Completeness of the Hypergeometric Solutions of the qKZ Equation at Level Zero

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Abstract. We discuss the hypergeometric solutions of the quantized Knizhnik-Zamolodchikov (qKZ) equation at level zero and show that they give all solutions of the qKZ equation. We completely describe linear relations between the hypergeometric solutions and give an interpretation of the relations in terms of the quantum group $U_q(\mathfrak{sl}_2)$ at $q = e^{\pi i/2}$.

1. Introduction

In this paper we study the hypergeometric solutions of the quantized Knizhnik-Zamolodchikov (qKZ) equation at level zero, cf. (2.1) – (2.3). We are going to address two problems. The first one is to show that the hypergeometric solutions give all solutions of the qKZ equation. The second one is to describe completely linear relations between the hypergeometric solutions, in other words, to say in what cases the construction of hypergeometric solutions produces zero solutions of the qKZ equation.

The qKZ equation (2.1) at level zero for $\mu = 0$ was introduced in [S1] as equations for form factors in the $SU(2)$-invariant Thirring model and a construction of he hypergeometric solutions was given therein. A simple counting of dimensions shows that the obtained hypergeometric solutions are not linearly independent. Linear relations between the hypergeometric solutions were found later in [S2] as a consequence of the deformed Riemann bilinear identity, and it was conjectured that there was no other linear relations between the hypergeometric solutions. This conjecture, in particular, implied that all solutions of the qKZ equation can be obtained as hypergeometric solutions.

For the rational $\mathfrak{sl}_2$-type qKZ equation at generic level the hypergeometric solutions were contsructed in [TV], [MV] and it was shown that they exhausted all solutions of the qKZ equation. The general construction was adapted to the case of level zero for $\mu = 0$ in [NPT] and Smirnov's solution were obtained from it, but completeness of the hypergeometric solutions still remained an open question.

In this paper we consider the qKZ equation (2.1) at level zero both for $\mu \neq 0$ and $\mu = 0$. The completeness of the hypergeometric solutions in the case $\mu \neq 0$ is claimed in Theorem 4.2, the main part of its proof being given by a formula for a determinant of a matrix formed by the hypergeometric integrals, see Theorem 5.3, which is analogous to Theorem 5.14 in [TV]. Theorem 4.2 also implies that in the case $\mu \neq 0$ the hypergeometric solutions are linearly independent.

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To prove the completeness of the hypergeometric solutions in the case $\mu = 0$, claimed in Theorem 4.3, we employ a new differential equation obeyed by the hypergeometric solutions of the qKZ equation, see Theorem 6.1. Linear relations between the hypergeometric solutions in the case $\mu = 0$ are described in Theorem 4.4, its proof being given in Section 7. In Section 8 we give an interpretation of the relations between the hypergeometric solutions in terms of the quantum group $U_q(\mathfrak{sl}_2)$ at $q = e^{\pi i/2}$.

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2. The qKZ equation at level zero

Let $\hbar$ be a nonzero complex number. Let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ and let $R(z) \in \mathrm{End}\,(V \otimes V)$ be the following R-matrix:

$$R(z) \;=\; \frac{z + \hbar P}{z + \hbar}\;,$$

where $P \in \mathrm{End}\,(V \otimes V)$ is the permutation operator. Let

$$\sigma^+ \,=\, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- \,=\, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^3 \,=\, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli matrices and

$$H \;=\; \frac{1 - \sigma^3}{2} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

As usual, for any $X \in \mathrm{End}\,(V)$ we set $X_i \,=\, \mathrm{id} \otimes \ldots \otimes \underset{i\text{-th}}{X} \otimes \ldots \otimes \mathrm{id}$, and similarly, if $R = \sum r \otimes r' \in \mathrm{End}\,(V \otimes V)$, then we set

$$R_{ij} \;=\; \sum \,\mathrm{id} \otimes \ldots \otimes \underset{i\text{-th}}{r} \otimes \ldots \otimes \underset{j\text{-th}}{r'} \otimes \ldots \otimes \mathrm{id}\,.$$

Fix a nonzero complex number p called a *step* and a number μ, such that $0 \leqslant \mathrm{Im}\,\mu < 2\pi$. We consider the qKZ equation for a $V^{\otimes n}$-valued function $\Psi(z_1, \ldots, z_n)$:

$$(2.1) \qquad \Psi(z_1, \ldots, z_m + p, \ldots, z_n) \;=\; K_m(z_1, \ldots, z_n)\, \Psi(z_1, \ldots, z_n)\,, \qquad m = 1, \ldots, n\,,$$

$$(2.2) \qquad K_m(z_1, \ldots, z_n) \;=\; R_{m,m-1}(z_m - z_{m-1} + p)\, \ldots\, R_{m,1}(z_m - z_1 + p) \;\times$$
$$\times\; \exp\,(\mu H_m)\, R_{m,n}(z_m - z_n)\, \ldots\, R_{m,m+1}(z_m - z_{m+1})\,.$$

The number $-2 + p/\hbar$ is called a *level* of the qKZ equation. The operators $K_1, \ldots, K_n$ are called the qKZ operators. Notice, that they depend on the parameter μ only via e^μ, so the assumption $0 \leqslant \mathrm{Im}\,\mu < 2\pi$ is not, in fact, a restriction.

Let $\Sigma^a = \sum\limits_{m=1}^{n} \sigma^a_m\,,\;\; a = \pm, 3$. The operators $\Sigma^\pm, \Sigma^3$ define an $\mathfrak{sl}_2$-action in $V^{\otimes n}$. Denote by $(V^{\otimes n})_\ell$ the weight subspace:

$$(V^{\otimes n})_\ell \;=\; \{ v \in V^{\otimes n} \mid \Sigma^3 v \,=\, (n - 2\ell)\, v \}\,,$$

which is nontrivial only for $\ell = 0, , \ldots, n$, and by $(V^{\otimes n})^{sing}_\ell$ the subspace of singular vectors:

$$(V^{\otimes n})^{sing}_\ell \;=\; \{ v \in (V^{\otimes n})_\ell \mid \Sigma^+ v \,=\, 0 \}\,,$$

which is nontrivial only for $2\ell \leqslant n$. For $\mu = 0$ the qKZ operators $K_1, \dots, K_n$ commute with the $\mathfrak{sl}_2$-action in $V^{\otimes n}$ while for $\mu \neq 0$ they commute only with the action of Σ^3. Therefore, the space of solutions of the qKZ equation inherits a weight decomposition of the $\mathfrak{sl}_2$-module $V^{\otimes n}$ in general, and all the structure of the $\mathfrak{sl}_2$-module if $\mu = 0$.

