6/3/23

J. Wildeshaus, "Shimura varieties and interior motives", Lecture 1: "Weight structures"

1 Basic definitions

Fix a triangulated category $\mathcal{C}$.

Definition 1.1: (Bondarko, Pauksztello.)

A weight structure on $\mathcal{C}$ is a pair $(\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0})$ of full sub-cat.s of $\mathcal{C}$, st., putting $\mathcal{C}_{w\le n} := \mathcal{C}_{w\le 0}[n]$, $\mathcal{C}_{w\ge n} := \mathcal{C}_{w\ge 0}[n]$ $\forall n \in \mathbb{Z}$, we have:

(1) $\mathcal{C}_{w\le 0}$ and $\mathcal{C}_{w\ge 0}$ are Karoubi-closed, i.e., closed under direct summands in $\mathcal{C}$,

(2) $\mathcal{C}_{w\le 0} \subset \mathcal{C}_{w\le 1}$, $\mathcal{C}_{w\ge 0} \supset \mathcal{C}_{w\ge 1}$,

(3) (Orthogonality.) $\forall M \in \mathcal{C}_{w\le 0}$, $N \in \mathcal{C}_{w\ge 1}$,

$$\mathrm{Hom}_{\mathcal{C}}(M, N) = 0,$$

(4) (Weight filtrations.) $\forall M \in \mathcal{C}$ $\exists$ exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, with $A \in \mathcal{C}_{w\le 0}$, $B \in \mathcal{C}_{w\ge 1}$.

Remarks: (a) Pauksztello: "co-t-structures".

(b) weight str. $\longleftrightarrow$ t-str.: same axioms, but

$$\mathcal{C}^{t\le n} = \mathcal{C}^{t\le 0}[-n], \quad \mathcal{C}^{t\ge n} = \mathcal{C}^{t\ge 0}[-n]$$

(and then (1) is redundant).

Definition 1.2: The heart of a weight structure $w = (\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0})$ is the full sub-cat. $\mathcal{C}_{w=0} := \mathcal{C}_{w\le 0} \cap \mathcal{C}_{w\ge 0}$.

Remark: Let $A \in \mathcal{C}_{w=0}$. Then

$$A \longrightarrow 0 \longrightarrow A[1] = A[1]$$

is a wt. filtration of the zero object!

$\rightsquigarrow$ wt filtrations are rarely unique!!

Proposition 1.3: Let $\mathcal{H} \subset \mathcal{C}$ be full. Assume $\mathcal{H}$ generates $\mathcal{C}$ (as a triang. cat.).
(Bondarko.)

(a) If there is a wt. str. on $\mathcal{C}$ whose heart $\mathcal{C}_{w=0}$ contains $\mathcal{H}$, then it is unique.

In this case, $\mathcal{C}_{w=0}$ is the Karoubi closure of $\mathcal{H}$ in $\mathcal{C}$.

(b) The following are equivalent:

(i) $\exists$ wt. str. on $\mathcal{C}$ whose heart contains $\mathcal{H}$,

(ii) $\mathcal{H}$ is <u>negative</u>, i.e.,

$$\mathrm{Hom}_{\mathcal{C}}(A, B[i]) = 0 \quad \forall A, B \in \mathcal{H},\ i \in \mathbb{Z}_{\geq 1}.$$

<u>Idea of proof:</u> (b) (i) $\Rightarrow$ (ii): obvious by def.

$\mathcal{C}_{w\leq 0} = \{$ success. ext.s of obj.s $A[n]$, $n \leq 0\}$, $\Big\}$ necess.^{ly}, as $\mathcal{C}_{w\leq 0}$

$\mathcal{C}_{w\geq 0} = \{$ ——— " ——— , $n \geq 0\}$. $\Big\}$ and $\mathcal{C}_{w\geq 0}$ are stable under ext.s

Bondarko: (ii) $\Rightarrow$ this defines a wt. str. (q.e.d.)

<u>Perspective:</u> (tomorrow...)

$\mathcal{C} =$ Voevodsky mot. $DM_{gm}(k)$, $\mathcal{H} =$ Chow motives $Mot_{CH}(k)$

$\xRightarrow{\text{(Bondarko)}}$ the <u>motivic weight structure</u>!

<u>2 Weight filtrations avoiding certain weights</u>

$\mathcal{C}$ as before, $w = (\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$ a wt. str. on $\mathcal{C}$.

<u>Definition 2.1:</u> $M \in \mathcal{C}$, $m \leq n$ integers. <u>A weight filtration of M avoiding weights</u> <u>$m, m+1, \ldots, n$</u> is an ex. tr.

$$M_{\leq m-1} \longrightarrow M \longrightarrow M_{\geq n+1} \longrightarrow M_{\leq m-1}[1],$$

with $M_{\leq m-1} \in \mathcal{C}_{\leq m-1}$, $M_{\geq n+1} \in \mathcal{C}_{\geq n+1}$.

<u>Exercise 2.2:</u> $m \leq n$ integers,

$$M_{\leq m-1} \longrightarrow M \longrightarrow M_{\geq n+1} \longrightarrow M_{\leq m-1}[1] \qquad \text{and}$$

$$N_{\leq m-1} \longrightarrow N \longrightarrow N_{\geq n+1} \longrightarrow N_{\leq m-1}[1]$$

two wt. filtr. avoiding wt.s $m, m+1, \ldots, n$. Then any morphism $M \longrightarrow N$ extends uniquely to a morph. of wt. filtr.

<u>Main points:</u> $\mathrm{Hom}_{\mathcal{C}}(M_{\leq m-1}, N_{\geq n+1}) = 0$, $\mathrm{Hom}_{\mathcal{C}}(M_{\leq m-1}, M_{\geq n+1}[-1]) = 0$ (orthogonality).

hence: if a wt filt. avoiding wt.s $m, m+1, \dots, n$ exists, then it is unique up to unique isomorphism!

<u>Definition 2.3:</u> $M \in \mathcal{C}$ <u>avoids weights $m, m+1, \dots, n$</u>, or <u>is without weights $m, m+1, \dots, n$</u> if M admits a wt filt. avoiding wt.s $m, m+1, \dots, n$.

<u>Definition 2.4:</u> (a) $\mathcal{C}_{w\le 0, \ne -1}$ is the full sub-cat. of $\mathcal{C}_{w\le 0}$ of objects without wt. -1.

