Quotients of unital A_∞-categories

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Abstract

Assuming that $\mathcal{B}$ is a full A_∞-subcategory of a unital A_∞-category $\mathcal{C}$ we construct the quotient unital A_∞-category $\mathcal{D}$ =‘$\mathcal{C}/\mathcal{B}$’. It represents the A_∞-2-functor $\mathcal{A} \mapsto A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$, which associates with a given unital A_∞-category $\mathcal{A}$ the A_∞-category of unital A_∞-functors $\mathcal{C} \to \mathcal{A}$, whose restriction to $\mathcal{B}$ is contractible. Namely, there is a unital A_∞-functor $e : \mathcal{C} \to \mathcal{D}$ such that the composition $\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{e} \mathcal{D}$ is contractible, and for an arbitrary unital A_∞-category $\mathcal{A}$ the restriction A_∞-functor $(e \boxtimes 1)M : A^u_\infty(\mathcal{D},\mathcal{A}) \to A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ is an equivalence.

Let C be the differential graded category of differential graded $\Bbbk$-modules. We prove that the Yoneda A_∞-functor $Y : \mathcal{A} \to A^u_\infty(\mathcal{A}^{\mathrm{op}},\mathsf{C})$ is a full embedding for an arbitrary unital A_∞-category $\mathcal{A}$. In particular, such $\mathcal{A}$ is A_∞-equivalent to a differential graded category with the same set of objects.

Let $\mathcal{A}$ be an Abelian category. The question: what is the quotient

$$\{\text{category of complexes in } \mathcal{A}\}/\{\text{category of acyclic complexes}\}?$$

admits several answers. The first answer – the derived category of $\mathcal{A}$ – was given by Grothendieck and Verdier [Ver77].

The second answer – a differential graded category $\mathcal{D}$ – is given by Drinfeld [Dri02]. His article is based on the work of Bondal and Kapranov [BK90] and of Keller [Kel99]. The derived category $D(\mathcal{A})$ can be obtained as $H^0(\mathcal{D}^{\text{pre-tr}})$.

The third answer – an A_∞-category of bar-construction type – is given by Lyubashenko and Ovsienko [LO02]. This A_∞-category might turn out useful in additional assumptions, e.g. when the basic ring $\Bbbk$ is a field. It is an A_∞-version of one of the constructions of Drinfeld [Dri02].

The fourth answer – an A_∞-category freely generated over the category of complexes in $\mathcal{A}$ – is given in this article. It is likely to be related to the first three answers, but the precise relationship is not yet established.

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1. Introduction

Since A_∞-algebras were introduced by Stasheff [Sta63, II] there existed a possibility to consider A_∞-generalizations of categories. It did not happen until A_∞-categories were encountered in studies of mirror symmetry by Fukaya [Fuk93] and Kontsevich [Kon95]. A_∞-categories may be viewed as generalizations of differential graded categories for which the binary composition is associative only up to a homotopy. The possibility to define A_∞-functors was mentioned by Smirnov [Smi89], who reformulated one of his results in the language of A_∞-functors between differential graded categories. The definition of A_∞-functors between A_∞-categories was published by Keller [Kel01], who studied their applications to homological algebra. Homomorphisms of A_∞-algebras (e.g. [Kad82]) are particular cases of A_∞-functors.

A_∞-transformations between A_∞-functors are certain coderivations. Given two A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, one can construct a third A_∞-category $A_\infty(\mathcal{A},\mathcal{B})$, whose objects are A_∞-functors $f:\mathcal{A}\to\mathcal{B}$, and morphisms are A_∞-transformations (Fukaya [Fuk02], Kontsevich and Soibelman [KS02, KS99], Lefèvre-Hasegawa [LH03], as well as [Lyu03]). For an A_∞-category $\mathcal{C}$ there is a homotopy invariant notion of unit elements (identity morphisms) [Lyu03]. They are cycles ${}_X\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{C}(X,X)$ of degree -1 such that the maps $(1\otimes\mathbf{i}_0^{\mathcal{C}})b_2, -(\mathbf{i}_0^{\mathcal{C}}\otimes 1)b_2 : s\mathcal{C}(X,Y)\to s\mathcal{C}(X,Y)$ are homotopic to the identity map. This allows to define the 2-category $\overline{A_\infty^u}$, whose objects are unital A_∞-categories (those which have units), 1-morphisms are unital A_∞-functors (their first components preserve the units up to a boundary) and 2-morphisms are equivalence classes of natural A_∞-transformations [Lyu03]. We continue to study this 2-category. Notations and terminology follow [Lyu03], complemented by [LO02] and [LM03].

Unital A_∞-categories and unital A_∞-functors can be considered as strong homotopy generalizations of differential graded categories and functors. Let us illustrate the notion of A_∞-transformations in a familiar context.

1.1. Differential for A_∞-transformations compared with the Hochschild differential. Let $\mathcal{A}$, $\mathcal{B}$ be ordinary $\Bbbk$-linear categories. We consider $\mathcal{A}(\_,\_)$ and $\mathcal{B}(\_,\_)$ as complexes of $\Bbbk$-modules concentrated in degree 0. This turns $\mathcal{A}$ and $\mathcal{B}$ into differential graded categories and, thereby, into unital A_∞-categories. An A_∞-functor between $\mathcal{A}$ and $\mathcal{B}$ is necessarily strict, for $(s\mathcal{A})^{\otimes k} = \mathcal{A}^{\otimes k}[k]$ and $s\mathcal{B} = \mathcal{B}[1]$ are concentrated in different degrees if $k>1$. Thus, a unital A_∞-functor $f:\mathcal{A}\to\mathcal{B}$ is the same as an ordinary $\Bbbk$-linear functor f. Let $f,g:\mathcal{A}\to\mathcal{B}$ be $\Bbbk$-linear functors. All complexes $\mathsf{C}((s\mathcal{A})^{\otimes k}(X,Y), s\mathcal{B}(Xf,Yg))$ are concentrated in degree $k-1$. Their direct product

$$\Psi_k = \prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \mathsf{C}((s\mathcal{A})^{\otimes k}(X,Y), s\mathcal{B}(Xf,Yg))$$

is the same, whether taken in the category of $\Bbbk$-modules or graded $\Bbbk$-modules or complexes of $\Bbbk$-modules. It is the module of k-th components of A_∞-transformations. The graded $\Bbbk$-module of A_∞-transformations $sA_\infty(\mathcal{A},\mathcal{B})(f,g)$ is isomorphic to the direct product $\prod_{k=0}^\infty \Psi_k$ taken in the category of graded $\Bbbk$-modules [Lyu03, Section 2.7]. That is, $[sA_\infty(\mathcal{A},\mathcal{B})(f,g)]^n = \prod_{k=0}^\infty \Psi_k^n$, where Ψ_k^n is the degree n part of Ψ_k. Therefore, in our case it simply coincides with the graded $\Bbbk$-module $\Psi[1] : \mathbb{Z} \ni n \mapsto \Psi_{n+1} \in \Bbbk\text{-mod}$. The graded $\Bbbk$-module $sA_\infty(\mathcal{A},\mathcal{B})(f,g)$ is equipped with the differential B_1, $rB_1 = rb^{\mathcal{B}} - (-)^r b^{\mathcal{A}} r$ [Lyu03, Proposition 5.1]. Since the only non-vanishing component of b (resp. f) is b_2 (resp. f_1), the explicit formula for components of rB_1 is the following:

$$(rB_1)_{k+1} = (f_1 \otimes r_k)b_2 + (r_k \otimes g_1)b_2 - (-)^{r_k} \sum_{a+c=k-1} (1^{\otimes a} \otimes b_2 \otimes 1^{\otimes c})r_k.$$

Recalling that $\deg r_k = k-1$, we get the differential in $\Psi[1]$ also denoted B_1:

$$r_k B_1 = (f_1 \otimes r_k)b_2 + (r_k \otimes g_1)b_2 + (-)^k \sum_{a+c=k-1} (1^{\otimes a} \otimes b_2 \otimes 1^{\otimes c})r_k,$$

where $r_k \in \Psi_k$, $r_k B_1 \in \Psi_{k+1}$. We consider an isomorphism of graded $\Bbbk$-modules $\Psi \to \Psi' : \mathbb{Z} \ni k \mapsto \Psi'_k$ given by

$$\Psi_k \ni r_k \mapsto (s \otimes \cdots \otimes s) r_k s^{-1} = s^{\otimes k} r_k s^{-1} \in \Psi'_k \overset{\text{def}}{=} \prod_{X,Y \in \operatorname{Ob}\mathcal{A}} \mathsf{C}(\mathcal{A}^{\otimes k}(X,Y), \mathcal{B}(Xf, Yg)).$$

Its inverse is $\Psi'_k \ni t_k \mapsto (s^{\otimes k})^{-1} t_k s \in \Psi_k$. This isomorphism induces the differential

$$d : \Psi'_k \to \Psi'_{k+1}, \qquad t_k d = s^{\otimes k+1} \cdot [(s^{\otimes k})^{-1} t_k s] B_1 \cdot s^{-1}.$$

The explicit formula for d is

$$t_k d = (f \otimes t_k)m_2 + \sum_{a+c=k-1} (-1)^{a+1} (1^{\otimes a} \otimes m_2 \otimes 1^{\otimes c}) t_k + (-1)^{k+1} (t_k \otimes g) m_2.$$

Up to an overall sign this coincides with the differential in the Hochschild cochain complex $C^\bullet(\mathcal{A}, {}_f\mathcal{B}_g)$ (cf. [Mac63, Section X.3]). The $\mathcal{A}$-bimodule ${}_f\mathcal{B}_g$ acquires its left $\mathcal{A}$-module structure via f and its right $\mathcal{A}$-module structure via g. Therefore, in our situation A_∞-transformations are nothing else but Hochschild cochains. Natural A_∞-transformations $r : f \to g : \mathcal{A} \to \mathcal{B}$ (such that $\deg r = -1$ and $rB_1 = 0$) are identified with the Hochschild cocycles of degree 0, that is, with natural transformations $t = rs^{-1} : f \to g : \mathcal{A} \to \mathcal{B}$ in the ordinary sense.

When $\mathcal{A}$, $\mathcal{B}$ are differential graded categories and $f, g : \mathcal{A} \to \mathcal{B}$ are differential graded functors, we may still interpret the complex $(sA_\infty(\mathcal{A},\mathcal{B})(f,g), B_1)$ as the complex of Hochschild cochains $C^\bullet(\mathcal{A}, {}_f\mathcal{B}_g)$ for the differential graded category $\mathcal{A}$ and the differential graded bimodule ${}_f\mathcal{B}_g$. Indeed, for a homogeneous element $r \in sA_\infty(\mathcal{A},\mathcal{B})(f,g)$ the components of rB_1 are

$$\begin{aligned}(rB_1)_k = r_k b_1 &+ (f_1 \otimes r_{k-1})b_2 + (r_{k-1} \otimes g_1)b_2 \\ &- (-)^r \sum_{a+1+c=k} (1^{\otimes a} \otimes b_1 \otimes 1^{\otimes c}) r_k - (-)^r \sum_{a+2+c=k} (1^{\otimes a} \otimes b_2 \otimes 1^{\otimes c}) r_{k-1}.\end{aligned}$$

For A_∞-functors between differential graded categories or A_∞-categories the differential B_1 is not interpreted as Hochschild differential any more. But we may view the complex of A_∞-transformations as a generalization of the Hochschild cochain complex.

1.2. Main result. By Definition 6.4 of [LO02] an A_∞-functor $g : \mathcal{B} \to \mathcal{A}$ from a unital A_∞-category $\mathcal{B}$ is *contractible* if for all objects X, Y of $\mathcal{B}$ the chain map $g_1 : s\mathcal{B}(X,Y) \to s\mathcal{A}(Xg,Yg)$ is null-homotopic. If $g : \mathcal{B} \to \mathcal{A}$ is a unital A_∞-functor, then it is contractible if and only if for any $X \in \operatorname{Ob}\mathcal{B}$ and any $V \in \operatorname{Ob}\mathcal{A}$ the complexes $s\mathcal{A}(Xg,V)$ and $s\mathcal{A}(V,Xg)$ are contractible. Equivalently, $g\mathbf{i}^{\mathcal{A}} \equiv 0 : g \to g : \mathcal{B} \to \mathcal{A}$ [LO02, Proposition 6.1(C5)]. Other equivalent conditions are listed in Propositions 6.1–6.3 of [LO02].

Let $\mathcal{B}$ be a full A_∞-subcategory of a unital A_∞-category $\mathcal{C}$. Let $\mathcal{A}$ be an arbitrary unital A_∞-category. Denote by $A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ the full A_∞-subcategory of $A^u_\infty(\mathcal{C},\mathcal{A})$, whose objects are unital A_∞-functors $\mathcal{C} \to \mathcal{A}$, whose restriction to $\mathcal{B}$ is contractible. We allow to consider A_∞-categories with the empty set of objects.

1.3 Main Theorem. *In the above assumptions there exists a unital A_∞-category $\mathcal{D}$ and a unital A_∞-functor $e : \mathcal{C} \to \mathcal{D}$ such that*

1) *the composition $\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{e} \mathcal{D}$ is contractible;*

2) *the strict A_∞-functor given by composition with e*

$$(e \boxtimes 1)M : A^u_\infty(\mathcal{D},\mathcal{A}) \to A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}, \qquad f \mapsto ef,$$

is an A_∞-equivalence for an arbitrary unital A_∞-category $\mathcal{A}$.

