GALOIS EXTENSIONS OF STRUCTURED RING SPECTRA

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August 31st 2005

Abstract. We introduce the notion of a Galois extension of commutative S-algebras (E_∞ ring spectra), often localized with respect to a fixed homology theory. There are numerous examples, including some involving Eilenberg–Mac Lane spectra of commutative rings, real and complex topological K-theory, Lubin–Tate spectra and cochain S-algebras. We establish the main theorem of Galois theory in this generality. Its proof involves the notions of separable and étale extensions of commutative S-algebras, and the Goerss–Hopkins–Miller theory for E_∞ mapping spaces. We show that the global sphere spectrum S is separably closed, using Minkowski's discriminant theorem, and we estimate the separable closure of its localization with respect to each of the Morava K-theories. We also define Hopf–Galois extensions of commutative S-algebras, and study the complex cobordism spectrum MU as a common integral model for all of the local Lubin–Tate Galois extensions.

Contents

1991 *Mathematics Subject Classification.* 13B05, 13B40, 55N15, 55N22, 55P43, 55P60.
Key words and phrases. Galois theory, commutative S-algebra.

1. Introduction

The present paper is motivated by (1) the "brave new rings" paradigm coined by Friedhelm Waldhausen, that structured ring spectra are an unavoidable generalization of discrete rings, with arithmetic properties captured by their algebraic K-theory, (2) the presumption that algebraic K-theory will satisfy an extended form of the étale- and Galois descent foreseen by Dan Quillen, and (3) the algebro-geometric perspective promulgated by Jack Morava, on how the height-stratified moduli space of formal group laws influences stable homotopy theory, by way of complex cobordism theory.

We here develop the arithmetic notion of a Galois extension of structured ring spectra, viewed geometrically as an algebraic form of a regular covering space, by always working intrinsically in a category of spectra, rather than at the naïve level of coefficient groups. The result is a framework that well accommodates much recent work in stable homotopy theory. We hope that this study will eventually lead to a conceptual understanding of objects like the algebraic K-theory of the sphere spectrum, which by Waldhausen's stable parametrized h-cobordism theorem bears on such seemingly unrelated geometric objects as the diffeomorphism groups of manifolds, in much the same way that we now understand the algebraic K-theory spectrum of the ring of integers.

Let E be any spectrum and G a finite group. We say that a map $A \to B$ of E-local commutative S-algebras is an *E-local G-Galois extension* if G acts on B through commutative A-algebra maps in such a way that the two canonical maps

$$i\colon A \to B^{hG}$$

and

$$h\colon B \wedge_A B \to \prod_G B$$

induce isomorphisms in E_*-homology (Definition 4.1.3). When $E = S$ this means that the maps i and h are weak equivalences, and we may talk of a *global* G-Galois extension. In more detail, the map i is the standard inclusion into the homotopy fixed points for the G-action on B and h is given in symbols by $h(b_1 \wedge b_2) = \{g \mapsto b_1 \cdot g(b_2)\}$. To make the definition homotopy invariant we also assume that A is a cofibrant commutative S-algebra and that B is a cofibrant commutative A-algebra.

There are many interesting examples of such "brave new" Galois extensions.

Examples 1.1.

(a) The Eilenberg–Mac Lane functor $R \mapsto HR$ takes each G-Galois extension $R \to T$ of commutative rings to a global G-Galois extension $HR \to HT$ of commutative S-algebras (Proposition 4.2.1).

(b) The complexification map $KO \to KU$ from real to complex topological K-theory is a global $\mathbb{Z}/2$-Galois extension (Proposition 5.3.1).

(c) For each rational prime p and natural number n the profinite extended Morava stabilizer group $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}$ acts on the even periodic Lubin–Tate spectrum E_n, with $\pi_0(E_n) = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$, so that $L_{K(n)}S \to E_n$ is a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension (see Notation 3.2.2 and Theorem 5.4.4(d)).

(d) For most regular covering spaces $Y \to X$ the map of cochain $H\mathbb{F}_p$-algebras $F(X_+, H\mathbb{F}_p) \to F(Y_+, H\mathbb{F}_p)$ is a Galois extension (Proposition 5.6.3(a)).

A map $A \to B$ of commutative S-algebras will be said to be *faithful* if for each A-module N with $N \wedge_A B \simeq *$ we have $N \simeq *$ (Definition 4.3.1). The map $A \to B$ is *separable* if the multiplication map $\mu\colon B \wedge_A B \to B$ admits a bimodule section up to homotopy (Definition 9.1.1). A commutative S-algebra B is *connected,* in the sense of algebraic geometry, if its space of idempotents $\mathcal{E}(B)$ is weakly equivalent to the two-point space $\{0, 1\}$ (Definition 10.2.1). There are analogous definitions in each E-local context.

In commutative ring theory each Galois extension is faithful, but it remains an open problem to decide whether each Galois extension of commutative S-algebras is faithful (Question 4.3.6). Rather conveniently, a commutative S-algebra B is connected if and only if the ring $\pi_0(B)$ is connected (Proposition 10.2.2).

Here is our version of the Main Theorem of Galois theory for commutative S-algebras. The first two parts (a) and (b) of the theorem are obtained by specializing Theorem 7.2.3 and Proposition 9.1.4 to the case of a finite, discrete Galois group G. The recovery in (c) of the Galois group is Theorem 11.1.1. The converse part (d) is the less general part of Theorem 11.2.2.

Theorem 1.2. *Let $A \to B$ be a faithful E-local G-Galois extension.*

(a) For each subgroup $K \subset G$ the map $C = B^{hK} \to B$ is a faithful E-local K-Galois extension, with $A \to C$ separable.

(b) For each normal subgroup $K \subset G$ the map $A \to C = B^{hK}$ is a faithful E-local G/K-Galois extension.

If furthermore B is connected, then:

(c) The Galois group G is weakly equivalent to the mapping space $\mathcal{C}_A(B, B)$ of commutative A-algebra self-maps of B.

(d) For each factorization $A \to C \to B$ of the G-Galois extension, with $A \to C$ separable and $C \to B$ faithful, there is a subgroup $K \subset G$ such that $C \simeq B^{hK}$ as an A-algebra over B.

In other words, for a faithful E-local G-Galois extension $A \to B$ with B connected there is a bijective contravariant Galois correspondence $K \leftrightarrow C = B^{hK}$ between the subgroups of G and the weak equivalence classes of separable A-algebras mapping faithfully to B. The inverse correspondence takes C to $K = \pi_0 \mathcal{C}_C(B, B)$.

The main theorem fully describes the intermediate extensions in a G-Galois extension $A \to B$, but what about the further extensions of B? We say that a connected E-local commutative S-algebra A is *separably closed* if there are no connected E-local G-Galois extensions $A \to B$ for non-trivial groups G (Definition 10.3.1). The following fundamental example is a consequence of Minkowski's discriminant theorem in number theory, and is proved as Theorem 10.3.3.

Theorem 1.3. *The global sphere spectrum S is separably closed.*

The absence of localization is crucial for this result. At the other extreme the Morava $K(n)$-local category is maximally localized, for each p and n. Here the Lubin–Tate spectrum E_n admits a $K(n)$-local pro-$n\hat{\mathbb{Z}}$-Galois extension $E_n \to E_n^{nr}$, with

$$\pi_0(E_n^{nr}) = \mathbb{W}(\bar{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]]$$

given by adjoining all roots of unity of order prime to p (§5.4.6). We expect that each further G-Galois extension $E_n^{nr} \to B$ of such a Landweber exact even periodic spectrum must again be Landweber exact and even periodic, and such that $\pi_0(E_n^{nr}) \to \pi_0(B)$ will be a G-Galois extension of commutative rings. But $\mathbb{W}(\bar{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]]$ is separably closed as a commutative ring, so such a $\pi_0(B)$ cannot be connected, and B the cannot be connected for non-trivial groups G. Therefore we expect:

Conjecture 1.4. *The extension E_n^{nr} of the Lubin–Tate spectrum E_n is $K(n)$-locally separably closed. In particular, the Galois group $\mathbb{G}_n^{nr} = \mathbb{S}_n \rtimes \hat{\mathbb{Z}}$ of $L_{K(n)}S \to E_n^{nr}$ is the $K(n)$-local absolute Galois group of the $K(n)$-local sphere spectrum $L_{K(n)}S$.*

Partial results supporting this conjecture have been obtained by Andy Baker and Birgit Richter [BR:r], for global Galois extensions that are furthermore assumed to be faithful and abelian.

The substantial supply of pro-Galois extensions in the $K(n)$-local category, like $L_{K(n)}S \to E_n$, is not available in the $E(n)$-local category (see §5.5.4). This draws extra attention to the non-smashing Bousfield localizations, and thus to the distinction between the whole category of modules over $L_{K(n)}S$ and its full subcategory of $K(n)$-local modules. A study of the sphere spectrum as an algebro-geometric scheme- or stack-like object, that only involves smashing localizations or only treats the whole module categories over the various Bousfield localizations, does thus not capture these very interesting examples of regular geometric covering spaces.

There are structured ring spectrum replacements for Kähler differentials, called topological Hochschild homology (= *THH*, see Section 9.2) and topological André–Quillen homology (= *TAQ*, see Section 9.4), in the context of associative and commutative S-algebras, respectively. These need not be $K(n)$-local when applied to $K(n)$-local S-algebras (see Example 9.2.3). Therefore the notions of (formally) étale extensions of associative or commutative S-algebras will again give a richer theory when considered within the $K(n)$-local subcategory, rather than in the whole module category over $L_{K(n)}S$. Thus also a study of the algebraic geometry of the sphere spectrum with respect to the étale topology will become more substantial by taking these Bousfield local subcategories into account. This phenomenon differs from that which is familiar in discrete algebraic geometry, since there all localizations are, indeed, smashing.

It therefore appears to be better to think of the algebraic geometry of the sphere spectrum as the "S-algebraic stack" of all Bousfield E-local subcategories $\mathcal{M}_{S,E}$ of spectra, for varying spectra E, rather than the "S-algebraic scheme" of the Bousfield E-local S-algebras $L_E S$ themselves. The former stack maps to the stack of module categories of the latter scheme, but it is the former that carries the most interesting closed symmetric monoidal structures. See Definition 3.2.1 for the notations used here, and Section 3.2, Chapter 9 and Section 12.2 for more on these S-algebro-geometric ideas.

The (mono-)chromatic localizations $L_{K(n)}S$ of the sphere are of course even more drastic than the p-localizations $S_{(p)}$, so that many of the principal examples studied in this paper are of an even more local nature than e.g. local number fields. But the arithmetic properties of a global number field can usefully be studied by adèlic means, in terms of the system of local number fields that can be obtained from it by the various completions that are available. We are therefore also interested in finding global models for the system of naturally occurring $K(n)$-local Galois extensions of $L_{K(n)}S$, for varying p and n.

The obvious candidate, given Quillen's discovery of the relation of formal group law theory to complex cobordism, is the unit map $S \to MU$ to the complex cobordism spectrum. The following statement is proved in Corollary 9.6.6, Proposition 12.2.1 and the discussion surrounding diagram (12.2.6). In the second part, $S[BU]$ is the commutative Hopf S-algebra $\Sigma^\infty BU_+$. In summary, MU is very close to such a global model, up to formal thickenings by Henselian maps. This makes the author inclined to think of $S \to MU$ as a kind of (large) ramified global Galois extension, with $S[BU]$ playing the part of the functional dual of its imaginary Galois group. To make good sense of this, we introduce the notion of a Hopf–Galois extension of commutative S-algebras in Section 12.1.

Theorem 1.5. *For each prime p and integer $n \geq 1$ the $K(n)$-local pro-$\mathbb{G}_n$-Galois extension $L_{K(n)}S \to E_n$ factors as the composite of the following maps of commutative S-algebras*

$$L_{K(n)}S \to \hat{L}^{MU}_{K(n)}MU \xrightarrow{q} \widehat{E(n)} \to E_n\,.$$

Here the first map admits the global model $S \to MU$, by Bousfield $K(n)$-localization in the category of S-modules and $K(n)$-nilpotent completion in the category of MU-modules, respectively. The second map q is a formal thickening, or more precisely, symmetrically (and possibly commutatively) Henselian. The third map is a finite Galois extension (and can be avoided by passing to the even periodic version MUP of MU and adjoining some roots of unity).

Furthermore, the global model $S \to MU$ is an $S[BU]$-Hopf–Galois extension of commutative S-algebras, with coaction $\beta\colon MU \to MU \wedge S[BU]$ given by the Thom diagonal. For each element $g \in \mathbb{G}_n$ its Galois action on E_n can be directly recovered from this $S[BU]$-coaction, up to the adjunction of some roots of unity.

Here are some more detailed references into the body of the paper.

Chapter 2 contains a review of the basic Galois theory for fields and for commutative rings, together with some algebraic facts that we will need for our examples. We also make a comparison with the theory of regular covering spaces, for the benefit of the topologically minded reader.

As hinted at above, we sometimes consider more general Galois groups G than finite (and profinite) groups. For the initial theory, all that is required is that the unreduced suspension spectrum $S[G] = L_E\Sigma^\infty G_+$ admits a good Spanier–Whitehead dual in the E-local stable homotopy category, i.e., that G is *stably dualizable* (Definition 3.4.1). We review the basic properties of stably dualizable groups and their actions on spectra in Chapter 3, referring to the author's companion paper [Rog:s] for most proofs. This chapter also contains a discussion of the various categories of E-local S-modules and (commutative) S-algebras in which we work.

The precise Definition 4.1.3 of a Galois extension of commutative S-algebras is given in Chapter 4, followed by a discussion showing that the Eilenberg–Mac Lane embedding from commutative rings preserves and detects Galois extensions (Proposition 4.2.1). We also consider the elementary properties of faithful modules over structured ring spectra, flatness being implicit in our homotopy invariant work. We shall often make use of how various algebro-geometric properties of S-algebras are preserved by base change, or are detected by suitable forms of faithful base change.

Chapter 5 is devoted to the many examples of Galois extensions mentioned above, including all the intermediate $K(n)$-local Galois extensions between $L_{K(n)}S$

and the maximal unramified extension E_n^{nr} of E_n. We also go through the $K(1)$-local case of the Lubin–Tate extensions in much detail, making explicit the close analogy with the classification of abelian extensions of the p-adic and rational fields $\mathbb{Q}_p$ and $\mathbb{Q}$. Finally we extend the example of cochain algebras of regular covering spaces to cochain algebras of principal G-bundles $P \to X$, for stably dualizable groups G.

Chapter 6 develops the formal consequences of the Galois conditions on $A \to B$, including the basic fact that B is a dualizable A-module (Proposition 6.2.1), two useful alternate characterizations of (faithful) Galois extensions (Propositions 6.3.1 and 6.3.2), and two further characterizations of faithfulness (Proposition 6.3.3 and Lemma 6.5.4). These let us prove in Chapter 7 that faithful Galois extensions are preserved by arbitrary base change (Lemma 7.1.1) and are detected by faithful and dualizable base change (Lemma 7.1.4(b)). From these results, in turn, the "forward" part of the Galois correspondence (Theorem 7.2.3) follows rather formally, saying that for a faithful G-Galois extension $A \to B$ the homotopy fixed point spectra $C = B^{hK}$ give rise to K-Galois extensions $C \to B$ for subgroups $K \subset G$, and to G/K-Galois extensions $A \to C$ when K is normal.

When this much of the Galois correspondence has been established, we can make sense of the notion of a pro-Galois extension, which we do somewhat informally in Section 8.1.

The "converse" part of the Galois correspondence (Theorem 11.2.2) relies on the possibility of recovering the Galois group G in a G-Galois extension $A \to B$ from the space $\mathcal{C}_A(B, B)$ of commutative A-algebra self-maps $B \to B$, or more generally, to recover the subgroup K from the mapping space $\mathcal{C}_C(B, B)$, when $C = B^{hK}$ is a fixed S-algebra of B (Proposition 11.2.1). This is achieved in Chapter 11, but relies on three preceding developments.

First of all, we use the commutative form of the Hopkins–Miller theory, as developed by Paul Goerss and Mike Hopkins [GH04], to study such mapping spaces. We use an extension of their work, from dealing with spaces of E_∞ ring spectrum maps, or commutative S-algebra maps, to spaces of commutative A-algebra maps. This is discussed in Section 10.1, where we also touch on the consequences for this theory of working E-locally. The main computational tool is the Goerss–Hopkins spectral sequence (10.1.4), whose E_2-term involves suitable André–Quillen cohomology groups, which fortunately vanish in all relevant cases for the Galois extensions that we consider.

Second, the recovery of the Galois group G from $\mathcal{C}_A(B, B)$ only has a chance, judging from the discrete algebraic case, when B is connected in the geometric sense that it has no non-trivial idempotents. For a commutative S-algebra B there is a space $\mathcal{E}(B)$ of idempotents, which in turn is a commutative B-algebra mapping space of the sort that can be studied by the Goerss–Hopkins spectral sequence. So in Section 10.2 we treat connectivity in this geometric sense for commutative S-algebras, reaching a convenient algebraic criterion in Proposition 10.2.2. This also lets us define separably closed commutative S-algebras in Section 10.3.

Thirdly, not all commutative A-algebras C mapping faithfully to B occur in the Galois correspondence as fixed S-algebras $C = B^{hK}$. As in the discrete algebraic case, the characteristic property is that C must be separable over A, and in

Section 9.1 we develop the basic theory of separable extensions of S-algebras. As further generalizations of separable maps we have the étale maps, which we discuss in three related contexts in Sections 9.2 through 9.4, leading to the notions of symmetrically (=thh-)étale, smashing and (commutatively) étale maps of S-algebras, respectively.

Topological Hochschild homology THH controls the Kähler differentials in the associative setting, while topological André–Quillen homology TAQ takes on the same role in the purely commutative setting. Our discussion here relies heavily on the work of Maria Basterra [Bas99] and Andrej Lazarev [La01]. There is much conceptual overlap between the triviality of the topological André–Quillen homology spectrum $TAQ(B/A)$ for (formally) étale maps $A \to B$, and the vanishing of the Goerss–Hopkins André–Quillen cohomology groups $D^s_{B_*T}(B^A_*(B), \Omega^t B)$ for finite Galois extensions $A \to B$, but the direct connection is not as well understood as might be desired.

The remainder of the text is concerned with the interpretation of $S \to MU$ as a Hopf–Galois extension that provides an integral model, up to Henselian maps, for all of the Lubin–Tate extensions $L_{K(n)}S \to E_n$. Thus we consider square-zero extensions, singular extensions and Henselian maps as various forms of infinitesimal and formal thickenings in Section 9.5. We then obtain a good supply of relevant examples in Section 9.6, using work of Baker and Lazarev on I-adic towers. We have already cited Corollary 9.6.6 as relevant for part of Theorem 1.5.

The idea of Hopf–Galois extensions is to replace the action by the Galois group G on a commutative A-algebra B by a coaction by the functional dual $DG_+ = F(G_+, S)$ of the Galois group, which is a commutative Hopf S-algebra. In the algebraic situation such coactions have been useful, e.g. to classify inseparable Galois extensions of fields [Cha71]. In the absence of an actual Galois group the condition that $i\colon A \to B^{hG}$ is a weak equivalence must be rewritten, by using a cosimplicial resolution for the coaction (the Hopf cobar complex) in place of the homotopy fixed points. This rewriting can naturally go through a second cosimplicial resolution associated to $A \to B$, which we know as the Amitsur complex. We discuss the Amitsur complex in Section 8.2, so as to have the accompanying notion of (nilpotent) completion of A along B available in Chapters 9 and 10, and give the definitions of the Hopf cobar complex and of Hopf–Galois extensions in Section 12.1.

To conclude the paper, in Section 12.2 we go through some of the details of how the inseparable extension $S \to MU$ is an $S[BU]$-Hopf–Galois extension, and how the Hopkins–Miller theory and the Lubin–Tate deformation theory work together to show that the global $S[BU]$-coaction on MU captures the Morava stabilizer group action on E_n, at all primes p and chromatic heights n.

Acknowledgments.

The study of idempotents in Chapter 10 got going during an Oberwolfach hike with Neil Strickland, and the right way to use separability in Chapter 11 was found in a discussion with Birgit Richter. I am very grateful to them, as well as to Andy Baker, Daniel G. Davis, Halvard Fausk, Jack Morava and the referee, for a number of helpful or encouraging comments.

Much of this work was done in the year 2000 and announced at various confer-

ences. I apologize for the long delay in publication, which for most of the time was due to the unresolved Question 4.3.6, on the faithfulness of Galois extensions.

2. Galois extensions in algebra

2.1. Galois extensions of fields.

We first recall the basics about Galois extensions of fields. Let G be a finite group acting effectively (only the unit element acts as the identity) from the left by automorphisms on a field E, and let $F = E^G$ be the fixed subfield. Let

$$j \colon E\langle G\rangle \to \mathrm{Hom}_F(E, E)$$

be the canonical associative ring homomorphism taking $e_1 g$ to the homomorphism $e_2 \mapsto e_1 \cdot g(e_2)$, from the *twisted group ring* of G over E to the F-module endomorphisms of E. Then j is an isomorphism, for by Dedekind's lemma j is injective, and $\dim_F(E)$ equals the order of G, so j is also surjective by a dimension count. See [Dr95, App.] for elementary proofs. Let

$$h \colon E \otimes_F E \to \prod_G E$$

be the canonical commutative ring homomorphism taking $e_1 \otimes e_2$ to the sequence $\{g \mapsto e_1 \cdot g(e_2)\}$, from the tensor product of two copies of E over F to the product of G copies of E. Then also h is an isomorphism, for it is the E-module dual of j, by way of the identifications $\mathrm{Hom}_E(E \otimes_F E, E) \cong \mathrm{Hom}_F(E, E)$ and $\mathrm{Hom}_E(\prod_G E, E) \cong E\langle G\rangle$ (using that G is finite).

2.2. Regular covering spaces.

There is a parallel geometric theory of regular (= normal) covering spaces [Sp66, 2.6.7], [Ha02, 1.39]. Let G be a finite discrete group acting from the right on a compact Hausdorff space Y. Let $X = Y/G$ be the orbit space, and let $\pi \colon Y \to X$ be the orbit projection. There is a canonical map

$$\xi \colon Y \times G \to Y \times_X Y$$

to the fiber product of π with itself, taking (y, g) to $(y, y \cdot g)$. This map is always surjective, by the definition of X as an orbit space, and it is injective if and only if G acts freely on Y. So ξ is a homeomorphism if and only if $Y \to X$ is a regular covering space, with G as its group of deck transformations, acting freely and transitively on each fiber. In general, the possible failure of ξ to be injective measures the extent to which G does not act freely on Y, which in turn can be interpreted as a measure of to what extent Y is *ramified* as a cover of X. The theory of Riemann surfaces provides numerous examples of the latter phenomenon.

Dually, let $R = C(X)$ and $T = C(Y)$ be the rings of continuous (real or complex) functions on X and Y, respectively. As usual the points of X can be recovered as the maximal ideals in R, and similarly for Y. The group G acts from the left on T, by the formula $g(t) = g * t \colon y \mapsto t(y \cdot g)$, and the natural map $R \to T$ dual to π

identifies R with the invariant ring T^G, by the isomorphism $C(Y)^G \cong C(Y/G)$. The map ξ above is dual to the canonical homomorphism

$$h\colon T \otimes_R T \to \prod_G T$$

taking $t_1 \otimes t_2$ to the function $g \mapsto t_1 \cdot g(t_2)$, considered as an element in the product $\prod_G T$. Then ξ is a homeomorphism if and only if h is an isomorphism, by the categorical anti-equivalence between compact Hausdorff spaces and their function rings. The surjectivity of ξ ensures that h is always injective, and in general the possible failure of h to be surjective measures the extent of ramification in $Y \to X$.

For a moment, let us also consider the more general case of a principal G-bundle $\pi\colon P \to X$ for a compact Hausdorff topological group G. The map $\xi\colon P \times G \to P \times_X P$ is a homeomorphism, now with respect to the given topology on G. Let $R = C(X)$, $T = C(P)$ and $H = C(G)$. Then H is a commutative Hopf algebra with coproduct $\psi\colon H \to H \otimes H$, if the map $H \to C(G \times G)$ dual to the group multiplication $G \times G \to G$ factors through the canonical map $H \otimes H \to C(G \times G)$. Likewise H coacts on T from the right by $\beta\colon T \to T \otimes H$, if the map $T \to C(P \times G)$ induced by the group action $P \times G \to P$ factors through $T \otimes H \to C(P \times G)$. These factorizations can always be achieved by using suitably completed tensor products, but we wish to refer to the algebraic tensor products here. Then the freeness of the group action on P is expressed by saying that the composite map

$$h\colon T \otimes_R T \xrightarrow{1\otimes\beta} T \otimes_R T \otimes H \xrightarrow{\mu\otimes 1} T \otimes H$$

is an isomorphism. We shall return to this dualized context in Chapter 12 on Hopf–Galois extensions.

2.3. Galois extensions of commutative rings.

Generalizing the two examples above, for finite Galois groups, Auslander and Goldman [AG60, App.] gave a definition of Galois extensions of commutative rings as part of their study of separable algebras over such rings. Chase, Harrison and Rosenberg [CHR65, §1] found several other equivalent definitions, and developed the Galois theory for commutative rings to also encompass the fundamental Galois correspondence. We now recall their basic results.

Let $R \to T$ be a homomorphism of commutative rings, making T a commutative R-algebra, and let G be a finite group acting on T from the left through R-algebra homomorphisms. Let

$$i\colon R \to T^G$$

be the inclusion into the fixed ring, let

$$h\colon T \otimes_R T \to \prod_G T$$

be the commutative ring homomorphism that takes $t_1 \otimes t_2$ to the sequence $\{g \mapsto t_1 \cdot g(t_2)\}$, and let

$$j\colon T\langle G\rangle \to \mathrm{Hom}_R(T, T)$$

be the associative ring homomorphism that takes $t_1 g$ to the R-module homomorphism $t_2 \mapsto t_1 \cdot g(t_2)$. We give $\prod_G T$ the pointwise product $(t_g)_g \cdot (t'_g)_g = (t_g t'_g)_g$ and $T\langle G\rangle$ the twisted product $t_1 g_1 \cdot t_2 g_2 = t_1 g_1(t_2) g_1 g_2$, using the left G-action $(g_1, t_2) \mapsto g_1(t_2)$ on T.

Definition 2.3.1. Let G act on T over R, as above. We say that $R \to T$ is a *G-Galois extension of commutative rings* if both $i\colon R \to T^G$ and $h\colon T \otimes_R T \to \prod_G T$ are isomorphisms.

Here we are following Greither [Gre92, 0.1.5]. Auslander and Goldman [AG60, p. 396] instead took the condition below on i, j and T to be the defining property, but Chase, Harrison and Rosenberg [CHR65, 1.3] proved that the two definitions are equivalent.

Proposition 2.3.2. *Let G act on T over R, as above. Then $R \to T$ is a G-Galois extension if and only if both $i\colon R \to T^G$ and $j\colon T\langle G\rangle \to \mathrm{Hom}_R(T,T)$ are isomorphisms and T is a finitely generated projective R-module.*

The condition that i is an isomorphism means that we can speak of R as the *fixed ring* of T. The homomorphism h measures to what extent the extension $R \to T$ is *ramified,* and Galois extensions are required to be unramified. The injectivity of j is a form of Dedekind's lemma, and ensures that the action by G is effective.

Example 2.3.3. If $K \to L$ is a G-Galois extension of number fields, then the corresponding extension $R = \mathcal{O}_K \to \mathcal{O}_L = T$ of rings of integers is a G-Galois extension of commutative rings if and only if $K \to L$ is unramified as an extension of number fields [AB59]. More generally, if Σ is a set of prime ideals in $\mathcal{O}_K$, and Σ' the set of primes in $\mathcal{O}_L$ above those in Σ, then the extension $\mathcal{O}_{K,\Sigma} \to \mathcal{O}_{L,\Sigma'}$ of rings of Σ-integers is G-Galois if and only if Σ contains all the primes that ramify in L/K [Gre92, 0.4.1]. Here $\mathcal{O}_{K,\Sigma}$ is defined as the ring of elements $x \in K$ that have non-negative valuation $v_{\mathfrak{p}}(x) \geq 0$ for all prime ideals $\mathfrak{p} \notin \Sigma$. Thus $\mathcal{O}_K \to \mathcal{O}_L$ becomes a G-Galois extension precisely upon localizing away from (= inverting) the ramified primes.

To see this, note that if $T = R\{t_1, \dots, t_n\}$ is a free R-module of rank n, then $T \otimes_R T$ is a free T-module on the generators $1 \otimes t_1, \dots, 1 \otimes t_n$, and h is represented as a T-module homomorphism by the square matrix $A = (g(t_i))_{g,i}$ of rank n, with $g \in G$ and $i = 1, \dots, n$. The discriminant of T/R is $d = \det(A)^2$, by definition, and the prime ideals in $\mathcal{O}_K$ that ramify in L/K are precisely the prime ideals dividing the discriminant. So h is an isomorphism if and only if $\det(A)$ and d are units in R, or equivalently, if there are no ramified primes. A local version of the same argument works when T is not free over R.

Here are some further basic properties of Galois extensions of commutative rings, which will be relevant to our discussion.

Proposition 2.3.4. *Let $R \to T$ be a G-Galois extension. Then:*

(a) T is faithfully flat as an R-module, i.e., the functor $(-) \otimes_R T$ preserves and detects (=reflects) exact sequences.

(b) The trace map $tr\colon T \to R$ (taking $t \in T$ to $\sum_{g \in G} g(t) \in T^G = R$) is a split surjective R-module homomorphism.

(c) T is invertible as an $R[G]$-module, i.e., a finitely generated projective $R[G]$-module of constant rank 1.

For proofs, see e.g. [Gre92, 0.1.9], [Gre92, 0.1.10] and [Gre92, 0.6.1]. Beware that part (b) does not extend well to the topological setting, as Example 6.4.4 demonstrates.

3. Closed categories of structured module spectra

3.1. Structured spectra.

We now adapt these ideas to the context of "brave new rings," i.e., of commutative S-algebras. These can in Chapters 2–9 and 12 be interpreted as the commutative monoids in either one of the popular symmetric monoidal categories of structured spectra, such as the S-modules of Elmendorf, Kriz, Mandell and May [EKMM97], the symmetric spectra in simplicial sets of Hovey, Shipley and Smith [HSS00], symmetric spectra in topological spaces or orthogonal spectra of Mandell, May, Schwede and Shipley [MMSS01] or the simplicial functors of Segal and Lydakis [Ly98], according to the reader's needs or preferences.

However, in Chapters 10 and 11 we make use of the Goerss–Hopkins obstruction theory for E_∞ mapping spaces [GH04], which presumes that one works in a category of spectra that satisfies Axioms 1.1 and 1.4 in *op. cit.* In particular, this theory is needed for the proof of parts (c) and (d) of our Theorem 1.2, and for Theorem 1.3. It is known that S-modules, symmetric spectra formed in topological spaces and orthogonal spectra all satisfy the required axioms, by [GH04, 1.5]. To be concrete, and to have a convenient source for the more technical references, we shall work with the S-modules of Peter May *et al.*

Let S be the sphere spectrum, and let $\mathcal{M}_S$ be the category of S-modules. Among other things, it is a topological category with all limits and colimits and all topological tensors and cotensors. A map $f\colon X \to Y$ of S-modules is called a *weak equivalence* if the induced homomorphism $\pi_*(f)\colon \pi_*(X) \to \pi_*(Y)$ of stable homotopy groups is an isomorphism. The category $\mathcal{D}_S$ obtained from $\mathcal{M}_S$ by inverting the weak equivalences is called the *stable homotopy category,* and is equivalent to the homotopy category of spectra constructed by Boardman [Vo70].

The smash product $X \wedge Y$ and function object $F(X, Y)$ make $\mathcal{M}_S$ a closed symmetric monoidal category, with S as the unit object. For each topological space T the topological tensor $X \wedge T_+$ equals the smash product $X \wedge S[T]$, and the topological cotensor $Y^T = F(T_+, Y)$ equals the function spectrum $F(S[T], Y)$, where $S[T] = \Sigma^\infty T_+$ denotes the unreduced suspension S-module on T.

An (associative) S-algebra A is a monoid in $\mathcal{M}_S$, i.e., an S-module A equipped with a unit map $\eta\colon S \to A$ and a unital and associative multiplication $\mu\colon A \wedge A \to A$. A commutative S-algebra A is a commutative monoid in $\mathcal{M}_S$, i.e., one such that the multiplication μ is also commutative. We write $\mathcal{A}_S$ and $\mathcal{C}_S$ for the categories of S-algebras and commutative S-algebras, respectively. More generally, for a commutative S-algebra A we write $\mathcal{M}_A$, $\mathcal{A}_A$ and $\mathcal{C}_A$ for the categories of A-modules, associative A-algebras and commutative A-algebras, respectively [EKMM97, VII.1].

3.2. Localized categories.

Our first examples of Galois extensions of structured ring spectra will be maps $A \to B$ of commutative S-algebras, with a finite group G acting on B through A-algebra maps, such that there are weak equivalences $i\colon A \simeq B^{hG}$ and $h\colon B \wedge_A B \simeq \prod_G B$. The formal definition appears in Section 4.1 below. However, there are interesting examples that only appear as Galois extensions to the eyes of weaker invariants than the stable homotopy groups $\pi_*(-)$. More precisely, for a fixed homology theory $E_*(-)$ we shall allow ourselves to work in the E-local stable homotopy category, where have arranged that each map $f\colon X \to Y$ such that $E_*(f)\colon E_*(X) \to E_*(Y)$ is an isomorphism, is in fact a weak equivalence. In particular, we will encounter situations where we only have that $E_*(i)$ and $E_*(h)$ are isomorphisms, in which case we shall interpret $A \to B$ as an E-local G-Galois extension.

Note the close analogy between the E-local theory and the case (Example 2.3.3) of rings of integers localized away from some set of primes. Doug Ravenel's influential treatise on the chromatic filtration of stable homotopy theory [Ra84, §5], brings emphasis to the tower of cases when $E = E(n)$, the n-th Johnson–Wilson spectrum. To us, the most interesting case is when $E = K(n)$ is the n-th Morava K-theory spectrum. The $K(n)$-local stable homotopy category is studied in detail in [HSt99, §§7–8], and captures the n-th layer, or stratum, in the chromatic filtration.

Definition 3.2.1. Let E be a fixed S-module, with associated homology theory $X \mapsto E_*(X) = \pi_*(E \wedge X)$. By definition, an S-module Z is said to be *E-acyclic* if $E \wedge Z \simeq *$, and an S-module Y is said to be *E-local* if $F(Z, Y) \simeq *$ for each E-acyclic S-module Z. Let $\mathcal{M}_{S,E} \subset \mathcal{M}_S$ be the full subcategory of *E-local S-modules.* A map $f\colon X \to Y$ of E-local S-modules is a weak equivalence if and only if it is an *E_*-equivalence,* i.e., if $E_*(f)$ is an isomorphism.

There is a Bousfield localization functor $L_E\colon \mathcal{M}_S \to \mathcal{M}_{S,E} \subset \mathcal{M}_S$ [Bo79], [EKMM97, VIII.1.6], and an accompanying natural E_*-equivalence $X \to L_E X$ for each S-module X. We may assume that this E_*-equivalence is the identity when X is already E-local, so that the localization functor L_E is idempotent. The homotopy category $\mathcal{D}_{S,E}$ of $\mathcal{M}_{S,E}$ is the *E-local stable homotopy category.*

More generally, for a commutative S-algebra A we let $\mathcal{M}_{A,E} \subset \mathcal{M}_A$ be the full subcategory of E-local A-modules, with homotopy category $\mathcal{D}_{A,E}$. To be precise, there is an A-module $\mathbb{F}_A E$ of the homotopy type of $A \wedge E$, and a localization functor $L^A_{\mathbb{F}_A E}\colon \mathcal{M}_A \to \mathcal{M}_{A,E}$, with respect to $\mathbb{F}_A E$ in the category of A-modules, which amounts to E-localization at the level of the underlying S-modules [EKMM97, VIII.1.7]. We shall allow ourselves to simply denote this functor by L_E.

Notation 3.2.2. We write
$$L_n X = L_{E(n)} X$$
for the Bousfield localization of X with respect to the Johnson–Wilson spectrum $E(n)$ [JW73], with $\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$, for each non-negative integer n, and
$$L_{K(n)} X$$
for the Bousfield localization of X with respect to the Morava K-theory spectrum $K(n)$ [JW75], with $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$, for each natural number n.

