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CANONICAL BASES OF QUANTUM SCHUBERT CELLS AND THEIR SYMMETRIES

ARKADY BERENSTEIN AND JACOB GREENSTEIN

Abstract. The goal of this work is to provide an elementary construction of the canonical basis $\mathbf{B}(w)$ in each quantum Schubert cell $U_q(w)$ and to establish its invariance under modified Lusztig's symmetries. To that effect, we obtain a direct characterization of the upper global basis $\mathbf{B}^{up}$ in terms of a suitable bilinear form and show that $\mathbf{B}(w)$ is contained in $\mathbf{B}^{up}$ and its large part is preserved by modified Lusztig's symmetries.

Contents

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1. Introduction and main results

Let $\mathfrak{g} = \mathfrak{b}^- \oplus \mathfrak{n}^+$ be a symmetrizable Kac-Moody Lie algebra. For any w in its Weyl group W, define the algebra $U_q(w)$ by

$$U_q(w) := T_w(U_q(\mathfrak{b}^-)) \cap U_q(\mathfrak{n}^+) \tag{1.1}$$

which we refer to as a quantum Schubert cell (see §2.1 for notation). This terminology is justified in Remark 4.7. The definition (1.1) for an infinite (affine) type first appeared in [1, Proposition 2.3]. In [2] we conjectured that this definition coincides with Lusztig's one which was proved for all Kac-Moody algebras by Tanisaki in [19, Proposition 2.10] and independently by Kimura ([8, Theorem 1.3]).

In a remarkable paper [7] Kimura proved that each $U_q(w)$ is compatible with the upper global basis $\mathbf{B}^{up}$ of $U_q(\mathfrak{n}^+)$. The aim of the present work is twofold:

- to construct the basis $\mathbf{B}(w)$ of $U_q(w)$ explicitly using a generalization of Lusztig's Lemma.
- to compute the action of Lusztig symmetries on these bases, thus partially verifying Conjecture 1.16 from [2].

To achieve the first goal, first we provide an independent definition (see §2.4 and §2.6) of the global crystal basis $\mathbf{B}^{up}$ (which coincides with the dual canonical basis). For reader's convenience, we put all necessary definitions and results in Section 2.

Let $\bar{\cdot}$ be the anti-linear anti-involution of $U_q(\mathfrak{g})$ which maps $q^{\frac{1}{2}}$ to $q^{-\frac{1}{2}}$ and fixes Chevalley generators. It should be noted that we use a slightly different presentation of $U_q(\mathfrak{g})$ (see [2] and §2.1) and accordingly modified T_w so that they commute with $\bar{\cdot}$ ([2]).

Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w)$. Generalizing [6, 13], we show (see §4.1) that the $\mathbb{A} := \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-subalgebra of $U_q(\mathfrak{n}^+)$ generated by the $X_{\mathbf{i},k} := T_{s_{i_1}\cdots s_{i_{k-1}}}(E_{i_k})$, $1 \le k \le m$ of $U_q(\mathfrak{n}^+)$ is in fact independent of $\mathbf{i}$, hence is denoted $U^{\mathbb{A}}(w)$, and has an $\mathbb{A}$-basis $\{X_{\mathbf{i}}^{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}_{\ge 0}^m\}$ where $X_{\mathbf{i}}^{\mathbf{a}} = q_{\mathbf{i},\mathbf{a}} X_{\mathbf{i},1}^{a_1} \cdots X_{\mathbf{i},m}^{a_m}$ and $q_{\mathbf{i},\mathbf{a}} \in q^{\frac{1}{2}\mathbb{Z}}$ is defined in (4.1). The importance of this choice of the $q_{\mathbf{i},\mathbf{a}}$ is highlighted by the following version of Lusztig's Lemma.

Theorem 1.1. *Let $w \in W$ and $\mathbf{i} \in R(w)$. For every $\mathbf{a} \in \mathbb{Z}_{\ge 0}^m$ there exists a unique $b_{\mathbf{a}} = b_{\mathbf{i},\mathbf{a}} \in U^{\mathbb{A}}(w)$ such that $\overline{b_{\mathbf{a}}} = b_{\mathbf{a}}$ and*

$$b_{\mathbf{a}} - X_{\mathbf{i}}^{\mathbf{a}} \in \sum_{\mathbf{a}' \ne \mathbf{a}} q^{-1}\mathbb{Z}[q^{-1}] X_{\mathbf{i}}^{\mathbf{a}'}.$$

We prove Theorem 1.1 in §4.2.

In particular, elements $b_{\mathbf{i},\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}_{\ge 0}^m$ form a basis $\mathbf{B}(\mathbf{i})$ of $U^{\mathbb{A}}(w)$ which a priori depends on $\mathbf{i}$. However, the following result implies that this is not the case.

Theorem 1.2. *Let $w \in W$ and $\mathbf{i} \in R(w)$. Then for all $\mathbf{a} \in \mathbb{Z}_{\ge 0}^{\ell(w)}$ we have $b_{\mathbf{i},\mathbf{a}} \in \mathbf{B}^{up}$.*

This Theorem implies that for any $\mathbf{i}, \mathbf{i}' \in R(w)$ we have $\mathbf{B}(\mathbf{i}) = \mathbf{B}(\mathbf{i}')$ and thus can introduce $\mathbf{B}(w)$. As a consequence, we recover the main result (Theorem 4.22) of [7].

Corollary 1.3. $\mathbf{B}(w) = \mathbf{B}^{up} \cap U_q(w)$ *for all $w \in W$.*

Remark 1.4. To obtain his result, Kimura used a rather elaborate theory of global crystal bases. By contrast, our proofs of Theorems 1.1 and 1.2 are quite elementary and short.

Now we turn our attention to the second goal, that is, to the action of Lusztig's symmetries on $U_q(w)$.

Theorem 1.5 ([2, Conjecture 1.17])**.** *Let $w, w' \in W$ be such that $\ell(ww') = \ell(w) + \ell(w')$. Then*

$$\mathbf{B}(w) \subset \mathbf{B}(ww'), \qquad T_w(\mathbf{B}(w')) \subset \mathbf{B}(ww').$$

Remark 1.6. (a) In [2] we constructed a basis $\mathbf{B}_{\mathfrak{g}}$ of $U_q(\mathfrak{g})$ containing $\mathbf{B}^{up}$ and conjectured ([2, Conjecture 1.16]) that $T_w(\mathbf{B}^{up}) \subset \mathbf{B}_{\mathfrak{g}}$. Thus, Theorem 1.5 provides supporting evidence for that conjecture.
(b) It would be interesting to compare the symmetries discussed above with the quantum twist computed in [9].

We deduce Theorem 1.5 from Theorem 1.1 in §4.4. All these results are obtained using the following striking property of $\mathbf{B}^{up}$ which is parallel to a highly non-trivial result of Lusztig ([17]).

Theorem 1.7. *$T_{s_i}(b) \in \mathbf{B}^{up}$ whenever $b \in \mathbf{B}^{up} \cap T_{s_i}^{-1}(U_q(\mathfrak{n}^+))$.*

We prove Theorem 1.7 in §3.5. Our proof, which is quite elementary and short, relies on the notion of *decorated algebras* (Definition 3.1) to which we generalize T_{s_i} and obtain an explicit formula (Theorem 3.6) for it.

We conclude this section with the following curious application of the above constructions. It is well-known (see e.g. Remark 2.14) that the natural linear anti-involution * on $U_q(\mathfrak{n}^+)$ fixing the Chevalley generators (see §2.1) preserves $\mathbf{B}^{up}$. Since $T_w \circ {}^* = {}^* \circ T_{w^{-1}}^{-1}$ (cf. [2]) it follows that

$$U_q(w)^* = T_{w^{-1}}^{-1}(U_q(\mathfrak{b}^-)) \cap U_q(\mathfrak{n}^+)$$

and Corollary 1.3 implies that $U_q(w)^*$ has a basis $\mathbf{B}(w)^* = U_q(w)^* \cap \mathbf{B}^{up}$. In particular, one can consider the algebras

$$U_q(w, w') := U_q(w) \cap U_q(w')^*, \qquad w, w' \in W$$

which is natural to call bi-Schubert algebras. The following is immediate.

Corollary 1.8. *For any $w, w' \in W$, the bi-Schubert algebra $U_q(w, w')$ has a basis $\mathbf{B}(w, w') := \mathbf{B}(w) \cap \mathbf{B}(w')^* = U_q(w, w') \cap \mathbf{B}^{up}$.*

Based on numerous examples (see §5.5) one can conjecture that bi-Schubert algebras are Poincaré-Birkhoff-Witt (PBW).

Remark 1.9. One can also consider intersections $U_q(w) \cap U_q(w')$; however, in this case it appears (and is probably well-known) that the corresponding algebra is always $U_q(w'')$ where w'' is less than both w and w' in the weak right Bruhat order and is maximal with that property.

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2. Definition and characterization of $\mathbf{B}^{up}$

2.1. **Preliminaries.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra with the Cartan matrix $A = (a_{ij})_{i,j\in I}$. Let $\{\alpha_i\}_{i\in I}$ be the standard basis of $Q = \mathbb{Z}^I$. Fix $d_i \in \mathbb{Z}_{>0}$, $i \in I$ such that the matrix $(d_i a_{ij})_{i,j\in I}$ is symmetric and define a symmetric bilinear form $(\cdot,\cdot) : Q \times Q \to \mathbb{Z}$ by $(\alpha_i, \alpha_j) = d_i a_{ij}$; clearly, $(\gamma,\gamma) \in 2\mathbb{Z}$ for any $\gamma \in Q$. We will write $(\alpha_i^\vee, \gamma)$, $\gamma \in Q$ as an abbreviation for $(\alpha_i, \gamma)d_i^{-1}$.

The quantized enveloping algebra $U_q(\mathfrak{g})$ is an associative algebra over $\Bbbk = \mathbb{Q}(q^{\frac{1}{2}})$ generated by the E_i, F_i, $K_i^{\pm 1}$, $i \in I$ subject to the relations

$$[E_i, F_j] = \delta_{ij}(q_i^{-1} - q_i)(K_i - K_i^{-1}), \quad K_iE_j = q_i^{a_{ij}} E_jK_i,\ K_iF_j = q_i^{-a_{ij}} F_jK_i,\ K_iK_j = K_jK_i \tag{2.1}$$

$$\sum_{r,s\ge 0,\, r+s=1-a_{ij}} (-1)^s E_i^{\langle r\rangle} E_j E_i^{\langle s\rangle} = \sum_{r,s\ge 0,\, r+s=1-a_{ij}} (-1)^s F_i^{\langle r\rangle} F_j F_i^{\langle s\rangle} = 0, \quad i \ne j \tag{2.2}$$

for all $i,j\in I$, where $q_i=q^{d_i}$, $X_i^{\langle k\rangle}:=\big(\prod_{s=1}^k\langle s\rangle_{q_i}\big)^{-1}X_i^k$ and $\langle s\rangle_v=v^s-v^{-s}$. We also set $(n)_v=\langle n\rangle_v/\langle 1\rangle_v$, $\langle n\rangle_v!=\prod_{t=1}^n\langle t\rangle_v$, $(n)_v!=\langle n\rangle_v!/(\langle 1\rangle_v)^n$,

$$\binom{n}{k}_v=\frac{\prod_{t=0}^{k-1}(n-t)_v}{(k)_v!}=\frac{\prod_{t=0}^{k-1}\langle n-t\rangle_v}{\langle k\rangle_v!}$$

and $X_i^{(n)}:=X_i^n/(n)_{q_i}!$.

We denote by $U_q(\mathfrak n^+)$ (respectively, $U_q(\mathfrak n^-)$) the subalgebra of $U_q(\mathfrak g)$ generated by the E_i (respectively, the F_i), $i\in I$. Let $\mathcal K$ be the subalgebra of $U_q(\mathfrak g)$ generated by the $K_i^{\pm1}$, $i\in I$ and set $U_q(\mathfrak b^\pm)=\mathcal K U_q(\mathfrak n^\pm)$.

It is easy to see from the presentation that $U_q(\mathfrak g)$ admits anti-involutions ^{t} and *, where ^{t} interchanges E_i and F_i for each $i\in I$ and preserves the $K_i^{\pm1}$ while * preserves the E_i and F_i while $K_i^*=K_i^{-1}$. Furthermore, $U_q(\mathfrak g)$ admits an anti-linear anti-involution $\bar{\cdot}$ which preserves all generators and maps $q^{\frac12}$ to $q^{-\frac12}$.

The algebra $U_q(\mathfrak n^+)$ is naturally graded by $Q^+:=\bigoplus_{i\in I}\mathbb Z_{\ge0}\alpha_i$ via $\deg E_i=\alpha_i$. We denote the homogeneous component of $U_q(\mathfrak n^+)$ of degree $\gamma\in Q^+$ by $U_q(\mathfrak n^+)_\gamma$. This can be extended to a Q-grading on $U_q(\mathfrak g)$ via $\deg F_i=-\alpha_i$, $\deg K_i=0$.

2.2. **Modified Lusztig symmetries.** Let W be the Weyl group of $\mathfrak g$. It is generated by the simple reflections s_i, $i\in I$ which act on Q via $s_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$. Given $w\in W$, denote $R(w)$ the set of reduced words for w, that is, the set of $\mathbf i=(i_1,\dots,i_m)\in I^m$ of minimal length $m:=\ell(w)$ such that $w=s_{i_1}\cdots s_{i_m}$. It is well-known that the form $(\cdot,\cdot)$ is W-invariant.

The following essentially coincides with Theorem 1.13 from [2].

Lemma 2.1. (a) *For each $i\in I$ there exists a unique automorphism T_i of $U_q(\mathfrak g)$ which satisfies $T_i(K_j)=K_jK_i^{-a_{ij}}$ and*

$$T_i(E_j)=\begin{cases}q_i^{-1}K_i^{-1}F_i, & i=j\\ \sum\limits_{r+s=-a_{ij}}(-1)^r q_i^{s+\frac12 a_{ij}}E_i^{\langle r\rangle}E_jE_i^{\langle s\rangle}, & i\ne j\end{cases}$$

$$T_i(F_j)=\begin{cases}q_i^{-1}K_iE_i, & i=j\\ \sum\limits_{r+s=-a_{ij}}(-1)^r q_i^{s+\frac12 a_{ij}}F_i^{\langle r\rangle}F_jF_i^{\langle s\rangle}, & i\ne j\end{cases}$$

(b) *For all $x\in U_q(\mathfrak g)$, $\overline{T_i(x)}=T_i(\overline{x})$, $(T_i(x))^*=T_i^{-1}(x^*)$ and $(T_i(x))^t=T_i^{-1}(x^t)$.*
(c) *The T_i, $i\in I$ satisfy the braid relations on $U_q(\mathfrak g)$, that is, they define a representation of the Artin braid group $\mathrm{Br}_{\mathfrak g}$ of $\mathfrak g$ on $U_q(\mathfrak g)$.*

2.3. **Bilinear forms.** Following [2, 16], we define a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $U_q(\mathfrak n^+)$. Let $V=\bigoplus_{i\in I}\Bbbk E_i$ and let $\langle\cdot,\cdot\rangle$ be the bilinear form on V defined by $\langle E_i,E_j\rangle=\delta_{ij}(q_i-q_i^{-1})$. Extend it naturally to $T(V)$ via

$$\langle v_1\otimes\cdots\otimes v_k,v_1'\otimes\cdots\otimes v_l'\rangle'=\delta_{kl}\prod_{r=1}^k\langle v_r,v_r'\rangle,\qquad v_r,\,v_r'\in V,\,1\le r\le k.$$

Define a linear map $\Psi:V\otimes V\to V\otimes V$ by $\Psi(E_i\otimes E_j)=q^{(\alpha_i,\alpha_j)}E_j\otimes E_i$. Finally, define $\langle\cdot,\cdot\rangle_\Psi$ via

$$\langle u,v\rangle_\Psi=\delta_{k,l}\langle [k]_\Psi!(u),v\rangle'=\delta_{k,l}\langle u,[k]_\Psi!(v)\rangle',\qquad u\in V^{\otimes k},\,v\in V^{\otimes l}$$

and $[k]_\Psi!\in\operatorname{End}_\Bbbk V^{\otimes k}$ is the standard notation for the braided k-factorial (see e.g. [2, §A.1]). It is well-known (see e.g. [16]) that the kernel J of the canonical map $T(V)\to U_q(\mathfrak n^+)$, $E_i\mapsto E_i$ is the radical of $\langle\cdot,\cdot\rangle_\Psi$. Thus, we have a well-defined non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $U_q(\mathfrak{n}^+)$ given by $\langle u+J, v+J\rangle = \langle u,v\rangle_\Psi$. This form in fact coincides with the form $\langle\cdot,\cdot\rangle$ we introduced in [2, §A.3] if we identify $U_q(\mathfrak{n}^-)$ with $U_q(\mathfrak{n}^+)$ via ^{*t}. We will often use the following obvious

Lemma 2.2. *Let $x, x' \in U_q(\mathfrak{n}^+)$ be homogeneous. Then $\langle x,x'\rangle \neq 0$ implies that $\deg x = \deg x'$.*

Define $(\!|\cdot,\cdot|\!) : U_q(\mathfrak{n}^+)\otimes U_q(\mathfrak{n}^+) \to \Bbbk$ by

$$(\!|x,y|\!) = \mu(\gamma) q^{-\frac{1}{2}(\gamma,\gamma)}\langle x,y\rangle, \qquad x,y\in U_q(\mathfrak{n}^+)_\gamma$$

where

$$\mu(\gamma) = q^{\frac{1}{4}(\gamma,\gamma)+\frac{1}{2}\eta(\gamma)}, \qquad \gamma\in Q \tag{2.3}$$

and $\eta\in \operatorname{Hom}_{\mathbb{Z}}(Q,\mathbb{Z})$ is defined by $\eta(\alpha_i) = d_i$. Note the following properties of μ which will be often used in the sequel

$$\mu(r\alpha_i) = q_i^{\binom{r+1}{2}}, \quad \mu(s_i\gamma) = \mu(\gamma)q^{-\frac{1}{2}(\alpha_i,\gamma)}, \quad \mu(\gamma+\gamma') = \mu(\gamma)\mu(\gamma')q^{\frac{1}{2}(\gamma,\gamma')} \tag{2.4}$$

Define an anti-linear automorphism $\tilde{\ }$ of $U_q(\mathfrak{g})$ by

$$\tilde{x} = (\operatorname{sgn}\gamma)\bar{x}^*, \qquad x\in U_q(\mathfrak{g})_\gamma$$

where $\operatorname{sgn} : Q \to \{\pm 1\}$ is the homomorphism of abelian groups defined by $\operatorname{sgn}(\alpha_i) = -1$. Then (cf. [2])

$$\overline{(\!|x,y|\!)} = (\!|\bar{x},\tilde{y}|\!) = (\!|\tilde{x},\bar{y}|\!). \tag{2.5}$$

2.4. **Lattices and signed basis in $U_q(\mathfrak{n}^+)$.** Let $\mathbb{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ which is a subring of $\mathbb{Q}(q^{\frac{1}{2}})$. Denote $\mathbb{A}_0 = \mathbb{Z}[q,q^{-1}]$ and $\mathbb{A}_1 = q^{\frac{1}{2}}\mathbb{A}_0$; clearly, $\mathbb{A} = \mathbb{A}_0\oplus\mathbb{A}_1$ as an $\mathbb{A}_0$-module. Following [2, §3.1], for any $J\subset I$, let $U_{\mathbb{Z}}(\mathfrak{n}^+)_J$ (respectively, $U_{\mathbb{Z}}(\mathfrak{n}^-)_J$) be the $\mathbb{A}_0$-subalgebra of $U_q(\mathfrak{n}^+)$ (respectively, $U_q(\mathfrak{n}^-)$) generated by the $E_i^{\langle n\rangle}$ (respectively, $F_i^{\langle n\rangle}$), $i\in J$, $n\in\mathbb{Z}_{\ge 0}$. We abbreviate $U_{\mathbb{Z}}(\mathfrak{n}^\pm) := U_{\mathbb{Z}}(\mathfrak{n}^+)_I$. Set

$$U^{\mathbb{Z}}(\mathfrak{n}^+) = \{x\in U_q(\mathfrak{n}^+) \,:\, (\!|x, U_{\mathbb{Z}}(\mathfrak{n}^+)|\!) \subset \mathbb{A}_0\}.$$

Clearly, $U^{\mathbb{Z}}(\mathfrak{n}^+)$ is an $\mathbb{A}_0$-submodule of $U_q(\mathfrak{n}^+)$.

Lemma 2.3. *We have $q^{\frac{1}{2}(\gamma,\gamma')}xx' \in U^{\mathbb{Z}}(\mathfrak{n}^+)$ for all $x\in U^{\mathbb{Z}}(\mathfrak{n}^+)_\gamma$, $x'\in U^{\mathbb{Z}}(\mathfrak{n}^+)_{\gamma'}$. In particular, all powers of a homogeneous element of $U^{\mathbb{Z}}(\mathfrak{n}^+)$ are in $U^{\mathbb{Z}}(\mathfrak{n}^+)$ and $U^{\mathbb{A}}(\mathfrak{n}^+) := U^{\mathbb{Z}}(\mathfrak{n}^+)\otimes_{\mathbb{A}_0}\mathbb{A}$ is an $\mathbb{A}$-algebra.*

Proof. Following [16, §1.2], let $\underline{\Delta} : U_q(\mathfrak{n}^+) \to U_q(\mathfrak{n}^+)\underline{\otimes} U_q(\mathfrak{n}^+)$ be the braided co-multiplication defined by $\underline{\Delta}(E_i) = E_i\otimes 1 + 1\otimes E_i$, where $U_q(\mathfrak{n}^+)\underline{\otimes} U_q(\mathfrak{n}^+) = U_q(\mathfrak{n}^+)\otimes U_q(\mathfrak{n}^+)$ endowed with an algebra structure via $(x\otimes y)(x'\otimes y') = q^{(\gamma,\gamma')}xx'\otimes yy'$ for all $x,y'\in U_q(\mathfrak{n}^+)$, $y\in U_q(\mathfrak{n}^+)_\gamma$, $x'\in U_q(\mathfrak{n}^+)_{\gamma'}$. Then $\langle xx',y\rangle = \langle x,\underline{y}_{(1)}\rangle\langle x',\underline{y}_{(2)}\rangle$ and so

$$(\!|xx',y|\!) = q^{-\frac{1}{2}(\gamma,\gamma')}(\!|x,\underline{y}_{(1)}|\!)(\!|x',\underline{y}_{(2)}|\!), \qquad x\in U_q(\mathfrak{n}^+)_\gamma,\ x'\in U_q(\mathfrak{n}^+)_{\gamma'},$$

where $\underline{\Delta}(y) = \underline{y}_{(1)}\otimes\underline{y}_{(2)}$ in Sweedler-like notation. It follows from [16, Lemma 1.4.2] that $\underline{\Delta}(U_{\mathbb{Z}}(\mathfrak{n}^+)) \subset U_{\mathbb{Z}}(\mathfrak{n}^+)\otimes_{\mathbb{A}_0} U_{\mathbb{Z}}(\mathfrak{n}^+)$, hence we can assume that $\underline{y}_{(1)},\underline{y}_{(2)}\in U_{\mathbb{Z}}(\mathfrak{n}^+)$ provided that $y\in U_{\mathbb{Z}}(\mathfrak{n}^+)$. All assertions are now immediate. □

Define, for any $\gamma\in Q^+$

$$\mathbf{B}^{\pm up}{}_\gamma = \{b\in U^{\mathbb{Z}}(\mathfrak{n}^+)_\gamma \,:\, \bar{b} = b,\ \mu(\gamma)^{-1}(\!|b,b|\!)\in 1+q^{-1}\mathbb{Z}[[q^{-1}]])\} \tag{2.6}$$

and set $\mathbf{B}^{\pm up} = \bigsqcup_{\gamma\in Q^+}\mathbf{B}^{\pm up}{}_\gamma$.

2.5. **Signed basis is (K_-,μ)-orthonormal.** Let R be a commutative unital subring of a field $\Bbbk$. Following [16, §14.2.1], a subset $\mathbf{B}^{\pm}$ of a free R-module L is a *signed basis* of L if $\mathbf{B}^{\pm}=\mathbf{B}\sqcup(-\mathbf{B})$ for some basis $\mathbf{B}$ of L.