(b) $\mathcal{C}_{w\ge 0, \ne +1}$ —— " —— $\mathcal{C}_{w\ge 0}$ —— " —— $+1$.

<u>3 The heart</u>

$\mathcal{C}$, w as before.

<u>Exercise 3.1:</u> (a) The inclusion $\iota_- : \mathcal{C}_{w=0} \hookrightarrow \mathcal{C}_{w\le 0, \ne -1}$ admits a left adjoint $\mathrm{Gr}_0 : \mathcal{C}_{w\le 0, \ne -1} \longrightarrow \mathcal{C}_{w=0}$.

On objects, it is given by sending M to the term $M_{\ge 0}$ of a wt filtr. $M_{\le -2} \longrightarrow M \longrightarrow M_{\ge 0} \longrightarrow M_{\le -2}[1]$ avoiding wt. -1. We have $\mathrm{Gr}_0 \circ \iota_- = \mathrm{id}_{\mathcal{C}_{w=0}}$.

(b) Similarly: $\mathrm{Gr}_0 : \mathcal{C}_{w\ge 0, \ne +1} \longrightarrow \mathcal{C}_{w=0}$ right adj. to $\iota_+ \dots$

<u>hence:</u> if $M \in \mathcal{C}_{w\le 0, \ne -1}$ carries the action of a group / an algebra / whatever (example: the Hecke algebra!), then so does $\mathrm{Gr}_0 M$.

<u>FIX:</u> $u : M_- \longrightarrow M_+$ morph. in $\mathcal{C}$, with $M_- \in \mathcal{C}_{w\le 0}$, $M_+ \in \mathcal{C}_{w\ge 0}$,

$$C \xrightarrow{v_-} M_- \xrightarrow{u} M_+ \xrightarrow{v_+} C[1] \quad \text{ex. tr. in } \mathcal{C}.$$

<u>ASSUMPTION:</u> C is without wt.s -1 and 0:

$$C_{\le -2} \xrightarrow{c_-} C \xrightarrow{c_+} C_{\ge 1} \xrightarrow{\delta_C} C_{\le -2}[1] \quad \text{ex. tr.}, \; C_{\le -2} \in \mathcal{C}_{w\le -2}, C_{\ge 1} \in \mathcal{C}_{w\ge 1}$$

<u>Perspective:</u> $M_- = M(X)^e$, $M_+ = M^c(X)^e$ direct factors of the <u>motive</u> and the <u>motive with compact support</u> of a smooth variety X over a field, $C = \partial M(X)^e$ the corresponding direct factor of the <u>boundary motive</u> of X.

($\rightsquigarrow$ The ASSUMPTION puts a restriction on the projector e on $\partial M(X)^e$. ($e = \mathrm{id}$ will not do i.g.!))

<u>Theorem 3.2:</u> In the above (general) situation, and under the ASSUMPTION N:

(a) $M_- \in \mathcal{C}_{w\le 0, \neq -1}$, $M_+ \in \mathcal{C}_{w\ge 0, \neq 1}$.

(b) The wt. filtr.s according wt -1, resp. wt. $+1$, are

$$C_{\le -2} \xrightarrow{v_-c_-} M_- \xrightarrow{\pi_0} \mathrm{Gr}_0 M_- \xrightarrow{\delta_-} C_{\le -2}[1] ,$$
$$C_{\ge 1} \xrightarrow{\delta_+} \mathrm{Gr}_0 M_+ \xrightarrow{i_0} M_+ \xrightarrow{(c_+[1])v_+} C_{\ge 1}[1] .$$

(c) There is a canonical isomorphism $\mathrm{Gr}_0 M_- \xrightarrow{\sim} \mathrm{Gr}_0 M_+$.

As a morphism, it is uniquely determined by the commutativity of

$$\begin{tikzcd} M_- \arrow[r, "u"] \arrow[d, "\pi_0"'] & M_+ \\ \mathrm{Gr}_0 M_- \arrow[r] & \mathrm{Gr}_0 M_+ \arrow[u, "i_0"'] \end{tikzcd}.$$

<u>Perspective:</u> $u: M(X)^e \longrightarrow M^c(X)^e$ $\rightsquigarrow$ $\mathrm{Gr}_0 M(X)^e = \mathrm{Gr}_0 M^c(X)^e =$ the <u>e-part of the interior motive</u>.

<u>Proof of Theorem 3.2:</u> Choose and fix cones G_- of v_-c_- and G_+ of $c_+(v_+[-1])$:

$$C_{\le -2} \xrightarrow{v_-c_-} M_- \xrightarrow{\pi_0'} G_- \xrightarrow{\delta_-} C_{\le -2}[1] ,$$
$$C_{\ge 1} \xrightarrow{\delta_+} G_+ \xrightarrow{i_0'} M_+ \xrightarrow{(c_+[1])v_+} C_{\ge 1}[1] .$$

hence: $G_- \in \mathcal{C}_{w=0}$, $G_+ \in \mathcal{C}_{w=0}$.

Let us show: $\exists\, \alpha: G_- \xrightarrow{\sim} G_+$ making the diagram

$$\begin{tikzcd} M_- \arrow[r, "u"] \arrow[d, "\pi_0'"'] & M_+ \\ G_- \arrow[r, "\alpha"] & G_+ \arrow[u, "i_0'"'] \end{tikzcd}$$

commute ($\Longrightarrow$ (a), (b), (c)):

consider

$$(*) \quad \begin{tikzcd} M_- \arrow[dd, "u"', "{[1]}"] & & C_{\le -2} \arrow[ll, "v_-c_-"'] \arrow[dl, "c_-"] \\ & C \arrow[ul, "v_-"'] \arrow[dr, "c_+", "{[1]}"'] & \\ M_+[-1] \arrow[ur, "{v_+[-1]}"] \arrow[rr, "{c_+(v_+[-1])}", "{[1]}"'] & & C_{\ge 1}[-1] \arrow[uu, "{\delta_c[-1]}"'] \end{tikzcd}$$

here:
- the 3 arrows marked $[1]$ link the source to the target, shifted by $[1]$,
- the upper and lower triangles are commutative,
- the left and right triangles are exact.

By definition, this means that $(*)$ is a <u>calotte inférieure</u>.