We will reserve the symbol $\hat{L}$ for the Bousfield nilpotent completion recalled in Definition 8.2.2, and shall therefore not use this notation for the functor $L_{K(n)}$, unlike e.g. [HSt99].

The smash product $X \wedge Y$ of two E-local S-modules will in general not be E-local, although this is the case when L_E is a so-called *smashing localization,* i.e., one that commutes with direct limits [Ra84, 1.28]. The Johnson–Wilson spectra $E = E(n)$ provide interesting examples of smashing localizations $L_n = L_{E(n)}$ [Ra92, 7.5.6], while localization $L_{K(n)}$ with respect to the Morava K-theories $E = K(n)$ is not smashing [HSt99, 8.1]. Likewise, the unit S for the smash product is rarely E-local. So in order to work with S-algebras and related constructions internally within $\mathcal{M}_{S,E}$, we first perform each construction as usual in $\mathcal{M}_S$, and then apply the Bousfield localization functor L_E.

Definition 3.2.3. We implicitly give $\mathcal{M}_{S,E}$ all colimits, topological tensors, smash products and a unit object by applying Bousfield localization to the constructions in $\mathcal{M}_S$. So $\operatorname{colim}_{i\in I} X_i$ means $L_E(\operatorname{colim}_{i\in I} X_i)$, $X \wedge Y$ means $L_E(X \wedge Y)$, S means $L_E S$ and $S[T]$ means $L_E \Sigma^\infty T_+$. All limits, topological cotensors and function objects formed from E-local S-modules are already E-local, so no Bousfield localization is required in these cases. With these conventions, $\mathcal{M}_{S,E}$ is a topological closed symmetric monoidal category with all limits and colimits. The same considerations apply for $\mathcal{M}_{A,E}$.

There is a natural map $L_E X \wedge L_E Y \to L_E(X \wedge Y)$, making L_E a lax monoidal functor, so that $L_E S$ is always a commutative S-algebra. When E is smashing, the category $\mathcal{M}_{S,E}$ of E-local S-modules is equivalent (at the level of homotopy categories) to the category $\mathcal{M}_{L_E S}$ of $L_E S$-modules, so the study of E-local S-modules is a special case of the study of modules over a general commutative S-algebra $A = L_E S$. However, when E is not smashing, as is the case for $E = K(n)$, the two homotopy categories are not equivalent, and we shall need to consider the more general notion.

When $E = S$, every S-module is E-local and $\mathcal{M}_{S,E} = \mathcal{M}_S$, etc., so the E-local context specializes to the "global", unlocalized situation. For brevity, we shall often simply refer to the E-local S-modules as S-modules, or even as spectra, but except where we explicitly assume that $E = S$, the discussion is intended to encompass also the general E-local case.

Remark 3.2.4. By analogy with algebraic geometry, we may heuristically wish to view A-modules M as suitable sheaves $M^\sim$ over some geometric "structure space" $\operatorname{Spec} A$. This structure space would come with a Zariski topology, with open subspaces $U_{A,E} \subset \operatorname{Spec} A$ corresponding to the various localization functors L_E on the category of A-modules, in such a way that the restriction of the sheaf $M^\sim$ over $\operatorname{Spec} A$ to the subspace $U_{A,E}$ would be the sheaf $(L_E M)^\sim$ corresponding to the E-local A-module $L_E M$. For smashing E this would precisely amount to an $L_E A$-module, so that $U_{A,E}$ could be identified with the structure space $\operatorname{Spec} L_E A$.

However, for non-smashing E the condition of being an E-local A-module is strictly stronger than being an $L_E A$-module. Therefore, the geometric structure on $\operatorname{Spec} A$ is not simply that of an "S-algebra'ed space" carrying the (commutative) S-algebra $L_E A$ over $U_{A,E}$, by analogy with the ringed spaces of algebraic geometry. If we wish to allow non-smashing localizations E to correspond to Zariski opens, then the geometric structure must also capture the additional restriction it is for an $L_E A$-module to be an E-local A-module. This exhibits a difference compared to the situation in commutative algebra, where localization at an ideal commutes with direct limits, and behaves as a smashing localization, while completions behave more like non-smashing localizations. It does not seem to be so common to do commutative algebra in such implicitly completed situations, however.

A continuation of this analogy would be to consider other Grothendieck-type topologies on $\operatorname{Spec} A$, with coverings built from E-local Galois extensions $L_E A \to B$ (Definition 4.1.3) or more general étale extensions (Definition 9.4.1), subject to a combined faithfulness condition (Definition 4.3.1). In the unlocalized cases, such a (big) étale site on the opposite category of $\mathcal{C}_S$, and associated small étale sites on the opposite category of each $\mathcal{C}_A$, have been developed by Bertrand Toën and Gabriele Vezzosi [TV05, §5.2]. However, the rich source of $K(n)$-local Galois extensions of $L_{K(n)}S$ discussed in Section 5.4 provides, by Lemma 9.4.4, an equally rich supply of $K(n)$-local étale maps from $L_{K(n)}S$. It appears, by extension from the case $n = 1$ discussed in Section 5.5, that these are not globally étale maps, in which case the étale topology proposed in [TV05] will be too coarse to encompass these examples. The author therefore thinks that a finer étale site, taking non-smashing localizations like $L_{K(n)}$ into account, would lead to a stronger and more interesting theory.

3.3. Dualizable spectra.

In each closed symmetric monoidal category there is a canonical natural map

$$\nu \colon F(X, Y) \wedge Z \to F(X, Y \wedge Z) .$$

It is right adjoint to a map $\epsilon \wedge 1 \colon X \wedge F(X, Y) \wedge Z \to Y \wedge Z$, where the adjunction counit $\epsilon \colon X \wedge F(X, Y) \to Y$ is left adjoint to the identity map on $F(X, Y)$.

Dold and Puppe [DP80] say that an object X is strongly dualizable if the canonical map $\nu \colon F(X, Y) \wedge Z \to F(X, Y \wedge Z)$ is an isomorphism for all Y and Z. Lewis, May and Steinberger [LMS86, III.1.1] say that a spectrum X is finite if it is strongly dualizable in the stable homotopy category, i.e., if the map ν is a weak equivalence. We shall instead follow Hovey and Strickland [HSt99, 1.5(d)] and briefly call such spectra dualizable. By [LMS86, III.1.3(ii)] it suffices to verify this condition in the special case when $Y = S$ and $Z = X$, so we take this simpler criterion as our definition.

Definition 3.3.1. Let $DX = F(X, S)$ be the *functional dual* of X. We say that X is *dualizable* if the canonical map $\nu \colon DX \wedge X \to F(X, X)$ is a weak equivalence. More generally, for an (implicitly E-local) module M over a commutative S-algebra A, let $D_A M = F_A(M, A)$ be the functional dual, and say that M is a dualizable A-module if the canonical map $\nu \colon D_A M \wedge_A M \to F_A(M, M)$ is a weak equivalence.

Lemma 3.3.2. *(a) If X or Z is dualizable, then the canonical map $\nu \colon F(X, Y) \wedge Z \to F(X, Y \wedge Z)$ is a weak equivalence.*

(b) If X is dualizable, then DX is also dualizable and the canonical map $\rho \colon X \to DDX$ is a weak equivalence.

(c) The dualizable spectra generate a thick subcategory, i.e., they are closed under passage to weakly equivalent objects, retracts, mapping cones and (de-)suspensions.

Here $\rho \colon X \to DDX = F(F(X, S), S)$ is right adjoint to $F(X, S) \wedge X \to S$, which is obtained by twisting the adjunction counit $\epsilon \colon X \wedge F(X, S) \to S$. For proofs, see [LMS86, III.1.2 and III.1.3]. We sometimes also use ν to label the conjugate map

$Y \wedge F(X, Z) \to F(X, Y \wedge Z)$. The corresponding results hold for E-local A-modules, by the same formal proofs.

One justification for the term "finite" is the following converse to Lemma 3.3.2(c), in the unlocalized setting $E = S$.

Proposition 3.3.3. *Let A be commutative S-algebra. A global A-module M is dualizable in $\mathcal{M}_A = \mathcal{M}_{A,S}$ if and only if it is weakly equivalent to a retract of a finite cell A-module. When A is connective, this is in turn equivalent to being a retract of a finite CW A-module spectrum.*

The proof [EKMM97, III.7.9] uses in an essential way that stable homotopy $X \mapsto \pi_*(X) = [A, X]^A_*$ commutes with coproducts, which amounts to A being small in the homotopy category $\mathcal{D}_A$ of A-modules. This fails in some E-local contexts. For example, the $K(n)$-local sphere spectrum $L_{K(n)}S$ is not small in the $K(n)$-local category [HSt99, 8.1], and consequently $\pi_*(X)$ is not a homology theory on this category. So in general there will be more dualizable E-local A-modules than the *semi-finite* ones, i.e., the retracts of the finite cell $L_E A$-modules. In this paper we shall prefer to focus on the notion of dualizability, rather than on being semi-finite, principally because of Proposition 6.2.1 and (counter-)Example 6.2.2 below.

3.4. Stably dualizable groups.

For our basic theory of G-Galois extensions of commutative S-algebras the group action by G appears through the module action by its suspension spectrum $S[G] = L_E \Sigma^\infty G_+$, and the finiteness condition on G only enters through the property that $S[G]$ is a dualizable spectrum. We then say that G is an E-locally stably dualizable group. Only when we turn to properties related to separability will it be relevant that G is discrete, and then usually finite. So we shall develop the basic theory in the greater generality of stably dualizable topological groups G.

Definition 3.4.1. A topological group G is *E-locally stably dualizable* if its suspension spectrum $S[G] = L_E \Sigma^\infty G_+$ is dualizable in $\mathcal{M}_{S,E}$. Writing $DG_+ = F(G_+, L_E S)$ for its functional dual, the condition is that the canonical map

$$\nu \colon DG_+ \wedge S[G] \to F(S[G], S[G])$$

is a weak equivalence in the E-local category.

Examples 3.4.2. (a) Each compact Lie group G admits the structure of a finite CW complex, so $S[G]$ is a finite cell spectrum and G is stably dualizable, for each E.

(b) The Eilenberg–Mac Lane spaces $G = K(\mathbb{Z}/p, q)$ are loop spaces and thus admit models as topological groups. They have infinite mod p homology for each $q \geq 1$, so $S[G]$ is never dualizable in $\mathcal{M}_S$ by Proposition 3.3.3. However, the Morava K-homology $K(n)_* K(\mathbb{Z}/p, q)$ is finitely generated over $K(n)_*$ by a calculation of Ravenel and Wilson [RaW80, 9.2], so $G = K(\mathbb{Z}/p, q)$ is in fact $K(n)$-locally stably dualizable by [HSt99, 8.6]. We are curious to see if these and similar topological Galois groups play any significant role in the $K(n)$-local Galois theory.

3.5. The dualizing spectrum.

The weak equivalence $S[G] = \bigvee_G S \to \prod_G S = DG_+$ for a finite group G generalizes to an E-local self-duality of the suspension spectrum $S[G]$, when G is an

E-locally stably dualizable group. The self-duality holds up to a twist by a so-called dualizing spectrum S^{adG}. When G is a compact Lie group this is the suspension spectrum on the one-point compactification of the adjoint representation adG of G, thus the notation, and so $S^{adG} = S$ for G finite. John Klein [Kl01, §1] introduced dualizing spectra S^{adG} for arbitrary topological groups, and Tilman Bauer [Bau04, 4.1] established the twisted self-duality of $S[G]$ in the p-complete category, when G is a p-compact group in the sense of Bill Dwyer and Clarence Wilkerson [DW94]. In [Rog:s] we have extended these results to all E-locally stably dualizable groups, as we now review.

Definition 3.5.1. Let G be an E-locally stably dualizable group. The group multiplication provides the suspension spectrum $S[G] = L_E\Sigma^\infty G_+$ with mutually commuting left and right G-actions. We define the *dualizing spectrum* S^{adG} to be the G-homotopy fixed point spectrum

$$S^{adG} = S[G]^{hG} = F(EG_+, S[G])^G$$

of $S[G]$, formed with respect to the right G-action [Rog:s, 2.5.1]. Here $EG = B(*, G, G)$ is the standard free, contractible right G-space. The remaining left action on $S[G]$ induces a left G-action on S^{adG}.

When G is finite, there is a natural weak equivalence

$$S^{adG} = S[G]^{hG} \simeq DG_+^{hG} \simeq S\,.$$

Here the last equivalence involves the collapsing homotopy equivalence $c\colon EG \to *$, which is a G-equivariant map, but not a G-equivariant homotopy equivalence. For general stably dualizable groups G, the dualizing spectrum is indeed dualizable and smash invertible [Rog:s, 3.2.3 and 3.3.4], so smashing with S^{adG} induces an equivalence of derived categories.

The left G-action on $S[G]$ functorially dualizes to a right G-action on DG_+, with associated module action map $\alpha\colon DG_+ \wedge S[G] \to DG_+$. The diagonal map on G induces a coproduct $\psi\colon S[G] \to S[G] \wedge S[G]$, using [EKMM97, II.1.2]. These combine to a *shear map*

$$sh\colon DG_+ \wedge S[G] \xrightarrow{1\wedge\psi} DG_+ \wedge S[G] \wedge S[G] \xrightarrow{\alpha\wedge 1} DG_+ \wedge S[G]\,,$$

which is equivariant with respect to each of three mutually commuting G-actions [Rog:s, 3.1.2] and is a weak equivalence [Rog:s, 3.1.3]. Taking homotopy fixed points with respect to the right action of G on $S[G]$ in the source and the diagonal right action on DG_+ and $S[G]$ in the target induces a natural *Poincaré duality equivalence* [Rog:s, 3.1.4]

$$DG_+ \wedge S^{adG} \xrightarrow{\simeq} S[G]\,. \tag{3.5.2}$$

This identification uses the stable dualizability of G, and expresses the twisted self-duality of $S[G]$. The weak equivalence is equivariant with respect to both a left and a right G-action. The left G-action is by the inverse of the right action on DG_+, the standard left action on S^{adG} and the standard left action on $S[G]$. The right G-action is by the inverse of the left action on DG_+, the trivial action on S^{adG} and the standard right action on $S[G]$.

3.6. The norm map.

Let X be any E-local S-module with left G-action, and equip it with the trivial right G-action. The smash product $X \wedge S[G]$ then has a diagonal left G-action, and a right G-action that only affects $S[G]$. Consider forming homotopy orbits $(-)_{hG}$ with respect to the left action and forming homotopy fixed points $(-)^{hG}$ with respect to the right action, in either order. There is then a canonical colimit/limit exchange map

$$\kappa\colon ((X \wedge S[G])^{hG})_{hG} \to ((X \wedge S[G])_{hG})^{hG} .$$

The source of κ receives a weak equivalence from $(X \wedge S^{adG})_{hG}$ (this uses the stable dualizability of G; see the proof of Lemma 6.4.2), and the target of κ maps by a weak equivalence to X^{hG} (this is easy). The composite of these three maps is the *(homotopy) norm map* [Rog:s, 5.2.2]

$$N\colon (X \wedge S^{adG})_{hG} \to X^{hG} . \tag{3.6.1}$$

If $X = W \wedge G_+ = W \wedge S[G]$ for some spectrum W with left G-action, with G acting in the standard way on $S[G]$, then the norm map for X is a weak equivalence [Rog:s, 5.2.5]. That reference only discusses the case when G acts trivially on W, but in general there is an equivariant shearing equivalence $\zeta\colon w \wedge g \mapsto g(w) \wedge g$ from $W \wedge S[G]$ with G acting only on $S[G]$ to $W \wedge S[G]$ with the diagonal G-action.

We can define the *G-Tate construction* X^{tG} to be the cofiber of the norm map

$$(X \wedge S^{adG})_{hG} \xrightarrow{N} X^{hG} \longrightarrow X^{tG} .$$

Then $X^{tG} \simeq *$ if and only if N is a weak equivalence, which in turn holds if and only if the exchange map κ is a weak equivalence. From this point of view X^{tG} is the obstruction to the commutation of the G-homotopy orbit and the G-homotopy fixed point constructions, when applied to $X \wedge S[G]$.

4. Galois extensions in topology

4.1. Galois extensions of E-local commutative S-algebras.

Fix an S-module E, and consider the categories $\mathcal{M}_{S,E}$ and $\mathcal{C}_{S,E}$ of E-local S-modules and E-local commutative S-algebras, respectively. These are full subcategories of the topological (closed) model categories $\mathcal{M}_S$ and $\mathcal{C}_S$, respectively, as explained in [EKMM97, VII.4].

The reader may, if preferred, alternatively work with the "convenient" S-model structures of Jeff Smith and Brooke Shipley [Sh04], but this will not be necessary. There is another E-local model structure on $\mathcal{M}_S$, with E_*-equivalences as the weak equivalences and the E-local S-modules as the fibrant objects, see [EKMM97, VIII.1], but there does not seem to be such an E-local model structure available in the case of $\mathcal{C}_S$.

Let $A \to B$ be a map of E-local commutative S-algebras, making B a commutative A-algebra, and let G be an E-locally stably dualizable group acting continuously on B from the left through commutative A-algebra maps. For example, G can be a finite discrete group.

Suppose that A is cofibrant as a commutative S-algebra, and that B is cofibrant as a commutative A-algebra. The commutative A-algebra B tends not to be cofibrant as an A-module, but the smash product functor $B \wedge_A (-)$ is still homotopically meaningful when applied to (other) cofibrant commutative A-algebras, as explained in [EKMM97, VII.6].

Let

$$i\colon A \to B^{hG} \tag{4.1.1}$$

be the map to the homotopy fixed point S-algebra $B^{hG} = F(EG_+, B)^G$ that is right adjoint to the composite G-equivariant map $A \wedge EG_+ \to A \to B$, collapsing the contractible free G-space EG to a point. Let

$$h\colon B \wedge_A B \to F(G_+, B) \tag{4.1.2}$$

be the canonical map to the product (cotensor) S-algebra $F(G_+, B)$ that is right adjoint to the composite map $B \wedge_A B \wedge G_+ \to B \wedge_A B \to B$, induced by the action $B \wedge G_+ \cong G_+ \wedge B \to B$ of G on B, followed by the A-algebra multiplication $B \wedge_A B \to B$ in B.

We consider $B \wedge_A B$ and $F(G_+, B)$ as B-modules by the multiplication in the first (left hand) copy of B in $B \wedge_A B$, and in the target of $F(G_+, B)$. Then h is a map of B-modules. The group G acts from the left on the second (right hand) copy of B in $B \wedge_A B$, and by right multiplication in the source of $F(G_+, B)$. Then h is also a G-equivariant map. These B- and G-actions clearly commute, and combine to a left module action by the group S-algebra $B[G]$.

Here is our key definition, which assumes that E, A, B and G are as above, and uses the maps i and h just introduced. We introduce the related map j in Section 6.1.

Definition 4.1.3. We say that $A \to B$ is an *E-local G-Galois extension of commutative S-algebras* if the two canonical maps $i\colon A \to B^{hG} = F(EG_+, B)^G$ and $h\colon B \wedge_A B \to F(G_+, B)$, formed in the category of E-local S-modules, are both weak equivalences.

The assumption that A and B are E-local ensures that B^{hG} and $F(G_+, B)$ are E-local, without any implicit localization. But $B \wedge_A B$ formed in S-modules needs not be E-local, unless E is smashing. The condition that h is a weak equivalence in $\mathcal{M}_{S,E}$ amounts to asking that the corresponding map $B \wedge_A B \to F(G_+, B)$ formed in $\mathcal{M}_S$ is an E_*-equivalence, i.e., that $E_*(h)$ is an isomorphism.

Lemma 4.1.4. *Subject to the cofibrancy conditions, the notion of an E-local G-Galois extension $A \to B$ is invariant under changes up to weak equivalence in A, B and the stabilized group $S[G] = L_E\Sigma^\infty G_+$.*

Proof. By [EKMM97, VII.6.7] the cofibrancy conditions ensure that the constructions A, B^{hG}, $B \wedge_A B$ and $F(G_+, B)$ preserve weak equivalences in A and B, whether implicitly E-localized or not.

The natural E_*-equivalences $\Sigma^\infty G_+ \to S[G]$ and $\Sigma^\infty EG_+ \to S[EG]$ induce a (not implicitly localized) map

$$F_{S[G]}(S[EG], B) \to F_{\Sigma^\infty G_+}(\Sigma^\infty EG_+, B) \cong F(EG_+, B)^G\,,$$

which is a weak equivalence when B is E-local. Thus the construction B^{hG} also preserves weak equivalences in $S[G]$.

Thus the E-local Galois conditions, that G is stably dualizable and the maps i and h are weak equivalences, are invariant under changes in A, B or G that amount to E-local weak equivalences of A, B and $S[G]$. □

When $E = S$, so there is no implicit E-localization, we may simply say that $A \to B$ is a G-Galois extension, or for emphasis, that $A \to B$ is a *global G-Galois extension.* However, most of the time we are implicitly working E-locally, for a general spectrum E, but omit to mention this at every turn. Hopefully no confusion will arise.

When G is discrete, we often prefer to write the target $F(G_+, B)$ of h as $\prod_G B$. When G is finite and discrete, we say that $A \to B$ is a *finite Galois extension.*

4.2. The Eilenberg–Mac Lane embedding.

The Eilenberg–Mac Lane functor H, which to a commutative ring R associates a commutative S-algebra HR with $\pi_* HR = R$ concentrated in degree 0, embeds the category of commutative rings into the category of commutative S-algebras. The two notions of Galois extension are compatible under this embedding. For this to make sense, we must assume that G is finite and that $E = S$.

Proposition 4.2.1. *Let $R \to T$ be a homomorphism of commutative rings, and let G be a finite group acting on T through R-algebra homomorphisms. Then $R \to T$ is a G-Galois extension of commutative rings if and only if the induced map $HR \to HT$ is a global G-Galois extension of commutative S-algebras.*

Proof. Suppose first that $R \to T$ is G-Galois. Then T is a finitely generated projective R-module by Proposition 2.3.2, hence flat, so $\mathrm{Tor}_s^R(T, T) = 0$ for $s \neq 0$. Furthermore, T is finitely generated projective (of constant rank 1) as an $R[G]$-module, by Proposition 2.3.4(c). There is an isomorphism of left $R[G]$-modules $R[G] \cong \mathrm{Hom}_R(R[G], R)$, since G is finite, so $\mathrm{Ext}^s_{R[G]}(R, R[G]) \cong \mathrm{Ext}^s_R(R, R) = 0$ for $s \neq 0$. Therefore $\mathrm{Ext}^s_{R[G]}(R, T) = 0$ for $s \neq 0$, by the finite additivity of Ext in its second argument.

It follows that the homotopy fixed point spectral sequence

$$E^2_{s,t} = H^{-s}(G; \pi_t HT) = \mathrm{Ext}^{-s,-t}_{R[G]}(R, T) \Longrightarrow \pi_{s+t}(HT^{hG})$$

derived from [EKMM97, IV.4.3], and the Künneth spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^R_{s,t}(T, T) \Longrightarrow \pi_{s+t}(HT \wedge_{HR} HT)$$

of [EKMM97, IV.4.2], both collapse to the origin $s = t = 0$. So $(HT)^{hG} \simeq H(T^G) = HR$ and $HT \wedge_{HR} HT \simeq H(T \otimes_R T) \cong H(\prod_G T) \simeq \prod_G HT$ are both weak equivalences. Thus $HR \to HT$ is a G-Galois extension of commutative S-algebras.

Conversely, suppose that $HR \to HT$ is G-Galois. Then by the same spectral sequences $T^G \cong \pi_0(HT^{hG}) \cong \pi_0 HR = R$ and $T \otimes_R T \cong \pi_0(HT \wedge_{HR} HT) \cong \pi_0(\prod_G HT) \cong \prod_G T$, so $R \to T$ is a G-Galois extension of commutative rings. □

4.3. Faithful extensions.

Galois extensions of commutative rings are always faithfully flat, and it will be convenient to consider the corresponding property for structured ring spectra. It remains an open problem whether Galois extensions of commutative S-algebras are in fact always faithful, but we shall verify that this is the case in most of our examples, with the possible exception of some cases in Section 5.6.

Definition 4.3.1. Let A be a commutative S-algebra. An A-module M is *faithful* if for each A-module N with $N \wedge_A M \simeq *$ we have $N \simeq *$. An A-algebra B, or G-Galois extension $A \to B$, is said to be faithful if B is faithful as an A-module.

A set of A-algebras $\{A \to B_i\}_i$ is a *faithful cover* of A if for each A-module N with $N \wedge_A B_i \simeq *$ for every i we have $N \simeq *$. In particular, a single faithful A-algebra B covers A in this sense.

By the following lemma, this corresponds well to the algebraic notion of a faithfully flat module [Gre92, 0.1.7]. Flatness (cofibrancy) is implicit in our homotopy invariant work, so we only refer to the faithfulness in our terminology.

Lemma 4.3.2. *Let M be a faithful A-module.*

(a) A map $f\colon X \to Y$ of A-modules is a weak equivalence if and only if $f \wedge 1\colon X \wedge_A M \to Y \wedge_A M$ is a weak equivalence.

(b) A diagram of A-modules $X \xrightarrow{f} Y \xrightarrow{g} Z$, with a preferred null-homotopy of gf, is a cofiber sequence if and only if $X \wedge_A M \to Y \wedge_A M \to Z \wedge_A M$, with the associated null-homotopy of $gf \wedge 1$, is a cofiber sequence.

Proof. (a) Consider the mapping cone C_f of f.

(b) Consider the induced map $C_f \to Z$. □

Faithful modules and extensions are preserved under base change, and are detected by faithful base change.

Lemma 4.3.3. *Let $A \to B$ be a map of commutative S-algebras and M a faithful A-module. Then $B \wedge_A M$ is a faithful B-module.*

Proof. Let N be a B-module such that $N \wedge_B (B \wedge_A M) \simeq *$. Then $N \wedge_A M \simeq *$, so $N \simeq *$ since M is faithful over A. □

Lemma 4.3.4. *Let $A \to B$ be a faithful map of commutative S-algebras and M an A-module such that $B \wedge_A M$ is a faithful B-module. Then M is a faithful A-module.*

Proof. Let N be an A-module such that $N \wedge_A M \simeq *$. Then $(N \wedge_A B) \wedge_B (B \wedge_A M) \cong N \wedge_A B \wedge_A M \cong (N \wedge_A M) \wedge_A B \simeq *$, so $N \wedge_A B \simeq *$ since $B \wedge_A M$ is faithful over B, and thus $N \simeq *$ since B is faithful over A. □

Lemma 4.3.5. *For each G-Galois extension $R \to T$ of commutative rings, the induced G-Galois extension $HR \to HT$ of commutative S-algebras is faithful.*

Proof. Recall that T is faithfully flat over R by Proposition 2.3.4(a). For each HR-module N we have $\pi_*(N \wedge_{HR} HT) \cong \pi_*(N) \otimes_R T$, by the Künneth spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^R_{s,t}(\pi_*(N), T) \Longrightarrow \pi_{s+t}(N \wedge_{HR} HT)$$

and the flatness of T. Therefore $N \wedge_{HR} HT \simeq *$ implies $\pi_*(N) \otimes_R T = 0$, which in turn implies that $\pi_*(N) = 0$ by the faithfulness of T. Thus $N \simeq *$ and $HR \to HT$ is faithful. □

Question 4.3.6. Is every E-local G-Galois extension $A \to B$ of commutative S-algebras faithful?

By Corollary 6.3.4 (or Lemma 6.4.3) the answer is yes when the order of G is invertible in $\pi_0(A)$, but in some sense this is the less interesting case.

In the case $E = K(n)$, it is very easy [HSt99, 7.6] to be faithful over $A = L_{K(n)}S$.

Lemma 4.3.7. *In the $K(n)$-local category, every non-trivial S-module is faithful over $L_{K(n)}S$.*

Proof. Let M and N be $K(n)$-local spectra, considered as modules over $L_{K(n)}S$. From the Künneth formula

$$K(n)_*(M \wedge_{L_{K(n)}S} N) \cong K(n)_*(M) \otimes_{K(n)_*} K(n)_*(N)$$

it follows that if $L_{K(n)}(M \wedge_{L_{K(n)}S} N) \simeq *$ then $K(n)_*(M) = 0$ or $K(n)_*(N) = 0$, since $K(n)_*$ is a graded field. So if M is non-trivial, we must have $N \simeq *$. Thus such an M is faithful. □

5. Examples of Galois extensions

In this chapter we catalog a variety of examples of Galois extensions, some global and some local, as indicated by the section headings.

5.1. Trivial extensions.

Let E be any S-module and work E-locally. For each cofibrant commutative S-algebra A and stably dualizable group G there is a *trivial G-Galois extension* from A to $B = F(G_+, A)$, given by the parametrized *diagonal map*

$$\pi^\#: A \to F(G_+, A)$$

that is functionally dual to the collapse map $\pi: G \to \{e\}$. Here G acts from the left on $F(G_+, A)$ by right multiplication in the source. More precisely, B is a functorial cofibrant replacement of $F(G_+, A)$ in the category of commutative A-algebras, which inherits the G-action by functoriality of the cofibrant replacement. When G is discrete we can write this extension as $\Delta: A \to \prod_G A$.

It is clear that $i: A \to B^{hG} = F(G_+, A)^{hG}$ is a weak equivalence, since $(G_+)_{hG} \simeq \{e\}_+$, and that $h: B \wedge_A B = F(G_+, A) \wedge_A F(G_+, A) \to F(G_+ \wedge G_+, A) \cong F(G_+, B)$ is a weak equivalence, since G is stably dualizable.

The trivial G-Galois extension admits an A-module retraction $F(G_+, A) \to A$ functionally dual to the inclusion $\{e\} \to G$, so $\pi^\#: A \to F(G_+, A)$ is always faithful.

For any G-Galois extension $A \to B$, there is an induced G-Galois extension $B \cong B \wedge_A A \to B \wedge_A B$ (see Proposition 6.2.1 and Lemma 7.1.3 below), and the map $h: B \wedge_A B \to F(G_+, B)$ exhibits an equivalence between this self-induced extension and the trivial G-Galois extension $\pi^\#: B \to F(G_+, B)$.

5.2. Eilenberg–Mac Lane spectra.

Let $E = S$. By Proposition 4.2.1 and Lemma 4.3.5, for each finite G-Galois extension $R \to T$ of commutative rings the induced map of Eilenberg–Mac Lane commutative S-algebras $HR \to HT$ is a faithful G-Galois extension. Proposition 4.2.1 also contains a converse to this statement.

5.3. Real and complex topological K-theory.

Let $E = S$, and let KO and KU be the real and complex topological K-theory spectra, respectively. Their connective versions ko and ku can be realized as the commutative S-algebras associated to the bipermutative topological categories of finite dimensional real and complex inner product spaces, respectively [May77, VI and VII]. The periodic commutative S-algebras KO and KU are obtained from these by Bousfield localization, in the ko- or ku-module categories, by [EKMM97, VIII.4.3].

The complexification functor from real to complex inner product spaces defines maps $c\colon ko \to ku$ and $c\colon KO \to KU$ of commutative S-algebras, and complex conjugation at the categorical level defines a ko-algebra self map $t\colon ku \to ku$ and a KO-algebra self map $t\colon KU \to KU$. Another name for t is the Adams operation ψ^{-1}. Complex conjugation is an involution, so $t^2 = 1$ is the identity in both cases. We therefore have an action by $G = \{e, t\} \cong \mathbb{Z}/2$ on KU through KO-algebra maps, and can make functorial cofibrant replacements to keep this property, while making KO cofibrant as a commutative S-algebra and KU cofibrant as a commutative KO-algebra.

Proposition 5.3.1. *The complexification map $c\colon KO \to KU$ is a faithful $\mathbb{Z}/2$-Galois extension, i.e., a global quadratic extension.*

See also Example 6.4.4 for more about this extension.

Proof. The claim that $i\colon KO \to KU^{h\mathbb{Z}/2}$ is a weak equivalence is well-known to follow from [At66]. We outline a proof in terms of the homotopy fixed point spectral sequence

$$E^2_{s,t} = H^{-s}(\mathbb{Z}/2; \pi_t KU) \Longrightarrow \pi_{s+t}(KU^{h\mathbb{Z}/2})\,.$$

Here $\pi_* KU = \mathbb{Z}[u^{\pm 1}]$ with $|u| = 2$, $t \in \mathbb{Z}/2$ acts by $t(u) = -u$ and

$$E^2_{**} = \mathbb{Z}[a, u^{\pm 2}]/(2a)$$

with $a \in E^2_{-1,2} = H^1(\mathbb{Z}/2; \mathbb{Z}\{u\}) \cong \mathbb{Z}/2$. A computation with the Adams e-invariant shows that i takes the generator $\eta \in \pi_1 KO$ to a class represented by $a \in E^\infty_{-1,2}$, so $\eta^3 = 0 \in \pi_3 KO$ implies that $a^3 \in E^2_{-3,6}$ is a boundary. The only possibility for this is that $d^3(u^2) = a^3$, leaving

$$E^4_{**} = E^\infty_{**} = \mathbb{Z}[a, b, u^{\pm 4}]/(2a, a^3, ab, b^2 = 4u^4)\,.$$

This abutment is isomorphic to $\pi_* KO$, and the graded ring structure implies that $\pi_*(i)$ is indeed an isomorphism.

To show that $h\colon KU \wedge_{KO} KU \to \prod_{\mathbb{Z}/2} KU$ is a weak equivalence, we use the Bott periodicity cofiber sequence

$$\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{\partial} \Sigma^2 KO \tag{5.3.2}$$

of KO-modules and module maps, up to an implicit weak equivalence between the homotopy cofiber of c and $\Sigma^2 KO$. It is the spectrum level version of the homotopy equivalence $\Omega(U/O) \simeq \mathbb{Z} \times BU$, and is sometimes stated as an equivalence $KU \simeq KO \wedge C_\eta$. Here η is given by smashing with the stable Hopf map $\eta \colon S^1 \to S^0$, and ∂ is characterized by $\partial \circ \beta \simeq \Sigma^2 r \colon \Sigma^2 KU \to \Sigma^2 KO$, where $\beta \colon \Sigma^2 KU \to KU$ is the Bott equivalence and $r \colon KU \to KO$ is the realification map. We could write $\partial = \Sigma^2 r \circ \beta^{-1}$ in $\mathcal{D}_{KO}$.

Inducing (5.3.2) up along $c \colon KO \to KU$, we obtain the upper row in the following map of horizontal cofiber sequences

$$\begin{tikzcd} KU \wedge_{KO} KO \arrow[r, "1\wedge c"] \arrow[d, "\cong"'] & KU \wedge_{KO} KU \arrow[r, "1\wedge\partial"] \arrow[d, "h"] & KU \wedge_{KO} \Sigma^2 KO \arrow[d, "\beta", "\simeq"'] \\ KU \arrow[r, "\Delta"] & \prod_{\mathbb{Z}/2} KU \arrow[r, "\delta"] & KU \end{tikzcd} \tag{5.3.3}$$

of KU-modules and module maps, up to another implicit identification of the homotopy cofiber of Δ with KU. Here h is the canonical map, Δ is the diagonal inclusion (so the lower row contains the trivial $\mathbb{Z}/2$-Galois extension of KU), β is the Bott equivalence $KU \wedge_{KO} \Sigma^2 KO \cong \Sigma^2 KU \to KU$, and the difference map δ is the difference of the two projections from $\prod_{\mathbb{Z}/2} KU$, indexed by the elements of $\{e, t\} \cong \mathbb{Z}/2$, written multiplicatively.

The left hand square commutes strictly, since $\mathbb{Z}/2$ acts on KU through KO-algebra maps. To see that the right hand square commutes up to KU-module homotopy, it suffices to prove this after precomposing with the weak equivalence $1 \wedge \beta \colon KU \wedge_{KO} \Sigma^2 KU \to KU \wedge_{KO} KU$. To show that the two resulting KU-module maps $KU \wedge_{KO} \Sigma^2 KU \to KU$ are homotopic, it suffices by adjunction to show that the restricted KO-module maps $\Sigma^2 KU \to KU$ are homotopic. This is then the computation

$$\beta \circ \Sigma^2 c \circ \Sigma^2 r = \delta \circ h \circ (c \wedge \beta)$$

in $\mathcal{D}_{KO}$, which follows directly from $\delta \circ h = \mu - \mu \circ (1 \wedge t) = \mu(1 \wedge (1 - t))$ and the well-known relations $c \circ r = 1 + t$ and $\beta \circ \Sigma^2(1 + t) = (1 - t) \circ \beta$.