Proof. There exists a differential graded category $\widetilde{\mathcal{C}}$ with $\operatorname{Ob}\widetilde{\mathcal{C}} = \operatorname{Ob}\mathcal{C}$, and quasi-inverse to each other A_∞-functors $\widetilde{Y} : \mathcal{C} \to \widetilde{\mathcal{C}}$, $\Psi : \widetilde{\mathcal{C}} \to \mathcal{C}$ such that $\operatorname{Ob}\widetilde{Y} = \operatorname{Ob}\Psi = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$ (by Remark A.13 it follows from the A_∞-version of Yoneda Lemma – Theorem A.11). Let $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{C}}$ be the full differential graded subcategory with $\operatorname{Ob}\widetilde{\mathcal{B}} = \operatorname{Ob}\mathcal{B}$. By considerations in Appendix B (Corollaries B.10, B.11) if a unital A_∞-category $\mathcal{D}$ and a unital A_∞-functor $\widetilde{e} : \widetilde{\mathcal{C}} \to \mathcal{D}$ represent the A_∞-2-functor $\mathcal{A} \mapsto A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\operatorname{mod}\widetilde{\mathcal{B}}}$ in the sense 1), 2), then the pair $\bigl(\mathcal{D}, e = (\mathcal{C} \xrightarrow{\widetilde{Y}} \widetilde{\mathcal{C}} \xrightarrow{\widetilde{e}} \mathcal{D})\bigr)$ represents $A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$. Indeed,

$$(e \boxtimes 1)M = \bigl(A^u_\infty(\mathcal{D},\mathcal{A}) \xrightarrow{(\widetilde{e}\boxtimes 1)M} A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\operatorname{mod}\widetilde{\mathcal{B}}} \xrightarrow{(\widetilde{Y}\boxtimes 1)M} A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}\bigr)$$

is a composition of two A_∞-equivalences.

For such $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{C}}$, and, more generally, for a full subcategory $\widetilde{\mathcal{B}}$ of an arbitrary strictly unital A_∞-category $\widetilde{\mathcal{C}}$ the representing A_∞-category $\mathcal{D} = \mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$ exists. It is constructed in Section 5.2 as an A_∞-category, freely generated over $\widetilde{\mathcal{C}}$ by application of contracting homotopies H to morphisms, whose source or target is in $\widetilde{\mathcal{B}}$. The strict A_∞-functor $\widetilde{e} : \widetilde{\mathcal{C}} \to \mathcal{D}$ is identity on objects and $\widetilde{e}_1$ is an embedding. Theorem 6.5 asserts unitality of $\mathcal{D} = \mathsf{Q}(\widetilde{\mathcal{C}}|\widetilde{\mathcal{B}})$. $\square$

The 2-category $\overline{A^u_\infty} = H^0 A^u_\infty$ has unital A_∞-categories as objects, unital A_∞-functors as 1-morphisms and equivalence classes of natural A_∞-transformations as 2-morphisms [Lyu03]. Thus, $\overline{A^u_\infty}(\mathcal{C},\mathcal{A})(f,g) = [H^0 A^u_\infty(\mathcal{C},\mathcal{A})](f,g) = H^0(A^u_\infty(\mathcal{C},\mathcal{A})(f,g),m_1)$.

A zero object of a category $\mathcal{E}$ is an object Z, which is simultaneously initial and terminal. For a linear (Ab-enriched, not necessarily additive) category $\mathcal{E}$ this can be formulated as follows: $\mathcal{E}(Z,X) = 0$ and $\mathcal{E}(X,Z) = 0$ for any object X of $\mathcal{E}$. This condition is equivalent to the equation $1_Z = 0 \in \mathcal{E}(Z,Z)$.

1.4 Corollary. *The unital A_∞-functor $e : \mathcal{C} \to \mathcal{D}$ from the main theorem has the following property: composition with e in the sequence of functors*

$$\overline{A^u_\infty}(\mathcal{D},\mathcal{A}) \xrightarrow{e\bullet} \overline{A^u_\infty}(\mathcal{C},\mathcal{A}) \longrightarrow \overline{A^u_\infty}(\mathcal{B},\mathcal{A})$$

is an equivalence of the category $\overline{A^u_\infty}(\mathcal{D},\mathcal{A})$ with the full subcategory

$$\operatorname{Ker}(\overline{A^u_\infty}(\mathcal{C},\mathcal{A}) \to \overline{A^u_\infty}(\mathcal{B},\mathcal{A})) = H^0(A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}, m_1) \subset \overline{A^u_\infty}(\mathcal{C},\mathcal{A}),$$

consisting of such unital A_∞-functors $f : \mathcal{C} \to \mathcal{A}$ that its restriction $f\big|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ is a zero object of $\overline{A^u_\infty}(\mathcal{B},\mathcal{A})$.

Proof. A unital A_∞-functor $g : \mathcal{B} \to \mathcal{A}$ is contractible if and only if $g\mathbf{i}^{\mathcal{A}} \equiv 0 : g \to g : \mathcal{B} \to \mathcal{A}$, that is, $1_g = 0 \in \overline{A^u_\infty}(\mathcal{B},\mathcal{A})(g,g)$. Thus, for unital g contractibility is equivalent to g being a zero object of $\overline{A^u_\infty}(\mathcal{B},\mathcal{A})$. □

The main theorem asserts that the chain map $e_\bullet = s(e\boxtimes 1)M_{01}s^{-1} : A^u_\infty(\mathcal{D},\mathcal{A})(f,g) \to A^u_\infty(\mathcal{C},\mathcal{A})(ef,eg)$ is a homotopy isomorphism, while Corollary 1.4 claims only that it induces isomorphism in 0-th homology.

1.5. Uniqueness of the representing A_∞-category. With each strict A_∞-2-functor $F : A^u_\infty \to A^u_\infty$ is associated an ordinary strict 2-functor $\overline{F} : \overline{A^u_\infty} \to \overline{A^u_\infty}$, $\overline{F}\mathcal{A} = F\mathcal{A}$ [LM03, Section 3.2]. With a strict A_∞-2-transformation $\lambda = (\lambda_{\mathcal{A}}) : F \to G : A^u_\infty \to A^u_\infty$ is associated an ordinary strict 2-transformation $\overline{\lambda} = (\overline{\lambda_{\mathcal{A}}}) : \overline{F} \to \overline{G} : \overline{A^u_\infty} \to \overline{A^u_\infty}$ in cohomology [*ibid*]. Assume that λ is a natural A_∞-2-equivalence. Since $\lambda_{\mathcal{A}} : F\mathcal{A} \to G\mathcal{A}$ are A_∞-equivalences, the 1-morphisms $\overline{\lambda_{\mathcal{A}}} : \overline{F}\mathcal{A} \to \overline{G}\mathcal{A}$ are equivalences in the 2-category $\overline{A^u_\infty}$. Composing $\overline{\lambda}$ with the 0-th cohomology 2-functor $H^0 : \overline{A^u_\infty} \to \mathcal{C}at$, we get a 2-natural equivalence $H^0\overline{\lambda} : H^0\overline{F} \to H^0\overline{G} : \overline{A^u_\infty} \to \mathcal{C}at$, which consists of equivalences of ordinary categories $H^0(\lambda_{\mathcal{A}}) : H^0(F\mathcal{A}) \to H^0(G\mathcal{A})$. In particular, if $F = A^u_\infty(\mathcal{D},\_)$ for some unital A_∞-category $\mathcal{D}$, then

$$H^0\overline{F} = H^0\overline{A^u_\infty(\mathcal{D},\_)} = \overline{A^u_\infty}(\mathcal{D},\_).$$

Indeed, the both categories $H^0\overline{A^u_\infty(\mathcal{D},\mathcal{A})}$ and $\overline{A^u_\infty}(\mathcal{D},\mathcal{A})$ have unital A_∞-functors $\mathcal{D} \to \mathcal{A}$ as objects and equivalence classes of natural A_∞-transformations as morphisms. If $\lambda : A^u_\infty(\mathcal{D},\_) \to G : A^u_\infty \to A^u_\infty$ is a natural A_∞-2-equivalence (G is *unitally representable* by $\mathcal{D}$), then $H^0(\lambda_{\mathcal{A}}) : \overline{A^u_\infty}(\mathcal{D},\mathcal{A}) \to H^0(G\mathcal{A}) : \overline{A^u_\infty} \to \mathcal{C}at$ is a 2-natural equivalence. Thus,

$H^0\overline{G}$ is represented by $\mathcal{D}$ in the 2-category sense and $\mathcal{D}$ is unique up to an equivalence by Section C.17.

In particular, if $G = A^u_\infty(\mathcal{C}, \_)_{\operatorname{mod}\mathcal{B}}$, then with each object e of $G\mathcal{D} = A^u_\infty(\mathcal{C},\mathcal{D})_{\operatorname{mod}\mathcal{B}}$ is associated a strict A_∞-2-transformation

$$\lambda = (e \boxtimes 1)M : A^u_\infty(\mathcal{D},\mathcal{A}) \to A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}} : A^u_\infty \to A^u_\infty.$$

We have identified $H^0\overline{G}(\mathcal{A})$ with $\operatorname{Ker}(\overline{A^u_\infty}(\mathcal{C},\mathcal{A}) \to \overline{A^u_\infty}(\mathcal{B},\mathcal{A}))$ in Corollary 1.4. The strict 2-natural equivalence $H^0\overline{\lambda} : H^0\overline{F} \to H^0\overline{G} : \overline{A^u_\infty} \to \mathcal{C}at$ identifies with the strict 2-transformation

$$^{H^0\overline{G}}\lambda^{\mathcal{D},e} : \overline{A^u_\infty}(\mathcal{D}, \_) \to H^0\overline{G} : \overline{A^u_\infty} \to \mathcal{C}at$$

from Proposition C.11, since

$$(f : \mathcal{D} \to \mathcal{A}) \xmapsto{(e\boxtimes 1)M} ef = (e)(H^0\overline{G}(f)) \overset{\text{def}}{=}: (f)^{H^0\overline{G}}\lambda^{\mathcal{D},e},$$
$$(r : f \to g : \mathcal{D} \to \mathcal{A}) \xmapsto{(e\boxtimes 1)M} er = (e)(H^0\overline{G}(r)) \overset{\text{def}}{=}: (r)^{H^0\overline{G}}\lambda^{\mathcal{D},e}.$$

Therefore, the pair $(\mathcal{D}, e)$ represents the strict 2-functor $H^0\overline{G} : \overline{A^u_\infty} \to \mathcal{C}at$ in the sense of Definition C.16.

1.6 Corollary. *The pair* $(\mathcal{D}, e : \mathcal{C} \to \mathcal{D})$ *is unique up to an equivalence in the sense of Section C.17.*

The proof immediately follows from results of Section C.17.

The unital A_∞-category $\mathcal{D}$ obtained in the main theorem can be replaced with a differential graded category by the A_∞-version of Yoneda Lemma (Theorem A.11). We may restrict Corollary 1.4 to differential graded categories $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}$ for the same reason. Then it becomes parallel to the second half of main Theorem 1.6.2 of Drinfeld's work [Dri02], which asserts exactness of the sequence of categories

$$T(\mathcal{D},\mathcal{A}) \to T(\mathcal{C},\mathcal{A}) \to T(\mathcal{B},\mathcal{A})$$

in the same sense as in Corollary 1.4. Here T is a certain 2-category whose objects are differential graded categories. It is not known in general whether categories $T(\mathcal{C},\mathcal{A})$ and $\overline{A^u_\infty}(\mathcal{C},\mathcal{A})$ are equivalent. If $\Bbbk$ is a field, then, as B. Keller explained to us, equivalence of $\overline{A^u_\infty}(\mathcal{C},\mathcal{A})$ and $T(\mathcal{C},\mathcal{A})$ can be deduced from results of Lefèvre-Hasegawa [LH03, Section 8.2].