One of the axioms of triang. cat.s (TR4' in BBD's formulation) implies:

$\exists$ calotte supérieure

$$\begin{tikzcd}
M_- \arrow[dd, "u"', "{[1]}"] \arrow[dr, "{[1]}"] & & C_{\leq -2} \arrow[ll, "v_- c_-"'] \\
& G \arrow[ur] \arrow[dl] & \\
M_+[-1] \arrow[rr, "c_+(v_+[1])", "{[1]}"'] & & C_{\geq 1}[-1] \arrow[ul] \arrow[uu, "\delta_c[-1]"']
\end{tikzcd}$$

· same meaning of [1],
· left and right: commutative
· upper and lower: exact.

"in particular": (i) the same object $G[1]$ can be chosen as cone of $v_- c_-$ and of $c_+(v_+[1])$,

(ii) $G[1]$ provides a factorisation of u. q.e.d.

<u>Corollary 3.3:</u> In the setting of Theorem 3.2, let $M_- \longrightarrow N \longrightarrow M_+$ be a factorisation of u through $N \in \mathcal{C}_{w=0}$. Then $Gr_0 M_- = Gr_0 M_+$ is canonically identified with a direct factor of N, admitting a canonical direct complement.

<u>Idea of proof:</u> Ex. 3.1 $\rightsquigarrow$ $M_- \xrightarrow{\pi_0} Gr_0 M_- \longrightarrow N \longrightarrow Gr_0 M_+ \xrightarrow{i_0} M_+$

Thm 3.2 (c) $\rightarrow$

this is the can. iso.! (q.e.d.)

4 Semi-primary categories

F: a coefficient ring $\rightsquigarrow$ F-categories, F-functors...

<u>Definition 4.1:</u> (Kelly.)

$\mathfrak{A}$ an F-cat. The <u>radical</u> of $\mathfrak{A}$ is the ideal $\mathrm{rad}_{\mathfrak{A}}$, which associates to each pair of objects $A, B \in \mathfrak{A}$

$$\mathrm{rad}_{\mathfrak{A}}(A,B) := \{ f \in \mathrm{Hom}_{\mathfrak{A}}(A,B) \mid \forall g \in \mathrm{Hom}_{\mathfrak{A}}(B,A),\ \mathrm{id}_A - gf \text{ invertible} \}$$
$$\subset \mathrm{Hom}_{\mathfrak{A}}(A,B).$$

(This <u>is</u> a two-sided ideal of $\mathfrak{A}$.)

Definition 4.2: (André-Kahn.)

(a) An F-functor $T : \mathfrak{A} \to \mathfrak{B}$ between F-cat.s is *radicial* if

$$T(\mathrm{rad}_{\mathfrak{A}}) \subset \mathrm{rad}_{\mathfrak{B}}.$$

(b) An F-cat. $\mathfrak{A}$ is *semi-primary* if

(1) $\forall A \in \mathfrak{A}$, $\mathrm{rad}_{\mathfrak{A}}(A,A)$ is nilpotent:

$$\exists N \in \mathbb{N},\ \mathrm{rad}_{\mathfrak{A}}(A,A)^N = 0 \subset \mathrm{End}_{\mathfrak{A}}(A),$$

(2) the quotient cat. $\overline{\mathfrak{A}} := \mathfrak{A}/\mathrm{rad}_{\mathfrak{A}}$ is semi-simple.

Proposition 4.3: (André-Kahn.)

Assume F to be a field. Let $T : \mathfrak{A} \to \mathfrak{B}$ be an F-functor.

Assume: (i) $\mathfrak{A}$ is semi-primary and pseudo-Abelian,

(ii) T is radicial,

(iii) $\forall A \in \mathfrak{A}$: $T(A) = 0 \Longrightarrow A = 0$.

Then T is conservative.

Idea of proof: $\overline{\mathfrak{A}}$ is Abelian semi-simple, and $\mathfrak{B} \to \overline{\mathfrak{B}}$ is conservative. (q.e.d.)

Perspective: T = the restriction of the (l-adic or Hodge theoretic) regulator to $\mathfrak{A}$ = Chow motives (of a certain type – in order to guarantee 4.3 (i), (ii); (iii)).

J. Wildeshaus, "Shimura varieties and interior motives",

Lecture 2: "Chow motives realizing to interior cohomology of Shimura varieties"

5 The motivic weight structure

recall from M. Levine's talks: · $Mot_{CH}^{eff}(k) \hookrightarrow DM_{gm}^{eff}(k)$, $\quad \cap \quad \cap$, $\quad Mot_{CH}(k) \hookrightarrow DM_{gm}(k)$

$\Big\}$ $\mathbb{Z}[1/p']$-categories, $p' = \begin{cases} 1, & \text{char } k = 0 \\ \text{char}(k), & \neq 0 \end{cases}$

· $M^c(X)$ + localisation

Theorem 5.1: (Bondarko.)

(a) There is a unique wt. str. on $DM_{gm}^{eff}(k)$, whose heart equals $Mot_{CH}^{eff}(k)$.

(b) There is a canonical extension of the wt. str. from (a) to one on $DM_{gm}(k)$.

It is unique wrt. the requirement that its heart equal $Mot_{CH}(k)$.

Proof: $DM_{gm}^{eff}(k) \supset DM^s$: full sub-cat. gen.^{d} by $\{M(X) \mid X \in Sm_k\}$,

$\cup$ J_0: full add. sub-cat. gen^d by $\{M(X) \mid X \in SmProj_k\}$.

$\rightsquigarrow$ $DM_{gm}^{eff}(k) = (DM^s)^\natural$,

$Mot_{CH}^{eff}(k) = J_0^\natural$.

(1): J_0 generates DM^s Δ-ly:

· local.^{tion} + ind. on dim: $\langle J_0 \rangle^\Delta \supset \{M^c(U) \mid U \underset{gen}{\subset} X \in SmProj_k\}$,

· alteration + —"—: $\langle J_0 \rangle^\Delta \supset \{M^c(U) \mid U \in Sm_k\}$.

· duality + —"—: $\langle J_0 \rangle^\Delta = DM^s$.

(2): J_0 is negative: $Hom_{DM_{gm}(k)}(M(X), M(Y)[i]) = 0 \ \forall X, Y \in SmProj_k, i \geq 1$:

M. Levine's talks.

(1), (2) $\underset{Prop. 1.31}{\Longrightarrow}$ unique wt str on DM^s whose heart contains J_0

(and $DM^s_{w=0} = J_0^{Kar}$).