Finally, $c \colon KO \to KU$ is faithful. For if N is a KO-module such that $N \wedge_{KO} KU \simeq *$, then applying $N \wedge_{KO} (-)$ to (5.3.2) gives a cofiber sequence

$$\Sigma N \xrightarrow{\eta} N \to N \wedge_{KO} KU \to \Sigma^2 N\,.$$

The assumption that $N \wedge_{KO} KU \simeq *$ implies that $\eta \colon \Sigma N \to N$ is a weak equivalence. But η is also nilpotent, since $\eta^4 = 0 \in \pi_4(S)$, so we must have $N \simeq *$. Therefore KU is faithful over KO. $\square$

The use of nilpotency in this argument may be suggestive of what could in general be required to answer Question 4.3.6. We note that the maps $i \colon ko \to ku^{h\mathbb{Z}/2}$ and $h \colon ku \wedge_{ko} ku \to \prod_{\mathbb{Z}/2} ku$ both fail to be weak equivalences. The homotopy cofiber of i is $\bigvee_{j<0} \Sigma^{4j} H\mathbb{Z}/2$, and the homotopy cofiber of h is $H\mathbb{Z}$, as is easily seen by adapting the arguments above. So $i \colon ko \to ku$ is not Galois.

5.4. The Morava change-of-rings theorem.

In this section we fix a rational prime p and a natural number n, and work locally with respect to the n-th p-primary Morava K-theory $K(n)$. The work of Devinatz and Hopkins [DH04] reinterprets the Morava change-of-rings theorem [Mo85, 0.3.3] as giving a weak equivalence

$$L_{K(n)}S \simeq E_n^{h\mathbb{G}_n}\,.$$

We will regard this as a fundamentally important example of a $K(n)$-local pro-Galois extension $L_{K(n)}S \to E_n$ of commutative S-algebras. See Definition 8.1.1 for the precise notion of a pro-Galois extension, which makes most sense after some of the basic Galois theory has been developed in Chapter 7.

5.4.1. The Lubin–Tate spectra.

Recall that E_n is the n-th p-primary even periodic Lubin–Tate spectrum, for which

$$\pi_0(E_n) = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$$

($\mathbb{W}(-)$ denotes the ring of p-typical Witt vectors) and $\pi_*(E_n) = \pi_0(E_n)[u^{\pm 1}]$. Related theories were studied by Morava [Mo79], Rudjak [Ru75] and Baker–Würgler [BW89], but in this precise form they seem to have been first considered by Hopkins and Miller [HG94], [Re98].

The height n Honda formal group law Γ_n is defined over $\mathbb{F}_p$ and is characterized by its p-series $[p]_n(x) = x^{p^n}$. Its Lubin–Tate deformation $\widetilde{\Gamma}_n$ over $\mathbb{F}_{p^n}$ is the universal formal group law over a complete local ring with residue field an extension of $\mathbb{F}_{p^n}$, whose reduction to the residue field equals the corresponding extension of Γ_n. In this case the universal complete local ring equals $\pi_0(E_n)$, with maximal ideal $(p, u_1, \dots, u_{n-1})$ and residue field $\mathbb{F}_{p^n}$. The Lubin–Tate spectrum E_n is (at first) the $K(n)$-local complex oriented commutative ring spectrum that represents the resulting Landweber exact homology theory $(E_n)_*(X) = \pi_*(E_n) \otimes_{\pi_*(MU)} MU_*(X)$.

More generally, we can consider Γ_n as a formal group law over the algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$. Its universal deformation is then defined over the complete local ring

$$\pi_0(E_n^{nr}) = \mathbb{W}(\bar{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]]$$

and there is a similar $K(n)$-local complex oriented commutative ring spectrum E_n^{nr} with $\pi_*(E_n^{nr}) = \pi_0(E_n^{nr})[u^{\pm 1}]$. The superscript "nr" is short for "non ramifiée", indicating that $\mathbb{W}(\bar{\mathbb{F}}_p)$ is the p-adic completion of the maximal unramified extension $\operatorname{colim}_f \mathbb{W}(\mathbb{F}_{p^f})$ of $\mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$. (The infinite product defining p-typical Witt vectors only commutes with the colimit over f after completion.)

5.4.2. The extended Morava stabilizer group.

The profinite Morava stabilizer group $\mathbb{S}_n = \operatorname{Aut}(\Gamma_n/\mathbb{F}_{p^n})$ of automorphisms defined over $\mathbb{F}_{p^n}$ of the formal group law Γ_n (see [Ra86, §A2.2, §6.2]), and the finite Galois group $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$ of the extension $\mathbb{F}_p \subset \mathbb{F}_{p^n}$, both act on the universal deformation $\widetilde{\Gamma}_n$, and thus on $\pi_*(E_n)$, by the universal property. These actions combine to one by the profinite semi-direct product $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}$. By the Hopkins–Miller [Re98] and Goerss–Hopkins theory [GH04, §7] the ring spectrum

E_n admits the structure of a commutative S-algebra, up to a contractible choice. Furthermore, the extended Morava stabilizer group $\mathbb{G}_n$ acts on E_n through commutative S-algebra maps, again up to contractible choice. However, these actions through commutative S-algebras do not take into account the profinite topology on $\mathbb{G}_n$, but rather treat $\mathbb{G}_n$ as a discrete group.

It is known by recent work of Daniel G. Davis [Da:h], that the profinite group $\mathbb{G}_n$ acts continuously on E_n in the category of $K(n)$-local S-modules, but only when E_n is reconsidered as a pro-object of discrete $\mathbb{G}_n$-module spectra, where the terms have coefficient groups of the form $\pi_*(E_n)/I_k$ for a suitable descending sequence of ideals $\{I_k\}_k$ with $\bigcap_k I_k = 0$. Presently, this kind of limit presentation is not available in the context of commutative S-algebras. Hopkins has suggested that a weaker form of structured commutativity, in terms of pro-spectra, may instead be available.

More generally, the Morava stabilizer group $\mathbb{S}_n$ and the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ (the Prüfer ring) of $\mathbb{F}_p$ both act on the universal deformation of Γ_n over the algebraic closure $\bar{\mathbb{F}}_p$, and thus on $\pi_*(E_n^{nr})$ by the universal property. These combine to an action by the profinite group

$$\mathbb{G}_n^{nr} = \mathbb{S}_n \rtimes \hat{\mathbb{Z}}\,.$$

Note that the (conjugation) action by $\hat{\mathbb{Z}}$ on $\mathbb{S}_n$ factors through the quotient $\hat{\mathbb{Z}} \to \mathbb{Z}/n = \mathrm{Gal}$, since all the automorphisms of the height n Honda formal group law are already defined over $\mathbb{F}_{p^n}$ [Ra86, A2.2.20(a)]. The Goerss–Hopkins theory cited above again implies that E_n^{nr} is a commutative S-algebra, and the extended Morava stabilizer group $\mathbb{G}_n^{nr}$ acts on E_n^{nr} through commutative S-algebra maps, up to contractible choices.

5.4.3. Intermediate S-algebras.

In the Galois theory for fields, the intermediate fields $F \subset E \subset \bar{F}$ correspond bijectively (via $E = (\bar{F})^K$ and $K = G_E$) to the closed subgroups $K \subset G_F$ of the absolute Galois group with the Krull topology, and the finite field extensions $F \subset E$ correspond to the open subgroups $U \subset G_F$. Note that in this topology, the open subgroups are exactly the closed subgroups of finite index. Furthermore, G_F acts continuously on $\bar{F}$ with the discrete topology, so $\bar{F}$ is the union over the open subgroups U of the fixed fields $(\bar{F})^U$.

By analogy, it is desirable to construct intermediate $K(n)$-local commutative S-algebras E_n^{hK} for every closed subgroup $K \subset \mathbb{G}_n$ in the profinite topology. If E_n were a discrete $\mathbb{G}_n$-module spectrum, this could be done by the usual definition $E_n^{hK} = F(EK_+, E_n)^K$, and indeed, for finite (and thus discrete) subgroups $K \subset \mathbb{G}_n$ the restricted K-action is continuous, E_n is a discrete K-module spectrum and E_n^{hK} can well be defined in this way. The maximal finite subgroups $M \subset \mathbb{G}_n$ were classified by Hewett [He95, 1.3, 1.4]. When M is unique up to conjugacy, E_n^{hM} is known as the *n-th higher real K-theory* spectrum EO_n of Hopkins and Miller. Such uniqueness holds for p odd when $n = (p-1)k$ with k prime to p, and for $p = 2$ when $n = 2k$ with k an odd natural number, by *loc. cit.*

However, as recalled in the previous subsection, the spectrum E_n is not itself a discrete $\mathbb{G}_n$-spectrum, but only an inverse limit of such, i.e., a pro-discrete $\mathbb{G}_n$-spectrum. The homotopy invariant way to form homotopy fixed points of such objects is to take the ordinary continuous homotopy fixed points for the profinite group acting discretely at each stage in the limit system, and then to pass to the homotopy limit, if desired. Note that the formation of continuous homotopy fixed points for profinite groups acting on discrete modules involves a colimit indexed over the finite quotients of the profinite group, and does therefore not generally commute with limits. This procedure describes the approach of [Da:h], but it only exhibits the homotopy fixed point spectrum $E_n^{h\mathbb{G}_n}$ as a module spectrum, and not as an algebra spectrum, precisely because we do not know how to realize E_n as a pro-object of $\mathbb{G}_n$-discrete associative or commutative S-algebras.

Devinatz and Hopkins circumvent this problem by defining $E_n^{h\mathbb{G}_n}$, and more generally E_n^{hU} for each open subgroup $U \subset \mathbb{G}_n$, in a "synthetic" way [DH04, Thm. 1], as the totalization of a suitably rigidified cosimplicial diagram, to obtain a $K(n)$-local commutative S-algebra of the desired homotopy type. In particular, $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S$. (See Section 8.2 for further discussion of the kind of cosimplicial diagram involved, namely the Amitsur complex.) For closed subgroups $K \subset \mathbb{G}_n$ they then define [DH04, Thm. 2]

$$E_n^{hK} = L_{K(n)}(\operatorname*{colim}_i E_n^{hU_iK})$$

where $\{U_i\}_{i=0}^{\infty}$ is a fixed descending sequence of open normal subgroups in $\mathbb{G}_n$ with $\bigcap_{i=0}^{\infty} U_i = \{e\}$, and the colimit is the homotopy colimit in commutative S-algebras. For finite subgroups $K \subset \mathbb{G}_n$ the synthetic construction agrees [DH04, Thm. 3] with the "natural" definition of E_n^{hK} as $F(EK_+, E_n)^K$.

Ethan Devinatz [De05] then proceeds to compare the commutative S-algebras E_n^{hK} and E_n^{hH} for closed subgroups K and H of $\mathbb{G}_n$ with H normal in K. There is a well-defined action by the quotient group K/H on E_n^{hH} through commutative S-algebra maps, in the $K(n)$-local category [De05, §3].

Theorem 5.4.4 (Devinatz–Hopkins). *(a) For each pair of closed subgroups $H \subset K \subset \mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}$ with H normal and of finite index in K, the map $E_n^{hK} \to E_n^{hH}$ is a $K(n)$-local K/H-Galois extension.*

(b) In particular, for each finite subgroup $K \subset \mathbb{G}_n$ the map $E_n^{hK} \to E_n$ is a $K(n)$-local K-Galois extension.

(c) Likewise, for each open normal subgroup $U \subset \mathbb{G}_n$ (necessarily of finite index) the map

$$L_{K(n)}S \to E_n^{hU}$$

is a $K(n)$-local $\mathbb{G}_n/U$-Galois extension.

(d) A choice of a descending sequence $\{U_i\}_i$ of open normal subgroups of $\mathbb{G}_n$, with $\bigcap_i U_i = \{e\}$, exhibits

$$L_{K(n)}S \to E_n$$

as a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension, in view of the weak equivalence

$$L_{K(n)}(\operatorname*{colim}_i E_n^{hU_i}) \xrightarrow{\simeq} E_n\,.$$

Proof. (a) Let $A = E_n^{hK}$, $B = E_n^{hH}$ and $G = K/H$ (which is finite and discrete). By [De05, Prop. 2.3, Thm. 3.1 and Thm. A.1] the homotopy fixed point spectral sequence for $\pi_*(B^{hG})$ agrees with a strongly convergent $K(n)_*$-local Adams spectral sequence converging to $\pi_*(A)$. So $i\colon A \to B^{hG}$ is a weak equivalence. By [De05, Cor. 3.9] the natural map $h\colon L_{K(n)}(B \wedge_A B) \to F(G_+, B)$ induces an isomorphism on homotopy groups.

Parts (b) and (c) are special cases of (a). Part (d) is contained in [DH04, Thm. 3(i)]. □

It would be nice to extend the statement of this theorem to the case when H is normal and closed, but not necessarily of finite index, in K.

For $n = 2$ and $p = 2$, the Morava stabilizer group $\mathbb{S}_2$ is the group of units in the maximal order in the quaternion algebra $\mathbb{Q}_2\{1, i, j, k\}$, and its maximal finite subgroup is the binary tetrahedral group $\hat{A}_4 = Q_8 \rtimes \mathbb{Z}/3$ of order 24, containing the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and the 16 other elements $(\pm 1 \pm i \pm j \pm k)/2$. See [CF67, pp. 137–138], [Ra86, 6.3.27]. The maximal finite subgroup of $\mathbb{G}_2$ is $G_{48} = \hat{A}_4 \rtimes \mathbb{Z}/2$, and $EO_2 = E_2^{hG_{48}}$ is the $K(2)$-localization of the connective spectrum eo_2 with $H^*(eo_2; \mathbb{F}_2) \cong A//A_2$ as a module over the Steenrod algebra, which is related to the topological modular forms spectrum *tmf* [Hop02, §3.5].

Proposition 5.4.5. *At $p = 2$, the natural map $EO_2 \to E_2$ is a $K(2)$-local faithful $G_{48} = \hat{A}_4 \rtimes \mathbb{Z}/2$-Galois extension.*

Proof. This follows from Theorem 5.4.4(b) above and Proposition 5.4.9(b) below, but we would also like to indicate a direct proof of faithfulness, using results of Hopkins and Mahowald [HM98]. There is a finite CW spectrum C_γ obtained as the mapping cone of a map

$$\gamma\colon \Sigma^5 C_\eta \wedge C_\nu \to C_\eta \wedge C_\nu\,,$$

such that $H^*(C_\gamma; \mathbb{F}_2) \cong DA(1) \cong A(2)//E(2)$ is the "double" of $A(1) = \langle Sq^1, Sq^2 \rangle$. The spectrum C_γ and the self-map γ can be obtained by a construction analogous to that of the spectrum A_1 and the map $v_1\colon \Sigma^2 Y \to Y$ in [DM81, pp. 619–620], but replacing all the real projective spaces occurring there by complex projective spaces. Furthermore, there is a weak equivalence $eo_2 \wedge C_\gamma \simeq BP\langle 2\rangle$ that realizes the isomorphism $A//A(2) \otimes A(2)//E(2) \cong A//E(2) \cong H^*(BP\langle 2\rangle; \mathbb{F}_2)$. Applying $K(2)$-localization yields

$$EO_2 \wedge C_\gamma \simeq \widehat{E(2)}\,,$$

in the notation of 5.4.7, using that $BP\langle 2\rangle \to v_2^{-1}BP\langle 2\rangle = E(2)$ is a $K(2)_*$-equivalence. Since η, ν and γ are all nilpotent (for $\eta \in \pi_1(S)$ and $\nu \in \pi_3(S)$ this is well-known; for γ it can be deduced from the Devinatz–Hopkins–Smith nilpotence theorem [DHS88, Cor. 2]), it follows as in the proof of Proposition 5.3.1 that $EO_2 \to \widehat{E(2)}$ is faithful. And $\widehat{E(2)} \to E_2$ is faithful by the elementary Proposition 5.4.9(a).

$$\begin{tikzcd}
 & E_2^{\mathrm{Gal}} \arrow[r, "\mathbb{Z}/2"] & E_2 \\
EO_2 \arrow[ur, "\hat{A}_4"] \arrow[r] & \widehat{E(2)} \arrow[u, "\mathbb{F}_4^*"'] \arrow[ur, "\Sigma_3"'] &
\end{tikzcd}$$

□

5.4.6. Adjoining roots of unity.

Including the maximal unramified extensions into this picture, we have the following diagram of $K(n)$-local extensions. The groups label Galois (or pro-Galois) extensions.

$$\begin{tikzcd}
E_n^{\mathrm{Gal}} \arrow[r, "\mathrm{Gal}"] \arrow[rr, bend left, "\hat{\mathbb{Z}}"] & E_n \arrow[r, "n\hat{\mathbb{Z}}"] & E_n^{nr} \\
EO_n \arrow[u] \arrow[ur, "M"] & & \\
L_{K(n)}S \arrow[u] \arrow[uur, "\mathbb{G}_n"'] \arrow[r, "\mathrm{Gal}"] \arrow[rr, bend right, "\hat{\mathbb{Z}}"'] & E_n^{h\mathbb{S}_n} \arrow[r, "n\hat{\mathbb{Z}}"] \arrow[uu, "\mathbb{S}_n"'] & (E_n^{nr})^{h\mathbb{S}_n} \arrow[uu, "\mathbb{S}_n"']
\end{tikzcd}$$

The maximal extension $L_{K(n)}S \to E_n^{nr}$ is $K(n)$-locally pro-$\mathbb{G}_n^{nr}$-Galois.

The $\hat{\mathbb{Z}}$-extension along the bottom is that obtained by adjoining all roots of unity of order prime to p to the p-complete commutative S-algebra $L_{K(n)}S$. We might write $E_n^{h\mathbb{S}_n} = L_{K(n)}S(\mu_{p^n-1})$ and $(E_n^{nr})^{h\mathbb{S}_n} = L_{K(n)}S(\mu_{\infty,p})$, where μ_m denotes the group of m-th order roots of unity and $\mu_{\infty,p} = \operatorname{colim}_{p\nmid m} \mu_m$ denotes the group of all roots of unity of order prime to p. Note that in the latter case, the infinite colimit of spectra must be implicitly $K(n)$-completed. Similarly, $E_n = E_n^{\mathrm{Gal}}(\mu_{p^n-1})$ and $E_n^{nr} = E_n^{\mathrm{Gal}}(\mu_{\infty,p})$.

The process of adjoining m-th roots of unity makes sense when applied to a p-local commutative S-algebra A, for $p \nmid m$, following Roland Schwänzl, Rainer Vogt and Waldhausen [SVW99], since $A(\mu_m)$ can be obtained from the group A-algebra $A[C_m] = A \wedge C_{m+}$ of the cyclic group of order m by localizing with respect to a p-locally defined idempotent. Likewise, adjoining an m-th root of unity to a p-complete commutative S-algebra A, for $m = p^f - 1$, can be achieved by localizing with respect to a further idempotent. The situation is analogous to how $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_m)$ splits as a product of copies of $\mathbb{Q}_p(\mu_m)$, when $m = p^f - 1$. For more on the process of adjoining roots of unity to S-algebras, see [La03, 3.4] in the associative case and [BR:r, 2.2.5 and 2.2.8] in the commutative case.

These observations may justify thinking of the projection $d\colon \mathbb{G}_n^{nr} = \mathbb{S}_n \rtimes \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$ as the *degree map* of a $K(n)$-local class field theory for structured ring spectra [Ne99, §IV.4].

5.4.7. Faithfulness.

Let $\widehat{E(n)} = L_{K(n)}E(n)$ be the $K(n)$-localization of the Johnson–Wilson spectrum $E(n)$ from 3.2.2, called Morava E-theory in [HSt99]. By [BW89, 4.1] or [HSt99, §1.1, 5.2] it has coefficients

$$\pi_* \widehat{E(n)} = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]^{\wedge}_{I_n} ,$$

where $I_n = (p, v_1, \dots, v_{n-1})$. The spectrum $\widehat{E(n)}$ was proved to be an associative S-algebra in [Bak91], and is in fact a commutative S-algebra by the homotopy fixed point description in Proposition 5.4.9(a) below.

Theorem 5.4.8 (Hovey–Strickland). *$L_{K(n)}S$ is contained in the thick subcategory of $K(n)$-local spectra generated by $\widehat{E(n)}$, so $\widehat{E(n)}$ is a faithful $L_{K(n)}S$-module in the $K(n)$-local category.*

Proof. The first claim is contained in the proof of [HSt99, 8.9], which relies heavily on the construction by Jeff Smith of a suitable finite p-local spectrum X, as explained in [Ra92, §8.3]. The second claim follows from the first, but also much more easily from Lemma 4.3.7. □

Proposition 5.4.9. *(a) The $K(n)$-local Galois extension $\widehat{E(n)} \to E_n$ and the $K(n)$-local pro-Galois extension $L_{K(n)}S \to E_n$ are both faithful.*

(b) For each pair of closed subgroups $H \subset K \subset \mathbb{G}_n$, with H normal and of finite index in K, the $K(n)$-local K/H-Galois extension $E_n^{hK} \to E_n^{hH}$ is faithful.

Proof. (a) There is a finite subgroup $\mathbb{F}_{p^n}^*$ of $\mathbb{S}_n$ such that for $K = \mathbb{F}_{p^n}^* \rtimes \mathrm{Gal}$ we have $\widehat{E(n)} \simeq E_n^{hK}$. In more detail, $\mathbb{S}_n$ contains the unit group $\mathbb{W}(\mathbb{F}_{p^n})^*$ [Ra86, A2.2.17], whose torsion subgroup reduces isomorphically to $\mathbb{F}_{p^n}^*$. For an element of finite order $\omega \in \mathbb{W}(\mathbb{F}_{p^n})^*$, with mod p reduction $\bar\omega \in \mathbb{F}_{p^n}^*$, the linear formal power series $g(x) = \bar\omega x$ defines an automorphism of Γ_n, i.e., an element $g \in \mathbb{S}_n$, which acts on $\pi_*(E_n)$ by $g(u) = \omega u$ and $g(uu_k) = \omega^{p^k} uu_k$ for $1 \le k < n$ by [DH95, 3.3, 4.4], leaving $v_n = u^{1-p^n}$ and $v_k = u^{1-p^k} u_k$ invariant. Thus $\pi_* E_n^{\mathrm{Gal}} = \mathbb{Z}_p[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$ and $\pi_* E_n^{hK}$ is the I_n-adic completion of $\pi_* E(n)$.

Then for any spectrum X,

$$(E_n)_*^\vee(X) \cong \pi_* E_n \otimes_{\pi_* \widehat{E(n)}} \widehat{E(n)}_*^\vee(X)$$

with $\pi_* E_n$ a free module of rank $|K| = (p^n - 1)n$ over $\pi_* \widehat{E(n)}$. Here we are using the notation $(E_n)_*^\vee(X) = \pi_* L_{K(n)}(E_n \wedge X)$, and similarly for $\widehat{E(n)}$, of [HSt99, 8.3]. It follows easily from this formula that $\widehat{E(n)} \to E_n$ is faithful in the $K(n)$-local category.

In combination with 5.4.8 this also shows that the composite extension $L_{K(n)}S \to \widehat{E(n)} \to E_n$ is faithful, but Lemma 4.3.7 provides a much easier argument.

(b) The second result follows by faithful base change along $\phi\colon L_{K(n)}S \to E_n$. There is a commutative diagram (for H and K as in the statement)

$$\begin{tikzcd}
E_n^{hH} \arrow[r] & L_{K(n)}(E_n \wedge E_n^{hH}) \\
E_n^{hK} \arrow[u, "\psi"] \arrow[r, "1\wedge\phi"] & L_{K(n)}(E_n \wedge E_n^{hK}) \arrow[u, "1\wedge\psi"'] \\
L_{K(n)}S \arrow[u] \arrow[r, "\phi"] & E_n \arrow[u]
\end{tikzcd}$$

where the squares are pushouts in the category of $K(n)$-local commutative S-algebras. By the Morava change-of-rings theorem and [DH04, Thm. 1(iii)],

$$\pi_* L_{K(n)}(E_n \wedge E_n^{hH}) \cong \mathrm{Map}(\mathbb{G}_n/H, \pi_* E_n)$$

and

$$\pi_* L_{K(n)}(E_n \wedge E_n^{hK}) \cong \mathrm{Map}(\mathbb{G}_n/K, \pi_* E_n)\,.$$

See also the proof of Theorem 7.2.3 below. Here Map denotes the unbased continuous maps with respect to the profinite topologies on $\mathbb{G}_n/H$, $\mathbb{G}_n/K$ and $\pi_* E_n$ (in each degree). Note that K/H is a finite group acting freely on the Hausdorff space $\mathbb{G}_n/H$, with orbit space $\mathbb{G}_n/K$, so $\pi\colon \mathbb{G}_n/H \to \mathbb{G}_n/K$ is a regular K/H-covering space. We claim that it admits a continuous section $\sigma\colon \mathbb{G}_n/K \to \mathbb{G}_n/H$, so that there is a homeomorphism $K/H \times \mathbb{G}_n/K \to \mathbb{G}_n/H$, and

$$\mathrm{Map}(\mathbb{G}_n/H, \pi_* E_n) \cong \prod_{K/H} \mathrm{Map}(\mathbb{G}_n/K, \pi_* E_n)\,.$$

Thus $\pi_* L_{K(n)}(E_n \wedge E_n^{hH})$ is a free module of rank $|K/H|$ over $\pi_* L_{K(n)}(E_n \wedge E_n^{hK})$, so that $L_{K(n)}(E_n \wedge E_n^{hH})$ is faithful over $L_{K(n)}(E_n \wedge E_n^{hK})$. The map $1 \wedge \phi$ is obtained by base change from ϕ, which is faithful by (a), and is therefore faithful by Lemma 4.3.3, so $\psi\colon E_n^{hK} \to E_n^{hH}$ is faithful by Lemma 4.3.4.

It remains to verify the claim. Let $\{U_i\}_{i=0}^\infty$ be a descending sequence of open normal subgroups of $\mathbb{G}_n$, with trivial intersection as above. Then $U_i H$ is normal of finite index in $U_i K$, K/H surjects to $U_i K/U_i H$ and there is a regular covering space $\pi_i\colon \mathbb{G}_n/U_i H \to \mathbb{G}_n/U_i K$, for each i. We have the following commutative diagram for $i < j$:

$$\begin{tikzcd}
K/H \arrow[r] \arrow[d] & U_j K/U_j H \arrow[r] \arrow[d] & U_i K/U_i H \arrow[d] \\
\mathbb{G}_n/H \arrow[r] \arrow[d, "\pi"'] & \mathbb{G}_n/U_j H \arrow[r] \arrow[d, "\pi_j"] & \mathbb{G}_n/U_i H \arrow[d, "\pi_i"] \\
\mathbb{G}_n/K \arrow[r] & \mathbb{G}_n/U_j K \arrow[r] & \mathbb{G}_n/U_i K
\end{tikzcd}$$

Since K/H is finite, the surjections $U_j K/U_j H \to U_i K/U_i H$ are isomorphisms for all sufficiently large i and j, say for $i, j \geq i_0$, and then π_j is the pullback of π_i along $\mathbb{G}_n/U_j K \to \mathbb{G}_n/U_i K$. Thus any choice of section σ_i to π_i pulls back to a section σ_j of π_j, so that the composite maps $\mathbb{G}_n/K \to \mathbb{G}_n/U_i K \to \mathbb{G}_n/U_i H$ are compatible for all $i \geq i_0$. Their limit defines the continuous section $\sigma\colon \mathbb{G}_n/K \to \mathbb{G}_n/H$. □

5.5. The $K(1)$-local case.

When $n = 1$, the discussion in Section 5.4 reduces to more classical statements about variants of topological K-theory, which we now make explicit, together with a comparison to the even more classical arithmetic theory of abelian extensions of $\mathbb{Q}_p$ and $\mathbb{Q}$.

5.5.1. p-complete topological K-theory.

Mod p complex topological K-theory, with $\pi_*(KU/p) = \mathbb{F}_p[u^{\pm 1}]$, splits as

$$KU/p \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} K(1)$$

where $\pi_* K(1) = \mathbb{F}_p[v_1^{\pm 1}]$. Bousfield $K(1)$-localization equals Bousfield KU/p-localization, which in turn equals Bousfield KU-localization followed by p-adic completion: $L_{K(1)}X = L_{KU/p}X = (L_{KU}X)^\wedge_p$ [Bo79, 2.11].

The height 1 Honda formal group law over $\mathbb{F}_p$ is the multiplicative one: $\Gamma_1(x,y) = x+y+xy$, its universal deformation $\widetilde{\Gamma}_1$ is the multiplicative formal group law over $\mathbb{Z}_p$, and the Lubin–Tate spectrum E_1 equals p-completed complex topological K-theory $KU^\wedge_p$ with $\pi_*(KU^\wedge_p) = \mathbb{Z}_p[u^{\pm 1}]$. The Morava stabilizer group $\mathbb{G}_1 = \mathbb{S}_1$ is the group of p-adic units $\mathbb{Z}_p^*$, with its profinite topology, and $k \in \mathbb{Z}_p^*$ acts on the commutative S-algebra $KU^\wedge_p$ by the p-adic Adams operation

$$\psi^k \colon KU^\wedge_p \to KU^\wedge_p \,.$$

On homotopy, $\psi^k(u) = ku$.

5.5.2. Subalgebras.

The homotopy fixed point spectrum $E_1^{h\mathbb{G}_n} = (KU^\wedge_p)^{h\mathbb{Z}_p^*}$ is the p-complete (non-connective) image-of-J spectrum $L_{K(1)}S = J^\wedge_p$, defined for $p = 2$ by the fiber sequence

$$J^\wedge_2 \to KO^\wedge_2 \xrightarrow{\psi^3 - 1} KO^\wedge_2$$

and for p odd by the fiber sequence

$$J^\wedge_p \to KU^\wedge_p \xrightarrow{\psi^r - 1} KU^\wedge_p$$

for r a topological generator of $\mathbb{Z}_p^*$. These identifications of the p-completed KU-localization of S with $J^\wedge_p$ are basically due to Mark Mahowald and Haynes Miller [Bo79, 4.2], respectively. (Adams–Baird and Ravenel went on to identify the p-local KU-localization of S, see [Bo79, 4.3].)

The Morava stabilizer group $\mathbb{S}_1 = \mathbb{Z}_p^*$ is isomorphic to the Galois group of the maximal (totally ramified) p-cyclotomic extension $\mathbb{Q}_p \subset \mathbb{Q}_p(\mu_{p^\infty})$, so the classification of intermediate commutative S-algebras $J_p \to C \to KU^\wedge_p$ of the form $C = (KU^\wedge_p)^{hK}$ for K closed in $\mathbb{Z}_p^*$ is identical to the classification of intermediate fields $\mathbb{Q}_p \subset E \subset \mathbb{Q}_p(\mu_{p^\infty})$. In this way $J^\wedge_p \to KU^\wedge_p$ provides a $K(1)$-local "realization" of the $K(0)$-local extension $\mathbb{Q}_p \to \mathbb{Q}_p(\mu_{p^\infty})$. There are similar $K(n)$-local realizations of the form $L_{K(n)}S \to E_n^{hK}$, when K is the kernel of the determinant/abelianization homomorphism $\mathbb{G}_n \to \mathbb{G}_n^{ab} \to \mathbb{Z}_p^*$ [Ra86, 6.2.6(b)].

When $p = 2$, $\mathbb{Z}_2^* \cong \mathbb{Z}_2 \times \mathbb{Z}/2$, where $\mathbb{Z}_2 \cong 1 + 4\mathbb{Z}_2$ is open of index 2, and $\mathbb{Z}/2 \cong \{\pm 1\} \subset \mathbb{Z}_2^*$ is closed. There are three different subgroups of index 2, namely the topologically generated subgroups $\langle 3\rangle$, $\langle 5\rangle$ and $\langle -1, 9\rangle$. The first of these corresponds to the complex image-of-J spectrum $JU_2^\wedge = (KU_2^\wedge)^{h\langle 3\rangle}$ given by the fiber sequence

$$JU_2^\wedge \to KU_2^\wedge \xrightarrow{\psi^3-1} KU_2^\wedge ,$$

and there is a $K(1)$-local (quadratic) $\mathbb{Z}/2$-Galois extension $c\colon J_2^\wedge \to JU_2^\wedge$, which is compatible with the complexification map $c\colon KO_2^\wedge \to KU_2^\wedge$. See Example 6.2.2 for more on this quadratic extension. The closed subgroup $\mathbb{Z}/2$ of $\mathbb{Z}_2^*$ corresponds to 2-complete real K-theory: $(KU_2^\wedge)^{h\mathbb{Z}/2} \simeq KO_2^\wedge$.

When p is odd, $\mathbb{Z}_p^* \cong \mathbb{Z}_p \times \mathbb{F}_p^*$ is pro-cyclic. Let $r \in \mathbb{Z}_p^*$ be a topological generator, chosen to be a natural number. Then $\mathbb{Z}_p^*$ has a unique open subgroup $\langle r^n\rangle$ of index n, for each integer n of the form $n = p^e d$ with $e \geq 0$ and $d \mid p-1$. In addition, it has the closed subgroups that appear as subgroups of $\mathbb{F}_p^*$. In particular, $\mathbb{Z}_p^*$ has an open subgroup $\mathbb{Z}_p \cong 1 + p\mathbb{Z}_p$ of index $(p-1)$, and a closed subgroup $\mathbb{F}_p^* \subset \mathbb{Z}_p^*$. The latter corresponds to the p-complete Adams summand $L_p^\wedge = (KU_p^\wedge)^{h\mathbb{F}_p^*}$ with $\pi_*(L_p^\wedge) = \mathbb{Z}_p[v_1^{\pm 1}]$. There are $K(1)$-local $\mathbb{F}_p^*$-Galois extensions $J_p^\wedge \to (KU_p^\wedge)^{h\mathbb{Z}_p}$ and $L_p^\wedge \to KU_p^\wedge$. Let us write $F\psi^{r^n} = (KU_p^\wedge)^{h\langle r^n\rangle}$ for the homotopy fixed point spectrum of ψ^{r^n}, which is equivalent to the homotopy fiber of $\psi^{r^n} - 1$. Then there is a $K(1)$-local $\mathbb{Z}/n$-Galois extension

$$J_p^\wedge = F\psi^r \to F\psi^{r^n}$$

for each integer $n = p^e d$ with $d \mid p-1$, as above.

5.5.3. Extensions.

Incorporating the roots of unity of order prime to p, we have the following diagram

$$\begin{tikzcd}
KU_p^\wedge \arrow[r, "\hat{\mathbb{Z}}"] & KU_p^\wedge(\mu_{\infty,p}) \\
J_p^\wedge \arrow[u, "\mathbb{Z}_p^*"] \arrow[ur, "\mathbb{G}_1^{nr}"] \arrow[r, "\hat{\mathbb{Z}}"] & J_p^\wedge(\mu_{\infty,p}) \arrow[u, "\mathbb{Z}_p^*"']
\end{tikzcd}$$

with $E_1^{nr} = KU_p^\wedge(\mu_{\infty,p})$. Here the maximal Galois group $\mathbb{G}_1^{nr} = \mathbb{Z}_p^* \times \hat{\mathbb{Z}}$ is abelian, since $\hat{\mathbb{Z}}$ acts trivially on $\mathbb{S}_1 = \mathbb{Z}_p^*$. It provides a $K(1)$-local realization of the Galois group of the maximal abelian extension $\mathbb{Q}_p \to \mathbb{Q}_p(\mu_\infty)$.