1.7. Basic properties of the main construction. The proof of universality of $\mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ is based on the fact that $\mathcal{D}$ is *relatively free* over $\mathcal{C}$, that is, it admits a filtration

$$\mathcal{C} = \mathcal{D}_0 \subset \mathcal{Q}_1 \subset \mathcal{D}_1 \subset \mathcal{Q}_2 \subset \mathcal{D}_2 \subset \mathcal{Q}_3 \subset \cdots \subset \mathcal{D}$$

by A_∞-subcategories $\mathcal{D}_j$ and differential graded subquivers $\mathcal{Q}_j$, such that the graded subquiver $\mathcal{D}_j \subset \mathcal{Q}_{j+1}$ has a direct complement $\mathcal{N}_{j+1}$ (a graded subquiver of $\mathcal{Q}_{j+1}$), and such that $\mathcal{D}_{j+1}$ is generated by $\mathcal{N}_{j+1}$ over $\mathcal{D}_j$. The precise conditions are given in Definition 5.1 (see also Proposition 2.1). This filtration allows to write down a sequence of restriction A_∞-functors

$$\begin{aligned} A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}} \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_0,\mathcal{Q}_1;\mathcal{A}) \longleftarrow A^{\psi u}_\infty(\mathcal{D}_1,\mathcal{A}) \\ \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_1,\mathcal{Q}_2;\mathcal{A}) \longleftarrow A^{\psi u}_\infty(\mathcal{D}_2,\mathcal{A}) \longleftarrow A^{\psi u}_{\infty 1}(\mathcal{D}_2,\mathcal{Q}_3;\mathcal{A}) \longleftarrow \dots \end{aligned} \tag{5.13.1}$$

and to prove that each of these A_∞-functors is an equivalence, surjective on objects (Theorem 4.7, Propositions 5.7, 5.10). The category $A^{\psi u}_{\infty 1}(\mathcal{D}_j,\mathcal{Q}_{j+1};\mathcal{A})$ is defined via pullback square (4.2.1)

$$\begin{tikzcd} A^{\psi u}_{\infty 1}(\mathcal{D}_j,\mathcal{Q}_{j+1};\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_1(\mathcal{Q}_{j+1},\mathcal{A}) \arrow[d] \\ A^{\psi u}_\infty(\mathcal{D}_j,\mathcal{A}) \arrow[r] & A_1(\mathcal{D}_j,\mathcal{A}) \end{tikzcd}$$

The A_∞-categories $\mathcal{D}_j$ are not unital, but only pseudounital – there are distinguished cycles $\mathbf{i}^{\mathcal{C}}_0 \in (s\mathcal{D}_j)^{-1}$, which are not unit elements of $\mathcal{D}_j$ if $j > 0$. The index ψu in $A^{\psi u}_\infty$ indicates that we consider pseudounital A_∞-functors – a generalization of unital ones (Definition 4.1). Their first components preserve the distinguished cycles up to a boundary. The A_∞-equivalence $A^u_\infty(\mathcal{D},\mathcal{A}) \to A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ is the limit case of (5.13.1) (Theorem 5.13).

The proof of unitality of $\mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ for strictly unital $\mathcal{C}$ is based on the study of multicategory of A_∞-operations and contracting homotopies operating in $\mathcal{D}$ (Theorem 6.5).

1.8. Description of various results. The proof of Theorem 4.7 is based on description of chain maps $P \to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\phi,\psi)$ to the complex of (ϕ,ψ)-coderivations (Proposition 2.6), where $\mathcal{F}\mathcal{Q}$ is the free A_∞-category, generated by a differential graded quiver $\mathcal{Q}$. A similar result for the quotient $\mathcal{F}\mathcal{Q}/s^{-1}I$ over an A_∞-ideal I is given in Proposition 2.9. Of course, any A_∞-category is a quotient of a free one (Proposition 3.2). We also describe homotopies between chain maps $P \to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\phi,\psi)$ (Corollary 2.7), and generalize the result to quotients $\mathcal{F}\mathcal{Q}/s^{-1}I$ (Corollary 2.10).

In Section 7 we consider the example of differential graded category $\mathcal{C} = \mathsf{C}(\mathcal{A})$ of complexes in a $\Bbbk$-linear Abelian category $\mathcal{A}$, and the full subcategory $\mathcal{B} \subset \mathcal{C}$ of acyclic complexes. The functor $H^0 e$ factors through a functor $D(\mathcal{A}) \to H^0(\mathsf{Q}(\mathcal{C}|\mathcal{B}))$ by Corollary 7.2.

In Appendix A we define, following Fukaya [Fuk02, Lemma 9.8], the Yoneda A_∞-functor $Y : \mathcal{A}^{\mathrm{op}} \to A_\infty(\mathcal{A},\mathsf{C})$, where C is the differential graded category of complexes of $\Bbbk$-modules. We prove for an arbitrary unital A_∞-category $\mathcal{A}$ that the Yoneda A_∞-functor Y is an equivalence of $\mathcal{A}^{\mathrm{op}}$ with its image – full differential graded subcategory of $A_\infty(\mathcal{A},\mathsf{C})$ (Theorem A.11). This is already proven by Fukaya in the case of strictly unital A_∞-category $\mathcal{A}$ [Fuk02, Theorem 9.1]. As a corollary we deduce that any $\mathscr{U}$-small unital A_∞-category $\mathcal{A}$ is A_∞-equivalent to a $\mathscr{U}$-small differential graded category (Corollary A.12).

In Appendix C we lift the classical Yoneda Lemma one dimension up – to strict 2-categories, weak 2-functors and weak 2-transformations.

An important result from another paper is recalled in simplified form, in which it is used in the present paper:

1.9 Corollary (to Theorem 8.8 [Lyu03]). *Let $\mathcal{C}$ be an A_∞-category and let $\mathcal{B}$ be a unital A_∞-category. Let $\phi : \mathcal{C} \to \mathcal{B}$ be an A_∞-functor such that for all objects X, Y of $\mathcal{C}$ the chain map $\phi_1 : (s\mathcal{C}(X,Y), b_1) \to (s\mathcal{B}(X\phi, Y\phi), b_1)$ is homotopy invertible. If* $\operatorname{Ob}\phi : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{B}$ *is surjective, then $\mathcal{C}$ is unital and ϕ is an A_∞-equivalence.*

Proof. Let $h : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ be an arbitrary mapping such that $h \cdot \operatorname{Ob}\phi = \mathrm{id}_{\operatorname{Ob}\mathcal{B}}$. The remaining data required in Theorem 8.8 of [Lyu03] can be chosen as ${}_Ur_0 = {}_Up_0 = {}_U\mathbf{i}_0^{\mathcal{B}} : \Bbbk \to (s\mathcal{B})^{-1}(U,U)$ for all objects U of $\mathcal{B}$. We conclude by this theorem that there exists an A_∞-functor $\psi : \mathcal{B} \to \mathcal{C}$ with $\operatorname{Ob}\psi = h$, quasi-inverse to ϕ. $\square$

Logical dependence of sections is the following. Appendices A and C do not depend on other sections. Appendix B depends on Appendix A. Sections 2–7 depend on appendices and on sections with smaller number (excluding the first section). The first section depends on all the rest of the article.

1.10. Conventions and preliminaries. We keep the notations and conventions of [Lyu03, LO02, LM03], sometimes without explicit mentioning. Some of the conventions are recalled here.

We assume that most quivers, A_∞-categories, etc. are small with respect to some universe $\mathscr{U}$. It means that the set of objects and the set of morphisms are $\mathscr{U}$-small, that is, isomorphic as sets to an element of $\mathscr{U}$ [GV73, Section 1.0]. The universe $\mathscr{U}$ is supposed to be an element of a universe $\mathscr{U}'$, which in its turn is an element of a universe $\mathscr{U}''$, and so on. All sets are supposed to be in bijection with some elements of some of the universes. Some differential graded categories in this paper will be $\mathscr{U}'$-small $\mathscr{U}$-categories. A category $\mathcal{C}$ is a $\mathscr{U}$-category if all its sets of morphisms $\mathcal{C}(X,Y)$ are $\mathscr{U}$-small [GV73, Definition 1.1].

The $\mathscr{U}$-small ground ring $\Bbbk$ is a unital associative commutative ring. A $\Bbbk$-module means a $\mathscr{U}$-small $\Bbbk$-module.

We use the right operators: the composition of two maps (or morphisms) $f : X \to Y$ and $g : Y \to Z$ is denoted $fg : X \to Z$; a map is written on elements as $f : x \mapsto xf = (x)f$. However, these conventions are not used systematically, and $f(x)$ might be used instead.

The set of non-negative integers is denoted $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$.

We consider only such A_∞-categories $\mathcal{C}$ that the differential $b : Ts\mathcal{C} \to Ts\mathcal{C}$ vanishes on $T^0s\mathcal{C}$, that is, $b_0 = 0$. We consider only those A_∞-functors $f : \mathcal{A} \to \mathcal{B}$, whose 0-th component f_0 vanishes.

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2. Quotients of free A_∞-categories

Let $\mathcal{Q}$ be a differential graded $\Bbbk$-quiver. There is a free A_∞-category $\mathcal{FQ}$, generated by $\mathcal{Q}$ [LM03, Section 2.1]. Let $R \subset s\mathcal{FQ}$ be a graded subquiver. Denote by $I = (R) \subset s\mathcal{FQ}$ the graded subquiver spanned by multiplying elements of R with some elements of $s\mathcal{FQ}$ via several operations $b_k^{\mathcal{FQ}}$, $k > 1$. It can be described as follows. Let $t \in \mathcal{T}^n_{\geqslant 2}$ be a plane rooted tree with $i(t) = n$ input leaves. Each decomposition

$$(t, \leqslant) = (1^{\sqcup\alpha_1} \sqcup \mathfrak{t}_{k_1} \sqcup 1^{\sqcup\beta_1}) \cdot (1^{\sqcup\alpha_2} \sqcup \mathfrak{t}_{k_2} \sqcup 1^{\sqcup\beta_2}) \cdot \dots \cdot \mathfrak{t}_{k_N}, \tag{2.0.1}$$

of t into the product of elementary forests gives a linear ordering $\leqslant$ of t. Here $\alpha_1+k_1+\beta_1 = n$ and $N = |t|$ is the number of internal vertices of t. An operation

$$b^{\mathcal{FQ}}_{(t,\leqslant)} = \bigl(s\mathcal{FQ}^{\otimes n} \xrightarrow{1^{\otimes\alpha_1}\otimes b^{\mathcal{FQ}}_{k_1}\otimes 1^{\otimes\beta_1}} s\mathcal{FQ}^{\otimes\alpha_1+1+\beta_1} \xrightarrow{1^{\otimes\alpha_2}\otimes b^{\mathcal{FQ}}_{k_2}\otimes 1^{\otimes\beta_2}} \dots \xrightarrow{b^{\mathcal{FQ}}_{k_N}} s\mathcal{FQ}\bigr) \tag{2.0.2}$$

is associated with the linearly ordered tree $(t, \leqslant)$. Different choices of the ordering of t change only the sign of the above map. In particular, one may consider the canonical linear ordering $t_<$ of t [LM03, Section 1.7] and the corresponding map $b^{\mathcal{FQ}}_{t_<}$. So the subquiver $I \subset s\mathcal{FQ}$ is defined as

$$I = (R) = \sum_{t\in\mathcal{T}^{\alpha+1+\beta}_{\geqslant 2}} \operatorname{Im}(b^{\mathcal{FQ}}_{t_<} : s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta} \to s\mathcal{FQ}),$$

where the summation goes over all $\alpha, \beta \in \mathbb{Z}_{\geqslant 0}$ and all trees with $\alpha + 1 + \beta$ input leaves.

2.1 Proposition. *Let $R \subset s\mathcal{FQ}$ be a graded subquiver such that $Rb_1^{\mathcal{FQ}} \subset (R) = I$. Then $Ib_1^{\mathcal{FQ}} \subset I$, I is an A_∞-ideal of $\mathcal{FQ}$, and $\mathcal{E} = \mathcal{FQ}/s^{-1}I$ is an A_∞-category.*

Proof. Clearly, I is closed under multiplications b_k, $k > 1$, with elements of $s\mathcal{FQ}$.

Let us prove that for all $t \in \mathcal{T}_{\geqslant 2}$

$$\operatorname{Im}(b^{\mathcal{FQ}}_{t_<} : s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta} \to s\mathcal{FQ})b_1^{\mathcal{FQ}} \subset I \tag{2.1.1}$$

using induction on $|t|$. This holds for $|t| = 0$, $t = \Big|$ by assumption. Let $|t| = N > 0$ and assume that (2.1.1) holds for all $t' \in \mathcal{T}_{\geqslant 2}$ with $|t'| < N$. The tree t can be presented as $t = (t_1 \sqcup \dots \sqcup t_k)\mathfrak{t}_k$ for some $k > 1$. We have $b^{\mathcal{FQ}}_{t<} = \pm(b^{\mathcal{FQ}}_{t_1} \otimes \dots \otimes b^{\mathcal{FQ}}_{t_k})b^{\mathcal{FQ}}_k$, $|t_i| < N$ and

$$b^{\mathcal{FQ}}_k b^{\mathcal{FQ}}_1 = -\sum_{a+q+c=k}^{a+c>0} (1^{\otimes a} \otimes b^{\mathcal{FQ}}_q \otimes 1^{\otimes c})b^{\mathcal{FQ}}_{a+1+c}.$$

One of the $a+1+c$ factors of

$$(s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta})(b^{\mathcal{FQ}}_{t_1} \otimes \dots \otimes b^{\mathcal{FQ}}_{t_k})(1^{\otimes a} \otimes b^{\mathcal{FQ}}_q \otimes 1^{\otimes c}) \tag{2.1.2}$$

is contained in I (for $q = 1$ this is the induction assumption). Hence,

$$(s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta})(b^{\mathcal{FQ}}_{t_1} \otimes \dots \otimes b^{\mathcal{FQ}}_{t_k})(1^{\otimes a} \otimes b^{\mathcal{FQ}}_q \otimes 1^{\otimes c})b^{\mathcal{FQ}}_{a+1+c} \subset I,$$

and the inclusion $Ib^{\mathcal{FQ}}_1 \subset I$ follows by induction.