Bondarko: w is compatible with $\natural$! Whence (a).

(b): Implied formally by (a). q.e.d.

Definition 5.2: The weight structures from Thm. 5.2 are called the motivic weight structure on $DM_{gm}^{eff}(k)$ and $DM_{gm}(k)$, respectively.

6 The boundary motive

recall from M. Levine's talks: · $NST(k)$ Nisnevich sheaves with transfers,

$$\begin{tikzcd} D(NST(k)) \arrow[r, "RC^{Sus}"] & DM^{eff}(k) \arrow[l, bend left] \\ & DM_{gm}^{eff}(k) \arrow[u, hook, "i"] \end{tikzcd}$$

s.t. $i(M(X)) = RC^{Sus}(\mathbb{Z}^{tr}(X)) \quad \forall X \in Sm_k$,

with $\mathbb{Z}^{tr}(X) := Hom_{Cor(k)}(\,\cdot\,, [X])$.

· $M^c(X) := RC^{Sus}(z_{qfin}(X))$, $X \in Sm_k$,

with $z_{qfin}(X) \subset z_{\dim X}(X \times \cdot\,)$

gen^d by quasi-finite cycles over $\cdot$.

$\mathbb{Z}^{tr}(X) \subset z_{qfin}(X)$ in $NST(k)$.

Definition 6.1: The boundary motive of X is defined as

$$\partial M(X) := RC^{Sus}\big(z_{qfin}(X)/\mathbb{Z}^{tr}(X)\big)[-1] \in DM^{eff}(k).$$

Proposition 6.2: (a) There is a canonical exact triangle

$$(*) \quad \partial M(X) \to M(X) \xrightarrow{u} M^c(X) \longrightarrow \partial M(X)[1].$$

(b) $\partial M(X) \in DM_{gm}^{eff}(k)$.

Proof: (a): by definition.

(b): M. Levine's talks: $M^c(X) \in DM_{gm}^{eff}(k)$. q.e.d.

The exact Δ $(*)$ is equivariant under

$$\mathbb{Z}^{tr}(X)(X) \cap (\mathbb{Z}^{tr}(X)(X))^t \subset z_{\dim X}(X \times X)$$

gen^d by cycles finite over both copies of X

$\to$ source of idempotents e on $(*)$.

7 The (e-part of the) interior motive

$X \in Sm_k$, e an idempotent acting on $(*)$.

Set up the situation from §3:

$u : M(X)^e \longrightarrow M^c(X)^e$. Indeed:

Proposition 7.1: $M(X) \in DM^{eff}_{gm}(k)_{w\le 0}$, $M^c(X) \in DM^{eff}_{gm}(k)_{w\ge 0}$.

Proof: duality $\Rightarrow$ suffices to show :

ind. on dim X: $X = pt$: $X \in Sm\,Proj_k$, so $M^c(X) = M(X) \in DM^{eff}_{gm}(k)_{w=0}$

if $X \hookrightarrow \overline{X}$, $\overline{X} \in Sm\,Proj_k$, then

localization: $M^c(\overline{X}) \longrightarrow M^c(X) \longrightarrow M^c(\overline{X} \setminus X)[1] \to M^c(\overline{X})[1]$

$$\begin{array}{ccc} M^c(\overline{X}) & & M^c(\overline{X}\setminus X)[1] \\ \| & & \uparrow \\ M(X) & & \underbrace{\text{wt} \ge 0 \text{ by ind.}} \\ \text{wt. } 0 & & \text{wt} \ge 1 \end{array}$$

... q.e.d.

ASSUMPTION: $\partial M(X)^e$ is without weights -1 and 0.

$\rightsquigarrow$ Thm. 3.2 applies. In particular:

Theorem 7.2: Under the ASSUMPTION,

$M(X)^e$ is without weight -1, $M^c(X)^e$ is without weight $+1$, and

$Gr_0 M(X)^e = Gr_0 M^c(X)^e$ (in $DM^{eff}_{gm}(k)_{w=0} = Mot_{CH}(k)^{eff}$).

Definition 7.3: $Gr_0 M(X)^e$ is <u>the e-part of the interior motive of X.</u>

recall from M. Levine's talks: realizations

- $R_B : DM_{gm}(k)^{op} \longrightarrow D(Vec_{\mathbb{Q}})$ (for $k \hookrightarrow \mathbb{C}$),

  refining to

  $R_{MHS} : DM_{gm}(k)^{op} \longrightarrow D(MHS_{\mathbb{Q}})$,

- $R_{\acute{e}t,\ell} : DM_{gm}(k)^{op} \longrightarrow D_{\acute{e}t,cfn}(\mathbb{Q}_\ell, k)$ (for $\ell \ne$ char k).

For $X \in Sm_k$, the cohomology objects of $R_?(M(X))$

compute (Betti or ℓ-adic) cohomology of X.

weights on the target of R_{MHS} / $R_{\acute{e}t,\ell}$ (if k is fin^{ly} gen^d over its prime field):

in the sense of Deligne! Meaning: $R(M(X))$ is of weight zero for $X \in Sm\,Proj_k$.

Theorem 7.4: Under the ASSUMPTION:

(a) $\mathcal{H}^n(R_?(Gr_0 M_{gm}(X)^e)) = H^n_!(X, \mathbb{Q}_{(\ell)})^e \quad \forall n \in \mathbb{Z}$.

$\uparrow$ im $(H^n_c \to H^n)$

(b) For $? =$ MHS or ($? =$ ét, ℓ and k fin.$^{\ell}$ gen.$^{\ell}$ over its prime field), $H^n_!(X, \mathbb{Q}_{(\ell)})^e$ equals the lowest weight filtr. step W_n of $H^n(X, \mathbb{Q}_{(\ell)})^e$, $\forall n \in \mathbb{Z}$.

Proof: Thm. 3.2(b): ex. seq.

$$C_{\le -2} \longrightarrow M(X)^e \longrightarrow Gr_0 M(X)^e \longrightarrow C_{\le -2}[1]$$

$C_{\le -2} \in \mathcal{C}_{w \le -2}$

$\Rightarrow \mathcal{H}^n(R_?(C_{\le -2}))$ of wts $\ge n+2$, $\forall n \in \mathbb{Z}$... q.e.d.

