3, D_i is an isolated irreducible component of D, i.e., $\Delta_i = D_i$. Then $\pi_1(S^o)$ is a finite group.*

Proof. We use $D_1^2 = -2, D_2^2 = -3, D_3^2 = -3, -4$ or -5 and Lemmas 1.10 and 1.13.

In view of Lemma 6.5, we may assume, from now on till the end of the section, that D_i is not an isolated irreducible component for $i = 1, 2$ or 3. Therefore, $K_{\tilde{S}} + 2C + D_1 + D_2 + D_3 \sim \Gamma$ by Lemma 6.3.

Lemma 6.6. (1) *There are no $(-n)$-curves in $D - D_2 - D_3$ with $n \geq 4$ and there are at most two (-3)-curves in $D - D_2 - D_3$.*

(2) *Each connected component of D contains at most one $(-n)$-curve with $n \geq 3$. In particular, $\Delta_i - D_i$ consists of (-2)-curves for $i = 2$ and 3, and Δ_1 consists of (-2)-curves and possibly one (-3)-curve.*

Proof. (1) Let B_i $(i = 1, \cdots, s)$ be all $(-n_i)$-curves in $D - D_1 - D_2 - D_3$ with $n_i \geq 3$. Note that $D^* \geq \sum_i (n_i - 2)/n_i B_i$, and by Lemma 1.1, $0 < -\Gamma.(K + D^*) = 1 - \Gamma.D^* \leq 1 - \sum_i (n_i - 2)/n_i \Gamma.B_i \leq 1 - \sum_i (n_i - 2)/n_i K_{\tilde{S}}.B_i$ (cf. Lemma 6.2) $= 1 - \sum_i (n_i - 2)^2/n_i$. Then the assertion (1) follows from this observation.

(2) Let Δ be a connected component of D. Suppose to the contrary that Δ contains two irreducible components of self intersection number ≤ -3. Take a linear chain $G = G_1 + \cdots + G_t$ $(t \geq 2)$ in Δ such that $G_1^2 \leq -3$, $G_t^2 \leq -3$, $G_i.G_{i+1} = 1$ $(i = 1, \cdots, t-1)$. Then $D^* \geq 1/2 \sum_i G_i$. Note that $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - 1/2 \sum_i \Gamma.G_i$. So, $\Gamma.\sum_i G_i = 0, 1$.

If $\Delta \neq \Delta_i$ for $i = 2$ and 3, then for $k = 1$ and t we have $\Gamma.G_k \geq K_{\tilde{S}}.G_k$ (cf. Lemma 6.2) ≥ 1. So, $\Gamma.\sum_i G_i \geq 2$. This is a contradiction.

Suppose $\Delta = \Delta_i$ for $i = 2$ or 3. We may assume that $G_1 = D_i$. If $t = 2$, then $\Gamma.G_2 \geq (K_{\tilde{S}} + D_i).G_2 \geq 2$, a contradiction. If $t \geq 3$, then $\Gamma.(G_2 + G_t) \geq (K_{\tilde{S}} + D_i).(G_2 + G_t) \geq 2$, a contradiction.

Now the following lemma follows from Lemmas 6.2 and 6.6.

Lemma 6.7. *Let B be an irreducible component of D. Then $B.\Gamma > 0$ if and only if one of the following cases occurs :*

(1) $B.D_i = 1$ for $i = 1, 2$ or 3, $B^2 = -2$ and $B.\Gamma = 1$.

(2) $B.D_1 = 1, B^2 = -3$ and $B.\Gamma = 2$.

(3) $B \leq \Delta_1, B.D_1 = 0, B^2 = -3$ and $B.\Gamma = 1$.

(4) B is contained in a connected component Δ of D other than Δ_i ($i = 1, 2, 3$), $B^2 = -3$, $\Delta - B = 0$ or consists of only (-2)-curves, and $B.\Gamma = 1$.

Lemma 6.8. $K_{\tilde{S}}^2 = 2 + D_3^2$.

Proof. Use Lemma 6.2.

Lemma 6.9. (1) *For* $i = 2$ *or* 3, Δ_i *is a linear chain with* D_i *as a tip.*

(2) *Suppose that* $D_3^2 \leq -4$. *Then for both* $i = 2$ *and* 3, Δ_i *is a linear chain with* D_i *as a tip.*

(3) *Suppose that* Δ_i *is a fork for* $i = 2$ *or* 3. *Then* D_i *is a tip.*

(4) *If* Δ_1 *is a fork, then* Δ_1 *consists of* (-2)*-curves.*

Proof. (1) Suppose to the contrary that for both $i = 2$ and 3, either Δ_i is a fork or Δ_i is a linear chain but D_i is not a tip. Then $D^* \geq 1/2D_2 + 1/2D_3$. This leads to $0 < -C.(K + D^*) \leq 1 - C.(1/2D_2 + 1/2D_3) = 0$, a contradiction.

(2) Assume $D_3^2 \leq -4$. Suppose that (2) is not true for $i = 2$ (resp. $i = 3$). Then $D^* \geq 1/2D_2 + 1/2D_3$ (resp. $D^* \geq 1/3D_2 + 2/3D_3$). We reach a contradiction as in (1). So, (2) is true.

(3) Suppose that Δ_i is a fork but D_i is not a tip for $i = 2$ or 3. Then $D^* \geq 2/3D_i + 1/3D_j$ where $\{i, j\} = \{2, 3\}$ as sets. We reach a contradiction as in (1).

(4) Suppose that Δ_1 is fork but does not consist of (-2)-curves. Then Δ_1 contains a (-3)-curve B and $\Delta_1 - B$ consists of (-2)-curves (cf. Lemma 6.6). Note that $D^* \geq 1/2B$.

Case(1) B is adjacent to D_1. Then $\Gamma.B = 2$ by Lemma 6.7. This leads to $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(1/2B) \leq 0$, a contradiction. So, B is not adjacent to D_1.

Let $B_1, B_2, \cdots, B_s$ be all irreducible components of Δ_1 adjacent to D_1. Then $\Gamma.B_i = \Gamma.B = 1$ by Lemma 6.7.

Case(2) D_1 is the central component. Then $s = 3$ and $D^* \geq 1/2B + 1/2D_1 + 1/2B_i + 1/4B_j + 1/4B_k$, where $\{i, j, k\} = \{1, 2, 3\}$ as sets and B_i and B are contained in one and the same twig of Δ_1. This leads to $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(1/2B + 1/2B_i + 1/4B_j + 1/4B_k) = -1/2$, a contradiction. So, D_1 is not the central component of Δ_1.

Case(3) B is the central component of Δ_1. So, D_1 is contained in a twig T_1 of Δ_1. Let $G = G_1 + \cdots + G_t$ $(t \geq 2)$ be a linear chain in $T_1 + B$ such that

$G_1 = D_1, G_i.G_{i+1} = 1 (i = 1, \cdots, t-1), G_t = B$. Then $D^* \geq \sum_{i=1}^{t} i/(t+1)G_i$. Note that $G_2 = B_j$ for $j = 1, \cdots, s-1$ or s. This leads to $0 < -\Gamma.(K_{\tilde{S}}+D^*) \leq 1 - \Gamma.(2/(t+1)B_j + t/(t+1)B) = -1/(t+1)$, a contradiction. So, B is not the central component of Δ_1.

Since the above cases (2) and (3) are impossible, D_1 and B are all contained in twigs of Δ_1. We shall see in the cases (4) and (5) below that this again leads to a contradiction.

Case(4) D_1 and B are in one and the same twig of Δ_1. Let $G = G_1 + \cdots + G_t$ $(t \geq 2)$ be a linear chain in the twig such that $G_1 = D_1, G_i.G_{i+1} = 1 (i = 1, \cdots, t-1), G_t = B$. Note that $G_2 = B_j$ for $j = 1, \cdots, s-1$ or s. If the distance from D_1 to the central component of Δ_1 is shorter than that from B to the central component, then $D^* \geq \sum_i 1/2G_i$. This leads to $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(1/2B + 1/2B_j) = 0$, a contradiction. If the distance from D_1 to the central component of Δ_1 is longer than that from B to the central component, then $D^* \geq \sum_i i/(t+1)G_i$. This leads to $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(2/(t+1)B_j + t/(t+1)B) = -1/(t+1)$, a contradiction. So, Case(4) is impossible.

Case(5) D_1 and B are contained in two different twigs T_1, T_2 of Δ_1. Let R be the central component of Δ_1. Let $G = G_1 + \cdots + G_t$ $(t \geq 2)$ be a linear chain in $T_1 + R$ such that $G_1 = D_1, G_i.G_{i+1} = 1 (i = 1, \cdots, t-1), G_t = R$. Then $G_2 = B_j$ for $j = 1, \cdots, s-1$ or s.

Case(5.a) T_2 has more than two irreducible components. Then $T_1 = D_1$ because Δ_1 is contractible to a quotient singularity, and $R = B_j$ for $j = 1, \cdots, s-1$ or s. This leads to $D^* \geq 1/2B + 1/2B_j + 1/4D_1$ and $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(1/2B + 1/2B_j) = 0$, a contradiction. So, Case(5.a) is impossible.

Case(5.b) $T_2 = B$. Then $D^* \geq (t+2)/(t+6)B + \sum_i 2i/(t+6)G_i$. This leads to $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.((t+2)/(t+6)B + 4/(t+6)B_j) = 0$, a contradiction. So, Case(5.b) is impossible.

