Tête-à-tête twists and geometric monodromy.

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Introduction. Let (Σ, Γ) be a pair consisting of a a compact connected oriented surface Σ with non empty boundary $\partial\Sigma$ and a finite graph Γ that is embedded in the interior of Σ. We assume that the surface Σ is a regular neihborhood of the graph Γ and that the embedded graph has the tête-à-tête property, which property we will define later in this paper. Moreover, we will construct for each pair (Σ, Γ) with the tête-à-tête property a mapping classe T_Γ on $(\Sigma, \partial\Sigma)$. We call the mapping classes resulting from this construction tête-à-tête twists.

A surface of genus g and with r boundary components carries up to congruence by homeomorphism of the surface only finite many graphs with the tête-à-tête property and hence for fixed (g, r) there are only finite many mapping classes, which are tête-à-tête twists.

The main theorem of this paper asserts:

Theorem. *The geometric monodromy diffeomorphism of a plane curve singularity is a tête-à-tête twist.*

As a corollary, we obtain a very strong topological restriction for mapping classes, that are geometric monodromies of plane curve singularities.

Section 1. Tête-à-tête twist.

Let Γ be a finite connected metric graph with $e(\Gamma)$ edges and no vertices of valency 1. We assume, that the edges are parametrized by continuous bijective maps $E_e : [0, L_e] \to \Gamma, L_e > 0, e = 1, \cdots, e(\Gamma)$, such that the distance from $E_e(t)$ to $E_e(s)$ is $|t - s|, t, s \in [0, L_e]$.

Let Σ be a smooth, connected and oriented surface with non empty boundary $\partial\Sigma$. We say, that a map π of Γ into Σ is regular if π is continuous, injective, $\pi(\Gamma) \cap \partial\Sigma = \emptyset$, the compositions $\pi \circ E_e, e = 1, \cdots, e(\Gamma)$, are smooth regular embeddings of intervals and moreover, at each vertex v of Γ all outgoing speed vectors of $\pi \circ E_e, v = E_e(0)$ or $v = E_e(L_e)$ are distinct.

We denote by abuse of language by the pair (Σ, Γ) the pair $(\Sigma, \pi(\Gamma))$.

A safe walk along Γ is a continuous injective path $\gamma : [0, 2] \to \Sigma$ with following properties:

– $\gamma(t) \in \Gamma, t \in [0,2]$,

– the speed, measured with the parametrization E_e at $t \in [0,2]$ equals ± 1 if $\gamma(t)$ is in the interior of edge e,

– if the path γ runs at $t \in (0,2)$ into the vertex v, the path γ makes the a sharpest possible right turn, i.e. the oriented angle at $v = \gamma(t) \in \Sigma$ in between the speed vectors $-\dot{\gamma}(t_-)$ and $\dot{\gamma}(t_+)$ is smallest possible.

It follows, that a save walk γ is determined by its starting point $\gamma(0)$ and its starting speed vector $\dot{\gamma}(0)$. Futhermore, if the metric graph $\Gamma \subset \Sigma$ is without cycles of length less are equal 2, from each interior point of an edge start two distinct save walks.

Definition: *Let (Σ, Γ) be the pair of a surface and regular embedded metric graph. We say that the tête-à-tête tête-à-tête property holds for the the pair if*

– *the graph Γ has no cycles of length ≤ 2,*

– *the graph Γ is a regular retract of the surface Σ,*

– *for each point $p \in \Gamma$, p not being a vertex, the two distinct safe walks $\gamma_p^+, \gamma_p^- : [0,2] \to \Sigma$ with $p = \gamma_p^+(0) = \gamma_p^-(0)$ satisfy to $\gamma_p^+(2) = \gamma_p^-(2)$.*

It follows that the underlying metric graph of a pair (Σ, Γ) with tête-à-tête property is the union of its cycles of length 4.

We give basic examples of pairs (Σ, Γ) with tête-à-tête property:

— the surface is the cylinder $[-1,1] \times S^1$ and the graph Γ is the cycle $\{0\} \times S^1$ subdivided by 4 vertices in edges of equal length. Here we think S^1 as a circle of length 4.