PROBLEM: How to verify the ASSUMPTION?

More generally, given $M \in DM_{gm}(k)$, how to recognize the absence of certain weights in M?

8 Weight conservativity

Definition 8.1: $\bar{k}$:= alg. closure of k.

(a) $Mot^{Ab}_{CH}(\bar{k}) \subset Mot_{CH}(\bar{k})$ full add. ps.-Abelian sub-cat. gen.^{d} by

- $\mathbb{Z}[m][2m] \quad \forall m \in \mathbb{Z}$,
- $M(A)$, $A/\bar{k}$ Abelian variety.

Chow motives of Abelian type over $\bar{k}$.

(b) $Mot^{Ab}_{CH}(k) \subset Mot_{CH}(k)$ full sub-cat. of obj. M s.t. $M \times_k \bar{k} \in Mot^{Ab}_{CH}(\bar{k})$.

Chow motives of Abelian type over k.

(c) $DM^{Ab}_{gm}(k) \subset DM_{gm}(k)$ full $\triangle$ sub-cat. gen.^{d} by $Mot^{Ab}_{CH}(k)$.

Voevodsky motives of Abelian type over k.

Proposition 8.2: The motivic weight structure induces a weight structure on $DM^{Ab}_{gm}(k)$.

Proof: apply Prop. 1.3. q.e.d.

Bondarko: as $Mot^{Ab}_{CH}(k)$ is pseudo-Abelian, so is $DM^{Ab}_{gm}(k)$.

from now on: need $\mathbb{Q}$-linear versions of everything:

$$\mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}, \quad \mathrm{DM}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}, \ldots$$

<u>Proposition 8.3:</u> All objects of $\mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}$ are <u>finite dimensional</u>, and $\mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}$ is semi-primary.

<u>Proof:</u> Künnemann: all objects in $\mathrm{Mot}^{\mathrm{ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}$ are f.d.

O'Sullivan: —— " —— $\mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}$ are f.d.

André-Kahn: all obj. f.d. $\Rightarrow$ the cat. is semi-primary. q.e.d.

<u>Corollary 8.4:</u> (Kimura.)

Let $M \in \mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}$ be such that $R_{\mathfrak{Z}}(M)$ or $R_{\sigma,\ell}(M)$ equals 0.

Then $M = 0$.

<u>Proof:</u> This holds actually as soon as M is f.d. (q.e.d.)

Let R be one of $R_{\mathfrak{Z}}$ or $R_{\sigma,\ell}$. $\mathcal{H}^* R := \bigoplus_{n\in\mathbb{Z}} \mathcal{H}^n \circ R$.

<u>Theorem 8.5:</u> The restriction of $\mathcal{H}^* R$ to $\mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}} \subset \mathrm{DM}^{\mathrm{Ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}} \subset \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ maps the radical $\mathrm{rad}_{\mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}}$ to zero. (In particular, it is radicial.)

<u>Corollary 8.6:</u> The restriction of $\mathcal{H}^* R$ to $\mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}$ is conservative.

<u>Proof:</u> apply Prop. 4.3: (i) = Prop. 8.3,

(iii) = Cor. 8.4,

(ii) = Thm. 8.5. q.e.d.

"<u>Corollary</u>" 8.7: The restriction of $(\mathcal{H}^*)R$ to $\mathrm{DM}^{\mathrm{Ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ is conservative.

(Proof omitted; a few technical efforts are needed to deduce the result from 8.5...)

<u>Corollary 8.8:</u> The restriction of $(\mathcal{H}^*)R$ to $\mathrm{DM}^{\mathrm{Ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ is <u>weight conservative</u>:

$M \in \mathrm{DM}^{\mathrm{Ab}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ is without weights $\alpha, \alpha+1, \ldots, \beta$ iff

$\mathcal{H}^n R(M)$ is without wt.s $n-\beta, \ldots, n-(\alpha+1), n-\alpha$, $\forall n \in \mathbb{Z}$.

<u>Proof:</u> "only if": ✓.

"if": $\mathrm{Mot}^{\mathrm{Ab}}_{\mathrm{CH}}(k)_{\mathbb{Q}}$ semi-primary $\Longrightarrow$ can find <u>minimal weight filtr.</u>

$$M_{\le \alpha-1} \longrightarrow M \longrightarrow M_{\ge \alpha} \xrightarrow{\delta_\alpha} M_{\le \alpha-1}[1],$$

$$M_{\le\beta} \longrightarrow M \longrightarrow M_{\ge \beta+1} \xrightarrow{\delta_{\beta+1}} M_{\le\beta}[1], \text{ i.e.,}$$

wt. filtr.s st. δ_α and $\delta_{\beta+1}$ belong to the radical.

Find morph.

$$\begin{tikzcd} M_{\le\alpha-1} \arrow[r] \arrow[d, "m"] & M \arrow[r] \arrow[d, equal] & M_{\ge\alpha} \arrow[r, "\delta_\alpha"] \arrow[d] & M_{\le\alpha-1}[1] \arrow[d, "m[1]"] \\ M_{\le\beta} \arrow[r] & M \arrow[r] & M_{\ge\beta+1} \arrow[r, "\delta_{\beta+1}"] & M_{\le\beta}[1] \end{tikzcd}$$

Thm. 8.5 $\Rightarrow \mathcal{H}^*R(\delta_\alpha)=0,\ \mathcal{H}^*R(\delta_{\beta+1})=0.$

⇝ ex. seq.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^*R(M_{\le\alpha-1}) \arrow[r] \arrow[d, "\mathcal{H}^*R(m)", "\wr"'] & \mathcal{H}^*R(M) \arrow[d, equal] \\ 0 \arrow[r] & \mathcal{H}^*R(M_{\le\beta}) \arrow[r] & \mathcal{H}^*R(M) \end{tikzcd}$$

our hyp. on wt.s: $\mathcal{H}^*R(m)$ is an iso.

$\underset{\text{Cor 8.7}}{\Longrightarrow}$ m is an iso. q.e.d.

<u>Proof of Theorem 8.5:</u> comparison: WLOG $R = R_B$.

O'Sullivan: $\chi_k\mathbb{C}$ is radicial ⇝ WLOG; $k=\mathbb{C}$.