Case(5.c) $T_2 = B + B'$ where B' is adjacent to R. Then $t = 2, 3$ because Δ_1 is contractible to a quotient singularity. Moreover, $D^* \geq 4/(10-t)B + (2+t)/(10-t)B' + \sum_i 2i/(10-t)G_i$. This implies that $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(4/(10-t)B + 4/(10-t)B_j) = (2-t)/(10-t) \leq 0$, a contradiction. So, Case(5.c) is impossible.

Case(5.d) $T_2 = B + B'$ where B is adjacent to R. Then we have also $D^* \geq (t+2)/(t+10)B' + 2(t+2)/(t+10)B + \sum_i 4i/(t+10)G_i$. We reach

$0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(2(t+2)/(t+10)B + 8/(t+10)B_j) = -(t+2)/(t+10) < 0$, a contradiction. So, Case(5.d) is impossible.

This proves Lemma 6.9.

Lemma 6.10. *Assume that Δ_1 does not consist of (-2)-curves. Then for both $i = 2$ and 3, Δ_i is a linear chain with D_i as a tip.*

Proof. Suppose Lemma 6.10 is false. Then for $k = 2$ or 3, either Δ_k is a fork or a linear chain but D_k is not a tip. Decompose D^* into the form :$D^* = \sum_{i=1}^{3} \Delta_i^* + D'^*$ such that Supp $\Delta_i^* \subseteq \Delta_i$ and Supp $D'^* \subseteq D' := D - \sum_{i=1}^{3} \Delta_i$. On the one hand, we have $0 < 1 - \Gamma.\sum_i \Delta_i^* - \Gamma.D'^*$. On the other hand, we shall show that $\Gamma.\Delta_1^* \geq 1/2$ and $\Gamma.\Delta_k^* \geq 1/2$. Thus, we would reach a contradiction and therefore prove Lemma 6.10.

Let $B_1, \cdots, B_s$ be all irreducible components of Δ_k adjacent to D_k. Then $\Gamma.B_i = 1$ (cf. Lemmas 6.6 and 6.7). If $s \geq 2$, then $\Delta_k^* \geq 1/2D_k + 1/4\sum_i B_i$ and $\Gamma.\Delta_k^* \geq 1/2$. If $s = 1$ then, by the additional assumption, Δ_k is a fork with D_k as a tip. Therefore, $\Delta_k^* \geq 1/2D_k + 1/2B_1$ and $\Gamma.\Delta_k^* \geq 1/2$.

Let $B_1, \cdots, B_s$ be all irreducible components of Δ_1 adjacent to D_1. If $B_i^2 \leq -3$ for some i, then $\Delta_1^* \geq 1/5D_1 + 2/5B_i$. This leads to $\Gamma.\Delta_1^* \geq 4/5 > 1/2$ because $B_i^2 = -3$ and $\Gamma.B_i = 2$ (cf. Lemmas 6.6 and 6.7). Suppose that $B_i^2 = -2$ for all i. By the hypothesis, $\Delta_1 - \sum_i B_i$ contains a (-3)- curve B (cf. Lemma 6.6). Let $G := G_1 + \cdots + G_t$ $(t \geq 3)$ be a linear chain in Δ_1 such that $G_1 = D_1, G_2 = B_1$ and $G_t = B$. Then $\Delta_1^* \geq \sum_i i/(2t+1)G_i$. This leads to $\Gamma.\Delta_1^* \geq \Gamma.(2/(2t+1)B_1 + t/(2t+1)B) = (t+2)/(2t+1) > 1/2$ (cf. Lemma 6.7).

This proves Lemma 6.10.

Lemma 6.11. *Assume that Δ_1 is linear chain but D_1 is not a tip of Δ_1. Then Δ_1 consists of (-2)-curves.*

Proof. Suppose Lemma 6.11 is false. Then by Lemma 6.11, Δ_2, Δ_3 are linear with D_2, D_3 as tips. Then Δ_1 contains a (-3)- curve B and $\Delta_1 - B$ consists of (-2)-curves (cf. Lemma 6.6). By the hypothesis, D_1 meets two irreducible components B_1, B_2 of Δ_1.

Claim(1). B is not adjacent to D_1.

If B is adjacent to D_1, say $B = B_2$, then $D^* \geq 3/7B + 2/7D_1 + 1/7B_1$.

By Lemma 6.7, $\Gamma.B = 2$ and $\Gamma.B_2 = 1$. This leads to $0 < -\Gamma.(K_{\widetilde{S}} + D^*) \leq 1 - \Gamma.(3/7B + 1/7B_2) = 0$, a contradiction. This proves Claim(1).

By Claim(1), $B_1^2 = B_2^2 = -2$. Let $S_0 := 2(C + D_1) + B_1 + B_2$ and let $\varphi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0 as a singular fiber. Then D_2 and D_3 are 2-sections of φ. Let S_1 be an arbitray singular fiber, let $E_i (i = 1, \cdots, s)$ be all (-1)-curves in S_1 and let a_i be the coefficient of E_i in S_1. By the minimality of $-C.(K_{\widetilde{S}} + D^*)$ and by noting that C has coefficient two in S_0, we see that $\sum_i a_i = 2$ and for all i, $-E_i.(K_{\widetilde{S}} + D^*) = -C.(K_{\widetilde{S}} + D^*)$. Thus S_1 has one of the following two dual graphs:

(1) $$(-1) - (-2) - (-2) - \cdots - (-2) - (-2) - (-2),$$

(2) $$(-1) - (-2) - (-2) - \cdots - (-2) - (-2) - (-1),$$

[Bwhere in the first (resp. second) graph S_1 has three or more (resp. two or more) irreducible components. So, no singular fiber contains a $(-n)$-curve with $n \geq 3$. In particular, B must be adjacent to B_1 or B_2, say B_2, and B is a cross-section of φ.

Claim(2). $\Delta_1 = B_1 + D_1 + B_2 + B$. In particular, $D^* = 1/9B_1 + 2/9D_1 + 3/9B_2 + 4/9B + \sum_{i=2}^{3} \Delta_i^* + D'^*$ where Supp $\Delta_i^* \subseteq \Delta_i$ and Supp $D'^* \subseteq D' := D - \sum_{i=1}^{3} \Delta_i$.

Claim(2) is equivalent to saying that B_1 and B are tips. If B_1 is not a tip then $D^* \geq 2/11B_1 + 3/11D_1 + 4/11B_2 + 5/11B$. This leads to $0 < -\Gamma.(K_{\widetilde{S}} + D^*) \leq 1 - \Gamma.(2/11B_1 + 4/11B_2 + 5/11B) = 0$ (cf. Lemma 6.7), a contradiction. If B is not a tip, then $D^* \geq 1/7B_1 + 2/7D_1 + 3/7B_2 + 4/7B$. This leads to $0 < -\Gamma.(K_{\widetilde{S}} + D^*) \leq 1 - \Gamma.(1/7B_1 + 3/7B_2 + 4/7B) = -1/7$, again a contradiction. This proves Claim(2).

Claim(3). (1) For both $i = 2$ and 3, Δ_i is a linear chain with D_i as a tip. (cf. Lemma 6.10.)

(2) For $i = 2$ or 3, $\Delta_i = D_i$.

If for both $i = 2$ and 3, $\Delta_i > D_i$, then $D^* \geq 2/9D_1 + 2/5D_2 + 2/5D_3$. This leads to $0 < -C.(K_{\widetilde{S}} + D^*) \leq 1 - C.(2/9D_1 + 2/5D_2 + 2/5D_3) = -1/45 < 0$, a contradiction. This proves (2). Thus, Claim(3) is proved.

Since B, D_2, D_3 are not contained in fibers of φ and since $\rho(S) = 1$, there are two singular fibers S_1, S_2 of φ each of which has the second type of the above picture. By Claim(3), both 2-sections D_2 and D_3 meet only (-1)-curves of S_i for $i = 1$ or 2, say $i = 1$. Let E_1, E_2 be two (-1)-curves in S_1.

Note that the cross-section B meets E_1 or E_2. Thus, $5 = (B+D_2+D_3).S_1 = (B+D_2+D_3).(E_1+E_2)$. Thus, $(B+D_2+D_3, E_k) \geq 3$ for $k = 1$ or 2. This, together with $D^* \geq 4/9B + 1/3D_2 + 1/3D_3$, implies that $0 < -E_k.(K_{\tilde{S}} + D^*) \leq 1 - E_k.(4/9B + 1/3D_2 + 1/3D_3) \leq 0$, a contradiction.

This proves Lemma 6.11.

Now we shall prove the following

Theorem 6.12. *Suppose that either Δ_1 is a fork, or Δ_1 is a linear chain but D_1 is not a tip of Δ_1. Then we reduce to the case in §3.1 with C replaced by a new minimal (-1)-curve.*

Proof. By the assumption, Δ_1 consists of only (-2)-curves (cf. Lemmas 6.9 and 6.11). So, there are irreducible components $B_1, \cdots, B_r$ $(r \geq 3)$ of Δ_1 such that $B_1 = D_1, B_i.B_{i+1} = B_{r-2}.B_r = 1 (i = 1, \cdots, r-2)$ and $S_0 := 2(C + \sum_{i=1}^{r-2} B_i) + B_{r-1} + B_r$ has the first type of the picture in Lemma 6.11. We see that $r = 3$ if Δ_1 is a linear chain or a fork with D_1 as the central component.