— the surface $\Sigma_{1,1}$ is of genus 1 with 1 boundary component and the metric graph $\Gamma \subset \Sigma$ is the biparted complet graph $K_{3,2}$.

— for $p, q \in \mathbf{N}, p > 0, q > 0$, the biparted complet graph $K_{p,q}$ is the spine of a surface $S_{g,r}, g = 1/2(p-1)(q-1), r = (p,q)$, such that the tête-à-tête property holds. For instance, let P and Q be two parallel lines in the plane and draw p points on P, q points on Q. We add pq edges and get a planar projection of the graph $K_{p,q}$. The surface $S_{g,r}$ is a regular thickening of that projection.

Let (Σ, Γ) a pair of a surface and graph with tête-à-tête property. Our purpose is to construct for this pair a well defined element T_Γ in the relative mapping class group of the surface Σ. For each edge e of Γ we embed relatively a copy $(I_e, \partial I_e)$ of the interval $[-1,1]$ into $(\Sigma, \partial\Sigma)$ such that alle copies are pairwise disjoint and such that each copy I_e intersects in its midpoint $0 \in I_e$ the graph Γ transversally in one point which is the midpoint of the edge e. We call I_e the dual arc of the edge e. Let Γ_e be the union of $\Gamma \cup I_e$. We consider Γ_e also as a metric graph. The graph Γ_e has 2 terminal vertices a, b.

Let $w_a, w_b : [-1,2] \to \Gamma_e$ be the only save walks along Γ_e with $w_a(-1) = a, w_b(-1) = b$. We displace by a small isotopy the walks w_a, w_b to smooth injektive path w'_a, w'_b, that keeps the points $w_a(-1), w_b(-1)$ and $w_a(2), w_b(2)$ fixed, such that $w'_a(t) \notin \Gamma_e$ for $t \in (-1,2)$. The walks w_a, w_b meet each other in the midpoint of the edge e. Hence by the tête-à-tête property we have $w_a(2) = w_b(2)$. Let w_e the juxtaposition of the pathes w'_a and $-w'_b$. We may assume that the path w_e is smooth and intersects Γ transversally. Let I'_e the image of the path w_e. We now claim that there exits up to isotopy a unic relative diffeomorphism ϕ_Γ of Σ with $\phi_\Gamma(I_e) = I'_e$. We define the tête-à-tête twist T_Γ as the class of ϕ_Γ.

For our first basic example we obtain back the classical right Dehn twist. The second example has as tête-à-tête twist the geometric monodromy of the plane curve singularity $x^3 - y^2$. The twist of the example $(S_{g,r}, K_{p,q})$ computes the geometric monodromy of for the singularity $x^p + y^q$.

Section 2. Relative tête-à-tête retracts.

We prepare material, that will allow us to glue the previous examples. Let S be a connected compact surface with boundary ∂S. The boundary $\partial S = A \cup B$ is decomposed as a partition of boundary components of the surface S. We assume $A \neq \emptyset, B \neq \emptyset$.

Definition. *A relative tête-à-tête graph (S, A, Γ) in (S, A) is an embedded metric graph Γ in S with $A \subset \Gamma$. Moreover, the following properties hold:*

– the graph Γ has no cycles of length ≤ 2,

– the graph Γ is a regular retract of the surface Σ,

– for each point $p \in \Gamma \setminus A$, p not being a vertex, the two distint safe walks $\gamma_p^+, \gamma_p^- : [0,2] \to \Sigma$ with $p = \gamma_p^+(0) = \gamma_p^-(0)$ satisfy to $\gamma_p^+(2) = \gamma_p^-(2)$.

— for each point $p \in A$, p not being a vertex, the only save walk γ_p^+ satisfies $\gamma_p^+(2) \in A$.

We call the subset A the boundary of the relative tête-à-tête graph (S, A, Γ). This boundary carries a self map $p \in A \mapsto \gamma_p^+(2) \in A$, which we call the boundary walk w.