It also appears to be possible to fit the various rational primes together, so as to obtain KU-local realizations of the abelian extensions of the rational field $\mathbb{Q}$ itself. The Galois group $G = \hat{\mathbb{Z}}^*$ of the maximal abelian extension $\mathbb{Q} \to \mathbb{Q}(\mu_\infty)$ contains the Galois group of $\mathbb{Q}_p \to \mathbb{Q}_p(\mu_\infty)$ as the decomposition group D_p of the prime ideal (p). Let $Z_p = \mathbb{Q}(\mu_\infty)^{D_p}$ be the corresponding decomposition field [Ne99, I.9.2].

$$\mathbb{Q} \xrightarrow{G/D_p} Z_p \xrightarrow{D_p} \mathbb{Q}(\mu_\infty) .$$

After base change along $\mathbb{Q} \to \mathbb{Q}_p$ there are weak product splittings [Ne99, II.8.3]

$$\mathbb{Q}_p \otimes_{\mathbb{Q}} Z_p \cong \prod_{G/D_p}{}' \mathbb{Q}_p \qquad \text{and} \qquad \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_\infty) \cong \prod_{G/D_p}{}' \mathbb{Q}_p(\mu_\infty) ,$$

i.e., as colimits of the products over the finite quotients of

$$G/D_p = \hat{\mathbb{Z}}^*/(\mathbb{Z}_p^* \times \hat{\mathbb{Z}}) \cong (\prod_{\ell \neq p} \mathbb{Z}_\ell^*)/\hat{\mathbb{Z}}\,.$$

In the latter profinite quotient, the unit of $\hat{\mathbb{Z}}$ maps diagonally to the class of p in each $\mathbb{Z}_\ell^*$. Hence $G = G_{\mathbb{Q}}^{ab}$ is realized as the Galois group of

$$\mathbb{Q}_p \xrightarrow{G/D_p} \prod_{G/D_p}{}' \mathbb{Q}_p \xrightarrow{D_p} \prod_{G/D_p}{}' \mathbb{Q}_p(\mu_\infty)\,,$$

where the first map is a (pro-)trivial Galois extension.

We can realize the same groups in the $K(1)$-local category, by the two pro-Galois extensions

$$J_p^\wedge \xrightarrow{G/D_p} \prod_{G/D_p}{}' J_p^\wedge \xrightarrow{D_p} \prod_{G/D_p}{}' E_1^{nr}\,.$$

Here the first is the implicitly $K(1)$-localized colimit of the trivial Galois extensions of $J_p^\wedge$, indexed over the finite quotients of G/D_p.

For brevity, let $B_p = \prod'_{G/D_p} E_1^{nr}$. Then $J_p^\wedge \to B_p$ is a $K(1)$-local realization of the maximal abelian extension of $\mathbb{Q}$. It seems plausible to find arithmetic pullback squares

$$\begin{array}{ccc} L_{KU}S & \longrightarrow & \prod_p J_p^\wedge \\ \downarrow & & \downarrow \\ L_0 S & \longrightarrow & L_0 \prod_p J_p^\wedge \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & \prod_p B_p \\ \downarrow & & \downarrow \\ L_0 B & \longrightarrow & L_0 \prod_p B_p \end{array}$$

of commutative S-algebras, so as to get an integral KU-local realization $L_{KU}S \to B$ of the same Galois group. It would be wonderful if analogous (non-abelian) $K(n)$-local constructions for $n \geq 2$ turn out to detect more of the absolute Galois group of $\mathbb{Q}_p$ in $\mathbb{G}_n^{nr}$, or of the absolute Galois group of $\mathbb{Q}$. The paper [Mo05] may be relevant.

5.5.4. p-local topological K-theory.

The p-local complex K-theory spectrum $KU_{(p)}$ is also a commutative S-algebra, and admits an action by the Adams operation ψ^r and its powers through commutative S-algebra maps [BR:g, 9.2]. However, in this case the $E(1)$-local extension

$$KU_{(p)}^{h\langle r\rangle} \to KU_{(p)}^{h\langle r^n\rangle}$$

is not a $\mathbb{Z}/n$-Galois extension. It even fails to be one rationally, i.e., $K(0)$-locally. For

$$\pi_*(KU_{(p)}^{h\langle r^n\rangle}) \otimes \mathbb{Q} \cong E_{\mathbb{Q}}(\zeta_{r^n})$$

is an exterior algebra over $\mathbb{Q}$ on one generator, and $E_{\mathbb{Q}}(\zeta_r) \to E_{\mathbb{Q}}(\zeta_{r^n})$ is an isomorphism, so $\{e\}$-Galois, but not $\mathbb{Z}/n$-Galois. In spite of the relatively rich source of $K(n)$-local Galois extensions, there are ramification phenomena that frequently enter when several chromatic strata are involved.

The idempotent operation $(p-1)^{-1}\sum_{k\in\mathbb{F}_p^*}\psi^k$ on $KU_p^\wedge$ that defines the p-complete Adams summand $L_p^\wedge$ is in fact p-locally defined [Ad69, p. 85], so as to split off the p-local Adams summand $L_{(p)}$ in

$$KU_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} L_{(p)} .$$

However, the p-adic Adams operations ψ^k of finite order, for k in the torsion subgroup $\mathbb{F}_p^* \subset \mathbb{Z}_p^*$, are not defined over $\mathbb{Z}_{(p)}$, since $\psi^k(u) = ku$ on homotopy. Therefore the extension $L_{(p)} \to KU_{(p)}$ only becomes Galois after p-adic completion. This provides an example of an $E(1)$-local étale extension (in the sense of Section 9.4) that does not extend to a Galois extension. Again, this is an instance of $K(0)$-local ramification of the $E(1)$-local prolongation of a, by definition unramified, $K(1)$-local Galois extension. These examples are meant as partial justification for the last paragraph of Remark 3.2.4.

5.6. Cochain S-algebras.

Let G be a topological group and consider a principal G-bundle $\pi\colon P \to X$. Fix a rational prime p and let $A = F(X_+, H\mathbb{F}_p)$ and $B = F(P_+, H\mathbb{F}_p)$ be the mod p cochain $H\mathbb{F}_p$-algebras on X and P, respectively. Note that $\pi_*(A) = H^{-*}(X;\mathbb{F}_p)$ and $\pi_*(B) = H^{-*}(P;\mathbb{F}_p)$. We think of A and B as models for the singular cochain algebras $C^*(X;\mathbb{F}_p)$ and $C^*(P;\mathbb{F}_p)$, in conformance with [DGI:d, §3]. The direct relation between the differential graded E_∞ structure on $C^*(X;\mathbb{F}_p)$ and the commutative S-algebra structure on $A = F(X_+, H\mathbb{F}_p)$ seems not to have been made explicit, however.

The projection π induces a map of commutative $H\mathbb{F}_p$-algebras $A \to B$, the right action of G on P induces a left action of G on B through commutative A-algebra maps, and the weak equivalence $P \times_G EG \to X$ makes its cochain dual $i\colon A \to F((P\times_G EG)_+, H\mathbb{F}_p) \cong B^{hG}$ a weak equivalence. We now investigate when $h\colon B \wedge_A B \to F(G_+, B)$ is a weak equivalence.

The Künneth spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*(A)}_{s,t}(\pi_*(B), \pi_*(B)) \Longrightarrow \pi_{s+t}(B \wedge_A B) \tag{5.6.1}$$

can be derived from the skeleton filtration of the (simplicial) two-sided bar construction

$$B^{H\mathbb{F}_p}(B, A, B)\colon [q] \mapsto B \wedge A^{\wedge q} \wedge B$$

with all smash products formed over $H\mathbb{F}_p$ [EKMM97, IV.7.7]. Dually, let

$$\Omega(P, X, P)\colon [q] \mapsto P \times X^q \times P$$

be the (cosimplicial) two-sided cobar construction, with totalization equal to the fiber product $P \times_X P$. There is a natural simplicial map

$$\wedge\colon B^{H\mathbb{F}_p}(B, A, B) \to F(\Omega(P, X, P), H\mathbb{F}_p)$$

which is a degreewise weak equivalence by the Künneth formula in mod p cohomology, under the assumption that $H_*(X;\mathbb{F}_p)$ and $H_*(P;\mathbb{F}_p)$ are finite in each degree. So the Künneth spectral sequence equals the one obtained by applying mod p cohomology to the cobar construction, i.e., the mod p Eilenberg–Moore spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^{H^*(X;\mathbb{F}_p)}_{s,t}(H^*(P;\mathbb{F}_p), H^*(P;\mathbb{F}_p)) \Longrightarrow H^{-(s+t)}(P \times_X P;\mathbb{F}_p) \tag{5.6.2}$$

[EM66]. By [Dw74], [Sh96, 3.1], the Eilenberg–Moore spectral sequence converges strongly if, for example, $\pi_0(G)$ is finite, X is path-connected, and $\pi_1(X)$ acts nilpotently on $H_*(G;\mathbb{F}_p)$. The Künneth spectral sequence is always strongly convergent, so this comparison implies that the upper horizontal map in

$$\begin{tikzcd}
B \wedge_A B \arrow[r, "\wedge"] \arrow[d, "h"'] & F((P \times_X P)_+, H\mathbb{F}_p) \arrow[d, "\xi^*"] \\
F(G_+, B) \arrow[r, "\cong"] & F((P \times G)_+, H\mathbb{F}_p)
\end{tikzcd}$$

is a weak equivalence. The right hand vertical map is induced by the homeomorphism $\xi\colon P \times G \to P \times_X P$, hence is an isomorphism, as is the lower horizontal map. Therefore these hypotheses ensure that the left hand vertical map h is a weak equivalence.

Proposition 5.6.3. *Let G be a stably dualizable group and $P \to X$ a principal G-bundle.*

(a) Suppose that $\pi_0(G)$ is finite, X is path-connected, $\pi_1(X)$ acts nilpotently on $H_*(G;\mathbb{F}_p)$, and that $H_*(X;\mathbb{F}_p)$ and $H_*(P;\mathbb{F}_p)$ are finite in each degree. Then the map of cochain $H\mathbb{F}_p$-algebras

$$F(X_+, H\mathbb{F}_p) \to F(P_+, H\mathbb{F}_p)$$

is a G-Galois extension.

(b) In particular, when G is a finite discrete group acting nilpotently on $\mathbb{F}_p[G]$ (this includes all finite p-groups), then there is a G-Galois extension

$$F(BG_+, H\mathbb{F}_p) \to F(EG_+, H\mathbb{F}_p) \simeq H\mathbb{F}_p$$

that exhibits $H\mathbb{F}_p$ as a Galois extension by each such group.

A similar argument applies for the map of rational cochain algebras

$$F(X_+, H\mathbb{Q}) \to F(P_+, H\mathbb{Q}),$$

when $H^*(X;\mathbb{Q})$ and $H^*(P;\mathbb{Q})$ are finite dimensional over $\mathbb{Q}$ in each degree.

For each natural number n the Morava K-theory spectrum $K(n)$ admits uncountably many associative S-algebra structures [Rob89, 2.5], none of which are strictly commutative (cf. Lemma 5.6.4). Therefore

$$F(X_+, K(n)) \to F(P_+, K(n))$$

is at best a kind of non-commutative G-Galois extension. As a further complication, the convergence of the $K(n)$-based Eilenberg–Moore spectral sequence, analogous to (5.6.2), is not yet well understood.

Lemma 5.6.4. *$K(n)$ does not admit the structure of a commutative S-algebra.*

Proof. Suppose that $K(n)$ is a commutative S-algebra. Then so is its connective cover $k(n)$, and there is a 1-connected commutative S-algebra map $u\colon k(n) \to H\mathbb{F}_p$. Then $u_*\colon H_*(k(n); \mathbb{F}_p) \to H_*(H\mathbb{F}_p; \mathbb{F}_p)$ is an injective algebra homomorphism, that commutes with the Dyer–Lashof operations on both sides [BMMS86, III.2.3]. The target equals the dual Steenrod algebra $A_* = E(\chi\tau_k \mid k \geq 0) \otimes P(\chi\xi_k \mid k \geq 1)$, and the image of u_* contains $\chi\tau_{n-1}$, but not $\chi\tau_n$. This contradicts the operation $Q^{p^k}(\chi\tau_k) = \chi\tau_{k+1}$ in A_*, in the case $k = n-1$. □

6. Dualizability and alternate characterizations

6.1. Extended equivalences.

Let $A \to B$ be a map of E-local commutative S-algebras, and let G be a topological group acting from the left on B through A-algebra maps, say by $\alpha\colon G_+ \wedge B \to B$. For example, $A \to B$ could be a G-Galois extension.

The twisted group S-algebra $B\langle G\rangle$ is defined to be $B \wedge G_+$ (implicitly E-localized, like $B[G]$), with the multiplication $B\langle G\rangle \wedge B\langle G\rangle \to B\langle G\rangle$ obtained from the composite map

$$G_+ \wedge B \xrightarrow{\Delta\wedge 1} G_+ \wedge G_+ \wedge B \xrightarrow{1\wedge\alpha} G_+ \wedge B \cong B \wedge G_+$$

and the multiplications on B and G. As usual, Δ is the diagonal map. The map $A \to B$ and the unit inclusion $\{e\} \to G$ induce a central map $\eta\colon A \to B\langle G\rangle$, which makes $B\langle G\rangle$ an associative A-algebra. Likewise, the endomorphism algebra $F_A(B, B)$ of B over A is an associative A-algebra with respect to the composition pairing.

Let

$$j\colon B\langle G\rangle \to F_A(B, B) \tag{6.1.1}$$

be the canonical map of A-algebras that is right adjoint to the composite map

$$B \wedge G_+ \wedge_A B \xrightarrow{1\wedge\alpha} B \wedge_A B \xrightarrow{\mu} B\,,$$

induced by the (A-linear) action of G on B and the multiplication on B. Note that $B\langle G\rangle$ and $F_A(B, B)$ are left B-modules, with respect to the action on the target in the latter case, and that j is a map of B-modules. There is also a diagonal left action by G on $B \wedge G_+$ and on the target in $F_A(B, B)$, and j is G-equivariant with respect to these actions. These B- and G-actions do not commute, but combine to a left module action by $B\langle G\rangle$.

For a map f of spectra, we will write $f_\#$ and $f^\#$ for various maps induced by left and right composition with f, respectively.

Lemma 6.1.2. *Let $A \to B$ be a map of commutative S-algebras, and let G be a stably dualizable group acting on B through A-algebra maps, such that $h\colon B \wedge_A B \to F(G_+, B)$ is a weak equivalence. For example, $A \to B$ could be a G-Galois extension. Then:*

(a) For each B-module M there is a natural weak equivalence

$$h_M \colon M \wedge_A B \to F(G_+, M)\,.$$

(b) The canonical map

$$j \colon B\langle G\rangle \to F_A(B, B)$$

is a weak equivalence.

(c) For each B-module M there is a natural weak equivalence

$$j_M \colon M \wedge G_+ \to F_A(B, M)\,.$$

Proof. (a) By definition, h_M is the composite map

$$M \wedge_A B \cong M \wedge_B B \wedge_A B \xrightarrow{1\wedge h} M \wedge_B F(G_+, B) \xrightarrow{\nu} F(G_+, M)\,,$$

which is a weak equivalence because h is a weak equivalence and G is stably dualizable.

(b) This is the special case of (c) below when $M = B$.

(c) By definition, j_M is right adjoint to the composite map $M \wedge G_+ \wedge_A B \to M \wedge_A B \to M$ induced by the group action of G on B and the module action of B on M. We can factor j_M in the stable homotopy category as the following chain of weak equivalences:

$$\begin{aligned} M \wedge G_+ \xrightarrow{1\wedge\rho} M \wedge DDG_+ \xrightarrow{\nu} F(DG_+, M) &\cong F_B(B \wedge DG_+, M) \\ \xleftarrow{\nu^\#} F_B(F(G_+, B), M) \xrightarrow{h^\#} F_B(B \wedge_A B, M) &\cong F_A(B, M)\,. \end{aligned}$$

Here the map $h^\#$ makes sense because h is a map of B-modules, and similarly for $\nu^\#$. Algebraically, $m \wedge g$ lifts over $\nu^\#$ to the map $f \mapsto f(g)\cdot m$ in $F_B(F(G_+, B), M)$, which $h^\#$ takes to $j_M(m \wedge g)$. $\square$

Lemma 6.1.3. *Let $A \to B$ be a G-Galois extension. For each B-module M the canonical map*

$$\nu' \colon M \wedge_A B^{hG} \to (M \wedge_A B)^{hG}$$

is a weak equivalence.

Proof. The weak equivalence $M \wedge_A A \cong M \to F(G_+, M)^{hG}$ factors as the composite

$$M \wedge_A A \underset{\simeq}{\xrightarrow{1\wedge i}} M \wedge_A B^{hG} \xrightarrow{\nu'} (M \wedge_A B)^{hG} \underset{\simeq}{\xrightarrow{h_M^{hG}}} F(G_+, M)^{hG}$$

where i and h_M are weak equivalences by hypothesis and the previous lemma, respectively. The G-equivariance of h_M, needed to make sense of h_M^{hG}, follows like that of h. $\square$

6.2. Dualizability.

For each G-Galois extension $R \to T$ of commutative rings, T is a finitely generated projective R-module. The following is the analogous statement for E-local commutative S-algebras.

Proposition 6.2.1. *Let $A \to B$ be a G-Galois extension. Then B is a dualizable A-module.*

Proof. We must show that the canonical map $\nu\colon D_A B \wedge_A B \to F_A(B, B)$ is a weak equivalence. To keep the different B's apart, we observe more generally that for each B-module M there is a commutative diagram

$$\begin{tikzcd}
M \wedge_A F_A(B, A) \arrow[r, "\nu"] \arrow[d, "1\wedge i_\#"'] & F_A(B, M \wedge_A A) \arrow[d, "(1\wedge i)_\#"] \\
M \wedge_A F_A(B, B^{hG}) \arrow[r, "\nu"] \arrow[d, "\cong"'] & F_A(B, M \wedge_A B^{hG}) \arrow[d, "\nu'_\#"] \\
M \wedge_A F_A(B, B)^{hG} \arrow[r, "\nu"] & F_A(B, M \wedge_A B)^{hG} \\
M \wedge_A (B \wedge G_+)^{hG} \arrow[r, "\nu"] \arrow[u, "1\wedge j^{hG}"] & (M \wedge_A B \wedge G_+)^{hG} \arrow[u, "j^{hG}_{M\wedge_A B}"'] \\
M \wedge_A (B \wedge G_+ \wedge S^{adG})_{hG} \arrow[r, "\cong"] \arrow[u, "1\wedge N"] & (M \wedge_A B \wedge G_+ \wedge S^{adG})_{hG} \arrow[u, "N"']
\end{tikzcd}$$

where $\nu'_\#$ is a weak equivalence by Lemma 6.1.3, the maps induced by $i\colon A \to B^{hG}$ are weak equivalences by hypothesis, the maps involving j are well-defined by the G-equivariance of j (and $j_{M\wedge_A B}$), and are weak equivalences by Lemma 6.1.2, and finally the norm maps N from (3.6.1) are weak equivalences because the spectra with G-action in question have the form $W \wedge G_+$, with G acting freely on itself [Rog:s, 5.2.5]. Thus all maps in this diagram are weak equivalences.

The special case when $M = B$ then verifies that B is dualizable over A. □

In the global case $E = S$ it follows from Propositions 3.3.3 and 6.2.1 that in any G-Galois extension $A \to B$, B is a semi-finite A-module, i.e., it is weakly equivalent to a retract of a finite cell A-module. For example, by Proposition 5.3.1 the complexification map $KO \to KU$ is a global quadratic extension, and indeed, $KU \simeq KO \wedge C_\eta$ is a finite 2-cell KO-module. However, in the localized cases the following counterexample shows that dualizability is probably the best one can hope for.

Example 6.2.2. Let $p = 2$, recall that $L_{K(1)}S = J_2^\wedge$, and consider the $K(1)$-local quadratic Galois extension $c\colon J_2^\wedge \to JU_2^\wedge$ from 5.5.2. We claim that $JU_2^\wedge$ is not a semi-finite $J_2^\wedge$-module, even if it is a dualizable $J_2^\wedge$-module, in the $K(1)$-local category. There is a diagram of horizontal and vertical fiber sequences:

$$\begin{tikzcd}
J_2^\wedge \arrow[r] \arrow[d, "c"] & KO_2^\wedge \arrow[r, "\psi^3-1"] \arrow[d, "c"] & KO_2^\wedge \arrow[d, "c"] \\
JU_2^\wedge \arrow[r] \arrow[d] & KU_2^\wedge \arrow[r, "\psi^3-1"] \arrow[d, "\partial"] & KU_2^\wedge \arrow[d, "\partial"] \\
\Sigma^2 X_3 \arrow[r] & \Sigma^2 KO_2^\wedge \arrow[r, "\Sigma^2(3^{-1}\psi^3-1)"] & \Sigma^2 KO_2^\wedge
\end{tikzcd}$$

The factor 3^{-1} in the lower row comes from the appearance of the inverse of the Bott equivalence $\beta\colon \Sigma^2 KU \to KU$ in the connecting map ∂, and the relation $\psi^k\beta = k\beta\psi^k$. By definition, following [HMS94, 2.6], but using real K-theory, X_3 is the homotopy fiber of $3^{-1}\psi^3 - 1\colon KO_2^\wedge \to KO_2^\wedge$.

We can compute the zero-th $E_1 = KU_2^\wedge$-cohomology of the spectra in the upper left hand square, as modules over the group $\mathbb{S}_1 = \mathbb{Z}_2^*$ of stable Adams operations, with $k \in \mathbb{Z}_2^*$ acting by ψ^k. First, $E_1^0(KU_2^\wedge) \cong \mathbb{Z}_2[[\mathbb{Z}_2^*]]$ (see also Example 8.1.4), and the remaining modules are the following quotients:

$$\begin{tikzcd}
\mathbb{Z}_2 & \mathbb{Z}_2[[\mathbb{Z}_2^*/\langle -1\rangle]] \arrow[l] \\
\mathbb{Z}_2[[\mathbb{Z}_2^*/\langle 3\rangle]] \arrow[u, "c^*"] & \mathbb{Z}_2[[\mathbb{Z}_2^*]] \arrow[l] \arrow[u]
\end{tikzcd}$$

Here $\langle 3\rangle \subset \mathbb{Z}_2^*$ is the subgroup topologically generated by 3. The map c^* takes $E_1^0(JU_2^\wedge) \cong \mathbb{Z}_2[[\mathbb{Z}_2^*/\langle 3\rangle]] \cong \mathbb{Z}_2\{1, \psi^{-1}\}$ to $E_1^0(J_2^\wedge) \cong \mathbb{Z}_2\{1\}$ by mapping both 1 and ψ^{-1} to the generator. Thus $E_1^0(\Sigma^2 X_3) = \ker(c^*) \cong \mathbb{Z}_2\{1 - \psi^{-1}\}$ is such that ψ^3 acts as the identity, but ψ^{-1} acts by reversing the sign.

We claim that there is no semi-finite spectrum with this Morava module, i.e., this E_1-cohomology as an $\mathbb{S}_1$-module. For each finite cell spectrum X the Atiyah–Hirzebruch spectral sequence

$$E_2^{s,t} = H^s(X; \pi_{-t}(E_1)) \Longrightarrow E_1^{s+t}(X)$$

is strongly convergent. After rationalization (inverting 2) it collapses at the E_2-term, yielding the Chern character isomorphism

$$ch\colon E_1^0(X)[2^{-1}] \cong \bigoplus_{i\in\mathbb{Z}} H^{2i}(X; \mathbb{Q}_2)$$

in degree zero. Here the i-th summand appears as the eigenspace of weight i, where ψ^k acts by multiplication by k^i for each $k \in \mathbb{Z}_2^*$. By naturality, there is also such an eigenspace decomposition of $E_1^0(X)[2^{-1}]$ for each semi-finite $J_2^\wedge$-module X. (For general spectra X, the Atiyah–Hirzebruch spectral sequence needs not converge.)

Now note that $E_1^0(\Sigma^2 X_3)[2^{-1}] \cong \mathbb{Q}_2$ has ψ^3 acting as the identity, and ψ^{-1} acting by sign, which means that it should lie both in the weight 0 eigenspace and in an eigenspace of odd weight. This contradicts the possibility that $\Sigma^2 X_3$ is semi-finite. It follows that also $JU_2^\wedge$ cannot be $K(1)$-locally semi-finite.

Dualizable modules are preserved under base change, and are detected by faithful and dualizable base change.

Lemma 6.2.3. *Let $A \to B$ be a map of commutative S-algebras and M a dualizable A-module. Then $B \wedge_A M$ is a dualizable B-module.*

Proof. We must verify that the canonical map

$$\nu\colon F_B(B \wedge_A M, B) \wedge_B (B \wedge_A M) \to F_B(B \wedge_A M, B \wedge_A M)$$

is a weak equivalence. It factors as the composite

$$\begin{aligned} F_B(B \wedge_A M, B) \wedge_B (B \wedge_A M) &\cong F_A(M, B) \wedge_A M \\ &\xrightarrow{\nu} F_A(M, B \wedge_A M) \cong F_B(B \wedge_A M, B \wedge_A M)\,, \end{aligned}$$

where the middle map is a weak equivalence by Lemma 3.3.2(a), since M is a dualizable A-module. $\square$

Lemma 6.2.4. *Let $A \to B$ be a faithful map of commutative S-algebras, with B dualizable over A, and let M be an A-module such that $B \wedge_A M$ is a dualizable B-module. Then M is a dualizable A-module.*

Proof. We must verify that $\nu\colon F_A(M, A) \wedge_A M \to F_A(M, M)$ is a weak equivalence. It suffices to show that the map $1 \wedge \nu$ in the commutative square below is a weak equivalence, since B is assumed to be faithful over A.

$$\begin{tikzcd} B \wedge_A F_A(M, A) \wedge_A M \arrow[r, "1\wedge\nu"] \arrow[d, "\nu\wedge 1"'] & B \wedge_A F_A(M, M) \arrow[d, "\nu"] \\ F_A(M, B) \wedge_A M \arrow[r] & F_A(M, B \wedge_A M) \end{tikzcd}$$

Here the lower horizontal map is isomorphic to

$$\nu\colon F_B(B \wedge_A M, B) \wedge_B (B \wedge_A M) \to F_B(B \wedge_A M, B \wedge_A M)\,,$$

which is a weak equivalence because $B \wedge_A M$ is assumed to be dualizable over B. The vertical maps are weak equivalences because B is dualizable over A, in view of Lemma 3.3.2(a). Therefore the upper horizontal map $1 \wedge \nu$ is also a weak equivalence. $\square$

Corollary 6.2.5. *If A is a commutative S-algebra and G is a stably dualizable group, so $S[G]$ is dualizable over S, then $A[G]$ is dualizable over A.*

Conversely, if A is a faithful commutative S-algebra, with A dualizable over S, and G is a topological group such that $A[G]$ is dualizable over A, then G is stably dualizable.

The following lemma gives the same conclusion as Lemma 6.1.3, but under different hypotheses, and will be often used.

Lemma 6.2.6. *Let $A \to B$ be a map of commutative S-algebras, let G be a topological group acting on B through A-algebra maps, and let M be a dualizable A-module. Then the canonical map*

$$\nu' \colon M \wedge_A B^{hG} \to (M \wedge_A B)^{hG}$$

is a weak equivalence.

Proof. In the commutative diagram

$$\begin{tikzcd}
M \wedge_A B^{hG} \arrow[r, "\rho\wedge 1"] \arrow[d, "\nu'"'] & D_A D_A M \wedge_A B^{hG} \arrow[r, "\nu"] \arrow[d] & F_A(D_A M, B^{hG}) \arrow[d, "\cong"] \\
(M \wedge_A B)^{hG} \arrow[r, "(\rho\wedge 1)^{hG}"] & (D_A D_A M \wedge_A B)^{hG} \arrow[r, "\nu^{hG}"] & F_A(D_A M, B)^{hG}
\end{tikzcd}$$

the horizontal maps derived from ν and ρ are weak equivalences because M is dualizable over A, and the right hand vertical map is an isomorphism. Thus the left hand vertical map ν' is a weak equivalence. □

6.3. Alternate characterizations.

The following alternate characterization of Galois extensions corresponds to the Auslander–Goldman definition. Compare Proposition 2.3.2. Implicit cofibrancy and localization at some S-module E is to be understood.

Proposition 6.3.1. *Let $A \to B$ be a map of commutative S-algebras, and let G be a stably dualizable group acting on B through A-algebra maps. Then $A \to B$ is a G-Galois extension if and only if both $i \colon A \to B^{hG}$ and $j \colon B\langle G\rangle \to F_A(B, B)$ are weak equivalences and B is a dualizable A-module.*

Proof. Lemma 6.1.2(b) and Proposition 6.2.1 establish one implication. For the converse, suppose that i and j are weak equivalences and that B is dualizable over A. We must show that $h \colon B \wedge_A B \to F(G_+, B)$ is a weak equivalence. Again, to keep the B's apart we shall observe that for each B-module M the map h_M factors in the stable homotopy category as the following chain of weak equivalences:

$$\begin{aligned}
M \wedge_A B \xrightarrow{1\wedge\rho} M \wedge_A D_A D_A B \xrightarrow{\nu} F_A(D_A B, M) \cong F_B(D_A B \wedge_A B, M) \\
\xleftarrow{\nu^{\#}} F_B(F_A(B, B), M) \xrightarrow{j^{\#}} F_B(B\langle G\rangle, M) \cong F(G_+, M)\,.
\end{aligned}$$

Algebraically, the forward image of $m \wedge b$ lifts over $\nu^{\#}$ to $f \mapsto f(b) \cdot m$, which maps by $j^{\#}$ to $h_M(m \wedge b) = \{g \mapsto g(b) \cdot m\}$. The hypotheses that B is dualizable over A and j is a weak equivalence thus imply that h_M is a weak equivalence. The special case $M = B$ lets us conclude that $A \to B$ is G-Galois. □

In the presence of faithfulness we have a third characterization of Galois extensions. See also Propositions 8.2.8 and 12.1.8.

Proposition 6.3.2. *Let $A \to B$ be a map of commutative S-algebras, and let G be a stably dualizable group acting on B through A-algebra maps. Then $A \to B$ is a faithful G-Galois extension if and only if $h\colon B \wedge_A B \to F(G_+, B)$ is a weak equivalence and B is faithful and dualizable as an A-module.*

Proof. Proposition 6.2.1 provides one implication. For the converse, suppose that h is a weak equivalence and that B is dualizable and faithful over A. We must show that $i\colon A \to B^{hG}$ is a weak equivalence, and by faithfulness it suffices to show that $1 \wedge i\colon B \cong B \wedge_A A \to B \wedge_A B^{hG}$ is a weak equivalence. In the stable homotopy category we can identify this map with the chain of weak equivalences

$$B \xrightarrow{\simeq} F(G_+, B)^{hG} \xleftarrow{h^{hG}} (B \wedge_A B)^{hG} \xleftarrow{\nu'} B \wedge_A B^{hG} .$$

Here ν' is a weak equivalence by Lemma 6.2.6, because B is dualizable over A. We are viewing h as a G-equivariant map with respect to the left G-actions specified in Section 4.1. □

Here is a characterization of faithfulness in terms of the norm map.

Proposition 6.3.3. *A G-Galois extension $A \to B$ is faithful if and only if the norm map $N\colon (B \wedge S^{adG})_{hG} \to B^{hG}$ is a weak equivalence, or equivalently, if the Tate construction B^{tG} is contractible.*

Proof. If the norm map is a weak equivalence, and Z is an A-module so that $Z \wedge_A B \simeq *$, then $Z \simeq Z \wedge_A B^{hG} \simeq Z \wedge_A (B \wedge S^{adG})_{hG} \cong (Z \wedge_A B \wedge S^{adG})_{hG} \simeq *$. Thus $A \to B$ is faithful.

For the converse, consider $B \wedge_A (-)$ applied to the norm map, appearing as the left hand vertical map in the following commutative diagram.

$$\begin{tikzcd}
B \wedge_A (B \wedge S^{adG})_{hG} \arrow[r, "\cong"] \arrow[d, "1\wedge N"'] & (B \wedge_A B \wedge S^{adG})_{hG} \arrow[r, "(h\wedge 1)_{hG}"] \arrow[d, "N"] & (F(G_+, B) \wedge S^{adG})_{hG} \arrow[d, "N"] \\
B \wedge_A B^{hG} \arrow[r, "\nu'"] & (B \wedge_A B)^{hG} \arrow[r, "h^{hG}"] & F(G_+, B)^{hG}
\end{tikzcd}$$

The map ν' is a weak equivalence because B is dualizable over A, by Lemma 6.2.6. The upper and lower right hand horizontal maps are weak equivalences since h is G-equivariant and a weak equivalence.

The right hand vertical map is the norm map for the spectrum with G-action $F(G_+, B)$. In the source,

$$(F(G_+, B) \wedge S^{adG})_{hG} \simeq (B \wedge DG_+ \wedge S^{adG})_{hG} \simeq (B \wedge S[G])_{hG} \simeq B$$

by the stable dualizability of G and the Poincaré duality equivalence (3.5.2). In the target, $F(G_+, B)^{hG} \simeq B$. A direct inspection (inducing up from the case $B = S$, where it suffices to check on π_0) verifies that these identifications are compatible under the norm map. Therefore the right hand vertical map N is a weak equivalence, and so the norm map for B must be a weak equivalence, assuming that B is faithful over A.

The second equivalence is obvious from the definition of B^{tG} as the homotopy cofiber of the norm map. □

Corollary 6.3.4. *Any finite G-Galois extension $A \to B$ is faithful if the order $|G|$ of G is invertible in $\pi_0(B)$.*

Proof. Under these hypotheses $\pi_*(B_{hG}) \cong \pi_*(B)/G$, $\pi_*(B^{hG}) \cong \pi_*(B)^G$ and the composite

$$\pi_*(B) \to \pi_*(B)/G \xrightarrow{N_*} \pi_*(B)^G \to \pi_*(B)$$

is multiplication by $|G|$, so the norm map N must induce an isomorphism in homotopy. $\square$

The same conclusion, under different hypotheses (allowing ramification) appears in Lemma 6.4.3.

6.4. The trace map and self-duality.

In this section we work principally in the derived category, i.e., in the stable homotopy category $\mathcal{D}_{A,E}$.

Let $A \to B$ be a map of E-local commutative S-algebras, and let G be a stably dualizable group acting on B through A-algebra maps. Suppose that $i\colon A \to B^{hG}$ is a weak equivalence.

Definition 6.4.1. The *trace map* $tr\colon B \wedge S^{adG} \to A$ in $\mathcal{D}_{A,E}$ is defined by the natural chain of maps

$$B \wedge S^{adG} \xrightarrow{in} (B \wedge S^{adG})_{hG} \xrightarrow{N} B^{hG} \underset{\simeq}{\xleftarrow{i}} A\,,$$

where in denotes the inclusion induced by $G \subset EG$, and the wrong-way map i is a weak equivalence.

When G is finite, the dualizing spectrum $S^{adG} = S$ can of course be ignored.

Lemma 6.4.2. *The trace map $tr\colon B \wedge S^{adG} \to A$ equals the composite map*

$$B \wedge S^{adG} = B \wedge S[G]^{hG} \underset{\simeq}{\xrightarrow{\nu}} (B \wedge S[G])^{hG} \xrightarrow{(\alpha')^{hG}} B^{hG} \underset{\simeq}{\xleftarrow{i}} A$$

where $\alpha'\colon B \wedge S[G] \to B$ is the right action derived from $\alpha\colon G_+ \wedge B \to B$ by way of the group inverse.

Proof. The canonical map $\nu\colon B \wedge S^{adG} \to (B \wedge S[G])^{hG}$ can be identified with the chain of weak equivalences

$$B \wedge S^{adG} \xrightarrow{\simeq} F(G_+, B \wedge S^{adG})^{hG} \underset{\simeq}{\xleftarrow{\nu^{hG}}} (B \wedge DG_+ \wedge S^{adG})^{hG} \xrightarrow{\simeq} (B \wedge S[G])^{hG}\,,$$

using that G is stably dualizable and the (right G-equivariant) Poincaré duality equivalence (3.5.2). In particular, ν itself is a weak equivalence.