Let K_- be a subring of $\Bbbk$ not containing 1. We say that $\mathbf{B}$ is (K_-,μ)-orthonormal for some $\mu\in R^\times$ with respect to a fixed symmetric bilinear pairing $(\!|\cdot,\cdot|\!):L\otimes_R L\to\Bbbk$ if

$$\mu\cdot(\!|b,b'|\!)\in\delta_{b,b'}+K_-,\qquad b,b'\in\mathbf{B}.$$

Accordingly, we say that a signed basis $\mathbf{B}^{\pm}$ is (K_-,μ)-orthonormal if it contains a (K_-,μ)-orthonormal basis of L.

The following result is parallel to [16, Theorem 14.2.3].

Theorem 2.4. *$\mathbf{B}^{\pm up}$ is a signed basis of $U^{\mathbb{Z}}(\mathfrak{n}^+)$ with $R=\mathbb{A}_0$. Moreover, for each $\gamma\in Q^+$, ${\mathbf{B}^{\pm up}}_\gamma$ is a $(K_-,\mu(\gamma)^{-1})$-orthonormal basis where μ is defined by (2.3) and $K_-=q^{-1}\mathbb{Z}[[q^{-1}]]\cap\mathbb{Q}(q)$.*

Proof. We need the following general setup.

We say that a domain R_0 is *strongly integral* if a sum of squares of its non-zero elements is never zero and if $c_1^2+\cdots+c_n^2=1$, $c_r\in R_0$, implies that for all $1\le i\le n$, $c_i=\pm\delta_{ij}$ for some $1\le j\le n$.

Let R be a domain with a subdomain R_0. Given a totally ordered additive monoid Γ, a map $\nu:R\to\Gamma\sqcup\{-\infty\}$ is called an *R_0-linear valuation* if the following hold for all $f,g\in R$

(V_1) $\nu(f)=-\infty$ if and only if $f=0$
(V_2) $\nu(R_0\setminus\{0\})=0$,
(V_3) $\nu(fg)=\nu(f)+\nu(g)$;
(V_4) $\nu(f+g)\le\max(\nu(f),\nu(g))$.

It follows that

$$\nu(f)\neq\nu(g)\implies\nu(f+g)=\max(\nu(f),\nu(g)).\tag{2.7}$$

Furthermore, for each $a\in\Gamma$, set $R_{\le a}=\{r\in R\,:\,\nu(r)\le a\}$ and $R_{<a}=\{r\in R\,:\,\nu(r)<a\}$. Clearly, $R_{\le a}$ and $R_{<a}$ are R_0-submodules of R and $R_{<a}\subset R_{\le a}$. In the spirit of [12, §2.1], we call the R_0-module $R_{\le a}/R_{<a}$ the *leaf of ν at a*; we say that ν *has one-dimensional leaves* if for each $a\in\nu(R)$, the leaf of ν at a is a non-zero cyclic R_0-module.

Let M be a free R-module with a basis $\mathbf{B}$. Then we can define $\nu_{\mathbf{B}}:M\to\Gamma\cup\{-\infty\}$ by

$$\nu_{\mathbf{B}}(\sum_{b\in\mathbf{B}}c_b b)=\max_b\nu(c_b).\tag{2.8}$$

Clearly (V_4) holds and we also have $\nu_{\mathbf{B}}(fx)=\nu(f)+\nu_{\mathbf{B}}(x)$, $f\in R$, $x\in M$. We will need the following Lemma.

Lemma 2.5. *Suppose that $\nu:R\to\Gamma\cup\{-\infty\}$ has one-dimensional leaves and let M be a free R module with a basis $\mathbf{B}$. Then every $x\in M$ with $\nu_{\mathbf{B}}(x)>0$ can be written as $x=fx_0+x_1$ where $f\in R$ with $\nu(f)=\nu(x)$, $0\neq x_0\in\sum_{b\in\mathbf{B}}R_0 b$ and $x_1\in M$ satisfies $\nu_{\mathbf{B}}(x_1)<\nu_{\mathbf{B}}(x)$.*

Proof. Let $x\in M$ with $a=\nu_{\mathbf{B}}(x)>0$ and write

$$x=\sum_{b\in\mathbf{B}}x_b b=\sum_{b\in\mathbf{B}\,:\,\nu(x_b)=a}x_b b+\sum_{b\in\mathbf{B}\,:\,\nu(x_b)<a}x_b b.$$

Since ν has one-dimensional leaves and $R_{\le a}\neq R_{<a}$, $R_{\le a}/R_{<a}$ is a non-zero cyclic R_0-module. Let $f\in R_{\le a}$ be any element whose image generates $R_{\le a}/R_{<a}$ as an R_0-module. Then $\nu(f)=a$ and for every $b\in\mathbf{B}$ with $\nu(x_b)=a$ there exists $r_b\in R_0$ such that $\nu(x_b-r_bf)<a$. Set

$$x_0=\sum_{b\in\mathbf{B}\,:\,\nu(x_b)=a}r_b b,\qquad x_1=x-fx_0.$$

Clearly, $\nu_{\mathbf{B}}(x_1)<a$, whence $x_0\neq 0$. $\square$

Henceforth

- R_0 is a strongly integral domain
- $\Bbbk$ is a field containing R_0;
- $R_0 \subset R \subset \Bbbk$ as subrings
- $\nu : \Bbbk \to \Gamma \cup \{-\infty\}$ is an R_0-linear valuation;
- K_- is an R_0-subalgebra of $\Bbbk$ such that $\nu(f) < 0$ for all $f \in K_-$ (note that this implies that $K_- \cap R_0 = \emptyset$) and $(1 + K_-)^{-1} \subset 1 + K_-$.
- There is a field involution $\bar{\cdot}$ of $\Bbbk$ which restricts to R and is identity on R_0, while $\overline{K_-} \cap K_- = \emptyset$
- $\nu(R^{\bar{}} \setminus R_0) > 0$ where $R^{\bar{}} = \{f \in R : \bar{f} = f\}$;
- The restriction of ν to $R^{\bar{}}$ is a valuation $\nu : R^{\bar{}} \to \Gamma \sqcup \{-\infty\}$ with one-dimensional leaves;

For an R-module L, an endomorphism of $\mathbb{Z}$-modules $\varphi : L \to L$ is called anti-linear if for all $r \in R$, $x \in L$ we have $\overline{r \cdot x} = \bar{r} \cdot \bar{x}$. Anti-linear endomorphisms of a $\Bbbk$-vector space V are defined similarly.

Let V be a $\Bbbk$-vector space with a non-degenerate symmetric bilinear form $(\!|\cdot,\cdot|\!)$. Suppose that φ, φ' are anti-linear involutions on V satisfying $\overline{(\!|x, y|\!)} = (\!|\varphi(x), \varphi'(y)|\!)$, $x, y \in V$. Let L be a free R-module such that $V = \Bbbk \otimes_R L$. Denote $L^\vee = \{x \in V : (\!|x, L|\!) \subset R\}$. Clearly, $L^\vee$ is a free R-module and $V = \Bbbk \otimes_R L^\vee$.

Given $\mu \in R^\times$ define

$$\mathbf{B}^\pm(\mu) = \{b \in L : \varphi'(b) = b,\ \mu \cdot (\!|b, b|\!) \in 1 + K_-\}$$

and

$$\mathbf{B}^\vee_\pm(\mu) = \{b \in L^\vee : \varphi(b) = b,\ \mu \cdot (\!|b, b|\!) \in 1 + K_-\}.$$

Proposition 2.6. *Suppose that* $\dim_\Bbbk V < \infty$. *The following are equivalent*
(a) $\mathbf{B}^\pm(\mu)$ *is a* (K_-, μ)*-orthonormal signed* R*-basis of* L.
(b) $\mathbf{B}^\vee_\pm(\mu^{-1})$ *is a* (K_-, μ^{-1})*-orthonormal signed* R*-basis of* $L^\vee$.
In that case, $\mathbf{B}^\pm(\mu)$ *and* $\mathbf{B}^\vee_\pm(\mu)$ *are dual to each other with respect to* $(\!|\cdot,\cdot|\!)$.

Proof. (a) $\Longrightarrow$ (b) Let $\underline{\mathbf{B}}^\pm(\mu)$ be any basis of L contained in $\mathbf{B}^\pm(\mu)$.

Since $(\!|\cdot,\cdot|\!)$ is non-degenerate, for each $b \in \underline{\mathbf{B}}^\pm(\mu)$ there exists a unique $\delta_b \in L^\vee$ such that $(\!|\delta_b, b'|\!) = \delta_{b,b'}$. Clearly, the set $\underline{\mathbf{B}}^\pm(\mu)^\vee := \{\delta_b : b \in \underline{\mathbf{B}}^\pm(\mu)\}$ is a basis of $L^\vee$. Note that $\varphi(\delta_b) = \delta_b$.

Lemma 2.7. *The set* $\underline{\mathbf{B}}^\pm(\mu)^\vee$ *is* (K_-, μ^{-1})*-orthonormal basis of* $L^\vee$. *In particular,* $\nu(\mu^{-1}(\!|\delta_b, \delta_{b'}|\!)) \leq 0$ *with the equality if and only if* $b = b'$.

Proof. We need the following

Lemma 2.8. *Let* $G = (G_{r,s})_{1 \leq r,s \leq n}$ *be a matrix over* $\Bbbk$ *such that* $\mu G_{rs} \in \delta_{rs} + K_-$. *Then* G *is invertible and* $M = (M_{rs})_{1 \leq r,s \leq n} = G^{-1}$ *satisfies* $\mu^{-1} M_{rs} \in \delta_{rs} + K_-$.

Proof. Let $\Delta_{r,s}(G)$ be the minor of G obtained by removing the rth row and the sth column. Then it is easy to see that $\mu^{n-1}\Delta_{r,s}(G) \in \delta_{rs} + K_-$. Similarly, $\mu^n \det G \in 1 + K_-$ hence G is invertible. Moreover, $\mu^{-n}(\det G)^{-1} \in 1 + K_-$. Since $M_{rs} = (-1)^{r+s}(\det G)^{-1}\Delta_{s,r}(G)$, the assertion follows. □

Since $\underline{\mathbf{B}}^\pm(\mu)$ is (K_-, μ)-orthogonal, the above Lemma applies to the (finite) Gram matrix $G = ((\!|b, b'|\!))_{b,b' \in \underline{\mathbf{B}}^\pm(\mu)}$ of $(\!|\cdot,\cdot|\!)$ with respect to the basis $\underline{\mathbf{B}}^\pm(\mu)$ and hence $\mu^{-1} M_{b,b'} \in \delta_{b,b'} + K_-$ where $M = G^{-1}$. Since $\mu^{-1}\delta_b = \sum_{b' \in \underline{\mathbf{B}}^\pm(\mu)} \mu^{-1} M_{b,b'} b'$, we have $\mu^{-1}\delta_b \in b + K_- \cdot \underline{\mathbf{B}}^\pm(\mu)$. This implies that for all $b, b' \in \underline{\mathbf{B}}^\pm(\mu)$ one has

$$\mu^{-1}(\!|\delta_b, \delta_{b'}|\!) = (\!|\mu^{-1}\delta_b, \delta_{b'}|\!) \in (\!|b, \delta_{b'}|\!) + K_-(\!|\underline{\mathbf{B}}^\pm(\mu), \delta_{b'}|\!) = \delta_{b,b'} + K_-.$$

This proves Lemma 2.7. □

Note that for any $x = \sum_b x_b \delta_b$, $y = \sum_{b'} y_{b'} \delta_{b'}$ in $L^\vee$ we have

$$\mu^{-1}(\!|x, y|\!) = \sum_{b,b'} x_b y_{b'} \mu^{-1}(\!|\delta_b, \delta_{b'}|\!).$$

Since $\nu(\mu^{-1}(\!|\delta_b, \delta_{b'}|\!)) \le 0$ for all b, b' by Lemma 2.7, it follows from (V_3) and (V_4) that

$$\nu(\mu^{-1}(\!|x, y|\!)) \le \nu(x) + \nu(y), \qquad x, y \in L^\vee. \tag{2.9}$$

Clearly, for $x \in L^\vee$ we have

$$\varphi(x) = x \iff x \in \sum_{b \in \underline{\mathbf{B}}^\pm(\mu)} \bar{R}\,\delta_b. \tag{2.10}$$

Thus, the set $(L^\vee)^\varphi$ of φ-invariant elements in $L^\vee$ is a free $\bar{R}$-module with a basis $\underline{\mathbf{B}}^\pm(\mu)^\vee$.

The following Lemma is the crucial point of our argument.

Lemma 2.9. *Let $x \in L^\vee$ and suppose that $\varphi(x) = x$. Define $\nu = \nu_{\underline{\mathbf{B}}^\pm(\mu)^\vee} : L^\vee \to \Gamma \cup \{-\infty\}$ as in (2.8). Then*

(a) *If $\nu(x) = 0$, that is, $x = \sum_b x_b \delta_b$ with $x_b \in R_0$, then $\mu^{-1}(\!|x, x|\!) - \sum_b x_b^2 \in K_-$ and $\nu(\mu^{-1}(\!|x, x|\!)) = 0$.*

(b) *If $\nu(x) > 0$ then $\nu(\mu^{-1}(\!|x, x|\!)) > 0$*

Proof. Write $x = \sum_b x_b \delta_b$, $x_b \in \bar{R}$.

To prove (a), note that $\nu(x) = 0$ and $\varphi(x) = x$ implies that $x_b \in \mathbb{Z}$ for all $b \in \underline{\mathbf{B}}^\pm(\mu)$. We have

$$\mu^{-1}(\!|x, x|\!) = \sum_b x_b^2 \mu^{-1}(\!|\delta_b, \delta_b|\!) + \sum_{b \ne b'} x_b x_{b'} \mu^{-1}(\!|\delta_b, \delta_{b'}|\!).$$

By Lemma 2.7, the first sum belongs to $\sum_b x_b^2 + K_-$ while the second sum belongs to K_-. Since R_0 is strongly integral, $\sum_b x_b^2 \ne 0$. Thus, $\nu(\mu^{-1}(\!|x, x|\!)) = 0$.

To prove (b), let $a = \nu(x) > 0$. Applying Lemma 2.5 to $M = (L^\vee)^\varphi$ and the ring $\bar{R}$, we can write $x = f x_0 + x_1$ where $f \in \bar{R}$, $\nu(f) = a$, $\varphi(x_0) = x_0$ (and so $\varphi(x_1) = x_1$), $\nu(x_0) = 0$ and $\nu(x_1) < a$. Then

$$\nu(\mu^{-1}(\!|x, x|\!)) = \nu(f^2 \mu^{-1}(\!|x_0, x_0|\!) + 2f \mu^{-1}(\!|x_0, x_1|\!) + \mu^{-1}(\!|x_1, x_1|\!)) = \nu(f^2 \mu^{-1}(\!|x_0, x_0|\!)) = 2a > 0$$

since $\nu(\mu^{-1}(\!|x_1, x_1|\!)), \nu(f \mu^{-1}(\!|x_0, x_1|\!)) < 2a$ by (2.9) and (V_3), (V_4) while $\nu(\mu^{-1}(\!|x_0, x_0|\!)) = 0$ by part (a). This proves (b). □

It follows from Lemma 2.9(a,b) that if $x \in L^\vee$ is fixed by φ and $\mu^{-1}(\!|x, x|\!) \in 1 + K_-$ then $x = \pm\delta_b$ for some $b \in \underline{\mathbf{B}}^\pm(\mu)$ by the strong integrality of R_0. Thus, $\mathbf{B}^\vee_\pm(\mu) = \underline{\mathbf{B}}^\pm(\mu)^\vee \bigsqcup(-\underline{\mathbf{B}}^\pm(\mu)^\vee)$. This completes the proof of the implication (a) $\Longrightarrow$ (b) and the last assertion. The opposite implication follows by the symmetry between L and $L^\vee$ and φ and φ'. □

We now apply Proposition 2.6 with $L = U_\mathbb{Z}(\mathfrak{n}^+)_\gamma$, $v = q^{\frac{1}{2}}$, $\varphi = \bar{\cdot}$, $\varphi' = \tilde{\cdot}$, $R = \mathbb{A}_0 = \mathbb{Z}[v^2, v^{-2}]$, $\Bbbk = \mathbb{Q}(v)$ and $K_- = v^{-2}\mathbb{Z}[[v^{-2}]] \cap \mathbb{Q}(v)$. We define $\nu : \mathbb{Q}(v) \to \mathbb{Z} \cup \{-\infty\}$ via

$$\nu\Big(c v^n \frac{1+f}{1+g}\Big) = n$$

where $c \in \mathbb{Q}^\times$, $n \in \mathbb{Z}$ and $f, v \in v^{-1}\mathbb{Z}[v^{-1}]$. Note that $\bar{R} = \mathbb{Z}[q + q^{-1}]$ and ν has one-dimensional leaves on $\bar{R}$ since $\nu((v + v^{-1})^n) = n$. By [16, Theorem 14.2.3], $\mathbf{B}^{\pm can} \cap U_\mathbb{Z}(\mathfrak{n}^+)$ is a $(K_-, \mu(\gamma))$-orthonormal signed basis of L. Since ${\mathbf{B}^{\pm up}}_\gamma = (\mathbf{B}^{\rm can} \cap U_\mathbb{Z}(\mathfrak{n}^+))^\vee_\pm$ in the notation of Proposition 2.6, it is a signed $(K_-, \mu(\gamma)^{-1})$-orthonormal basis of $L^\vee = U^\mathbb{Z}(\mathfrak{n}^+)_\gamma$. This completes the proof of Theorem 2.4. □

2.6. **Choosing $\mathbf{B}^{up}$ inside the signed basis.** It remains to describe a canonical way to choose $\mathbf{B}^{up}$ inside $\mathbf{B}^{\pm up}$. Needless to say, it can be taken as the dual basis of $\mathbf{B}^{can}$ with respect to $(\!|\cdot,\cdot|\!)$. However, it more instructive to provide an intrinsic definition.

To that effect, following [2, §3.5] and also [16, Proposition 3.1.6], define $\Bbbk$-linear endomorphisms $\partial_i, \partial_i^{op}$, $i\in I$ of $U_q(\mathfrak{n}^+)$ by

$$[F_i,x]=(q_i-q_i^{-1})(q^{-\frac12(\alpha_i,\gamma-\alpha_i)}K_i\partial_i(x)-q^{\frac12(\alpha_i,\gamma-\alpha_i)}K_i^{-1}\partial_i^{op}(x)),\qquad x\in U_q(\mathfrak{n}^+)_\gamma. \tag{2.11}$$

We need the following properties of these operators (cf. [2, Lemmata 3.18 and 3.20]).

Lemma 2.10. *For all $x\in U_q(\mathfrak{n}^+)_\gamma$ and $i\in I$ we have*

(a) $\overline{\partial_i(x)}=\partial_i(\overline{x})$, $\overline{\partial_i^{op}(x)}=\partial_i^{op}(\overline{x})$, $\partial_i(x^*)^*=\partial_i^{op}(x)$ *and* $\partial_i\partial_i^{op}(x)=\partial_i^{op}\partial_i(x)$.

(b) *for all $y\in U_q(\mathfrak{n}^+)$, $n\in\mathbb{Z}_{\ge0}$*

$$(\!|x,yE_i^{\langle n\rangle}|\!)=(\!|\partial_i^{(n)}(x),y|\!),\qquad (\!|x,E_i^{\langle n\rangle}y|\!)=(\!|(\partial_i^{op})^{(n)}(x),y|\!),$$

where $f_i^{(n)}=(q_i-q_i^{-1})^nf_i^{\langle n\rangle}$.

(c) ∂_i, ∂_i^{op} *are quasi-derivations. Namely, for $x\in U_q(\mathfrak{n}^+)_\gamma$, $y\in U_q(\mathfrak{n}^+)_{\gamma'}$ we have*

$$\begin{aligned}\partial_i(xy)&=q^{\frac12(\alpha_i,\gamma')}\partial_i(x)y+q^{-\frac12(\alpha_i,\gamma)}x\partial_i(y),\\ \partial_i^{op}(xy)&=q^{-\frac12(\alpha_i,\gamma')}\partial_i^{op}(x)y+q^{\frac12(\alpha_i,\gamma)}x\partial_i^{op}(y).\end{aligned}\tag{2.12}$$

It is easy to see that

$$\partial_i^{(n)}(E_i^r)=\binom{r}{n}_{q_i}E_i^{r-n}=(\partial_i^{op})^{(n)}(E_i^r)\tag{2.13}$$

whence

$$((q_i-q_i^{-1})\partial_i)^n(E_i^{\langle r\rangle})=E_i^{\langle r-n\rangle}=((q_i-q_i^{-1})\partial_i^{op})^n(E_i^{\langle r\rangle})\tag{2.14}$$

The following is an immediate consequence of this identity and Lemma 2.10.

Corollary 2.11. *For all $i\in I$ we have*

(a) *if $x\in U_{\mathbb{Z}}(\mathfrak{n}^+)_\gamma$ then $\langle1\rangle_{q_i}\partial_i(x),\langle1\rangle_{q_i}\partial_i^{op}(x)\in q^{\frac12(\alpha_i,\gamma)}U_{\mathbb{Z}}(\mathfrak{n}^+)$;*

(b) *the $\partial_i^{(n)}$, $(\partial_i^{op})^{(n)}$, $n\in\mathbb{Z}_{\ge0}$ restrict to operators on $U^{\mathbb{Z}}(\mathfrak{n}^+)$.*

By degree considerations it is clear that ∂_i, ∂_i^{op} are locally nilpotent, that is, for any $x\in U_q(\mathfrak{n}^+)$ we have $\partial_i^k(x)=(\partial_i^{op})^k(x)=0$ for $k\gg0$. Thus, for each $x\in U_q(\mathfrak{n}^+)\setminus\{0\}$ we can define $\ell_i(x)$ as the maximal $k>0$ such that $\partial_i^k(x)\ne0$. Define $\partial_i^{(top)},(\partial_i^{op})^{(top)}:U_q(\mathfrak{n}^+)\setminus\{0\}\to U_q(\mathfrak{n}^+)\setminus\{0\}$ by $\partial_i^{(top)}(x)=\partial_i^{(\ell_i(x))}(x)$ and $(\partial_i^{op})^{(top)}(x)=(\partial_i^{(top)}(x^*))^*=(\partial_i^{op})^{(\ell_i(x^*))}(x)$. Similar notation will be used for other locally nilpotent operators in the sequel.

For any sequence $\mathbf{i}=(i_1,\dots,i_m)\in I^m$ set $\partial_{\mathbf{i}}^{(top)}=\partial_{i_m}^{(top)}\cdots\partial_{i_1}^{(top)}$.

Proposition 2.12. *For every $b\in\mathbf{B}^{\pm up}$ there exists $\mathbf{i}=(i_1,\dots,i_m)$ such that $\partial_{\mathbf{i}}^{(top)}(b)\in\{\pm1\}$. Moreover, if $\mathbf{i}'=(i'_1,\dots,i'_{m'})$ also satisfies $\partial_{\mathbf{i}'}^{(top)}(b)\in\{\pm1\}$ then $\partial_{\mathbf{i}}^{(top)}(b)=\partial_{\mathbf{i}'}^{(top)}(b)\in\{\pm1\}$.*

Thus, we can define $\mathbf{B}^{up}$ to be the set of all $b\in\mathbf{B}^{\pm up}$ such that $\partial_{\mathbf{i}}^{(top)}(b)=1$ for some $\mathbf{i}=(i_1,\dots,i_m)$.

Proof. By Proposition 2.6, $\mathbf{B}^{\pm up}$ contains the dual basis $\mathbf{B}'$ of $\mathbf{B}^{\rm can}$. Our goal is to prove that $\mathbf{B}^{up}=\mathbf{B}'$. We need the following result.