In what follows we discuss solutions of the qKZ equation (2.1) taking values in a weight subspace $(V^{\otimes n})_\ell$. For $\mu = 0$ we write down only solutions taking values in the subspace of singular vectors $(V^{\otimes n})_\ell^{sing}$. All other solutions can be obtained from them by the action of Σ^-.

In this paper we consider the case of the qKZ equation at level zero, that is, all over the paper we assume that

$$p \,=\, 2\hbar\,, \tag{2.3}$$

unless otherwise stated. We also assume that $\operatorname{Im} p < 0$. This assumption is used in the definition of the hypergeometric integral, cf. (3.3).

3. Hypergeometric solutions of the qKZ equation at level zero

The hypergeometric solutions of the qKZ equation at level zero in the case $\mu = 0$ are discussed in detail in [NPT]. The case $\mu \neq 0$ can be considered similarly. For this reason we give here only the most essential points of the construction. In general, the hypergeometric solutions of the rational qKZ equation are obtained in [TV]. Here we adapt the general construction from [TV] to the case $p = 2\hbar$.

For any function $f(t_1, \dots, t_\ell)$ we set

$$\operatorname{Asym} f(t_1, \dots, t_\ell) \,=\, \sum_{\sigma \in \mathbf{S}_\ell} \operatorname{sgn}(\sigma)\, f(t_{\sigma_1}, \dots, t_{\sigma_\ell})\,.$$

Let M be a subset of $\{1, \dots, n\}$ such that $\#M = \ell$. We write $M = \{m_1, \dots, m_\ell\}$ assuming that $m_1 < \ldots < m_\ell$. Introduce a function w_M as follows:

$$w_M \,=\, \operatorname{Asym} g_M\,, \tag{3.1}$$

$$g_M(t_1, \dots, t_\ell) \,=\, \prod_{a=1}^{\ell} \Bigl(\frac{1}{t_a - z_{m_a}} \prod_{1 \leqslant j < m_a} \frac{t_a - z_j - \hbar}{t_a - z_j} \Bigr) \prod_{1 \leqslant a < b \leqslant \ell} (t_a - t_b - \hbar)\,,$$

and define a vector $v_M \in (V^{\otimes n})_\ell$ by the rule:

$$v_M \,=\, v_{\varepsilon_1} \otimes \ldots \otimes v_{\varepsilon_n}\,, \qquad\qquad \varepsilon_m \,=\, \begin{cases} + & \text{if } m \notin M \\ - & \text{if } m \in M \end{cases} \tag{3.2}$$

Let $\bar{\mathcal{F}}_q^{\wedge \ell}$ be a space of functions $F(t_1, \dots, t_\ell; z_1, \dots, z_n)$ p-periodic in $z_1, \dots, z_n$:

$$F(t_1, \dots, t_\ell; z_1, \dots, z_m + p, \dots, z_n) \,=\, F(t_1, \dots, t_\ell; z_1, \dots, z_n)\,,$$

and such that the product

$$F(t_1, \dots, t_\ell; z_1, \dots, z_n) \prod_{m=1}^{n} \prod_{a=1}^{\ell} \bigl(1 - \exp\bigl(2\pi i (t_a - z_m)/p\bigr)\bigr)$$

is an antisymmetric polynomial in the exponentials $e^{2\pi i t_1/p}, \ldots, e^{2\pi i t_\ell/p}$ of degree less than n in each of the variables. Notice that any $F \in \bar{\mathcal{F}}_q^{\wedge\ell}$ is a p-periodic function of $t_1, \ldots, t_\ell$:

$$F(t_1, \ldots, t_a + p, \ldots, t_\ell; z_1, \ldots, z_n) \;=\; F(t_1, \ldots, t_\ell; z_1, \ldots, z_n)\,.$$

Denote by $\mathcal{F}_q^{\wedge\ell} \subset \bar{\mathcal{F}}_q^{\wedge\ell}$ a subspace of functions F such that the product

$$F(t_1, \ldots, t_\ell; z_1, \ldots, z_n)\, \exp\bigl(-2\pi i \sum_{a=1}^{\ell} t_a/p\bigr) \prod_{m=1}^{n} \prod_{a=1}^{\ell} \bigl(1 - \exp\bigl(2\pi i (t_a - z_m)/p\bigr)\bigr)$$

is a polynomial in the exponentials $e^{2\pi i t_1/p}, \ldots, e^{2\pi i t_\ell/p}$.

Remark. Here we defined a minimal space $\bar{\mathcal{F}}_q^{\wedge\ell}$ of periodic functions required to obtain the hypergeometric solutions of the *qKZ* equation. In [NPT] a larger space of periodic functions is considered. But it is easy to see that it produces the same space of the hypergeometric solutions of the *qKZ* equation (2.1) for $\mu = 0$.

Let $\phi(t)$ be the *phase function*:

$$\phi(t) \;=\; \exp\,(\mu t/p) \prod_{m=1}^{n} \frac{\Gamma\bigl((t - z_m - \hbar)/p\bigr)}{\Gamma\bigl((t - z_m)/p\bigr)}\,.$$

Lemma 3.1. *For any* $\varepsilon > 0$ *the phase function* $\phi(t)$ *has the following asymptotics*

$$\phi(t) \;=\; (t/p)^{-n/2} \exp\,(\mu t/p)\,\bigl(1 + o(1)\bigr)$$

as $t \to \infty\,, \quad \varepsilon < |\arg\,(t/p)| < \pi - \varepsilon\,.$

The statement follows from the Stirling formula.

Let C be a simple curve separating the sets $\bigcup_{m=1}^{n} (z_m + \hbar + p\mathbb{Z}_{\leqslant 0})$ and $\bigcup_{m=1}^{n} (z_m + p\mathbb{Z}_{\geqslant 0})$, and going from $-\infty$ to $+\infty$. More precisely, the contour C admits a parametrization $\rho : \mathbb{R} \to \mathbb{C}$ such that $\rho(u) \to \pm\infty$ and $\mathrm{Im}\,\rho(u)$ has finite limits as $u \to \pm\infty$. Recall that we assume $\mathrm{Im}\,p < 0$ and $p = 2\hbar$.