Let $\varphi : \tilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$- fibration with S_0 as a singular fiber. Then D_2 and D_3 are 2-sections of φ. By the same reasoning as in Lemma 6.11, every singular fiber of φ has one of two types in Lemma 6.11. Moreover, $-E.(K_{\tilde{S}} + D^*) = -C.(K_{\tilde{S}} + D^*)$ for every (-1)-curve E in a singular fiber of φ. Let $S_0, S_1, \cdots, S_s$ (resp. $T_1, \cdots, T_t$) be all singular fibers of the first (resp. second) type in Lemma 6.11. Then those $s+t+1$ ones are all singular fibers of φ. Let E_i (resp. E_{j1}, E_{j2}) be the (-1)-curve(s) in S_i (resp. T_j). Let G_{im}, H_{jn} be irreducible components of D. We can write S_i, T_j in the following forms :

$$S_i = 2(E_i + \sum_{k=1}^{s_i-2} G_{i,k}) + G_{i,s_i-1} + G_{i,s_i},$$

$$T_j = E_{j1} + \sum_{k=1}^{t_j} H_{j,k} + E_{j2},$$

where $E_0 = C, E_i.G_{i,1} = G_{i,k}.G_{i,k+1} = G_{i,s_i-2}.G_{i,s_i} = 1 (k = 1, \cdots, s_i - 2), E_{j,1}.H_{j,1} = H_{j,k}.H_{j,k+1} = H_{j,t_j}.E_{j,2} = 1 (k = 1, \cdots, t_j - 1)$.

Let $\sigma : \tilde{S} \to \Sigma_d$ be a smooth blowing-down of all irreducible components in S_i's and T_j's except for G_{i,s_i}'s and $E_{j,2}$'s. Here Σ_d is a Hirzebruch surface of degree d. Let M_d be the minimal section of Σ_d. Then $\sigma(D_k) \sim 2M_d + b_k\sigma(S_0)$ for $k = 2,3$. In particular, $\sigma(D_k)^2 \equiv \sigma(D_2)^2 - \sigma(D_3)^2 \equiv 0 \pmod 4$ for $k = 2,3$.

Claim(1). (1) Suppose that $D_k.E_{ja} = 0$ for some j in $\{1, \cdots, t\}$, some k in $\{2,3\}$ and some a in $\{1,2\}$. Then we are reduced to the case in §4 with C replaced by E_{jb} where $\{a,b\} = \{1,2\}$ as sets.

(2) Suppose that for $k = 2$ or 3 and for some i in $\{1, \cdots, s\}$, we have $D_k.E_i = 0$ in the case $s_i = 2$ and $D_k.E_i = D_k.G_{i,1} = 0$ in the case $s_i \geq 3$. Then we are reduced to the case in §4 with C replaced by E_i.

By the assumption, $2 = D_k.S_j = D_k.(S_j - E_{ja})$. So, $E_{jb} + D$ contains a loop and $|K_{\tilde{S}} + E_{jb} + D| \neq \phi$ (cf. Lemma 1.8). The first assertion of Claim(1) is proved.

In the case $s_i = 2$, we have $2 = D_k.S_i = D_k.(G_{i,1} + G_{i,2})$. Hence $E_i + D$ contains a loop and the claim is proved. In the case $s_i \geq 3$, we have $D_k.G_{i,n} = 1$ for some $2 \leq n \leq s_i - 2$ or $D_k.(G_{i,s_i-1} + G_{i,s_i}) = 2$ by the assumption and by $D_k.S_i = 2$. Then Δ_k can not be contracted to a quotient singularity, a contradiction.

This proves Claim(1).

By Claim(1), we may assume that for both $k = 2$ and 3, we have $D_k.E_{j1} = D_k.E_{j2} = 1$ for all j's, that $D_k.E_i = 1$ for all i's with $s_i = 2$ and that $D_k.(E_i + G_{i,1}) = 1$ for all i's with $s_i \geq 3$.

Case(1). $D_k.G_{i,1} = 1$ for some k in $\{2,3\}$ and some i in $\{1, \cdots, s\}$ with $s_i \geq 3$, say $i = 1$. Then Δ_k is a fork with G_{1,s_1-2} as the central component. Thus, $\Delta_k.E_i = 1$ for all $i \neq 1$ because Δ_k is contractible to a quotient singularity. By Lemma 6.9, $\Delta_{k'}$ is a linear chain with $D_{k'}$ as a tip and $D_3^2 = -3$, where $\{k, k'\} = \{2,3\}$ as sets. Hence $D_{k'}.E_i = 1$ for all i for otherwise $\Delta_{k'}$ would be a fork. But then $\sigma(D_k)^2 = -3 + \sum_i s_i + \sum_j (t_j + 1) - 1$ and $\sigma(D_{k'})^2 = -3 + \sum_i s_i + \sum_j (t_j + 1)$. This contradicts $\sigma(D_2)^2 - \sigma(D_3)^2 \equiv 0 \pmod 4$. So, Case(1) is impossible.

Case(2) $D_k.E_i = 1$ for both $k = 2$ and 3 and for all i in $\{1, \cdots, s\}$. Then $\sigma(D_k)^2 = D_k^2 + \sum_i s_i + \sum_j (t_j + 1)$ for both $k = 2$ and 3. Since $\sigma(D_2)^2 - \sigma(D_3)^2 \equiv 0 \pmod 4$, we must have $D_3^2 = -3$. Then $\sigma(D_2)^2 = \sigma(D_3)^2$. Hence $\sigma(D_2) \sim \sigma(D_3)$. Thus, $\sigma(D_2).\sigma(D_3) = \sigma(D_2)^2$. But $\sigma(D_2).\sigma(D_3) = \sum_i s_i + \sum_j (t_j + 1)$. We reach a contradiction. So, Case(2) is impossible.

This proves Theorem 6.12.

Theorem 6.13. *Suppose that Δ_1 is a linear chain with D_1 as a tip.*

Then for both $i = 2$ and 3, Δ_i is a linear chain with D_i as a tip.

Proof. We consider, througout the proof, the case where Δ_1 is a linear chain with D_1 as a tip and for $k = 2$ or 3, Δ_k is either a fork or a linear chain but D_k is not a tip. We want to get a contradiction. By Lemma 6.10, Δ_1 consists of (-2)-curves. By Lemma 6.9, we have $D_3^2 = -3$ and $\Delta_{k'}$ is a linear chain with $D_{k'}$ as a tip, where $\{k', k\} = \{2, 3\}$ as sets. So, we may assume that $k' = 2, k = 3$ because $D_2^2 = D_3^2 = -3$. Let G_i $(1 \le i \le s; s \ge 3)$ be irreducible components of Δ_3 such that $G_1 = D_3, G_i.G_{i+1} = G_{s-2}.G_s = 1 (1 \le i \le s-2)$. If Δ_3 is a linear chain then $s = 3$, and if Δ_3 is a fork then $s \ge 4$ because D_3 is then a tip by Lemma 6.9.

Claim(1). $D - \sum_{i=1}^3 \Delta_i$ consists of (-2)-curves.

Suppose to the contrary that $D - \sum_{i=1}^3 \Delta_i$ contains a $(-n)$-curve B with $n \ge 3$. Then $B^2 = -3$ by Lemma 6.6. Moreover, $\Delta_2 = D_2$ for otherwise $D^* \ge 2/5D_2 + 1/5D_2' + 1/3B + 1/2(D_3 + G_2 + \cdots + G_{s-2}) + 1/4G_{s-1} + 1/4G_s$ and $0 < -\Gamma.(K_{\widetilde{S}} + D^*) \le 1 - \Gamma.(1/5D_2' + 1/3B + 1/2\sum_{i=1}^{s-2} G_i + 1/4G_{s-1} + 1/4G_s) = 1 - 1/5 - 1/3 - 1/2 = -1/30$ (cf. Lemma 6.7), a contradiction. Here D_2' is an irreducible component of Δ_2 adjacent to D_2.

Let $R_0 := 2(\Gamma + \sum_{i=2}^{s-2} G_i) + G_{s-1} + G_s$ and let $\psi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with R_0 as a singular fiber. Let R_1 be the singular fiber of ψ containing $C + D_1 + D_2$. Then there exists a (-1)-curve E such that $E.D_2 = 1$ and $R_1 = 2C + D_1 + D_2 + E$. Note that B is a 2-section of ψ because $R_0.B = 2\Gamma.B = 2$ (cf. Lemma 6.7). Hence $2 = R_1.B = E.B$. This leads to $0 < -E.(K_{\widetilde{S}} + D^*) \le 1 - E.(1/3B + 1/3D_2) = 0$ because $D^* \ge 1/3B + 1/3D_2$. We reach a contradiction. This proves Claim(1).

Let $S_0 := 4C + 2(D_1 + G_1 + \cdots + G_{s-2}) + G_{s-1} + G_s$. Let $\varphi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_0 as a singular fiber. Then D_2 is a 4-section. By Claim(1), every singular fiber S_1 of φ other than S_0 consists of (-1)-curves and (-2)-curves. So, it is easy to see that S_1 has one of two types in Lemma 6.11. Let $S_i (i = 1, \cdots, m)$, (resp. $T_j (j = 1, \cdots, n)$) be all singular fibers of the first (resp. second) type. Then S_i's, T_j's are all singular fibers of φ. Let E_i (resp. E_{i1}, E_{i2}) be the (-1)-curve(s) in S_i (resp. T_j).

Claim(2). $\Delta_2 = D_2$.

Suppose to the contrary that $\Delta_2 = H_1 + \cdots + H_t$ with $t \ge 2, H_1 = D_2, H_i.H_{i+1} = 1 (i = 1, \cdots, t-1)$. Let L_1 be the singular fiber of φ containing $H_2 + \cdots + H_t$. Then $L_1 = T_j$ for some j, say $j = 1$ because D_2 is a 4-section and D_2 is a tip of the linear chain Δ_2. Then $T_1 = E_{11} + H_2 + \cdots + H_t + E_{12}$

with, say $E_{11}.H_2 = E_{12}.H_t = 1$. Since D_2 is a 4-section, we have $D_2.E_{1k} \geq 2$ for $k = 1$ or 2. This leads to $0 < -E_k.(K_{\widetilde{S}} + D^* \leq 1 - E_k.\sum_i i/(2t+1)H_i \leq 1 - 2t/(2t+1) - 1/(2t+1) = 0$ because $D^* = \sum_{i=1}^{t} i/(2t+1)H_i +$ (other terms). We reach a contradiction. This proves Claim(2).