Therefore, I is stable under all $b^{\mathcal{FQ}}_k$, $k \geqslant 1$, so it is an A_∞-ideal, and $\mathcal{E} = \mathcal{FQ}/s^{-1}I$ is an A_∞-category. □

2.2. A_∞-functors from a quotient of a free A_∞-category. The following statement is Proposition 2.3 from [LM03].

2.3 Proposition. *Let $\mathcal{Q}$ be a differential graded quiver, and let $\mathcal{A}$ be an A_∞-category. A_∞-functors $f : \mathcal{FQ} \to \mathcal{A}$ are in bijection with sequences $(f'_1, f_k)_{k>1}$, where $f'_1 : s\mathcal{Q} \to (s\mathcal{A}, b_1)$ is a chain morphism of differential graded quivers with the underlying mapping of objects $\operatorname{Ob} f : \operatorname{Ob}\mathcal{Q} \to \operatorname{Ob}\mathcal{A}$ and $f_k : T^k s\mathcal{FQ} \to s\mathcal{A}$ are $\Bbbk$-quiver morphisms of degree 0 with the same underlying map $\operatorname{Ob} f$ for all $k > 1$. The morphisms f_k are components of f for $k > 1$. The component $f_1 : s\mathcal{FQ} \to s\mathcal{A}$ is an extension of f'_1.*

We are going to extend this description to quotients of free A_∞-categories. Let a graded subquiver $R \subset s\mathcal{FQ}$ satisfy the assumption $Rb^{\mathcal{FQ}}_1 \subset (R) = I$. Denote by $\pi : \mathcal{FQ} \longrightarrow \mathcal{E} = \mathcal{FQ}s^{-1}I$ the natural projection strict A_∞-functor.

2.4 Proposition. *An A_∞-functor $f : \mathcal{FQ} \to \mathcal{A}$ factorizes as $f = \big(\mathcal{FQ} \xrightarrow{\pi} \mathcal{E} \xrightarrow{\tilde f} \mathcal{A}\big)$ for some (unique) A_∞-functor $\tilde f : \mathcal{E} \to \mathcal{A}$ if and only if the following two conditions are satisfied:*

1. *$Rf_1 = 0$;*
2. *$(s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})f_{\alpha+1+\beta} = 0$ for all $\alpha, \beta \in \mathbb{Z}_{\geqslant 0}$ such that $\alpha + \beta > 0$.*

Proof. If $f = \pi\tilde f$, then $f_k = \pi_1^{\otimes k}\tilde f_k$ and the above conditions are necessary.

Assume that the both conditions are satisfied. Let us prove that $If_1 = 0$. We are going to prove that for all $t \in \mathcal{T}_{\geqslant 2}$

$$\big(s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta} \xrightarrow{b^{\mathcal{FQ}}_{t<}} s\mathcal{FQ} \xrightarrow{f_1} s\mathcal{A}\big) = 0 \tag{2.4.1}$$

by induction on $|t|$. This holds for $|t| = 0$, $t = \big|$ by assumption. Let $|t| = N > 0$ and assume that (2.4.1) holds for all $t' \in \mathcal{T}_{\geqslant 2}$ with $|t'| < N$. The tree t can be presented as $t = (t_1 \sqcup \dots \sqcup t_k)\mathfrak{t}_k$ for some $k > 1$. We have $b^{\mathcal{FQ}}_{t_<} = \pm(b^{\mathcal{FQ}}_{t_1} \otimes \dots \otimes b^{\mathcal{FQ}}_{t_k})b^{\mathcal{FQ}}_k$, $|t_i| < N$ and

$$b^{\mathcal{FQ}}_k f_1 = \sum_{i_1+\dots+i_l=k} (f_{i_1} \otimes \dots \otimes f_{i_l}) b^{\mathcal{A}}_l - \sum_{a+q+c=n}^{a+c>0} (1^{\otimes a} \otimes b^{\mathcal{FQ}}_q \otimes 1^{\otimes c}) f_{a+1+c}. \tag{2.4.2}$$

One of the $a + 1 + c$ factors of (2.1.2) is contained in I and also one of the l factors of

$$(s\mathcal{Q}^{\otimes\alpha} \otimes R \otimes s\mathcal{Q}^{\otimes\beta})(b^{\mathcal{FQ}}_{t_1} \otimes \dots \otimes b^{\mathcal{FQ}}_{t_k})(f_{i_1} \otimes \dots \otimes f_{i_l})$$

is contained in I (for $i_j = 1$ this is the induction assumption). Hence, the right hand side of (2.4.2) is contained in I, and $If_1 \subset I$ follows by induction. Therefore,

$$(s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})f \subset \sum_{i_1+\dots+i_l=\alpha+1+\beta} (s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})(f_{i_1} \otimes \dots \otimes f_{i_l}) = 0$$

for all $\alpha, \beta \geqslant 0$. Clearly, f factorizes as $f = \pi\tilde{f}$. □

2.5. Transformations from a free A_∞-category. The following statement is Proposition 2.8 from [LM03].

2.6 Proposition. *Let $\phi, \psi : \mathcal{FQ} \to \mathcal{A}$ be A_∞-functors. For an arbitrary complex P of $\Bbbk$-modules chain maps $u : P \to sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$ are in bijection with the following data: $(u', u_k)_{k>1}$*

1. *a chain map $u' : P \to sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)$,*
2. *$\Bbbk$-linear maps*

$$u_k : P \to \prod_{X,Y \in \operatorname{Ob}\mathcal{Q}} \mathsf{C}\big((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi)\big)$$

of degree 0 for all $k > 1$.

The bijection maps u to $u_k = u \cdot \mathrm{pr}_k$,

$$u' = \big(P \xrightarrow{u} sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)\big). \tag{2.6.1}$$

The inverse bijection can be recovered from the recurrent formula

$$\begin{aligned}(-)^p b^{\mathcal{FQ}}_k(pu_1) = -(pd)u_k + \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b^{\mathcal{A}}_{\alpha+1+\beta} \\ - (-)^p \sum_{\alpha+q+\beta=k}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b^{\mathcal{FQ}}_q \otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta}) : (s\mathcal{FQ})^{\otimes k} \to s\mathcal{A},\end{aligned} \tag{2.6.2}$$

where $k > 1$, $p \in P$, and $\phi_{a\alpha}$, $\psi_{c\beta}$ are matrix elements of ϕ, ψ.

The following statement is Corollary 2.10 from [LM03].

2.7 Corollary. *Let $\phi, \psi : \mathcal{FQ} \to \mathcal{A}$ be A_∞-functors. Let P be a complex of $\Bbbk$-modules. Let $w : P \to sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$ be a chain map. The set (possibly empty) of homotopies $h : P \to sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$, $\deg h = -1$, such that $w = dh + hB_1$ is in bijection with the set of data $(h', h_k)_{k>1}$, consisting of*

1. *a homotopy $h' : P \to sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)$, $\deg h' = -1$, such that $dh' + h'B_1 = w'$, where*

$$w' = \bigl(P \xrightarrow{w} sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)\bigr);$$

2. *$\Bbbk$-linear maps*

$$h_k : P \to \prod_{X,Y \in \mathrm{Ob}\,\mathcal{Q}} \mathsf{C}\bigl((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi)\bigr)$$

of degree -1 for all $k > 1$.

The bijection maps h to $h_k = h \cdot \mathrm{pr}_k$,

$$h' = \bigl(P \xrightarrow{h} sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)\bigr). \quad (2.7.1)$$

The inverse bijection can be recovered from the recurrent formula

$$\begin{aligned}(-)^p b_k(ph_1) = pw_k - (pd)h_k - \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes ph_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta} \\ - (-)^p \sum_{a+q+c=k}^{a+c>0} (1^{\otimes a} \otimes b_q \otimes 1^{\otimes c})(ph_{a+1+c}) : (s\mathcal{FQ})^{\otimes k} \to s\mathcal{A}, \quad (2.7.2)\end{aligned}$$

where $k > 1$, $p \in P$, and $\phi_{a\alpha}$, $\psi_{c\beta}$ are matrix elements of ϕ, ψ.

2.8. Transformations from a quotient of a free A_∞-category. We are going to extend the above description to quotients of free A_∞-categories. Assume that a graded subquiver $R \subset s\mathcal{FQ}$ satisfies $Rb_1^{\mathcal{FQ}} \subset (R) = I$. Let $\mathcal{A}$ be an A_∞-category. Composition with the projection A_∞-functor $\pi : \mathcal{FQ} \to \mathcal{E} = \mathcal{FQ}/s^{-1}I$ gives a strict A_∞-functor $L^\pi = (\pi \boxtimes 1)M : A_\infty(\mathcal{E}, \mathcal{A}) \to A_\infty(\mathcal{FQ}, \mathcal{A})$. It is injective on objects and morphisms, that is, both maps $\mathrm{Ob}\, L^\pi : \phi \mapsto \pi\phi$ and $L_1^\pi : sA_\infty(\mathcal{E}, \mathcal{A})(\phi, \psi) \to sA_\infty(\mathcal{FQ}, \mathcal{A})(\pi\phi, \pi\psi)$, $r \mapsto \pi r$ are injective. We are going to characterize the subcomplex $sA_\infty(\mathcal{E}, \mathcal{A})(\phi, \psi) \hookrightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(\pi\phi, \pi\psi)$.

2.9 Proposition. *Let P be a complex of $\Bbbk$-modules, and let $u : P \to sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)$ be a chain map. Denote*

$$u_k = u\cdot \mathrm{pr}_k : P \to \prod_{X,Y\in\mathrm{Ob}\,\mathcal{Q}} \mathsf{C}\big((s\mathcal{FQ})^{\otimes k}(X,Y), s\mathcal{A}(X\phi,Y\psi)\big), \qquad k\geqslant 0.$$

Then the image of u is contained in the subcomplex $sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$ if and only if the following two conditions are satisfied:

1. *$R(pu_1) = \mathrm{Im}(pu_1 : R(X,Y)\to s\mathcal{A}(X\phi,Y\psi)) = 0$ for all $p\in P$;*
2. *$(s\mathcal{FQ}^{\otimes\alpha}\otimes I\otimes s\mathcal{FQ}^{\otimes\beta})(pu_k) = 0$ for all $p\in P$ and all $\alpha,\beta\geqslant 0$ such that $\alpha+1+\beta = k>1$.*

Proof. The conditions are obviously necessary. Let us prove that they are sufficient. First of all, we are going to show that $I(pu_1)=0$. Namely, we claim that

$$(s\mathcal{Q}^{\otimes\alpha}\otimes R\otimes s\mathcal{Q}^{\otimes\beta})b^{\mathcal{FQ}}_{t<}(pu_1) = 0 \tag{2.9.1}$$

for all trees $t\in\mathcal{T}^{\alpha+1+\beta}_{\geqslant 2}$. We prove it by induction on $|t|$. For $|t|=0$, $t = \big|$ this holds by assumption 1. Let $t\in\mathcal{T}_{\geqslant 2}$ be a tree with $|t| = N>0$ internal vertices. Assume that (2.9.1) holds for all $t'\in\mathcal{T}_{\geqslant 2}$ with $|t'|<N$. The tree t can be presented as $t = (t_1\sqcup\dots\sqcup t_k)\mathfrak{t}_k$ for some $k>1$, so $b^{\mathcal{FQ}}_{t<} = \pm(b^{\mathcal{FQ}}_{t_1}\otimes\dots\otimes b^{\mathcal{FQ}}_{t_k})b^{\mathcal{FQ}}_k$, and $|t_i|<N$. Formula (2.6.2) in the form

$$\begin{aligned}(-)^p b^{\mathcal{FQ}}_k(pu_1) = -(pd)u_k + \sum_{a+q+c=k}^{m,n} (\pi_1^{\otimes a}\phi_{am}\otimes pu_q\otimes\pi_1^{\otimes c}\psi_{cn})b^{\mathcal{A}}_{m+1+n} \\ -(-)^p\sum_{a+q+c=k}^{a+c>0}(1^{\otimes a}\otimes b^{\mathcal{FQ}}_q\otimes 1^{\otimes c})(pu_{a+1+c}) : s\mathcal{F}_{t_1}\mathcal{Q}\otimes\dots\otimes s\mathcal{F}_{t_k}\mathcal{Q}\to s\mathcal{A}\end{aligned} \tag{2.9.2}$$

implies that

$$\begin{aligned}(s\mathcal{Q}^{\otimes\alpha}\otimes R\otimes s\mathcal{Q}^{\otimes\beta})b^{\mathcal{FQ}}_{t<}(pu_1) &= (s\mathcal{Q}^{\otimes\alpha}\otimes R\otimes s\mathcal{Q}^{\otimes\beta})(b^{\mathcal{FQ}}_{t_1}\otimes\dots\otimes b^{\mathcal{FQ}}_{t_k})b^{\mathcal{FQ}}_k(pu_1)\\ &\subset (s\mathcal{FQ}^{\otimes\gamma}\otimes I\otimes s\mathcal{FQ}^{\otimes\delta})[(pd)u_k]\\ &+\sum(s\mathcal{Q}^{\otimes\alpha}\otimes R\otimes s\mathcal{Q}^{\otimes\beta})(b^{\mathcal{FQ}}_{t_1}\otimes\dots\otimes b^{\mathcal{FQ}}_{t_k})(\pi_1^{\otimes a}\phi_{am}\otimes pu_q\otimes\pi_1^{\otimes c}\psi_{cn})b^{\mathcal{A}}_{m+1+n}\\ &+\sum_{\lambda+\mu>0}(s\mathcal{FQ}^{\otimes\lambda}\otimes I\otimes s\mathcal{FQ}^{\otimes\mu})(pu_{\lambda+1+\mu}) = 0.\end{aligned}$$

Indeed, summands with $q=0$ vanish due to $I\pi_1 = 0$, summands with $q=1$ vanish due to induction assumption (2.9.1), and other summands vanish due to condition 2.