Denote by $I^{\otimes\ell}(w, W)$ the following integral:

$$I^{\otimes\ell}(w, W) \;=\; \frac{1}{\ell!} \int_{C^\ell} w(t_1, \ldots, t_\ell)\, W(t_1, \ldots, t_\ell) \prod_{a=1}^{\ell} \phi(t_a)\, dt_a\,. \tag{3.3}$$

We are interested in the integrals $I^{\otimes\ell}(w_M, W)$ where a function w_M, $\#M = \ell$, is defined by (3.1) and $W \in \bar{\mathcal{F}}_q^{\wedge\ell}$, and call them the hypergeometric integrals. If $0 < \mathrm{Im}\,\mu < 2\pi$, then these integrals are clearly convergent. If $\mathrm{Im}\,\mu = 0$, then the integral $I^{\otimes\ell}(w_M, W)$ is convergent provided that either $2\ell \leqslant n$ or $W \in \mathcal{F}_q^{\wedge\ell}$. In fact, the integral $I^{\otimes\ell}(w_M, W)$ for arbitrary ℓ, w_M and $W \in \bar{\mathcal{F}}_q^{\wedge\ell}$ considered as a function of μ has a finite limit if μ tends from inside the strip $0 < \mathrm{Im}\,\mu < 2\pi$ to any point on the boundary lines $\mathrm{Im}\,\mu = 0$ and $\mathrm{Im}\,\mu = 2\pi$ except 0 and $2\pi i$. Further if necessary, the integrals $I^{\otimes\ell}(w_M, W)$ should be understood in the described regularized sense.

For any $W \in \bar{\mathcal{F}}_q^{\wedge\ell}$ define a function $\Psi_W(z_1, \ldots, z_n)$ with values in $(V^{\otimes n})_\ell$ as follows:

$$\Psi_W(z_1, \ldots, z_n) = \sum_{\#M=\ell} I^{\otimes\ell}(w_M, W)\, v_M\,, \tag{3.4}$$

where the vectors v_M are given by (3.2).

Theorem 3.2. *Let $0 \leqslant \operatorname{Im}\mu < 2\pi$ and $\mu \neq 0$. Then for any $W \in \bar{\mathcal{F}}_q^{\wedge\ell}$ the function Ψ_W is a solution of the qKZ equation (2.1) taking values in $(V^{\otimes n})_\ell$.*

Theorem 3.3. *Let $\mu = 0$ and either $2\ell \leqslant n$ or $W \in \mathcal{F}_q^{\wedge\ell}$. Then the function Ψ_W is a solution of the qKZ equation (2.1) taking values in $(V^{\otimes n})_\ell^{sing}$.*

Corollary 3.4. *Let $\mu = 0$, $2\ell > n$ and $W \in \mathcal{F}_q^{\wedge\ell}$. Then $\Psi_W = 0$.*

Proof. The subspace $(V^{\otimes n})_\ell^{sing}$ is trivial if $2\ell > n$, which implies the claim. □

Theorems 3.2 and 3.3 follow from the results on formal integral representations for solutions of the *qKZ* equation [R], [V]. Details of the proof in the case $\mu = 0$ are available in [NPT]. The case $\mu \neq 0$ is similar.

Solutions Ψ_W of the *qKZ* equation are called the *hypergeometric solutions*.

4. Completeness of the hypergeometric solutions

Let us formulate accurately the meaning of completeness of the hypergeometric solutions of the *qKZ* equation. We will use the following compact notation: $z = (z_1, \ldots, z_n)$.

Consider a vector space $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$ and a union of hyperplanes $\mathbb{D} \subset \mathbb{C}^n$:

$$\mathbb{D} = \{z \in \mathbb{C}^n \mid z_k - z_m + \hbar \in p\mathbb{Z}\,, \quad k, m = 1, \ldots, n\,, \quad k \neq m\}\,,$$

called a *discriminant*. Say that $z_1, \ldots, z_n$ are generic if $(z_1, \ldots, z_n) \notin \mathbb{D}$. The complement of the discriminant is a natural definition region for solutions of the *qKZ* equation (2.1).

Indeed, the *qKZ* operators $K_1(z), \ldots, K_n(z)$ are invertible for generic z since the R-matrix $R(x)$ is invertible for $x \neq \pm\hbar$. Therefore, for generic z values of a solution Ψ of the *qKZ* equation (2.1) at points of a lattice $z + p\mathbb{Z}$, are uniquely determined by its value $\Psi(z)$ at the initial point; the initial value can be an arbitrary vector in $V^{\otimes n}$ because for any $j, m = 1, \ldots, n$

$$K_j(z_1, \ldots, z_m + p, \ldots, z_n)\, K_m(z_1, \ldots, z_n) = K_m(z_1, \ldots, z_j + p, \ldots, z_n)\, K_j(z_1, \ldots, z_n)\,.$$

The contour C used in the definition of the hypergeometric integral (3.3) exists for generic z. Therefore, if a periodic function $W \in \bar{\mathcal{F}}_q^{\wedge\ell}$ is regular in a vicinity of a generic point z, so is the corresponding hypergeometric solution Ψ_W of the *qKZ* equation. Hence, we can formulate completeness of the hypergeometric solutions of the *qKZ* equation as follows.

Theorem 4.1. *Let $z_1, \ldots, z_n$ be generic. If $0 \leqslant \operatorname{Im} \mu < 2\pi$ and $\mu \neq 0$, then*

$$\{\Psi_W(z_1, \ldots, z_n) \mid W \in \bar{\mathcal{F}}_q^{\wedge\ell}\} = (V^{\otimes n})_\ell .$$

If $\mu = 0$ and $2\ell \leqslant n$, then

$$\{\Psi_W(z_1, \ldots, z_n) \mid W \in \bar{\mathcal{F}}_q^{\wedge\ell}\} = (V^{\otimes n})_\ell^{sing}.$$

Given a point $z \in \mathbb{C}^n$ consider a space $\bar{\mathcal{F}}_q^{\wedge\ell}(z)$ of functions in $t_1, \ldots, t_\ell$ which have the form $W(\cdot\,; z)$ for some $W \in \bar{\mathcal{F}}_q^{\wedge\ell}$. In other words, $\bar{\mathcal{F}}_q^{\wedge\ell}(z)$ is a space of functions $F(t_1, \ldots, t_\ell)$ such that the product

$$F(t_1, \ldots, t_\ell) \prod_{m=1}^{n} \prod_{a=1}^{\ell} \bigl(1 - \exp\bigl(2\pi i (t_a - z_m)/p\bigr)\bigr)$$

is an antisymmetric polynomial in the exponentials $e^{2\pi i t_1/p}, \ldots, e^{2\pi i t_\ell/p}$ of degree less than n in each of the variables. The space $\bar{\mathcal{F}}_q^{\wedge\ell}(z)$ is finite-dimensional and $\dim \bar{\mathcal{F}}_q^{\wedge\ell}(z) = \dim (V^{\otimes n})_\ell = \binom{n}{\ell}$.