By Claim(2), D_2 meets only (-1)-curves in singular fibers. So, $D_2.E_i = D_2.E_{j1} = D_2.E_{j2} = 2$ for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$ because $-E_{ik}.(K_{\widetilde{S}} + D^*) > 0$ for $k = 1, 2$ and $D^* \geq 1/3D_2$. Let $\sigma : \widetilde{S} \to \Sigma_d$ be a smooth blowing-down of curves in singular fibers. Here Σ_d is a Hirzebruch surface of degree d. Let M_d be a minimal section on Σ_d. Then we have $\sigma(D_2) \sim 4M_d + b\sigma(S_0)$ and $\sigma(D_2)^2 \equiv 0 \pmod 8$. On the other hand, by the above description on the intersection of D_2 with singular fibers, we have $\sigma(D_2)^2 \equiv D_2^2 + 2 \pmod 4$. We reach a contradiction. This proves Theorem 6.13.

Next we prove the following Theorem 6.14.

Theorem 6.14. *Assume the same hypothesis as in Theorem 6.13 and assume further that Δ_1 is not a (-2)-chain. Then either Theorem 6.1 is true, or there is a minimal (-1)-curve E such that Case (II-3) in Remark 3.11, with C, Δ_i replaced by E, $\widetilde{\Delta_i}$, is true and that $\widetilde{\Delta_1}$ consists of exactly two (-2)-curves. Here, $\widetilde{\Delta_i}$ for i = 1, 2, 3 are all the connected components of D meeting E.*

Proof. In view of Lemma 6.6, Δ_1 consists of one (-3)-curve and several (-2)-curves (cf. Lemma 6.6). By Theorem 6.13, Δ_i is a linear chain with D_i as a tip for $i = 2$ and 3. Let D'_i be the irreducible component of D adjacent to D_i $(i = 1, 2, 3)$. Let $D' := D - \sum_{i=1}^{3} \Delta_i$. Write $\Delta_1 = R_1 + \cdots + R_r + \cdots + R_d$ such that $R_1 = D_1, R_r^2 = -3, R_i.R_{i+1} = 1 (i = 1, \cdots, d-1)$. So, $R_2 = D'_1$. We have :

$$D^* = \sum_{i=1}^{r} i(d-r+1)/(d+1+r(d-r+1))R_i$$

$$+ \sum_{i=r+1}^{d} r(d-i+1)/(d+1+r(d-r+1))R_i + \Delta_2^* + \Delta_3^* + D'^*,$$

where Supp $\Delta_i^* \subseteq \Delta_i$ and Supp $D'^* \subseteq D'$.

Case(1). $\Delta_i = D_i$ for $i = 2$ and 3.

Before we consider the Case(1), we will make some remarks which will be used often till the end of the proof of Theorem 6.1.

We will often use a different minimal (-1)-curve E instead of the original curve C. Let $\widetilde{D_i}$ be all the irreducible components of D which intersect E and let $\widetilde{\Delta_i}$ be the connected component of D containing $\widetilde{D_i}$. By repeated use of results in §3, §4, §5, Lemmas 6.2, 6.3, 6.5 and Theorems 6.12. 6.13 we see that one of the following situations takes place :

(a) Case (II-3) in Remark 3.11, with C, Δ_i replaced by $E, \widetilde{\Delta_i}$, is true. Moreover, $\widetilde{\Delta_i}$ is a linear chain with $\widetilde{D_i}$ as a tip, $\widetilde{\Delta_1}, \widetilde{\Delta_2} - \widetilde{D_2}, \widetilde{\Delta_3} - \widetilde{D_3}$ are (-2)-chains and $\widetilde{D_2}^2 = -3, \widetilde{D_3}^2 = -3, -4, -5$.

(b) Case (II-4) in Remark 3.11, with C, Δ_i replaced by $\cdots$, is true.

(c) $\pi_1(S^o)$ is finite.

To prove Theorem 6.14, we can always assume assume that every minimal curve E fits case (a).

Consider the case where $\Delta_i = D_i$ for $i = 2$ and 3. If D' contains two (-3)-curves B_1, B_2, then $D^* \geq 1/3B_1 + 1/3B_2 + \sum_{i=1}^{r} i/(2r+1)R_i$ and $0 < -\Gamma.(K_{\widetilde{S}}+D^*) \leq 1-\Gamma.(1/3B_1+1/3B_2+r/(2r+1)R_r) \leq 1-1/3-1/3-2/5 < 0$, a contradiction. So, D' consists of (-2)-curves and possibly one (-3)-curve (cf. Lemma 6.6).

First assume that $r > 2$.

Consider the $\mathbf{P}^1-$ fibration φ with $S_0 := 3C + 2D_1 + R_2 + D_2$ as one of the singular fibers. Then Γ and R_3 are cross-sections, D_3 is a 3−section and $D - (D_3 + R_3)$ is contained in fibers.

Case(1.1). $r > 2$ and D' contains a (-3)-curve B.

Consider the fiber S_1 containing B. Since $\Gamma \cdot R_r = 1 = \Gamma \cdot B, R_r$ cannot lie in S_1. Hence S_1 has a unique (-3)-curve and all the components are (-1) or (-2) curves. By Lemma 1.6 of [18], the sum of the coefficients of all the (-1) curves in S_1 is at least 3. As C is minimal, we see that each (-1) curve E_i in S_1 is minimal and the sum of the coefficients of the E_i's is precisely 3.

Case(1.1)(1). S_1 contains a unique (-1) curve E. Then the multiplicity of E is 3. Since E fits the Case (a) above, we have $E \cdot B = E \cdot D_3 = E \cdot G_1 = 1$ for some (-2)-curve G_1. Hence $\widetilde{D_1} = G_1, \widetilde{D_2} = B, \widetilde{D_3} = D_3$. Thus, S_1 has the configuration :$B - E - G_1 - G_2$ for some (-2)-curve G_2.

Now $\Delta_1 = R_1 + \cdots + R_5, r = 3, G_2 = R_4, G_1 = R_5$.

Now $D^* \geq 1/3B + \Sigma_{i=1}^{3} i/5R_i$, leading to a contradiction $0 < -\Gamma \cdot (K + D^*) \leq 1 - 1/3 - (2/5 + 3/5) < 0$ (cf. Lemma 6.7).

Case(1.1)(2). Suppose S_1 has exactly two (-1) curves E_1, E_2 with multiplicity of E_2 equal to 2. Now $R_3 \cap E_2 = \phi$. Since $E_1 and E_2$ fit Case (a) above, we have $E_2 \cdot B = E_2 \cdot D_3 = 1$ and S_1 has exactly two possible configurations :

(α) $S_1 = G - E_2 - B - E_1$, where G is a (-2) curve. As E_1 has to intersect some (-2) curve, we see that R_3 is a (-2) curve, $R_3 \cdot E_1 = 1$ and hence $r > 3$. But then R_3 is not a tip of Δ_1 which intersects E_1, a contradiction.

(β) $d = r = 3, S_1 = (-2) - E_2 - B_1 - \cdots - B_m - E_1, B_1 = B, E_1 \cdot D_3 = E_1 \cdot R_3 = E_1 \cdot B_m = E_2 \cdot D_3 = E_2 \cdot B_1 = 1$. However, $-C \cdot (K + D^*) = -E_2 \cdot (K + D^*)$ implies that $m = 10$. But as in the assertion (3) in Case(2) below, we can see that the $\rho(\widetilde{S}) \leq 13$. This is a contradiction.

Case(1.1)(3). S_1 has three (-1) curves E_1, E_2, E_3. Since each E_i fits Case (a) above, using Lemma 1.6 of [18] we can assume that E_2 meets only the curve B from S_1. Again since E_2 has to meet some (-2) curve, R_3 is a (-2) curve meeting E_2. But again in that case R_3 is not a tip of Δ_1, a contradiction.

Case(1.2). $r > 2$ and D' has only (-2). Hence $\Gamma \cap D' = \phi$.

Case(1.2)(1). $r > 3$.

Let S_1 be the singular fiber containing the (-3) curve R_r. We consider three cases as in Case(1.1) above. We are easily reduced to considering the case when S_1 has the configuration :

$R_r - E - G_1 - G_2$, where $E^2 = -1, G_1^2 = G_2^2 = -2$. As in the assertion in Case 2, part (3) below, $\# D = 7 + a, D_3^2 = -a$. Since E fits Case (a) above, we have $E \cdot D_3 = 1$ and $r = d = 4$. By taking E as the minimal curve, we have the case (a) in the statement of Theorem 6.14. Indeed, $\widetilde{\Delta_1} = G_1 + G_2, \widetilde{\Delta_2} = R_1 + \cdots + R_4, \widetilde{\Delta_3} = D_3$.

Case(1.2)(2). $r = 3$.

Let $\lambda = d - r$. Then the determinant of the intersection matrix of $\Delta_1 = \pm(4\lambda + 7)$.

Now Γ meets only R_2, R_3. We will apply Lemma 1.14.

Let U be a tubular neighborhood of $\Delta_1 \cup \Gamma$. Let a small loop around D_1 be denoted by γ. If γ_2, γ_3 are small loops around R_2 and R_3 respectively, then in $\pi_1(U - \Delta_1), \gamma_2 = \gamma^2, \gamma_3 = \gamma^3$.