Lemma 2.13. $\partial_i^{(top)}(b)\in\mathbf{B}'$ *for all $b\in\mathbf{B}'$, $i\in I$. Moreover, if $\partial_i^{(top)}(b)=\partial_i^{(top)}(b')$ and $\ell_i(b)=\ell_i(b')$ for some $b'\in\mathbf{B}'$ then $b=b'$.*

Proof. Following [16, §14.3], denote $\mathbf{B}^{\rm can}_{i;\ge r}=\mathbf{B}^{\rm can}\cap E_i^r U_q(\mathfrak{n}^+)$ and $\mathbf{B}^{\rm can}_{i;r}=\mathbf{B}^{\rm can}_{i;\ge r}\setminus\mathbf{B}^{\rm can}_{i;\ge r+1}$. It follows from [16, §14.3] that for all $i\in I$,

$$\mathbf{B}^{\rm can}=\bigsqcup_{r\ge 0}\mathbf{B}^{\rm can}_{i;r}. \tag{2.15}$$

Let $b\in\mathbf{B}^{\rm can}$ and let $n=\ell_i(\delta_b)$, $u=\partial_i^{(top)}(\delta_b)=\partial_i^{(n)}(\delta_b)$, where δ_b is the element of $\mathbf{B}'$ satisfying $(\!|\delta_b,b'|\!)=\delta_{b,b'}$. Then $u\in\ker\partial_i$ which, by Lemma 2.10(c), is orthogonal to $\mathbf{B}^{\rm can}_{i;s}$, $s>0$. Thus, we can write

$$u=\sum_{b'\in\mathbf{B}^{\rm can}_{i;0}}(\!|u,b'|\!)\delta_{b'}=\sum_{b'\in\mathbf{B}^{\rm can}_{i;0}}(\!|\delta_b,E_i^{\langle n\rangle}b'|\!)\delta_{b'}.$$

By [16, Theorem 14.3.2], for each $b'\in\mathbf{B}^{\rm can}_{i;0}$ there exists a unique $\pi_{i,n}(b')\in\mathbf{B}^{\rm can}_{i;n}$ such that $E_i^{\langle n\rangle}b'-\pi_{i;n}(b')\in\sum_{r>n}\mathbb{Z}[q,q^{-1}]\mathbf{B}^{\rm can}_{i;r}$. Using Lemma 2.10(c) again, we conclude that for any $b''\in\mathbf{B}^{\rm can}_{i;r}$ with $r>n$, $(\!|\delta_b,b''|\!)\in(\!|\delta_b,E_i^{\langle r\rangle}U_q(\mathfrak{n}^+)|\!)=(\!|\partial_i^{(r)}(\delta_b),U_q(\mathfrak{n}^+)|\!)=0$. Thus,

$$u=\sum_{b'\in\mathbf{B}^{\rm can}_{i;0}}(\!|\delta_b,\pi_{i;n}(b')|\!)\delta_{b'}.$$

Note that, since $u\ne 0$, we cannot have $(\!|\delta_b,\pi_{i;n}(b')|\!)=0$ for all $b'\in\mathbf{B}^{\rm can}_{i;0}$. Since $(\!|\delta_b,b''|\!)=\delta_{b,b''}$, we conclude that there exists a unique $b'\in\mathbf{B}^{\rm can}_{i;0}$ such that $\pi_{i;n}(b')=b$ and then $u=\partial_i^{(top)}(\delta_b)=\delta_{b'}$. Since $\pi_{i;n}:\mathbf{B}^{\rm can}_{i;0}\to\mathbf{B}^{\rm can}_{i;n}$ is a bijection by [16, Theorem 14.3.2], the first assertion follows. The second assertion is immediate from (2.15). □

This implies that for every element $b\in\mathbf{B}'$, there exists $\mathbf{i}=(i_1,\dots,i_m)$ such that $\partial_{\mathbf{i}}^{(top)}(b)=1$. Since 1 is the unique element of $\mathbf{B}'$ of degree 0, for any sequence $\mathbf{i}'$ such that $\partial_{\mathbf{i}'}^{(top)}(b)=c\in\Bbbk^\times$, one has $c=1$. This completes the proof of Proposition 2.12. □

Remark 2.14. Since $\mathbf{B}^{can}$ is preserved by * by [16, Theorem 14.4.3] and * is self-adjoint with respect to $(\!|\cdot,\cdot|\!)$, it follows that $\mathbf{B}^{up}$ is preserved by *. In particular, we can replace ∂_i by ∂_i^{op} in Lemma 2.13 and Proposition 2.12.

Note that Lemma 2.13 and Remark 2.14 immediately yield the following well-known fact.

Corollary 2.15. *Let $x\in U_q(\mathfrak{n}^+)$ and write $x=\sum_{b\in\mathbf{B}^{up}}c_b(x)b$. Then $c_b(x)\ne 0$ implies that $\ell_i(b)\le\ell_i(x)$ and $\ell_i(b^*)\le\ell_i(x^*)$ and*

$$\partial_i^{(top)}(x)=\sum_{b\in\mathbf{B}^{up}\,:\,\ell_i(b)=\ell_i(x)}c_b(x)\partial_i^{(top)}(b),\quad(\partial_i^{op})^{(top)}(x)=\sum_{b\in\mathbf{B}^{up}\,:\,\ell_i(b^*)=\ell_i(x^*)}c_b(x)(\partial_i^{op})^{(top)}(b)$$

are the decompositions of $(\partial_i)^{(top)}(x)$ and $(\partial_i^{op})^{(top)}(x)$, respectively, in the basis $\mathbf{B}^{up}$.

3. Decorated algebras and proof of Theorem 1.7

3.1. **Decorated algebras.** Let $\mathcal{A}$ be an associative $\mathbb{Z}$-graded algebra over $\Bbbk=\mathbb{Q}(v^{\frac12})$. Denote the degree of a homogeneous $u\in\mathcal{A}$ by $|u|$.

Definition 3.1. We say that $\mathcal{A}=\mathcal{A}(E,\underline{F}_+,\underline{F}_-)$ is *decorated* if it contains an element E with $|E|=2$ and admits mutually commuting locally nilpotent $\Bbbk$-linear endomorphisms $\underline{F}_+,\underline{F}_-:\mathcal{A}\to\mathcal{A}$ of degree -2 satisfying $\underline{F}_\pm(E)=1$ and

$$\underline{F}_\pm(xy)=v^{\pm\frac12|y|}\underline{F}_\pm(x)y+v^{\mp\frac12|x|}x\underline{F}_\pm(y) \tag{3.1}$$

for $x,y\in\mathcal{A}$ homogeneous.

Denote $\mathcal{A}_\pm = \ker D_\mp$ and set $\mathcal{A}_0 = \mathcal{A}_+ \cap \mathcal{A}_-$. Clearly, $\underline{F}_\pm$ restricts to an endomorphism of $\mathcal{A}_\pm$ which will also be denoted by $\underline{F}_\pm$. Since $F_\pm$ are skew derivations, $\mathcal{A}_\pm$ are subalgebras of $\mathcal{A}$.

The following is a basic example of a decorated algebra. Let $\mathcal{F}_{m,n} = \Bbbk\langle E, x, y\rangle$ with the $\mathbb{Z}$-grading defined by $|x| = -m$, $|y| = -n$, $|E| = 2$. The following is immediate

Lemma 3.2. *There exists unique operators $\underline{F}_\pm \in \operatorname{End}_\Bbbk \mathcal{F}_{m,n}$ such that $\underline{F}_\pm(E) = 1$, $\underline{F}_\pm(x) = \underline{F}_\pm(y) = 0$ and (3.1) holds. In particular, $\mathcal{F}_{m,n}$ is a decorated algebra and for any decorated algebra $\mathcal{A}$ and any $x', y' \in \mathcal{A}_0$ homogeneous the natural homomorphism of graded associative algebras $\phi_{|x'|,|y'|} : \mathcal{F}_{|x'|,|y'|} \to \mathcal{A}$, $x \mapsto x'$, $y \mapsto y'$ is a homomorphism of decorated algebras, that is, it commutes with $\underline{F}_\pm$.*

Define $\underline{K}^{\frac{1}{2}}, \underline{E}_\pm \in \operatorname{End}_\Bbbk \mathcal{A}$ by

$$\underline{K}^{\frac{1}{2}}(x) = v^{\frac{1}{2}|x|}u, \quad \underline{E}_\pm(x) = \pm\langle 1\rangle_v^{-1}(v^{\pm\frac{1}{2}|x|}Ex - v^{\mp\frac{1}{2}|x|}xE),$$

for $x \in \mathcal{A}$ homogeneous. Clearly $\underline{E}_\pm$ are of degree 2. The following is easily checked.

Lemma 3.3. (a) *For $x, y \in \mathcal{A}$ homogeneous we have*

$$\begin{aligned} \underline{E}_+^{(r)}(x) &= \sum_{r'+r''=r} (-1)^{r'} v^{\frac{1}{2}(r+|x|-1)(r'-r'')} E^{\langle r'\rangle} x E^{\langle r''\rangle} \\ \underline{E}_-^{(r)}(x) &= \sum_{r'+r''=r} (-1)^{r''} v^{\frac{1}{2}(r+|x|-1)(r''-r')} E^{\langle r'\rangle} x E^{\langle r''\rangle} \end{aligned} \tag{3.2}$$

$$\underline{E}_\pm^{(r)}(xy) = \sum_{r'+r''=r} v^{\pm\frac{1}{2}(r'|y|-r''|x|)} \underline{E}_\pm^{(r')}(x)\underline{E}_\pm^{(r'')}(y). \tag{3.3}$$

and

$$[\underline{E}_\pm, \underline{F}_\pm] = \langle 1\rangle_v{}^{-1}(\underline{K} - \underline{K}^{-1}).$$

In particular, $\underline{E}_\pm$, $\underline{F}_\pm$ and $\underline{K}$ provide actions of Chevalley generators of $U_v(\mathfrak{sl}_2)$ in its standard *presentation on $\mathcal{A}$;*

(b) *$\underline{E}_\pm$ restrict to endomorphisms of $\mathcal{A}_\pm$. In particular, $\mathcal{A}_\pm$ is a $U_v(\mathfrak{sl}_2)_\pm$-submodule of $\mathcal{A}$ and $\mathcal{A}_0$ is the space of lowest weight vectors for both actions.*

(c) *A homomorphism of decorated algebras $\mathcal{A} \to \mathcal{A}'$ is a homomorphism of $U_v(\mathfrak{sl}_2)_+$- and $U_v(\mathfrak{sl}_2)_-$-modules.*

Remark 3.4. Suppose that $y \in \mathcal{A}_0$. The following is rather standard an is an obvious consequence of say [16, Corollary 3.1.9].

$$\underline{F}_\pm^{(a)}\underline{E}_\pm^{(b)}(y) = \begin{cases} \binom{a-b-|y|}{a}_v \underline{E}_\pm^{(b-a)}(y), & 0 \le a \le b \\ 0, & a > b \end{cases} \tag{3.4}$$

Suppose that $\underline{E}_\pm$ are locally nilpotent on $\mathcal{A}_\pm$. Then $\mathcal{A}_\pm$ are direct sums of finite dimensional $U_v(\mathfrak{sl}_2)_\pm$-modules and if $x \in \mathcal{A}_0$ is homogeneous then $|x| \le 0$. We need following

Lemma 3.5. (a) *There exists unique isomorphisms of $U_v(\mathfrak{sl}_2)$-modules $\sigma_\pm : \mathcal{A}_\pm \to \mathcal{A}_\mp$ such that $\sigma_\pm|_{\mathcal{A}_0} = \mathrm{id}_{\mathcal{A}_0}$, where $\mathcal{A}_\pm$ is regarded as a $U_v(\mathfrak{sl}_2)_\pm$-module. In particular, $\sigma_\pm \circ \sigma_\mp = \mathrm{id}_{\mathcal{A}_\mp}$.*

(b) *There exists unique $\Bbbk$-linear involution $\eta_\pm : \mathcal{A}_\pm \to \mathcal{A}_\pm$ such that*

$$\eta_\pm \circ \underline{E}_\pm = \underline{F}_\pm \circ \eta_\pm, \quad \eta_\pm \circ \underline{F}_\pm = \underline{E}_\pm \circ \eta_\pm, \quad \eta_\pm \circ \underline{K} = \underline{K}^{-1} \circ \eta_\pm \tag{3.5}$$

and $\eta_\pm(x) = \underline{E}_\pm^{(top)}(x) = \underline{E}_\pm^{(-|x|)}(x)$ for $x \in \mathcal{A}_0$ homogeneous.

Proof. Part (a) is immediate from the semi-simplicity of $\mathcal{A}_\pm$ as $U_v(\mathfrak{sl}_2)_\pm$-modules and the fact that any endomorphism of any lowest weight $U_v(\mathfrak{sl}_2)$-module fixing all lowest weight vectors is identity on that module. To prove (b) recall that every simple finite dimensional $U_v(\mathfrak{sl}_2)$-module V_λ of type 1 has a basis $\{z_k\}_{0\le k\le}$ such that $\underline{E}(x_k)=(k)_v z_{k-1}$, $\underline{F}(z_k)=(\lambda-k)_v z_{k+1}$, $\underline{K}(z_k)=v^{\lambda-2k}z_k$. Then it is easy to see that $\eta_\lambda\in\operatorname{End}_\Bbbk V_\lambda$ defined by $\eta(z_k)=z_{\lambda-k}$ is the unique linear map satisfying (3.5) and such that $\eta_\lambda(z)=\underline{E}^{(\lambda)}(z)$ for any lowest weight vector z of V_λ. It remains to observe that η_λ can be extended uniquely to any semi-simple $U_v(\mathfrak{sl}_2)$-module. $\square$

3.2. **An isomorphism between $\mathcal{A}_-$ and $\mathcal{A}_+$.** The following is quite surprising.

Theorem 3.6. *Let $\mathcal{A}$ be a decorated algebra such that the operators $\underline{E}_\pm$ are locally nilpotent on $\mathcal{A}_\pm$. Then the map $\tau:=\eta_+\circ\sigma_-:\mathcal{A}_-\to\mathcal{A}_+$ is an isomorphism of algebras.*

Proof. Let $x,y\in\mathcal{A}_0$ be homogeneous and let $m=-|x|$, $n=-|y|$. For $r\ge 0$, define $x*_r y\in\mathcal{A}$ by

$$x*_ry=\sum_{\substack{r',r''\ge 0\\ r'+r''\le r}}(-1)^{r'+r''}\prod_{t=1}^{r'}(n-r+t)_v\prod_{t=1}^{r''}(m-r+t)_v\prod_{t=r'+r''+1}^{r}(m+n-2r+t+1)_vE^{\langle r'\rangle}xE^{\langle r-r'-r''\rangle}yE^{\langle r''\rangle}.$$

Clearly $x*_r y$ is homogeneous of degree $2r-m-n$.

Proposition 3.7. *Let $\mathcal{A}$ be a decorated algebra and let $x,y\in\mathcal{A}_0$ be homogeneous with $|x|=-m$, $|y|=-n$. For all $r\ge 0$ we have $x*_r y\in\mathcal{A}_0$ and*

$$\begin{aligned}x*_r y&=\sum_{t'+t''=r}(-1)^{t''}v^{\frac12(mt'-nt''+(r-1)(t''-t'))}\frac{(m-t')_v!(n-t'')_v!}{(n-r)_v!(m-r)_v!}\underline{E}_+^{(t')}(x)\underline{E}_+^{(t'')}(y)\\&=\sum_{t'+t''=r}(-1)^{t'}v^{\frac12(nt''-mt'+(r-1)(t'-t''))}\frac{(m-t')_v!(n-t'')_v!}{(n-r)_v!(m-r)_v!}\underline{E}_-^{(t')}(x)\underline{E}_-^{(t'')}(y)\end{aligned}\tag{3.6}$$

Proof. By Lemma 3.2 it suffices to prove the proposition for the decorated algebra $\mathcal{F}_{m,n}$. Let $\mathcal{V}_{m,n}$ be the subspace of $\mathcal{F}_{m,n}$ with the basis $\{E^{\langle a\rangle}xE^{\langle b\rangle}yE^{\langle c\rangle}\,:\,a,b,c\in\mathbb{Z}_{\ge0}\}$. Clearly, $\underline{E}_\pm(\mathcal{V}_{m,n})$, $\underline{F}_\pm(\mathcal{V}_{m,n})\subset\mathcal{V}_{m,n}$ and $x*_r y\in\mathcal{V}_{m,n}$. We need the following

Lemma 3.8. *$(\mathcal{F}_{m,n})_0\cap\mathcal{V}_{m,n}$ is spanned by the $x*_r y$, $r\ge0$ as a $\Bbbk$-vector space.*

Proof. It is easy to check that $x*_r y\in(\mathcal{F}_{m,n})_0$. Conversely, let $u\in(\mathcal{F}_{m,n})_0\cap\mathcal{V}_{m,n}$ be homogeneous of degree $2r-m-n$ and write

$$u=\sum_{r',r''\ge0,r'+r''\le r}c_{r',r''}E^{\langle r'\rangle}xE^{\langle r-r'-r''\rangle}yE^{\langle r''\rangle}.$$

Then

$$\begin{aligned}\langle 1\rangle_v\underline{F}_\pm(u)=&\sum_{r'\ge1,\,r'\ge0,\,r'+r''\le r}c_{r',r''}v^{\pm\frac12(|x|+|y|+2(r-r'))}E^{\langle r'-1\rangle}xE^{\langle r-r'-r''\rangle}yE^{\langle r''\rangle}\\&+\sum_{r',r''\ge0,r'+r''\le r-1}c_{r',r''}v^{\pm\frac12(|y|-|x|+2(r''-r'))}E^{\langle r'\rangle}xE^{\langle r-r'-r''-1\rangle}yE^{\langle r''\rangle}\\&+\sum_{r'\ge0,\,r''\ge1,\,r'+r''\le r}c_{r',r''}v^{\mp\frac12(|x|+|y|+2(r-r''))}E^{\langle r'\rangle}xE^{\langle r-r'-r''\rangle}yE^{\langle r''-1\rangle}\\=&\sum_{r',r''\ge0,\,r'+r''\le r-1}\big(c_{r'+1,r''}v^{\pm\frac12(|x|+|y|+2(r-r'-1))}+c_{r',r''}v^{\pm\frac12(|y|-|x|+2(r''-r'))}\\&+c_{r',r''+1}v^{\mp\frac12(|x|+|y|+2(r-r''-1))}\big)E^{\langle r'\rangle}xE^{\langle r-r'-r''\rangle}yE^{\langle r''\rangle}.\end{aligned}$$

It follows that

$$c_{r'+1,r''}\langle |x|+|y|+2r-r'-r''-2\rangle_v + c_{r',r''}\langle |y|-r'+r-1\rangle_v = 0$$

and

$$c_{r',r''+1}\langle |x|+|y|+2r-r'-r''-2\rangle_v + c_{r',r''}\langle |x|-r''+r-1\rangle_v = 0.$$

Thus,

$$c_{r',r''} = (-1)^{r'}\frac{\prod_{t=1}^{r'}(n-r+t)_v}{\prod_{t=1}^{r'}(m+n-2r+r''+t)_v}c_{0,r''} \\ = (-1)^{r'+r''}\frac{\prod_{t=1}^{r'}(n-r+t)_v\prod_{t=1}^{r''}(m-r+t)_v}{\prod_{t=1}^{r'+r''}(m+n-2r+t+1)_v}c_{0,0}.$$

Therefore, $u = \prod_{t=1}^{r}(m+n-2r-t+1)_v{}^{-1}c_{0,0}x *_r y$. □

Thus, $x *_r y \in \mathcal{A}_0$. Furthermore, it is easy to check, using (3.1), that right hand sides of (3.6) are in $(\mathcal{F}_{m,n})_0 \cap \mathcal{V}_{m,n}$ and hence proportional to $x *_r y$ by Lemma 3.8. It remains then to compare the coefficient of $E^{\langle r\rangle}xy$ in both expressions, which is easily calculated using (3.2). □

Remark 3.9. Clearly, there exist unique injective homomorphisms $j_{\pm,m}$ and $j_{\pm,n}$ from the lowest weight Verma $U_v(\mathfrak{sl}_2)_\pm$-modules $M^\pm_{-m}$, $M^\pm_{-n}$ of lowest weight $-m$ (respectively, $-n$) to $\mathcal{V}_{m,n}$ sending a fixed lowest weight vector to x (respectively, to y). This yields natural injective homomorphisms of $U_v(\mathfrak{sl}_2)_\pm$-modules $j_{\pm,m,n} : M^\pm_{-m}\otimes M^\pm_{-n} \to \mathcal{V}_{m,n}$ where the comultiplication on $U_v(\mathfrak{sl}_2)_\pm$ is defined by $\Delta_\pm(\underline{E}_\pm) = \underline{E}_\pm\otimes\underline{K}^{\pm\frac{1}{2}}+\underline{K}^{\pm\frac{1}{2}}\otimes\underline{E}_\pm$. In particular, Proposition 3.7 implies that $j_{\pm,m,n}(M^\pm_{-m}\otimes M^\pm_{-n})$ share lowest weight vectors of weight $-m-n+2r$.

The following Lemma is essentially concerned with quantum Clebsch-Gordan coefficients (also known as $3j$-symbols, see e.g. [11, Chapter VII]).

Lemma 3.10. *Let $\mathcal{A}$, $x, y \in \mathcal{A}_0$ be as in Proposition 3.7.*

(a) *For $r \le \min(m,n)$ we have*

$$\begin{aligned}\underline{E}_+^{(a)}(x *_r y) &= \sum_{t'+t''=a+r} C_{r;t',t''}(v)\underline{E}_+^{(t')}(x)\underline{E}_+^{(t'')}(y),\\ \underline{E}_-^{(a)}(x *_r y) &= \sum_{t'+t''=a+r} (-1)^r C_{r;t',t''}(v^{-1})\underline{E}_-^{(t')}(x)\underline{E}_-^{(t'')}(y)\end{aligned}\tag{3.7}$$

where

$$C_{r;t',t''}(v) = v^{\frac{1}{2}(mt''-nt')}\sum_{k+l=r}(-1)^l v^{lt'-kt''+\frac{1}{2}(k-l)(1+m+n-r)}\frac{(n-l)_v!(m-k)_v!}{(n-r)_v!(m-r)_v!}\binom{t'}{k}_v\binom{t''}{l}_v \in \mathbb{Z}[v,v^{-1}].$$

(b) *The $C_{r;t',t''}(v)$ satisfy the following recurrence relations*

$$\begin{aligned}(m+n-r-t'-t'')_v C_{r;t',t''}(v) &= v^{t''-\frac{1}{2}n}(m-t')_v C_{r;t'+1,t''}(v)\\ &\quad + v^{\frac{1}{2}m-t'}(n-t'')_v C_{r;t',t''+1}(v),\end{aligned}\tag{3.8}$$

$$(t'+t''-r)_v C_{r;t',t''}(v) = v^{t''-\frac{1}{2}n}(t')_v C_{r;t'-1,t''}(v) + v^{\frac{1}{2}m-t'}(t'')_v C_{r;t',t''-1}(v).\tag{3.9}$$

(c) *For all $0 \le t' \le m$, $0 \le t'' \le n$, $0 \le r \le \min(m,n)$ we have*

$$C_{r;m-t',n-t''}(v) = (-1)^r C_{r;t',t''}(v^{-1}).\tag{3.10}$$

Proof. Let $a = 0$. Then

$$C_{r;t',t''}(v) = (-1)^{t''} v^{\frac{1}{2}(mt'-nt''+(r-1)(t''-t'))} \frac{(m-t')_v!(n-t'')_v!}{(n-r)_v!(m-r)_v!}$$

and the assertion follows from (3.6). The case of arbitrary a is then easily deduced by applying $\underline{E}_{\pm}^{(a)}$ to (3.6) and using Lemma 3.3(a). To prove (3.8) (respectively, (3.9)) it suffices to apply $\underline{F}$ (respectively, $\underline{E}$) to both sides of the first identity in (3.7). We leave the details of these computations as an exercise for the reader.