Thus condition 2 holds not only for $k>1$ but for $k=1$ as well. At last

$$(s\mathcal{FQ}^{\otimes\alpha}\otimes I\otimes s\mathcal{FQ}^{\otimes\beta})(pu)\subset\sum_{a+q+c=\alpha+1+\beta}^{m,n}(s\mathcal{FQ}^{\otimes\alpha}\otimes I\otimes s\mathcal{FQ}^{\otimes\beta})(\pi_1^{\otimes a}\phi_{am}\otimes pu_q\otimes\pi_1^{\otimes c}\psi_{cn}) = 0,$$

and the proof is finished. □

Let us extend the description of homotopies between chain maps to the case of quotient of a free A_∞-category. We keep the assumptions of Section 2.8.

2.10 Corollary. *Let P be a complex of $\Bbbk$-modules, and let $v : P \to sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$ be a chain map. Denote*

$$w = \bigl(P \xrightarrow{v} sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi) \hookrightarrow sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)\bigr).$$

Let $h : P \to sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)$ be a homotopy, $\deg h = -1$, $w = dh + hB_1$. Then the image of h is contained in the subcomplex $sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$ if and only if it factorizes as

$$h = \bigl(P \xrightarrow{\eta} sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi) \hookrightarrow sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)\bigr),$$

where $\deg\eta = -1$, $v = d\eta + \eta B_1$, or if and only if the following two conditions are satisfied:

1. *$R(ph_1) = \operatorname{Im}(ph_1 : R(X,Y) \to s\mathcal{A}(X\phi,Y\psi)) = 0$ for all $p \in P$;*
2. *$(s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})(ph_k) = 0$ for all $p \in P$ and all $\alpha,\beta \geqslant 0$ such that $\alpha + 1 + \beta = k > 1$.*

Proof. Given pair (w,h) defines a chain map $\overline{w} : \operatorname{Cone}(\mathrm{id}_P) \to sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)$, $(q,ps) \mapsto qw + ph$, such that

$$w = \bigl(P \xrightarrow{\mathrm{in}_1} P \oplus P[1] = \operatorname{Cone}(\mathrm{id}_P) \xrightarrow{\overline{w}} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi\phi,\pi\psi)\bigr).$$

The image of h is contained in $sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$ if and only if the image of $\overline{w}$ is contained in $sA_\infty(\mathcal{E},\mathcal{A})(\phi,\psi)$. By Proposition 2.9 this is equivalent to conditions:

$1'$. $R(qw_1) = 0$;	$1''$. $R(ph_1) = 0$;
$2'$. $(s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})(qw_k) = 0$;	$2''$. $(s\mathcal{FQ}^{\otimes\alpha} \otimes I \otimes s\mathcal{FQ}^{\otimes\beta})(ph_k) = 0$

for all $q,p \in P$ and all $\alpha,\beta \geqslant 0$ such that $\alpha + 1 + \beta = k > 1$. However, the conditions $1'$ and $2'$ are satisfied automatically by Proposition 2.9 applied to w and v. ☐

3. A simple example

We want to consider an example, which is almost tautological. The non-trivial part of it is the concrete choice of a system of relations R generating an ideal. Let $\mathcal{D}$ be an A_∞-category. We view it as a differential graded quiver and construct the free A_∞-category $\mathcal{FD}$ out of it. We choose the following subquiver of relations: $R_\mathcal{D} = \sum_{n\geqslant 2} \operatorname{Im}(b_n^{\mathcal{FD}} - b_n^{\mathcal{D}} : s\mathcal{D}^{\otimes n} \to s\mathcal{FD})$. More precisely, a map $\delta_n : (s\mathcal{D})^{\otimes n} \to s\mathcal{FD}$, $n \geqslant 2$, is defined as the difference

$$\delta_n = \bigl((s\mathcal{D})^{\otimes n} = (s\mathcal{F}_{|}\mathcal{D})^{\otimes n} \xrightarrow[s^{-1}]{b_n^{\mathcal{FD}}} s\mathcal{F}_{\mathsf{t}_n}\mathcal{D} \hookrightarrow s\mathcal{FD}\bigr) - \bigl((s\mathcal{D})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}}} s\mathcal{D} = s\mathcal{F}_{|}\mathcal{D} \hookrightarrow s\mathcal{FD}\bigr),$$

and $R_\mathcal{D} = \sum_{n\geqslant 2} \operatorname{Im}(\delta_n)$.

3.1 Lemma. *The subquiver $I = (R_{\mathcal{D}}) \subset s\mathcal{F}\mathcal{D}$ is an A_∞-ideal.*

Proof. According to Proposition 2.1 it suffices to check that $R_{\mathcal{D}} b_1^{\mathcal{F}\mathcal{D}} \subset (R_{\mathcal{D}})$. As

$$(b_n^{\mathcal{F}\mathcal{D}} - b_n^{\mathcal{D}}) b_1^{\mathcal{F}\mathcal{D}} = -b_n^{\mathcal{D}} b_1^{\mathcal{F}\mathcal{D}} - \sum_{a+k+c=n}^{a+c>0} (1^{\otimes a} \otimes b_k^{\mathcal{F}\mathcal{D}} \otimes 1^{\otimes c}) b_{a+1+c}^{\mathcal{F}\mathcal{D}},$$

we have

$$\begin{aligned}\operatorname{Im}(b_n^{\mathcal{F}\mathcal{D}} - b_n^{\mathcal{D}}) b_1^{\mathcal{F}\mathcal{D}} &\subset -\operatorname{Im}\Big(b_n^{\mathcal{D}} b_1^{\mathcal{F}\mathcal{D}} + \sum_{a+k+c=n}^{a+c>0} (1^{\otimes a} \otimes b_k^{\mathcal{D}} \otimes 1^{\otimes c}) b_{a+1+c}^{\mathcal{F}\mathcal{D}}\Big) + (R_{\mathcal{D}}) \\ &\subset -\operatorname{Im}\Big(b_n^{\mathcal{D}} b_1^{\mathcal{D}} + \sum_{a+k+c=n}^{a+c>0} (1^{\otimes a} \otimes b_k^{\mathcal{D}} \otimes 1^{\otimes c}) b_{a+1+c}^{\mathcal{D}}\Big) + (R_{\mathcal{D}}) = (R_{\mathcal{D}}),\end{aligned}$$

and the lemma is proven. □

Let us describe the ideal $I = (R_{\mathcal{D}}) \subset s\mathcal{F}\mathcal{D}$. Let $t' \in \mathcal{T}$ be a tree, and let $t = (1^{\sqcup\alpha} \sqcup \mathfrak{t}_k \sqcup 1^{\sqcup\beta}) \cdot t'_<$. Define a map $\delta_{\alpha,k,t'}$ as the difference

$$\begin{aligned}\delta_{\alpha,k,t'} = &\Big((s\mathcal{D})^{\otimes n} \xrightarrow{1^{\otimes\alpha} \otimes b_k^{\mathcal{F}\mathcal{D}} \otimes 1^{\otimes\beta}} (s\mathcal{F}_|\mathcal{D})^{\otimes\alpha} \otimes s\mathcal{F}_{\mathfrak{t}_k}\mathcal{D} \otimes (s\mathcal{F}_|\mathcal{D})^{\otimes\beta} \xrightarrow{b_{t'_<}^{\mathcal{F}\mathcal{D}}} s\mathcal{F}_t\mathcal{D} \longrightarrow s\mathcal{F}\mathcal{D}\Big) \\ &- \Big((s\mathcal{D})^{\otimes n} \xrightarrow{1^{\otimes\alpha} \otimes b_k^{\mathcal{D}} \otimes 1^{\otimes\beta}} (s\mathcal{D})^{\otimes\alpha+1+\beta} = (s\mathcal{F}_|\mathcal{D})^{\otimes\alpha+1+\beta} \xrightarrow{b_{t'_<}^{\mathcal{F}\mathcal{D}}} s\mathcal{F}_{t'}\mathcal{D} \longrightarrow s\mathcal{F}\mathcal{D}\Big)\end{aligned}$$

for $k > 1$, $\alpha + k + \beta = n$. Clearly, $\sum \operatorname{Im}(\delta_{\alpha,k,t'}) = I$. These compositions can be simplified. Let h be the height of the distinguished vertex $\mathfrak{t}_k$ in $t_<$. Then the above difference equals

$$\begin{aligned}\delta_{\alpha,k,t'} = &\Big((s\mathcal{D})^{\otimes\alpha+k+\beta} = (s\mathcal{F}_|\mathcal{D})^{\otimes n} \xrightarrow{(-)^{|t|-h} s^{-|t|}} s\mathcal{F}_t\mathcal{D} \longrightarrow s\mathcal{F}\mathcal{D}\Big) \\ &- \Big((s\mathcal{D})^{\otimes\alpha+k+\beta} \xrightarrow{1^{\otimes\alpha} \otimes b_k^{\mathcal{D}} \otimes 1^{\otimes\beta}} (s\mathcal{D})^{\otimes\alpha+1+\beta} = (s\mathcal{F}_|\mathcal{D})^{\otimes\alpha+1+\beta} \xrightarrow{s^{-|t'|}} s\mathcal{F}_{t'}\mathcal{D} \longrightarrow s\mathcal{F}\mathcal{D}\Big).\end{aligned}$$

With the identity map $\mathrm{id} : \mathcal{D} \to \mathcal{D}$ is associated a strict A_∞-functor $\widehat{\mathrm{id}} : \mathcal{F}\mathcal{D} \to \mathcal{D}$ [LM03, Section 2.6]. Its first component equals

$$\widehat{\mathrm{id}}_1 = \Big(s\mathcal{F}_t\mathcal{D} \xrightarrow{[b_{(t,\leqslant)}^{\mathcal{F}\mathcal{D}}]^{-1}} T^{i(t)} s\mathcal{D} \xrightarrow{b_{(t,\leqslant)}^{\mathcal{D}}} s\mathcal{D}\Big) \tag{3.1.1}$$

for each linear ordering $(t, \leqslant)$ of a plane rooted tree $t \in \mathcal{T}^n_{\geqslant 2}$ with $i(t) = n$ input leaves. Note that in the above formula $b_{(t,\leqslant)}^{\mathcal{F}\mathcal{D}} = \pm s^{-|t|}$. In particular, for the canonical linear ordering $t_<$ the formula becomes

$$\widehat{\mathrm{id}}_1 = \Big(s\mathcal{F}_t\mathcal{D} \xrightarrow{s^{|t|}} T^{i(t)} s\mathcal{D} \xrightarrow{b_{t_>}^{\mathcal{D}}} s\mathcal{D}\Big).$$

Here

$$b_{(t,\leqslant)}^{\mathcal{D}} = \Big(s\mathcal{D}^{\otimes n} \xrightarrow{1^{\otimes\alpha_1} \otimes b_{k_1}^{\mathcal{D}} \otimes 1^{\otimes\beta_1}} s\mathcal{D}^{\otimes\alpha_1+1+\beta_1} \xrightarrow{1^{\otimes\alpha_2} \otimes b_{k_2}^{\mathcal{D}} \otimes 1^{\otimes\beta_2}} \dots \xrightarrow{b_{k_N}^{\mathcal{D}}} s\mathcal{D}\Big)$$

corresponds to the ordered decomposition

$$(t,\leqslant) = (1^{\sqcup\alpha_1} \sqcup \mathsf{t}_{k_1} \sqcup 1^{\sqcup\beta_1}) \cdot (1^{\sqcup\alpha_2} \sqcup \mathsf{t}_{k_2} \sqcup 1^{\sqcup\beta_2}) \cdot \ldots \cdot \mathsf{t}_{k_N},$$

of $(t,\leqslant)$ into the product of forests, $\alpha_1 + k_1 + \beta_1 = n$, and $b^{\mathcal{FD}}_{(t,\leqslant)}$ has a similar meaning.

3.2 Proposition. *The A_∞-category $\mathcal{FD}/s^{-1}(R_{\mathcal{D}})$ is isomorphic to $\mathcal{D}$.*

Proof. Let $\mathcal{E} = \mathcal{FD}/s^{-1}(R_{\mathcal{D}})$ be the quotient category. The projection map $\pi_1 : s\mathcal{FD} \to s\mathcal{E}$ with the underlying map of objects $\operatorname{Ob}\pi = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ determines a strict A_∞-functor $\pi : \mathcal{FD} \to \mathcal{E}$. The embedding $\iota_1 = \big(s\mathcal{D} \hookrightarrow s\mathcal{FD} \xrightarrow{\pi_1} s\mathcal{E}\big)$ with the underlying identity map of objects $\operatorname{Ob}\iota = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ determines a strict A_∞-functor $\iota : \mathcal{D} \to \mathcal{E}$. Indeed, $\iota_1^{\otimes n} b^{\mathcal{E}}_n = b^{\mathcal{D}}_n \iota_1 : s\mathcal{D}^{\otimes n} \to s\mathcal{E}$, for $\operatorname{Im}(b^{\mathcal{FD}}_n - b^{\mathcal{D}}_n) = \operatorname{Im}\delta_n \subset I \subset s\mathcal{FD}$.