Then $\pi_1(U - D) = < \gamma, u | u^{-1}\gamma^2 u = \gamma^{-3} >$.

Since $(\gamma) = (\gamma^2)$ as the order of γ is $4\lambda + 7$, the group generated by γ is normal in $\pi_1(U - D)$. Now the same argument as in §4 shows that $\pi_1(S^0)$ is finite.

Case(1.2)(3). Now we are left with the case $r = 2$.

Again first assume that D' has a (-3) curve B. Then $D^* \geq 1/5R_1 + 2/5R_2 + 1/3B$ and $0 < -\Gamma \cdot (K + D^*) \leq 1 - (4/5 + 1/3) < 0$, a contradiction.

Hence D' consists of only (-2) curves and $\Gamma \cap D' = \phi$.

Now again the same argument above using Lemma 1.14 shows that $\pi_1(S^0)$ is finite.

Case(2). $\Delta_k \neq D_k$ for $k = 2$ or 3. Then the following assertions are true.

(1) D' consists of (-2)-curves. Hence $D^* = \sum_{i=1}^{3} \Delta_i^*$.

(2) $D_3^2 = -3, -4$ and $r \geq 3$, i.e., $(D_1')^2 = -2$.

(3) $7 - D_3^2 = \sum_{i=1}^{3} \#(\Delta_i) + \#(D')$, where $\#(\Delta)$ denotes the number of irreducible component in Δ.

(4) Suppose $D_3^2 = -4$. Then $D_3.E \leq 1$ for every (-1)-curve E.

If D' contains a $(-n)$-curve B with $n \geq 3$, then $n = 3$ (cf. Lemma 6.6) and $D^* \geq 1/3B + \sum_{i=1}^{r} i/(2r+1)R_i + 1/5D_k' + 2/5D_k$. This leads to $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(1/3B + \sum_{i=1}^{r} i/(2r+1)R_i + 1/5D_k') = 1 - 1/3 - (r+2)/(2r+1) - 1/5 < 0$ (cf. Lemma 6.7), a contradiction. This proves the first assertion of Case(2).

If $D_3^2 = -5$, then $D^* \geq 1/5D_2' + 2/5D_2 + 3/5D_3$ (resp. $D^* \geq 1/3D_2 + 1/3D_3' + 2/3D_3$) in the case $k = 2$ (resp. $k = 3$). This leads to $-C.(K_{\tilde{S}} + D^*) = 1 - C.D^* \leq 0$, a contradiction. So, $D_3^2 = -3, -4$.

If $r = 2$, then $D^* \geq 1/5D_1 + 2/5D_1' + 1/5D_k' + 2/5D_k$ and $0 < -\Gamma.(K_{\tilde{S}} + D^*) \leq 1 - \Gamma.(2/5D_1' + 1/5D_k') = 0$ (cf. Lemma 6.7), a contradiction. So, $r \geq 3$.

(3) follows using Noether's equality and from the following observation : $10-(2+D_3^2)=10-K_{\tilde{S}}^2=\rho(\tilde{S})=1+\#(D)$ (cf. Lemma 6.8).

(4) follows from the fact that $D^* \geq 1/2D_3$ and $-E.(K_{\tilde{S}}+D^*)>0$.

This proves all the assertions of Case(2).

From now on until the end of the proof of Theorem 6.14 we will assume that we are in the situation of Case(2).

Claim(1). It is impossible that $\Delta_i \neq D_i$ for $i=2$ and 3.

Consider the case where $\Delta_i \neq D_i$ for $i=2$ and 3. Then $D^* \geq 1/5D_2'+2/5D_2+1/5D_3'+2/5D_3+\sum_{i=1}^r i/(2r+1)R_i$. Note that $0<-\Gamma.(K_{\tilde{S}}+D^*) \leq 1-\Gamma.(1/5D_2'+1/5D_3'+2/(2r+1)R_2+r/(2r+1)R_r)=1-1/5-1/5-(r+2)/(2r+1)$ (cf. Lemma 6.7). Hence, $r \geq 8$. On the other hand, by the assertion (3)in Case(2), $11 \geq 7-D_3^2=\sum_{i=1}^3 \#(\Delta_i)+\#(D') \geq r+2+2 \geq 12$. We get a contradiction.

Therefore Claim(1) is true.

Now we have either $\Delta_2=D_2$ or $\Delta_3=D_3, D_3^2=-3$ or -4. Further, D' has only (-2) curves by the assertions in Case(2).

Claim(2) $r>4$.

Now we may assume that $\Delta_k \neq D_k, \Delta_{k'}=D_{k'}$ for some $\{k,k'\}=\{2,3\}$ as sets. Write $D_k^2=-a, D_{k'}^2=-b$. Then $(a,b)=(3,3),(4,3),(3,4)$.

Write $\Delta_k=\sum_{i=1}^t T_i$ such that $T_1=D_k, T_i.T_{i+1}=1 (i=1,\cdots,t-1)$. Then we have

$$D^*=\sum_{i=1}^r i(d-r+1)/(d+1+r(d-r+1))R_i+\sum_{i=r+1}^d r(d-i+1)/(d+1+r(d-r+1))R_i$$

$$+\sum_{i=1}^t (a-2)(t-i+1)/((a-1)t+1)T_i+(b-2)/bD_{k'}.$$

We now calculate (cf. Lemma 6.7) :

$$-C.(K_{\tilde{S}}+D^*)=1-(d-r+1)/(d+1+r(d-r+1))-(a-2)t/((a-1)t+1)-(b-2)/b,$$

$$-\Gamma.(K_{\tilde{S}} + D^*) = 1 - 2(d-r+1)/(d+1+r(d-r+1))$$
$$-r(d-r+1)/(d+1+r(d-r+1)) - (a-2)(t-1)/((a-1)t+1).$$

Since $-C.(K_{\tilde{S}} + D^*) \leq -\Gamma.(K_{\tilde{S}} + D^*)$, we get

$$(r+1)(d-r+1)/(d+1+r(d-r+1)) \leq (b-2)/b + (a-2)/((a-1)t+1),$$

and

(14) $$2/b \leq (a-2)/((a-1)t+1) + r/(d+1+r(d-r+1)).$$

On the other hand, by the assertions in Case(2), one has

$$6 - D_3^2 = d + t + \#(D').$$

Consider the case where $(a,b) = (3,3)$. May assume $\Delta_2 = D_2, D_3^2 = -3$ and $\Delta_3 = \sum_{i=1}^t T_i$. By (14), we get

(14.1) $7/15 \leq 2/3 - 1/(2t+1) \leq r/((d+1+r(d-r+1)) < 1/(d-r+1)$.

Hence $d-r \leq 1$. If $d-r = 1$, then (14.1) implies that $7/15 \leq r/(3r+2) < 1/3$, a contradiction.

So, $d = r$. Then (14.1) implies that $7/15 \leq r/(2r+1)$ and $r \geq 7$.

Consider the case where $(a,b) = (4,3)$. Then $\Delta_2 = D_2, D_3^2 = -4$ and $\Delta_3 = \sum_{i=1}^t T_i$. (14) implies that

(15.2) $8/21 \leq 2/3 - 2/(3t+1) \leq r/(d+1+r(d-r+1)) < 1/(d-r+1)$.

Hence, $d-r \leq 1$. If $d-r = 1$, then (15.2) implies that $8/21 \leq r/(3r+2) < 1/3$, a contradiction.

So, $d = r$. Then (15.2) implies that $2/3 - 2/(3t+1) \leq r/(2r+1) < 1/2$ and $t = 2, 3$. On the other hand, $0 < -C.(K_{\tilde{S}} + D^*) = 1 - 1/(2r+1) - 2t/(3t+1) - 1/3 \leq 1 - 1/(2r+1) - 4/7 - 1/3$. Hence $r \geq 5$ and $r \geq 8$ if $t = 3$.

Consider the case where $(a,b) = (3,4)$. Then $\Delta_3 = D_3$ with $D_3^2 = -4$ and $\Delta_2 = \sum_{i=1}^{t} T_i$. (14) implies that

(15.3) $3/10 \leq 1/2 - 1/(2t+1) \leq r/(d+1+r(d-r+1)) < 1/(d-r+1)$.

Hence $d-r \leq 2$. If $d-r = 2$, then (15.3) implies that $3/10 \leq r/(4r+3) < 1/4$, a contradiction. Thus, $d - r = 0, 1$.

If $d = r+1$, then

$0 < -C \cdot (K + D^*) = 1 - 2/(3r+2) - t/(2t+1) - 1/2 \leq 1 - 2/(3r+2) - 2/5 - 1/2 = 1/10 - 2/(3r+2)$.

Hence $r > 6$ and $r > 8$ if $t \geq 3$.

If $d = r$, then $0 < -C \cdot (K + D^*) = 1 - 1/(2r+1) - t/(2t+1) - 1/2 \leq 1 - 1/(2r+1) - 2/5 - 1/2 = 1/10 - 1/(2r+1)$. Hence $r > 4$.

Thus Claim(2) is proved.

Let $S_0 = 3C + 2D_1 + R_2 + D_2$ and $\varphi : \tilde{S} \to \mathbf{P}^1$ the fibration as before with S_0 as one of the fibers. Recall that $r > 4$ and consider the fiber S_1 containing the (-3) curve R_r. Now D_3 is a 3-section, R_3 is a cross-section, $R_4 + \cdots + R_d$ is contained in S_1.
Since each E_i fits Case (a) above, $E_i \cdot R_r = E_i \cdot D_3 = 1$. Clearly $l < 4$ and if $l = 3$ then $S_1 = R_r + E_1 + E_2 + E_3$ and $r = 4$ which is not true.