The claim is then clear from the commutative diagram

$$\begin{tikzcd}
B \wedge S^{adG} \arrow[r, "\nu", "\simeq"'] \arrow[d, "in"'] & (B \wedge S[G])^{hG} \arrow[r, "="] \arrow[d, "in"'] & (B \wedge S[G])^{hG} \arrow[d, "(in)^{hG}"'] \arrow[dr, "(\alpha')^{hG}"] & \\
(B \wedge S^{adG})_{hG} \arrow[r, "\nu_{hG}", "\simeq"'] & ((B \wedge S[G])^{hG})_{hG} \arrow[r, "\kappa"] & ((B \wedge S[G])_{hG})^{hG} \arrow[r, "\simeq"'] & B^{hG}
\end{tikzcd}$$

where κ is the canonical hocolim/holim exchange map and the bottom row defines the norm map N, as in [Rog:s, 5.2.2]. The right hand triangle uses that the homotopy orbits $(B \wedge S[G])_{hG}$ are formed with respect to the diagonal left G-action, so the identification with B extends the right action map α'. Algebraically, $b \wedge g$ in $B \wedge S[G]$ is identified with $g^{-1}b \wedge e$ in the homotopy orbits, which maps to $\alpha'(b \wedge g) = g^{-1}b$ in B. $\square$

Lemma 6.4.3. *When G is finite the composite $B \xrightarrow{tr} A \to B$ is homotopic to the sum over all $g \in G$ of the group action maps $g \colon B \to B$, and the composite $A \to B \xrightarrow{tr} A$ is homotopic to the map multiplying by the order $|G|$ of G.*

Thus, if $|G|$ is invertible in $\pi_0(A)$ then tr is a split surjective map of A-modules, up to homotopy, and B is a faithful A-module. In particular, every G-Galois extension $A \to B$ with $|G|$ invertible in $\pi_0(A)$ is faithful.

Proof. When G is finite, the composite $B \xrightarrow{tr} A \to B$ can be expressed by continuing the factorization in Lemma 6.4.2 with the map $B^{hG} \to B$ that forgets homotopy invariance, and therefore factors as

$$B \xrightarrow{1 \wedge \Delta} B \wedge S[G] \xrightarrow{\alpha'} B\,,$$

where $\Delta \colon S \to S[G] \simeq \prod_G S$ is the diagonal map. Clearly this is the sum over the elements $g \in G$ of the group action maps $g \colon B \to B$, up to homotopy.

On the other hand, the composite $A \to B \xrightarrow{tr} A$ is the map of G-homotopy fixed points induced by the same composite displayed above. Since the action of each group element is homotopic to the identity when restricted to the homotopy fixed points, their sum equals multiplication by the group order $|G|$, up to homotopy. $\square$

Example 6.4.4. In the $\mathbb{Z}/2$-Galois extension $c \colon KO \to KU$ the trace map tr is homotopic to the realification map $r \colon KU \to KO$, as a KO-module map, and therefore also as an S-module map. For $c^\# \colon \mathcal{D}_{KO}(KU, KO) \to \mathcal{D}_{KO}(KO, KO)$ is injective, and both $tr \circ c$ and $r \circ c$ are homotopic to the multiplication by 2 map $KO \to KO$, by Lemma 6.4.3.

To justify the claim just made, that $c^\#$ is injective, we use the equivalence $KU \simeq KO \wedge C_\eta$ and adjunction to identify $c^\#$ with $i^\#$ in the exact sequence

$$\pi_1(KO) \xrightarrow{\eta^\#} \pi_2(KO) \xrightarrow{j^\#} [C_\eta, KO] \xrightarrow{i^\#} \pi_0(KO)$$

induced by the cofiber sequence $S^0 \xrightarrow{i} C_\eta \xrightarrow{j} S^2 \xrightarrow{\eta} S^1$. Here $i^\#$ is injective because $\eta^\#$ is well-known to be surjective.

In particular, the trace map $tr = r \colon KU \to KO$ is not split surjective up to homotopy (it is not even surjective on homotopy groups), so the analog of the algebraic Proposition 2.3.4(b) does not hold in topology.

Recall from Section 3.6 the shearing equivalence $\zeta \colon B \wedge S[G] \to B \wedge S[G]$ that takes the left action on $S[G]$ to the diagonal left action on B and $S[G]$.

Definition 6.4.5. The *trace pairing* $B \wedge_A B \wedge S^{adG} \to A$ in $\mathcal{D}_{A,E}$ is defined as the composite

$$B \wedge_A B \wedge S^{adG} \xrightarrow{\mu \wedge 1} B \wedge S^{adG} \xrightarrow{tr} A\,.$$

The *discriminant map* $\mathfrak{d}_{B/A} \colon B \wedge S^{adG} \to D_A B$ in $\mathcal{D}_{A,E}$ is defined as the composite

$$\begin{aligned} B \wedge S^{adG} = B \wedge S[G]^{hG} &\underset{\simeq}{\xrightarrow{\nu}} (B \wedge S[G])^{hG} \underset{\simeq}{\xrightarrow{\zeta^{hG}}} (B \wedge S[G])^{hG} \\ &\xrightarrow{j^{hG}} F_A(B,B)^{hG} \cong F_A(B, B^{hG}) \underset{\simeq}{\xleftarrow{i_\#}} F_A(B,A) = D_A B\,. \end{aligned}$$

Here j is G-equivariant with respect to the left G-action from Section 6.1.

We define $\mathrm{Pic}_E = \mathrm{Pic}_E(S)$ in Definition 6.5.1 below to be the group of weak equivalence classes of E-locally smash invertible spectra. The dualizing spectrum S^{adG} is one such [Rog:s, 3.3.4]. By the Pic_E-graded homotopy groups $\pi_*(Y)$ of a spectrum Y we mean the collection of groups $\pi_X(Y) = [X, Y]$, where X ranges through Pic_E. See [HSt99, 14.1]. This includes the ordinary stable homotopy groups as the cases $X = S^n$, $n \in \mathbb{Z}$, as well as the possibly exceptional case $X = S^{adG}$.

Lemma 6.4.6. *The trace pairing $B \wedge_A B \wedge S^{adG} \to A$ is left adjoint to the discriminant map $\mathfrak{d}_{B/A} \colon B \wedge S^{adG} \to D_A B$. Thus $\mathfrak{d}_{B/A}$ is in fact a map in $\mathcal{D}_{B,E}$, and represents a Pic_E-graded class in $\pi_* D_A(B)$.*

Proof. The first claim is a chase of definitions. The multiplications by B in the two copies of B in the source of the trace pairing get equalized by μ, so the adjoint (weak) map $\mathfrak{d}_{B/A}$ commutes with the obvious B-module actions on $B \wedge S^{adG}$ and $D_A B$. □

Proposition 6.4.7. *If $A \to B$ is a G-Galois extension, then the discriminant map $\mathfrak{d}_{B/A} \colon B \wedge S^{adG} \to D_A B$ is a weak equivalence. In particular, B is self-dual as an A-module, up to an invertible shift by S^{adG}.*

Proof. When $A \to B$ is G-Galois, $j \colon B \wedge G_+ \to F_A(B,B)$ is a weak equivalence by Lemma 6.1.2(b), so the discriminant map is defined as a composite of weak equivalences. □

In general, we think of the discriminant map $\mathfrak{d}_{B/A}$ as a measure of the extent to which $A \to B$ is ramified. When it is an equivalence, we think of the trace pairing as a perfect pairing.

6.5. Smash invertible modules.

The $K(n)$-local Picard group $\mathrm{Pic}_n = \mathrm{Pic}_{K(n)}(S)$ was introduced in [HMS94]. Here is a slight generalization.

Definition 6.5.1. Let A be a commutative S-algebra, and work locally with respect to the fixed spectrum E. An A-module M is *smash invertible* if there exists an A-module N such that $N \wedge_A M \simeq A$ as (implicitly E-local) A-modules.

Let $\mathrm{Pic}_E(A)$ be the class of weak equivalence classes of E-locally smash invertible A-modules. When $\mathrm{Pic}_E(A)$ is a set we call it the *E-local Picard group* of A, with the group structure induced by the (implicitly E-local) smash product of A-modules.

The following proof of the analog of Proposition 2.3.4(c) is close to one found by Andy Baker and Birgit Richter in the case of a finite abelian group G.

Proposition 6.5.2. *Let $A \to B$ be a faithful abelian G-Galois extension, i.e., one with G an (E-locally stably dualizable) abelian group. Then B is smash invertible as an $A[G]$-module.*

Proof. We consider B as an $A[G]$-module by way of the given left G-action. The smash inverse of B over $A[G]$ will be its functional dual $D_{A[G]}(B) = F_{A[G]}(B, A[G])$ in the category $\mathcal{M}_{A[G],E}$. There is a natural counit map

$$\epsilon \colon F_{A[G]}(B, A[G]) \wedge_{A[G]} B \to A[G] \,,$$

that is left adjoint to the identity map on $F_{A[G]}(B, A[G])$ in the category of $A[G]$-modules. In symbols, $\epsilon \colon f \wedge x \mapsto f(x)$. The claim is that ϵ is a weak equivalence. By assumption B is faithful over A, so it suffices to verify that ϵ becomes an equivalence after inducing up along $A \to B$. We factor the resulting map $1 \wedge \epsilon$ as

$$\begin{aligned} B \wedge_A F_{A[G]}(B, A[G]) \wedge_{A[G]} B &\xrightarrow{\nu'} F_{A[G]}(B, B[G]) \wedge_{A[G]} B \\ &\cong F_{B[G]}(B \wedge_A B, B[G]) \wedge_{B[G]} (B \wedge_A B) \xrightarrow{\epsilon_1} B[G] \,. \end{aligned}$$

Here ν' is a weak equivalence because B is dualizable over A (cf. Lemma 6.2.6), the middle isomorphism is a composite of two standard adjunctions, and ϵ_1 is a counit of the same sort as ϵ, now in the category of $B[G]$-modules. We have left to prove that ϵ_1 is a weak equivalence.

There is a chain of left $B[G]$-module maps

$$\begin{aligned} (B \wedge_A B) \wedge S^{adG} &\xrightarrow{h \wedge 1} F(G_+, B) \wedge S^{adG} \\ &\xleftarrow{\nu \wedge 1} B \wedge DG_+ \wedge S^{adG} \xrightarrow{\simeq} B[G] \xrightarrow{\chi} B[G] \,, \end{aligned} \tag{6.5.3}$$

each of which is a weak equivalence. Here h is a weak equivalence because $A \to B$ is G-Galois, ν is a weak equivalence because G is stably dualizable, and the unnamed weak equivalence is the identity on B smashed with the Poincaré duality equivalence from (3.5.2). The latter is left G-equivariant with respect to the inverse of the right G-action mentioned in Section 3.5, i.e., with respect to the left action on DG_+ given by right multiplication in the source, the trivial action on S^{adG}, and the inverse of the standard right action on $B[G]$. The map χ is induced by the group inverse in G, and takes the inverse of the standard right action on $B[G]$ to the standard left action on $B[G]$.

(When G is finite, the chain simplifies to

$$B \wedge_A B \xrightarrow{h} F(G_+, B) \xleftarrow{\kappa} B[G] \xrightarrow{\chi} B[G] \,,$$

where κ is the usual inclusion and weak equivalence $B[G] \cong \bigvee_G B \to \prod_G B = F(G_+, B)$. Again, the right hand $B[G]$ has the standard left $B[G]$-module structure.)

By [Rog:s, 3.3.4, 3.2.3] the dualizing spectrum S^{adG} is smash invertible (in the E-local stable homotopy category), with smash inverse its functional dual $S^{-adG} = (DG_+)_{hG}$. It follows that the counit map ϵ_1 for the $B[G]$-module $B \wedge_A B$ is the composite of a weak equivalence and the counit map ϵ_2 for $(B \wedge_A B) \wedge S^{adG}$. Furthermore, it follows by naturality with respect to the chain (6.5.3) of $B[G]$-module weak equivalences that the counit map ϵ_2 is related by a chain of weak equivalences to the counit map

$$\epsilon_3 \colon F_{B[G]}(B[G], B[G]) \wedge_{B[G]} B[G] \to B[G]\,,$$

for $B[G]$ considered as a left $B[G]$-module in the standard way. The latter map ϵ_3 is obviously an isomorphism. □

So each (implicitly E-local) abelian G-Galois extension $A \to B$ exhibits B as a possibly interesting element in the Picard group $\mathrm{Pic}_E(A[G])$.

The following converse to Proposition 6.5.2 does not require that G is abelian, but for abelian G it follows that the smash invertibility of B over $A[G]$ is equivalent to B being faithful over A.

Lemma 6.5.4. *Let $A \to B$ be a (not necessarily abelian) G-Galois extension. If B is smash invertible as an $A[G]$-module, i.e., if there exists an $A[G]$-module C and a weak equivalence $B \wedge_{A[G]} C \simeq A[G]$ of A-modules, then B is faithful over A.*

Proof. If $N \wedge_A B \simeq *$ then $N[G] \cong N \wedge_A A[G] \simeq N \wedge_A B \wedge_{A[G]} C \simeq *$, and N is a retract of $N[G]$, so $N \simeq *$. □

7. Galois theory I

We continue to work locally with respect to some S-module E.

7.1. Base change for Galois extensions.

Faithful G-Galois extensions $A \to C$ are preserved by base change along arbitrary maps $A \to B$,

$$\begin{tikzcd} C \arrow[r] & B \wedge_A C \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

and all Galois extensions are preserved by dualizable base change. Conversely, (faithful) Galois extensions are detected by faithful and dualizable base change. We do not know whether these dualizability hypotheses are necessary.

Lemma 7.1.1. *Let $A \to B$ be a map of commutative S-algebras and $A \to C$ a faithful G-Galois extension. Then $B \to B \wedge_A C$ is a faithful G-Galois extension.*

Proof. The action by G on C through A-algebra maps extends uniquely to an action on $B \wedge_A C$ through B-algebra maps, taking $g \colon C \to C$ to $1 \wedge g \colon B \wedge_A C \to B \wedge_A C$ on the point set level, for $g \in G$. The group G remains stably dualizable, irrespective of whether it is being regarded as acting on C or $B \wedge_A C$.

We show that $B \to B \wedge_A C$ is a faithful G-Galois extension by appealing to Proposition 6.3.2. We know that C is a dualizable A-module by Proposition 6.2.1, and it is faithful by hypothesis. Therefore $B \wedge_A C$ is a dualizable and faithful B-module by the base change lemmas 6.2.3 and 4.3.3. It remains to verify that the canonical map $h\colon (B \wedge_A C) \wedge_B (B \wedge_A C) \to F(G_+, B \wedge_A C)$ is a weak equivalence. It is the lower horizontal map in the commutative square

$$\begin{tikzcd} B \wedge_A C \wedge_A C \arrow[r, "1\wedge h"] \arrow[d, "\cong"'] & B \wedge_A F(G_+, C) \arrow[d, "\nu"] \\ (B \wedge_A C) \wedge_B (B \wedge_A C) \arrow[r, "h"] & F(G_+, B \wedge_A C)\,, \end{tikzcd} \tag{7.1.2}$$

where the upper horizontal map $1 \wedge h$ is a weak equivalence because $A \to C$ is G-Galois, and the right hand vertical map ν is a weak equivalence because G is stably dualizable. This verifies the hypotheses of Proposition 6.3.2, so $B \to B \wedge_A C$ is a faithful G-Galois extension. □

Lemma 7.1.3. *Let $A \to B$ be a map of commutative S-algebras, with B dualizable over A, and let $A \to C$ be a G-Galois extension. Then $B \to B \wedge_A C$ is a G-Galois extension.*

Proof. The group G is stably dualizable, acts on $B \wedge_A C$ through B-algebra maps, and makes the canonical map $h\colon (B \wedge_A C) \wedge_B (B \wedge_A C) \to F(G_+, B \wedge_A C)$ a weak equivalence, just as in the previous proof. In order to verify the conditions in Definition 4.1.3 of a G-Galois extension, it remains to show that the canonical map $i\colon B \to (B \wedge_A C)^{hG}$ is a weak equivalence. But $B \cong B \wedge_A A \simeq B \wedge_A C^{hG}$, so we can identify i with $\nu'\colon B \wedge_A C^{hG} \to (B \wedge_A C)^{hG}$, which is a weak equivalence by Lemma 6.2.6 because B is dualizable over A. □

Lemma 7.1.4. *Let $A \to B$ and $A \to C$ be maps of commutative S-algebras, with B a faithful and dualizable A-module, and let G be a stably dualizable group acting on C through A-algebra maps.*

(a) If $B \to B \wedge_A C$ is a G-Galois extension, then $A \to C$ is a G-Galois extension.

(b) If $B \to B \wedge_A C$ is a faithful G-Galois extension, then $A \to C$ is a faithful G-Galois extension.

Proof. We must verify that the two maps $i\colon A \to C^{hG}$ and $h\colon C \wedge_A C \to F(G_+, C)$ are weak equivalences. For the first map we factor the weak equivalence $i\colon B \to (B \wedge_A C)^{hG}$ for the G-Galois extension $B \cong B \wedge_A A \to B \wedge_A C$ as the composite

$$B \wedge_A A \xrightarrow{1\wedge i} B \wedge_A C^{hG} \xrightarrow{\nu'} (B \wedge_A C)^{hG}\,.$$

Here the right hand map ν' is a weak equivalence because B is dualizable over A, by Lemma 6.2.6. Therefore the left hand map $1 \wedge i$ is a weak equivalence, and so $i\colon A \to C^{hG}$ is a weak equivalence because B is faithful over A.

For the second map we use the commutative square (7.1.2) again. The right hand vertical map ν is a weak equivalence because G is stably dualizable, and the lower horizontal map h is a weak equivalence because $B \to B \wedge_A C$ is assumed to be G-Galois. So the upper horizontal map $1 \wedge h$ is a weak equivalence, and so $h\colon C \wedge_A C \to F(G_+, C)$ is a weak equivalence because B is faithful over A.

Finally, if $B \to B \wedge_A C$ is faithful, then we know that $A \to C$ is faithful by Lemma 4.3.4. □

7.2. Fixed S-algebras.

Let G be a stably dualizable group and let $A \to B$ be a G-Galois extension. We consider the sub-extensions that occur as the homotopy fixed points $C = B^{hK}$, for suitable subgroups K of G.

Definition 7.2.1. Let $K \subset G$ be a topological subgroup. We say that K is an *allowable subgroup* if (a) K is stably dualizable, (b) the collapse map $c\colon G \times_K EK \to G/K$ induces a stable equivalence

$$S[G \times_K EK] \xrightarrow{\simeq} S[G/K] \,,$$

and (c) as a continuous map of spaces, the projection $\pi\colon G \to G/K$ admits a section up to homotopy.

We consider two allowable subgroups K and K' to be *equivalent* if $K \subset K'$ and $S[K] \to S[K']$ is a weak equivalence, or more generally, if K and K' are related by a chain of such (elementary) equivalences. We say that K is an *allowable normal subgroup* if, furthermore, K is a normal subgroup of G.

It follows immediately from (c) above that the orbit space G/K is stably dualizable, since $S[G/K]$ is a retract up to homotopy of $S[G]$, and that there is a homotopy equivalence $G \simeq K \times G/K$ compatible with the obvious projections π and pr_2 to G/K. If K is an allowable normal subgroup then G/K is a stably dualizable group.

Example 7.2.2. When G is discrete the allowable subgroups of G are just the subgroups of G in the usual sense, for then G is a disjoint union of free K-orbits, so $c\colon G \times_K EK \to G/K$ is already a weak equivalence, and there is no difficulty in finding a continuous section to $\pi\colon G \to G/K$.

For $A \to B$ a G-Galois extension and $K \subset G$ an allowable subgroup, we can form the following maps of commutative A-algebras

$$F(EG_+, B)^G \to F(EG_+, B)^K \to F(EG_+, B) \,.$$

In view of the natural weak equivalences $A \to F(EG_+, B)^G$ and $F(EG_+, B) \to B$, we will keep the notation simple by writing the maps above as

$$A \to B^{hK} \to B \,.$$

So to be precise, we interpret B as $F(EG_+, B)$, which then admits a K-action through B^{hK}-algebra maps. Likewise, if K is normal in G then B^{hK} admits a G/K-action through B^{hG}-algebra maps, which in turn are A-algebra maps. An implicit cofibrant replacement is also necessary at this stage.

Here is the forward part of the Galois correspondence for E-local commutative S-algebras.

Theorem 7.2.3. *Let $A \to B$ be a faithful G-Galois extension and $K \subset G$ any allowable subgroup. Then $C = B^{hK} \to B$ is a faithful K-Galois extension.*

If furthermore $K \subset G$ is an allowable normal subgroup, then $A \to C = B^{hK}$ is a faithful G/K-Galois extension.

Proof. We shall detect that $C \to B$ (resp. $A \to C$) is faithfully Galois by applying Lemma 7.1.4 to the case of faithful and dualizable base change along $C \to B \wedge_A C$ (resp. $A \to B$). Here B is faithful and dualizable as an A-module by hypothesis and Proposition 6.2.1, so $B \wedge_A C$ is faithful and dualizable as a C-module by Lemma 4.3.3 and Lemma 6.2.3. In the commutative diagram

$$(7.2.4)\qquad \begin{tikzcd}
B \arrow[r] & B \wedge_A B \arrow[r, "h", "\simeq"'] & F(G_+, B) & F(G_+, B) \arrow[l, "="'] \\
C \arrow[u] \arrow[r] & B \wedge_A C \arrow[u] \arrow[r, "h'", "\simeq"'] & F(G_+, B)^{hK} \arrow[u] & F(G/K_+, B) \arrow[l, "c^\#"', "\simeq"] \arrow[u, "\pi^\#"'] \\
A \arrow[u] \arrow[r] & B \arrow[u] \arrow[r, "="'] & B \arrow[u] & B \arrow[l, "="] \arrow[u, "\pi^\#"']
\end{tikzcd}$$

the left hand squares are base change pushouts in the category of commutative S-algebras.

The middle horizontal maps are weak equivalences. For h is a weak equivalence by the assumption that $A \to B$ is G-Galois. The map $h' \colon B \wedge_A C = B \wedge_A B^{hK} \to F(G_+, B)^{hK}$ factors as a composite weak equivalence

$$B \wedge_A B^{hK} \xrightarrow[\simeq]{\nu'} (B \wedge_A B)^{hK} \xrightarrow[\simeq]{h^{hK}} F(G_+, B)^{hK}$$

using that B is dualizable over A (and Lemma 6.2.6) and that h is a weak equivalence. Here K acts from the left on $B \wedge_A B$ and $F(G_+, B)$ by restriction of the actions by G, i.e., on the second copy of B in $B \wedge_A B$ and by right multiplication in the source in $F(G_+, B)$, so in particular h is K-equivariant.

Likewise, the right hand horizontal maps are weak equivalences. For $c^\#$ is the composite map

$$F(G/K_+, B) \xrightarrow[\simeq]{c^\#} F((G_+)_{hK}, B) \cong F(G_+, B)^{hK}$$

functionally dual to the collapse map $c \colon (G_+)_{hK} = (G \times_K EK)_+ \to G/K_+$, which is a stable equivalence by part (b) of the hypothesis that K is allowable.

Therefore, the induced extension $B \wedge_A C \to B \wedge_A B$ is weakly equivalent to the map $\pi^\# \colon F(G/K_+, B) \to F(G_+, B)$ functionally dual to the projection $\pi \colon G \to G/K$. By part (c) of the hypothesis that K is allowable there is a weak equivalence

$$F(G_+, B) \simeq F((K \times G/K)_+, B) \cong F(K_+, F(G/K_+, B)),$$

compatible with the commutative S-algebra maps $\pi^\#$ and $pr_2^\#$ from $F(G/K_+, B)$, so that $\pi^\#$ is indeed weakly equivalent to the trivial K-Galois extension (Section 5.1) of $F(G/K_+, B)$. In particular, $B \wedge_A C \to B \wedge_A B$ is faithfully K-Galois, and so by the faithful and dualizable detection result Lemma 7.1.4 it follows that $C \to B$ is faithfully K-Galois.

If furthermore K is normal in G, then the induced extension $B \to B \wedge_A C$ is weakly equivalent to the map $\pi^\#\colon B \to F(G/K_+, B)$ functionally dual to the collapse map $\pi\colon G/K \to \{e\}$, i.e., to the trivial G/K-Galois extension of B. So $B \to B \wedge_A C$ is faithfully G/K-Galois, and by Lemma 7.1.4 we can conclude that $A \to C$ is faithfully G/K-Galois. □

The following lemma will be applied in Section 9.1, when we discuss separable extensions.

Lemma 7.2.5. *Let $A \to B$ be a faithful G-Galois extension and $K \subset G$ an allowable subgroup. Then $C = B^{hK}$ is faithful and dualizable over A, and the canonical map $\kappa\colon B^{hK} \wedge_A B^{hK} \to (B \wedge_A B)^{h(K\times K)}$ is a weak equivalence.*

Proof. It is formal that $A \to C$ is faithful when the composite $A \to C \to B$ is faithful. For if $N \in \mathcal{M}_A$ has $N \wedge_A C \simeq *$ then $N \wedge_A B \cong N \wedge_A C \wedge_C B \simeq *$, so $N \simeq *$.

The extension $A \to B$ is faithful with B dualizable over A by Proposition 6.2.1, and $B \wedge_A C \simeq F(G/K_+, B)$ as in (7.2.4) is dualizable over B, since $S[G/K]$ is assumed to be a retract up to homotopy of $S[G]$ and therefore is dualizable over S. Thus C is dualizable over A by Lemma 6.2.4.

The map κ factors as the composite of two weak equivalences

$$B^{hK} \wedge_A B^{hK} \xrightarrow[\simeq]{\nu'} (B \wedge_A B^{hK})^{hK} \xrightarrow[\simeq]{(\nu')^{hK}} (B \wedge_A B)^{h(K\times K)}$$

derived from Lemma 6.2.6, where the first uses that $C = B^{hK}$ (on the right hand side of the smash product) is dualizable over A, and the second uses that B (on the left hand side of the smash product) is dualizable over A. □

8. Pro-Galois extensions and the Amitsur complex

We continue to let E be a fixed S-module and to work entirely in the E-local category.

8.1. Pro-Galois extensions.

Definition 8.1.1. Let A be an E-local cofibrant commutative S-algebra, and consider a directed system of E-local finite G_α-Galois extensions $A \to B_\alpha$, such that $B_\alpha \to B_\beta$ is a cofibration of commutative A-algebras for each $\alpha \le \beta$. Suppose further that each $A \to B_\alpha$ is an E-local sub-Galois extension of $A \to B_\beta$, so/such that there is a preferred surjection $G_\beta \to G_\alpha$ with kernel $K_{\alpha\beta}$, and a natural weak equivalence $B_\alpha \simeq B_\beta^{hK_{\alpha\beta}}$. Let $B = \operatorname{colim}_\alpha B_\alpha$, where the colimit is formed in $\mathcal{C}_{A,E}$, and let $G = \lim_\alpha G_\alpha$, with the (profinite) limit topology. Then, by definition, $A \to B$ is an *E-local pro-G-Galois extension.*

More generally, one might consider a directed system of (E-local) Galois extensions with stably dualizable (rather than finite) Galois groups G_α, arranging that each normal subgroup $K_{\alpha\beta}$ is stably dualizable. We prefer to wait for some relevant examples before discussing the analog of the Krull topology on the resulting limit group G, but compatibility with the "natural topology" on E-local Hom-sets (see [HPS97, §4.4] and [HSt99, §11]) is certainly desirable.

For each α the weak equivalence $h_\alpha \colon B_\alpha \wedge_A B_\alpha \to F(G_{\alpha+}, B_\alpha)$ extends by Lemma 6.1.2(a) to a weak equivalence $h_{\alpha,B} \colon B \wedge_A B_\alpha \to F(G_{\alpha+}, B)$. The colimit of these over α is a weak equivalence

$$h \colon B \wedge_A B \to F(\!(G_+, B)\!)\,, \tag{8.1.2}$$

where by definition $F(\!(G_+, B)\!) = \operatorname{colim}_\alpha F(G_{\alpha+}, B)$ is the "continuous" mapping spectrum with respect to the Krull topology, and $\operatorname{colim}_\alpha B \wedge_A B_\alpha = B \wedge_A \operatorname{colim}_\alpha B_\alpha = B \wedge_A B$, since pushout with B commutes with colimits in the category of commutative A-algebras.

Likewise, for each α the weak equivalence $j_\alpha \colon B_\alpha\langle G_\alpha\rangle \to F_A(B_\alpha, B_\alpha)$ extends by Lemma 6.1.2(c) to a weak equivalence $j_{\alpha,B} \colon B\langle G_\alpha\rangle \to F_A(B_\alpha, B)$. The limit of these over α is a weak equivalence

$$j \colon B\langle\!\langle G\rangle\!\rangle \to F_A(B, B)\,, \tag{8.1.3}$$

where by definition $B\langle\!\langle G\rangle\!\rangle = \lim_\alpha B\langle G_\alpha\rangle$ is the "completed" twisted group A-algebra, and $\lim_\alpha F_A(B_\alpha, B) \cong F_A(\operatorname{colim}_\alpha B_\alpha, B) = F_A(B, B)$.

Example 8.1.4. In the case of the $K(n)$-local pro-$\mathbb{G}_n$-Galois extension $L_{K(n)}S \to E_n$, these weak equivalences induce the isomorphism

$$\Phi \colon E_{n*}^\vee(E_n) \cong \operatorname{Map}(\mathbb{G}_n, \pi_*(E_n))$$

that is implicit in [Mo85] and explicit in [St00, Thm. 12] and [Hov04, 4.11], and the isomorphism

$$\Psi \colon E_{n*}\langle\!\langle \mathbb{G}_n\rangle\!\rangle \cong E_n^*(E_n)$$

from [St00, p. 1029] and [Hov04, 5.1]. The appearance of the continuous mapping space and the completed twisted group ring corresponds to the spectrum level colimits and limits above, combined with the I_n-adic completion at the level of homotopy groups induced by the implicit $K(n)$-localization [HSt99, 7.10(e)].

The pro-Galois formalism thus accounts for the first steps in a proof of Gross–Hopkins duality [HG94], following [St00]. The next step would be to study the $K(n)$-local functional dual of E_n as the continuous homotopy fixed point spectrum

$$L_{K(n)}DE_n = F(E_n, L_{K(n)}S) \simeq F(E_n, E_n)^{h\mathbb{G}_n} \simeq (E_n\langle\!\langle \mathbb{G}_n\rangle\!\rangle)^{h\mathbb{G}_n}\,,$$

but here technical issues related to the continuous cohomology of profinite groups arise, which are equivalent to those handled by Strickland.

8.2. The Amitsur complex.

As usual, let A be a cofibrant commutative S-algebra and B a cofibrant commutative A-algebra.

Definition 8.2.1. The (additive) *Amitsur complex* [Am59, §5], [KO74, §II.2] is the cosimplicial commutative A-algebra

$$C^\bullet(B/A) \colon [q] \mapsto B \otimes_A [q] = B \wedge_A \cdots \wedge_A B$$

($(q+1)$ copies of B), coaugmented by $A \to B = C^0(B/A)$. Here $B \otimes_A [q]$ refers to the tensored structure in $\mathcal{C}_{A,E}$, and the cosimplicial structure is derived from the functoriality of this construction. In particular, the i-th coface map is induced by smashing with $A \to B$ after the i first copies of B, and the j-th codegeneracy map is induced by smashing with $B \wedge_A B \to B$ after the j first copies of B.

Let the *completion of A along B* be the totalization $A^\wedge_B = \operatorname{Tot} C^\bullet(B/A)$ of this cosimplicial resolution. The coaugmentation induces a natural completion map $\eta \colon A \to A^\wedge_B$ of commutative A-algebras.

Gunnar Carlsson has considered this form of completion in his work on the descent problem for the algebraic K-theory of fields [Ca:d, §3], and Bendersky–Thompson have considered an unstable analog in [BT00]. It compares perfectly with Bousfield's B-nilpotent completion [Bo79, §5], as extended from spectra to the context of A-modules.

Definition 8.2.2. The canonical B-based Adams resolution of A in A-modules is the diagram below, inductively defined from $D_0 = A$ by letting D_{s+1} be the homotopy fiber of the natural map $D_s = A \wedge_A D_s \to B \wedge_A D_s$ for all $s \geq 0$.

$$\begin{tikzcd}
A \arrow[d] & D_1 \arrow[l] \arrow[d] & D_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
B \arrow[ur, dashed] & B \wedge_A D_1 \arrow[ur, dashed] & B \wedge_A D_2 \arrow[ur, dashed] &
\end{tikzcd}$$

Continuing, K_s is defined to be the homotopy cofiber of the composite map $D_s \to A$, and the *B-nilpotent completion* of A in A-modules is the homotopy limit $\hat{L}^A_B A = \operatorname{holim}_s K_s$.

Lemma 8.2.3. *The completion $A^\wedge_B$ of A along B is weakly equivalent to the Bousfield B-nilpotent completion $\hat{L}^A_B A$ of A formed in A-modules.*

Proof. One proof uses Bousfield's paper [Bo03] on cosimplicial resolutions. The functor $\Gamma(M) = B \wedge_A M$ defines a triple, or monad, on $\mathcal{M}_A$, and $A \to C^\bullet(B/A)$ is the corresponding triple resolution of A [Bo03, §7]. The B-module spectra define a class $\mathcal{G}$ of injective models in $\mathcal{D}_A$, whose $\mathcal{G}$-completion is the B-nilpotent completion in A-modules, by [Bo79, 5.8] and [Bo03, 5.7]. It agrees with the totalization of the triple resolution by [Bo03, 6.5], which by definition is the completion of A along B, in the sense above.

A more computational proof follows the unstable case of [BK73, §3–5], especially 5.3. There is an "iterated boundary isomorphism" from the E_1-term of the Bousfield–Kan spectral sequence associated to the Tot-tower of the cosimplicial spectrum $C^\bullet(B/A)$, to the E_1-term of the Adams spectral sequence associated to the tower of derived spectra $\{D_s\}_s$. The isomorphisms persist, with a shift in indexing, upon passage to the tower of cofibers $\{K_s\}_s$. Since $\operatorname{Tot}_0 C^\bullet(B/A) = B \simeq K_1$, it follows that the homotopy limits $A^\wedge_B$ and $\hat{L}^A_B A$ are also weakly equivalent. $\square$

More generally, for each functor F from commutative A-algebras to a category of spaces or spectra, like the units functor $U = GL_1$, the Amitsur complex $C^\bullet(B/A; F)$ is the cosimplicial object $[q] \mapsto F(B \otimes_A [q])$. It is natural to consider the colimit of its totalization, as B ranges over a class of A-algebras. When F is the identity functor, this is the completion defined above. When $A \to B$ is Galois, or ranges through all Galois extensions, we obtain forms of Amitsur cohomology [Am59] and Galois cohomology [CHR65, §5]. Note that if $\operatorname{Spec} B$ is thought of as a covering of $\operatorname{Spec} A$, then $\operatorname{Spec}(B \wedge_A B)$ consists of the covering of $\operatorname{Spec} A$ by double intersections, or fiber products, from the first covering, and likewise for $\operatorname{Spec} C^q(B/A)$ and $(q+1)$-fold intersections. We are therefore recovering a form of Čech cohomology. In general, the appropriate context for what classes of extensions $A \to B$ to consider is that of a *Grothendieck model topology* on the category of commutative A-algebras, or a *model site.* We simply refer to [TV05] for a detailed exposition on this matter.

The following is a form of faithfully projective descent.

Lemma 8.2.4. *If B is faithful and dualizable over A, then $\eta\colon A \to A^\wedge_B$ is a weak equivalence, i.e., A is complete along B.*

Proof. It suffices to prove that $1 \wedge \eta\colon B \wedge_A A \to B \wedge_A A^\wedge_B$ is a weak equivalence. Here $B \wedge_A A^\wedge_B \simeq F_A(D_A B, \operatorname{Tot} C^\bullet(B/A)) \cong \operatorname{Tot} F_A(D_A B, C^\bullet(B/A)) \simeq \operatorname{Tot} B \wedge_A C^\bullet(B/A)$, and

$$B \wedge_A C^\bullet(B/A)\colon [q] \to B \wedge_A (B \otimes_A [q]) \cong B \otimes_A [q]_+$$

admits a cosimplicial contraction to B, so $1 \wedge \eta$ is indeed a weak equivalence. $\square$

Let G be a topological group acting from the left on an S-module M, and let

$$EG_\bullet = B(G, G, *)\colon [q] \mapsto \operatorname{Map}([q], G) \cong G^{q+1}$$

be the usual free contractible simplicial left G-space.

Definition 8.2.5. The (group) *cobar complex* for G acting on M is the cosimplicial S-module

$$C^\bullet(G; M) = F(EG_{\bullet +}, M)^G\colon [q] \mapsto F(G_+^{q+1}, M)^G \cong F(G_+^q, M)\,.$$

Its totalization is the homotopy fixed point spectrum $M^{hG} = \operatorname{Tot} C^\bullet(G; M)$.