We claim that the A_∞-functor $\widehat{\mathrm{id}} : \mathcal{FD} \to \mathcal{D}$ factorizes as $\widehat{\mathrm{id}} = \big(\mathcal{FD} \xrightarrow{\pi} \mathcal{E} \xrightarrow{\widetilde{\mathrm{id}}} \mathcal{D}\big)$ for some A_∞-functor $\widetilde{\mathrm{id}} : \mathcal{E} \to \mathcal{D}$. Indeed, both conditions of Proposition 2.4 are satisfied. The second is obvious since $\widehat{\mathrm{id}}$ is strict. The first condition $R_{\mathcal{D}}\widehat{\mathrm{id}}_1 = 0$ follows from the computation $(b^{\mathcal{FD}}_n - b^{\mathcal{D}}_n)\widehat{\mathrm{id}}_1 = \mathrm{id}_1^{\otimes n} b^{\mathcal{D}}_n - b^{\mathcal{D}}_n \mathrm{id}_1 = 0 : s\mathcal{D}^{\otimes n} \to s\mathcal{D}$. Thus, $\widetilde{\mathrm{id}} : \mathcal{E} \to \mathcal{D}$ is a strict A_∞-functor.

Both ι and $\widetilde{\mathrm{id}}$ induce the identity map on objects. Clearly, $\iota \cdot \widetilde{\mathrm{id}} = \mathrm{id}_{\mathcal{D}}$. Furthermore, ι is surjective on morphisms because every element of $\mathcal{F}_t\mathcal{D}$ reduces to an element of $\mathcal{D}$ modulo $(R_{\mathcal{D}})$. Therefore, ι is invertible and A_∞-functors ι and $\widetilde{\mathrm{id}}$ are strictly inverse to each other. $\square$

3.3 Corollary. $(R_{\mathcal{D}})\widehat{\mathrm{id}}_1 = 0$.

4. A_∞-categories and quivers

4.1 Definition (Pseudounital A_∞-categories). A *pseudounital* structure of an A_∞-category $\mathcal{D}$ is a choice of an element $[\iota_X] \in H^{-1}(s\mathcal{D}(X,X), b_1)$ for each object X of $\mathcal{D}$. By ι_X we mean a representative of the chosen cohomology class, $\deg\iota_X = -1$. An A_∞-functor $f : \mathcal{D} \to \mathcal{A}$ between two pseudounital A_∞-categories is called *pseudounital* if it preserves the distinguished cohomology classes, that is, $\iota^{\mathcal{A}}_{Xf} - \iota^{\mathcal{D}}_X f_1 \in \operatorname{Im} b_1$ for all objects X of $\mathcal{D}$.

The composition of pseudounital A_∞-functors is pseudounital as well. The full subcategory of pseudounital A_∞-functors is denoted $A^{\psi u}_\infty(\mathcal{D},\mathcal{A}) \subset A_\infty(\mathcal{D},\mathcal{A})$.

A unital A_∞-category $\mathcal{A}$ has a canonical pseudounital structure: $\iota^{\mathcal{A}}_X = {}_X\mathbf{i}^{\mathcal{A}}_0$. An A_∞-functor $f : \mathcal{D} \to \mathcal{A}$ between unital A_∞-categories is unital if and only if it is pseudounital for the canonical pseudounital structures of $\mathcal{D}$ and $\mathcal{A}$ [Lyu03, Definition 8.1].

Let $\mathcal{D}$ be a pseudounital A_∞-category, let $\mathcal{Q}$ be a differential graded quiver with $\operatorname{Ob}\mathcal{Q} = \operatorname{Ob}\mathcal{D}$, and let $\mathrm{in}^{\mathcal{D}} : \mathcal{D} \to \mathcal{Q}$ be an A_1-functor, such that $\operatorname{Ob}\mathrm{in}^{\mathcal{D}} = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ and $\mathrm{in}^{\mathcal{D}}_1 : s\mathcal{D} \hookrightarrow s\mathcal{Q}$ is an embedding. Let $\mathcal{A}$ be a pseudounital A_∞-category.

4.2. **$A_{\infty 1}$-functors and transformations.** We define A_∞-category $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ via pull-back square

$$\begin{tikzcd} A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_1(\mathcal{Q},\mathcal{A}) \arrow[d, "A_1(\mathrm{in}^{\mathcal{D}},\mathcal{A})"] \\ A^{\psi u}_{\infty}(\mathcal{D},\mathcal{A}) \arrow[r, "\mathrm{restr}_{\leqslant 1}"] & A_1(\mathcal{D},\mathcal{A}) \end{tikzcd} \tag{4.2.1}$$

In details, the objects of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ are pairs (f,f'), where $f:\mathcal{D}\to\mathcal{A}$ is a pseudounital A_∞-functor, $f':\mathcal{Q}\to\mathcal{A}$ is an A_1-functor such that $\operatorname{Ob} f=\operatorname{Ob} f'$ and $f_1=f'_1\big|_{s\mathcal{D}}$. Morphisms of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ from (f,f') to (g,g') are pairs (r,r'), where $r\in A_\infty(\mathcal{D},\mathcal{A})(f,g)$, $r'\in A_1(\mathcal{Q},\mathcal{A})(f',g')$ are such that $\deg r=\deg r'$, $r_0=r'_0$ and $r_1=r'_1\big|_{s\mathcal{D}}$. For any n-tuple of composable morphisms $(r^j,p^j)\in sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})((f^{j-1},g^{j-1}),(f^j,g^j))$, $1\leqslant j\leqslant n$, their n-th product is defined as

$$[(r^1,p^1)\otimes\dots\otimes(r^n,p^n)]B_n \overset{\text{def}}{=} \bigl((r^1\otimes\dots\otimes r^n)B_n,(p^1\otimes\dots\otimes p^n)B_n\bigr).$$

It is well-defined, because formulas for B_k agree for all A_N-categories, $1\leqslant N\leqslant\infty$. The identity $B^2=0$ for $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ follows from that for $A^{\psi u}_{\infty}(\mathcal{D},\mathcal{A})$ and $A_1(\mathcal{Q},\mathcal{A})$. Thus, $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ is an A_∞-category.

4.3 Proposition. *If $\mathcal{A}$ is unital, then the A_∞-category $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ is unital as well.*

Proof. Let us denote by $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ the A_∞-transformation $\mathrm{id}\to\mathrm{id}:A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\to A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$, whose n-th component is

$$(r^1,p^1)\otimes\dots\otimes(r^n,p^n)\mapsto\bigl((r^1\otimes\dots\otimes r^n\boxtimes\mathbf{i}^{\mathcal{A}})M_{n1},(p^1\otimes\dots\otimes p^n\boxtimes\mathbf{i}^{\mathcal{A}})M_{n1}\bigr).$$

It is well-defined, since formulas for multiplication M agree for all A_N-categories, $1\leqslant N\leqslant\infty$. In a sense, $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ for $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ is determined by a pair of A_∞-transformations: $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ for $A^{\psi u}_{\infty}(\mathcal{D},\mathcal{A})$ and for $A_1(\mathcal{Q},\mathcal{A})$. Since the latter two A_∞-transformations are natural and satisfy

$$[(1\boxtimes\mathbf{i}^{\mathcal{A}})M\otimes(1\boxtimes\mathbf{i}^{\mathcal{A}})M]B_2\equiv(1\boxtimes\mathbf{i}^{\mathcal{A}})M,$$

the A_∞-transformation $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ for $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ has the same properties.

We claim that $(1\boxtimes\mathbf{i}^{\mathcal{A}})M$ is the unit transformation of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ as defined in [Lyu03, Definition 7.6]. Indeed, it remains to prove that chain endomorphisms

$$a\overset{\text{def}}{=}\bigl(1\otimes{}_{(g,g')}[(1\boxtimes\mathbf{i}^{\mathcal{A}})M]_0\bigr)B_2,\quad c\overset{\text{def}}{=}\bigl({}_{(f,f')}[(1\boxtimes\mathbf{i}^{\mathcal{A}})M]_0\otimes1\bigr)B_2:$$
$$\Phi\overset{\text{def}}{=}\bigl(sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})((f,f'),(g,g')),B_1\bigr)\to\Phi$$

are homotopy invertible for all pairs (f,f'), (g,g') of objects of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$. We have

$$(r,r')a=\bigl((r\otimes g\mathbf{i}^{\mathcal{A}})B_2,(r'\otimes g'\mathbf{i}^{\mathcal{A}})B_2\bigr),$$
$$(r,r')c=\bigl(r(f\mathbf{i}^{\mathcal{A}}\otimes1)B_2,r'(f'\mathbf{i}^{\mathcal{A}}\otimes1)B_2\bigr).$$

As a graded $\Bbbk$-module $\Phi = \prod_{n=0}^{\infty} V_n$, where

$$V_0 = \prod_{X \in \operatorname{Ob} \mathcal{D}} s\mathcal{A}(Xf, Xg),$$
$$V_1 = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} \mathsf{C}(s\mathcal{Q}(X,Y), s\mathcal{A}(Xf, Yg)),$$
$$V_n = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} \mathsf{C}((s\mathcal{D})^{\otimes n}(X,Y), s\mathcal{A}(Xf, Yg)) \qquad \text{for} \qquad n \geqslant 2. \tag{4.3.1}$$

Consider the decreasing filtration $\Phi = \Phi_0 \supset \Phi_1 \supset \cdots \supset \Phi_n \supset \Phi_{n+1} \supset \dots$ of the complex Φ, defined by $\Phi_n = 0 \times \cdots \times 0 \times \prod_{m=n}^{\infty} V_m$. We may write

$$\Phi_1 = \{(r, r') \in \Phi \mid r_0 = 0\},$$
$$\Phi_n = \{(r, 0) \in \Phi \mid \forall l < n \quad r_l = 0\} \qquad \text{for} \qquad n \geqslant 2.$$

The differential B_1 preserves the submodules Φ_n. It induces the differential

$$d = \mathsf{C}(1, b_1) - \sum_{\alpha+1+\beta=n} \mathsf{C}(1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta}, 1)$$

in the quotient $V_n = \Phi_n/\Phi_{n+1}$. Due to

$$[(r \otimes g\mathbf{i}^{\mathcal{A}})B_2]_k = \sum_l (r \otimes g\mathbf{i}^{\mathcal{A}})\theta_{kl} b_l = \sum_{a+p+c+q+e=k}^{\alpha,\gamma,\varepsilon} (f_{a\alpha} \otimes r_p \otimes g_{c\gamma} \otimes (g\mathbf{i}^{\mathcal{A}})_q \otimes g_{e\varepsilon}) b_{\alpha+\gamma+\varepsilon+2}$$

and similar formulas, the endomorphisms $a, c : \Phi \to \Phi$ preserve the subcomplexes Φ_n. They induce the endomorphisms $\operatorname{gr} a, \operatorname{gr} c : V_n \to V_n$ in the quotient complex V_n:

$$(r_1') \operatorname{gr} a = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} ({}_{X,Y}r_1' \otimes {}_{Yg}\mathbf{i}_0^{\mathcal{A}}) b_2^{\mathcal{A}}, \qquad r_1' \in V_1,$$
$$(r_1') \operatorname{gr} c = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} {}_{X,Y}r_1' ({}_{Xf}\mathbf{i}_0^{\mathcal{A}} \otimes 1) b_2^{\mathcal{A}},$$
$$(r_n) \operatorname{gr} a = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} ({}_{X,Y}r_n \otimes {}_{Yg}\mathbf{i}_0^{\mathcal{A}}) b_2^{\mathcal{A}}, \qquad r_n \in V_n, \quad n = 0 \text{ or } n \geqslant 2.$$
$$(r_n) \operatorname{gr} c = \prod_{X,Y \in \operatorname{Ob} \mathcal{D}} {}_{X,Y}r_n ({}_{Xf}\mathbf{i}_0^{\mathcal{A}} \otimes 1) b_2^{\mathcal{A}}.$$

Due to unitality of $\mathcal{A}$, for each pair X, Y of objects of $\mathcal{D}$ there exist $\Bbbk$-linear maps ${}_{X,Y}h, {}_{X,Y}h' : s\mathcal{A}(Xf, Yg) \to s\mathcal{A}(Xf, Yg)$ of degree -1 such that

$$(1 \otimes {}_{Yg}\mathbf{i}_0^{\mathcal{A}}) b_2^{\mathcal{A}} = 1 + {}_{X,Y}h \cdot d + d \cdot {}_{X,Y}h, \tag{4.3.2}$$
$$({}_{Xf}\mathbf{i}_0^{\mathcal{A}} \otimes 1) b_2^{\mathcal{A}} = -1 + {}_{X,Y}h' \cdot d + d \cdot {}_{X,Y}h'. \tag{4.3.3}$$

We equip the $\Bbbk$-modules $\Phi^d = \prod_{n=0}^{\infty} V_n^d$ with the topology of the product of discrete Abelian groups V_n^d. Thus the $\Bbbk$-submodules $\Phi_m^d = 0^{m-1} \times \prod_{n=m}^{\infty} V_n^d$ form a basis of neighborhoods of 0 in Φ^d. Continuous maps $A : V^d \to V^{d+p}$ are identified in the proof of Proposition A.9 with $\mathbb{N} \times \mathbb{N}$-matrices of linear maps $A_{nm} : V_n^d \to V_m^{d+p}$ with finite number of non-vanishing elements in each column.