Hence $l < 3$.

From Lemma 1.6 of [18], if $l = 1$ then $r = 4$. Thus $l = 2$ and $S_1 = E_1 - R_r - E_2 - (-2)$. This again means $r = 4$ and Theorem 6.14 is proved.

Remark. By using a more detailed argument we can prove the following more precise result : *With the same hypothesis as in Theorem 6.13, the connected component Δ_1 consists of only (-2) curves.*

Now we can prove the following Theorem 6.15 which consists of Lemmas 6.16, 6.18, 6.21, 6.22, 6.23 and 6.24 below.

Theorem 6.15. *Assume the hypothesis as in Theorem 6.13 and assume further that Δ_1 is a (-2)-chain. Then Theorem 6.1 is true.*

Now we consider the case where Δ_1 is a linear chain with D_1 as a tip. By Theorem 6.13, for both $i = 2$ and 3, Δ_i is a linear chain with D_i as a tip. By Lemma 6.6, $\Delta_1, \Delta_2 - D_2$ and $\Delta_3 - D_3$ consist of only (-2)-curves. Write $f(C) \equiv -cK_S, f(\Gamma) \equiv -\gamma K_S$. Then $-C.(K_{\tilde{S}}+D^*) = c(K_{\tilde{S}}+D^*)^2$ and $-\Gamma.(K_{\tilde{S}}+D^*) = \gamma(K_{\tilde{S}}+D^*)^2$. By the choice of C, we have $\gamma \geq c$. By Lemma 6.2, we have $f(K_{\tilde{S}}+2C+D_1+D_2+D_3) \equiv f(\Gamma)$ and hence $-1+2c = \gamma \geq c$. Therefore, $c \geq 1$ and $c = 1$ if and only if $\gamma = 1$, if and only if $\gamma = c$. We can write $f^*f(C) \equiv C + D_c^*$ where $D_c^* \geq 0$ and Supp $D_c^* \subseteq D$. It is easy to see that $K_S + f(C) \equiv (c-1)(-K_S)$ and $K_{\tilde{S}} + C + D^* + D_c^* \equiv P_1$. Here $P_1 := -(c-1)(K_{\tilde{S}} + D^*)$ which is zero (resp. a nef and big divisor) in the case $c = 1$ (resp. $c > 1$).

Write $\Delta_1 = \sum_{i=1}^r R_i$ such that $R_r = D_1, R_i.R_{i+1} = 1 (i = 1, \cdots, r-1)$, $\Delta_2 = \sum_{i=1}^s S_i$ such that $S_s = D_2, S_i.S_{i+1} = 1 (i = 1, \cdots, s-1)$ and $\Delta_3 = \sum_{i=1}^t T_i$ such that $T_t = D_3, T_i.T_{i+1} = 1 (i = 1, \cdots, t-1)$.

Lemma 6.16. (1) $D^* = \sum_i i/(2s+1)S_i + \sum_i (a-2)i/((a-1)t+1)T_i + D'^*$ *where* $a := -D_3^2$ *and Supp* $D'^* \subseteq D' := D - \sum_{i=1}^3 \Delta_i$.

(2) $N := D - D^* - D_c^* \geq 0$ *and Supp* $N = D$.

(3) $\kappa(\tilde{S}, K_{\tilde{S}} + C + D) \geq 0$ *and* $K_{\tilde{S}} + C + D = P + N$ *is the Zariski decomposition where* $P := K_{\tilde{S}} + C + D^* + D_c^*$. *Moreover,* $P \equiv -(c-1)f^*(K_{\tilde{S}})$ *and hence either* $c > \gamma > 1$ *and* $\kappa(\tilde{S}, K_{\tilde{S}} + C + D) = 2$ *or* $c = \gamma = 1$ *and* $\kappa(\tilde{S}, K_{\tilde{S}} + C + D) = 0$.

Proof. (1) follows from that $B.(K_{\tilde{S}} + D^*) = 0$ for every $B \leq D$. By a similar reasoning, one obtains $D_c^* = \sum_i i/(r+1)R_i + \sum_i i/(2s+1)S_i + \sum_i i/((a-1)t+1)T_i$. So, $N = \sum_i (r+1-i)/(r+1)R_i + \sum_i (2(s-i)+1)/(2s+1)S_i + \sum_i ((a-1)(t-i)+1)/((a-1)t+1)T_i + D' - D'^*$. Then (2) follows (cf. Lemma 1.1,(1)).

(3) Note that $\kappa(\tilde{S}, K_{\tilde{S}} + C + D) \geq \kappa(\tilde{S}, P) = \kappa(\tilde{S}, P_1)$ (because $\tilde{S}$ is a rational surface) $= 0$ (resp. 2) if $c = 1$ (resp. $c > 1$). So, $\kappa(\tilde{S}, K_{\tilde{S}} + C + D) \geq 0$. So, there is a Zariski decomposition for $K_{\tilde{S}} + C + D$. Since P ($\equiv P_1$) is nef, $N \geq 0$ and $P.N_i = 0$ for every irreducible component N_i of D, the decomposition given in (2) above is the Zariski decomposition. Therefore, $\kappa(\tilde{S}, K_{\tilde{S}} + C + D) = \kappa(\tilde{S}, P) = 0, 2$.

This proves Lemma 6.16.

Remark 6.17. Note that every twig of $C + D$ is admissible. Since Supp$N = D \subseteq C + D$, $N = Bk^*(C + D)$ by Fujita [6 , 6.17 and 6.18]. In particular, if $\kappa(\widetilde{S}, K_{\widetilde{S}}+C+D)=0$, then $P \equiv P_1 = 0$, $K_{\widetilde{S}}+C+D \equiv N = Bk^*(C+D)$ and hence $(K_{\widetilde{S}}+C+D).C = Bk^*(C+D).C$.

Lemma 6.18. *Assume $\kappa(\widetilde{S}, K_{\widetilde{S}}+C+D)=0$. Then Theorem 6.1 is true.*

Proof. Let $D' := D - \sum_{i=1}^{3}\Delta_i$. By Remark 6.17, we can apply Fujita [6 , 8.7] to the pair $(\widetilde{S}, C+D)$. Since in our case $\beta(C)=3$, $D_i^2 \le -3$ and $D_i.C = 1$ for both $i=2$ and 3, only the case(4) there takes place. Therefore, $\sum_{i=1}^{3} d(\Delta_i) = 1$. This, together with $D_2^2=-3, D_3^2=-3,-4,-5$, implies that $D_3^2=-3, \Delta_2 = D_2, \Delta_3 = D_3, \Delta_1 = D_1 + R_1$ where R_1 is a (-2)-curve. By Lemma 6.7, $\Gamma.\sum_{i=1}^{3}\Delta_i = \Gamma.R_1 = 1$. Moreover, $D^* = 1/3D_2 + 1/3D_3 + D'^*$, where Supp $D'^* \subsetneq D'$. Hence $C.D^* = 2/3$.

By Lemma 6.16, $\gamma = c = 1$. Hence $-\Gamma\cdot(K_{\widetilde{S}}+D^*) = -C\cdot(K_{\widetilde{S}}+D^*), \Gamma\cdot D^* = C\cdot D^* = 2/3 > 0$ and Γ is a minimal curve.
By the arguments in the beginning of Theorem 6.14, we may assume that Γ fits Case (a) there. So by Theorem 6.14 and Lemma 1.5 of [18], Γ meets two (-3)-curves B_1, B_2 of D'.

Suppose first that B_1, B_2 are not both isolated components of D. This will lead to $\Gamma.D^* = \Gamma.D'^* > 1/3+1/3$ because at least one of the connected components which contains B_1 or B_2 has more than two irreducible components. This is a contradiction.

Now we assume that both B_1, B_2 are isolated. We consider again the fibration φ given by $S_0 := 3C + 2D_1 + R_1 + D_2$. The curve Γ is a section of this fibration and B_1, B_2 lie in different singular fibers, say S_1, S_2 respectively. Since $\rho(\widetilde{S}) = \#(D)+1, S_i$ contains only one (-1)-curve E_i which is also minimal. We may assume that each E_i fits Case (a) in Theorem 6.14. So we reduce to the situation (cf. Theorem 6.14) :

$S_1 = B_1 - E_1 - G_1 - G_2, S_2 = B_2 - E_2 - G_3 - G_4$ where $G_j^2 = -2$ for $j = 1,2,3,4$.

The 3-section D_3 meets only the (-1) curves from the three singular fibers. The triple cover $\varphi : D_3 \to \mathbf{P}^1$ has at least 3 ramification points with ramification index 3. This clearly contradicts Hurwitz formula.

This completes the proof of Lemma 6.18.

Remark. $\kappa(\widetilde{S}, K_{\widetilde{S}}+C+D)=0$ is impossible.

Now we consider, till the end of the proof for Theorem 6.15, the case $\kappa(\tilde{S}, K_{\tilde{S}} + C + D) = 2$.

Lemma 6.19. $\Delta_4 := D - \sum_{i=1}^{3} \Delta_i$ *is zero or a single connected component of* D.

Proof. We will need the following result from [7].

'Let V *be an affine surface with atmost quotient singularities. Assume that* $H_0(V; \mathbf{Q}) \approx \mathbf{Q}$ *and* $H_i(V; \mathbf{Q}) = (0)$ *for* $i > 0$. *If* $\overline{\kappa}(V - SingV) = 2$, *then* V *does not contain any irreducible curve homeomorphic to* $\mathbf{C}$ *and* V *has at most one singular point.'* (In [7], the assertion about V having atmost one singular point is not made but it follows very easily from the Lemma 8 of [7].)