We now give a family of examples of relative tête-à-tête graphs.

— Consider the previous example $(S_{g,r}, K_{p,q}), g = 1/2(p-1)(q-1), r = (p,q)$. We blow up in the real oriented sense the p vertices of valency q, so we replace such a vertex $v_i, 1 \leq i \leq p$ by a circle A_i and attach the edges of $K_{p,q}$ that are incident with v_i to the circle in the cyclic order given by the embedding of $K_{p,q}$ in $S_{g,r}$. We get a surface $S_{g,r+p}$ and its boundary is partitioned in $A := \cup A_i$ and $B = \partial S_{g,r}$. The new graph is the union of A with the strict transform of $K_{p,q}$. So the new graph is in fact the total transform $K'_{p,q}$. We think this graph as a metric graph. The metric will be such that all edges have a positive length and that the tête-à-tête property remains for all points of $K'_{p,q} \setminus A$. We achieve this by giving the edges of A the length $2\epsilon, \epsilon > 0, \epsilon$ small and by giving the edges of $K'_{p,q} \setminus A$ the length $1 - \epsilon$. The boundary walk is an interval exchange map from $w : A \to A$. We denote by the pair $(S_{g,r+p}, K'_{p,q})$ this relative tête-à-tête graph together with its boundary walk.

Section 3. Gluing and closing of relative tête-à-tête graphs.

First we discribe the procedure of closing. We do it by an example. Consider $(S_{6,1+2}, K'_{2,13})$. We have two A boundary components A_1 and A_2. In oder to close the A components, we choose a piece-wise linear orientation reversing selfmap $s_1 : A_1 \to A_1$ of order 2. The boundary component A_1 will be closed if we identify the pieces using the map s_1. In order to get the tête-à-tête property we do the same with the component A_2, but we take care such that the involution $s_2 : A_2 \to A_2$ is equivariant via the boundary walk w to the involution s_1. Hence we take $p \in A_2 \mapsto s_2(p) := w \circ s_1 \circ w^{-1}(p) \in A_2$. More concretely, we can choose for $s_1 : A_1 \to A_1$ an involution that exchange in an orientation reversing way the opposite edges of an hexagon. If we do so, we get a surface $S_{8,1}$ with tête-à-tête graph. The corresponding twist is the geometric monodromy of the singularity $(x^3 - y^2)^2 - x^5y$. If we make our choices generically, the resulting graph will have 51 vertices, 36 edges, 6 vertices of valency 2, 45 vertices of valency 3.

Now an example of gluing. We glue in an walk equivariant way to copies of $(S_{2,1}, K'_{2,5})$. We get a tête-à-tête graph on the surface $S_{5,2}$. The corresponding twist is the monodromy of the singularity $(x^3 - y^2)(x^2 - y^3)$.

This is work in progress. A futher constuction for isolated singularities $f : \mathbf{C}^{n+1} \to \mathbf{C}$ provides its Milnor fiber with a spine, that consists of lagrangian strata. Again the monodromy is concentrated at the spine. The monodromy diffeomorphism is a generalized tête-à-tête twist. The case of plane curves is already interesting for we are aiming progress in restricting the adjacency tables. Thanks for your interest.

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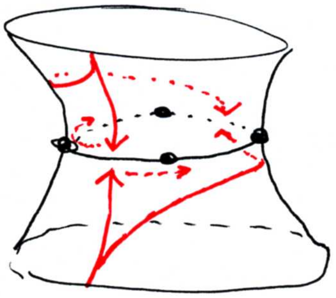
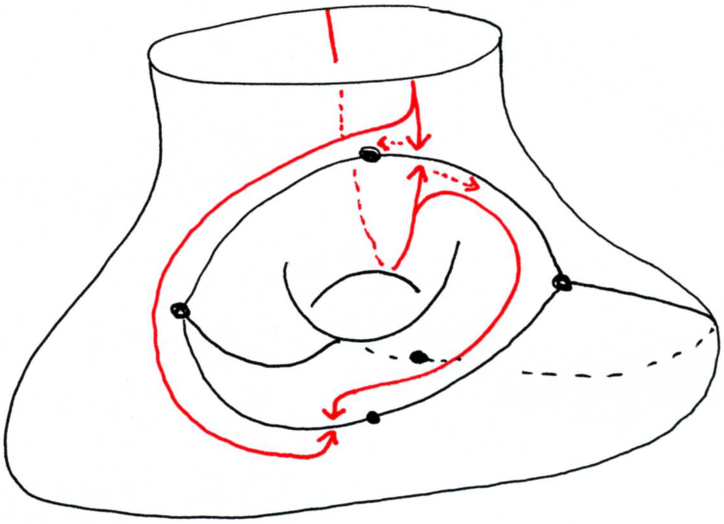
Example of pairs (S, Γ) with tête-à-tête property