We now prove part (c). Note first that for all $0 \le t' \le m$, $0 \le t'' \le n$

$$\begin{aligned} C_{r;t',0}(v) &= v^{\frac{1}{2}(r(1+m+n-r)-nt')} \frac{(n)_v!}{(n-r)_v!} \binom{t'}{r}_v, \\ C_{r;0,t''}(v) &= (-1)^r v^{\frac{1}{2}(mt''-r(1+m+n-r))} \frac{(m)_v!}{(m-r)_v!} \binom{t''}{r}_v. \end{aligned} \tag{3.11}$$

Using (3.8) with $t'' = n$ we obtain

$$C_{r;t'+1,n}(v) = \frac{(m-r-t')_v}{(m-t')_v} v^{-\frac{1}{2}n} C_{r;t',n}$$

whence by (3.11)

$$\begin{aligned} C_{r;t',n}(v) &= v^{-\frac{1}{2}t'n} \frac{(m-r)_v!(m-t')_v!}{(m)_v!(m-r-t')_v!} C_{r;0,n}(v) \\ &= (-1)^r v^{\frac{1}{2}(n(m-t')-r(1+m+n-r))} \frac{(n)_v!}{(n-r)_v!} \binom{m-t'}{r}_v = (-1)^r C_{r;m-t',0}(v^{-1}). \end{aligned}$$

Thus, (3.10) holds for all $0 \le t' \le m$ and for $t'' = n$. Suppose that (3.10) was established for all $0 \le t' \le m$ and for all $s+1 \le t'' \le n$. We have by (3.11)

$$\begin{aligned} (-1)^r(n-r-s)_v C_{r;m,s}(v^{-1}) &= (-1)^r C_{r;m,s+1}(v^{-1})(n-s)_v v^{\frac{1}{2}m} = v^{\frac{1}{2}m}(n-s)_v C_{r;0,n-s-1}(v) \\ &= (-1)^r v^{\frac{1}{2}(m(n-s)-r(1+m+n-r))} \frac{(m)_v!(n-s)_v}{(m-r)_v!} \binom{n-s-1}{r}_v = (n-r-s)_v C_{r;0,n-s}(v). \end{aligned}$$

Finally, assume that (3.10) is established for $k+1 \le t' \le m$ and for $t'' = s$. Then using (3.8) and (3.9) we obtain

$$\begin{aligned} &(-1)^r(m+n-r-k-s)_v C_{r;k,s}(v^{-1}) \\ &\quad = (-1)^r C_{r;k+1,s}(v^{-1})(m-k)_v v^{-s+\frac{1}{2}n} + (-1)^r C_{r;k,s+1}(v^{-1})(n-s)_v v^{-\frac{1}{2}m+k} \\ &\quad = C_{r;m-k-1,n-s}(v)(m-k)_v v^{-s+\frac{1}{2}n} + C_{r;m-k,n-s-1}(n-s)_v v^{-\frac{1}{2}m+k} \\ &\quad = (m+n-r-k-s)C_{r;m-k,n-s}(v). \end{aligned}$$

This proves the inductive step and completes the proof of the Lemma. □

We can now complete the proof of Proposition 3.6. By construction, τ is an isomorphism of $U_v(\mathfrak{sl}_2)$-modules. Explicitly, if $z \in \mathcal{A}_0$ then $\tau(\underline{E}_-^{(r)}(z)) = \underline{E}_+^{(-|z|-r)}(z)$. It suffices to prove that for any $x, y \in \mathcal{A}_0$ homogeneous with $|x| = -m$, $|y| = -n$ we have

$$\tau(E_-^{(k)}(x))\tau(E_-^{(l)}(y)) = \underline{E}_+^{(m-k)}(x)\underline{E}_+^{(n-k)}(y) = \tau(E_-^{(k)}(x)E_-^{(l)}(y)).$$

It is immediate from the Remark 3.9 that $C_{r;t',t''}(v)$ (respectively, $(-1)^r C_{r;t',t''}(v^{-1})$) provide the transition matrix between the two bases of $U_v(\mathfrak{sl}_2)_+$-) (respectively, $U_v(\mathfrak{sl}_2)_-$) modules $V_m \otimes V_n =$

$\bigoplus_{0\le k\le \min(m,n)} V_{m+n-2k}$. In particular, there exists $\tilde C_{r;k,l}(v)\in\Bbbk$, $0\le k\le m$, $0\le l\le n$, $0\le r\le \min(m,n,k+l)$ such that

$$\sum_{r=0}^{\min(m,n,k+l)} (-1)^r \tilde C_{r;k,l}(v) C_{r;t',t''}(v^{-1}) = \delta_{k,t'}\delta_{l,t''}. \tag{3.12}$$

Then

$$\underline{E}_-^{(k)}(x)\underline{E}_-^{(l)}(y) = \sum_{r=0}^{\min(m,n,k+l)} \tilde C_{r;k,l}(v)\underline{E}_-^{(k+l-r)}(x *_r y)$$

and so

$$\begin{aligned}
\tau(\underline{E}_-^{(k)}(x)\underline{E}_-^{(l)}(y)) &= \sum_{r=0}^{\min(m,n,k+l)} \tilde C_{r;k,l}(v)\underline{E}_+^{(m+n-r-k-l)}(x *_r y)\\
&= \sum_{r=0}^{\min(m,n,k+l)} \tilde C_{r;k,l}(v) \sum_{s'+s''=m+n-k-l} C_{r;s',s''}(v)\underline{E}_+^{(s')}(x)\underline{E}_+^{(s'')}(y)\\
&= \sum_{t'+t''=k+l}\Big(\sum_{r=0}^{\min(m,n,k+l)} \tilde C_{r;k,l}(v) C_{r;m-t',n-t''}(v)\Big)\underline{E}_+^{(m-t')}(x)\underline{E}_+^{(n-t'')}(y)\\
&= \sum_{t'+t''=k+l}\Big(\sum_{r=0}^{\min(m,n,k+l)} (-1)^r\tilde C_{r;k,l}(v) C_{r;t',t''}(v^{-1})\Big)\underline{E}_+^{(m-t')}(x)\underline{E}_+^{(n-t'')}(y)\\
&= \underline{E}_+^{(m-k)}(x)\underline{E}_+^{(n-l)}(y) = \tau(E_-^{(k)}(x))\tau(E_-^{(l)}(y)),
\end{aligned}$$

where we used (3.10) and (3.12). ∎

Note that for $x\in\mathcal{A}_-$ homogeneous, $\tau(x)$ can be calculated explicitly in the following way. First, if $y\in\mathcal{A}_0$ is homogeneous and $x=\underline{E}_-^{(r)}(y)$ then

$$\tau(x) = \underline{E}_+^{(-|y|-r)}(y) = \underline{E}_+^{(r-|x|)}(y) = \binom{2r-|x|}{r}_v^{-1} \underline{E}_+^{(r-|x|)}\underline{F}_-^{(r)}(x). \tag{3.13}$$

By linearity, it remains to observe that any homogeneous element of $\mathcal{A}_-$ can be written, uniquely, as $x=\sum_{r\ge\max(0,|x|)}\underline{E}_-^{(r)}(x_r)$ where $x_r\in\mathcal{A}_0$ and $|x_r|=|x|-2r$.

We will also need the following property of τ.

Lemma 3.11. $\underline{F}_+^{(top)}\circ\tau = \underline{F}_-^{(top)}$.

Proof. Given $x\in\mathcal{A}_-$, write $x=\sum_{r\ge\max(0,|x|)}\underline{E}_-^{(r)}(x_r)$ where $x_r\in\mathcal{A}_0$ and $|x_r|=|x|-2r$. Then $\tau(x)=\sum_{r\ge\max(0,|x|)}\underline{E}_+^{(r-|x|)}(x_r)$. Let $r_0=\max\{r\ge\max(0,|x|)\,:\,x_r\ne 0\}$. Then by (3.4)

$$\underline{F}_-^{(top)}(x) = \underline{F}_-^{(r_0)}(x) = \underline{F}_-^{(r_0)}\underline{E}_-^{(r_0)}(x_{r_0}) = \binom{2r_0-|x|}{r_0}_v x_{r_0}.$$

On the other hand,

$$\underline{F}_+^{(top)}\tau(x) = \underline{F}_+^{(r_0-|x|)}\tau(x) = \underline{F}_+^{(r_0-|x|)}\underline{E}_+^{(r_0-|x|)}(x_{r_0}) = \binom{2r_0-|x|}{r_0}_v x_{r_0}.$$ ∎

3.3. $U_q(\mathfrak{n}^+)$ **as a decorated algebra.** Define a comultiplication Δ on $U_q(\mathfrak{g})$ by

$$\Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, \qquad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i, \qquad i \in I.$$

Then (cf. [16, §3.1.5])

$$\Delta(E_i^{\langle r\rangle}) = \sum_{r'+r''=r} q_i^{-r'r''} E_i^{\langle r'\rangle} K_i^{-r''} \otimes E_i^{\langle r''\rangle}, \quad \Delta(F_i^{\langle r\rangle}) = \sum_{r'+r''=r} q_i^{r'r''} F_i^{\langle r'\rangle} \otimes K_i^{r'} F_i^{\langle r''\rangle}. \tag{3.14}$$

For any $J, J' \subset I$ denote $U_{\mathbb{Z}}(\mathfrak{g})_{J,J'}$ the $\mathbb{A}_0$-subalgebra of $U_q(\mathfrak{g})$ generated by $U_{\mathbb{Z}}(\mathfrak{n}^-)_J$, $U_{\mathbb{Z}}(\mathfrak{n}^+)_{J'}$, the $K_i^{\pm 1}$ and the $\binom{K_i;c}{a}_{q_i}$, $a \in \mathbb{Z}_{\ge 0}$, $c \in \mathbb{Z}$ and $i \in J \cap J'$, where

$$\binom{K;c}{a}_v = \prod_{k=0}^{a-1} \frac{K^{-1}v^{k-c} - Kv^{c-k}}{v^{k+1} - v^{-k-1}}.$$

We also abbreviate $U_{\mathbb{Z}}(\mathfrak{g})_J = U_{\mathbb{Z}}(\mathfrak{g})_{J,I}$ and $U_{\mathbb{Z}}(\mathfrak{g}) := U_{\mathbb{Z}}(\mathfrak{g})_{I,I}$. The corresponding $\Bbbk$-subalgebras of $U_q(\mathfrak{g})$ will be denoted $U_q(\mathfrak{g})_{J,J'}$. It follows from (3.14) that $U_{\mathbb{Z}}(\mathfrak{g})$ is a Hopf $\mathbb{A}_0$-algebra.

Let ad be the corresponding adjoint action of $U_q(\mathfrak{g})$ on itself. Consider the extension $\widetilde{U}_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$ obtained by adjoining $K_i^{\pm\frac{1}{2}}$, $i \in I$. Define operators $\underline{E}_i$, $\underline{F}_i$ on $\widetilde{U}_q(\mathfrak{g})$ via

$$\begin{aligned} \underline{E}_i(x) &= -(\operatorname{ad} E_i^{\langle 1\rangle} K_i^{\frac{1}{2}}) = \frac{E_i K_i^{\frac{1}{2}} x K_i^{-\frac{1}{2}} - K_i^{-\frac{1}{2}} x K_i^{\frac{1}{2}} E_i}{q_i^{-1} - q_i}, \\ \underline{F}_i(x) &= (\operatorname{ad} K_i^{-\frac{1}{2}} F_i^{\langle 1\rangle})(x) = \partial_i(x) - K_i^{-1} \partial_i^{op}(x) K_i^{-1}. \end{aligned} \tag{3.15}$$

Clearly, $\underline{E}_i$ and $\underline{F}_i$ restrict to operators on $U_q(\mathfrak{g})$ and we have

$$[\underline{E}_i, \underline{F}_i] = -\langle 1\rangle_{q_i}{}^{-2} \operatorname{ad}([E_i, F_i]) = \langle 1\rangle_{q_i}{}^{-1}(\underline{K}_i - \underline{K}_i^{-1}), \tag{3.16}$$

where $\underline{K}_i(x) = K_i x K_i^{-1}$. We will also need operators $\underline{E}_i^{op}$, $\underline{F}_i^{op}$ defined by

$$\underline{E}_i^{op}(x) = (\underline{E}_i(x^*))^*, \qquad \underline{F}_i^{op}(x) = (\underline{F}_i(x^*))^*. \tag{3.17}$$

We collect some properties of these operators in the following Lemma.

Lemma 3.12. (a) *$U_q(\mathfrak{n}^+)$ is a decorated algebra with $E = E_i$, $\underline{F}_+ = \partial_i$, $\underline{F}_- = \partial_i^{op}$, $v = q_i$ and $|x| = (\alpha_i^\vee, \gamma)$ for $x \in U_q(\mathfrak{n}^+)_\gamma$; in particular, $\underline{E}_+ = \underline{E}_i$ and $\underline{E}_- = \underline{E}_i^{op}$.*
(b) *$\underline{E}_i$, $\underline{F}_i$ commute with $\bar{\cdot}$.*
(c) *If $x \in U_{\mathbb{Z}}(\mathfrak{g})_\gamma$ then $\underline{E}_i^{(r)}(x), \underline{F}_i^{(r)}(x) \in q^{\frac{1}{2}(r\alpha_i,\gamma)} U_{\mathbb{Z}}(\mathfrak{g})$ for all $r \in \mathbb{Z}_{\ge 0}$;*
(d) *For all $x \in U_q(\mathfrak{n}^+)_\gamma$, $y \in U_q(\mathfrak{n}^+)$ we have*

$$(\!(\underline{E}_i^{(r)}(x), y)\!) = \sum_{r'+r''=r} (-1)^{r'} q_i^{-\frac{1}{2}(r+(\alpha_i^\vee,\gamma)-1)(r''-r')} (\!(x, \partial_i^{(r'')}(\partial_i^{op})^{(r')}(y))\!)$$

(e) *$T_i \circ \underline{E}_i^{op} = \underline{F}_i \circ T_i$, $T_i \circ \underline{F}_i^{op} = \underline{E}_i \circ T_i$.*

Proof. Parts (a) and (b) are obvious from the definitions. Since $U_{\mathbb{Z}}(\mathfrak{g})$ is a Hopf $\mathbb{A}_0$-algebra, the first assertion in (c) follows from

$$\underline{E}_i^{(r)} = (-1)^r q_i^{\binom{r}{2}} \operatorname{ad}(E_i^{\langle r\rangle} K_i^{\frac{r}{2}}), \quad \underline{F}_i^{(r)} = q_i^{\binom{r}{2}} (\operatorname{ad} K_i^{-\frac{r}{2}} F_i^{\langle r\rangle}),$$

while the second is immediate from the above formulae and (3.14). Part (d) is immediate from part (a), (3.2) and Lemma 2.10(b). Part (e) is easy to check using Lemma 2.1. ☐

3.4. **A new formula for T_i.** Using the notation from [2], denote by $U_i := T_i^{-1}(U_q(\mathfrak{n}^+)) \cap U_q(\mathfrak{n}^+)$ and ${}_iU := T_i(U_q(\mathfrak{n}^+)) \cap U_q(\mathfrak{n}^+)$. It follows from [16, Proposition 38.1.6] that $U_i = \ker \partial_i$ and ${}_iU = \ker \partial_i^{op}$. Let $U_i^{\mathbb{Z}} = U_i \cap U^{\mathbb{Z}}(\mathfrak{n}^+)$ and ${}_iU^{\mathbb{Z}} = {}_iU \cap U^{\mathbb{Z}}(\mathfrak{n}^+)$.

Lemma 3.13. *Let $i \in I$.*

(a) *For all $x \in U_q(\mathfrak{n}^+)_\gamma$, $y \in {}_iU$, $z \in U_i$ and $r \ge 0$ we have*

$$(\!|\underline{E}_i^{(r)}(x), y|\!) = q_i^{-\frac{1}{2}(r+(\alpha_i^\vee,\gamma)-1)r}(\!|x, \underline{F}_i^{(r)}(y)|\!), \quad (\!|(\underline{E}_i^{op})^{(r)}(x), z|\!) = q_i^{-\frac{1}{2}(r+(\alpha_i^\vee,\gamma)-1)r}(\!|x, (\underline{F}_i^{op})^{(r)}(z)|\!).$$

(b) *$\underline{E}_i$, $\underline{F}_i$ (respectively, $\underline{E}_i^{op}$, $\underline{F}_i^{op}$) restrict to locally nilpotent operators on ${}_iU$ (respectively, on U_i).*

(c) *$\underline{E}_i^{(n)}({}_iU^{\mathbb{Z}}), \underline{F}_i^{(n)}({}_iU^{\mathbb{Z}}) \subset {}_iU^{\mathbb{Z}}$ (respectively, $(\underline{E}_i^{op})^{(n)}(U_i^{\mathbb{Z}}), (\underline{F}_i^{op})^{(n)}(U_i^{\mathbb{Z}}) \subset U_i^{\mathbb{Z}}$) for all $n \ge 0$.*

Proof. We only prove the assertion for $\underline{E}_i$ and $\underline{F}_i$. The assertion for $\underline{E}_i^{op}$ and $\underline{F}_i^{op}$ is proved similarly using (3.17) and the fact that * is self-adjoint with respect to $(\!|\cdot,\cdot|\!)$. Since by (3.15) $\underline{F}_i|_{{}_iU} = \partial_i|_{{}_iU}$, in particular, $\underline{F}_i$ is a locally nilpotent operator on ${}_iU$. Part (a) is now immediate from Lemma 3.12(d).

Suppose that $x \in U_q(\mathfrak{n}^+)_\gamma$ and $\underline{E}_i^{(n)}(x) \neq 0$ for all $n \ge 0$. Then $T_i(\underline{E}_i^{(n)}(x))$ is homogeneous of degree $s_i(\gamma + n\alpha_i) = \gamma - ((\alpha_i^\vee, \gamma) + n)\alpha_i \notin Q^+$ for $n \gg 0$. Since $T_i({}_iU) \subset U_q(\mathfrak{n}^+)$, this is a contradiction. This proves (b).

To prove (c), note that the assertion for $\underline{F}_i$ follows from Corollary 2.11. Since $U_{\mathbb{Z}}(\mathfrak{n}^+) = {}_iU_{\mathbb{Z}} \oplus (E_iU_q(\mathfrak{n}^+) \cap U_{\mathbb{Z}}(\mathfrak{n}^+))$, it suffices to prove that for $x \in {}_iU^{\mathbb{Z}} \cap U_q(\mathfrak{n}^+)_\gamma$, $y \in U_i^{\mathbb{Z}} \cap U_q(\mathfrak{n}^+)_{\gamma+n\alpha_i}$ we have $(\!|\underline{E}_i^{(n)}(x), y|\!) \in \mathbb{A}_0$. But for such y we have by part (a) and (2.4)

$$(\!|\underline{E}_i^{(n)}(x), y|\!) = q_i^{-\frac{1}{2}n(n+(\alpha_i^\vee,\gamma)-1)}(\!|x, \underline{F}_i^{(n)}(y)|\!) = q_i^{\binom{n}{2}}(\!|x, q_i^{-\frac{1}{2}n(\alpha_i^\vee,\gamma+n\alpha_i)}\underline{F}_i^{(n)}(y)|\!) \in \mathbb{A}_0$$

and $q_i^{-\frac{1}{2}n(\alpha_i^\vee,\gamma+n\alpha_i)}\underline{F}_i^{(n)}(y) \in U_{\mathbb{Z}}(\mathfrak{n}^+)$ by Lemma 3.12(c). □

Thus, given $x \in U_i \cap U_q(\mathfrak{n}^+)_\gamma$, $y \in {}_iU \cap U_q(\mathfrak{n}^+)_\gamma$ we can write uniquely

$$x = \sum_{r \ge \max(0,(\alpha_i^\vee,\gamma))} (\underline{E}_i^{op})^{(r)}(x_r), \quad y = \sum_{r \ge \max(0,(\alpha_i^\vee,\gamma))} \underline{E}_i^{(r)}(y_r), \quad x_r, y_r \in {}_iU \cap U_i \cap U_q(\mathfrak{n}^+)_{\gamma - r\alpha_i}, \tag{3.18}$$

and $x_r, y_r = 0$ for $r \gg 0$.

Corollary 3.14. *Let $x, x' \in U_i \cap U_q(\mathfrak{n}^+)_\gamma$, $y, y' \in {}_iU \cap U_q(\mathfrak{n}^+)_\gamma$ and write x, x' and y, y' as in (3.18). Then*

$$\begin{aligned} (\!|x, x'|\!) &= \sum_{r \ge \max(0,(\alpha_i^\vee,\gamma))} q_i^{\frac{1}{2}r(r+1-(\alpha_i^\vee,\gamma))} \binom{2r - (\alpha_i^\vee,\gamma)}{r}_{q_i} (\!|x_r, x'_r|\!), \\ (\!|y, y'|\!) &= \sum_{r \ge \max(0,(\alpha_i^\vee,\gamma))} q_i^{\frac{1}{2}r(r+1-(\alpha_i^\vee,\gamma))} \binom{2r - (\alpha_i^\vee,\gamma)}{r}_{q_i} (\!|y_r, y'_r|\!). \end{aligned} \tag{3.19}$$

Proof. Let $z, z' \in {}_iU \cap U_i$, $z' \in U_q(\mathfrak{n}^+)_{\gamma'}$. Let $a \ge b \ge 0$. Then we have by Lemma 3.13(a) and (3.4)

$$\begin{aligned} &(\!|\underline{E}_i^{(a)}(z), \underline{E}_i^{(b)}(z')|\!) = q_i^{-\frac{1}{2}b(b+(\alpha_i^\vee,\gamma')-1)}(\!|\underline{F}_i^{(b)}\underline{E}_i^{(a)}(z), z'|\!) \\ &= q_i^{-\frac{1}{2}b(b+(\alpha_i^\vee,\gamma')-1)}\binom{b - a - (\alpha_i^\vee,\gamma')}{b}_{q_i}(\!|\underline{E}_i^{(a-b)}(z), z'|\!) = \delta_{a,b} q_i^{-\frac{1}{2}a(a+(\alpha_i^\vee,\gamma')-1)}\binom{-(\alpha_i^\vee,\gamma')}{a}_{q_i}(\!|z, z'|\!). \end{aligned}$$

This yields the second identity in (3.19). The first follows from the second one by applying *. □

We now establish a formula for the action of T_i on U_i in terms of $\underline{E}_i^{op}$ and $\underline{E}_i$.

Theorem 3.15. *Write $x \in U_i \cap U_q(\mathfrak{n}^+)_\gamma$ as in (3.18). Then*

$$T_i(x) = \sum_{r \ge \max(0,(\alpha_i^\vee,\gamma))} \underline{E}_i^{(r-(\alpha_i^\vee,\gamma))}(x_r).$$

In particular, $\partial_i^{(top)} T_i(x) = (\partial_i^{op})^{(top)}(x)$.

Proof. We apply Proposition 3.6 to $\mathcal{A} = U_q(\mathfrak{n}^+)$ which is a decorated algebra by Lemma 3.12(a) with locally nilpotent $\underline{E}_\pm$ on $\mathcal{A}_\pm$ by Lemma 3.13(b). We claim that $T_i = \tau$. Since both τ and T_i are isomorphisms of algebras $U_i \to {}_iU$, it is enough to check that they coincide on generators of U_i.

Let $j \ne i$ and define for all $0 \le l \le -a_{ij}$

$$E_{ji^l} = \binom{-a_{ij}}{l}_{q_i}^{-1} (\underline{E}_i^{op})^{(l)}(E_j), \qquad E_{i^lj} = (E_{ji^l})^* = \binom{-a_{ij}}{l}_{q_i}^{-1} \underline{E}_i^{(l)}(E_j). \tag{3.20}$$

Clearly, $\overline{E_{ji^l}} = E_{ji^l}$ and it is immediate from (3.2) that

$$E_{ji^l} = \binom{-a_{ij}}{l}_{q_i}^{-1} \sum_{r+s=l} (-1)^r q_i^{\frac{1}{2}(r-s)(l+a_{ij}-1)} E_i^{\langle s\rangle} E_j E_i^{\langle r\rangle}. \tag{3.21}$$

Lemma 3.16. *Let $i \ne j \in I$, $0 \le m \le -a_{ij}$. Then*

(a) $T_i(E_{ji^m}) = E_{i^{-a_{ij}-m}j} = \tau(E_{ji^m})$

(b) *The elements E_{ji^l} (respectively, E_{i^lj}), $j \ne i$, $0 \le l \le -a_{ij}$ generate the algebra U_i (respectively ${}_iU$).*

Proof. To prove (a), note that by Lemma 2.1 and (3.21) we have $T_i(E_j) = E_{i^{-a_{ij}}j}$. On the other hand, $\tau(E_j) = \underline{E}_i^{(-a_{ij})}(E_j) = E_{i^{-a_{ij}}j}$. Then by Lemma 3.12(e) and (3.4)

$$T_i(E_{ji^l}) = \binom{-a_{ij}}{l}_{q_i}^{-1} \underline{F}_i^{(l)}(E_{i^{-a_{ij}}j}) = \binom{-a_{ij}}{l}_{q_i}^{-1} \underline{F}_i^{(l)} \underline{E}_i^{(-a_{ij})}(E_j) = E_{i^{-a_{ij}-l}j}.$$

Since by construction τ also satisfies Lemma 3.12(e), it follows that $\tau(E_{ji^l}) = T_i(E_{ji^l})$.