Define the *hypergeometric map* $\bar{I}_{z,\ell} : \bar{\mathcal{F}}_q^{\wedge\ell}(z) \to (V^{\otimes n})_\ell$ by the rule:

$$\bar{I}_{z,\ell} W = \sum_{\#M=\ell} I^{\otimes\ell}(w_M, W)\, v_M , \tag{4.1}$$

so that $\Psi_W(z) = \bar{I}_{z,\ell} W(\cdot\,; z)$, cf. (3.4). Then we write down Theorem 4.1 in terms of the hypergeometric map.

Theorem 4.2. *Let $\mu \neq 0$. Then for generic $z_1, \ldots, z_n$ the hypergeometric map $\bar{I}_{z,\ell} : \bar{\mathcal{F}}_q^{\wedge\ell}(z) \to (V^{\otimes n})_\ell$ is nondegenerate.*

The proof is given in the next section.

Theorem 4.3. *Let $\mu = 0$. Then for generic $z_1, \ldots, z_n$ we have* $\operatorname{im} \bar{I}_{z,\ell} = (V^{\otimes n})_\ell^{sing}$.

The proof is given in Section 6.

Since $\dim \bar{\mathcal{F}}_q^{\wedge\ell}(z) = \dim (V^{\otimes n})_\ell$, the hypergeometric map I_z has a trivial kernel if $\mu \neq 0$. So any nonzero periodic function W produces a nonzero solution Ψ_W, that is, the space of solutions of the *qKZ* equation (2.1) with values in $(V^{\otimes n})_\ell$ is isomorphic to the space $\bar{\mathcal{F}}_q^{\wedge\ell}$ of periodic functions. On the contrary, for $\mu = 0$ the hypergeometric map $\bar{I}_{z,\ell}$ has a nontrivial kernel, which means that for certain periodic functions the corresponding hypergeometric solutions vanish, and we have a problem of describing those periodic functions which produces zero solutions of the *qKZ* equation.

Introduce the following functions:

$$\Xi^{(1)}(t) = \Theta(t) - 1 , \qquad \Theta(t) = \prod_{m=1}^{n} \frac{\exp\bigl(2\pi i (t - z_m)/p\bigr) + 1}{\exp\bigl(2\pi i (t - z_m)/p\bigr) - 1} , \tag{4.2}$$

$$\Xi^{(2)}(t_1, t_2) = \bigl(\Theta(t_1)\,\Theta(t_2) - 1\bigr) \frac{\exp\bigl(2\pi i (t_1 - t_2)/p\bigr) - 1}{\exp\bigl(2\pi i (t_1 - t_2)/p\bigr) + 1} + \Theta(t_1) - \Theta(t_2) .$$

It is easy to see that $\Xi^{(1)} \in \bar{\mathcal{F}}_q^{\wedge 1}$ and $\Xi^{(2)} \in \bar{\mathcal{F}}_q^{\wedge 2}$. Define linear maps

$$X^{(1)}_{z,\ell} : \bar{\mathcal{F}}_q^{\wedge(\ell-1)}(z) \to \bar{\mathcal{F}}_q^{\wedge\ell}(z) ,$$

$$X^{(1)}_{z,\ell} F(t_1,\ldots,t_\ell) = \operatorname{Asym}\bigl(\Xi^{(1)}(t_1)\, F(t_2,\ldots,t_\ell)\bigr) ,$$

$$X^{(2)}_{z,\ell} : \bar{\mathcal{F}}_q^{\wedge(\ell-2)}(z) \to \bar{\mathcal{F}}_q^{\wedge\ell}(z) ,$$

$$X^{(2)}_{z,\ell} F(t_1,\ldots,t_\ell) = \operatorname{Asym}\bigl(\Xi^{(2)}(t_1,t_2)\, F(t_3,\ldots,t_\ell)\bigr) .$$

They obviously induce linear maps $X^{(1)}_\ell : \bar{\mathcal{F}}_q^{\wedge(\ell-1)} \to \bar{\mathcal{F}}_q^{\wedge\ell}$ and $X^{(2)}_\ell : \bar{\mathcal{F}}_q^{\wedge(\ell-2)} \to \bar{\mathcal{F}}_q^{\wedge\ell}$.

Theorem 4.4. *Let $\mu = 0$ and $2\ell \leqslant n$. Then for generic $z_1,\ldots,z_n$ we have* $\ker \bar{I}_{z,\ell} = \operatorname{im} X^{(1)}_{z,\ell} + \operatorname{im} X^{(2)}_{z,\ell}$. *For $\ell < a$, we assume by convention that* $\operatorname{im} X^{(a)}_{z,\ell} = 0$.

The proof is given in Section 7.

The last theorem means that if $\mu = 0$ and a periodic function W has the form $W = X^{(1)}_\ell W_1 + X^{(2)}_\ell W_2$ where W_1, W_2 are some periodic functions in less numbers of variables, then $\Psi_W = 0$, that is, the space of solutions of the qKZ equation (2.1) at $\mu = 0$ with values in $(V^{\otimes n})^{sing}_\ell$ is isomorphic to the quotient space $\bar{\mathcal{F}}_q^{\wedge\ell}/\bigl(\operatorname{im} X^{(1)}_\ell + \operatorname{im} X^{(2)}_\ell\bigr)$. An interesting question is to see whether there is a distinguished subspace in $\bar{\mathcal{F}}_q^{\wedge\ell}$ complementary to $\operatorname{im} X^{(1)}_\ell + \operatorname{im} X^{(2)}_\ell$.

Remark. The relation $\bar{I}_{z,\ell}\, X^{(2)}_{z,\ell}\, W = 0$ was originally discovered by F. Smirnov in [S2] as a consequence of the deformed Riemann bilinear identity. In Section 8 we give another interpretation of this relation in terms of the representation theory of the quantum group $U_q(\mathfrak{sl}_2)$ at $q = e^{\pi i/2}$.