In particular, the maps $B_1 : \Phi^d \to \Phi^{d+1}$ are continuous for all $d \in \mathbb{Z}$. Let us introduce continuous $\Bbbk$-linear maps $H, H' : \prod_{n=0}^{\infty} V_n^d \to \prod_{n=0}^{\infty} V_n^{d-1}$ by diagonal matrices ${}_{X,Y}r_n \mapsto {}_{X,Y}r_n {}_{X,Y}h$, ${}_{X,Y}r_n \mapsto {}_{X,Y}r_n {}_{X,Y}h'$. We may view $\operatorname{gr} a$, $\operatorname{gr} c$ as diagonal matrices and as the corresponding continuous endomorphisms of $\prod_{n=0}^{\infty} V_n^d$. Equations (4.3.2), (4.3.3) can be written as

$$\operatorname{gr} a = 1 + Hd + dH, \qquad \operatorname{gr} c = -1 + H'd + dH'.$$

The continuous chain maps

$$N = a - HB_1 - B_1H - 1, \quad N' = c - H'B_1 - B_1H' + 1 : \prod_{n=0}^{\infty} V_n^d \to \prod_{n=0}^{\infty} V_n^d$$

have strictly upper triangular matrices. Therefore, the endomorphisms $1 + N, -1 + N' : \prod_{n=0}^{\infty} V_n^d \to \prod_{n=0}^{\infty} V_n^d$ are invertible. Their inverse maps correspond to well-defined $\mathbb{N} \times \mathbb{N}$-matrices $\sum_{i=0}^{\infty} (-N)^i$ and $-\sum_{i=0}^{\infty} (N')^i$. Since a, c are homotopic to invertible maps, they are homotopy invertible and the proposition is proven. □

4.4. A strict A_∞-2-functor. Let us describe a strict A_∞-2-functor $F : A_\infty^u \to A_\infty^u$. It maps a unital A_∞-category $\mathcal{A}$ to the unital A_∞-category $A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A})$. The strict unital A_∞-functor

$$F_{\mathcal{A},\mathcal{B}} = A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \_) : A_\infty^u(\mathcal{A}, \mathcal{B}) \to A_\infty^u\big(A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A}), A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{B})\big)$$

is specified by the following data. It maps an object $f : \mathcal{A} \to \mathcal{B}$ (a unital A_∞-functor) to the object

$$A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; f) = (A_\infty^{\psi u}(\mathcal{D}, f), A_1(\mathcal{Q}, f)) = ((1 \boxtimes f)M, (1 \boxtimes f)M),$$

a unital A_∞-functor $A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A}) \to A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{B})$, which sends $(g, g') \in \operatorname{Ob} A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{A})$ to $(gf, g'f) \in \operatorname{Ob} A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \mathcal{B})$. Its n-th component is

$$[A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; f)]_n : (r^1, p^1) \otimes \cdots \otimes (r^n, p^n) \mapsto \big((r^1 \otimes \cdots \otimes r^n \boxtimes f)M_{n0}, (p^1 \otimes \cdots \otimes p^n \boxtimes f)M_{n0}\big).$$

An A_∞-transformation $q : f \to g : \mathcal{A} \to \mathcal{B}$ is mapped by $[A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; \_)]_1$ to the A_∞-transformation

$$A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; q) = (A_\infty^{\psi u}(\mathcal{D}, q), A_1(\mathcal{Q}, q)) = ((1 \boxtimes q)M, (1 \boxtimes q)M),$$

whose n-th component is

$$[A_{\infty 1}^{\psi u}(\mathcal{D}, \mathcal{Q}; q)]_n : (r^1, p^1) \otimes \cdots \otimes (r^n, p^n) \mapsto \big((r^1 \otimes \cdots \otimes r^n \boxtimes q)M_{n1}, (p^1 \otimes \cdots \otimes p^n \boxtimes q)M_{n1}\big).$$

Thus, a strict A_∞-functor $F_{\mathcal{A},\mathcal{B}} = A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\_)$ is constructed. It is unital, because the unit element $f\mathbf{i}^{\mathcal{B}}$ of $f \in \operatorname{Ob} A^u_\infty(\mathcal{A},\mathcal{B})$ is mapped to the unit element $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};f\mathbf{i}^{\mathcal{B}}) = ((1\boxtimes f\mathbf{i}^{\mathcal{B}})M,(1\boxtimes f\mathbf{i}^{\mathcal{B}})M)$ of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};f) \in \operatorname{Ob} A^u_\infty(F\mathcal{A},F\mathcal{B})$.

Necessary equation, given by diagram (3.1.1) of [LM03] follows from the same equation written for $A^{\psi u}_\infty(\mathcal{D},\_)$ and for $A_1(\mathcal{Q},\_)$. Therefore, the strict A_∞-2-functor F is constructed.

4.5. An A_∞-category freely generated over an A_∞-category. Let $\mathcal{D}$ be a pseudounital A_∞-category, let $\mathcal{Q}$ be a differential graded quiver with $\operatorname{Ob}\mathcal{Q} = \operatorname{Ob}\mathcal{D}$, equipped with a chain embedding $s\mathcal{D} \hookrightarrow s\mathcal{Q}$, identity on objects. Assume that there exists a graded $\Bbbk$-subquiver $\mathcal{N} \subset \mathcal{Q}$, $\operatorname{Ob}\mathcal{N} = \operatorname{Ob}\mathcal{D}$, which is a direct complement of $\mathcal{D}$. Thus, $\mathcal{D}\oplus\mathcal{N} = \mathcal{Q}$ is an isomorphism of graded $\Bbbk$-linear quivers. Then there exists a differential in the graded quiver $s\mathcal{M} = \mathcal{N}$ and a chain map $\alpha : s\mathcal{M} \to s\mathcal{D}$ such that $\operatorname{Ob}\alpha = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ and $\mathcal{Q} = \operatorname{Cone}\alpha$. Indeed, the embedding $\mathrm{in}^{\mathcal{D}}$ in the exact sequence

$$0 \to s\mathcal{D} \xrightarrow{\mathrm{in}^{\mathcal{D}}} s\mathcal{Q} \xrightarrow{\mathrm{pr}^{\mathcal{N}}} \mathcal{M}[2] \to 0$$

is a chain map. Thus the graded $\Bbbk$-quiver $\mathcal{M}[2] = s\mathcal{N} = \operatorname{Coker}(\mathrm{in}^{\mathcal{D}})$ acquires a differential $d^{\mathcal{M}[2]}$, such that $\mathrm{pr}^{\mathcal{N}}$ is a chain map. The differential in $s\mathcal{Q}$ has the form

$$(t,ms)b^{\mathcal{Q}}_1 = (tb^{\mathcal{D}}_1 + m\alpha, msd^{\mathcal{M}[2]}) = (tb^{\mathcal{D}}_1 + m\alpha, -md^{\mathcal{M}[1]}s) \tag{4.5.1}$$

for $t \in s\mathcal{D}(X,Y)$, $m \in \mathcal{M}[1](X,Y)$, where $\alpha : s\mathcal{M}(X,Y) \to s\mathcal{D}(X,Y)$ are $\Bbbk$-linear maps of degree 0. The condition $(b^{\mathcal{Q}}_1)^2 = 0$ is equivalent to α being a chain map. Therefore, $s\mathcal{Q} = \operatorname{Cone}(\alpha : s\mathcal{M} \to s\mathcal{D})$.

Define a pseudounital A_∞-category $\mathcal{E} = \mathcal{F}\mathcal{Q}/s^{-1}(R_{\mathcal{D}})$, where $R_{\mathcal{D}} = \sum_{n\geqslant 2}\operatorname{Im}(\delta_n)$ for

$$\delta_n = \big((s\mathcal{D})^{\otimes n} \hookrightarrow (s\mathcal{F}\mathcal{Q})^{\otimes n} \xrightarrow{b^{\mathcal{F}\mathcal{Q}}_n} s\mathcal{F}\mathcal{Q}\big) - \big((s\mathcal{D})^{\otimes n} \xrightarrow{b^{\mathcal{D}}_n} s\mathcal{D} \hookrightarrow s\mathcal{F}\mathcal{Q}\big).$$

Repeating word by word the proof of Lemma 3.1 we deduce that $J = (R_{\mathcal{D}}) \subset s\mathcal{F}\mathcal{Q}$ is an A_∞-ideal. The distinguished elements $\iota_X \in (s\mathcal{E})^{-1}(X,X)$ are those of $\mathcal{D} \subset \mathcal{E}$. Let $\mathcal{A}$ be a pseudounital A_∞-category. There is the restriction strict A_∞-functor

$$\mathrm{restr} : A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A}), \qquad x \mapsto (x\big|_{\mathcal{D}}, x\big|_{\mathcal{Q}}).$$

If $\mathcal{A}$ is unital, then the above A_∞-functor is unital, because the unit element $f\mathbf{i}^{\mathcal{A}}$ of $f \in \operatorname{Ob} A^{\psi u}_\infty(\mathcal{E},\mathcal{A})$ is mapped to the unit element $\big((f\mathbf{i}^{\mathcal{A}})\big|_{\mathcal{D}}, (f\mathbf{i}^{\mathcal{A}})\big|_{\mathcal{Q}}\big) = \big(f\big|_{\mathcal{D}}\mathbf{i}^{\mathcal{A}}, f\big|_{\mathcal{Q}}\mathbf{i}^{\mathcal{A}}\big)$ of $(f\big|_{\mathcal{D}}, f\big|_{\mathcal{Q}}) \in \operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$.

4.6 Proposition. *The map* $\operatorname{Ob}\mathrm{restr} : \operatorname{Ob} A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to \operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ *is surjective. An object* (f,f') *of* $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ *is the restriction of a unique pseudounital* A_∞*-functor* $\widetilde{f} : \mathcal{E} \to \mathcal{A}$ *such that* $\operatorname{Ob}\widetilde{f} = \operatorname{Ob} f$, $\widetilde{f}\big|_{\mathcal{Q}} = f'$, $\widetilde{f}_k\big|_{s\mathcal{D}^{\otimes k}} = f_k$, *and* $\widetilde{f}_k$ *vanishes on all summands of* $T^k s\mathcal{E}$ *containing a factor* $s\mathcal{N}$ *for* $k > 1$.

Proof. Let us define an A_∞-functor $\hat{f} : \mathcal{FQ} \to \mathcal{A}$ via Proposition 2.3 by the following data. On objects it is $\operatorname{Ob}\hat{f} = \operatorname{Ob} f = \operatorname{Ob} f'$, the restriction to $s\mathcal{Q}$ of the first component is $\hat{f}_1\big|_{s\mathcal{Q}} = f'$. On each direct summand of $T^k s\mathcal{FQ}$, $k > 1$, containing the factor $s\mathcal{N}$ we set $\hat{f}_k = 0$. On the direct summand $T^k s\mathcal{FD}$ of $T^k s\mathcal{FQ}$ we define

$$\hat{f}_k = \bigl(T^k s\mathcal{FD} \xrightarrow{\widehat{\mathrm{id}}_1^{\otimes k}} T^k s\mathcal{D} \xrightarrow{f_k} s\mathcal{A}\bigr)$$

for $k > 1$, where $\widehat{\mathrm{id}}_1$ is defined by (3.1.1). These requirements specify $\hat{f}$ completely. It is pseudounital, since $\hat{f}_1\big|_{s\mathcal{D}} = f_1$ and f is pseudounital.

Let us prove that the A_∞-functor $\hat{f}$ factors as $\hat{f} = \bigl(\mathcal{FQ} \xrightarrow{\pi} \mathcal{E} \xrightarrow{\widetilde{f}} \mathcal{A}\bigr)$ for some unique A_∞-functor $\widetilde{f}$. Denote $J = (R_\mathcal{D}) \subset s\mathcal{FQ}$. We have to check conditions of Proposition 2.4. The second condition, $(s\mathcal{FQ}^{\otimes\alpha} \otimes J \otimes s\mathcal{FQ}^{\otimes\beta})\hat{f}_{\alpha+1+\beta} = 0$ if $\alpha + \beta > 0$, holds on direct summands of $s\mathcal{FQ}^{\otimes\alpha} \otimes J \otimes s\mathcal{FQ}^{\otimes\beta}$, which contain a factor $s\mathcal{N}$ in some of $\mathcal{FQ}$. It holds also on summands of J of the form $\operatorname{Im}(b_t^{\mathcal{FQ}} : \dots \otimes s\mathcal{N} \otimes \dots \otimes R_\mathcal{D} \otimes \dots \to s\mathcal{FQ})$ or $\operatorname{Im}(b_t^{\mathcal{FQ}} : \dots \otimes R_\mathcal{D} \otimes \dots \otimes s\mathcal{N} \otimes \dots \to s\mathcal{FQ})$. We have to verify that $(s\mathcal{FD}^{\otimes\alpha} \otimes I \otimes s\mathcal{FD}^{\otimes\beta})\hat{f}_{\alpha+1+\beta} = 0$ if $\alpha + \beta > 0$, where $I \subset s\mathcal{FD}$ is the ideal described in Lemma 3.1. This equation holds true because

$$(s\mathcal{FD}^{\otimes\alpha} \otimes I \otimes s\mathcal{FD}^{\otimes\beta})\hat{f}_{\alpha+1+\beta} = (s\mathcal{FD}^{\otimes\alpha} \otimes I \otimes s\mathcal{FD}^{\otimes\beta})\widehat{\mathrm{id}}_1^{\alpha+1+\beta} f_{\alpha+1+\beta} = 0$$

due to equation $I\widehat{\mathrm{id}}_1 = 0$, obtained in Corollary 3.3. Therefore, the second condition of Proposition 2.4 is verified.