Part (b) can be easily deduced from [16, §38.1.1]. □

This implies that $\tau = T_i$ on U_i. The second assertion of Theorem 3.15 follows from Lemma 3.11. □

We now prove the following

Proposition 3.17. *For all $i \in I$, $T_i(U_i^{\mathbb{Z}}) = {}_iU^{\mathbb{Z}}$.*

Proof. We need the following

Lemma 3.18. *Any element $x \in U_{\mathbb{Z}}(\mathfrak{n}^+)$ can be written as $x = \sum_{r,s\ge 0} E_i^{\langle r\rangle} x_{rs} E_i^{\langle s\rangle}$, where $x_{rs} \in {}_iU \cap U_i \cap U_{\mathbb{Z}}(\mathfrak{n}^+)$ and only finitely many of them are non-zero.*

Proof. We need the following elementary fact.

Lemma 3.19. *Let V be a finite dimensional $\Bbbk$-vector space with a non-degenerate bilinear form $(\!|\cdot,\cdot|\!) : V \otimes V \to \Bbbk$. Assume that we have two orthogonal direct sum decompositions $V = V_1 \oplus W_1 = V_2 \oplus W_2$ with respect to that form. Then $V = (V_1 \cap V_2) \oplus (W_1 + W_2) = (W_1 \cap W_2) \oplus (V_1 + V_2)$ (orthogonal direct sum decompositions).*

Proof. Clearly $(V_1\cap V_2)$ is orthogonal to W_1+W_2 and $(W_1\cap W_2)$ is orthogonal to V_1+V_2. Note that for any $v\in V_i$, $(\!|v,v|\!)=0$ if and only if $v=0$. This implies that the sums $U_1=(V_1\cap V_2)+(W_1+W_2)$, $U_2=(W_1\cap W_2)+(V_1+V_2)$ are direct. It remains to prove that $\dim U_1=\dim U_2=\dim V$. Since

$$\begin{aligned}\dim U_1+\dim U_2 &= (\dim W_1+\dim W_2-\dim W_1\cap W_2)+\dim V_1\cap V_2\\ &\qquad+(\dim V_1+\dim V_2-\dim V_1\cap V_2)+\dim W_1\cap W_2=2\dim V,\end{aligned}$$

and $\dim U_1,\dim U_2\le \dim V$ the assertion follows. □

Given $\gamma\in Q^+$, let $n_i(\gamma)$ be the coefficient of α_i in γ. For any $\gamma\in Q^+$ we have two orthogonal direct sum decompositions $U_q(\mathfrak{n}^+)_\gamma=(\ker\partial_i|_{U_q(\mathfrak{n}^+)_\gamma}\oplus U_q(\mathfrak{n}^+)_{\gamma-\alpha_i}E_i)=(\ker\partial_i^{op}|_{U_q(\mathfrak{n}^+)_\gamma}\oplus E_iU_q(\mathfrak{n}^+)_{\gamma-\alpha_i})$. Since $U_q(\mathfrak{n}^+)_\gamma$ is finite dimensional, it follows from Lemma 3.19 that $U_q(\mathfrak{n}^+)_\gamma=({}_iU\cap U_i\cap U_q(\mathfrak{n}^+)_\gamma)\oplus(U_q(\mathfrak{n}^+)_{\gamma-\alpha_i}E_i+E_iU_q(\mathfrak{n}^+)_{\gamma-\alpha_i})$. Then an obvious induction on $n_i(\gamma)$ implies that every $x\in U_q(\mathfrak{n}^+)$ can be written in $x=\sum_{r,s\ge0}E_i^{\langle r\rangle}x_{rs}E_i^{\langle s\rangle}$, where $x_{rs}\in{}_iU\cap U_i$ and only finitely many of the x_{rs} are non-zero.

We now prove by induction on $n_i(\gamma)$ that if $x\in U_{\mathbb Z}(\mathfrak{n}^+)_\gamma$ then $x_{rs}\in U_{\mathbb Z}(\mathfrak{n}^+)\cap{}_iU\cap U_i$. If $n_i(\gamma)=0$ then $x=x_{00}$ and there is nothing to prove. For the inductive step, we have

$$\sum_{r,s\ge0}E_i^{\langle r\rangle}(q_i^{-r-1}x_{r+1,s}+q_i^{-(\alpha_i^\vee,\gamma)+s+1}x_{r,s+1})E_i^{\langle s\rangle}=q_i^{-\frac12(\alpha_i^\vee,\gamma)}\langle 1\rangle_{q_i}\partial_i(x)\in U_{\mathbb Z}(\mathfrak{n}^+),$$

where we used Lemma 2.10(c) and Corollary 2.11(a). Then $x_{r+1,s}+q_i^{-(\alpha_i^\vee,\gamma)+r+s+2}x_{r,s+1}\in U_{\mathbb Z}(\mathfrak{n}^+)$ by the induction hypothesis. Let s_0 be such that $x_{rs}=0$ for all r and for all $s>s_0$. It follows then that $x_{r,s_0}\in U_{\mathbb Z}(\mathfrak{n}^+)$ for all $r\ge0$. Suppose now that $x_{rt}\in U_{\mathbb Z}(\mathfrak{n}^+)$ for all r and for all $s+1\le t\le s_0$. Since $x_{r,s}=-q_i^{-(\alpha_i^\vee,\gamma)+s+r+1}x_{r-1,s+1}$ it follows that $x_{r,s}\in U_{\mathbb Z}(\mathfrak{n}^+)$ for all $r,s\ge0$ with $r+s>0$. It remains to observe that $x_{00}=x-\sum_{r,s\ge0,r+s>0}E_i^{\langle r\rangle}x_{rs}E_i^{\langle s\rangle}$. □

Lemma 3.20. *Let $x\in U_i^{\mathbb Z}\cap U_q(\mathfrak{n}^+)_\gamma$ and write $x=\sum_{r\ge\max(0,(\alpha_i^\vee,\gamma))}(\underline{E}_i^{op})^{(r)}(x_r)$ where $x_r\in{}_iU\cap U_i\cap U_q(\mathfrak{n}^+)_{\gamma-r\alpha_i}$. Then $\binom{-(\alpha_i^\vee,\gamma-r\alpha_i)}{r}_{q_i}x_r\in U_i^{\mathbb Z}$.*

Proof. The argument is by induction on $\ell_i(x^*)$. If $\ell_i(x^*)=0$, that is $\partial_i^{op}(x)=0$, then $x=x_0$ and there is nothing to do. If $\ell_i(x^*)=n$ then $x_r=0$ for all $r>n$. We have $(\partial_i^{op})^{(top)}(x)=(\partial_i^{op})^{(n)}(x)\in U_i^{\mathbb Z}$ by Corollary 2.11(b). On the other hand, $(\partial_i^{op})^{(n)}(x)=(\partial_i^{op})^{(n)}(\underline{E}_i^{op})^{(n)}(x_n)=\binom{2n-(\alpha_i^\vee,\gamma)}{n}_{q_i}x_n$ by (3.4). Thus, $\binom{2n-(\alpha_i^\vee,\gamma)}{n}_{q_i}x_n\in U_i^{\mathbb Z}\cap{}_iU^{\mathbb Z}$. It remains to observe that the induction hypothesis applies to $x-(\underline{E}_i^{op})^{(n)}(x_n)$. □

Thus, is suffices to consider $x=(\underline{E}_i^{op})^{(r)}(z)\in U_i^{\mathbb Z}$ where $z\in{}_iU\cap U_i\cap U_q(\mathfrak{n}^+)_\gamma$. We claim that $T_i(x)=\underline{E}_i^{(-(\alpha_i^\vee,\gamma)-r)}(z)\in{}_iU^{\mathbb Z}$. Given $y\in U_{\mathbb Z}(\mathfrak{n}^+)_{\gamma+(-(\alpha_i^\vee,\gamma)-r)\alpha_i}$, use Lemma 3.18 to write $y=\sum_{s,s'\ge0}E_i^{\langle s'\rangle}y_{s's}E_i^{\langle s\rangle}$ with $y_{s's}\in{}_iU\cap U_i\cap U_{\mathbb Z}(\mathfrak{n}^+)$. Then by Lemma 2.10(b) and (3.4)

$$\begin{aligned}(\!|\underline{E}_i^{(-(\alpha_i^\vee,\gamma)-r)}(z),y|\!) &= \sum_{s',s\ge0}(\!|\underline{E}_i^{(-(\alpha_i^\vee,\gamma)-r)}(z),E_i^{\langle s'\rangle}y_{s's}E_i^{\langle s\rangle}|\!)=\sum_{s\ge0}(\!|\partial_i^{(s)}\underline{E}_i^{(-(\alpha_i^\vee,\gamma)-r)}(z),y_{0s}|\!)\\ &=\sum_{s\ge0}(\!|\underline{F}_i^{(s)}\underline{E}_i^{(-(\alpha_i^\vee,\gamma)-r)}(z),y_{0s}|\!)=\sum_{s=0}^{-(\alpha_i^\vee,\gamma)-r}\binom{s+r}{r}_{q_i}(\!|\underline{E}_i^{(-(\alpha_i^\vee,\gamma)-r-s)}(z),y_{0s}|\!)\\ &=\binom{-(\alpha_i^\vee,\gamma)}{r}_{q_i}(\!|z,y_{0,-(\alpha_i^\vee,\gamma)-r}|\!),\end{aligned}$$

since $(\!|{}_iU, E_iU_q(\mathfrak{n}^+)|\!) = 0$ and $(\!|\underline{E}_i^{(a)}(z), y_{0s}|\!) = 0$ if $a > 0$ by Lemma 3.12(d). Since $\binom{-(\alpha_i^\vee,\gamma)}{r}_{q_i} z \in U^{\mathbb{Z}}(\mathfrak{n}^+)$ by Lemma 3.20, it follows that $(\!|\underline{E}_i^{(-(\alpha_i^\vee,\gamma)-r)}(z), y|\!) \in \mathbb{A}_0$. □

3.5. **Proof of Theorem 1.7.** We need the following result which can also be deduced from [16, Proposition 38.2.1]. However, our argument is much shorter.

Lemma 3.21. *For all $x, x' \in U_q(\mathfrak{n}^+)_\gamma \cap U_i$, $a, a' \in \mathbb{Z}_{\ge 0}$ we have*

$$(\!|E_i^a T_i(x), E_i^{a'} T_i(x')|\!) = q^{\frac{1}{2}(a-1)(\alpha_i,\gamma)} \delta_{a,a'} \langle a\rangle_{q_i}! (\!|x,x'|\!) = q^{\frac{1}{2}(a-1)(\alpha_i,\gamma)} \delta_{a,a'} \mu(a\alpha_i) \prod_{t=1}^{a} (1-q_i^{-2t}) (\!|x,x'|\!),$$

where μ is defined as in Theorem 2.4.

Proof. It follows immediately from Lemma 2.10(b) and (2.13) that if $y, y' \in \ker \partial_i^{op}$, $y \in U_q(\mathfrak{n}^+)_{\gamma'}$ and $a, a' \in \mathbb{Z}_{\ge 0}$ then

$$(\!|E_i^a y, E_i^{a'} y'|\!) = \delta_{a,a'} \langle a\rangle_{q_i}! q^{-\frac{1}{2}a(\alpha_i,\gamma')} (\!|y,y'|\!) = \delta_{a,a'} q_i^{\binom{a+1}{2} - \frac{1}{2}a(\alpha_i^\vee,\gamma')} \prod_{t=1}^{a} (1-q_i^{-2t}) (\!|y,y'|\!).$$

Let $\gamma_i = (\alpha_i^\vee, \gamma)$. Since $T_i(x), T_i(x') \in \ker \partial_i^{op} \cap U_q(\mathfrak{n}^+)_{s_i\gamma}$, it remains to prove the assertion for $a = a' = 0$, $x = (\underline{E}_i^{op})^{(a)}(z)$ and $x' = (\underline{E}_i^{op})^{(b)}(z')$ where $z, z' \in {}_iU \cap U_i$ and $a \ge b \ge \max(0, \gamma_i)$. Since $z \in U_q(\mathfrak{n}^+)_{\gamma - a\alpha_i}$, $z' \in U_q(\mathfrak{n}^+)_{\gamma - b\alpha_i}$ we have, by Corollary 3.14

$$(\!|x,x'|\!) = \delta_{a,b} q_i^{\frac{1}{2}a(1+a-\gamma_i)} \binom{2a-\gamma_i}{a}_{q_i} (\!|z,z'|\!).$$

On the other hand, using Theorem 3.15 and Corollary 3.14 we obtain

$$(\!|T_i(x), T_i(y)|\!) = (\!|\underline{E}_i^{(a-\gamma_i)}(z), \underline{E}_i^{(b-\gamma_i)}(z')|\!) = q_i^{\frac{1}{2}(a-\gamma_i)(a+1)} \delta_{a,b} \binom{2a-\gamma_i}{a-\gamma_i}_{q_i} (\!|z,z'|\!) = q_i^{-\frac{1}{2}\gamma_i} (\!|x,y|\!). \qquad \square$$

Let $b \in \mathbf{B}^{up}{}_\gamma \cap T_i^{-1}(U_q(\mathfrak{n}^+))$. Since T_i commutes with $\bar{\cdot}$ we have $\overline{T_i(b)} = T_i(\bar{b}) = T_i(b)$. By Proposition 3.17 we have $T_i(b) \in U^{\mathbb{Z}}(\mathfrak{n}^+)$. Furthermore, by Lemma 3.21 and (2.4)

$$\mu(s_i\gamma)^{-1} (\!|T_i(b), T_i(b)|\!) = \mu(\gamma)^{-1} q^{\frac{1}{2}(\alpha_i,\gamma)} (\!|T_i(b), T_i(b)|\!) = \mu(\gamma)^{-1} (\!|b,b|\!) \in 1 + K_-.$$

Thus, $T_i(b) \in \mathbf{B}^{\pm up}$ by (2.6).

It remains to prove that $T_i(b) \in \mathbf{B}^{up}$. Since $(\partial_i^{op})^{(top)}(b) \in \mathbf{B}^{up}$ by Remark 2.14, there exists a sequence $\mathbf{i}' = (i_1, \dots, i_m) \in I^m$ such that $\partial_{\mathbf{i}'}^{(top)}((\partial_i^{op})^{(top)}(b)) = 1$. Let $\mathbf{i} = (i, i_1, \dots, i_m)$. Then $\partial_{\mathbf{i}}^{(top)}(T_i(b)) = \partial_{\mathbf{i}'}^{(top)} \partial_i^{(top)} T_i(b) = \partial_{\mathbf{i}'}^{(top)}((\partial_i^{op})^{(top)}(b)) = 1$ by Theorem 3.15. Thus, $T_i(b) \in \mathbf{B}^{up}$. □

4. Proofs of mains results

4.1. **Properties of quantum Schubert cells.** Let $w \in W$ and $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$. Set $X_{\mathbf{i},k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k})$, $1 \le k \le m$, and let $U_q(\mathbf{i})$ be the subalgebra of $U_q(\mathfrak{n}^+)$ generated by the $X_{\mathbf{i},k}$, $1 \le k \le m$ and set $U^{\mathbb{Z}}(\mathbf{i}) = U_q(\mathbf{i}) \cap U^{\mathbb{Z}}(\mathfrak{n}^+)$, $U_{\mathbb{Z}}(\mathbf{i}) = U_q(\mathbf{i}) \cap U_{\mathbb{Z}}(\mathfrak{n}^+)$. The following is well-known.

Lemma 4.1 ([16, Propositions 40.2.1 and 41.1.4])**.** *The elements $X_{\mathbf{i}}^{\langle \mathbf{a}\rangle} := X_{\mathbf{i},1}^{\langle a_1\rangle} \cdots X_{\mathbf{i},m}^{\langle a_m\rangle}$, $\mathbf{a} \in \mathbb{Z}_{\ge 0}^m$ form an $\mathbb{A}_0$-basis of $U_{\mathbb{Z}}(\mathbf{i})$ and a $\Bbbk$-basis of $U_q(\mathbf{i})$.*

Set $\alpha^{(k)} = \alpha_{\mathbf{i}}^{(k)} := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \deg X_{\mathbf{i},k}$ and given $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$ denote $|\mathbf{a}| = |\mathbf{a}|_{\mathbf{i}} = \sum_{k=1}^m a_k \deg X_{\mathbf{i},k} = \sum_{k=1}^m a_k \alpha_{\mathbf{i}}^{(k)}$. Define

$$X_{\mathbf{i}}^{\mathbf{a}} = q_{\mathbf{i},\mathbf{a}} X_{\mathbf{i},1}^{a_1} \cdots X_{\mathbf{i},m}^{a_m}, \qquad \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}_{\ge 0}^m,$$

where

$$q_{\mathbf{i},\mathbf{a}} = q^{\frac{1}{2}\sum_{1\le k<l\le m}(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(l)})a_k a_l}. \tag{4.1}$$

This choice is justified by the following

Proposition 4.2. *For all* $\mathbf{i}\in R(w)$, $\mathbf{a},\mathbf{a}'\in\mathbb{Z}^m_{\ge 0}$ *we have* $X_{\mathbf{i}}^{\mathbf{a}}\in U^{\mathbb{Z}}(\mathfrak{n}^+)$ *and*

$$\mu(|\mathbf{a}|)^{-1}(\!| X_{\mathbf{i}}^{\mathbf{a}}, X_{\mathbf{i}}^{\mathbf{a}'} |\!) = \delta_{\mathbf{a},\mathbf{a}'}\prod_{r=1}^{m}\prod_{t=1}^{a_m}(1-q_{i_r}^{-2t}). \tag{4.2}$$

Thus, the set $\{X_{\mathbf{i}}^{\mathbf{a}} : |\mathbf{a}|_{\mathbf{i}}=\gamma\}$ *is a* $(K_-,\mu(\gamma)^{-1})$*-orthonormal basis of* $U^{\mathbb{Z}}(\mathbf{i})_\gamma$ *and*

$$(\!| X_{\mathbf{i}}^{\mathbf{a}}, X_{\mathbf{i}}^{\langle\mathbf{a}'\rangle} |\!) = \delta_{\mathbf{a},\mathbf{a}'}, \qquad \mathbf{a},\mathbf{a}'\in\mathbb{Z}^m_{\ge 0}. \tag{4.3}$$

Proof. We need the following

Lemma 4.3. *For all* $w\in W$, $j\in I$ *such that* $\ell(ws_j)=\ell(w)+1$ *we have* $T_w(E_j)\in U^{\mathbb{Z}}(\mathfrak{n}^+)$.

Proof. The argument is by induction on $\ell(w)$. If $\ell(w)=0$ there is nothing to prove. Suppose that $w=s_iw'$ with $\ell(w)=\ell(w')+1$. Clearly, $\ell(w's_j)=\ell(w')+1$. Then $T_{w'}(E_j)\in\ker\partial_i$ by [16, Lemma 40.1.2] and also $T_{w'}(E_j)\in U^{\mathbb{Z}}(\mathfrak{n}^+)$ by the induction hypothesis. Then by Proposition 3.17, $T_w(E_j)=T_i(T_{w'}(E_j))\in U^{\mathbb{Z}}(\mathfrak{n}^+)$. □

This implies that $X_{\mathbf{i},k}\in U^{\mathbb{Z}}(\mathfrak{n}^+)$ and hence $X_{\mathbf{i}}^{\mathbf{a}}\in U^{\mathbb{Z}}(\mathfrak{n}^+)$ by Lemma 2.3.

To prove (4.2) we use induction on $\ell(w)$. The case $\ell(w)=0$ is trivial. For the inductive step, assume that $\ell(s_iw)=\ell(w)+1$ and note that we have

$$X_{(i,\mathbf{i})}^{(a,\mathbf{a})} = q^{\frac{1}{2}a(\alpha_i,s_i|\mathbf{a}|_{\mathbf{i}})}E_i^aT_i(X_{\mathbf{i}}^{\mathbf{a}}) = q^{-\frac{1}{2}a(\alpha_i,|\mathbf{a}|_{\mathbf{i}})}E_i^aT_i(X_{\mathbf{i}}^{\mathbf{a}}).$$

Since $T_i(X_{\mathbf{i}}^{\mathbf{a}})\in {}_iU$, we have by Lemmata 2.10, 3.21 and (2.4)

$$\begin{aligned}(\!| X_{(i,\mathbf{i})}^{(a,\mathbf{a})}, X_{(i,\mathbf{i})}^{(a',\mathbf{a}')} |\!) &= q^{-\frac{1}{2}(a(\alpha_i,|\mathbf{a}|_{\mathbf{i}})+a'(\alpha_i,|\mathbf{a}'|_{\mathbf{i}}))}(\!| E_i^aT_i(X_{\mathbf{i}}^{\mathbf{a}}), E_i^{a'}T_i(X_{\mathbf{i}}^{\mathbf{a}'}) |\!)\\
&= \delta_{a,a'}\mu(a\alpha_i)q^{-\frac{1}{2}(a+1)(\alpha_i,|\mathbf{a}|_{\mathbf{i}})}\prod_{t=1}^{a}(1-q_i^{-2t})(\!| X_{\mathbf{i}}^{\mathbf{a}}, X_{\mathbf{i}}^{\mathbf{a}'} |\!)\\
&= \delta_{a,a'}\delta_{\mathbf{a},\mathbf{a}'}\mu(|\mathbf{a}|)\mu(a\alpha_i)q^{-\frac{1}{2}(a+1)(\alpha_i^\vee,|\mathbf{a}|_{\mathbf{i}})}\prod_{t=1}^{a}(1-q_i^{-2t})\prod_{r=1}^{m}\prod_{t=1}^{a_r}(1-q_{i_r}^{-2r})\\
&= \delta_{a,a'}\delta_{\mathbf{a},\mathbf{a}'}\mu(|(a,\mathbf{a})|_{(i,\mathbf{i})})\prod_{t=1}^{a}(1-q_i^{-2t})\prod_{r=1}^{m}\prod_{t=1}^{a_r}(1-q_{i_r}^{-2r}),\end{aligned}$$

since $|(a,\mathbf{a})|_{(i,\mathbf{i})}=a\alpha_i+s_i(|\mathbf{a}|_{\mathbf{i}})$. Finally, (4.3) is immediate from (4.2) and (2.4). □

Set $U^{\mathbb{Z}}(w)=U^{\mathbb{Z}}(\mathfrak{n}^+)\cap U_q(w)$ where $U_q(w)$ is defined by (1.1).

Proposition 4.4. *For each* $\mathbf{i}\in R(w)$, $\{X_{\mathbf{i}}^{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{Z}^m_{\ge 0}}$ *is an* $\mathbb{A}_0$*-basis of* $U^{\mathbb{Z}}(w)$*. In particular,* $U^{\mathbb{Z}}(w)=U^{\mathbb{Z}}(\mathbf{i})$ *for all* $w\in W$, $\mathbf{i}\in R(w)$.