To apply this, we notice that $V = S - f(C)$ satisfies the hypothesis in the result above. Hence Δ_4 is connected.

Lemma 6.20. (1) $\Delta_4 = D - \sum_{i=1}^{3} \Delta_i$ *is a single connected component of* D *and consists of one* (-3)*-curve* B *and several* (-2)*-curves.*

(2) *Write* $D^* = \alpha B + \sum_i i/(2s+1)S_i + \sum_i (a-2)i/((a-1)t+1)T_i +$ *(other terms) in notations of Lemma 6.16. Then* $1 > s/(2s+1) + (a-2)t/((a-1)t+1)$ *and* $\alpha < 1/(2s+1) + (a-2)/((a-1)t+1)$.

(3) $r + s + t + u = 7 - D_3^2$ *where* r, s, t, u *are respectively the numbers of irreducible components in* Δ_i $(i = 1, 2, 3, 4)$.

Proof. (1) In view of Lemmas 6.6 and 6.19, it suffices to show the assertion that Δ_4 contains a (-3)-curve. Suppose to the contrary that this assertion is false. Then $D - (D_2 + D_3)$ consists of (-2)-curves (cf. Lemma 6.6 and Theorem 6.14). By Lemma 6.8, we have $K_{\tilde{S}}^2 = 2 - a$ where $a := -D_3^2$. Write $D^* = \alpha D_2 + \beta D_3 +$ (other terms). First one has $0 < -C.(K_{\tilde{S}} + D^*) = 1 - \alpha - \beta$. So, one can calculate as follows :

$$0 < (K_{\tilde{S}} + D^*)^2 = K_{\tilde{S}}.(K_{\tilde{S}} + D^*) = K_{\tilde{S}}^2 + \alpha + (a-2)\beta = 2 - a + \alpha + (a-2)\beta$$

$$= (2 - a) + (\alpha + \beta) + (a-3)\beta < (3 - a) + (a-3)\beta = (a-3)(\beta - 1) \leq 0.$$

We reach a contradiction. So, Δ_4 contains a (-3)-curve. Thus, (1) is proved.

(2) First $0 < -C.(K_{\tilde{S}} + D^*) = 1 - s/(2s+1) - (a-2)t/((a-1)t+1)$. So, the first inequality follows. Next, by (1) and Lemma 6.7, $-\Gamma.(K_{\tilde{S}} + D^*) = 1 - \Gamma.(\alpha B + (s-1)/(2s+1)S_{s-1} + (a-2)(t-1)/((a-1)t+1)T_{t-1}$. Now the second inequality in (2) follows from that $-C.(K_{\tilde{S}} + D^*) = c(K_{\tilde{S}} + D^*)^2 < \gamma(K_{\tilde{S}} + D^*)^2 = -\Gamma.(K_{\tilde{S}} + D^*)$.

(3) By Lemma 6.8, $K_{\tilde{S}}^2 = 2 + D_3^2$. Hence $\rho(\tilde{S}) = 8 - D_3^2$. Now (3) follows from that the number of irreducible components in D is equal to $\rho(\tilde{S}) - 1$.

We have proved Lemma 6.20.

Lemma 6.21. *It is impossible that $\Delta_i = D_i$ for two of i's in $\{1,2,3\}$. In particular, $D_3^2 = -3, -4$.*

Proof. Assume that $\Delta_i = D_i, \Delta_j = D_j$ for some distinct i, j in $\{1, 2, 3\}$. Then $\Gamma \cdot (\Delta_1 + \Delta_2 + \Delta_3) = 1$ and $\Gamma \cdot \Delta_4 = 1$ by Lemma 6.20(1). From the proof of Lemma 6.19, we know that $S - f(C)$ does not contain any curve homeomorphic to $\mathbf{C}$. But the image of Γ in S gives rise to such a curve. This is a contradiction.

Suppose that $D_3^2 = -5$. By the first assertion $D_2 < \Delta_2$ or $D_3 < \Delta_3$. This leads to $D^* \geq 2/5D_2 + 3/5D_3$ or $D^* \geq 1/3D_2 + 2/3D_3$. Either of the two cases leads to $-C.(K_{\tilde{S}} + D^*) \geq 0$, a contradiction. So, $D_3^2 = -3, -4$.

This proves Lemma 6.21.

Lemma 6.22. *Suppose that $D_3^2 = 4$. Then either Theorem 6.1 is true or there is a $\mathbf{P}^1$-fibration $\varphi : \tilde{S} \to \mathbf{P}^1$ such that all singular fibers and irreducible components of D are as described in the proof of Claim 2 below.*

Proof. Consider the case $D_3^2 = -4$. We use the notations in Lemmas 6.16 and 6.20. We also let $D_1' = R_{r-1}, D_2' = S_{s-1}, D_3' = T_{t-1}$. These are (-2)-curves and adjacent to D_1, D_2, D_3 respectively. By the first inequality in Lemma 6.20,(2), we obtain the following :

Claim(1). $(s,t) = (2,2), (1,t), (s,1)$.

Consider first the case $(s,t) = (2,2)$. Then, one has $\Delta_i = D_i + D_i'$ for $i = 2$ and 3. By Lemma 6.20, one has $\alpha < 1/5 + 2/7 < 1/2$. Hence Δ_4 is a linear chain with B as a tip. Write $\Delta_4 = \sum_{i=1}^{u} B_i$ such that $B_i.B_{i+1} = 1$ and

$B_u = B$. By the same lemma, one has $r + u = 7$.

Claim(2). Suppose that $(s, t) = (2, 2)$ and $r = 1$. Then there is a minimal (-1)-curve E_2 on $\tilde{S}$ and two connected components Δ_3, Δ_4 of D, both linear chains, $\Delta_3 = D_3 + D_3', \Delta_4 = B_1 + \cdots + B_6$ such that $D_3^2 = -4, B_6^2 = -3, D_3'^2 = -2 = B_1^2 = \cdots = B_5^2$. Further, $E_2 \cdot \Delta_3 = E_2 \cdot D_3 = E_2 \cdot \Delta_4 = E_2 \cdot B_5 = 1$.

Consider the case where $(s, t) = (2, 2)$ and $r = 1$. In the present case, we have $\Delta_1 = D_1$ and $u = 6$. Hence

$$\Delta_1 = D_1,\ \Delta_2 = D_2 + D_2',\ \Delta_3 = D_3 + D_3',\ \Delta_4 = \sum_{i=1}^{6} B_i\, (B_6 = B),$$

$$\Delta_i\ (i = 1, 2, 3, 4)$$

are all connected components of D. Then $\Gamma.D = \Gamma.(B + D_2' + D_3') = 3$ (cf. Lemma 6.7).

Let

$$F_0 := 2\Gamma + D_2' + D_3',$$
$$\varphi : \tilde{S} \to \mathbf{P}^1$$

the $\mathbf{P}^1$-fibration with F_0 as a singular fiber. Let F_1 (resp. F_2) be the singular fiber containing $C + D_1$ (resp. $B_1 + \cdots + B_5$). Then there exists a (-1)-curve E such that $E.D_1 = 1, E.B_6 = 2$ and

$$F_1 = C + D_1 + E$$

because B_6 is a 2-section. Note that F_2 consists of only (-1) and (-2)-curves (cf. Lemma 1.1,(4)). So, F_2 has the second type in Lemma 6.11. Thus, there are two (-1)-curves E_1, E_2 such that $E_1.B_1 = B_5.E_2 = 1$ and

$$F_2 = E_1 + B_1 + \cdots + B_5 + E_2.$$

Since $\rho(S) = 1$ and since $D - (D_2 + D_3 + B_6)$ is contained in singular fibers of φ,

$$F_0,\ F_1,\ F_2$$

are all singular fibers of φ for otherwise the cross-section D_2 would meet an irreducible component of D in a singular fiber ($\neq F_0, F_1, F_2$) which contains only one (-1)-curve.

Let $\tau : \widetilde{S} \to \Sigma_3$ be a smooth blowing-down of curves in singualr fibers such that $\tau(D_2)^2 = -3$. Then $\tau(D_3) \sim \tau(D_2) + 3\tau(F_0), \tau(B_6) \sim 2\tau(D_2) + 6\tau(F_0)$. Hence $\tau(D_3)^2 = 3$ and $\tau(B_6)^2 = 12$. Thus, $D_2.E_1 = D_3.E_2 = B_6.E_1 = 1$.

Now all singular fibers of φ and $C + D$ are precisely described above.

This proves Claim(2).

Now we have only to consider the case $r \geq 2$. Indeed, if $(s,t) = (1,t)$ or $(s,1)$ then $r \geq 2$ by Lemma 21.

Claim(3). It is impossible that $(s,t) = (2,2)$ and $r \geq 2$.

We consider the case where $(s,t) = (2,2), r \geq 2$. Then $u = 7 - r \leq 5$. Let $F_0 := 3C + 2D_1 + D_1' + D_2$ and let $\psi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with F_0 as a singular fiber. Since Γ is a cross-section of ψ with $\Gamma.D_3' = \Gamma.B = 1$, D_3' and Δ_4 are contained in two distinct singular fibers, say F_1, F_2. So, F_1 consists of (-1) and (-2)-curves (cf. Lemma 1.1, (4)). Hence F_1 has one of two types in Lemma 6.11. Since D_2' is a cross-section, F_1 has two (-1)-curves E_1, E_2 such that $F_1 = E_1 + D_3' + E_2$ and $D_2'.E_1 = 1$. Since D_3 is a 3-section, one has $D_3.E_1 = D_3.E_2 = 1$ or $D_3.E_i = 2$ for $i = 1$ or 2. This is a contradiction to the fact that $-E_i.(K_{\widetilde{S}} + D^*) > 0$. Indeed, note that $D^* = 4/7D_3 + 2/7D_3' + 1/5D_2' +$ (other terms).