5. Proof of Theorem 4.2

Recall that we assume $p = 2\hbar$, so that $z_1,\ldots,z_n$ are generic if and only if

$$\prod_{1\leqslant k<m\leqslant n} \bigl(e^{2\pi i z_k/p} + e^{2\pi i z_m/p}\bigr) \neq 0.$$

Let M be a subset of $\{1,\ldots,n\}$, $M = \{m_1 < \ldots < m_\ell\}$. Introduce a function $W_M(t_1,\ldots,t_\ell; z_1,\ldots,z_n)$ as follows:

$$W_M = \operatorname{Asym} G_M , \tag{5.1}$$

$$G_M(t_1,\ldots,t_\ell) = \prod_{a=1}^{\ell} \Bigl(\frac{1}{\exp\bigl(2\pi i (t_a - z_{m_a})/p\bigr) - 1} \prod_{1\leqslant j<m_a} \frac{\exp\bigl(2\pi i (t_a - z_j)/p\bigr) + 1}{\exp\bigl(2\pi i (t_a - z_j)/p\bigr) - 1} \Bigr) ,$$

cf. (3.1). It is clear that $W_M \in \bar{\mathcal{F}}_q^{\wedge\ell}$.

Proposition 5.1. *For generic $z_1,\ldots,z_n$ functions $W_M(\cdot, z)$, $\# M = \ell$, form a basis in the space $\bar{\mathcal{F}}_q^{\wedge\ell}(z)$.*

Proof. First observe that it suffices to prove the statement for $\ell = 1$. For $\ell = 1$ the statement is implied by Lemma 5.2 after a substitution: $u = e^{2\pi i t/p}$, $x_m = y_m = -e^{2\pi i z_m/p}$, $m = 1,\ldots,n$. $\square$

Lemma 5.2. *Define a matrix M as follows:* $\sum_{m=1}^{n} M_{jk}\, u^{k-1} = \prod_{1\leqslant j<k} (u-x_j) \prod_{k<j\leqslant n} (u+y_j)$. *Then*

$$\det M = \prod_{1\leqslant j<k\leqslant n} (x_j+y_k)\,.$$

The proof is similar to the proof of the Vandermonde determinant.

Theorem 5.3. *Let* $0 \leqslant \operatorname{Im}\mu < 2\pi$ *and* $z_1,\dots,z_n$ *be generic. Then*

$$\det\bigl[I^{\otimes\ell}(w_M,W_N)\bigr]_{\substack{M,N\subset\{1,\dots,n\}\\ \#M=\#N=\ell}} =$$

$$= \Bigl((ip)^{n(n-1)/2}\,\bigl(\Gamma(-1/2)\bigr)^n\,(1-e^{-\mu})^{n/2}\,\exp\bigl(\mu\sum_{m=1}^{n} z_m/p\bigr)\Bigr)^{\binom{n-1}{\ell-1}}\times$$

$$\times \prod_{1\leqslant k<m\leqslant n} (z_k-z_m-\hbar)^{-\binom{n-2}{\ell-1}}\,.$$

Here $-2\pi < \arg\,(1-e^{-\mu}) \leqslant 0$.

Proof (idea). The proof of the theorem is in common with the proof of Theorem 5.14 in [TV]. Let us indicate here only the main points.

Denote by $D(z)$ the determinant in question and by $E(z)$ the right hand side of the formula to be proved. The determinant obeys a system of difference equations following from the qKZ equation:

$$D(z_1,\dots,z_m+p,\dots,z_n) = \overset{(\ell)}{\det} K_m(z_1,\dots,z_n)\,D(z_1,\dots,z_n)\,, \qquad m=1,\dots,n\,,$$

where

$$\overset{(\ell)}{\det} K_m = e^{\mu\binom{n-1}{\ell-1}}\Bigl(\prod_{1\leqslant j<m}\frac{z_m-z_j-\hbar+p}{z_m-z_j+\hbar+p}\prod_{m<j\leqslant n}\frac{z_m-z_j-\hbar}{z_m-z_j+\hbar}\Bigr)^{\binom{n-2}{\ell-1}}$$

is a determinant of the restriction of the operator K_m to the weight subspace $(V^{\otimes n})_\ell$. This shows that the ratio $D(z)/E(z)$ is a p-periodic function of $z_1,\dots,z_n$ (quasiconstant).

To find this quasiconstant we look at the behaviour of the hypergeometric integrals $I^{\otimes\ell}(w_M,W_N)$ as $\operatorname{Im}(z_m-z_{m+1})$ tends to $-\infty$ for all $m=1,\dots,n-1$. We observe, that the leading term of the asymptotics of the matrix $\bigl[I^{\otimes\ell}(w_M,W_N)\bigr]$ is a diagonal matrix. Therefore, the leading term of the asymptotics of the determinant $\det\bigl[I^{\otimes\ell}(w_M,W_N)\bigr]$ is given by the product of asymptotics of the diagonal entries $I^{\otimes\ell}(w_M,W_M)$. They can be calculated using the formula

$$\int_{\widetilde{C}} e^{u\,(\mu-\pi i)}\,\Gamma(u-1/2)\,\Gamma(-u)\,du = 2\pi i\,\Gamma(-1/2)\,(1-e^{-\mu})^{1/2}$$

where the contour $\widetilde{C}$ is going from $-i\infty$ to $+i\infty$ separating the sets $\mathbb{Z}_{\geqslant 0}$ and $1/2+\mathbb{Z}_{\leqslant 0}$. The formula is equivalent to the formula for the Barnes integral [WW].

Finally, we obtain that the ratio $D(z)/E(z)$ is a quasiconstant which tends to 1 as $\operatorname{Im}(z_m - z_{m+1}) \to -\infty$ for all $m = 1, \dots, n-1$. Such a quasiconstant necesserily equals 1. The theorem is proved. $\square$

Remark. Below we describe a differential equation with respect to μ obeyed by the hypergeometric solutions of the *qKZ* equation, see (6.1). It implies a differential equation with respect to μ for the determinant $\det\bigl[I^{\otimes\ell}(w_M, W_N)\bigr]$, which allows to give another proof of Theorem 5.3 using the differential equation and asymptotics of the hypergeometric integrals $I^{\otimes\ell}(w_M, W_N)$ as $\operatorname{Re}\mu \to \pm\infty$.

Theorem 4.2 follows from the definition (4.1) of hypergeometric map, Proposition 5.1 and Theorem 5.3.