Proof. Since $U_q(w)=U_q(\mathbf{i})$ by [19, Proposition 2.10], for any $x\in U^{\mathbb{Z}}(w)$ we can write $x=\sum_{\mathbf{a}'}c_{\mathbf{a}'}X_{\mathbf{i}}^{\mathbf{a}'}$ where $c_{\mathbf{a}'}\in\Bbbk$. Since $(\!| x, X_{\mathbf{i}}^{\mathbf{a}} |\!)\in\mathbb{A}_0$, it follows from (4.3) that $c_{\mathbf{a}}\in\mathbb{A}_0$ for all $\mathbf{a}\in\mathbb{Z}^m_{\ge 0}$. Thus, the $X_{\mathbf{i}}^{\mathbf{a}}$, $\mathbf{a}\in\mathbb{Z}^m_{\ge 0}$ generate $U^{\mathbb{Z}}(w)$ as an $\mathbb{A}_0$-module. Since they are already linearly independent over $\Bbbk$, they form its $\mathbb{A}_0$-basis. □

Theorem 4.5. *Let $w \in W$, $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$. Then the algebra $U^{\mathbb{A}}(w) = U^{\mathbb{Z}}(w) \otimes_{\mathbb{A}_0} \mathbb{A}$ has the following presentation*

$$q^{-\frac{1}{2}(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(l)})} X_{\mathbf{i},l} X_{\mathbf{i},k} - q^{\frac{1}{2}(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(l)})} X_{\mathbf{i},k} X_{\mathbf{i},l} \in (q_{i_k} - q_{i_k}^{-1}) \sum_{\mathbf{a}=(0,\dots,0,a_{k+1},\dots,a_{l-1},0,\dots,0)\in\mathbb{Z}_{\geq 0}^m} \mathbb{A}_0 X_{\mathbf{i}}^{\mathbf{a}}, \qquad (4.4)$$

for all $1 \le k < l \le m$

Proof. We need the following

Lemma 4.6. *Let $w' \in W$ and $i, j \in I$ be such that $\ell(s_i w' s_j) = \ell(w') + 2$. Then*

$$T_{s_i w'}(E_j) E_i - q^{-(\alpha_i, w'\alpha_j)} E_i T_{s_i w'}(E_j) \in (q_i - q_i^{-1}) T_i(U^{\mathbb{A}}(w')). \qquad (4.5)$$

Proof. First we prove that

$$K_i[F_i, T_{w'}(E_j)] \in \langle 1\rangle_{q_i} U^{\mathbb{A}}(w'). \qquad (4.6)$$

Our assumption implies that $\ell(s_i w') = \ell(w') + 1$, $\ell(w' s_j) = \ell(w') + 1$, whence $T_{w'}(E_j), T_{s_i w'}(E_j) \in U_q(\mathfrak{n}^+)$ by [16, Lemma 40.1.2] and so $T_{w'}(E_j) \in \ker \partial_i$ by [16, Proposition 38.1.6]. Moreover, by Proposition 4.2 we have $T_{w'}(E_j), T_{s_i w'}(E_j) \in U^{\mathbb{A}}(\mathfrak{n}^+)$. Then

$$K_i[F_i, T_{w'}(E_j)] = -(1 - q_i^{-2}) q^{-\frac{1}{2}(\alpha_i, w'\alpha_j)} \partial_i^{op}(T_{w'}(E_j)) \in \langle 1\rangle_{q_i} U^{\mathbb{A}}(\mathfrak{n}^+),$$

where we used (2.11), Lemma 4.3, Proposition 3.17 and Corollary 2.11(b). On the other hand, $T_{w'}^{-1}(F_i) \in U_q(\mathfrak{n}^-)$, whence $[T_{w'}^{-1}(F_i), E_j] \in U_q(\mathfrak{b}^-)$. Therefore,

$$T_{w'}^{-1}(K_i[F_i, T_{w'}(E_j)]) = T_{w'}^{-1}(K_i)[T_{w'}^{-1}(F_i), E_j] \in U_q(\mathfrak{b}^-).$$

Thus,

$$K_i[F_i, T_{w'}(E_j)] \in \langle 1\rangle_{q_i} T_{w'}(U_q(\mathfrak{b}^-)) \cap U^{\mathbb{A}}(\mathfrak{n}^+) = \langle 1\rangle_{q_i} U^{\mathbb{A}}(w').$$

This proves (4.6). Since $T_i(K_i F_i) = q_i^{-1} E_i$ sand $K_i T_{s_i w'}(E_j) K_i^{-1} = q^{-(\alpha_i, w'\alpha_j)} T_{s_i w'}(E_j)$, (4.5) follows by applying T_i to both sides of (4.6). ☐

Now we use induction on $\ell(w)$, the induction base being trivial. Applying $T_{i_1} \cdots T_{i_{k-1}}$ to (4.5) with $w' = s_{i_{k+1}} \cdots s_{i_{l-1}}$, $i = i_k$, $j = i_l$ we obtain

$$X_l X_k - q^{-(\alpha_{i_k}, s_{i_{k+1}} \cdots s_{i_{l-1}} \alpha_{i_l})} X_k X_l \in \langle 1\rangle_{q_{i_k}} T_{i_1} \cdots T_{i_k}(U^{\mathbb{A}}(w')).$$

By Proposition 4.4, $U^{\mathbb{A}}(w')$ has an $\mathbb{A}$-basis $\{X_{\mathbf{i}',1}^{a_{k+1}} \dots X_{\mathbf{i}',l-1}^{a_{l-1}} : a_{k+1}, \dots, a_{l-1} \in \mathbb{Z}_{\geq 0}\}$ where $\mathbf{i}' = (i_{k+1}, \dots, i_{l-1})$. Applying $T_{i_1} \cdots T_{i_k}$ we conclude that $\{X_{\mathbf{i},k+1}^{a_{k+1}} \dots X_{\mathbf{i},l-1}^{a_{l-1}} : a_{k+1}, \dots, a_{l-1} \in \mathbb{Z}_{\geq 0}\}$ is an $\mathbb{A}$-basis of $T_{i_1} \cdots T_{i_k}(U^{\mathbb{A}}(w'))$. Note that $(\alpha_{i_k}, s_{i_{k+1}} \cdots s_{i_{l-1}} \alpha_{i_l}) = -(\alpha_{\mathbf{i}}^{(k)}, \alpha_{\mathbf{i}}^{(l)})$ and so we can write

$$q^{-\frac{1}{2}(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(l)})} X_{\mathbf{i},l} X_{\mathbf{i},k} - q^{\frac{1}{2}(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(l)})} X_{\mathbf{i},k} X_{\mathbf{i},l} \in \sum_{\mathbf{a}=(0,\dots,0,a_{k+1},\dots,a_{l-1},0,\dots,0)\in\mathbb{Z}_{\geq 0}^m} \langle 1\rangle_{q_{i_k}} c_{\mathbf{a}} X_{\mathbf{i}}^{\mathbf{a}},$$

where $c_{\mathbf{a}} \in \mathbb{A}$. Repeating the argument from the proof of Proposition 4.4 we conclude that $\langle 1\rangle_{q_{i_k}} c_{\mathbf{a}} \in \mathbb{A}_0$. Thus, $c_{\mathbf{a}} \in \mathbb{A} \cap (\langle 1\rangle_{q_{i_k}})^{-1} \mathbb{A}_0 = \mathbb{A}_0$. Since relations (4.4) imply that $U^{\mathbb{A}}(w)$ is generated, as an $\mathbb{A}$-module, by the $X_{\mathbf{i}}^{\mathbf{a}}$, it follows that (4.4) is a presentation. ☐

Remark 4.7. Let $A(w)$ be the $\mathbb{Z}$-algebra defined by $A(w) = U^{\mathbb{Z}}(w)/(q-1)U^{\mathbb{Z}}(w)$. Clearly, $A(w)$ is commutative and identifies with the coordinate algebra $\mathbb{Z}[U(w)]$, where $U(w) = U \cap w(U^-)w^{-1}$ is the Schubert cell in the maximal unipotent subgroup U of the Kac-Moody group G corresponding to $\mathfrak{g}$. This justifies (1.1) and the name quantum Schubert cell used for $U_q(w)$.

4.2. **Lusztig's Lemma and proof of Theorem 1.1.** Let $w \in W$, $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$ and let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be the standard basis of $\mathbb{Z}^m$. For each pair $1 \le k < l \le m$ with $k+1 \le l-1$ define $\mathcal{A}_{k,l} = \mathcal{A}_{k,l}(\mathbf{i})$ to be the finite set of all tuples $(a_{k+1}, \dots, a_{l-1})$ such that $X_{\mathbf{i},k+1}^{a_{k+1}} \cdots X_{\mathbf{i},l-1}^{a_{l-1}}$ occurs in the right hand side of (4.4) with a non-zero coefficient. Let $C_{\mathbf{i}}$ be the submonoid of $\mathbb{Z}^m$ generated by elements

$$\mathbf{e}_k + \mathbf{e}_l - \sum_{r=k+1}^{l-1} a_r \mathbf{e}_r$$

for all $1 \le k, l \le m$ with $k+1 \le l-1$ such that $\mathcal{A}_{kl} \ne \emptyset$ and for all $(a_{k+1}, \dots, a_{l-1}) \in \mathcal{A}_{kl}$.

Proposition 4.8. *$C_{\mathbf{i}}$ is pointed, that is, if $\mathbf{x}, -\mathbf{x} \in C_{\mathbf{i}}$ then $\mathbf{x} = 0$. In particular, the relation $\prec$ on $\mathbb{Z}^m_{\ge 0}$ defined by*

$$\mathbf{a} \preceq \mathbf{a}' \iff \mathbf{a}' - \mathbf{a} \in C_{\mathbf{i}}$$

is a partial order.

Proof. The first assertion is a special case of the following

Lemma 4.9. *For each $k < l$ fix $\mathcal{A}_{k,l} \subset \Big(\bigoplus_{i=k+1}^{l-1} \mathbb{Z}_{\ge 0} \mathbf{e}_i \Big) \setminus \{0\}$. Let Γ be the submonoid of $\mathbb{Z}^m$ generated by all elements of the form $\mathbf{e}_k + \mathbf{e}_l - \mathbf{a}$, $\mathbf{a} \in \mathcal{A}_{k,l}$ for all $k < l$ such that $\mathcal{A}_{k,l} \ne \emptyset$. Then Γ is pointed.*

Proof. Let $\mathbf{y} = \sum_{k<l} \sum_{\mathbf{a} \in \mathcal{A}_{k,l}} n_{k,l,\mathbf{a}_{k,l}} (\mathbf{e}_k + \mathbf{e}_l - \mathbf{a}_{k,l})$ where $n_{k,l,\mathbf{a}_{k,l}} \in \mathbb{Z}_{\ge 0}$ and are not all zero. Let k be minimal such that $n_{k,l,\mathbf{a}} \ne 0$ for some $l > k$, $\mathbf{a} \in \mathcal{A}_{k,l}$. Then the coefficient of $\mathbf{e}_k$ in $\mathbf{y}$ is positive. This immediately implies that 0 admits a unique presentation in Γ. □

To prove the second assertion, note that the relation $\prec$ is clearly transitive. Furthermore, if $\mathbf{a}' \prec \mathbf{a}$ and $\mathbf{a} \prec \mathbf{a}'$ then $\mathbf{a}' - \mathbf{a}, \mathbf{a} - \mathbf{a}' \in C_{\mathbf{i}}$ which implies that $\mathbf{a} = \mathbf{a}'$. □

Since T_w commutes with $\bar{\cdot}$-anti-involution, $\overline{U_q(w)} = U_q(w)$ and $\overline{X_{\mathbf{i},k}} = X_{\mathbf{i},k}$. Since also $\overline{U^{\mathbb{Z}}(\mathfrak{n}^+)} = U^{\mathbb{Z}}(\mathfrak{n}^+)$, it follows that $\overline{U^{\mathbb{Z}}(w)} = U^{\mathbb{Z}}(w)$. Thus, the restriction of $\bar{\cdot}$ to $U^{\mathbb{Z}}(\mathbf{i})$ is the unique anti-linear anti-involution of that algebra fixing its generators $X_{\mathbf{i},k}$.

Note that for each $\gamma \in Q^+$, the set $\{\mathbf{a} \in \mathbb{Z}^m_{\ge 0} : |\mathbf{a}|_{\mathbf{i}} = \gamma\}$ is finite. The following result is crucial for the proof of Theorem 1.1.

Proposition 4.10. *For all $\mathbf{a} \in \mathbb{Z}^m_{\ge 0}$ we have*

$$\overline{X_{\mathbf{i}}^{\mathbf{a}}} - X_{\mathbf{i}}^{\mathbf{a}} \in \sum_{\mathbf{a}' \prec \mathbf{a}} \mathbb{A}_0 X_{\mathbf{i}}^{\mathbf{a}'}.$$

Proof. We need some notation. Let $\mathcal{U} = U^{\mathbb{Z}}(\mathbf{i})$ and let $\mathcal{I} = [1, m]$. Let $\mathcal{B}$ be the set of all finite non-decreasing sequences in $\mathcal{I}$. Given a sequence $\mathbf{k} = (k_1, \dots, k_N) \in \mathcal{I}^N$, let $\mathbf{e}_{\mathbf{k}} = \sum_{r=1}^N \mathbf{e}_{k_r}$ and define

$$X(\mathbf{k}) = q^{\frac{1}{2} \sum_{1 \le r < s \le N} \operatorname{sign}(k_s - k_r)(\alpha^{(k_r)}, \alpha^{(k_s)})} X_{\mathbf{i},k_1} \cdots X_{\mathbf{i},k_N}.$$

In particular, if $\mathbf{k} = (k_1, \dots, k_N) \in \mathcal{B}$ and $a_k = \#\{1 \le r \le N : k_r = k\}$ then $X(\mathbf{k}) = X_{\mathbf{i}}^{\mathbf{a}}$. Given $\mathbf{a} \in \mathbb{Z}^m_{\ge 0}$, set

$$\mathcal{U}_{\prec \mathbf{a}} = \sum_{\mathbf{a}' \prec \mathbf{a}} \mathbb{A}_0 \cdot X_{\mathbf{i}}^{\mathbf{a}'} = \sum_{\mathbf{k} \in \mathcal{B} \,:\, \mathbf{e}_{\mathbf{k}} \prec \mathbf{a}} \mathbb{A}_0 \cdot X(\mathbf{k}), \qquad \mathcal{U}'_{\prec \mathbf{a}} := \sum_{N \ge 0,\, \mathbf{k} \in \mathcal{I}^N : \mathbf{e}_{\mathbf{k}} \prec \mathbf{a}} \mathbb{A}_0 \cdot X(\mathbf{k})$$

with the convention that $\mathcal{U}_{\prec \mathbf{a}} = \mathcal{U}'_{\prec \mathbf{a}} = \{0\}$ if $\mathbf{a}$ is minimal with respect to $\prec$. Clearly, both are increasing filtration on $\mathcal{U}$. Note following immediate

Lemma 4.11. *If $\mathbf{a}' \prec \mathbf{a}$, $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}_{\ge 0}^m$ then $|\mathbf{a}'|_{\mathbf{i}} = |\mathbf{a}|_{\mathbf{i}}$. In particular, $\mathcal{U}_{\prec\mathbf{a}}$, $\mathcal{U}'_{\prec\mathbf{a}}$ are finite dimensional.*

Lemma 4.12. *For any sequence $\mathbf{k} = (k_1, \dots, k_N) \in \mathcal{I}^N$, $N \ge 0$ and for any $\sigma \in S_N$ we have*

$$X(\sigma(\mathbf{k})) - X(\mathbf{k}) \in \mathcal{U}'_{\prec\mathbf{e_k}}, \tag{4.7}$$

where $\sigma(\mathbf{k}) = (k_{\sigma(1)}, \dots, k_{\sigma(N)})$.

Proof. Clearly, it suffices to prove the assertion for a transposition $\sigma = (r, r+1)$. Without loss of generality we may assume that $k_r < k_{r+1}$. Let $\mathbf{k}_r^- = (k_1, \dots, k_{r-1})$, $\mathbf{k}_r^+ = (k_{r+2}, \dots, k_N)$. Then the relation (4.4) taken with $k = k_r$, $l = k_{r+1}$ implies

$$X_{\sigma(\mathbf{k})} = X_{(\mathbf{k}_r^-, k_{r+1}, k_r, \mathbf{k}_r^+)} = X_{\mathbf{k}} + \sum_{\mathbf{k}' \in \mathcal{B}\,:\, \mathbf{e}_{\mathbf{k}'} \prec \mathbf{e}_{i_r} + \mathbf{e}_{i_{r+1}}} c_{\mathbf{k}'} X_{(\mathbf{k}_r^-, \mathbf{k}', \mathbf{k}_r^+)}, \qquad c_{\mathbf{k}'} \in \mathbb{A}_0. \tag{4.8}$$

Clearly, $\mathbf{e}_{(\mathbf{k}_r^-, \mathbf{k}', \mathbf{k}_r^+)} = \mathbf{e}_{\mathbf{k}_r^-} + \mathbf{e}_{\mathbf{k}'} + \mathbf{e}_{\mathbf{k}_r^+} \prec \mathbf{e}_{\mathbf{k}}$ for all $\mathbf{k}' \in \mathcal{B}$ such that $\mathbf{e}_{\mathbf{k}'} \prec \mathbf{e}_{k_r} + \mathbf{e}_{k_{r+1}}$. This implies that each $X_{(\mathbf{k}_r^-, \mathbf{k}', \mathbf{k}_r^+)}$ in the right hand side of (4.8) belongs to $\mathcal{U}'_{\prec e_{\mathbf{k}}}$ and we obtain (4.7) for $\sigma = (r, r+1)$. □

Lemma 4.13. $\mathcal{U}_{\prec\mathbf{a}} = \mathcal{U}'_{\prec\mathbf{a}}$ *for all* $\mathbf{a} \in \mathbb{Z}_{\ge 0}^m$.

Proof. The inclusion $\mathcal{U}_{\prec\mathbf{a}} \subseteq \mathcal{U}'_{\prec\mathbf{a}}$ is obvious. To prove the opposite inclusion, we use induction on the partial order $\prec$ which is applicable since $\{\mathbf{a}' \in \mathbb{Z}_{\ge 0}^m : \mathbf{a}' \prec \mathbf{a}\}$ is finite for all $\mathbf{a} \in \mathbb{Z}_{\ge 0}^m$.

If $\mathbf{a} \in \mathbb{Z}_{\ge 0}^m$ is minimal with respect to $\prec$, then $\mathcal{U}_{\prec\mathbf{a}} = \{0\}$ and we have nothing to prove. Assume now that $\mathbf{a}$ is not minimal. Then for each $\mathbf{k} \in \mathcal{I}^N$, $N \ge 0$ such that $\mathbf{e_k} \prec \mathbf{a}$ we have $\mathcal{U}'_{\prec\mathbf{e_k}} = \mathcal{U}_{\prec\mathbf{e_k}}$ by the induction hypothesis.

Using this and Lemma 4.12, we conclude that for any $\sigma \in S_N$

$$X(\mathbf{k}) - X(\sigma(\mathbf{k})) \in \mathcal{U}_{\prec\mathbf{e_k}}.$$

Taking σ such that $\sigma(\mathbf{k}) \in \mathcal{B}$, that is, is non-decreasing, implies that $X(\mathbf{k}) \in \mathcal{U}_{\prec\mathbf{a}}$. □

Combining Lemmata 4.12 and 4.13 we obtain the following obvious corollary:

Corollary 4.14. *For any $\mathbf{k} \in \mathcal{B}$ and any $\sigma \in S_N$, we have $X(\sigma(\mathbf{k})) - X(\mathbf{k}) \in \mathcal{U}_{\prec\mathbf{e_k}}$.*

Note that $\overline{X(\mathbf{k})} = X(\mathbf{k}^{op})$ for any $\mathbf{k} \in \mathcal{I}^N$ where $\mathbf{k}^{op}$ is $\mathbf{k}$ written in the reverse order, and $X(\mathbf{k}) = X_{\mathbf{i}}^{\mathbf{e_k}}$ for $\mathbf{k} \in \mathcal{B}$. Since for any $\mathbf{a} \in \mathbb{Z}_{\ge 0}$ there exists a unique $\mathbf{k} \in \mathcal{B}$ such that $\mathbf{e_k} = \mathbf{a}$, these observations together with the above Corollary complete the proof Proposition 4.10. □

Proposition 4.10 implies that for each $\gamma \in Q^+$ the assumptions of [5, Theorem 1.1] with $(L, \prec) = (\{\mathbf{a} \in \mathbb{Z}_{\ge 0}^m : |\mathbf{a}|_{\mathbf{i}} = \gamma\}, \prec)$ and $v = q^{-1}$ are satisfied. The assertion of Theorem 1.1 now follows. □

Note the following useful fact, which is immediate from the proof of Proposition 4.10.

Corollary 4.15. *Define $\Lambda = \Lambda_{\mathbf{i}} : \mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{Z}^m \to \mathbb{Z}$ by $\Lambda(\mathbf{e}_k, \mathbf{e}_l) = \operatorname{sign}(l-k)(\alpha_{\mathbf{i}}^{(k)}, \alpha_{\mathbf{i}}^{(l)})$. Then for all $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}_{\ge 0}^m$*

$$X_{\mathbf{i}}^{\mathbf{a}} X_{\mathbf{i}}^{\mathbf{b}} - q^{-\frac{1}{2}\Lambda(\mathbf{a},\mathbf{b})} X_{\mathbf{i}}^{\mathbf{a}+\mathbf{b}} \in q^{-\frac{1}{2}\Lambda(\mathbf{a},\mathbf{b})} \sum_{\mathbf{a}' \prec \mathbf{a}+\mathbf{b}} \mathbb{A}_0 X_{\mathbf{i}}^{\mathbf{a}'}$$

and also

$$X_{\mathbf{i}}^{\mathbf{b}} X_{\mathbf{i}}^{\mathbf{a}} - q^{\Lambda(\mathbf{a},\mathbf{b})} X_{\mathbf{i}}^{\mathbf{a}} X_{\mathbf{i}}^{\mathbf{b}} \in q^{\frac{1}{2}\Lambda(\mathbf{a},\mathbf{b})} \sum_{\mathbf{a}' \prec \mathbf{a}+\mathbf{b}} \mathbb{A}_0 X_{\mathbf{i}}^{\mathbf{a}'}.$$

We note an obvious property of Λ which will be used in the sequel.

Lemma 4.16. *For any $1 \le k \le m$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$ we have*

$$\Lambda_{\mathbf{i}}(\mathbf{e}_k, \mathbf{a}) = (\alpha_{\mathbf{i}}^{(k)}, |\mathbf{a}_{>k}|_{\mathbf{i}} - |\mathbf{a}_{<k}|_{\mathbf{i}}),$$

where $\mathbf{a}_{<k} = \sum_{t=1}^{k-1} a_t \mathbf{e}_t$, $\mathbf{a}_{>k} = \sum_{t=k+1}^{m} a_t \mathbf{e}_t$.

4.3. **Containment of $\mathbf{B}(\mathbf{i})$ in $\mathbf{B}^{up}$ and proof of Theorem 1.2.** Let $w \in W$, $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$. Let $\gamma = |\mathbf{a}|_\mathbf{i}$. Since $b_{\mathbf{i},\mathbf{a}} \in X_\mathbf{i}^\mathbf{a} + \sum_{\mathbf{a}' \neq \mathbf{a},\, |\mathbf{a}'|_\mathbf{i} = \gamma} K_- X_\mathbf{i}^{\mathbf{a}'}$, it follows from (4.2) that

$$\mu(\gamma)^{-1} (\!| b_{\mathbf{i},\mathbf{a}}, b_{\mathbf{i},\mathbf{a}} |\!) \in \mu(\gamma)^{-1} (\!| X_\mathbf{i}^\mathbf{a}, X_\mathbf{i}^\mathbf{a} |\!) + \sum_{\mathbf{a}' \in \mathbb{Z}_{\geq 0}^m \setminus \{\mathbf{a}\} \,:\, |\mathbf{a}'| = \gamma} K_- \mu(\gamma)^{-1} (\!| X_\mathbf{i}^{\mathbf{a}'}, X_\mathbf{i}^{\mathbf{a}'} |\!) \in 1 + K_-.$$

Since $b_{\mathbf{i},\mathbf{a}} \in U^\mathbb{Z}(\mathfrak{n}^+)$ and $\overline{b_{\mathbf{i},\mathbf{a}}} = b_{\mathbf{i},\mathbf{a}}$, it follows from (2.6) that $b_{\mathbf{i},\mathbf{a}} \in \mathbf{B}^{\pm up}$.

To prove that $b_{\mathbf{i},\mathbf{a}} \in \mathbf{B}^{up}$, we use induction on m. The induction base is trivial. For the inductive step, write $X_\mathbf{i}^\mathbf{a} = \sum_{b \in \mathbf{B}^{up}} c_{\mathbf{a},b} b$. Since $\pm b_{\mathbf{i},\mathbf{a}} \in \mathbf{B}^{up}$, it follows that $c_{\mathbf{a},b} \in K_-$ for all $b \neq b_0 = \pm b_{\mathbf{i},\mathbf{a}}$ and $c_{\mathbf{a},b_0} = \pm 1$. Thus, we only need to prove that $c_{\mathbf{a},b_0} = 1$ for some $b_0 \in \mathbf{B}^{up}$.