André-Kahn : $B \in \mathrm{SmProj}_{\mathbb{C}}$ st. $H(B)$ is f.d., then $\mathrm{rad}_{\mathrm{Mot}_{CH}(k)}(H(B),H(B))$ = classes of num.^{ly} trivial cycles on $B\times_{\mathbb{C}} B$.

Lieberman : B Abelian var. $\Rightarrow$ (num^{ly} trivial = hom.^{ly} trivial).

q.e.d.

J. Wildeshaus, "Shimura varieties and interior motives", Lecture 3: "Verifications and applications"

§ The canonical construction

$(G, \mathcal{H})$: (pure) Shimura data à la Pink:

(among others:) $G/\mathbb{Q}$ connected reductive,

$\mathcal{H}$ homogeneous under $G(\mathbb{R})$,

$h: \mathcal{H} \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}})$ with finite fibres,
$\mathbb{S} := \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ $\quad G(\mathbb{R})$-equiv.

$\rightsquigarrow V \in \mathrm{Rep}_{\mathbb{Q}}(G)$ yields: $\forall x \in \mathcal{H}$, $h_x: \mathbb{S} \longrightarrow G_{\mathbb{R}} \longrightarrow GL(V_{\mathbb{R}})$
fin. dim. alg. rep.s $\quad \hat{=}\ V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ s.t. $V^{q,p} = \overline{V^{p,q}}$

$\rightsquigarrow \mu: \mathrm{Rep}_{\mathbb{Q}}(G) \longrightarrow \mathrm{VHS}_{\mathbb{Q}}(\mathcal{H})$.
variation of $\mathbb{Q}$-HS

If V is irreducible, then $\mu(V)$ is pure
$V = V_{\underline{\lambda}}$, $\underline{\lambda}$ dominant weight. } say of wt. $r(\underline{\lambda})$

$K \subset G(\mathbb{A}_f)$ open, compact

$\rightsquigarrow M^K(G,\mathcal{H})(\mathbb{C}) := G(\mathbb{Q}) \backslash (\mathcal{H} \times (G(\mathbb{A}_f)/K))$ $\mathbb{C}$-valued pt.s of
$M^K(G,\mathcal{H})$: Shimura variety of level K.
- def.^{d} over a number field;
- normal,
- smooth if K is neat.

$V \in \mathrm{Rep}_{\mathbb{Q}}(G) \rightsquigarrow \mu(V)/\mathcal{H}$ descends to a $\mathrm{VHS}_{\mathbb{Q}}$ on $M^K(G,\mathcal{H})_{\mathbb{C}}$
if K is neat.

$\rightsquigarrow$ canonical construction $\mu_{K,\mathrm{HS}}: \mathrm{Rep}_{\mathbb{Q}}(G) \longrightarrow \mathrm{VHS}_{\mathbb{Q}}(M^K(G,\mathcal{H})_{\mathbb{C}})$
for K neat.
- exact
- $\otimes$-comp.
- $V = V_{\underline{\lambda}} \Rightarrow \mu_{K,\mathrm{HS}}(V)$ pure of wt. $r(\underline{\lambda})$.

A similar construction is possible in the l-adic context:

$$\mu_{K,\ell}: \mathrm{Rep}_{\mathbb{Q}_\ell}(G) \longrightarrow \mathrm{Et}^{\text{lisse}}_{\mathbb{Q}_\ell}(M^K(G,\mathcal{H})) \qquad \text{for } K \text{ neat.}$$

<u>Example 9.1:</u> $(G,\mathcal{H}) = (GL_{2,\mathbb{Q}}, \mathcal{H}_1 := \mathbb{C}\setminus\mathbb{R})$.

$$\underline{\alpha} \in \{(k,r) \in \mathbb{N}_0 \times \mathbb{Z} \mid k \equiv r \bmod 2\} \rightsquigarrow V_{\underline{\alpha}} = \mathrm{Sym}^k \underbrace{V}_{\text{stand. rep. of } GL_2} \otimes \det^{-\frac{r+k}{2}}, \quad r(\underline{\alpha}) = r$$

$$\rightsquigarrow \mu_{K,?}(V_{\underline{\alpha}}) = \mathrm{Sym}^k(R^1\pi_* \mathbb{Q}_{(\ell)})^\vee\left(-\frac{r+k}{2}\right), \text{ for}$$

$$\pi: \mathcal{E} \longrightarrow \underbrace{M^K(G,\mathcal{H})}_{\text{mod. curve!}} \text{ the univ. ell. curve.}$$

In this context, recall: Eichler-Shimura:

$$H^1_!\big(M^K(G,\mathcal{H})(\mathbb{C}), \underbrace{\mathrm{Sym}^k(R^1\pi_*\mathbb{Q})^\vee(-k)}_{=\mathrm{Sym}^k(R^1\pi_*\mathbb{Q})}\big)\otimes_{\mathbb{Q}}\mathbb{C} = \underbrace{S_{k+2}(K)}_{\text{cusp forms of level } K \text{ and "weight } k+2\text{"}} \oplus \overline{S_{k+2}(K)}.$$

Denote by $j: M^K(G,\mathcal{H}) \hookrightarrow M^K(G,\mathcal{H})^*$ the open immersion into the <u>Baily-Borel compactification</u>, and by

$$i: \partial M^K(G,\mathcal{H})^* \hookrightarrow M^K(G,\mathcal{H})^* \qquad \text{the closed complement.}$$

<u>Conjecture 9.2:</u> Assume $(G,\mathcal{H})$ to be <u>of Hodge type</u> (meaning:

$\exists$ morph. of Shimura data: $(G,\mathcal{H}) \longrightarrow (CSp_{2g}, \mathcal{H}_g)$

for some $g \geq 1$, st. $G \to CSp_{2g}$ is a monomorphism).

Let $V_{\underline{\alpha}} \in \mathrm{Rep}_{\mathbb{Q}}(G)$ irred., st. $\underline{\alpha}$ is <u>regular</u>. Then

$$i^* Rj_* \mu_{K,?}(V_{\underline{\alpha}}) \,/\, \partial M^K(G,\mathcal{H})^*$$

avoids weights $r(\underline{\alpha})$ and $r(\underline{\alpha})+1$: $\forall n \in \mathbb{Z}$,

$$i^* R^n j_* \mu_{K,?}(V_{\underline{\alpha}}) \text{ is a } \left\langle \begin{array}{l}\text{Hodge module}\\ \ell\text{-adic sheaf}\end{array} \right\rangle \text{ on } \partial M^K(G,\mathcal{H})^*_{(\mathbb{C})}$$

without weights $n + r(\underline{\alpha})$ and $n + r(\underline{\alpha}) + 1$.