Here the standard identification $F(G_+^{q+1}, M)^G \cong F(G_+^q, M)$ takes the left G-map $f\colon G_+^{q+1} \to M$ to the map $\phi\colon G_+^q \to M$ that satisfies

$$\begin{aligned} f(g_0, \dots, g_q) &= g_0 \cdot \phi([g_0^{-1} g_1| \dots |g_{q-1}^{-1} g_q]) \\ \phi([h_1| \dots |h_q]) &= f(e, h_1, \dots, h_1 \dots h_q) \end{aligned}$$

(adapted as needed to make sense when the target is a spectrum).

In the presence of a left G-action on B through commutative A-algebra maps, these two cosimplicial constructions can be compared.

Definition 8.2.6. There is a natural map of cosimplicial commutative A-algebras $h^\bullet \colon C^\bullet(B/A) \to C^\bullet(G; B)$ given in codegree q by the map

$$h^q \colon B \wedge_A \cdots \wedge_A B \to F(G_+^{q+1}, B)^G \cong F(G_+^q, B)$$

given symbolically by

$$\begin{aligned} b_0 \wedge \cdots \wedge b_q \mapsto \big(f \colon (g_0, \dots, g_q) \mapsto g_0(b_0) \cdot \dots \cdot g_q(b_q)\big) \\ \cong \big(\phi \colon [h_1| \dots |h_q] \mapsto b_0 \cdot h_1(b_1) \cdot \dots \cdot (h_1 \dots h_q)(b_q)\big) . \end{aligned}$$

On totalizations, $h^\bullet$ induces a natural map of commutative A-algebras $h' \colon A^\wedge_B \to B^{hG}$.

In codegree 1, we can recognize $h^1 \colon B \wedge_A B \to F(G_+, B)$ as the canonical map h from (4.1.2). It is not hard to give a formal definition of h^q as the right adjoint of a G-equivariant map $B \otimes_A [q] \wedge \mathrm{Map}([q], G)_+ \to B$.

Lemma 8.2.7. *Let G be a stably dualizable group acting on B through A-algebra maps, and suppose that $h \colon B \wedge_A B \to F(G_+, B)$ is a weak equivalence. Then $h^\bullet$ is a codegreewise weak equivalence that induces a weak equivalence $h' \colon A^\wedge_B \to B^{hG}$.*

Proof. In each codegree q, the map h^q factors as a composite of weak equivalences of the form

$$\begin{aligned} B^{\wedge_A i} \wedge_A F(G_+^j, B) \xrightarrow{\simeq} B^{\wedge_A (i-1)} \wedge_A F(G_+^j, B \wedge_A B) \\ \xrightarrow{\simeq} B^{\wedge_A (i-1)} \wedge_A F(G_+^j, F(G_+, B)) \cong B^{\wedge_A (i-1)} \wedge_A F(G_+^{(j+1)}, B) \end{aligned}$$

with $j = 0, \dots, q-1$ and $i + j = q$. Here the first map is a weak equivalence because G, and thus G^j, is stably dualizable, and the second map is a weak equivalence because $h \colon B \wedge_A B \to F(G_+, B)$ is assumed to be one. The claim follows by induction. □

The following is close to Proposition 6.3.2. See also Proposition 12.1.8.

Proposition 8.2.8. *Let G be a stably dualizable group acting on B through commutative A-algebra maps, and suppose that $h \colon B \wedge_A B \to F(G_+, B)$ is a weak equivalence. Then $A \to B$ is G-Galois if and only if A is complete along B*

Proof. We have $i = h' \circ \eta$, with h' a weak equivalence, so $i \colon A \to B^{hG}$ is a weak equivalence if and only if $\eta \colon A \to A^\wedge_B$ is a weak equivalence. □

9. Separable and étale extensions

We now address structured ring spectrum analogs of the unique lifting properties in covering spaces, continuing to work implicitly in some E-local category. Throughout, we let A be a cofibrant commutative S-algebra and B a cofibrant associative or cofibrant commutative A-algebra. (There appear to be interesting intermediate theories of E_n A-ring spectra for $1 \leq n \leq \infty$, in the operadic sense, but we shall focus on the extreme cases of $E_1 = A_\infty$ A-ring spectra, i.e., associative A-algebras, and E_∞ A-ring spectra, i.e., commutative A-algebras.)

Our main observations are that G-Galois extensions $A \to B$ with G discrete are necessarily separable and dualizable, hence symmetrically étale (= thh-étale) and étale (= taq-étale). In most cases of current interest, including $E = S$ and $E = K(n)$ for $0 \le n \le \infty$, a discrete group G is stably dualizable if and only if it is finite.

9.1. Separable extensions.

The algebraic definition [KO74, p. 74] of a separable extension of commutative rings can be adapted to stable homotopy theory as follows.

Definition 9.1.1. We say that $A \to B$ is *separable* if the A-algebra multiplication map $\mu\colon B \wedge_A B^{op} \to B$, considered as a map in the stable homotopy category $\mathcal{D}_{B\wedge_A B^{op}}$ of B-bimodules relative to A, admits a section $\sigma\colon B \to B \wedge_A B^{op}$. Equivalently, there is a map $\sigma\colon B' \to B \wedge_A B^{op}$ of B-bimodules relative to A, such that the composite $\mu\sigma\colon B' \to B$ is a weak equivalence.

Here B^{op} is B with the opposite A-algebra multiplication $\mu\gamma\colon B \wedge_A B \cong B \wedge_A B \to B$. It equals B precisely when B is commutative. Since B will rarely be cofibrant as a B-bimodule relative to A, it is only reasonable to ask for the existence of a bimodule section σ in the stable homotopy category. The condition for $A \to B$ to be separable only involves the bimodule structure on B, so it is quite accessible to verification by calculation. For example, it is equivalent to the condition that the algebra multiplication μ induces a surjection

$$\mu_\#\colon THH_A^0(B, B \wedge_A B^{op}) \to THH_A^0(B, B)$$

of zero-th topological Hochschild cohomology groups. See [La01, 9.3] for a spectral sequence computing the latter in many cases.

Lemma 9.1.2. *Let $A \to B$ be a G-Galois extension, with G a discrete group. Then $A \to B$ is separable.*

Proof. Let $d\colon G_+ \to \{e\}_+$ be the continuous (Kronecker delta) map given by $d(e) = e$ (the unit element in G) and $d(g) = *$ (the base point) for $g \ne e$. Its functional dual

$$in_e = d^\#\colon B \cong F(\{e\}_+, B) \to F(G_+, B)$$

and the canonical weak equivalence h define the required weak B-bimodule section $\sigma = h^{-1} \circ in_e$ to μ, as a morphism in the stable homotopy category.

(9.1.3)
$$\begin{tikzcd} B \arrow[r, dashed, "\sigma"] \arrow[dr, "in_e"'] & B \wedge_A B \arrow[r, "\mu"] \arrow[d, "h", "\simeq"'] & B \\ & \prod_G B \arrow[ur, "pr_e"'] & \end{tikzcd}$$

□

Proposition 9.1.4. *Let $A \to B$ be a faithful G-Galois extension, with G a discrete group and $K \subset G$ any subgroup. Then $A \to C = B^{hK}$ is separable.*

Proof. By Example 7.2.2, any subgroup K of G is allowable. We are therefore in the situation of Lemma 7.2.5.

The map $h \colon B \wedge_A B \to \prod_G B$ is $(K \times K)$-equivariant with respect to the action $(k_1, k_2) \cdot (b_1 \wedge b_2) = k_1(b_1) \wedge k_2(b_2)$ in the source, and the action that takes a sequence $\{g \mapsto \phi(g)\}$ to the sequence $\{g \mapsto k_1(\phi(k_1^{-1} g k_2))\}$ in the target. There are maps

$$\prod_K B \xrightarrow{in_K} \prod_G B \xrightarrow{pr_K} \prod_K B$$

functionally dual to a characteristic map $d_K \colon G_+ \to K_+$ (taking $G \setminus K$ to the base point) and the inclusion $K_+ \subset G_+$, whose composite is the identity. We give $\prod_K B$ the $(K \times K)$-action that takes $\{k \mapsto \phi(k)\}$ to $\{k \mapsto k_1(\phi(k_1^{-1} k k_2))\}$, so that in_K and pr_K are $(K \times K)$-equivariant. The weak equivalence $B \to (\prod_K B)^{hK}$ induces a natural weak equivalence $B^{hK} \to (\prod_K B)^{h(K \times K)}$ that makes the following diagram commute:

$$\begin{tikzcd}
B^{hK} \arrow[r, dashed] \arrow[d, "="'] & B^{hK} \wedge_A B^{hK} \arrow[r, "\mu"] \arrow[d, "\simeq"', "\kappa"] & B^{hK} \arrow[d, "="] \\
B^{hK} \arrow[r, dashed] \arrow[d, "\simeq"'] & (B \wedge_A B)^{h(K \times K)} \arrow[r] \arrow[d, "\simeq"', "h_\#"] & B^{hK} \arrow[d, "\simeq"] \\
(\prod_K B)^{h(K \times K)} \arrow[r, "in_{K\#}"] & (\prod_G B)^{h(K \times K)} \arrow[r, "pr_{K\#}"] & (\prod_K B)^{h(K \times K)}
\end{tikzcd}$$

The vertical map κ is a weak equivalence by Lemma 7.2.5, and the maps $h_\#$ and $pr_{K\#} \circ in_{K\#}$ are obtained from weak equivalences by passage to $(K \times K)$-homotopy fixed points, so a little diagram chase shows that $\mu \colon B^{hK} \wedge_A B^{hK} \to B^{hK}$ does indeed admit a weak bimodule section. □

Remark 9.1.5. It is easy to see that separable extensions are preserved by base change. To detect separable extensions by faithful base change will require some additional hypotheses, as in [KO74, III.2.2].

9.2. Symmetrically étale extensions.

The *topological Hochschild homology* $THH^A(B)$ of B relative to A is the geometric realization of a simplicial A-module

$$B \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; B \wedge_A B \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; B \wedge_A B \wedge_A B \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots$$

with the smash product of $(q+1)$ copies of B in degree q. See [EKMM97, IX.2]. Alternatively, $THH^A(B)$ can be computed in the stable homotopy category as

$$\mathrm{Tor}^{B \wedge_A B^{op}}(B, B) = B \wedge^L_{B \wedge_A B^{op}} B \,.$$

In the case $A = S$, we will often write $THH(B)$ for $THH^S(B)$, which agrees with the topological Hochschild homology introduced by Marcel Bökstedt [BHM93]. The inclusion of 0-simplices defines a natural map $\zeta\colon B \to THH^A(B)$. When B is commutative, $THH^A(B)$ can be expressed in terms of the topologically tensored structure on $\mathcal{C}_A$ as $B \otimes_A S^1$.

It is also possible to define $THH^A(B)$ for non-commutative A, by analogy with the definition of Hochschild homology over a non-commutative ground ring [Lo98, 1.2.11], but we have found no occasion to make use of this more general definition.

Definition 9.2.1. We say that $A \to B$ is *formally symmetrically étale* (= *formally thh-étale*) if the map $\zeta\colon B \to THH^A(B)$ is a weak equivalence. If furthermore B is dualizable as an A-module, then we say that $A \to B$ is *symmetrically étale* (= *thh-étale*).

Remark 9.2.2. This definition of an (symmetrically) étale map does not quite conform to the algebraic case, in that it may be too restrictive to ask that B is dualizable as an A-module. Instead, it is likely to be more appropriate to only impose the dualizability condition locally with respect to some Zariski open cover of $\operatorname{Spec} A$. This may be taken to mean that for some set of (smashing, Bousfield) localization functors $\{L_{E_i}\}_i$, such that the collection $\{A \to L_{E_i}A\}_i$ is a faithful cover in the sense of Definition 4.3.1, each localization $L_{E_i}B$ is dualizable as an $L_{E_i}A$-module. The author is undecided about exactly which localization functors to allow. However, for Galois extensions the stronger (global) dualizability hypothesis will always be satisfied, and this may permit us to leave the issue open.

Example 9.2.3. Note that Definition 9.2.1 implicitly takes place in an E-local category. By McClure–Staffeldt [MS93, 5.1] at odd primes p, and Angeltveit–Rognes [AnR:h, 8.10] at $p = 2$, the inclusion $\zeta\colon \ell \to THH(\ell)$ is a $K(1)$-local equivalence, where $\ell = BP\langle 1\rangle$ is the p-local connective Adams summand of topological K-theory, so $S \to \ell$ is $K(1)$-locally formally symmetrically étale. It also follows that the localization of this map, $J_p^\wedge = L_{K(1)}S \to L_{K(1)}\ell = L_p^\wedge$ is $K(1)$-locally formally symmetrically étale. Here $L_p^\wedge$ is the p-complete periodic Adams summand, as in 5.5.2.

These maps are not $K(1)$-locally symmetrically étale, because $L_p^\wedge$ is not dualizable as a $J_p^\wedge$-module. More globally, $S \to L_p^\wedge$ fails to be $E(1)$-locally formally symmetrically étale. For by [MS93, 8.1], $THH(L_p^\wedge) \simeq L_p^\wedge \vee L_0(\Sigma L_p^\wedge)$, so ζ has a rationally non-trivial cofiber.

Similarly, $\zeta\colon ku \to THH(ku)$ is a $K(1)$-homology equivalence by Christian Ausoni's calculation [Au:t, 6.5] for p odd, and [AnR, 8.10] again for $p = 2$, so the map $S \to ku$ to connective topological K-theory, and its $K(1)$-localization $J_p^\wedge \to KU_p^\wedge$, are $K(1)$-locally formally symmetrically étale. The map $L_p^\wedge \to KU_p^\wedge$ is $K(1)$-locally $\mathbb{F}_p^*$-Galois, as noted in 5.5.2, so by Lemma 9.2.6 below $L_p^\wedge \to KU_p^\wedge$ is $K(1)$-locally symmetrically étale. In other words, $\zeta\colon ku \to THH^\ell(ku)$ and $\zeta\colon KU_p^\wedge \to THH^{L_p^\wedge}(KU_p^\wedge)$ are $K(1)$-local equivalences.

The terminology "thh-étale" is that of Randy McCarthy and Vahagn Minasian [MM03, 3.2], except that for brevity they suppress the distinction between the formal and non-formal cases. The author's lengthier term "symmetrically étale" was motivated by the following definitions and result.

Definition 9.2.4. Let M be a B-bimodule relative to A, i.e., a $B \wedge_A B^{op}$-module. The space of *associative A-algebra derivations of B with values in M* is defined to be the derived mapping space

$$\mathcal{A}\mathrm{Der}_A(B, M) := (\mathcal{A}_A/B)(B, B \vee M)$$

in the topological model category of associative A-algebras over B, where $pr_1 \colon B \vee M \to B$ is the square-zero A-algebra extension of B with fiber M. We say that a B-bimodule relative to A is *symmetric* if it has the form $\mu^! N$ for some B-module N, i.e., if the bimodule action is obtained by composing with the A-algebra multiplication map $\mu \colon B \wedge_A B^{op} \to B$.

Proposition 9.2.5. *$A \to B$ is formally symmetrically étale if and only if the space of associative derivations $\mathcal{A}\mathrm{Der}_A(B, M)$ is contractible for each symmetric B-bimodule M.*

Proof. Let $\Omega_{B/A}$ be a cofibrant replacement of the homotopy fiber of $\mu \colon B \wedge_A B^{op} \to B$ in the category of B-bimodules relative to A. There is a cofiber sequence

$$B \wedge_{B \wedge_A B^{op}} \Omega_{B/A} \to B \xrightarrow{\zeta} THH^A(B)$$

and for each B-module N, with associated symmetric B-bimodule $M = \mu^! N$, there is an adjunction equivalence

$$\mathcal{M}_{B \wedge_A B^{op}}(\Omega_{B/A}, M) \simeq \mathcal{M}_B(B \wedge_{B \wedge_A B^{op}} \Omega_{B/A}, N)\,.$$

Furthermore, there is an equivalence (for each $B \wedge_A B^{op}$-module M)

$$\mathcal{A}\mathrm{Der}_A(B, M) = (\mathcal{A}_A/B)(B, B \vee M) \simeq \mathcal{M}_{B \wedge_A B^{op}}(\Omega_{B/A}, M)$$

obtained by Lazarev [La01, 2.2]. So ζ is an equivalence if and only if $B \wedge_{B \wedge_A B^{op}} \Omega_{B/A} \simeq *$, which is equivalent to $\mathcal{A}\mathrm{Der}_A(B, M) \simeq \mathcal{M}_B(B \wedge_{B \wedge_A B^{op}} \Omega_{B/A}, N)$ being contractible for each symmetric B-bimodule $M = \mu^! N$.

In the E-local context, this argument shows that $E_*(\zeta)$ is an isomorphism if and only if $\mathcal{A}\mathrm{Der}_A(B, M) \simeq *$ for each E-local symmetric B-module M. For $\mathcal{A}_{A,E}/B$ is a full subcategory of $\mathcal{A}_A/B$, and likewise for the homotopy categories. □

Lemma 9.2.6. *Each separable extension $A \to B$ of commutative S-algebras is formally symmetrically étale. In particular, each G-Galois extension $A \to B$ with G discrete is symmetrically étale.*

Proof. By assumption there is a bimodule section σ so that the composite $B \xrightarrow{\sigma} B \wedge_A B^{op} \xrightarrow{\mu} B$ is homotopic to the identity. Smashing with B over $B \wedge_A B^{op}$ tells us that the composite

$$THH^A(B) \xrightarrow{\sigma \wedge 1} B \xrightarrow{\zeta} THH^A(B)$$

is an equivalence. Furthermore, there is a retraction $\rho\colon THH^A(B) \to B$ given in simplicial degree q by the iterated multiplication map $\mu^{(q)}\colon B \wedge_A \cdots \wedge_A B \to B$, since we are assuming that B is commutative. Therefore ζ admits a right and a left inverse, up to homotopy, and is therefore a weak equivalence.

When $A \to B$ is G-Galois with G discrete, we showed in Lemma 9.1.2 that $A \to B$ is separable and in Proposition 6.2.1 that B is a dualizable A-module. The above argument then implies that $A \to B$ is symmetrically étale. □

9.3. Smashing maps.

Maps $A \to B$ having the corresponding property to the conclusion of Proposition 9.2.5 for associative derivations into arbitrary (not necessarily symmetric) B-bimodules relative to A, also have a familiar characterization. This material is not needed for our Galois theory, but nicely illustrates the relation of smashing localizations (and Zariski open sub-objects) to étale and symmetrically étale maps.

Definition 9.3.1. We say that $A \to B$ is *smashing* if the algebra multiplication map $\mu\colon B \wedge_A B^{op} \to B$ is a weak equivalence.

In view of the following proposition, smashing maps could also be called formally associatively étale extensions.

Proposition 9.3.2. *$A \to B$ is smashing if and only if $\mathcal{A}\mathrm{Der}_A(B, M)$ is contractible for each B-bimodule M relative to A.*

Proof. This is immediate from the equivalence

$$\mathcal{A}\mathrm{Der}_A(B, M) \simeq \mathcal{M}_{B\wedge_A B^{op}}(\Omega_{B/A}, M)$$

from [La01], since $A \to B$ is smashing if and only if $\Omega_{B/A} \simeq *$. □

The terminology is explained by the following result, one part of which the author learned from Mark Hovey.

Proposition 9.3.3. *$A \to B$ is smashing if and only if $LM = B \wedge_A M$ defines a smashing Bousfield localization functor on $\mathcal{M}_A$, in which case $B = LA$. In particular, B will be a commutative A-algebra.*

Proof. Let $B^A_*(-)$ be the homotopy functor on $\mathcal{M}_A$ defined by $B^A_*(M) = \pi_*(B \wedge_A M)$. The natural map $M \to B \wedge_A M$ is a B^A_*-equivalence, since $A \to B$ is smashing, and $B \wedge_A M$ is B^A_*-local by the prototypical ring spectrum argument of Adams [Ad71]: if $B \wedge_A Z \simeq *$ then any map $f\colon Z \to B \wedge_A M$ factors as

$$Z \to B \wedge_A Z \xrightarrow{1\wedge f} B \wedge_A B \wedge_A M \xrightarrow{\mu\wedge 1} B \wedge_A M$$

and is therefore null-homotopic. So $LM = B\wedge_A M$ defines a (Bousfield) localization functor L on $\mathcal{M}_A$.

Conversely, a smashing localization functor L on $\mathcal{M}_A$ produces an associative A-algebra $B = LA$, by [EKMM97, VIII.2.1], such that $LM \simeq B\wedge_A M$ (since L is assumed to be smashing). The idempotency of L then ensures that the multiplication map $B \wedge_A B^{op} \to B$ is a weak equivalence. □

Lemma 9.3.4. *Each smashing map $A \to LA$ is separable, hence formally symmetrically étale.*

Proof. If $A \to B = LA$ is smashing, then $\mu\colon B \wedge_A B^{op} \to B$ is an equivalence. It therefore admits a bimodule section σ up to homotopy, so $A \to B$ is separable. □

In general, LA is not dualizable as an A-module, as easy algebraic examples illustrate ($\mathbb{Z} \subset \mathbb{Z}_{(p)}$). Instead, the local dualizability of Remark 9.2.2 is more appropriate.

9.4. Étale extensions.

We keep on working implicitly in an E-local category, now with B a cofibrant commutative A-algebra.

For a map $A \to B$ of commutative S-algebras, the *topological André–Quillen homology* $TAQ(B/A)$ is defined in [Bas99, 4.1] as

$$TAQ(B/A) := (LQ_B)(RI_B)(B \wedge_A^L B)\,,$$

i.e., as the B-module of (left derived) indecomposables in the non-unital B-algebra given by the (right derived) augmentation ideal in the augmented B-algebra defined by the (left derived) smash product $B \wedge_A^L B$, augmented over B by the A-algebra multiplication μ.

Definition 9.4.1. Let $A \to B$ be a map of commutative S-algebras. We say that $A \to B$ is *formally étale* (= *formally taq-étale*) if $TAQ(B/A)$ is weakly equivalent to $*$. If furthermore B is dualizable as an A-module, then we say that $A \to B$ is *étale* (= *taq-étale*).

Like in Remark 9.2.2, the condition that B is dualizable over A is likely to be stronger than necessary for B to qualify as étale over A, and should eventually be replaced with a local condition over each subobject in an open cover of A. The apologetic discussion from the associative/symmetric case applies in the same way here.

The terminology is justified by the following definition and result from [Bas99]. The vanishing of $TAQ(B/A)$ gives a unique infinitesimal lifting property, up to contractible choice, for geometric maps into the affine covering represented (in the opposite category) by a formally étale map $A \to B$.

$$\begin{tikzcd} B \arrow[r, "="] \arrow[dr, dashed] & B \\ A \arrow[u] \arrow[r] & B \vee M \arrow[u] \end{tikzcd}$$

Compare [Mil80, I.3.22].

Definition 9.4.2. Let $A \to B$ be a map of commutative S-algebras and let M be a B-module. The space of *commutative A-algebra derivations of B with values in M* is defined to be the derived mapping space

$$\mathcal{C}\mathrm{Der}_A(B, M) := (\mathcal{C}_A/B)(B, B \vee M)$$

in the topological model category of commutative A-algebras over B, where $pr_1\colon B \vee M \to B$ is the square-zero extension of B with fiber M.

Proposition 9.4.3. *A map $A \to B$ of commutative S-algebras is formally étale if and only if $\mathcal{C}\mathrm{Der}_A(B, M)$ is contractible for each B-module M.*

Proof. There is an equivalence

$$\mathcal{C}\mathrm{Der}_A(B, M) = (\mathcal{C}_A/B)(B, B \vee M) \simeq \mathcal{M}_B(TAQ(B/A), M)$$

for each B-module M, by [Bas99, 3.2]. By considering the universal example $M = TAQ(B/A)$, we conclude that $TAQ(B/A) \simeq *$ if and only if $\mathcal{C}\mathrm{Der}_A(B, M) \simeq *$ for each B-module M. In the implicitly local context only E-local M occur, so we can conclude that $TAQ(B/A)$ is E-acyclic, i.e., E-locally weakly equivalent to $*$. □

For a finite commutative R-algebra T, the two conditions $T \cong HH_*^R(T)$ and $D_*(T/R) = AQ_*(T/R) = 0$ are logically equivalent [Gro67, 18.3.1(ii)], where HH_*^R denotes Hochschild homology and $D_* = AQ_*$ denotes André–Quillen homology. In the context of commutative S-algebras this is only true subject to a connectivity hypothesis [Min03, 2.8], due to a convergence issue in the analog of the Quillen spectral sequence from André–Quillen homology to Hochschild homology. However, one implication (from symmetrically étale to étale) does not depend on the connectivity hypothesis stated there. In other words, if $\zeta\colon B \to THH^A(B)$ is a weak equivalence, then $TAQ(B/A) \simeq *$. We discuss a proof below, based on [BMa05].

There is a counterexample to the opposite implication, due to Mike Mandell, which is discussed in [MM03, 3.5]. For $n \geq 2$ let $X = K(\mathbb{Z}/p, n)$ be an Eilenberg–Mac Lane space and let $B = F(X_+, H\mathbb{F}_p)$ be its mod p cochain $H\mathbb{F}_p$-algebra, with $\pi_*(B) = H^{-*}(K(\mathbb{Z}/p, n); \mathbb{F}_p)$. Then $H\mathbb{F}_p \to B$ is formally étale, but not symmetrically (=thh-)étale. So, any converse statement deducing that an étale map is symmetrically étale must contain additional hypotheses to exclude this example.

Lemma 9.4.4. *Each (formally) symmetrically étale extension $A \to B$ of commutative S-algebras is (formally) étale. In particular, each G-Galois extension $A \to B$ with G discrete is étale, and each smashing localization $A \to LA = B$ is formally étale.*

Proof. Recall that $THH^A(B) \simeq B \otimes_A S^1$ as commutative A-algebras. Here $\otimes_A$ denotes the tensored structure on $\mathcal{C}_A$ over unbased topological spaces. To describe the commutative B-algebra structure on $THH^A(B)$ in similar terms, and to relate it to the B-module $TAQ(B/A)$, we will need a tensored structure over based topological spaces. This makes sense when we replace $\mathcal{C}_A$ by the pointed category $\mathcal{C}_B/B$ of commutative B-algebras augmented over B. There is then a (reduced) tensor structure $(-)\widetilde{\otimes}_B X$ on $\mathcal{C}_B/B$ over based topological spaces X, with

$$(\mathcal{C}_B/B)(C\widetilde{\otimes}_B X, C') \cong \mathrm{Map}_*(X, (\mathcal{C}_B/B)(C, C')) ,$$

where Map_* denotes the base-point preserving mapping space. It follows that $(C\widetilde{\otimes}_B X)\widetilde{\otimes}_B Y \cong C\widetilde{\otimes}_B(X \wedge Y)$. The unbased and based tensored structures are related by $C\widetilde{\otimes}_B X \cong B \wedge_C (C \otimes_B X)$ and $C \otimes_B T \cong C\widetilde{\otimes}_B(T_+)$, for unbased spaces T.

There is a pointed model structure on $\mathcal{C}_B/B$, and the associated Quillen suspension functor E is given on cofibrant objects by the reduced tensor $\mathrm{E}(C) = C\widetilde{\otimes}_B S^1$ with the based circle. For each $n \geq 0$ we can form the n-fold iterated suspension

$$\mathrm{E}^n(C) = C\widetilde{\otimes}_B S^n$$

in $\mathcal{C}_B/B$, so that $\mathrm{E}(\mathrm{E}^n(C)) \cong \mathrm{E}^{n+1}(C)$, and these objects assemble to a sequential suspension spectrum $\mathrm{E}^\infty(C)$, in this category. By [BMa05, Thm. 3], the homotopy category of such spectra, up to stable equivalence, is equivalent to the homotopy category $\mathcal{D}_B$ of B-modules, up to weak equivalence.

Base change along $A \to B$ takes B to $B \wedge_A B$, which is a cofibrant commutative B-algebra, augmented over B by the multiplication map $\mu\colon B \wedge_A B \to B$. Hereafter, write $C = B \wedge_A B$ for brevity. By [BMa05, Thm. 4], the cited equivalence takes $\mathrm{E}^\infty(C)$ to the topological André–Quillen homology spectrum $TAQ(B/A)$. So $\mathrm{E}^\infty(C)$ is stably trivial if and only if $TAQ(B/A) \simeq *$, i.e., if and only if $A \to B$ is formally étale.

On the other hand,

$$\mathrm{E}(C) = C \widetilde{\otimes}_B S^1 \simeq B \wedge_C THH^B(C) \cong THH^A(B)\,,$$

now as commutative B-algebras. So $\mathrm{E}(C)$ is weakly trivial, i.e., weakly equivalent to the base point B in $\mathcal{C}_B/B$, if and only if $\zeta\colon B \to THH^A(B)$ is a weak equivalence.

The proof of the lemma is now straightforward. If $A \to B$ is formally symmetrically étale, then $\mathrm{E}(C)$ is weakly trivial, and therefore so is each of its suspensions $\mathrm{E}^n(C) = \mathrm{E}^{n-1}(\mathrm{E}(C))$ for $n \geq 1$. Thus the suspension spectrum $\mathrm{E}^\infty(C)$ is stably trivial (in a very strong sense), and so $TAQ(B/A)$ is weakly equivalent to the trivial B-module. □

In the notation of the above proof: $C = B \wedge_A B$ is weakly trivial in $\mathcal{C}_B/B$ if and only if $A \to B$ is smashing, $\mathrm{E}(C) = THH^A(B)$ is weakly trivial if and only if $A \to B$ is formally symmetrically étale, and $\mathrm{E}^\infty(C)$ is stably trivial if and only if $A \to B$ is formally étale.

9.5. Henselian maps.

By definition, an étale map $A \to B$ has the unique lifting property up to contractible choice for each square-zero extension of commutative A-algebras $B \vee M \to B$, and satisfies a finiteness condition. In this chapter we conversely ask which extensions $D \to C$ of commutative A-algebras are such that each étale map $A \to B$, with B mapping to C, has this homotopy unique lifting property with respect to $D \to C$.

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dashed] & C \\ A \arrow[u] \arrow[r] & D \arrow[u] \end{tikzcd}$$

We shall refer to such $D \to C$ as Henselian maps. Section 9.6 will exhibit some interesting examples of Henselian maps.

In the opposite category to that of commutative A-algebras, of affine algebro-geometric objects in a homotopy-theoretic sense [TV05, §5.1], we can view the square-zero extensions as infinitesimal thickenings of a special kind, forming a generating class of acyclic cofibrations. The étale extensions then correspond to smooth and unramified covering maps, and constitute a class of fibrations characterized by their right lifting property with respect to these generating acyclic cofibrations, together with a finiteness hypothesis. The Henselian maps, in turn characterized by their left lifting property with respect to these fibrations, then form a class of thickenings that contains all composites of the generating acyclic cofibrations of the theory, i.e., all infinitesimal thickenings, but which also encompasses many other maps. By comparison, in the algebraic context Hensel's lemma applies to a complete local ring mapping to its residue field, but also to many other cases.

For a fixed commutative S-algebra A, this discussion could take place as above in the context of commutative A-algebras, with maps from (taq-)étale extensions $A \to B$, but also in the alternate context of associative A-algebras, with maps from symmetrically (= thh-)étale extensions. To be concrete we shall focus on the commutative case, although all of the formal arguments carry over to the associative category and extensions by symmetric bimodules.

Throughout this section we continue to work E-locally, and let A be a cofibrant commutative S-algebra, $B \to C$ a map of commutative A-algebras and M any C-module. We sometimes consider M as a B-module by pull-back along $B \to C$. We always make the cofibrant and fibrant replacements required for homotopy invariance, implicitly.

Lemma 9.5.1. *The square-zero extension $B \vee M \to B$ is the pull-back in $\mathcal{C}_A$ of the square-zero extension $C \vee M \to C$ along $B \to C$,*

$$\begin{tikzcd}
B \arrow[r, "="] \arrow[dr, dashed] \arrow[drr, dashed] & B \arrow[r] & C \\
A \arrow[u] \arrow[r] & B \vee M \arrow[u] \arrow[r] & C \vee M \arrow[u, "pr_1"']
\end{tikzcd}$$

so there is a weak equivalence

$$(\mathcal{C}_A/B)(B, B \vee M) \simeq (\mathcal{C}_A/C)(B, C \vee M)\,.$$

In particular, both of these spaces are contractible whenever $A \to B$ is formally étale.

Proof. The pullback along $B \to C$ of a fibrant replacement for $C \vee M \to C$ is a fibrant replacement for $B \vee M \to B$, and forming mapping spaces from a cofibrant replacement for B in $\mathcal{C}_A$ has a left adjoint given by the tensored structure, hence commutes with pullbacks and other limits. So the homotopy fiber at the identity of B of $\mathcal{C}_A(B, B \vee M) \to \mathcal{C}_A(B, B)$ is weakly equivalent to the homotopy fiber at $B \to C$ of $\mathcal{C}_A(B, C \vee M) \to \mathcal{C}_A(B, C)$. □

Lemma 9.5.2. *The commutative diagram*

$$\begin{tikzcd}
B \arrow[rr] \arrow[dr, dashed] \arrow[drr, dashed] & & C \\
A \arrow[u] \arrow[r] & C \arrow[ur, "="] \arrow[r, "in_1"'] & C \vee M \arrow[u, "pr_1"']
\end{tikzcd}$$

yields a homotopy fiber sequence

$$(\mathcal{C}_A/C \vee M)(B, C) \to (\mathcal{C}_A/C)(B, C) \to (\mathcal{C}_A/C)(B, C \vee M)$$

for which the middle space is contractible. In particular, all three spaces are contractible whenever $A \to B$ is formally étale.

Proof. After replacing first pr_1 and then in_1 by fibrations, the mapping spaces in $\mathcal{C}_A$ from a cofibrant replacement for B to these fibrations sit in two fibrations p and i, whose composite $p \circ i$ is also a fibration. The fibers of the i, $p \circ i$ and p above $B \to C$ then form the desired fiber sequence. □

The following definition is the commutative analog of that in [La01, 3.3].

Definition 9.5.3. A map $\pi\colon D \to C$ of commutative A-algebras is a *singular extension* if there is an A-linear derivation of C with values in M, i.e., a commutative A-algebra map $d\colon C \to C \vee M$ over C, and a homotopy pull-back square

$$\begin{tikzcd} C \arrow[r, "d"] & C \vee M \\ D \arrow[u, "\pi"] \arrow[r] & C \arrow[u, "in_1"'] \end{tikzcd}$$

of commutative A-algebras.

For example, the square-zero extension $C \vee \Sigma^{-1} M \to C$ is the singular extension pulled back from the trivial derivation $d = in_1\colon C \to C \vee M$. So the class of singular extensions contains the class of square-zero extensions.

Lemma 9.5.4. *For each singular extension $\pi\colon D \to C$ the commutative diagram*

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dashed] \arrow[drr, dashed] & C \arrow[r, "d"] & C \vee M \\ A \arrow[u] \arrow[r] & D \arrow[u, "\pi"] \arrow[r] & C \arrow[u, "in_1"'] \end{tikzcd}$$

induces a weak equivalence

$$(\mathcal{C}_A/C)(B, D) \simeq (\mathcal{C}_A/C \vee M)(B, C)\,.$$

In particular, both of these spaces are contractible whenever $A \to B$ is formally étale.

Proof. The first part of the proof is like that of Lemma 9.5.1. The second claim follows by using the definition of formally étale maps to deduce that both mapping spaces are contractible for all formally étale maps $A \to B$, in the special case when $\pi\colon D = C \vee \Sigma^{-1} M \to C$ is the square-zero extension pulled back from the trivial derivation $d = in_1\colon C \to C \vee M$. The right hand mapping space does not depend on the particular singular extension, so it follows from the first claim applied to a general singular extension $\pi\colon D \to C$ that also the left hand mapping space is contractible for arbitrary singular extensions $D \to C$ and formally étale maps $A \to B$. □

In view of [Gro67, 18.5.5] or [Mil80, I.4.2(d)] we can make the following definition.