Let $i = i_1$ and $a = a_1$. Since $X_\mathbf{i}^\mathbf{a} = q^{-\frac{1}{2} a (\alpha_i, |\mathbf{a}'|_{\mathbf{i}'})} E_i^a T_i(X_{\mathbf{i}'}^{\mathbf{a}'})$ where $\mathbf{i}' = (i_2, \dots, i_m)$, $\mathbf{a}' = (a_2, \dots, a_m)$, $T_i(X_{\mathbf{i}'}^{\mathbf{a}'}) \in \ker \partial_i^{op}$ and $(\partial_i^{op})^{(top)}(E^a) = (\partial_i^{op})^{(a)}(E^a) = 1$, we have

$$(\partial_i^{op})^{(top)}(X_\mathbf{i}^\mathbf{a}) = (\partial_i^{op})^{(a)}(X_\mathbf{i}^\mathbf{a}) = T_i(X_{\mathbf{i}'}^{\mathbf{a}'}) = \sum_{b \in \mathbf{B}^{up} \,:\, \ell_i(b^*) = a} c_{\mathbf{a},b} (\partial_i^{op})^{(top)}(b),$$

where we used Corollary 2.15. Since $T_i^{-1}((\partial_i^{op})^{(top)}(b)) \in \mathbf{B}^{up}$ for any $b \in \mathbf{B}^{up}$ by Theorem 1.7, we obtain from the above that

$$X_{\mathbf{i}'}^{\mathbf{a}'} = T_i^{-1}((\partial_i^{op})^{(top)}(X_\mathbf{i}^\mathbf{a})) = \sum_{b \in \mathbf{B}^{up} \,:\, \ell_i(b^*) = a} c_{\mathbf{a},b} T_i^{-1}((\partial_i^{op})^{(top)}(b))$$

is the decomposition of $X_{\mathbf{i}'}^{\mathbf{a}'}$ with respect to $\mathbf{B}^{up}$. By the induction hypothesis, $b_{\mathbf{i}',\mathbf{a}''} \in \mathbf{B}^{up}$ for all $\mathbf{a}'' \in \mathbb{Z}_{\geq 0}^{m-1}$ and therefore precisely one of the $c_{\mathbf{a},b}$, $\ell_i(b^*) = a$ is not in K_- and is equal to 1. □

Remark 4.17. Note that for any $w \in W$, $\mathbf{i} \in R(w)$, $1 \leq k \leq \ell(w)$ and $a \geq 0$ we have $X_{\mathbf{i},k}^a \in \mathbf{B}^{up}$.

4.4. **Embeddings of bases and proof of Theorem 1.5.** Note that $U_q(w) \subset U_q(ww')$. Since $\mathbf{B}(w) = U_q(w) \cap \mathbf{B}^{up}$ and $\mathbf{B}(ww') = U_q(ww') \cap \mathbf{B}^{up}$, the first assertion follows. To establish the second assertion, it suffices to prove that for $i \in I$ such that $\ell(s_i w) = \ell(w) + 1$ we have $T_i(\mathbf{B}(w)) \subset \mathbf{B}(s_i w)$. The assumption implies that $T_i(\mathbf{B}(w)) \subset U_q(\mathfrak{n}^+)$ and therefore is contained in $\mathbf{B}^{up}$ by Theorem 1.7. Since $T_i(U_q(w)) \subset U_q(s_i w)$, it follows that $T_i(\mathbf{B}(w)) \subset U_q(s_i w) \cap \mathbf{B}^{up} = \mathbf{B}(s_i w)$. □

5. Examples

In this section we compute bases $\mathbf{B}(w)$ for various Schubert cells $U_q(w)$. We denote by $E_{i_1^{a_1} \cdots i_r^{a_r}}$ the unique element b of $\mathbf{B}^{up}$ for which $\partial_\mathbf{i}^{(top)}(b) = \partial_{i_r}^{(a_r)} \cdots \partial_{i_1}^{(a_1)}(b) = 1$ where $\mathbf{i} = (i_1, \dots, i_r)$. Note that this element also satisfies $(\partial_{\mathbf{i}^{op}}^{op})^{(top)}(b) = (\partial_{i_1}^{op})^{(a_1)} \cdots (\partial_{i_r}^{op})^{(a_r)}(b) = 1$. We use the notation from §4.2.

5.1. **Repetition free elements.** We say that $w \in W$ is repetition-free if $w = s_{i_1} \dots s_{i_m}$ where $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$ is repetition free. Clearly, if w is repetition free then so is each $\mathbf{i} \in R(w)$. Such an element is called a *Coxeter element* if $\ell(w) = |I|$, that is, any $\mathbf{i} \in R(w)$ is an ordering of I.

Lemma 5.1. *Let $w \in W$ be repetition free and let $\mathbf{i} \in R(w)$. Then in the notation of §4.1:*

(a) *$U_q(w)$ is a quantum plane of rank $\ell(w)$ with presentation*

$$q^{-\frac{1}{2}(\alpha_\mathbf{i}^{(k)}, \alpha_\mathbf{i}^{(l)})} X_{\mathbf{i},l} X_{\mathbf{i},k} = q^{\frac{1}{2}(\alpha_\mathbf{i}^{(k)}, \alpha_\mathbf{i}^{(l)})} X_{\mathbf{i},k} X_{\mathbf{i},l}, \qquad 1 \leq k < l \leq \ell(w). \tag{5.1}$$

(b) $\mathbf{B}(w) = \{X_\mathbf{i}^\mathbf{a} \,:\, \mathbf{a} \in \mathbb{Z}_{\geq 0}^{\ell(w)}\}$.

(c) $X_{\mathbf{i},k} = E_{i_1^{m_{k1}} \cdots i_{k-1}^{m_{k,k-1}} i_k} = \underline{E}_{i_1}^{(m_{k1})} \cdots \underline{E}_{i_{k-1}}^{(m_{k,k-1})}(E_{i_k})$ *where* $m_{kr} = -(\alpha_{i_r}^\vee, s_{i_{r+1}} \cdots s_{i_{k-1}}(\alpha_{i_k})) = d_{i_r}^{-1}(\alpha_\mathbf{i}^{(k)}, \alpha_\mathbf{i}^{(r)})$.

Proof. Note that the coefficient of α_{i_k} in every element of the submonoid of Q^+ generated by $\alpha_{\mathbf{i}}^{(r)}$, $k < r < l$ is zero. Since the algebra $U_q(w)$ is Q^+-graded, it follows that the right hand side of (4.4) is zero. This proves part (a). In particular, it follows that $\overline{X_{\mathbf{i}}^{\mathbf{a}}} = X_{\mathbf{i}}^{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, hence $b_{\mathbf{i},\mathbf{a}} = X_{\mathbf{i}}^{\mathbf{a}}$. To prove (b) it remains to apply Theorems 1.1 and 1.2. To prove part (c), let $u_r = T_{i_{r+1}} \cdots T_{i_{k-1}}(E_{i_k})$ and observe that the coefficient of α_{i_r} in $\deg u_r = s_{i_{r+1}} \cdots s_{i_{k-1}}(\alpha_{i_k})$ is zero if $\mathbf{i}$ is repetition free. Therefore, $u_r \in {}_{i_r}U \cap U_{i_r}$, $T_{i_r}(u_r) = \underline{E}_i^{(-(\alpha_{i_r}^\vee, \deg u_r))}(u_r)$ by Theorem 3.15 and so $\ell_i(T_{i_r}(u_r)) = -(\alpha_{i_r}^\vee, \deg u_r)$. The assertion now follows by induction on $k - r$. □

Remark 5.2. The assertion of Lemma 5.1(a) holds for any $w \in W$, $\mathbf{i} \in R(w)$ and $1 \leq k < l \leq \ell(w)$ such that the subsequence $(i_k, \dots, i_l)$ is repetition free.

5.2. **Elements with a single repetition.** We say that $w \in W$ is an element with a single repetition if there exists $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$ with $i_k \neq i_l$, $k < l$ unless $k = r$ and $l = r'$ for some $1 \leq r < r' \leq m$.

Proposition 5.3. *Let $w \in W$ be an element with a single repetition and let $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$, where the i_k, $k \neq r, r'$, $1 \leq k \leq m$ are distinct and $i_r = i_{r'} = i$, $1 \leq r < r' \leq m$. Then $U_q(w)$ is generated by the $X_{\mathbf{i},k}$, $1 \leq k \leq m$ where*

$$X_{\mathbf{i},k} = \begin{cases} E_{i_1^{m_{k,1}} \cdots i_{k-1}^{m_{k,k-1}} i_k}, & k \neq r' \\ E_{i_1^{m_{r',1}} \cdots i_{r-1}^{m_{r',r-1}} i_r^{1+m_{r',r}} \cdots i_{r'-1}^{m_{r',r'-1}}}, & k = r' \end{cases} \tag{5.2}$$

with $m_{kl} = -(\alpha_{i_l}^\vee, s_{i_{l+1}} \cdots s_{i_{l-1}}(\alpha_{i_l})) = d_{i_l}^{-1}(\alpha_{\mathbf{i}}^{(k)}, \alpha_{\mathbf{i}}^{(l)})$, subject to the relations

$$\begin{aligned} &q^{-\frac{1}{2}(\alpha_{\mathbf{i}}^{(k)}, \alpha_{\mathbf{i}}^{(l)})} X_{\mathbf{i},l} X_{\mathbf{i},k} = q^{\frac{1}{2}(\alpha_{\mathbf{i}}^{(k)}, \alpha_{\mathbf{i}}^{(l)})} X_{\mathbf{i},k} X_{\mathbf{i},l}, \qquad 1 \leq k < l \leq \ell(w),\, k \neq r,\, l \neq r' \\ &q^{-\frac{1}{2}(\alpha_{\mathbf{i}}^{(r')}, \alpha_{\mathbf{i}}^{(r)})} X_{\mathbf{i},r'} X_{\mathbf{i},r} = q^{\frac{1}{2}(\alpha_{\mathbf{i}}^{(r)}, \alpha_{\mathbf{i}}^{(r')})} X_{\mathbf{i},r} X_{\mathbf{i},r'} + (q_i - q_i^{-1}) X_{\mathbf{i}}^{\mathbf{n}(r,r')}, \quad \mathbf{n}(r,r') = -\sum_{k=r+1}^{r'-1} a_{i_k i} \mathbf{e}_k. \end{aligned} \tag{5.3}$$

Proof. Clearly, the sequences $(i_1, \dots, i_{r'-1})$ and $(i_{r+1}, \dots, i_m)$ are repetition free. In particular, for $1 \leq k \leq r' - 1$ we have $X_{\mathbf{i},k} = E_{i_1^{m_{k1}} \cdots i_{k-1}^{m_{k,k-1}} i_k}$ by Lemma 5.1(c). Furthermore,

$$X_{\mathbf{i},r'} = T_{i_1} \cdots T_{i_{r-1}} T_i T_{i_{r+1}} \cdots T_{i_{r'-1}}(E_i) = T_{i_1} \cdots T_{i_{r-1}} T_i (E_{i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}} i})$$

where $i = i_r = i_{r'}$. Clearly, $(\partial_i^{op})^2 (E_{i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}} i}) = 0$, hence

$$E_{i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}} i} = (2 + m_{r',r})_{q_i}{}^{-1} \underline{E}_i^{op}(E_{i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}}}) + x_0$$

where $x_0 \in {}_iU \cap U_i$. This implies that

$$T_i(E_{i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}} i}) = (2 + m_{r',r})_{q_i}^{-1} \underline{E}_i^{(1+m_{r',r})}(E_{i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}}}) + \underline{E}_i^{(m_{r',r})}(x_0),$$

and so $T_i(E_{i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}} i}) = E_{i^{1+m_{r',r}} i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}}}$, whence

$$X_{\mathbf{i},r'} = E_{i_1^{m_{r',1}} \cdots i_{r-1}^{m_{r',r-1}} i^{1+m_{r',r}} i_{r+1}^{m_{r',r+1}} \cdots i_{r'-1}^{m_{r',r'-1}}}.$$

Since the sequence $(i_{r+1}, \dots, i_k)$, $r' + 1 \leq k \leq m$ is repetition free, we have $T_{i_{r+1}} \cdots T_{i_{k-1}}(E_{i_k}) = E_{i_{r+1}^{m_{k,r+1}} \cdots i_{k-1}^{m_{k,k-1}} i_k}$ by Lemma 5.1(c) and hence is in ${}_iU \cap U_i$. Then $X_{\mathbf{i},k} = E_{i_1^{m_{k,1}} \cdots i_{k-1}^{m_{k,k-1}} i_k}$ by Theorem 3.15. This proves (5.2). The first identity in (5.3) is proved similarly to (5.1). To prove the second, we need the following combinatorial fact similar to [3, Lemma 4.8].

Lemma 5.4. *Let $w \in W$ and suppose that $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$ has a single repetition $i_r = i_{r'} = i$. Then*

$$\alpha_{\mathbf{i}}^{(r)} + \alpha_{\mathbf{i}}^{(r')} = -\sum_{k=r+1}^{r'-1} a_{i_k i}\alpha_{\mathbf{i}}^{(k)} \tag{5.4}$$

and any proper subset of $\{\alpha_{\mathbf{i}}^{(k)}\}_{r \le k \le r'}$ is linearly independent.

Proof. Fix $r < k \le r'$. Then

$$-\sum_{t=r+1}^{k-1} a_{i_t,i}\alpha_{\mathbf{i}}^{(t)} = -\sum_{t=r+1}^{k-1} (\alpha_{i_t}^\vee, \alpha_i) s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}) = \sum_{t=r+1}^{k-1} s_{i_1} \cdots s_{i_{t-1}}(s_{i_t}(\alpha_i) - \alpha_i)$$
$$= s_{i_1} \cdots s_{i_{k-1}}(\alpha_i) - s_{i_1} \cdots s_{i_r}(\alpha_i) = s_{i_1} \cdots s_{i_{k-1}}(\alpha_i) + \alpha_{\mathbf{i}}^{(r)}. \tag{5.5}$$

The first assertion of the Lemma is now immediate. To prove the second, suppose that $\sum_{t=r}^{r'} c_t \alpha_{\mathbf{i}}^{(t)} = 0$. Using (5.4) we may assume that $c_{r'} = 0$ and let $r < k < r'$ be maximal such that $c_k \ne 0$. Then α_{i_k} occurs with coefficient 1 in $\alpha_{\mathbf{i}}^{(k)}$ and does not occur in $\alpha_{\mathbf{i}}^{(t)}$ with $t < k$, whence $c_k = 0$ which contradicts with the choice of k. □

It follows from (4.4) and Lemma 5.4 that

$$q^{-\frac{1}{2}(\alpha_{\mathbf{i}}^{(r')}, \alpha_{\mathbf{i}}^{(r)})} X_{\mathbf{i},r'} X_{\mathbf{i},r} - q^{\frac{1}{2}(\alpha_{\mathbf{i}}^{(r)}, \alpha_{\mathbf{i}}^{(r')})} X_{\mathbf{i},r} X_{\mathbf{i},r'} = (q_i - q_i^{-1}) c X_{\mathbf{i}}^{\mathbf{n}(r,r')}, \tag{5.6}$$

for some $c \in \mathbb{A}_0$. We may assume, without loss of generality, that $r = 1$. Then $\ell_i(X_{\mathbf{i},k}) = m_{k,1}$, $2 \le k \le r' - 1$, hence by (5.4)

$$\ell_i(X_{\mathbf{i}}^{\mathbf{n}(r,r')}) = -\sum_{k=2}^{r'-1} m_{k,1} a_{i_k,i} = -\sum_{k=2}^{r'-1} (\alpha_i^\vee, \alpha_{\mathbf{i}}^{(k)}) a_{i_k,i} = (\alpha_i^\vee, \alpha_i + \alpha_{\mathbf{i}}^{(r')}) = m_{r',1} + 2.$$

Applying $\partial_i^{(m_{r',1}+2)}$ to both sides of (5.6) and taking into account that $\ell_i(X_{\mathbf{i},r'}) = m_{r',1} + 1$ we obtain

$$\partial_i^{(top)} X_{\mathbf{i},r'} = c \partial_i^{(top)} X_{\mathbf{i}}^{\mathbf{n}(r,r')}.$$

Since $X_{\mathbf{i},r'}$ and $X_{\mathbf{i}}^{\mathbf{n}(r,r')}$ are in $\mathbf{B}^{up}$ this implies that $c = 1$. □

Theorem 5.5. *Let $w \in W$ and suppose that $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$ has a single repetition $i_r = i_{r'} = i$. Then*

$$\mathbf{B}(w) = \{q^{\frac{1}{2}a\Lambda(\mathbf{a}, \mathbf{e}_r + \mathbf{e}_{r'})} X_{\mathbf{i}}^{\mathbf{a}} Y_{\mathbf{i}}^a \,:\, \mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}_{\ge 0}^m,\ \min(a_r, a_{r'}) = 0,\ a \in \mathbb{Z}_{\ge 0}\}$$

where

$$Y_{\mathbf{i}} = q^{\frac{1}{2}(\alpha_{\mathbf{i}}^{(r)}, \alpha_{\mathbf{i}}^{(r')})} X_{\mathbf{i},r} X_{\mathbf{i},r'} - q_i^{-1} X_{\mathbf{i}}^{\mathbf{n}(r,r')} = E_{i_1^{m_{r',1}} \cdots i_{r-1}^{m_{r',r-1}} i_r^{1+m_{r',r}} \cdots i_{r'-1}^{m_{r',r'-1}} i_1^{m_{r,1}} \cdots i_{r-1}^{m_{r,r-1}} i_r} \tag{5.7}$$

and $\Lambda = \Lambda_{\mathbf{i}}$ is defined as in Corollary 4.15.

Proof. By (5.4) $Y_{\mathbf{i}} \in U_q(\mathfrak{n}^+)_{\alpha_{\mathbf{i}}^{(r)} + \alpha_{\mathbf{i}}^{(r')}}$. It is immediate from (5.3) that $\overline{Y_{\mathbf{i}}} = Y_{\mathbf{i}}$, whence $Y_{\mathbf{i}} \in \mathbf{B}(w)$ by Theorems 1.1 and 1.2. It is easy to see that $(\partial_{i_1}^{op})^{(m_{r,1})} \cdots (\partial_{i_{r-1}}^{op})^{(m_{r,r-1})} \partial_{i_r}^{op}(Y_{\mathbf{i}}) = X_{\mathbf{i},r'}$ whence $Y_{\mathbf{i}} = E_{i_1^{m_{r',1}} \cdots i_{r-1}^{m_{r',r-1}} i_r^{1+m_{r',r}} \cdots i_{r'-1}^{m_{r',r'-1}} i_1^{m_{r,1}} \cdots i_{r-1}^{m_{r,r-1}} i_r}$.

Furthermore, we need the following

Lemma 5.6. $X_{\mathbf{i}}^{\mathbf{a}} Y_{\mathbf{i}} = q^{-\Lambda(\mathbf{a}, \mathbf{e}_r + \mathbf{e}_{r'})} Y_{\mathbf{i}} X_{\mathbf{i}}^{\mathbf{a}}$ *for all $\mathbf{a} \in \mathbb{Z}_{\ge 0}^m$.*

Proof. It suffices to prove the assertion for $\mathbf{a}=\mathbf{e}_k$, $1\le k\le m$. By Corollary 4.15 and Lemma 4.16 we have for $k\ne r,r'$

$$X_{\mathbf{i},k}X_{\mathbf{i}}^{\mathbf{a}}=q^{-\Lambda(\mathbf{e}_k,\mathbf{a})}X_{\mathbf{i}}^{\mathbf{a}}X_{\mathbf{i},k}=q^{(\alpha_{\mathbf{i}}^{(k)},|\mathbf{a}_{<k}|_{\mathbf{i}}-|\mathbf{a}_{>k}|_{\mathbf{i}})}X_{\mathbf{i}}^{\mathbf{a}}X_{\mathbf{i},k}. \tag{5.8}$$

This immediately yields the assertion for $k<r$ or $k>r'$. If $r<k<r'$ then

$$\begin{aligned}(\alpha_{\mathbf{i}}^{(k)},|\mathbf{n}(r,r')_{<k}|_{\mathbf{i}}-|\mathbf{n}(r,r')_{>k}|_{\mathbf{i}})&=(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(r)}-\alpha_{\mathbf{i}}^{(r')}+s_{i_1}\cdots s_{i_{k-1}}(s_{i_k}(\alpha_i)+\alpha_i))\\&=(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(r)}-\alpha_{\mathbf{i}}^{(r')})+(\alpha_{i_k},s_{i_k}(\alpha_i)+\alpha_i)=(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(r)}-\alpha_{\mathbf{i}}^{(r')}),\end{aligned} \tag{5.9}$$

where we used (5.4) and (5.5). Thus, $X_{\mathbf{i},k}X_{\mathbf{i}}^{\mathbf{n}(r,r')}=q^{(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(r)}-\alpha_{\mathbf{i}}^{(r')})}X_{\mathbf{i}}^{\mathbf{n}(r,r')}X_{\mathbf{i},k}$. Since we also have

$$X_{\mathbf{i},k}X_{\mathbf{i}}^{\mathbf{e}_r+\mathbf{e}_{r'}}=q^{(\alpha_{\mathbf{i}}^{(k)},\alpha_{\mathbf{i}}^{(r)}-\alpha_{\mathbf{i}}^{(r')})}X_{\mathbf{i}}^{\mathbf{e}_r+\mathbf{e}_{r'}}X_{\mathbf{i},k},$$

we conclude that the assertion holds in this case. Furthermore,

$$\begin{aligned}X_{\mathbf{i},r'}Y_{\mathbf{i}}&=q^{\frac12(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')})}X_{\mathbf{i},r'}X_{\mathbf{i},r}X_{\mathbf{i},r'}-q_i^{-1}X_{\mathbf{i},r'}X_{\mathbf{i}}^{\mathbf{n}(r,r')}\\&=q^{\frac12(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')})}(q^{(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')})}X_{\mathbf{i},r}X_{\mathbf{i},r'}+q^{\frac12(\alpha_{\mathbf{i}}^{(r')},\alpha_{\mathbf{i}}^{(r)})}(q_i-q_i^{-1})X_{\mathbf{i}}^{\mathbf{n}(r,r')})X_{\mathbf{i},r'}-q_i^{-1}q^{(\alpha_{\mathbf{i}}^{(r')},\alpha_{\mathbf{i}}^{(r)}+\alpha_{\mathbf{i}}^{(r')})}X_{\mathbf{i}}^{\mathbf{n}(r,r')}X_{\mathbf{i},r'}\\&=q^{(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')})}Y_{\mathbf{i}}X_{\mathbf{i},r'}\end{aligned}$$

and similarly $Y_{\mathbf{i}}X_{\mathbf{i},r}=q^{(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')})}X_{\mathbf{i},r}Y_{\mathbf{i}}$. Since $(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')})=\Lambda(\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_{r'})=-\Lambda(\mathbf{e}_{r'},\mathbf{e}_r+\mathbf{e}_{r'})$ this completes the proof of Lemma 5.6. □

Proposition 5.7. *For all* $\mathbf{a}\in\mathbb{Z}_{\ge0}^m$ *we have*

$$X_{\mathbf{i}}^{\mathbf{a}}=\sum_{k+l=\min(a_r,a_{r'})}q_i^{-k(k+|a_r-a_{r'}|)}\begin{bmatrix}\min(a_r,a_{r'})\\k\end{bmatrix}_{q_i^{-2}}b(\mathbf{a}-\min(a_r,a_{r'})(\mathbf{e}_r+\mathbf{e}_{r'})+k\mathbf{n}(r,r'),l)$$

where for $\mathbf{n}\in\mathbb{Z}_{\ge0}^m$ *with* $\min(n_r,n_r')=0$ *we set*

$$b(\mathbf{n},l)=q^{\frac12 l\Lambda(\mathbf{n},\mathbf{e}_r+\mathbf{e}_{r'})}X_{\mathbf{i}}^{\mathbf{n}}Y_{\mathbf{i}}^{l}$$

and $\begin{bmatrix}m\\n\end{bmatrix}_v\in1+v\mathbb{Z}[v]$ *is the Gaussian binomial coefficient defined by*

$$\begin{bmatrix}m\\n\end{bmatrix}_v=\prod_{t=0}^{n-1}\frac{[m-t]_v}{[t+1]_v},\qquad [k]_v=\sum_{l=0}^{k-1}v^l.$$

Proof. We need the following

Lemma 5.8. $X_{\mathbf{i}}^{m(\mathbf{e}_r+\mathbf{e}_{r'})}=\sum_{k+l=m}q_i^{-k^2}\begin{bmatrix}m\\k\end{bmatrix}_{q_i^{-2}}X_{\mathbf{i}}^{k\mathbf{n}(r,r')}Y_{\mathbf{i}}^{l}$ *for all* $m\ge0$.