<u>Theorem 9.3</u>: The conjecture is true for the following Shimura data:

(1) $(GL_{2,\mathbb{Q}}, \mathcal{H}_1)$: mod. curves (well known),

(2) $(\mathrm{Res}^F_{\mathbb{Q}} GL_{2,F}, \mathcal{H}_1^{[F:\mathbb{Q}]})$, $F|\mathbb{Q}$ tot.[ly] real, $F\neq\mathbb{Q}$: Hilbert-Blumenthal var.s (W.),

(3) $(GU(2,1), \mathcal{H})$, for $GU(2,1) \subset \mathrm{Res}^E_{\mathbb{Q}} GL_{3,E}$, E quadr. imag.: Picard surfaces (Ancona),

(4) $(\mathrm{Res}^{E^+}_{\mathbb{Q}} GU(2,1), \mathcal{H})$, for $GU(2,1) \subset \mathrm{Res}^E_{E^+} GL_{3,E}$, E CM-field, $E^+\neq\mathbb{Q}$: Picard varieties (Cloître),

(5) $(CSp_4, \mathcal{H}_2)$: Siegel threefolds (W.),

(6) $(\mathrm{Res}^F_{\mathbb{Q}} CSp_4, \mathcal{H}_2^{[F:\mathbb{Q}]})$, $F|\mathbb{Q}$ tot.[ly] real, $F\neq\mathbb{Q}$: Hilbert-Siegel var.s (Lavicchia),

(7) $(CSp_6, \mathcal{H}_3)$: Siegel sixfolds (W.).

<u>Indication for (1):</u> $\underline{\lambda} = (k,r)$, $V_{\underline{\lambda}} = \mathrm{Sym}^k V \otimes \det^{-\frac{r+k}{2}}$.

Then

$$i^* R^n j_* \mu_{n,!}(V_{\underline{\lambda}}) = \begin{cases} 0, & n \neq 0,1 \\ \mathbb{Q}_{\ell}(k) \otimes \mathbb{Q}_{\ell}\left(-\frac{r+k}{2}\right) = \mathbb{Q}_\ell\left(-\frac{r-k}{2}\right), & n=0 \\ \mathbb{Q}_{\ell}(-1) \otimes \mathbb{Q}_{\ell}\left(-\frac{r+k}{2}\right) = \mathbb{Q}_\ell\left(-\frac{r+k+2}{2}\right), & n=1 \end{cases}.$$

$\rightsquigarrow$ wt. $r-k$ for $n=0$, $\quad \left.\begin{matrix} < r \\ > r+2 \end{matrix}\right\}$ iff $k>0 \Leftrightarrow \underline{\lambda}$ regular.
wt. $r+k+2$ for $n=1$.

<u>Remarks</u>: (a) In cases (2), (4) and (6): certain (mildly) irregular $\underline{\lambda}$ give rise to $V_{\underline{\lambda}}$ still satisfying the conclusion of Conj. 9.2.

(b) The analogue of Conj. 9.2 is (very) false e.g. when the BB-comp. is replaced by a toroidal comp.

10 The relative situation

<u>Black box 10.1:</u> (Cisinski–Déglise.)

$DM_{gm}(\cdot)_{\mathbb{Q}}$ can be defined for rather general base schemes $\cdot$, and there is a formalism of 6 functors.

<u>Black box 10.2:</u> (Hébert.)

There is a motivic weight structure on $DM_{gm}(S)_{\mathbb{Q}}$, uniquely determined by the following:

$$\pi : X \longrightarrow S \text{ proper}, \ X \text{ regular}$$
$$\Rightarrow \pi_* \mathbb{Q}_X \in DM_{gm}(S)_{\mathbb{Q}, w=0}.$$

W.r.t. this wt. str.: $f : S \longrightarrow T$ ⇝

f_* and $f^!$ respect $DM_{gm}(\cdot)_{\mathbb{Q}, w\geq 0}$,

f^* and $f_!$ respect $DM_{gm}(\cdot)_{\mathbb{Q}, w\leq 0}$.

<u>Definition 10.3:</u> $Mot_{CH}(S)_{\mathbb{Q}} := DM_{gm}(S)_{\mathbb{Q}, w=0}$.

<u>Black box 10.4:</u> Let $X \in Sm_k$, $j : X \hookrightarrow \overline{X}$ a compactification,

$$\begin{tikzcd} X \arrow[r, hook, "j"] & \overline{X} \arrow[d, "\overline{a}"] & \partial X := \overline{X} \setminus X \arrow[l, hook', "i"'] \\ & \operatorname{Spec} k & \end{tikzcd}$$

Then the image under $\overline{a}_*$ of the exact triangle

$$j_! \mathbb{Q}_X \longrightarrow j_* \mathbb{Q}_X \longrightarrow i_* i^* j_* \mathbb{Q}_X \longrightarrow j_! \mathbb{Q}_X[1]$$

(in $DM_{gm}(\overline{X})_{\mathbb{Q}}$) is canonically dual to

$$(*) \qquad M^c(X) \longleftarrow M(X) \longleftarrow \partial M(X) \longleftarrow M^c(X)[-1].$$

<u>observe:</u> $j_! \longrightarrow j_* \longrightarrow i_* i^* j_* \longrightarrow j_![1]$ is exact, whatever coeff. we take

⇝ may replace $\mathbb{Q}_X$ by some other object of $Mot_{CH}(X)_{\mathbb{Q}}$

⇝ (*) "with coefficients":

$$\partial M(V) \longrightarrow M(V) \longrightarrow M^c(V) \longrightarrow \partial M(V)[1],$$

$V \in Mot_{CH}(X)_{\mathbb{Q}}$ $\Longrightarrow$ same wt. estimates as usual for $M(V)$ and $M^c(V)$.

Theorem 9.4: (Ancona.) K neat.

Assume $(G, \mathcal{H})$ is <u>of PEL-type</u> ($\Rightarrow$ of Hodge type).