Definition 9.5.5. Let $D \to C$ be a map of commutative S-algebras. We say that $D \to C$ is *Henselian* if for each étale map $A \to B$, with B and D commutative A-algebras over C,

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dashed] & C \\ A \arrow[u] \arrow[r] & D \arrow[u] \end{tikzcd}$$

the derived mapping space

$$(\mathcal{C}_A/C)(B, D) \simeq *$$

is contractible, i.e., if $A \to B$ has the unique lifting property up to contractible choice with respect to $D \to C$.

If D is a commutative S-algebra and C is an associative D-algebra, we say that $D \to C$ is *symmetrically (= thh-)Henselian* if for each symmetrically (= thh-)étale map $A \to B$, in a diagram as above, the associative A-algebra mapping space $(\mathcal{A}_A/C)(B, D)$ is contractible.

By the following lemma it suffices (in the commutative case) to verify the homotopy unique lifting property for the étale maps $A \to B$ with $A = D$. For $A \to B$ étale implies $D \to B \wedge_A D$ étale by the base change formula $TAQ(B \wedge_A D/D) \simeq TAQ(B/A) \wedge_A D$ [Bas99, 4.6] and Lemma 6.2.3.

Lemma 9.5.6. *Let $B \to C$ and $D \to C$ be maps of commutative A-algebras, with pushout $B \wedge_A D \to C$. The commutative diagram*

$$\begin{tikzcd} B \arrow[r] \arrow[drr, dashed] & B \wedge_A D \arrow[r] \arrow[dr, dashed] & C \\ A \arrow[u] \arrow[r] & D \arrow[u] \arrow[r, "="'] & D \arrow[u] \end{tikzcd}$$

induces a weak equivalence

$$(\mathcal{C}_A/C)(B, D) \simeq (\mathcal{C}_D/C)(B \wedge_A D, D)\,.$$

Proof. Dual to the proof of Lemma 9.5.1. □

Proposition 9.5.7. *The class of Henselian maps $D \to C$ contains the square-zero extensions $C \vee M \to C$ and the singular extensions $\pi \colon D \to C$. It is closed under weak equivalences, compositions, retracts and filtered homotopy limits (for diagrams of maps to a fixed C).*

Proof. The first claims follow from Lemma 9.5.4 and the remark that square-zero extensions are trivial examples of singular extensions. The closure claims are clear, perhaps except for the the last one. If $\alpha \mapsto (D_\alpha \to C)$ is a diagram of Henselian maps to C, then let $D = \operatorname{holim}_\alpha D_\alpha$. For each étale map $A \to B$ (mapping to $D \to C$ as above) there is a weak equivalence

$$(\mathcal{C}_A/C)(B, D) \simeq \operatorname*{holim}_\alpha (\mathcal{C}_A/C)(B, D_\alpha) \simeq *\,,$$

since each $D_\alpha \to C$ is Henselian and the limit category is assumed to be filtering. $\square$

In fact, the Henselian maps that we will encounter in the following section are sequential homotopy limits of towers of singular extensions, and thus of a rather special form. If desired, the reader can view them as the residue maps of complete local rings, and refer to them as formal thickenings, rather than as general Henselian maps.

9.6. *I*-adic towers.
In this section we let R be a commutative S-algebra and R/I an R-ring spectrum, i.e., an R-module with homotopy unital and homotopy associative maps $R \to R/I$ and $R/I \wedge_R R/I \to R/I$. Define the R-module I by the cofiber sequence $I \to R \to R/I$, and let

$$I^{(s)} = I \wedge_R \cdots \wedge_R I$$

(s copies of I) be its s-fold smash power over R, for each $s \geq 1$. Define the R-module $R/I^{(s)}$ by the cofiber sequence $I^{(s)} \to R \to R/I^{(s)}$. There is then a tower of R-modules

$$R \to \cdots \to R/I^{(s)} \to \cdots \to R/I \tag{9.6.1}$$

that Baker and Lazarev [BL01, §4] refer to as the *external I-adic tower.*

Angeltveit [An:a, 4.2] recently showed that when R is even graded, i.e., the homotopy ring $\pi_*(R)$ is concentrated in even degrees, and R/I is a regular quotient, i.e., $\pi_*(I)$ is an ideal in $\pi_*(R)$ that can be generated by a regular sequence, then each R-ring spectrum multiplication on R/I can be rigidified to an associative R-algebra structure.

Given that R/I is an associative R-algebra, Lazarev [La01, 7.1] proved earlier on that the whole I-adic tower can be given the structure of a tower of associative R-algebras, and that each cofiber sequence

$$I^{(s)}/I^{(s+1)} \to R/I^{(s+1)} \xrightarrow{\pi} R/I^{(s)}$$

is a singular extension of associative R-algebras.

It remains an open problem to decide when the diagram (9.6.1) can be realized as a tower of commutative R-algebras, and whether each map $R/I^{(s+1)} \to R/I^{(s)}$ can be taken to be a singular extension in the commutative context. See [La04, 4.5] for a remark on a similar problem for square-zero extensions.

The homotopy limit

$$\hat{L}^R_{R/I} R = \operatorname*{holim}_s R/I^{(s)}$$

of the I-adic tower is the Bousfield *R/I-nilpotent completion* of R, formed in the category of R-modules (or R-algebras), which we introduced in Definition 8.2.2. It is in general not the same as the Bousfield *R/I-localization* of R, formed in the category of R-modules, which we denote by $L^R_{R/I} R$.

However, Baker and Lazarev [BL01, 6.3] use an *internal I-adic tower* to prove that when R is even graded and R/I is a homotopy commutative regular quotient R-algebra, then the R/I-nilpotent completion has the expected homotopy ring

$$\pi_* \hat{L}^R_{R/I} R \cong \pi_*(R)^\wedge_{\pi_*(I)} .$$

If the regular sequence generating $\pi_*(I)$ is finite, then they also show that the R/I-localization and the R/I-nilpotent completion of R, both formed in R-modules, do in fact agree

$$L^R_{R/I} R \simeq \hat{L}^R_{R/I} R ,$$

but we shall most be interested in cases when the regular ideal $\pi_*(I)$ is not finitely generated.

Proposition 9.5.7 therefore has the following consequence, which admits some fairly obvious algebraically localized generalizations that we shall also make use of.

Proposition 9.6.2 (Baker–Lazarev). *Let R be an even graded commutative S-algebra, and R/I a homotopy commutative regular quotient R-algebra. Then the limiting map*

$$\hat{L}^R_{R/I} R = \operatorname*{holim}_s R/I^{(s)} \to R/I$$

is symmetrically (=thh-)Henselian, and induces the canonical surjection

$$\pi_*(R)^\wedge_{\pi_*(I)} \to \pi_*(R)/\pi_*(I)$$

of homotopy rings. In particular, if $\pi_*(R)$ is already $\pi_*(I)$-adically complete, so that $R \simeq \hat{L}^R_{R/I} R$, then $R \to R/I$ is symmetrically Henselian. $\square$

We now claim that the complex cobordism spectrum MU can be viewed as a global model, up to Henselian maps, of each of the commutative S-algebras $\widehat{E(n)} = L_{K(n)}E(n)$ that occur as fixed S-algebras in the p-primary $K(n)$-local pro-Galois extensions $L_{K(n)}S \to E_n \to E_n^{nr}$. So, even if there is ramification between the expected maximal unramified Galois extensions (covering spaces) over the different chromatic strata, reflected in the changing pro-Galois groups $\mathbb{G}_n$ and $\mathbb{G}_n^{nr}$ for varying n and p, these can all be compensated for by appropriate Henselian maps (formal thickenings), and unified into one global model, namely MU.

For the sphere spectrum S, the chromatic stratification we have in mind is first branched over the rational primes p, and then $S_{(p)}$ is filtered by the Bousfield localizations $L_nS = L_{E(n)}S$ for each $n \geq 0$. The associated (Zariski) stack has the category $\mathcal{M}_{S,E(n)}$ of $E(n)$-local S-modules over the n-th open subobject in the filtration, and the n-th *monochromatic category* of $E(n)$-local $E(n{-}1)$-acyclic S-modules over the n-th half-open stratum. The latter category is equivalent to the category $\mathcal{M}_{S,K(n)}$ of $K(n)$-local S-modules, at least in the sense that their homotopy categories are equivalent [HSt99, 6.19].

The latter $K(n)$-local module category is in turn equivalent to the category of $K(n)$-local $L_{K(n)}S$-modules, and we propose to understand it better by way of Galois descent from the related categories $\mathcal{M}_{B,K(n)}$ of $K(n)$-local B-modules, for the various $K(n)$-local Galois extensions $L_{K(n)}S \to B$. The limiting case of pro-Galois descent from $K(n)$-local modules over $B = E_n^{nr}$, or over the separable closure $B = \bar{E}_n$ (cf. Section 10.3), can optimistically be hoped to be particularly transparent.

This decomposition of the sphere spectrum, appearing in the lower row in the diagram below, can be paralleled for MU by applying the same localization functors in spectra. However, the proposition above indicates that it may be more appropriate to nilpotently complete MU, in the category of MU-modules. In other words, we are led to focus attention on the upper row, rather than the middle row, in the following commutative diagram.

$$\begin{tikzcd}
MU \arrow[r] & \hat{L}^{MU}_{E(n)} MU \arrow[r] & \hat{L}^{MU}_{K(n)} MU \\
MU \arrow[r] \arrow[u, "="] & L_{E(n)} MU \arrow[r] \arrow[u] & L_{K(n)} MU \arrow[u] \\
S \arrow[r] \arrow[u] & L_{E(n)} S \arrow[r] \arrow[u] & L_{K(n)} S \arrow[u]
\end{tikzcd} \tag{9.6.3}$$

In the middle column we have $L_{E(n)}S \simeq \hat{L}_{E(n)}S$, since every $E(n)$-local spectrum is $E(n)$-nilpotent [HSa99, 5.3]. However, in the right hand column $L_{K(n)}S \not\simeq \hat{L}_{K(n)}S$, since $K(n)$-localization is not smashing [Ra92, 8.2.4] and [HSt99, 8.1].

The coefficient rings of the various localizations and nilpotent completions of MU occurring in the diagram above are mostly understood. See [Ra92, 8.1.1] for $\pi_* L_{E(n)} MU$ (or rather, its BP-version). Let $J_n \subset \pi_* MU_{(p)}$ be the kernel of the ring homomorphism $\pi_* MU_{(p)} \to \pi_* E(n)$, i.e., the regular ideal generated by the kernel of $\pi_* MU_{(p)} \to \pi_* BP$ and the infinitely many classes v_k for $k > n$. Let $I_n = (p, v_1, \dots, v_{n-1})$, also considered as an ideal in $\pi_* MU_{(p)}$, so that the sum of ideals $I_n + J_n$ is the kernel of the ring homomorphism $\pi_* MU_{(p)} \to \pi_* K(n)$. Then

$$\pi_* L_{K(n)} MU = \pi_* MU_{(p)}[v_n^{-1}]^\wedge_{I_n}$$

by [HSt99, 7.10(e)]. By [HSa99, Thm. B], $L_{K(n)}BP$ splits as the $K(n)$-localization of an explicit countable wedge sum of suspensions of $\widehat{E(n)}$. It follows that $L_{K(n)}MU$ splits in a similar way.

By Proposition 9.6.2, applied to $R = MU_{(p)}[v_n^{-1}]$ and $R/I = E(n)$, we find that $\hat{L}^{MU}_{E(n)} MU \simeq \hat{L}^R_{R/I} R \to E(n)$ is symmetrically Henselian, with

$$\pi_* \hat{L}^{MU}_{E(n)} MU = \pi_* MU_{(p)}[v_n^{-1}]^\wedge_{J_n} .$$

By the same proposition applied to $R = MU_{(p)}[v_n^{-1}]$ and $R/I = K(n)$, at least for $p \neq 2$ to ensure that $K(n)$ is homotopy commutative, we also find that $\hat{L}^{MU}_{K(n)} MU \simeq \hat{L}^R_{R/I} R \to K(n)$ is symmetrically Henselian, with

$$\pi_* \hat{L}^{MU}_{K(n)} MU = \pi_* MU_{(p)}[v_n^{-1}]^\wedge_{I_n + J_n} . \tag{9.6.4}$$

This differs from the $K(n)$-localization of MU in S-modules by the additional completion along J_n.

This $K(n)$-nilpotently complete part, in MU-modules, of the global commutative S-algebra MU, can now be related by a symmetrically Henselian map to the extension $L_{K(n)}S \to \widehat{E(n)}$, which is closely related to the $K(n)$-locally pro-Galois extension $L_{K(n)}S \to E_n$.

(9.6.5)
$$\begin{tikzcd}
E_n \arrow[r, "\simeq?"] & \hat{L}^{MU}_{K(n)} E_n \\
\widehat{E(n)} \arrow[u] \arrow[r, "\simeq?"] & \hat{L}^{MU}_{K(n)} E(n) \arrow[u] \\
L_{K(n)} MU \arrow[u] \arrow[r] & \hat{L}^{MU}_{K(n)} MU \arrow[u] \arrow[ul, dashed, "q"'] \\
L_{K(n)} S \arrow[u] \arrow[r, "="'] & L_{K(n)} S \arrow[u]
\end{tikzcd}$$

Here the horizontal map $\widehat{E(n)} = L_{K(n)}E(n) \to \hat{L}^{MU}_{K(n)}E(n)$, and its analog for E_n, are both plausibly weak equivalences. For instance, the corresponding map of nilpotent completions of MU induces completion along J_n at the level of homotopy groups, and $\pi_*\widehat{E(n)}$ and π_*E_n are already J_n-adically complete in a trivial way.

We shall now apply Proposition 9.6.2 with $R = \hat{L}^{MU}_{K(n)}MU$. Formula (9.6.4) exhibits R as an even graded commutative S-algebra. Considering J_n as an ideal in π_*R, it is still generated by a regular sequence and $(\pi_*R)/J_n \cong \pi_*\widehat{E(n)}$. So we can form $R/I \simeq \widehat{E(n)}$ as a homotopy commutative regular quotient R-algebra. Then $\pi_*(I) = J_n$, and $\pi_*(R)$ is J_n-adically complete, so by the last clause of Proposition 9.6.2 the map $q\colon R \to R/I$, labeled q in the diagram (9.6.5) above, is symmetrically Henselian.

Corollary 9.6.6. *Each $K(n)$-local pro-Galois extension $L_{K(n)}S \to E_n$ factors as the composite map of commutative S-algebras*

$$L_{K(n)}S \to \hat{L}^{MU}_{K(n)}MU \xrightarrow{q} \widehat{E(n)} \to E_n\,,$$

where the first map admits the global model $S \to MU$, the second map is symmetrically (= thh-)Henselian, and the third map is a $K(n)$-local pro-Galois extension.

In other words, each $K(n)$-local stratum of S is related by a chain of pro-Galois covers $L_{K(n)}S \to E_n \leftarrow \widehat{E(n)}$ to a formal thickening $q\colon \hat{L}^{MU}_{K(n)}MU \to \widehat{E(n)}$ of the corresponding $K(n)$-nilpotently complete stratum of MU, formed in MU-modules.

We shall argue in Section 12.2 that there is a Hopf–Galois structure on this global model $S \to MU$ that also encapsulates all the known Galois symmetries over $L_{K(n)}S$, at least up to the adjunction of roots of unity, i.e., up to the passage from $\widehat{E(n)}$ to E_n (or to E_n^{nr}). The question remains whether q is (commutatively) Henselian, which would follow if the diagram (9.6.1) could be realized by singular extensions of commutative S-algebras.

After this discussion of $K(n)$-localization and $K(n)$-nilpotent completion in MU-modules, we make some remarks on the chromatic filtration in MU-modules. The study of the chromatic filtration and the monochromatic category of S-modules relies on the basic fact [JY80, 0.1] that $E(n)_*(X) = 0$ implies $E(n-1)_*(X)$ for S-modules X, so that there is a natural map $L_{E(n)}X \to L_{E(n-1)}X$. The analogous claim in the context of MU-modules is false, i.e., that $E(n)^{MU}_*(X) = 0$ implies $E(n-1)^{MU}_*(X) = 0$, as the easy example $X = MU_{(p)}/(v_n)$ illustrates. Thus there is no natural map $L^{MU}_{E(n)}X \to L^{MU}_{E(n-1)}X$.

For brevity, let $K[0,n] = K(0)\vee\cdots\vee K(n)$. It is well-known that $L_{K[0,n]} = L_{E(n)}$ in the category of S-modules [Ra84, 2.1(d)]. For any MU-module X it is obvious that $K[0,n]^{MU}_*(X) = 0$ implies $K[0,n-1]^{MU}_*(X) = 0$, so that there is a natural map $L^{MU}_{K[0,n]}X \to L^{MU}_{K[0,n-1]}X$. Therefore the example above shows that the two localization functors $L^{MU}_{K[0,n]}$ and $L^{MU}_{E(n)}$ in MU-modules cannot be equivalent.

We therefore think that it will be more appropriate to filter the category of MU-modules by the essential images

$$\mathcal{M}_{MU} \supset \cdots \supset \mathcal{M}^{MU}_{MU,K[0,n]} \supset \mathcal{M}^{MU}_{MU,K[0,n-1]} \supset \dots$$

of the Bousfield localization functors $L^{MU}_{K[0,n]}$, i.e., the full subcategories of $K[0,n]$-local MU-modules, within MU-modules, or the corresponding essential images

$$\mathcal{M}_{MU} \supset \cdots \supset \hat{\mathcal{M}}^{MU}_{MU,K[0,n]} \supset \hat{\mathcal{M}}^{MU}_{MU,K[0,n-1]} \supset \dots$$

of the nilpotent completion functors $\hat{L}^{MU}_{K[0,n]}$, i.e., the full subcategories of $K[0,n]$-nilpotently complete MU-modules, within MU-modules. Then we can consider the MU-chromatic towers

$$X \to \cdots \to L^{MU}_{K[0,n]}X \to L^{MU}_{K[0,n-1]}X \to \dots$$

and

$$X \to \cdots \to \hat{L}^{MU}_{K[0,n]}X \to \hat{L}^{MU}_{K[0,n-1]}X \to \dots$$

for each MU-module X. We then suspect that $L^{MU}_{K[0,n]}$ is a smashing localization, and that there is an equivalence of homotopy categories between the n-th monochromatic category of MU-modules and the $K(n)$-local category of MU-modules, like that of [HSt99, 6.19], but we have not verified this expectation. To be precise, the monochromatic category in question has objects the MU-modules that are $L^{MU}_{K[0,n]}$-local and $L^{MU}_{K[0,n-1]}$-acyclic. The $K(n)$-local category has objects the MU-modules that are $L^{MU}_{K(n)}$-local.

The thrust of Corollary 9.6.6 is now that the chromatic filtration on S-modules is related to a chromatic filtration on MU-modules, by a chain of pro-Galois extensions and Henselian maps with geometric content. The chromatic filtration on MU-modules is likely to be much easier to understand algebraically, in terms of the theory of formal group laws. Taken together, these two points of view may clarify the chromatic filtration on S-modules.

10. Mapping spaces of commutative S-algebras

We turn to the computation of the mapping space $\mathcal{C}_A(B, B)$ for a G-Galois extension $A \to B$, and related mapping spaces of commutative S-algebras, using the Hopkins–Miller obstruction theory in the commutative form presented by Goerss and Hopkins [GH04]. For the more restricted problem of the classification of commutative S-algebra structures, the related obstruction theory of Alan Robinson [Rob03] is also relevant.

10.1. Obstruction theory.

Let A be a cofibrant commutative S-algebra and let E be an S-module. We shall need an extension of the Goerss–Hopkins theory to the context of (simplicial algebras over simplicial operads in) the category $\mathcal{M}_{A,E}$ of E-local A-modules. The base change to A-modules is harmless, but in working E-locally we may loose the identification of the dualizable A-modules with the (homotopy retracts of) finite cell A-modules, recalled in Proposition 3.3.3 above. It seems clear that only the formal properties of dualizable modules are important to the Goerss–Hopkins theory, so that the whole extension can be carried through in full generality. However, for our specific purposes the only dualizable A-modules we must consider will in fact be finite cell A-modules, so we do not actually need to carry the generalization through.

Next, consider a fixed (cofibrant, E-local) commutative A-algebra B. The Goerss–Hopkins spectral sequence [GH04, Thm. 4.3 and Thm. 4.5] for the computation of the homotopy groups of commutative A-algebra mapping spaces like $\mathcal{C}_A(C, B)$, for various commutative A-algebras C, is based on working with a fixed homology theory given by a commutative A-algebra that they call E, but which we will take to be B. In particular, the target B in the mapping space is then equivalent to its completion along the given homology theory (cf. Definition 8.2.1), as required for the convergence of the spectral sequence.

This commutative A-algebra B is required to satisfy the so-called *Adams conditions* [Ad69, p. 28], [GH04, Def. 3.1], which in our notation asks that B is weakly equivalent to a homotopy colimit of finite cell A-module spectra B_α, satisfying two conditions. For our purposes it will suffice that B itself satisfies the two conditions, i.e., that there is only a trivial colimit system. The conditions are then:

Adams conditions 10.1.1. The commutative A-algebra B is weakly equivalent to a finite cell A-module, such that

(a) $B^A_*(D_A B)$ is finitely generated and projective as a B_*-module.

(b) For each B-module M the Künneth map

$$[D_A B, M]^A_* \to \mathrm{Hom}_{B_*}(B^A_*(D_A B), M_*)_*$$

is an isomorphism.

In the E-local situation we expect that it suffices to assume that B is a dualizable A-module, but in our applications the stronger finite cell hypothesis will always be satisfied.

Lemma 10.1.2. *The Adams conditions (a) and (b) are satisfied when $A \to B$ is an E-local G-Galois extension, with G a finite discrete group.*

Proof. From Lemma 6.1.2 we know that $j\colon B\langle G\rangle \to F_A(B,B)$ is a weak equivalence, and that $h_M\colon B \wedge_A M \to F(G_+, M)$ is a weak equivalence for each B-module M. By Proposition 6.2.1, B is dualizable over A, so $B \wedge_A D_A B \simeq F_A(B,B)$. So $B^A_*(D_A B) \cong \pi_* F_A(B,B) \cong B_*\langle G\rangle$ is a finitely generated free B_*-module, and $B^A_* M \cong [D_A B, M]^A_*$ is isomorphic to

$$\mathrm{Hom}_{B_*}(B^A_*(D_A B), M_*) \cong \mathrm{Hom}_{B_*}(B_*\langle G\rangle, M_*) \cong \prod_G M_* \cong \pi_* F(G_+, M)\,.$$

A diagram chase verifies that the Künneth map equals the composite of this chain of isomorphisms. □

The more general situation, with G an indiscrete stably dualizable group, will lead to much more complicated spectral sequence calculations, which we will not try to address here.

Goerss and Hopkins proceed to consider an E_2- or *resolution model structure* on spectra, which is suitably generated by a class $\mathcal{P}$ of finite cellular spectra. This class is required to satisfy a list of conditions [GH04, Def. 3.2.(1)–(5)]. Following the proof of [BR:r, 2.2.4], by Baker and Richter, we take $\mathcal{P}$ to be the smallest set of dualizable A-modules that contains A and B, and is closed under (de-)suspensions and finite wedge sums. This immediately takes care of conditions (3) and (4).

Lemma 10.1.3. *The resolution model category conditions* [GH04, Def. 3.2.(1)–(5)] *are satisfied when $A \to B$ is a finite E-local G-Galois extension.*

Proof. (1) $B^A_*(X)$ is a finite sum of shifted copies of $B^A_*(A) = B_*$ and $B^A_*(B) \cong \prod_G B_*$, for each A-module $X \in \mathcal{P}$, hence is projective as a B_*-module. (2) $D_A B$ is represented in $\mathcal{P}$, since B is self-dual as an A-module by Proposition 6.4.7. (5) The Künneth map

$$[X, M]^A_* \to \mathrm{Hom}_{B_*}(B^A_*(X), M_*)_*$$

is an isomorphism for all B-module spectra M when $X = D_A B$, by the Adams condition (b), and trivially for $X = A$, so the same follows for all $X \in \mathcal{P}$ by passage to (de-)suspensions and finite wedge sums. □

To sum up, a finite Galois extension $A \to B$ satisfies the Adams conditions and has an associated resolution model structure on A-modules, as required by [GH04, §3], whenever B is weakly equivalent to a finite cell A-module. It seems likely that the cited theory also extends to cover all finite Galois extensions, by replacing all references to finite cell objects by dualizable objects. However, in the following applications we shall always make use of the identification

$$\mathcal{C}_A(C, B) \cong \mathcal{C}_B(B \wedge_A C, B)$$

and only apply the Goerss–Hopkins spectral sequence in the case of commutative B-algebra maps to B. This is the very special case of Lemmas 10.1.2 and 10.1.3 when $A = B$ and G is the trivial group, in which case B is certainly a finite cell A-module. So we are only using the straightforward extension of [GH04] to a more general (cofibrant, commutative) ground S-algebra, namely B. Note also that $B^B_*(B \wedge_A C) \cong B^A_*(C)$, so the two equivalent mapping spaces above will have the same associated spectral sequences, which we now review.

Goerss and Hopkins define André–Quillen cohomology groups D^s of algebras and modules over a simplicially resolved E_∞-operad [GH04, (4.1)], as non-abelian right derived functors of algebra derivations. They then construct a convergent spectral sequence of Bousfield–Kan type [GH04, Thm. 4.5], which in our notation appears as

$$E_2^{s,t} \Longrightarrow \pi_{t-s}\mathcal{C}_A(C, B) \tag{10.1.4}$$

(based at a given commutative A-algebra map $C \to B$), with E_2-term

$$E_2^{0,0} = \mathrm{Alg}_{B_*}(B^A_*(C), B_*)$$

and

$$E_2^{s,t} = D^s_{B_*T}(B^A_*(C), \Omega^t B_*)$$

for $t > 0$. Here $\Omega^t B_*$ is the t-th desuspension of the module B_*. As usual for Bousfield–Kan spectral sequences, this spectral sequence is concentrated in the wedge-shaped region $0 \le s \le t$.

The subscript B_*T refers to a (Reedy cofibrant, etc.) simplicial E_∞ operad T that resolves the commutative algebra operad in the sense of [GH04, Thm. 2.1], and B_*T is the associated simplicial E_∞ operad in the category of B_*-modules. The Goerss–Hopkins André–Quillen cohomology groups D^s are the right derived functors of derivations of B_*T-algebras in B_*-modules, in the sense of Quillen's homotopical algebra. As surveyed by Basterra and Richter [BR04, 2.6], these groups D^s do not depend on the choice of resolving simplicial E_∞ operad T, and agree with the André–Quillen cohomology groups $AQ^s_{sE_\infty}$ defined by Mandell in [Man03, 1.1] for E_∞ simplicial B_*-algebras. These do in turn agree with the André–Quillen cohomology groups $AQ^s_{dgE_\infty}$ defined by Mandell for E_∞ differential graded B_*-algebras [Man03, 1.8], and with Basterra's topological André–Quillen cohomology groups TAQ^s of the Eilenberg–Mac Lane spectra associated to these algebras and modules [Man03, §7]. By the comparison result of Basterra and McCarthy [BMc02, 4.2], these are finally isomorphic to the Γ-cohomology groups $H\Gamma^s$ of Robinson and Sarah Whitehouse [RoW02], when $B^A_*(C)$ is projective over B_*, or more generally, when $B^A_*(C)$ is flat over B_* and the universal coefficient spectral sequence from homology to cohomology collapses. So in these cases the Goerss–Hopkins groups can be rewritten as

$$D^s_{B_*T}(B^A_*(C), \Omega^t B_*) = H\Gamma^{s,-t}(B^A_*(C)|B_*; B_*)\,.$$

It is not quite obvious from the above references that this chain of identifications preserves the internal t-grading of these cohomology groups, since this grading could be lost by the passage through Eilenberg–Mac Lane spectra. However, Birgit

Richter has checked that both gradings are indeed respected, up to the sign indicated above. In our applications all of these cohomology groups will in fact be zero, so the finer point about the internal grading is not so important.

If $B^A_*(C)$ is an étale commutative B_*-algebra (thus flat over B_*), then by [RoW02, 6.8(3)] all Γ-homology and Γ-cohomology groups of $B^A_*(C)$ over B_* are zero, so by the sequence of comparison results above (and the universal coefficient spectral sequence for TAQ), all the Goerss–Hopkins André–Quillen cohomology groups $D^s_{B_*T}(B^A_*(C), \Omega^t B_*)$ vanish. Therefore one can conclude:

Corollary 10.1.5. *Let $C \to B$ be a map of commutative A-algebras. If $B^A_*(C)$ is étale over B_*, then the Goerss–Hopkins spectral sequence for*

$$\pi_*\mathcal{C}_A(C, B) \cong \pi_*\mathcal{C}_B(B \wedge_A C, B)$$

collapses to the origin at the E_2-term, so $\mathcal{C}_A(C, B)$ is homotopy discrete (each path component is weakly contractible) with

$$\pi_0\mathcal{C}_A(C, B) \cong \mathrm{Alg}_{B_*}(B^A_*(C), B_*)\,.$$

10.2. Idempotents and connected S-algebras.

The converse part of the Galois correspondence, begun in Theorem 7.2.3, should intrinsically characterize the intermediate extensions $A \to C \to B$ that occur as K-fixed S-algebras $C = B^{hK}$ by allowable subgroups $K \subset G$. Already in the algebraic case of a G-Galois extension $R \to T$ of discrete rings there are additional complications (compared to the field case) when T admits non-trivial idempotents, i.e., when the spectrum of T is not connected in the sense of algebraic geometry. See [Mag74] for a general treatment of these complications. We do not expect that these issues are so central to the extension of the theory from discrete rings to S-algebras, so we prefer to focus on the analog of the situation when T is connected.

We can identify the idempotents of a commutative ring T with the non-unital T-algebra endomorphisms $T \to T$, taking an idempotent e (with $e^2 = e$) to the homomorphism $t \mapsto et$. The forgetful functor from T-algebras to non-unital T-algebras has a left adjoint, taking a non-unital T-algebra N to $T \oplus N$, with the multiplication $(t_1, n_1) \cdot (t_2, n_2) = (t_1t_2, t_1n_2 + n_1t_2 + n_1n_2)$ and unit $(1, 0)$. In particular, we can identify the set of idempotents $E(T) = \{e \in T \mid e^2 = e\}$ with the set of T-algebra maps

$$E(T) \cong \mathrm{Alg}_T(T \oplus T, T)\,.$$

Here $T \oplus T \cong T[x]/(x^2 - x)$ is finitely generated and free as a T-module. It is étale as a commutative T-algebra by [Mil80, I.3.4], since $(x^2 - x)' = 2x - 1$ is its own multiplicative inverse in $T[x]/(x^2 - x)$.

This leads us to the following definitions.

Definition 10.2.1. Let B be a (cofibrant) commutative S-algebra. Let the *space of idempotents*

$$\mathcal{E}(B) = \mathcal{N}_B(B, B)$$

be the mapping space of non-unital commutative B-algebra [Bas99, §1] endomorphisms $B \to B$. We say that B is *connected* if the map $\{0, 1\} \to \mathcal{E}(B)$ taking 0 and 1 to the constant map and the identity map $B \to B$, respectively, is a weak equivalence.

We shall not have need to do so, but if we wanted to express that the spectrum B has the property that $\pi_*(B) = 0$ for all $* \le 0$, we would say that B is 0-connected, reserving the term "connected" for the algebro-geometric interpretation just introduced. A spectrum B with $\pi_*(B) = 0$ for $* < 0$ will be called (-1)-connected or connective.

There is a homeomorphism

$$\mathcal{E}(B) \cong \mathcal{C}_B(B \vee B, B)\,,$$

where $B \vee B$ is defined as the split commutative S-algebra extension of B with fiber the underlying non-unital commutative S-algebra of B. Its unit $B \to B \vee B$ is the inclusion on the first wedge summand, and its multiplication is the composite

$$(B \vee B) \wedge_B (B \vee B) \cong B \vee (B \vee B \vee B) \xrightarrow{1 \vee \nabla} B \vee B$$

where ∇ folds the last three wedge summands together.

Proposition 10.2.2. *Let B be any commutative S-algebra. The space of idempotents $\mathcal{E}(B)$ is homotopy discrete, with $\pi_0\mathcal{E}(B) \cong E(\pi_0(B))$. In particular, the commutative S-algebra B is connected if and only if the commutative ring $\pi_0(B)$ is connected.*

Proof. We compute the homotopy groups of $\mathcal{E}(B) \cong \mathcal{C}_B(B \vee B, B)$ by means of the Goerss–Hopkins spectral sequence (10.1.4), in the almost degenerate case when $A = B$ and $C = B \vee B$. Here $A \to B$ is of course a G-Galois extension, in the trivial case $G = 1$, so our discussion in Section 10.1 justifies the use of this spectral sequence. It specializes to

$$E_2^{s,t} \Longrightarrow \pi_{t-s}\mathcal{E}(B)$$

with

$$E_2^{0,0} = \mathrm{Alg}_{B_*}(B_* \oplus B_*, B_*) = E(B_*)$$

and

$$E_2^{s,t} = D^s_{B_*T}(B_* \oplus B_*, \Omega^t B_*)$$

for $t > 0$. Here $B_* \oplus B_* = B_*[x]/(x^2 - x)$ is étale over B_*, so all the André–Quillen cohomology groups $D^s = H\Gamma^s$ vanish [RoW02, 6.8(3)], and we deduce that $\mathcal{E}(B)$ is homotopy discrete, with $\pi_0\mathcal{E}(B) \cong E(B_*)$ equal to the set of idempotents in the graded ring B_*, which of course are the same as the idempotents in the ring $\pi_0(B)$. In short, we have applied Corollary 10.1.5. □

The following argument, explained by Neil Strickland, shows that the above definition of connectedness for structured ring spectra is equivalent to another definition originally proposed by the author. We say that an S-algebra B is *trivial* if it is weakly contractible, i.e., if $\pi_*(B) = B_* = 0$, and *non-trivial* otherwise.

Lemma 10.2.3. *A non-trivial commutative S-algebra B is either connected, or weakly equivalent to a product $B_1 \times B_2$ of non-trivial commutative B-algebras, but not both.*

Proof. If B is non-trivial and not connected then there exists an idempotent $e \in \pi_0(B)$ different from 0 and 1. Let f_1 and $f_2 \colon B \to B$ be B-module maps inducing multiplication by e and $1 - e$ on $\pi_*(B)$, respectively. (These could also be taken to be non-unital commutative B-algebra maps by the previous proposition.) For $i = 1, 2$ let $B[f_i^{-1}]$ be the mapping telescope for the iterated self-map f_i, and let

$$B_i = L^B_{B[f_i^{-1}]} B$$

be the Bousfield $B[f_i^{-1}]$-localization of B in the category of B-modules. Then there are commutative B-algebra maps $B \to B_1$ and $B \to B_2$ inducing isomorphisms $e\pi_*(B) \cong \pi_*(B_1)$ and $(1 - e)\pi_*(B) \cong \pi_*(B_2)$, of nontrivial groups, and their product $B \to B_1 \times B_2$ is the asserted weak equivalence.

Conversely, if $B \simeq B_1 \times B_2$ as commutative B-algebras (or even just as ring spectra), with B_1 and B_2 non-trivial, then $\pi_0(B)$ is not connected as a commutative ring, so B is not connected as a commutative S-algebra. □

10.3. Separable closure.

The following terminology presumes, in some sense, that each finite separable extension can be embedded in a finite Galois extension, i.e., a kind of normal closure. We will not prove this in our context, but keep the terminology, nonetheless.

Definition 10.3.1. Let A be a connected commutative S-algebra. We say that A is *separably closed* if there are no G-Galois extensions $A \to B$ with G finite and non-trivial and B connected, i.e., if each finite G-Galois extension $A \to B$ has $G = \{e\}$ or B not connected.

A *separable closure* of A is a pro-G_A-Galois extension $A \to \bar{A}$ such that $\bar{A}$ is connected and separably closed. The pro-finite Galois group G_A of $\bar{A}$ over A is the *absolute Galois group* of A.

The existence of a separable closure follows from Zorn's lemma. However, we have not yet proved that two separable closures of A are weakly equivalent, so talking of "the" absolute Galois group is also a bit presumptive.

By Minkowski's theorem on the discriminant [Ne99, III.2.17], for every number field K different from $\mathbb{Q}$ the inclusion $\mathbb{Z} \to \mathcal{O}_K$ is ramified at one or more primes. In particular, there are no Galois extensions $\mathbb{Z} \to \mathcal{O}_K$ other than the identity. The following inference appears to be well-known.