Proof. The argument is by induction on m. The case $m=0$ is obvious. For the inductive step, note that we have, by the definition of $Y_{\mathbf{i}}$

$$\begin{aligned}X_{\mathbf{i}}^{(m+1)(\mathbf{e}_r+\mathbf{e}_{r'})}=q^{\frac12(m+1)^2(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')})}X_{\mathbf{i},r}^{m}X_{\mathbf{i},r}X_{\mathbf{i},r'}X_{\mathbf{i},r'}^{m}&=q^{(\frac12m^2+m)(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')})}X_{\mathbf{i},r}^{m}(Y_{\mathbf{i}}+q_i^{-1}X_{\mathbf{i}}^{\mathbf{n}(r,r')})X_{\mathbf{i},r'}^{m}\\&=X_{\mathbf{i}}^{m(\mathbf{e}_r+\mathbf{e}_{r'})}Y_{\mathbf{i}}+q_i^{-1-2m}X_{\mathbf{i}}^{\mathbf{n}(r,r')}X_{\mathbf{i}}^{m(\mathbf{e}_r+\mathbf{e}_{r'})}.\end{aligned}$$

By Corollary 4.15 we have $X_{\mathbf{i}}^{\mathbf{a}}X_{\mathbf{i}}^{k\mathbf{a}} = X_{\mathbf{i}}^{(k+1)\mathbf{a}}$ if $\mathbf{a} \in \sum_{t=r+1}^{r'-1}\mathbb{Z}_{\ge 0}\mathbf{e}_t$, whence by the induction hypothesis

$$\begin{aligned} X_{\mathbf{i}}^{(m+1)(\mathbf{e}_r+\mathbf{e}_{r'})} &= \sum_{k+l=m} q_i^{-k^2}\begin{bmatrix} m\\ k\end{bmatrix}_{q_i^{-2}} X_{\mathbf{i}}^{k\mathbf{n}(r,r')}Y_{\mathbf{i}}^{l+1} + \sum_{k+l=m} q_i^{-k^2-1-2m}\begin{bmatrix} m\\ k\end{bmatrix}_{q_i^{-2}} X_{\mathbf{i}}^{(k+1)\mathbf{n}(r,r')}Y_{\mathbf{i}}^{l}\\ &= \sum_{k+l=m+1} q_i^{-k^2}\Big(\begin{bmatrix} m\\ k\end{bmatrix}_{q_i^{-2}} + q_i^{-2(m+1-k)}\begin{bmatrix} m\\ k-1\end{bmatrix}_{q_i^{-2}}\Big)X_{\mathbf{i}}^{k\mathbf{n}(r,r')}Y_{\mathbf{i}}^{l}\\ &= \sum_{k+l=m+1} q_i^{-k^2}\begin{bmatrix} m+1\\ k\end{bmatrix}_{q_i^{-2}} X_{\mathbf{i}}^{k\mathbf{n}(r,r')}Y_{\mathbf{i}}^{l}. \end{aligned}$$ □

Using Corollary 4.15 we can write

$$X_{\mathbf{i}}^{\mathbf{a}} = q^{\frac12\Lambda(\mathbf{a},a_r\mathbf{e}_r+a_{r'}\mathbf{e}_{r'})}X_{\mathbf{i}}^{\mathbf{a}-a_r\mathbf{e}_r-a_{r'}\mathbf{e}_{r'}}X_{\mathbf{i}}^{a_r\mathbf{e}_r+a_{r'}\mathbf{e}_{r'}} = q^{-\frac12\Lambda(\mathbf{a},a_r\mathbf{e}_r+a_{r'}\mathbf{e}_{r'})}X_{\mathbf{i}}^{a_r\mathbf{e}_r+a_{r'}\mathbf{e}_{r'}}X_{\mathbf{i}}^{\mathbf{a}-a_r\mathbf{e}_r-a_{r'}\mathbf{e}_{r'}}.$$

If $a_r \ge a_{r'}$ then

$$\begin{aligned} X_{\mathbf{i}}^{\mathbf{a}} &= q^{\frac12(\Lambda(\mathbf{a},a_r\mathbf{e}_r+a_{r'}\mathbf{e}_{r'})+\Lambda((a_r-a_{r'})\mathbf{e}_r,a_{r'}\mathbf{e}_{r'}))}X_{\mathbf{i}}^{\mathbf{a}-a_r\mathbf{e}_r-a_{r'}\mathbf{e}_{r'}}X_{\mathbf{i}}^{(a_r-a_{r'})\mathbf{e}_r}X_{\mathbf{i}}^{a_{r'}(\mathbf{e}_r+\mathbf{e}_{r'})}\\ &= q^{\frac12 a_{r'}\Lambda(\mathbf{a},\mathbf{e}_r+\mathbf{e}_{r'})}X_{\mathbf{i}}^{\mathbf{a}-a_{r'}(\mathbf{e}_r+\mathbf{e}_{r'})}X_{\mathbf{i}}^{a_{r'}(\mathbf{e}_r+\mathbf{e}_{r'})}. \end{aligned}$$

Note that $\Lambda(\mathbf{e}_r+\mathbf{e}_{r'},\mathbf{n}(r,r')) = 0$. Then for $0 \le k \le a'_r$

$$\begin{aligned} &q^{\frac12 a_{r'}\Lambda(\mathbf{a},\mathbf{e}_r+\mathbf{e}_{r'})}X_{\mathbf{i}}^{\mathbf{a}-a_{r'}(\mathbf{e}_r+\mathbf{e}_{r'})}X_{\mathbf{i}}^{k\mathbf{n}(r,r')}Y_{\mathbf{i}}^{a_{r'}-k}\\ &\qquad = q^{\frac12\Lambda(\mathbf{a},a_{r'}(\mathbf{e}_r+\mathbf{e}_{r'})-k\mathbf{n}(r,r'))}X_{\mathbf{i}}^{\mathbf{a}-a_{r'}(\mathbf{e}_r+\mathbf{e}_{r'})+k\mathbf{n}(r,r')}Y_{\mathbf{i}}^{a_{r'}-k}\\ &\qquad = q^{\frac12 k\Lambda(\mathbf{a},\mathbf{e}_r+\mathbf{e}_{r'}-\mathbf{n}(r,r'))}b(\mathbf{a}-a_{r'}(\mathbf{e}_r+\mathbf{e}_{r'})+k\mathbf{n}(r,r'),a_{r'}-k). \end{aligned}$$

If $t < r$ or $t > r'$ then

$$\Lambda(\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_{r'}-\mathbf{n}(r,r')) = \pm(\alpha_i^{(t)},\alpha_{\mathbf{i}}^{(r)}+\alpha_{\mathbf{i}}^{(r')}-|\mathbf{n}(r,r')|_{\mathbf{i}}) = 0$$

by (5.4). For $r < t < r'$ it follows from (5.9) that

$$\Lambda(\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_{r'}-\mathbf{n}(r,r')) = (\alpha_{\mathbf{i}}^{(t)},\alpha_{\mathbf{i}}^{(r')}-|\mathbf{n}(r,r')_{>t}|_{\mathbf{i}}-\alpha_{\mathbf{i}}^{(r)}+|\mathbf{n}(r,r')_{<t}|_{\mathbf{i}}) = 0.$$

Since by Lemma 4.16

$$\Lambda(\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_{r'}-\mathbf{n}(r,r')) = (\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r')}-\mathbf{n}(r,r')) = -(\alpha_{\mathbf{i}}^{(r)},\alpha_{\mathbf{i}}^{(r)}) = -(\alpha_i,\alpha_i)$$

while

$$\Lambda(\mathbf{e}_{r'},\mathbf{e}_r+\mathbf{e}_{r'}-\mathbf{n}(r,r')) = (\alpha_{\mathbf{i}}^{(r')},-\alpha_{\mathbf{i}}^{(r)}+\mathbf{n}(r,r')) = (\alpha_{\mathbf{i}}^{(r')},\alpha_{\mathbf{i}}^{(r')}) = (\alpha_i,\alpha_i)$$

we conclude that $\Lambda(\mathbf{a},\mathbf{e}_r+\mathbf{e}_{r'}-\mathbf{n}(r,r')) = (\alpha_i,\alpha_i)(a_{r'}-a_r)$. Thus, by Lemma 5.8 we have

$$X_{\mathbf{i}}^{\mathbf{a}} = \sum_{k+l=a_{r'}} q_i^{-k(k+a_r-a_{r'})}\begin{bmatrix} a_{r'}\\ k\end{bmatrix}_{q_i^{-2}} b(\mathbf{a}-a_{r'}(\mathbf{e}_r+\mathbf{e}_{r'})+k\mathbf{n}(r,r'),l).$$

For $a_r \le a_{r'}$ we obtain in a similar way

$$X_{\mathbf{i}}^{\mathbf{a}} = q^{-\frac12\Lambda(\mathbf{a},a_r\mathbf{e}_r+a_{r'}\mathbf{e}_{r'})}X_{\mathbf{i}}^{a_r\mathbf{e}_r+a_{r'}\mathbf{e}_{r'}}X_{\mathbf{i}}^{\mathbf{a}-a_r\mathbf{e}_r-a_{r'}\mathbf{e}_{r'}} = q^{-\frac12 a_r\Lambda(\mathbf{a},\mathbf{e}_r+\mathbf{e}_{r'})}X_{\mathbf{i}}^{a_r(\mathbf{e}_r+\mathbf{e}_{r'})}X_{\mathbf{i}}^{\mathbf{a}-a_r(\mathbf{e}_r+\mathbf{e}_{r'})}.$$

Since for $0 \le k \le a_r$

$$\begin{aligned} &q^{-\frac12 a_r\Lambda(\mathbf{a},\mathbf{e}_r+\mathbf{e}_{r'})}X_{\mathbf{i}}^{k\mathbf{n}(r,r')}Y_{\mathbf{i}}^{a_r-k}X_{\mathbf{i}}^{\mathbf{a}-a_r(\mathbf{e}_r+\mathbf{e}_{r'})}\\ &\qquad = q^{-\frac12 k\Lambda(\mathbf{a},\mathbf{e}_r+\mathbf{e}_{r'}-\mathbf{n}(r,r'))}b(\mathbf{a}-a_r(\mathbf{e}_r+\mathbf{e}_{r'})+k\mathbf{n}(r,r'),a_r-k), \end{aligned}$$

it follows that

$$X_{\mathbf{i}}^{\mathbf{a}} = \sum_{k+l=a_r} q_i^{-k(k+a_{r'}-a_r)} \begin{bmatrix} a_r \\ k \end{bmatrix}_{q_i^{-2}} b(\mathbf{a} - a_r(\mathbf{e}_r + \mathbf{e}_{r'}) + k\mathbf{n}(r,r'), l).$$

Proposition 5.7 is proved. □

By Lemma 5.6, $\overline{\mathbf{b}(\mathbf{n},l)} = \mathbf{b}(\mathbf{n},l)$ provided that $\min(n_r, n_{r'}) = 0$. Then Proposition 5.7 and Theorems 1.1, 1.2 imply that $b(\mathbf{a} - \min(a_r, a'_r)(\mathbf{e}_r + \mathbf{e}_{r'}), \min(a_r, a'_r)) = b_{\mathbf{i},\mathbf{a}} \in \mathbf{B}(w)$. Clearly this gives the $b_{\mathbf{i},\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}_{\geq 0}^m$, which completes the proof of Theorem 5.5. □

5.3. **Type** A_3**.** Let $w_\circ$ be the longest element in W. We have $E_{ij} = T_i(E_j)$, $\{i,j\} = \{1,2\}$ or $\{2,3\}$, $E_{123} = T_1T_2(E_3) = T_3^{-1}T_2^{-1}(E_1)$, $E_{321} = T_3T_2(E_1) = T_1^{-1}T_2^{-1}(E_3)$, $E_{132} = T_1T_3(E_2)$, $E_{213} = E_{132}^* = T_1^{-1}T_3^{-1}(E_2) = T_2T_1T_3(E_2)$ and $E_{2132} = Y_{(2,1,3,2)} = E_2E_{213} - q^{-1}E_{21}E_{23}$ as defined in Theorem 5.5. The following was essentially proved in [4], although with a slightly different definition of $\bar{\cdot}$ and hence with different powers of q (see also Theorems 1.4.1 and 3.1.3 in a recent work [18]).

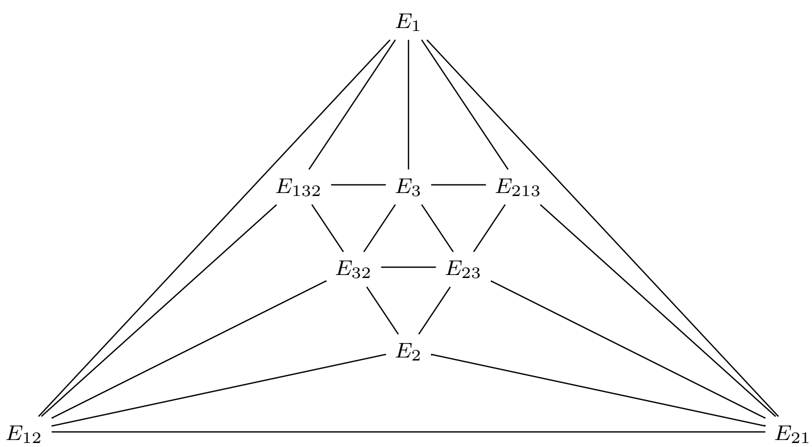
Theorem 5.9. $\mathbf{B}^{up} = \mathbf{B}(w_\circ)$ *consists of monomials*

$$q^{\frac{1}{2}f(\mathbf{a})} E_1^{m_1} E_2^{m_2} E_3^{m_3} E_{12}^{m_{12}} E_{21}^{m_{21}} E_{23}^{m_{23}} E_{32}^{m_{32}} E_{213}^{m_{213}} E_{132}^{m_{132}} E_{123}^{m_{123}} E_{321}^{m_{321}} E_{2132}^{m_{2132}}$$

where

$$\begin{aligned} f(\mathbf{a}) &= (m_1 - m_2)(m_{12} - m_{21}) + (m_3 - m_2)(m_{32} - m_{23}) + (m_1 + m_3)(m_{132} - m_{213}) \\ &+ (m_1 + m_{12} + m_{21} - m_3 - m_{23} - m_{32})(m_{123} - m_{321}) - (m_{12} + m_{32})m_{132} + (m_{21} + m_{23})m_{213}, \end{aligned}$$

and $\min(m_\alpha, m_\beta) = 0$ *if* E_α, $E_\beta \notin \{E_{123}, E_{321}, E_{2132}\}$ *and are not connected by an edge in the following graph (see* [4, §9.4, Fig 2]*)*



We have the following table for the action of the T_i^{-1}, $1 \leq i \leq 3$ on the E_α

	E_1	E_2	E_3	E_{12}	E_{21}	E_{23}	E_{32}	E_{132}	E_{213}	E_{123}	E_{321}	E_{2132}
T_1^{-1}		E_{21}	E_3	E_2		E_{213}	E_{321}	E_{32}		E_{23}		E_{2132}
T_2^{-1}	E_{12}		E_{32}		E_1	E_3			E_{132}	E_{123}	E_{321}	
T_3^{-1}	E_1	E_{23}		E_{123}	E_{213}		E_2	E_{12}			E_{21}	E_{2132}

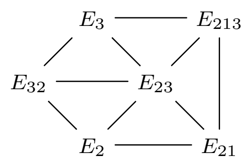
where the entry is empty if $T_i^{-1}(E_\alpha) \notin U_q(\mathfrak{n}^+)$. Using Theorem 1.5 we conclude that $\mathbf{B}(s_1w_\circ)$ (respectively, $\mathbf{B}(s_2w_\circ)$) consists of monomials of the form

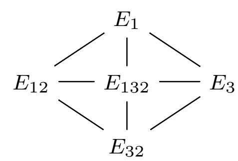
$$q^{\frac{1}{2}(m_2m_{21} + (m_3-m_2)(m_{32}-m_{23}) - (m_{21}-m_3-m_{23}-m_{32})m_{321} - m_3m_{213} + (m_{21}+m_{23})m_{213})} \times$$
$$E_2^{m_2} E_3^{m_3} E_{21}^{m_{21}} E_{23}^{m_{23}} E_{32}^{m_{32}} E_{213}^{m_{213}} E_{321}^{m_{321}} E_{2132}^{m_{2132}}$$

and, respectively

$$q^{\frac{1}{2}(m_1m_{12}+m_3m_{32}+(m_1+m_{12}-m_3-m_{32})(m_{123}-m_{321})+(m_1+m_3-m_{12}+m_{32})m_{132})}\times E_1^{m_1}E_3^{m_3}E_{12}^{m_{12}}E_{32}^{m_{32}}E_{132}^{m_{132}}E_{123}^{m_{123}}E_{321}^{m_{321}}$$

where $\min(m_\alpha, m_\beta) = 0$ if $E_\alpha, E_\beta \notin \{E_{123}, E_{321}, E_{2132}\}$ are not connected by an edge in the following respective graphs





The basis $\mathbf{B}(s_3w_\circ)$ is easy to obtain from $\mathbf{B}(s_1w_\circ)$ using the diagram automorphism which interchanges E_1 and E_3, E_{12} and E_{32}, E_{21} and E_{23} and E_{123}, E_{321} and fixes all other elements E_α.

Thus, $U_q(s_1w_\circ)$ is generated by E_2, E_3, E_{21} subject to the relations

$$[E_i,[E_i,E_j]_q]_{q^{-1}} = 0, \quad [E_2,E_{21}]_{q^{-1}} = 0, \quad [E_3,[E_3,E_{21}]_q]_{q^{-1}} = 0 = [E_{21},[E_{21},E_3]_q]_{q^{-1}},$$

where $[x,y]_t = xy - tyx$, $x,y \in U_q(\mathfrak{n}^+)$, $t \in \Bbbk^\times$ and $\{i,j\} = \{2,3\}$, while $U_q(s_2w_\circ)$ is generated by E_1, E_3, E_{12}, E_{32} subject to the relations

$$[E_1,E_3] = 0,\ [E_i,E_{i2}]_{q^{-1}} = 0,\ [E_{12},E_{32}] = 0, \quad [E_i,[E_i,E_{j2}]_q]_{q^{-1}} = 0,\ [E_{i2},[E_{i2},E_j]_q]_{q^{-1}} = 0,$$

where $\{i,j\} = \{1,3\}$.

Since all elements $w \in W$ with $\ell(w) \le 4$ are either repetition free or with a single repetition, all remaining Schubert cells have already been described in §5.1 and §5.2. For example,

$$\mathbf{B}(s_2s_1s_3s_2) = \{q^{\frac{1}{2}(m_2+m_{213})(m_{21}+m_{23})}E_2^{m_2}E_{21}^{m_{21}}E_{23}^{m_{23}}E_{213}^{m_{213}}E_{2132}^{m_{2132}} \,:\, \min(m_2,m_{213}) = 0\}$$

and $U_q(s_2s_1s_3s_2)$ is generated by E_2, E_{21}, E_{23}, E_{213} subject to the relations

$$[E_2,E_{2i}]_{q^{-1}} = 0, \quad [E_{21},E_{23}] = 0, \quad [E_{2i},E_{213}]_{q^{-1}} = 0, \quad [E_2,E_{213}] = (q^{-1}-q)E_{21}E_{23},\ i \in \{1,3\}$$

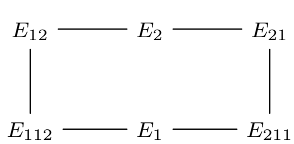
and coincides with the algebra of quantum 2×2-matrices.

5.4. **Type C_2.** We have $E_{12} = T_2^{-1}(E_1)$, $E_{1^22} = T_1(E_2)$, $E_{21} = T_2(E_1)$, $E_{21^2} = T_1^{-1}(E_2)$, $E_{121} = Y_{(1,2,1)}$ and $E_{21^22} = Y_{(2,1,2)}$ as defined in Theorem 5.5. The following is apparently well-known (and can be deduced for instance from [18, Theorems 1.4.1 and 3.1.3]).

Theorem 5.10. $\mathbf{B}^{up}$ *consists of all monomials*

$$q^{m_1(m_{1^22}-m_{21^2})+m_2(m_{21}-m_{12})-m_{12}m_{1^22}+m_{21}m_{21^2}}E_1^{m_1}E_2^{m_2}E_{12}^{m_{12}}E_{21}^{m_{21}}E_{1^22}^{m_{1^22}}E_{21^2}^{m_{21^2}}E_{121}^{m_{121}}E_{21^22}^{m_{21^22}}$$

where $\min(m_\alpha, m_\beta) = 0$ *if* $E_\alpha, E_\beta \notin \{E_{121}, E_{21^22}\}$ *are not connected by an edge in the following graph*



All other Schubert cells have already been described in §5.1 and §5.2.

5.5. **Bi-Schubert algebras.** Let $\mathfrak{g} = \mathfrak{sl}_4$. Using the computations from §5.3 we obtain

$$\mathbf{B}(s_1w_\circ, s_1w_\circ) = \{q^{\frac{1}{2}(m_3-m_2)(m_{32}-m_{23})} E_2^{m_2} E_3^{m_3} E_{23}^{m_{23}} E_{32}^{m_{32}} E_{2132}^{m_{2132}} \,:\, \min(m_2, m_3) = 0\}$$

and $U_q(s_1w_\circ, s_1w_\circ) \cong U_q(\mathfrak{sl}_3^+) \otimes \Bbbk[E_{2132}]$,

$$\mathbf{B}(s_1w_\circ, s_2w_\circ) = \{q^{\frac{1}{2}(-m_3(m_{23}+m_{213})-(m_{21}-m_3-m_{23})m_{321}+(m_{21}+m_{23})m_{213})} E_3^{m_3} E_{21}^{m_{21}} E_{23}^{m_{23}} E_{213}^{m_{213}} E_{321}^{m_{321}} \,:\, \min(m_3, m_{21}) = 0\}$$

and $U_q(s_1w_\circ, s_2w_\circ)$ is generated by E_3, E_{21} and E_{23} subject to the relations

$$[E_3, E_{23}]_q = 0, \quad [E_3, [E_3, E_{21}]_q]_{q^{-1}} = 0 = [E_{21}, [E_{21}, E_3]_q]_{q^{-1}},$$

$$\mathbf{B}(s_1w_\circ, s_3w_\circ) = \{q^{\frac{1}{2}(m_2-m_{321})(m_{21}-m_{32})} E_2^{m_2} E_{21}^{m_{21}} E_{32}^{m_{32}} E_{321}^{m_{321}} E_{2132}^{m_{2132}} \,:\, \min(m_{21}, m_{32}) = 0\}$$

and $U_q(s_1w_\circ, s_3w_\circ)$ is generated by E_2, E_{21}, E_{32}, E_{321} subject to the relations

$$[E_2, E_{21}]_{q^{-1}} = [E_2, E_{32}]_q = [E_{21}, [E_{21}, E_{32}]]_{q^2} = [E_{32}, [E_{32}, E_{21}]]_{q^{-2}} = 0$$

and

$$[E_2, E_{321}] = [E_{21}, E_{321}]_q = [E_{32}, E_{321}]_{q^{-1}} = 0,$$

$$\mathbf{B}(s_2w_\circ, s_2w_\circ) = \{q^{\frac{1}{2}(m_1-m_3)(m_{123}-m_{321})} E_1^{m_1} E_3^{m_3} E_{123}^{m_{123}} E_{321}^{m_{321}}\}$$

and $U_q(s_2w_\circ, s_2w_\circ)$ is a quantum plane. In particular, all these algebras are PBW.

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