$x^2 - y^2$

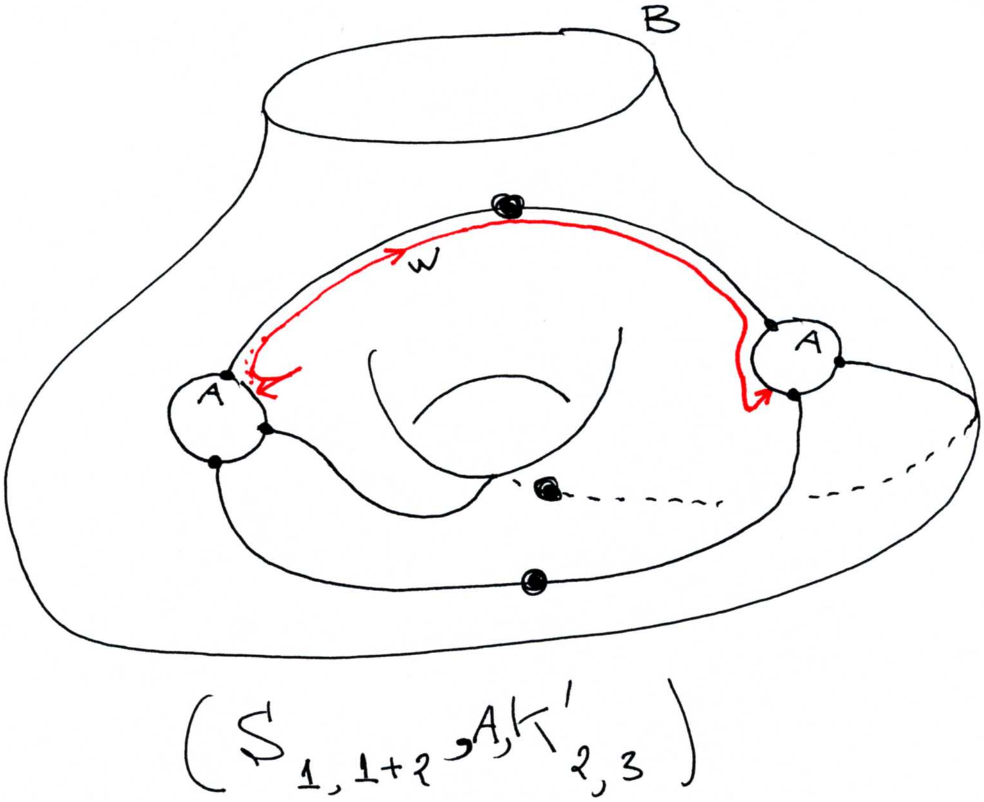
$(S_{0,2}, K_{2,2})$



$x^3 - y^2$

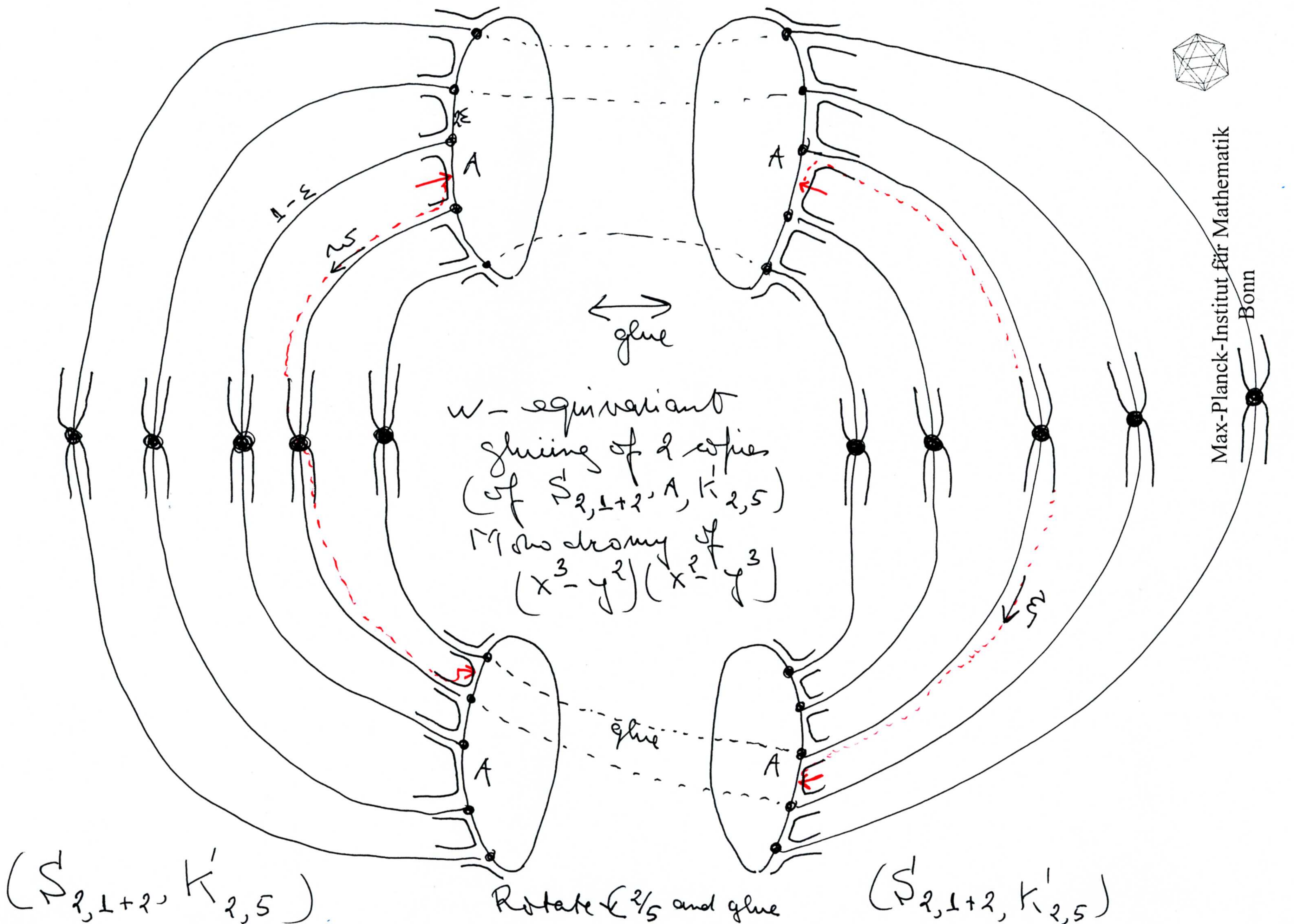
$(S_{1,1}, K_{2,3})$

Example of a relative pair (S, A, Γ)



$(S_{1,1+2}, A, K'_{2,3})$

(Length •——•) $= 2\varepsilon$, (Length •——●) $= 1 - \varepsilon$


glue
w-equivariant
gluing of 2 copies
(of $S'_{2,1+2}$, A, $K'_{2,5}$)
Monodromy of
$(x^3-y^2)(x^2-y^3)$
$1-\varepsilon$
ε
A
A
A
A
glue
$(S_{2,1+2}, K'_{2,5})$
Rotate $\measuredangle\, 2/5$ and glue
$(S'_{2,1+2}, K'_{2,5})$