Proposition 10.3.2. *The only connected Galois extension of the integers is $\mathbb{Z}$ itself, so $\mathbb{Z} = \bar{\mathbb{Z}}$ is separably closed.*

Proof. Let $\mathbb{Z} \to T$ be a G-Galois extension of commutative rings, so T is a finitely generated free $\mathbb{Z}$-module. Then $\mathbb{Q} \to \mathbb{Q} \otimes T$ is also a G-Galois extension, so $\mathbb{Q} \otimes T \cong \prod_i K_i$ is a product of number fields [KO74, III.4.1]. Then T is contained in the integral closure of $\mathbb{Z}$ in $\mathbb{Q} \otimes T$, which is a product $\prod_i \mathcal{O}_{K_i}$ of number rings. The condition $T \otimes T \cong \prod_G T$ and an index count imply, in combination, that

$T = \prod_i \mathcal{O}_{K_i}$ and that each $\mathcal{O}_{K_i}$ is unramified over $\mathbb{Z}$. By Minkowski's theorem, this only happens when each $K_i = \mathbb{Q}$, so $T = \prod_G \mathbb{Z}$. If T is connected, this implies that G is the trivial group and that $T = \mathbb{Z}$. $\square$

In other words, to have interesting Galois extensions of $\mathbb{Z}$ one must localize away from one or more primes. We have the following analog in the context of commutative S-algebras. The examples in Section 5.4 demonstrate that after localization there are indeed interesting examples of (local) Galois extensions of S.

Theorem 10.3.3. *The only (global, finite) connected Galois extension of the sphere spectrum S is S itself, so $S = \bar{S}$ is separably closed.*

Proof. Let $S \to B$ be any finite G-Galois extension of global, i.e., unlocalized, commutative S-algebras (Definition 4.1.3). Then B is a dualizable S-module (Proposition 6.2.1), hence of the homotopy type of (a retract of) a finite CW spectrum (Proposition 3.3.3). Thus $H_*(B) = H_*(B; \mathbb{Z})$ is finitely generated in each degree, and non-trivial only in finitely many degrees.

Let k be minimal such that $H_k(B) \neq 0$ and let ℓ be maximal such that $H_\ell(B) \neq 0$. The condition $B \wedge B \simeq \prod_G B$ implies that $k = \ell = 0$. For if $k < 0$ then $H_k(B) \otimes H_k(B)$ is isomorphic to $H_{2k}(B \wedge B) \cong \prod_G H_{2k}(B) = 0$, which contradicts $H_k(B) \neq 0$ and finitely generated. If $\ell > 0$ then $H_\ell(B) \otimes H_\ell(B)$ injects into $H_{2\ell}(B \wedge B) \cong \prod_G H_{2\ell}(B) = 0$, which again contradicts $H_\ell(B) \neq 0$ and finitely generated. Thus $H_*(B) = T$ is concentrated in degree 0.

By the Hurewicz theorem, B is a connective spectrum with $\pi_0(B) \cong H_0(B) = T$. The Künneth formula then implies that $T \otimes T \cong \prod_G T$ and $\mathrm{Tor}_1^{\mathbb{Z}}(T, T) = 0$, so the unit map $\mathbb{Z} \to T$ makes T a free abelian $\mathbb{Z}$-module of rank equal to the order of G. In particular, T is a faithfully flat $\mathbb{Z}$-module.

The result of inducing B up along the Hurewicz map $S \to H\mathbb{Z}$ has homotopy $\pi_*(H\mathbb{Z} \wedge B) = H_*(B) = T$ concentrated in degree 0, so there is a pushout square

$$\begin{tikzcd} B \arrow[r] & HT \\ S \arrow[u] \arrow[r] & H\mathbb{Z} \arrow[u] \end{tikzcd}$$

of commutative S-algebras. By a variation on the proof of Lemma 7.1.1, we shall now show that $H\mathbb{Z} \to HT$ is G-Galois.

The map $HT \wedge_{H\mathbb{Z}} HT \to \prod_G HT$ is induced up from the weak equivalence $B \wedge B \to \prod_G B$, cf. diagram (7.1.2), and is therefore a weak equivalence. Next, $S \to B$ is dualizable, so $H\mathbb{Z} \to HT$ is dualizable (Lemma 6.2.3). Finally, T is faithfully flat over $\mathbb{Z}$ and so HT is faithful over $H\mathbb{Z}$ by the proof of Lemma 4.3.5. Thus $H\mathbb{Z} \to HT$ is a faithful G-Galois extension (Proposition 6.3.2).

From Proposition 4.2.1 we deduce that $\mathbb{Z} \to T$ is a G-Galois extension of commutative rings. By the classical theorem of Minkowski, this is only possible if $G = \{e\}$ is the trivial group or T is not connected. And $\pi_0(B) \cong T$, so either G is trivial or B is not connected (Proposition 10.2.2). Thus S is separably closed. $\square$

Note that we did not have to (possibly) restrict attention to faithful G-Galois extensions $S \to B$ in this proof.

Question 10.3.4. Can the absolute Galois group G_A, or its maximal abelian quotient G_A^{ab}, be expressed in terms of arithmetic invariants of A, such as its algebraic K-theory $K(A)$? This would constitute a form of class field theory for commutative S-algebras. The author expects that there is a better hope for a simple answer in the maximally localized category of $K(n)$-local commutative S-algebras, than for general commutative S-algebras.

Question 10.3.5. If an E-local commutative S-algebra A is an even periodic Landweber exact spectrum, and $A \to B$ is a finite E-local G-Galois extension, does it then follow that B is also an even periodic Landweber exact spectrum, and that $\pi_0(A) \to \pi_0(B)$ is a G-Galois extension of commutative rings?

In the case of $E = K(n)$ and $A = E_n^{nr}$, for which $\pi_0(A) = \mathbb{W}(\bar{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]]$ is separably closed, there are no non-trivial such algebraic extensions to a connected ring $\pi_0(B)$, so it would follow that E_n^{nr} is $K(n)$-locally separably closed. In particular, E_n^{nr} would be the $K(n)$-local separable closure of $L_{K(n)}S$, with absolute Galois group $\mathbb{G}_n^{nr} = \mathbb{S}_n \rtimes \hat{\mathbb{Z}}$. This amounts to Conjecture 1.3 in the introduction.

Baker and Richter [BR:r] have partial results in this direction, in the global category. They are able to show that E_n^{nr} does not admit any non-trivial connected faithful abelian G-Galois extensions. So $E_n^{nr} = E_n^{ab}$ is the maximal global faithful abelian extension of E_n.

11. Galois theory II

As before, we are implicitly working E-locally, for some spectrum E.

11.1. Recovering the Galois group.

The space of commutative A-algebra endomorphisms of B in a G-Galois extension $A \to B$ can be rewritten as

$$\mathcal{C}_A(B, B) \cong \mathcal{C}_B(B \wedge_A B, B) \simeq \mathcal{C}_B(F(G_+, B), B)\,,$$

in view of the weak equivalence $h\colon B \wedge_A B \to F(G_+, B)$. When G is finite and discrete, and B admits no non-trivial idempotents, we can compute the homotopy groups of this mapping space by the Goerss–Hopkins spectral sequence.

When G is not discrete, these spectral sequence computations appear to be much harder, and we will not attempt them. We are therefore principally working in the context of the separable/étale extensions from Chapter 9.

Theorem 11.1.1. *Let $A \to B$ be a finite G-Galois extension of commutative S-algebras, with B connected. Then the natural map*

$$G \to \mathcal{C}_A(B, B)\,,$$

giving the action of G on B through commutative A-algebra maps, is a weak equivalence. In particular, $\mathcal{C}_A(B, B)$ is a homotopy discrete grouplike monoid, so each commutative A-algebra endomorphism of B is an automorphism, up to a contractible choice.

Proof. This time we compute the homotopy groups of $\mathcal{C}_A(B,B) \simeq \mathcal{C}_B(\prod_G B, B)$ by means of (10.1.4), once again in the almost degenerate case when $A = B$ and $C = F(G_+, B) = \prod_G B$. The E_2-term has

$$E_2^{0,0} = \mathrm{Alg}_{B_*}(\prod_G B_*, B_*) \cong G$$

since $B_* = \pi_*(B)$ is connected in the graded sense, or equivalently, $\pi_0(B)$ has no non-trivial idempotents. The remainder of the E_2-term is

$$E_2^{s,t} = D^s_{B_*T}(\prod_G B_*, \Omega^t B_*) = 0\,,$$

since $\prod_G B_*$ is étale over B_*. We are therefore in the collapsing situation of Corollary 10.1.5, and $\mathcal{C}_A(B,B) \simeq G$ follows. □

The extension to profinite pro-Galois extensions is straightforward.

Proposition 11.1.2. *Let $A \to B = \mathrm{colim}_\alpha B_\alpha$ be a pro-G-Galois extension, with each $A \to B_\alpha$ a finite G_α-Galois extension and $G = \lim_\alpha G_\alpha$. Suppose that B is connected. Then $\mathcal{C}_A(B,B)$ is homotopy discrete, and the natural map $G \to \pi_0\mathcal{C}_A(B,B)$ is a group isomorphism.*

Proof. Using (8.1.2), we rewrite the commutative A-algebra mapping space as

$$\begin{aligned}\mathcal{C}_A(B,B) &\cong \mathcal{C}_B(B \wedge_A B, B)\\ &\simeq \mathcal{C}_B(\operatorname*{colim}_\alpha F(G_{\alpha+}, B), B) \simeq \operatorname*{holim}_\alpha \mathcal{C}_B(F(G_{\alpha+}, B), B)\,.\end{aligned}$$

By the finite case, each $\mathcal{C}_B(F(G_{\alpha+}, B), B)$ is homotopy discrete with

$$\pi_0\mathcal{C}_B(F(G_{\alpha+}, B), B) \cong \mathrm{Alg}_{B_*}(\prod_{G_\alpha} B_*, B_*) \cong G_\alpha\,,$$

when B is connected. So $\mathcal{C}_A(B,B) \simeq \mathrm{holim}_\alpha G_\alpha$ is homotopy discrete, with $\pi_0\mathcal{C}_A(B,B) \cong \lim_\alpha G_\alpha \cong G$. □

11.2. The brave new Galois correspondence.

We now turn to the converse part of the Galois correspondence. The proper role of the separability condition in the following result was found in a conversation with Birgit Richter.

Proposition 11.2.1. *Let $A \to B$ be a G-Galois extension, with B connected and G finite and discrete, and let*

$$A \to C \to B$$

be a factorization of this map through a separable commutative A-algebra C. Then $\mathcal{C}_C(B,B)$ is homotopy discrete, and the natural map $\mathcal{C}_C(B,B) \to \mathcal{C}_A(B,B)$ identifies $K = \pi_0\mathcal{C}_C(B,B)$ with a subgroup of $G = \pi_0\mathcal{C}_A(B,B)$. Furthermore, the action of $\mathcal{C}_C(B,B) \simeq K$ on B induces a weak equivalence

$$h\colon B \wedge_C B \to \prod_K B\,.$$

Proof. By assumption $A \to C$ is separable, so there are maps

$$C' \xrightarrow{\sigma} C \wedge_A C \xrightarrow{\mu} C$$

of C-bimodules relative to A such that $\mu\sigma \colon C' \to C$ is a weak equivalence. Inducing these maps and modules up along $C \to B$, both as left and right modules, we get maps

$$B \wedge_C C' \wedge_C B \xrightarrow{\bar{\sigma}} B \wedge_A B \xrightarrow{\bar{\mu}} B \wedge_C B$$

of B-bimodules relative to A, such that the composite is a weak equivalence. We consider $C \wedge_A C$ as a commutative C-algebra via the left unit $C \cong C \wedge_A A \to C \wedge_A C$, and similarly for $B \wedge_A B$ over B. Then μ is a map of commutative C-algebras and $\bar{\mu}$ is a map of commutative B-algebras.

At the level of homotopy groups, we get a diagram

$$B^C_*(B) \xrightarrow{\bar{\sigma}_*} B^A_*(B) \xrightarrow{\bar{\mu}_*} B^C_*(B)$$

of $B^A_*(B)$-module homomorphisms, whose composite is the identity. Furthermore, $\bar{\mu}_*$ is a B_*-algebra homomorphism. It follows from the $B^A_*(B)$-linearity of the homomorphism $\bar{\sigma}_*$ that it is also a B_*-algebra homomorphism. For if $x, y \in B^C_*(B)$ then $\bar{\sigma}_* x \in B^A_*(B)$ acts on y through multiplication by its image $\bar{\mu}_*\bar{\sigma}_* x = x$ and on $\bar{\sigma}_* y \in B^A_*(B)$ by multiplication by $\bar{\sigma}_* x$. The $B^A_*(B)$-linearity of $\bar{\sigma}_*$ now provides the left hand equality below:

$$\bar{\sigma}_* x \cdot \bar{\sigma}_* y = \bar{\sigma}_*(\bar{\mu}_*\bar{\sigma}_* x \cdot y) = \bar{\sigma}_*(x \cdot y)\,.$$

Thus $\bar{\sigma}_*$ is a B_*-algebra homomorphism, so that $B^C_*(B)$ is a retract of $B^A_*(B)$, both in the category of $B^A_*(B)$-modules and, more importantly to us, in the category of commutative B_*-algebras.

Recall that $B^A_*(B) \cong \prod_G B_*$, since $A \to B$ is G-Galois. Here $\prod_G B_* \cong \bigoplus_G B_*$ is a finitely generated free B_*-module, since G is finite, so the retraction above implies that $B^C_*(B)$ is a finitely generated projective B_*-module. We may therefore once more consider the Goerss–Hopkins spectral sequence (10.1.4), now for the mapping space

$$\mathcal{C}_C(B, B) \cong \mathcal{C}_B(B \wedge_C B, B)\,.$$

The E_2-term has

$$E_2^{s,t} = D^s_{B_*T}(B^C_*(B), \Omega^t B_*)$$

for $t > 0$. The commutative B_*-algebra retraction $\bar{\mu}_* \colon B^A_*(B) \to B^C_*(B)$ induces a split injection from each of these cohomology groups to

$$D^s_{B_*T}(B^A_*(B), \Omega^t B_*)\,,$$

which we saw was zero in the proof of Theorem 11.1.1, since $B^A_*(B) = \prod_G B_*$ is étale over B_*. We therefore have $E_2^{s,t} = 0$ away from the origin, also in the Goerss–Hopkins spectral sequence for $\pi_{t-s}\mathcal{C}_C(B, B)$.

Thus $\mathcal{C}_C(B, B)$ is homotopy discrete, in the sense that each path component is weakly contractible, with set of path components

$$K = \pi_0 \mathcal{C}_C(B, B) \cong \mathrm{Alg}_{B_*}(B^C_*(B), B_*)\,.$$

The B_*-algebra retraction $\bar{\mu}_*$ induces a split injection from this set to

$$\mathrm{Alg}_{B_*}(B^A_*(B), B_*) \cong G\,.$$

It is clear that the natural map $\mathcal{C}_C(B, B) \to \mathcal{C}_A(B, B)$, viewing a map $B \to B$ of commutative C-algebras as a map of commutative A-algebras, is a monoid map with respect to the composition of maps. Therefore the injection $K \to G$ identifies K as a sub-monoid of G. But G is a finite group, so K is in fact a subgroup of G. This completes the proof of the first claims of the proposition.

The tautological action by $\mathcal{C}_C(B, B)$ on B through commutative C-algebra maps can be converted to an action by K on a commutative C-algebra B' weakly equivalent to B. We hereafter implicitly make this replacement, so as to have K acting directly on B over C, and turn to the proof of the final claim.

The composite $\bar{\sigma}_* \bar{\mu}_* \colon B^A_*(B) \to B^A_*(B)$ is an idempotent B_*-algebra map. Under the isomorphism $B^A_*(B) \cong \prod_G B_*$ it corresponds to an idempotent B_*-algebra endomorphism of $\prod_G B_*$. Since B_* is connected, it must be the retraction of $\prod_G B_*$ onto the subalgebra $\prod_{K'} B_*$, for some subset $K' \subset G$ (containing $e \in G$). Thus $B^C_*(B) \cong \prod_{K'} B_*$, which implies

$$K = \mathrm{Alg}_{B_*}(B^C_*(B), B_*) \cong \mathrm{Alg}_{B_*}(\prod_{K'} B_*, B_*) \cong K'\,.$$

Thus $K = K'$ as subsets of G, and the weak equivalence $B \wedge_A B \simeq \prod_G B$ retracts to a weak equivalence

$$h \colon B \wedge_C B \to \prod_K B\,.$$

It is quite clearly given by the action of K on B through commutative C-algebra maps, as in (4.1.2). $\square$

This leads us to the converse part of the Galois correspondence for E-local commutative S-algebras, in the case of finite, faithful Galois extensions.

Theorem 11.2.2. *Let $A \to B$ be a G-Galois extension, with B connected and G finite and discrete. Furthermore, let*

$$A \to C \to B$$

be a factorization of this map through a separable commutative A-algebra C such that $C \to B$ is faithful, and let $K = \pi_0 \mathcal{C}_C(B, B) \subset G$.

If $A \to B$ is faithful, or more generally, if B is dualizable over C, then $C \simeq B^{hK}$ as commutative C-algebras, and $C \to B$ is a faithful K-Galois extension.

Proof. We first prove that $A \to B$ faithful implies that B is dualizable over C.

By hypothesis, $A \to C$ is separable, so the multiplication map μ and its weak section σ make C a retract up to homotopy of $C \wedge_A C$, as a $C \wedge_A C$-module. Therefore μ makes C a dualizable $C \wedge_A C$-module, by Lemma 3.3.2(c). Similarly, for each $g \in G$ the twisted multiplication map

$$\mu(1 \wedge g) \colon C \wedge_A C \to C$$

and its weak section $(1 \wedge g^{-1})\sigma$ make C a dualizable $C \wedge_A C$-module. Inducing up along $C \to B$, Lemma 6.2.3 implies that each map $B \wedge_A C \to B$, given algebraically as $b \wedge c \mapsto b \cdot g(c)$, makes B a dualizable $B \wedge_A C$-module. By Lemma 3.3.2(c) again, it follows that the natural map $B \wedge_A C \to B \wedge_A B$ makes $B \wedge_A B \simeq \prod_G B \simeq \bigvee_G B$ a dualizable $B \wedge_A C$-module. By Proposition 6.2.1 and hypothesis, B is dualizable and faithful over A, so by Lemma 6.2.3 and Lemma 4.3.3 we know that $B \wedge_A C$ is faithful and dualizable over C. Thus by Lemma 6.2.4 it follows that the natural map $C \to B$ makes B a dualizable C-module.

By Proposition 11.2.1, $K = \pi_0 \mathcal{C}_C(B, B) \subset G$ acts on (a weakly equivalent replacement for) B through C-algebra maps, so that $h \colon B \wedge_C B \to \prod_K B$ is a weak equivalence. By hypothesis (and the argument above), $C \to B$ is faithful and dualizable. Then by Proposition 6.3.2 the natural map $i \colon C \to B^{hK}$ is a weak equivalence, and so $C \to B$ is a faithful K-Galois extension. □

12. Hopf–Galois extensions in topology

In this final chapter we work globally, i.e., not implicitly localized at any spectrum (other than at $E = S$).

12.1. Hopf–Galois extensions of commutative S-algebras.

Let $A \to B$ be a G-Galois extension of commutative S-algebras, with G stably dualizable, as usual. The right adjoint $\tilde{\alpha} \colon B \to F(G_+, B)$ of the group action map $\alpha \colon G_+ \wedge B \to B$ can be lifted up to homotopy through the weak equivalence $\nu\gamma \colon B \wedge DG_+ \to F(G_+, B)$, to a map $\beta \colon B \to B \wedge DG_+$. The group multiplication $G \times G \to G$ induces a functionally dual map $DG_+ \to D(G \times G)_+$, which likewise can be lifted up to homotopy through the weak equivalence $\wedge \colon DG_+ \wedge DG_+ \to D(G \times G)_+$ to a coproduct $\psi \colon DG_+ \to DG_+ \wedge DG_+$. We shall require rigid forms of these structure maps.

Definition 12.1.1. A *commutative Hopf S-algebra* is a cofibrant commutative S-algebra H equipped with a counit $\epsilon \colon H \to S$ and a coassociative and counital coproduct $\psi \colon H \to H \wedge_S H$, in the category of commutative S-algebras.

Note that we are not assuming that the coproduct ψ is (strictly) cocommutative, nor that it admits a strict antipode/conjugation $\chi \colon H \to H$. This would severely limit the number of interesting examples.

Example 12.1.2. Let X be an infinite loop space. The E_∞ structure on X makes $S[X] = S \wedge X_+$ an E_∞ ring spectrum. The diagonal map $\Delta \colon X \to X \times X$ and $X \to *$ induce a coproduct $\psi \colon S[X] \to S[X \times X] \cong S[X] \wedge S[X]$ and counit $\epsilon \colon S[X] \to S$, which altogether can be rigidified to make $H \simeq S[X]$ a commutative Hopf S-algebra. The rigidification takes the coassociative and counital coproduct and counit on $S[X]$ to a corresponding co-A_∞ structure on H, which in turn can be rigidified to strictly coassociative and counital operations, by working entirely within commutative S-algebras. It is in general not possible to make a similar rigidification of co-E_∞ structures, within commutative S-algebras.

Definition 12.1.3. Let A be a cofibrant commutative S-algebra, let B be a cofibrant commutative A-algebra and let H be a commutative Hopf S-algebra. We say that H *coacts* on B over A if there is a coassociative and counital map

$$\beta\colon B \to B \wedge H$$

of commutative A-algebras. In this situation, let

$$h\colon B \wedge_A B \to B \wedge H$$

be the composite map $(\mu \wedge 1)(1 \wedge \beta)$ of commutative B-algebras.

Definition 12.1.4. The (Hopf) *cobar complex* $C^\bullet(H; B)$, for H coacting on B over A, is the cosimplicial commutative A-algebra with

$$C^q(H; B) = B \wedge H \wedge \cdots \wedge H$$

(q copies of H) in codegree q. The coface maps are $d_0 = \beta \wedge 1^{\wedge q}$, $d_i = 1^{\wedge i} \wedge \psi \wedge 1^{\wedge(q-i)}$ for $0 < i < q$ and $d_q = 1^{\wedge q} \wedge \eta$, where $\eta\colon S \to H$ is the unit map. The codegeneracy maps involve the counit $\epsilon\colon H \to S$. Let $C(H; B) = \operatorname{Tot} C^\bullet(H; B)$ be its totalization. The algebra unit $A \to B$ induces a coaugmentation $A \to C^\bullet(H; B)$, and a map

$$i\colon A \to C(H; B)\,.$$

Definition 12.1.5. A map $A \to B$ of commutative S-algebras is an *H-Hopf–Galois extension* if H is a commutative Hopf S-algebra that coacts on B over A, so that the maps $i\colon A \to C(H; B)$ and $h\colon B \wedge_A B \to B \wedge H$ are both weak equivalences.

Note that there is no finiteness/dualizability condition on H in this definition. See [Chi00] for a recent text on Hopf–Galois extensions in the algebraic setting.

Example 12.1.6. Let G be a stably dualizable topological group. The weak coproduct on $DG_+ = F(G_+, S)$, derived from the group multiplication, can be rigidified to give $H \simeq DG_+$ the structure of a commutative Hopf S-algebra. If G acts on B over A, then the weak coaction of DG_+ on B can be rigidified to a coaction of H on B over A. Then the (Hopf) cobar complex $C^\bullet(H; B)$ maps by a degreewise weak equivalence to the (group) cobar complex $C^\bullet(G; B)$ from Definition 8.2.5. In codegree q it is weakly equivalent to the composite natural map

$$B \wedge DG_+ \wedge \cdots \wedge DG_+ \xrightarrow[\simeq]{\wedge} B \wedge DG_+^q \xrightarrow[\cong]{\gamma} DG_+^q \wedge B \xrightarrow[\simeq]{\nu} F(G_+^q, B)\,.$$

On totalizations, we obtain a weak equivalence $C(H; B) \simeq B^{hG}$. In this case, the definition of an H-Hopf–Galois extension $A \to B$ generalizes that of a G-Galois extension $A \to B$, since $i\colon A \to B^{hG}$ factors as

$$A \xrightarrow{i} C(H; B) \xrightarrow{\simeq} B^{hG}\,,$$

and $h\colon B \wedge_A B \to \prod_G B$ factors as

$$B \wedge_A B \xrightarrow{h} B \wedge H \xrightarrow{\simeq} F(G_+, B)\,.$$

Recall the Amitsur complex $C^\bullet(B/A)$ from Definition 8.2.1.

Definition 12.1.7. There is a natural map of cosimplicial commutative A-algebras $h^\bullet \colon C^\bullet(B/A) \to C^\bullet(H; B)$ given in codegree q by the map

$$h^q \colon B \wedge_A B \wedge_A \cdots \wedge_A B \to B \wedge H \wedge \cdots \wedge H$$

that is the composite of the maps

$$\begin{aligned} B^{\wedge_A (i+1)} \wedge H^{\wedge j} &\cong B^{\wedge_A (i-1)} \wedge_A (B \wedge_A B) \wedge H^{\wedge j} \\ &\xrightarrow{1^{\wedge(i-1)} \wedge h \wedge 1^{\wedge j}} B^{\wedge_A (i-1)} \wedge_A (B \wedge H) \wedge H^{\wedge j} \cong B^{\wedge_A i} \wedge H^{\wedge (j+1)} \end{aligned}$$

for $j = 0, \dots, q-1$ and $i + j = q$. Upon totalization, it induces a map $h' \colon A^\wedge_B \to C(H; B)$ of commutative A-algebras.

The diagram chase needed to verify that $h^\bullet$ indeed is cosimplicial uses the strict coassociativity and counitality of the Hopf S-algebra structure on H.

Proposition 12.1.8. *Suppose that H coacts on B over A, as above, and that $h \colon B \wedge_A B \to B \wedge H$ is a weak equivalence. Then $h' \colon A^\wedge_B \to C(H; B)$ is a weak equivalence. As a consequence, $A \to B$ is an H-Hopf–Galois extension if and only if A is complete along B.*

Proof. The cosimplicial map $h^\bullet$ is a weak equivalence in each codegree, so the induced map of totalizations h' is a weak equivalence. Therefore the composite $i = h' \circ \eta$, of the two maps

$$A \xrightarrow{\eta} A^\wedge_B \xrightarrow{h'} C(H; B) ,$$

is a weak equivalence if and only if η is one. $\square$

12.2. Complex cobordism.

Let $A = S$ be the sphere spectrum, $B = MU$ the complex cobordism spectrum and $H = S[BU] = \Sigma^\infty BU_+$ the unreduced suspension spectrum of BU. Bott's infinite loop space structure on BU makes H a commutative S-algebra, and the diagonal map $\Delta \colon BU \to BU \times BU$ induces the Hopf coproduct $\psi \colon S[BU] \to S[BU] \wedge S[BU]$. The Thom diagonal

$$\beta \colon MU \to MU \wedge BU_+$$

defines a coaction by $S[BU]$ on MU over S. The induced map

$$h \colon MU \wedge MU \to MU \wedge BU_+$$

is the weak equivalence known as the Thom isomorphism. The Bousfield–Kan spectral sequence associated to the cosimplicial commutative S-algebra $C^\bullet(MU/S)$ is the Adams–Novikov spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{MU_* MU}(MU_*, MU_*) \Longrightarrow \pi_{t-s}(S) .$$

The convergence of this spectral sequence is the assertion that the coaugmentation

$$i \colon S \to S^\wedge_{MU} = \mathrm{Tot}\, C^\bullet(MU/S)$$

is a weak equivalence. In view of Proposition 12.1.8 we can summarize these facts as follows:

Proposition 12.2.1. *The unit map $S \to MU$ is an $S[BU]$-Hopf–Galois extension of commutative S-algebras.* $\square$

Remark 12.2.2. There is no topological group G such that $S \to MU$ is a G-Galois extension, but $S[BU]$ is taking on the role of its functional dual DG_+, as in Example 12.1.6. So the commutative Hopf S-algebra $S[BU]$ is trying to be the ring of functions on the non-existent Galois group of MU over S. Note that there is no bimodule section to the multiplication map $\mu\colon MU \wedge MU \to MU$, since the left and right units $\eta_L, \eta_R\colon MU_* \to MU_*MU$ are really different, so $S \to MU$ is not separable in the sense of Section 9.1.

Remark 12.2.3. There are similar $S[X]$-Hopf–Galois extensions $S \to Th(\gamma)$ to the Thom spectrum induced by any infinite loop map $\gamma\colon X \to BGL_1(S)$. For example, there is such an extension $S \to MUP$ to the even periodic version MUP of MU, which is the Thom spectrum of the tautological virtual bundle over $X = \mathbb{Z} \times BU = \Omega^\infty ku$. More generally, for any commutative S-algebra R and infinite loop map $\gamma\colon X \to BGL_1(R)$ there is an R-based Thom spectrum $Th^R(\gamma)$, i.e., a γ-twisted form of $R[X] = R \wedge X_+$, and an $R[X]$-Hopf–Galois extension $R \to Th^R(\gamma)$.

Remark 12.2.4. The extension $S \to MU$ is known not to be faithful, since by [Ra84, §3] or [Ra92, 7.4.2] $MU_*(cY) = 0$ for every finite complex Y with trivial rational cohomology. Here cY denotes the Brown–Comenetz dual of Y. This faithlessness leaves the telescope conjecture [Ra84, 10.5] or [Ra92, 7.5.5] a significant chance to be false. Recall that if $F(n)$ is a finite complex of type n (with a v_n-self map), and $T(n) = v_n^{-1}F(n)$ its mapping telescope, the conjecture is that the natural map $\lambda\colon T(n) \to L_nF(n)$ is a weak equivalence. After inducing up to MU, $1 \wedge \lambda\colon MU \wedge T(n) \to MU \wedge L_nF(n)$ is an equivalence, by the localization theorem $v_n^{-1}MU \wedge F(n) \simeq L_nMU \wedge F(n)$ [Ra92, 7.5.2]. Positive information about the faithfulness of Galois- or Hopf–Galois extensions (Question 4.3.6) might conceivably reflect back on this conjecture.

To conclude this paper, we wish to discuss how the Hopf–Galois extension $S \to MU$ provides a global, integral object whose p-primary $K(n)$-localization and nilpotent completion $L_{K(n)}S \to \hat{L}^{MU}_{K(n)}MU$ governs the pro-Galois extensions $L_{K(n)}S \to E_n$, for each rational prime p and integer $n \geq 0$.

(12.2.5)
$$\begin{tikzcd}
MU & & \hat{L}^{MU}_{K(n)}MU \arrow[r, "t"] & E_n \\
S \arrow[u, "{S[BU]}"] & & L_{K(n)}S \arrow[u] \arrow[ur, "\mathbb{G}_n"'] &
\end{tikzcd}$$

This suggests that $S \to MU$ is a kind of near-maximal ramified Galois extension, and that its weak Galois group ("weak" in the analytic sense that it is only realized through its functional dual $DG_+ = S[BU]$ that coacts on MU) is a kind of near-absolute ramified Galois group of the sphere. More precisely, the maximal extension may be the one obtained from the even periodic theory MUP by tensoring with the ring $\mathcal{O}_{\bar{\mathbb{Q}}}$ of algebraic integers.

Even if $S \to MU$ does not admit many Galois automorphisms, the Hopf coaction $\beta\colon MU \to MU \wedge BU_+$ still determines the Galois action of each element $g \in \mathbb{G}_n$ on E_n. By the Hopkins–Miller theory, each commutative S-algebra map $g\colon E_n \to E_n$ is uniquely determined by the underlying map of (commutative) ring spectra, so it is the description of the latter that we shall review.

Recall from 5.4.2 that Γ_n is the Honda formal group law over $\mathbb{F}_{p^n}$ and $\widetilde{\Gamma}_n$ its universal deformation, defined over $\pi_0(E_n)$. By the Lubin–Tate theorem [LT66, 3.1], each automorphism $g \in \mathbb{S}_n \subset \mathbb{G}_n$ of Γ_n determines a unique pair $(\phi, \tilde{g})$, where $\phi\colon \pi_0(E_n) \to \pi_0(E_n)$ is a ring automorphism and $\tilde{g}\colon \widetilde{\Gamma}_n \to \phi_*\widetilde{\Gamma}_n$ is an isomorphism of formal group laws over $\pi_0(E_n)$, whose expansion $\tilde{g}(x) \in \pi_0(E_n)[[x]] \cong E_n^0(\mathbb{C}P^\infty)$ reduces modulo $(p, u_1, \dots, u_{n-1})$ to the expansion $g(x) \in \mathbb{F}_{p^n}[[x]]$ of g. Then $\phi = \pi_0(g)$, when g is considered as a self-map of E_n. Furthermore,

$$\begin{aligned} \tilde{g}(x) \in E_n^0(\mathbb{C}P^\infty) &\cong \mathrm{Hom}_{E_{n*}}(E_{n*}(\mathbb{C}P^\infty), E_{n*}) \\ &\cong \mathrm{Alg}_{E_{n*}}(E_{n*}(BU), E_{n*}) \subset E_n^0(BU) \end{aligned}$$

corresponds to a unique map of ring spectra $\tilde{g}\colon S[BU] \to E_n$. Let $t\colon MU \to E_n$ be the usual complex orientation, corresponding to the graded version of $\widetilde{\Gamma}_n$. Then the following diagram commutes up to homotopy:

(12.2.6)
$$\begin{tikzcd} MU \arrow[r, "\beta"] \arrow[d, "t"'] & MU \wedge BU_+ \arrow[d, "\mu(t\wedge\tilde{g})"] \\ E_n \arrow[r, "g"] & E_n \end{tikzcd}$$

The composite $g \circ t = \mu(t \wedge \tilde{g})\beta$ determines g, in view of $t^*\colon E_n^*(E_n) \to E_n^*(MU)$ being nearly injective. Only the Galois automorphisms in $\mathrm{Gal} \subset \mathbb{G}_n$ are missing, but these may be ignored if we are focusing on $\widehat{E(n)}$, or can be detected by passing to MUP and adjoining some roots of unity.

By analogy, for number fields $K \subset L$ and primes $\mathfrak{p} \in \mathcal{O}_K$, a factorization $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdot \ldots \cdot \mathfrak{P}_r^{e_r}$ leads to a splitting of completions $K_\mathfrak{p} \to L \otimes_K K_\mathfrak{p} \cong \prod_i L_{\mathfrak{P}_i}$. If the field extension $K \to L$ is G-Galois, then each local extension $K_\mathfrak{p} \to L_{\mathfrak{P}_i}$ is $G_{\mathfrak{P}_i}$-Galois, where $G_{\mathfrak{P}_i} \subset G$ is the decomposition group of $\mathfrak{P}_i$, and G acts transitively on the finite set of primes over $\mathfrak{p}$. Thus when the global extension $K \to L$ is localized (i.e., completed), it splits as a product of smaller local extensions, in a way that depends on the place of localization.

(12.2.7)
$$\begin{tikzcd} L & L \otimes_K K_\mathfrak{p} \arrow[r, "pr_i"] & L_{\mathfrak{P}_i} \\ K \arrow[u, "G"] & K_\mathfrak{p} \arrow[u] \arrow[ur, "G_{\mathfrak{P}_i}"'] & \end{tikzcd}$$

In the algebraic case of a pro-Galois extension $K \to \bar{K}$ there is a profinite set of places over each prime $\mathfrak{p}$, still forming a single orbit for the action by the absolute Galois group G_K.

In the topological setting of $S \to MU$ there is likewise a single orbit of chromatic primes of MU over the one of S that corresponds to the localization functor $L_{K(n)}$ on $\mathcal{M}_S$, namely those corresponding to the nilpotent completion functors $\hat{L}^{MU}_{K(n)}$ on $\mathcal{M}_{MU}$, for all the various possible complex orientations $MU \to K(n)$. In the absence of a real Galois group of automorphisms of MU these do not form a geometric orbit of places, but the next-best thing is available, namely the S-algebraic coaction by $S[BU]$ via the Thom diagonal, a sub-coaction of which indeed links the various complex orientations of $K(n)$ into one "weak" orbit.

Jack Morava [Mo05] has developed this Galois theoretic perspective on the stable homotopy category further.

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