6. Proof of Theorem 4.3

Our proof of Theorem 4.3 is based on an additional differential equation which takes place for hypergeometric solutions of the *qKZ* equation.

Theorem 6.1. *Let Ψ be a hypergeometric solution of the qKZ equation* (2.1)*, see* (3.4)*. Then*

$$p\,\partial_\mu \Psi \,=\, \Bigl(\,\sum_{m=1}^{n} z_m H_m \,+\, \frac{\hbar}{e^\mu - 1}\,\bigl(\,\sum_{m=1}^{n} H_m + \sum_{1\leqslant k<m\leqslant n} (e^\mu\,\sigma_k^-\sigma_m^+ + \sigma_k^+\sigma_m^-)\bigr)\Bigr)\,\Psi\,. \tag{6.1}$$

The proof of the theorem will appear separately. The theorem is a particular specialization of a more general result about the hypergeometric solutions of the *qKZ* equations and the Knizhnik-Zamolodchikov differential equations [FTV].

Remark. One can check directly that the equation (6.1) is compatible with the *qKZ* equation (2.1):

$$L(z_1, \dots, z_m + p, \dots, z_n)\,K_m(z_1, \dots, z_n) \,=\, p\,\partial_\mu K_m(z_1, \dots, z_n)\,L(z_1, \dots, z_n)\,,$$

where L is the operator in the right hand side of equation (6.1).

Let U be a finite-dimensional vector space and $A(\mu)$ be an $\operatorname{End}(U)$-valued function holomorphic in a neighbourhood of $\mu = 0$. Consider a differential equation:

$$\mu\,\partial_\mu \Psi \,=\, A(\mu)\,\Psi\,. \tag{6.2}$$

Set $A_0 = A(0)$. Assume that the operator A_0 is diagonalizable and let $\psi \in U$ be an eigenvector of A_0: $A_0\,\psi = \alpha\psi$.

Proposition 6.2. *There is a solution Ψ of the equation* (6.2) *which have the form*

$$\Psi(\mu) \,=\, \mu^\alpha\,\bigl(\psi + \sum_{k=1}^{\infty}\,\sum_{j=0}^{i_k} \mu^k\,(\log\mu)^j\,\psi_{jk}\bigr)$$

where $0 = i_0 \leqslant i_1 \leqslant i_2 \leqslant \dots$ are integers such that $i_k + 1 \geqslant i_{k+1}$ for any k and $i_k = i_{k+1}$ for large k, and $\psi_{jk} \in U$. The series is convergent for μ in a punctured neighbourhood of $\mu = 0$.

The proof is an exercise on the analytic theory of differential equations and we leave it to a reader.

Corollary 6.3. *Assume that all nonzero eigenvalues of A_0 have positive real parts. Let $\Psi(\mu) \in \mathrm{End}(U)$ be a fundamental solution of the equation* (6.2), $\det \Psi(\mu) \neq 0$, *in a punctured neighbourhood of $\mu = 0$. Then there is a limit $\Psi_0 = \lim_{\mu \to 0} \Psi(\mu)$ and* $\mathrm{im}\, \Psi_0 = \ker A_0$.

Proof. According to Proposition 6.2 we can start from an eigenbasis of A_0 in U and lift it to a fundamental solution of the equation (6.2). For this fundamental solution the claim of the corollary clearly holds. Since two fundamental solutions of the same differential equation differ only by multiplication from the right by a nondegenerate constant matrix, the corollary is proved. □

For the equation (6.1) taking into account that $p = 2\hbar$ and $H_m = \sigma_m^- \sigma_m^+$ we have

$$A_0 = \frac{1}{2} \Sigma^- \Sigma^+ = \frac{1}{8} \left(\Sigma^3 (\Sigma^3 + 2) + 4 \Sigma^- \Sigma^+ \right) - \frac{1}{8} \Sigma^3 (\Sigma^3 + 2)$$

where $\Sigma^a = \sum_{m=1}^{n} \sigma_m^a$, $a = \pm, 3$. Since the first term of the last expression is the Casimir operator for $\mathfrak{sl}_2$, the operator A_0 restricted to the weight subspace $(V^{\otimes n})_\ell$ has the following eigenvalues:

$$\frac{(n-2k)(n-2k+2) - (n-2\ell)(n-2\ell+2)}{4} = \frac{(\ell - k)(n - k - \ell + 1)}{2},$$

$k = 0, \ldots, \min(\ell, n - \ell)$, which are nonnegative half-integers. Besides, $\ker A_0 \cap (V^{\otimes n})_\ell = (V^{\otimes n})_\ell^{sing}$.

Consider the hypergeometric map $\bar{I}_{z,\ell}(\mu)$, cf. (4.1), where we write its dependence of μ explicitly. Theorem 6.1 means that $\bar{I}_{z,\ell}(\mu)$ obeys the equation (6.1), and according to Theorem 4.2 it is a fundamental solution of the equation (6.1) with values in $(V^{\otimes n})_\ell$. Hence, by Corollary 6.3 there is a limit $\lim_{\mu \to 0} \bar{I}_{z,\ell}(\mu)$ and $\mathrm{im} \big(\lim_{\mu \to 0} \bar{I}_{z,\ell}(\mu) \big) = (V^{\otimes n})_\ell^{sing}$. If $2\ell \leqslant n$, then $\lim_{\mu \to 0} \bar{I}_{z,\ell}(\mu) = \bar{I}_{z,\ell}(0)$, which completes the proof of Theorem 4.3.

Remark. The fact that $\mathrm{im}\, \bar{I}_{z,\ell}(0) \subset (V^{\otimes n})_\ell^{sing}$ is well known, see [NPT] for details of its proof.