There is a $\mathbb{Q}$-linear $\otimes$-functor $\mu_{K,\mathrm{mot}}: \mathrm{Rep}_{\mathbb{Q}}(G) \longrightarrow \mathrm{Mod}^{s}_{CH}(M^K(G,\mathcal{H}))$.

Smooth Chow motives

It has the following properties:

(a) $\mathcal{H}^* R_? \circ \mu_{K,\mathrm{mot}} = \mu_{K,?}$, $? \in \{MHS, \ell\}$. More precisely:

$V = V_{\underline{\alpha}}$ irred. $\Rightarrow R_? \mu_{K,\mathrm{mot}}(V_{\underline{\alpha}})$ conc.^{d} in degree $r(\underline{\alpha})$

and its value equals $\mu_{K,?}(V_{\underline{\alpha}})$.

(b) $\mu_{K,\mathrm{mot}}$ maps the restr. of the stand. rep. $V \in \mathrm{Rep}_{\mathbb{Q}}(CSp_{2g})$ to $h^1(\mathcal{A}/M^K(G,\mathcal{H}))^\vee$: $\mathcal{A}$ the pull-back of the univ. ab. scheme over $M^L(CSp_{2g}, \mathcal{H}_g)$.

$\rightarrow$ for $\underline{\alpha}$ arbitrary (dominant), $\mu_{K,\mathrm{mot}}(V_{\underline{\alpha}})$ is a direct factor of a Lefschetz twist of some $h(\mathcal{A}^{m_{\underline{\alpha}}}/M^K(G,\mathcal{H}))$:

we'll ignore the twist $\mu_{K,\mathrm{mot}}(V_{\underline{\alpha}}) = h(\mathcal{A}^{m_{\underline{\alpha}}}/M^K(G,\mathcal{H}))^{e_{\underline{\alpha}}}$

($m_{\underline{\alpha}}$ and $e_{\underline{\alpha}}$ depending on $\underline{\alpha}$).

$\rightarrow \quad H^*(M^K(G,\mathcal{H}), \mu_{K,?}(V_{\underline{\alpha}})) = H^{*+r(\underline{\alpha})}(\mathcal{A}^{m_{\underline{\alpha}}}, \mathbb{Q}(e))^{e_{\underline{\alpha}}}$,

and likewise for H^*_c, ∂H^* and $H^*_!$.

Theorem 9.5: K neat, $(G,\mathcal{H})$ of PEL-type. Assume that Conj. 9.2 holds for $(G,\mathcal{H})$. Then for any regular $\underline{\alpha}$,

$\partial M(\mathcal{A}^{m_{\underline{\alpha}}})^{e_{\underline{\alpha}}}$ is without weights -1 and 0.

(Hence the $e_{\underline{\alpha}}$-part of the interior motive of $\mathcal{A}^{m_{\underline{\alpha}}}$ is defined.)

Proof: our normalizations: $\mathcal{H}^* R_?(\partial M(\mathcal{A}^{m_{\underline{\alpha}}})^{e_{\underline{\alpha}}}) = \partial H^{*-r(\underline{\alpha})}(M^K(G,\mathcal{H}), \mu_{K,?}(V_{\underline{\alpha}}))$,

which is the image of $i^* R j_* \mu_{K,?}(V_{\underline{\alpha}})[-r]$ under $R a_*$, for

$a: \partial M^K(G,\mathcal{H})^* \longrightarrow *$

a is proper, hence Ra_* preserves weights.

Our claim would thus follow from Cor. 8.8 if $\partial M(\mathcal{A}^{m_{\underline{\alpha}}})^{e_{\underline{\alpha}}}$ were of Abelian type.

Problem: $\partial M(A^{m_{\underline{\alpha}}})^{e_{\underline{\alpha}}}$ is of Abelian type if $\partial M^{K}(G,\mathcal{H})^*$ is of dimension zero.

If $\dim \partial M^{K}(G,\mathcal{H})^* \geq 1$, then $\partial M(A^{m_{\underline{\alpha}}})$ is i.g. not of Abelian type.

Solution: (1) redo <u>everything</u> for relative motives:

- $DM(\cdot)_{\mathbb{Q}}$: Cisinski–Déglise,
- wt. st. on $DM(\cdot)_{\mathbb{Q}}$: Hébert,
- notion of motives of Abelian type (over a stratified scheme),
- weight conservativity in that relative context,

(2) prove: $V \in Rep_{\mathbb{Q}}(G)$, then

$$i^* Rj_* \mu_{K,mot}(V) \in DM(\partial M^{K}(G,\mathcal{H})^*)_{\mathbb{Q}}$$

is of Abelian type. (This uses Pink's thesis...)

Hence Conj. 9.2 actually implies: $i^* Rj_* \mu_{K,mot}(V_{\underline{\alpha}})$ is without

wts 0 and 1 as soon as $\underline{\alpha}$ is regular

(and $\partial M(A^{m_{\underline{\alpha}}})^{e_{\underline{\alpha}}}$ is dual to its image under Ra_* ...) q.e.d.

10. How to verify Conjecture 9.2?

The boundary $\partial M^{K}(G,\mathcal{H})^*$ is stratified; to control $i^* Rj_* \mu_{K,\ell}(V)$, one is led to try to control $i_Z^* Rj_* \mu_{K,\ell}(V)$ for all strata Z (and the way these restrictions glue together...). Let Z be one such.

Pink: up to the free action, $Z = M^{\pi_1(K)}(G_1,\mathcal{H}_1)$ is a Shimura variety,

for "smaller" Shimura data $(G_1,\mathcal{H}_1)$. G_1 is obtained as follows:

Q : a (certain) parabolic sub-gr. of G,

W_1 : its unip. radical,

$Q \xrightarrow{\pi_1} Q/W_1 = G_1 \cdot G_\ell$ almost direct product.

The inertia group of Z in $M^{K}(G,\mathcal{H})^*$ is an arithmetic sub-group of $\pi_1^{-1}(G_\ell)(\mathbb{Q})$.

<u>Theorem:</u> (Pink, proving a conjecture of Harder (1989).)

$$\forall V \in Rep_{\mathbb{Q}}(G), \quad i_Z^* R^n j_* \mu_{K,\ell}(V) = \bigoplus_{p+q=n} \mu_{\pi_1(K),\ell}\left(H^p(\pi_1(H_C), H^q(W_1, Res^G_Q V))\right).$$

can (in principle) be controlled using Kostant's formula...