This proves Claim(3).

Next we consider the case where one of s,t is equal to 1. In view of Lemma 6.21, we have $r \geq 2$ and that only one of s,t is equal to 1. By Lemma 6.20, Δ_4 has a (-3) curve B. Let $F_0 := 3C + 2D_1 + D_1' + D_2$ and let $\varphi : \widetilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with F_0 as a singular fiber and F_1 the singular fiber containing B. By the arguments in Theorem 6.14, we may assume that every (-1)-curve E_i in F_1 fits Case (a) there. In particular, $E_i \cdot B = E_i \cdot D_3 = E_i \cdot H = 1$, where H is a (-2)-curve. Thus, D_3 meets only E_i's in F_1 because $D_3 \cdot F_1 = 3$ (cf. [18, Lemma 1.6]).

Case(1) There are three (-1) curves E_1, E_2, E_3 in F_1.

Then $F_1 = B + \Sigma E_i$. There are at most 2 more horizontal irreducible components of D viz. D_2' and R_{r-2} and they are sections of φ. Hence for at least one E_i, E_i meets no (-2)-curve, a contradiction.

Case(2) There are two (-1) curves E_1, E_2 in F_1.

Then $F_1 = E_1 - B - E_2 - G$, where G is a (-2) curve lying in Δ_1 or Δ_2.

If $G < \Delta_1$, then the section R_{r-2} does not meet either of E_1, E_2. Since E_1 meets a (-2)-curve, D_2' exists and $D_2' \cdot E_1 = 1$. Taking E_1 as a minimal curve we have a situation treated in the proof of Theorem 6.14. Hence Theorem 6.1 is true because in the present case D has no connected component of two (-2)-curves.

Suppose now that $G < \Delta_2$. Then the section D_2' does not meet either of E_1, E_2. By the same reasoning as in the case $G < \Delta_1$, but with E_1 replaced by E_2, Theorem 6.1 is true.

Case(3) There is a unique (-1) curve E in F_1.

Since E fits Case (a) in Theorem 6.14, F_1 has the configuration : $B - E - G_1 - G_2$, where G_1, G_2 are (-2) curves.

Now if $G_1 + G_2 < \Delta_2$, then Theorem 6.1 is true as above.

So assume $G_1 + G_2 < \Delta_1$. Now D_2' cannot exist as it cannot meet any curve in F_1. Hence $s = 1, r = 5, u = 1, t = 4$ by Lemma 6.20.

Since $\rho(S) = 1$, the number of horizontal irreducible components of D is one more than the difference between the number of (-1)-curves in singular fibers and the number of singular fibers of φ (cf. Lemma 1.5 of [18]).

Now Γ is a section of φ and $\Delta_3 - D_3$ is contained in a singular fiber, say F_2, which consists of only (-1) and (-2) curves. So, F_2 has the second type in Lemma 6.11 because $R_3 \cdot F_2 = 1$.

Then every singular fiber other than F_2 has exactly one (-1)-curve for $\rho(S) = 1$. We can write $F_2 = E_1 + T_1 + T_2 + T_3 + E_2$ with two (-1)-curves E_1, E_2 such that $E_1.T_1 = T_3.E_2 = 1$. Now $D^* = 1/3D_2 + \sum_{i=1}^{4} 2i/13T_i + 1/3B$. Since $D_3 \cdot F_2 = 3, (D_3 \cdot E_1, D_3 \cdot E_2) = (1,1), (0,2) or (2,0)$. This contradicts $-E_i \cdot (K + D^*) > 0$.

This completes the proof of Lemma 6.22.

Lemma 6.23. *Suppose that $D_3^2 = -3$. Then Theorem 6.1 is true.*

Proof. We use the notations in Lemmas 6.16 and 6.20. We also let $D_1' = R_{r-1}, D_2' = S_{s-1}, D_3' = T_{t-1}$. One may assume that $s \leq t$. By Lemma 6.21, it is impossible that $(s,t) = (1,1)$. So, $t \geq 2$.

Consider first the case where Δ_4 is a fork or a linear chain but B is not a tip. Then $\alpha \geq 1/2$. So,

$$1/2 \leq \alpha < 1/(2s+1) + 1/(2t+1)$$

by Lemma 6.20 and hence $(s,t) = (1,2)$. By Lemma 6.21 have $r > 1$.

We use arguments after Claim (3) in Lemma 6.22. In that proof we have used $D_3^2 = -4$ only in the last part.

In the present situation, by Lemma 6.20 (3), $r + s + t + u = 7 - (-3) = 10$. The arguments in the above Lemma reduced to considering the case $r = 5, s = 1, u = 1$. But then in the present situation $t = 3$, contradicting the assumption $t = 2$ above.

Hence the case when Δ_4 is a fork or a linear chain but B is not a tip can not to occur.

Next we consider the case where Δ_4 is a linear chain with B as a tip. Write $\Delta_4 = \sum_{i=1}^{u} B_i$ such that $B_u = B$. Then

$$D^* = \sum_i i/(2u+1)B_i + (other\, terms).$$

By Lemma 6.20, one has

$$u/(2u+1) < 1/(2s+1) + 1/(2t+1).$$

Therefore, by virtue of Lemma 6.20,(3),

$$(r, s, t, u) = (r, 1, t, 9 - r - t),\ (5, 2, 2, 1),\ (4, 2, 3, 1)$$

because $s \leq t$ by the additional assumption. If $r = 1$ or $t = 1$ then $(r, s, t, u) = (r, 1, t, u)$, a contradiction to Lemma 6.21. So, $r \geq 2, t \geq 2$.

Once $r > 1$, we obtained $r = 5, s = 1, u = 1$ and hence $t = 3$ by the arguments after Claim (3) in Lemma 6.22.

Let $F_0 := 3C + 2D_0 + D_0' + D_2$, and $\varphi : \tilde{S} \to \mathbf{P}^1$ the $\mathbf{P}^1$-fibration with F_0 as a singular fiber. Then D_3 is a 3-section. Let F_1 be the singular fiber containing Δ_4. By the same reasoning as in Lemma 6.22, we deduce that $F_1 = 3E + B + 2R_1 + R_2$, where E is a minimal (-1) curve and $E \cdot B = E \cdot R_1 = E \cdot D_3 = 1$. We also see easily that φ has precisely one more singular fiber $F_2 = E_1 + T_1 + T_2 + E_2$, where E_1, E_2 are (-1) curves (cf. [18, Lemma 1.5 and Lemma 1.1 (4)).

Now $D^* = 1/3D_2 + \sum_{i=1}^{3} i/7T_i + 1/3B$. Since $-E_i.(K_{\tilde{S}} + D^*) > 0, D_3.E_i = 1$ for $i = 1$ and 2 because $D_3.F_2 = 3$. We see also that $\tilde{E}_1 \cdot R_3 = 1$.

Let $S_0' := 3C + 2D_1 + D_1' + D_3$ and let $\psi : \tilde{S} \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with S_1' as a singular fiber. Then D_2, D_3', R_3 are 3-section, cross-section, cross-section, respectively. Since $\rho(S) = 1$ and since D_3' is a cross-section, one can find (-1)-curves $L_1, \cdots, L_4$ such that $S'_0, S'_1 := 2L_1+T_1+B+L_2, S'_2 := L_3 + R_1 + R_2 + L_4$ are all singular fibers of ψ. Moreover, $L_1.T_1 = L_1.B = L_2.B = L_2.R_3 = L_3.R_1 = L_4.R_2 = (L_3 + L_4).D'_3 = 1$ and $(D_2.L_1, D_2.L_2) = (1,1)$ or $(0,3)$.

If $(D_2.L_1, D_2.L_2) = (1,1)$, then $-C.(K_{\tilde{S}}+D^*) = 1-C.(1/3D_2+3/7D_3) = 1-(1/3+3/7) > 1-(1/3+1/3+1/7) = 1-L_1.(1/3D_2+1/3B+1/7T_1) = -L_1.(K_{\tilde{S}}+D^*)$, a contradiction to the choice of C. If $(D_2.L_1, D_2.L_2) = (0,3)$, then $0 < -L_2.(K_{\tilde{S}}+D^*) = 1-L_2.(1/3D_2+1/3B) = 1-(1/3)\times 3-1/3 < 0$, a contradiction.

So this Case is impossible.

This completes the proof of Lemma 6.23.

Lemma 6.24 *In the situation of Lemma 6.22, $\pi_1(S^0)$ is finite.*

Proof. Recall that in this case the curve E_2 meets only the irreducible components D_3 and B_5 of D, tranversally in one point and Δ_4 is linear with the (-3) curve B_6 as a tip. Hence, $A := E_2 + \Delta_4$ supports a divisor with positive self-intersection. We will now apply Lemma 1.10.

Let U be a nice tubular neighborhood of A. Since $E_2 - D$ is isomorphic to $\mathbf{C}^*$, we see easily that $U - D$ deforms to a tubular neighborhood of Δ_4. In particular, $\pi_1(U-D)$ is a finite cyclic group. By Lemma 1.10 we have a surjection of this group onto $\pi_1(S^0)$. Hence the latter group is finite.

This completes the proof of Lemma 6.24.

In view of the results in this part **I** of the paper, the proof of the Main Theorem will be complete once we have shown the finiteness of $\pi_1(S^0)$ in the "2-component" case i.e., Case (II-4). This will be accomplished in part **II** of the paper.

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