7. Proof of Theorem 4.4

It is known that $\mathrm{im}\, X_{z,\ell}^{(1)} \subset \ker \bar{I}_{z,\ell}$, see [NPT] for details. The proof of the second inclusion $\mathrm{im}\, X_{z,\ell}^{(2)} \subset \ker \bar{I}_{z,\ell}$ is similar, though slightly more complicated. Due to Theorem 4.3 what remains to show is

$$\dim \big(\mathrm{im}\, X_{z,\ell}^{(1)} + \mathrm{im}\, X_{z,\ell}^{(2)} \big) = \dim (V^{\otimes n})_\ell - \dim (V^{\otimes n})_\ell^{sing} = \dim \Sigma^- (V^{\otimes n})_{\ell - 1}.$$

The second equality is due to the common knowledge:

$$(V^{\otimes n})_\ell = (V^{\otimes n})_\ell^{sing} \oplus \Sigma^- (V^{\otimes n})_{\ell - 1}.$$

Let $\xi_1, \ldots, \xi_n$ be the Grassman variables: $\xi_k \xi_m = -\xi_m \xi_k$ for any $k, m = 1, \ldots, n$. Consider the group algebra $\mathbb{C}[\xi_1, \ldots, \xi_n]$. Set $\deg \xi_1 = \ldots = \deg \xi_n = 1$, and denote by $\mathbb{C}[\xi_1,\dots,\xi_n]_k$ the subspace of elements of degree k. By convention, set $\mathbb{C}[\xi_1,\dots,\xi_n]_k = 0$ for $k<0$. Introduce elements $\varphi^{(1)},\varphi^{(2)} \in \mathbb{C}[\xi_1,\dots,\xi_n]$:

$$\varphi^{(1)} = \sum_{m=1}^{n} \xi_m\,, \qquad \varphi^{(2)} = \sum_{1\leqslant k<m\leqslant n} \xi_k\xi_m\,. \tag{7.1}$$

Consider isomorphisms of vector spaces:

$$\bar{\mathcal{F}}_q^{\wedge k}(z) \to \mathbb{C}[\xi_1,\dots,\xi_n]_k\,, \qquad W_M(\cdot,z) \mapsto \xi_{m_1}\dots\xi_{m_k}\,, \tag{7.2}$$

where $M=\{m_1<\dots<m_k\}$. Under these isomorphisms the operator $X^{(a)}_{z,\ell}$ translates modulo proportionality into multiplication by $\varphi^{(a)}$ acting on $\mathbb{C}[\xi_1,\dots,\xi_n]_{(\ell-a)}$, because the functions $\Xi^{(1)}$, $\Xi^{(2)}$, cf. (4.2), can be written as follows:

$$\Xi^{(1)} = 2\sum_{m=1}^{n} W_{\{m\}}\,, \qquad \Xi^{(2)} = 4\sum_{1\leqslant k<m\leqslant n} W_{\{k,m\}}\,.$$

Here the functions $W_{\{m\}}$ and $W_{\{k,m\}}$ are given by (5.1) for $M=\{m\}$ and $M=\{k,m\}$, respectively.

Therefore, the problem is to show that

$$\dim\bigl(\varphi^{(1)}\mathbb{C}[\xi_1,\dots,\xi_n]_{(\ell-1)} + \varphi^{(2)}\mathbb{C}[\xi_1,\dots,\xi_n]_{(\ell-2)}\bigr) = \dim\Sigma^-(V^{\otimes n})_{\ell-1}. \tag{7.3}$$

The cases $\ell=0,1$ are trivial. The case $\ell=n=2$ is very easy. From now on we assume that $\ell\geqslant 2$ and $n\geqslant 3$ (recall that $\ell\leqslant n$). It is easy to see that

$$\mathbb{C}[\xi_1,\dots,\xi_n]_k = \mathbb{C}[\xi_1,\dots,\xi_{n-1}]_k \oplus \varphi^{(1)}\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_k$$

and

$$\varphi^{(1)}\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_k = \varphi^{(1)}\mathbb{C}[\xi_1,\dots,\xi_n]_k\,.$$

Set $\widetilde{\varphi} = \sum_{1\leqslant k<m<n} \xi_k\xi_m$. Since $\varphi^{(2)} = \widetilde{\varphi} + \varphi^{(1)}\xi_n$, it is a simple exercise to replace relation (7.3) by

$$\dim\varphi^{(1)}\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_{(\ell-1)} + \dim\widetilde{\varphi}\,\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_{(\ell-2)} = \dim\Sigma^-(V^{\otimes n})_{\ell-1}. \tag{7.4}$$

Calculating dimensions explicitly, we further reduce (7.4) to the following relation:

$$\dim\widetilde{\varphi}\,\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_{(\ell-2)} = \begin{cases} \dim\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_{(\ell-2)} & \text{if } 2\ell\leqslant n \\ \dim\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_{\ell} & \text{if } 2\ell > n \end{cases},$$

which another look is

$$\dim\widetilde{\varphi}\,\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_{(\ell-2)} = \tag{7.5}$$

$$= \min\bigl(\dim\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_{(\ell-2)}\,,\, \dim\mathbb{C}[\xi_1,\dots,\xi_{n-1}]_{\ell}\bigr)\,.$$

Now it is convenient to change the variables. Set

$$\zeta_k = \xi_k + \ldots + \xi_{n-k-1}\,, \qquad \zeta_{n-k} = \xi_{k+1} + \ldots + \xi_{n-k}\,, \qquad 1 \leqslant k < n/2\,,$$

and $\zeta_{n/2} = \xi_{n/2}$ if n is even. Then $\widetilde{\varphi} = \sum_{1 \leqslant k < n/2} \zeta_k \zeta_{n-k}$.

Denote by $\hat{\zeta}_m$ and $\hat{\varphi}$ the operators of multiplication by ζ_m and $\widetilde{\varphi}$, respectively. Introduce the Grassman derivations $\partial_1, \ldots, \partial_{n-1}$ corresponding to $\zeta_1, \ldots, \zeta_{n-1}$:

$$\partial_k 1 = 0\,, \qquad \partial_k \hat{\zeta}_m + \hat{\zeta}_m \partial_k = \delta_{km}\,, \qquad k, m = 1, \ldots, n-1\,.$$

Set $D = \sum_{1 \leqslant k < n/2} \partial_k \partial_{n-k}$. Then the operators D, $\hat{\varphi}$ and $[D, \hat{\varphi}]$ constitute an $\mathfrak{sl}_2$ action in $\mathbb{C}[\xi_1, \ldots, \xi_{n-1}]$, and the relation (7.5) results from the representation theory of $\mathfrak{sl}_2$.

8. Kernel of the hypergeometric map at $\mu = 0$ and $U_q(\mathfrak{sl}_2)$ at $q = e^{\pi i/2}$

Consider the quantum group $U_q(\mathfrak{sl}_2)$ with generators e, f, k subject to relations

$$ke = q^2 ek\,, \qquad kf = q^{-2} fk\,, \qquad [e, f] = \frac{k - k^{-1}}{q - q^{-1}}\,,$$

and a coproduct Δ:

$$\Delta(e) = 1 \otimes e + e \otimes k\,, \qquad \Delta(f) = k^{-1} \otimes f + f \otimes 1\,, \qquad \Delta(k) = k \otimes k\,,$$

Make the space V into a $U_q(\mathfrak{sl}_2)$-module, the generators e, f, k acting as σ^+, σ^-, q^{σ^3}, respectively. Then the generator f is represented in the $U_q(\mathfrak{sl}_2)$-module $V^{\otimes n}$ by the operator $F(q) \in \operatorname{End}(V^{\otimes n})$:

$$F(q) = \sum_{m=1}^{n} q^{-\sigma_1^3} \ldots q^{-\sigma_{m-1}^3} \sigma_m^-\,.$$

Looking at the limit $q \to e^{\pi i/2}$ define operators $F^{(1)}, F^{(2)} \in \operatorname{End}(V^{\otimes n})$:

$$F^{(1)} = -\,i F(e^{\pi i/2}) = \sum_{m=1}^{n} (-i)^m \sigma_1^3 \ldots \sigma_{m-1}^3 \sigma_m^-\,,$$

$$F^{(2)} = -\lim_{q \to i} \frac{F(q)^2}{1 + q^2} = \sum_{1 \leqslant k < m \leqslant n} (-i)^{k+m} \sigma_k^- \sigma_{k+1}^3 \ldots \sigma_{m-1}^3 \sigma_m^-\,,$$

the overall normalization factors being fixed just for the aesthetic reason. There is a faithful representation of $\mathbb{C}[\xi_1, \ldots, \xi_n]$ in $V^{\otimes n}$ given by

$$\xi_m \mapsto (-i)^m \sigma_1^3 \ldots \sigma_{m-1}^3 \sigma_m^-\,, \qquad m = 1, \ldots, n\,, \tag{8.1}$$

the operators $F^{(1)}, F^{(2)}$ representing the elements $\varphi^{(1)}, \varphi^{(2)} \in \mathbb{C}[\xi_1, \ldots, \xi_n]$, cf. (7.1).

This representation is equivalent to the left regular representation of $\mathbb{C}[\xi_1,\ldots,\xi_n]$, the intertwiner being given as follows:

$$\text{(8.2)}\qquad \xi_{m_1}\ldots\xi_{m_k}\,\mapsto\,(-i)^{\sum_{a=1}^{k}m_a}\,v_M\,,\qquad 1\leqslant m_1<\ldots<m_k\leqslant n\,,$$

where $M=\{m_1,\ldots,m_k\}$ and the vector v_M is defined by (3.2). Actually, the representation (8.1) is the Jordan-Wigner transformation.

Remark. For generic q it is known that $\dim F(q)\,(V^{\otimes n})_k=\dim\Sigma^-(V^{\otimes n})_k$ for any $k=0,\ldots,n$. Consider a subspace $\lim\limits_{q\to i}\bigl(F(q)\,(V^{\otimes n})_{\ell-1}\bigr)$. It clearly contains both $F^{(1)}(V^{\otimes n})_{\ell-1}$ and $F^{(2)}(V^{\otimes n})_{\ell-2}$. Hence, the equality (7.3) implies that

$$\text{(8.3)}\qquad F^{(1)}(V^{\otimes n})_{\ell-1}+F^{(2)}(V^{\otimes n})_{\ell-2}\,=\,\lim_{q\to i}\bigl(F(q)\,(V^{\otimes n})_{\ell-1}\bigr)\,.$$

Denote by C_z the composition of isomorphisms (7.2) and (8.2):

$$C_z\,:\,\bigoplus_{k=0}^{n}\bar{\mathcal{F}}_q^{\wedge k}(z)\,\to\,V^{\otimes n}\,,\qquad\qquad C_z\,:\,W_M(\cdot\,,z)\,\mapsto\,(-i)^{\sum_{a=1}^{k}m_a}\,v_M\,,$$

Under this map the operator $X^{(a)}_{z,\ell}$ translates modulo proportionality into the operator $F^{(a)}$ acting on $(V^{\otimes n})_{(\ell-a)}$. So, by Theorem 4.4 and relation (8.3) the kernel of the hypergeometric map $\bar I_{z,\ell}$ corresponds to the limit of the subspace $F(q)\,(V^{\otimes n})_{\ell-1}$ as q tends to i.

Appearance of the maps $C_{z,k}$ in the descriptiopn of solutions of the qKZ equation is not accidental. Utilizing the general picture developed in [TV], [MV] one can see that for generic step p the space of solutions of the qKZ equation (2.1) at $\mu\neq 0$ with values in $V^{\otimes n}$ is naturally isomorphic over the field of quasiconstants to the $U_q(\mathfrak{sl}_2)$-module $V^{\otimes n}$ desribed at the beginning of this section, q being equal to $e^{\pi i\hbar/p}$. The isomorphism is based on the construction of hypergeometric solutions of the qKZ equation and its realization in the case $p=2\hbar$, studied here, is the map C_z.

At the same time, for generic step p, the space of solutions of the qKZ equation (2.1) at $\mu=0$ with values in $(V^{\otimes n})^{sing}$ is isomorphic over quasiconstants to either the subspace $(V^{\otimes n})^{sing,q}$ of singular vectors with respect to the quantum group $U_q(\mathfrak{sl}_2)$ or the quotient space $V^{\otimes n}/F(q)V^{\otimes n}$. These models of the space of solutions are equivalent since $V^{\otimes n}=(V^{\otimes n})^{sing,q}\oplus F(q)V^{\otimes n}$ for generic q.

The results of this paper means that for $p=2\hbar$, so that $q=e^{\pi i/2}$, the second model survives with the minimal required modification; namely, the space of solutions of the qKZ equation (2.1) at $\mu=0$ is isomorphic over quasiconstants to the quotient space

$$V^{\otimes n}\big/\lim_{q\to i}\bigl(F(q)\,V^{\otimes n}\bigr)\,=\,V^{\otimes n}\big/\bigl(F^{(1)}V^{\otimes n}+F^{(2)}V^{\otimes n}\bigr)\,.$$

Unfortunately, the first model does not survive at $p=2\hbar$, at least for even n, because in this case the subspaces $\lim\limits_{q\to i}(V^{\otimes n})^{sing,q}$ and $\lim\limits_{q\to i}\bigl(F(q)\,V^{\otimes n}\bigr)$ intersect nontrivially, and therefore, their sum does not coincide with $V^{\otimes n}$.

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