Let us observe that the restriction of $\hat{f}$ to $\mathcal{FD}$ coincides with $\mathcal{FD} \xrightarrow{\widehat{\mathrm{id}}} \mathcal{D} \xrightarrow{f} \mathcal{A}$. Indeed, their components are equal, $\hat{f}_k\big|_{\mathcal{FD}} = \widehat{\mathrm{id}}_1^{\otimes k} \cdot f_k$ for $k > 1$, and $\hat{f}_1\big|_{s\mathcal{F}_{|}\mathcal{D}} = f_1 = \widehat{\mathrm{id}}_1 \cdot f_1 : s\mathcal{D} \to s\mathcal{A}$. Hence, $\hat{f}\big|_{\mathcal{FD}} = \widehat{\mathrm{id}} \cdot f$ by Proposition 2.3.

In particular, $\hat{f}_1\big|_{s\mathcal{FD}} = \bigl(s\mathcal{FD} \xrightarrow{\widehat{\mathrm{id}}_1} s\mathcal{D} \xrightarrow{f_1} s\mathcal{A}\bigr)$. Hence, $R_\mathcal{D}\hat{f}_1 = R_\mathcal{D}\widehat{\mathrm{id}}_1 f_1 = 0$ due to Corollary 3.3. Therefore, the first condition of Proposition 2.4 is also verified and $\widetilde{f}$ exists. Uniqueness of such $\widetilde{f}$ is obvious. □

The projection map $\pi_1 : s\mathcal{FQ} \to s\mathcal{E}$ with the underlying map of objects $\operatorname{Ob}\pi = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ determines a strict A_∞-functor $\pi : \mathcal{FQ} \to \mathcal{E}$. The embedding $\iota_1 = \bigl(s\mathcal{D} \xhookrightarrow{i} s\mathcal{FQ} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$ with the underlying identity map of objects $\operatorname{Ob}\iota = \mathrm{id}_{\operatorname{Ob}\mathcal{D}}$ determines a strict A_∞-functor $\iota : \mathcal{D} \to \mathcal{E}$. Indeed, $\iota_1^{\otimes n} b_n^\mathcal{E} = b_n^\mathcal{D}\iota_1 : s\mathcal{D}^{\otimes n} \to s\mathcal{E}$, for $\operatorname{Im}(b_n^{\mathcal{FD}} - b_n^\mathcal{D}) = \operatorname{Im}\delta_n \subset J \subset s\mathcal{FQ}$. These A_∞-functors produce other strict A_∞-functors for an arbitrary A_∞-category $\mathcal{A}$. For instance, the functor

$$(\pi \boxtimes 1)M : A_\infty(\mathcal{E}, \mathcal{A}) \to A_\infty(\mathcal{FQ}, \mathcal{A}), \qquad x \mapsto \pi x,$$

injective on objects and morphisms, and the restriction A_∞-functor

$$\mathrm{restr} = (\iota \boxtimes 1)M : A_\infty(\mathcal{E}, \mathcal{A}) \to A_\infty(\mathcal{D}, \mathcal{A}), \qquad y \mapsto \iota y = \overline{y}.$$

4.7 Theorem. *Let $\mathcal{E} = \mathcal{F}\mathcal{Q}/s^{-1}(R_\mathcal{D})$, where $\mathcal{D}$, $\mathcal{Q}$ satisfy assumptions of Section 4.5. Then the restriction A_∞-functor*

$$\mathrm{restr} : A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A}) \tag{4.7.1}$$

is an A_∞-equivalence, surjective on objects. The chain surjections restr_1 *admit a chain splitting.*

Proof. Let us prove that the chain map

$$\mathrm{restr}_1 : sA^{\psi u}_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \tag{4.7.2}$$

is homotopy invertible for all pairs of pseudounital A_∞-functors $f,g:\mathcal{E}\to\mathcal{A}$. This will be achieved in a sequence of Lemmata.

The graded $\Bbbk$-quiver decomposition $\mathcal{Q} = \mathcal{D}\oplus\mathcal{N}$ implies that the graded $\Bbbk$-quiver $\mathcal{F}\mathcal{D}$ is a direct summand of $\mathcal{F}\mathcal{Q}$. The projection $\mathrm{pr}^{\mathcal{F}\mathcal{D}} : s\mathcal{F}\mathcal{Q} \to s\mathcal{F}\mathcal{D}$ annihilates all summands with factors $s\mathcal{N}$. Define a degree 0 map

$$\varpi = \big(s\mathcal{F}\mathcal{Q} \xrightarrow{\mathrm{pr}^{\mathcal{F}\mathcal{D}}} s\mathcal{F}\mathcal{D} \xrightarrow{\widehat{\mathrm{id}}_1} s\mathcal{D}\big).$$

4.8 Lemma. *There exists a unique chain map*

$$u : sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g)$$

such that u' obtained from u via formula (2.6.1) *equals*

$$u' = \mathrm{restr}_\mathcal{Q} : sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \to sA_1(\mathcal{Q},\mathcal{A})(f|_\mathcal{Q}, g|_\mathcal{Q}), \quad (p,p')\mapsto p',$$

and for $k>1$ the maps u_k are

$$u_k = \Big[sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \xrightarrow{\mathrm{pr}_k} \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \mathsf{C}(s\mathcal{D}^{\otimes k}(X,Y), s\mathcal{A}(Xf,Yg))$$
$$\xrightarrow{\prod \mathsf{C}(\varpi^{\otimes k},1)} \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \mathsf{C}(s\mathcal{F}\mathcal{Q}^{\otimes k}(X,Y), s\mathcal{A}(Xf,Yg))\Big], \qquad (p,p')\mapsto p_k \mapsto \varpi^{\otimes k}p_k.$$

Proof. Apply Proposition 2.6 to $P = sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big)$. □

4.9 Lemma. *The map u from Lemma 4.8 takes values in*

$$sA_\infty(\mathcal{E},\mathcal{A})(f,g) \subset sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g).$$

Proof. Let us verify conditions of Proposition 2.9. We have $J\varpi = 0$. Indeed, ϖ vanishes on summands of $J = (R_\mathcal{D})$ of the form $\mathrm{Im}(b^{\mathcal{F}\mathcal{Q}}_t : \dots \otimes s\mathcal{N} \otimes \dots \otimes R_\mathcal{D} \otimes \dots \to s\mathcal{F}\mathcal{Q})$ or $\mathrm{Im}(b^{\mathcal{F}\mathcal{Q}}_t : \dots \otimes R_\mathcal{D} \otimes \dots \otimes s\mathcal{N} \otimes \dots \to s\mathcal{F}\mathcal{Q})$. Looking at $I = J\cap\mathcal{F}\mathcal{D}$ we find that $J\varpi = I\varpi = I\widehat{\mathrm{id}}_1 = 0$ by Corollary 3.3. Therefore,

$$(s\mathcal{F}\mathcal{Q}^{\otimes\alpha}\otimes J\otimes s\mathcal{F}\mathcal{Q}^{\otimes\beta})((p,p')u_k) = (s\mathcal{F}\mathcal{Q}^{\otimes\alpha}\otimes J\otimes s\mathcal{F}\mathcal{Q}^{\otimes\beta})\varpi^{\otimes k}p_k = 0, \tag{4.9.1}$$

and the second condition of Proposition 2.9 is verified.

Let us check now that $R_{\mathcal{D}}((p,p')u_1) = 0$ for any element

$$(p,p') \in sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_{\mathcal{D}}, f|_{\mathcal{Q}}),(g|_{\mathcal{D}}, g|_{\mathcal{Q}})\big).$$

That is,

$$(-)^p(i^{\otimes k}b_k^{\mathcal{FQ}} - b_k^{\mathcal{D}}i)((p,p')u_1) = 0 : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg) \tag{4.9.2}$$

for $k>1$, where $i : s\mathcal{D} \hookrightarrow s\mathcal{FQ}$ is the embedding of differential graded $\Bbbk$-quivers. By definition (2.6.2)

$$\begin{aligned}(-)^p i^{\otimes k}b_k^{\mathcal{FQ}}((p,p')u_1) = {}& -i^{\otimes k}[(pB_1,p'B_1)u_k] \\ &+ \sum_{a+q+c=k}^{m,n} (i^{\otimes a}\pi_1^{\otimes a}f_{am} \otimes i^{\otimes q}((p,p')u_q) \otimes i^{\otimes c}\pi_1^{\otimes c}g_{cn})b^{\mathcal{A}}_{m+1+n} \\ &- (-)^p i^{\otimes k}\sum_{a+q+c=k}^{a+c>0}(1^{\otimes a}\otimes b_q^{\mathcal{FQ}}\otimes 1^{\otimes c})((p,p')u_{a+1+c}) : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg).\end{aligned}$$

For an arbitrary $(t,t') \in sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_{\mathcal{D}}, f|_{\mathcal{Q}}),(g|_{\mathcal{D}}, g|_{\mathcal{Q}})\big)$, in particular for (p,p') or $(pB_1,p'B_1)$, we have for $k>1$

$$i^{\otimes k}((t,t')u_k) = (t,t')u_k\mathsf{C}(i^{\otimes k},1) = t_k\mathsf{C}(\varpi^{\otimes k},1)\mathsf{C}(i^{\otimes k},1) = t_k\mathsf{C}(i^{\otimes k}\varpi^{\otimes k},1) = t_k \tag{4.9.3}$$

due to relation $i\varpi = \mathrm{id}_{s\mathcal{D}}$. For $k=0$ or 1 we also have $i^{\otimes k}((t,t')u_k) = t_k$. Indeed,

$$\begin{gathered}(t,t')u_0 = (t,t')u'\operatorname{pr}_0 = t'_0 = t_0, \\ i((t,t')u_1) = \operatorname{in}^{\mathcal{D}}((t,t')u')\operatorname{pr}_1 = \operatorname{in}^{\mathcal{D}} t'_1 = t_1.\end{gathered}$$

Notice also that $\big(s\mathcal{D} \xhookrightarrow{i} s\mathcal{FQ} \xrightarrow{\pi_1} s\mathcal{E}\big) = \iota_1$, hence, $i^{\otimes a}\pi_1^{\otimes a}f_{am} = \iota_1^{\otimes a}f_{am} = \overline{f}_{am} \overset{\text{def}}{=} (f|_{\mathcal{D}})_{am}$. Due to already proven property (4.9.1) we may replace $i^{\otimes q}b_q^{\mathcal{FQ}}$ in the last sum with $b_q^{\mathcal{D}}i$. Therefore,

$$\begin{aligned}(-)^p i^{\otimes k}b_k^{\mathcal{FQ}}((p,p')u_1) = {}& -(pB_1)_k + \sum_{a+q+c=k}^{m,n}(\overline{f}_{am}\otimes p_q \otimes \overline{g}_{cn})b^{\mathcal{A}}_{m+1+n} \\ & - (-)^p\sum_{a+q+c=k}^{a+c>0}(1^{\otimes a}\otimes b_q^{\mathcal{D}}\otimes 1^{\otimes c})i^{\otimes a+1+c}((p,p')u_{a+1+c}) \\ = {}& -(pB_1)_k + (pb^{\mathcal{A}})_k - (-)^p\sum_{a+q+c=k}(1^{\otimes a}\otimes b_q^{\mathcal{D}}\otimes 1^{\otimes c})p_{a+1+c} + (-)^p b_k^{\mathcal{D}}p_1 \\ = {}& -\big(pb^{\mathcal{A}} - (-)^p b^{\mathcal{D}}p\big)_k + (pb^{\mathcal{A}})_k - (-)^p(b^{\mathcal{D}}p)_k + (-)^p b_k^{\mathcal{D}}i((p,p')u_1) \\ = {}& (-)^p b_k^{\mathcal{D}}i((p,p')u_1) : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg)\end{aligned}$$

and (4.9.2) is proven. We conclude by Proposition 2.9 that there is a chain map Φ such that

$$u = \bigl(sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr) \xrightarrow{\Phi} sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) \xhookrightarrow{(\pi\boxtimes 1)M_{01}} sA_{\infty}(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g)\bigr), \tag{4.9.4}$$

so the lemma is proven. □

4.10 Lemma. *The map Φ from* (4.9.4) *is a one-sided inverse to* restr_1*:*

$$\Bigl(sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr) \xrightarrow{\Phi} sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\operatorname{restr}_1} sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr)\Bigr) = \mathrm{id}\,.$$

Proof. Recall that restr_1 is the componentwise map

$$\begin{aligned}\bigl((\iota\boxtimes 1)M_{01},(j\boxtimes 1)M_{01}\bigr) : sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) &\to sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr)\\ r=(r_k)_k &\mapsto (\iota r, jr) = \bigl((\iota_1^{\otimes k}r_k)_{k\geqslant 0},(j_1^{\otimes k}r_k)_{k=0,1}\bigr)\end{aligned}$$

Also $(\pi\boxtimes 1)M_{01} : sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) \hookrightarrow sA_{\infty}(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g)$, $r=(r_k)_k \mapsto \pi r = (\pi_1^{\otimes k}r_k)_k$ is componentwise. Introduce another componentwise map of degree 0

$$\begin{aligned}L^i : sA_{\infty}(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) &\to sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr),\\ q=(q_k)_k &\mapsto \bigl((i^{\otimes k}q_k)_{k\geqslant 0},(\mathrm{in}^{\mathcal{Q}\otimes k}\,q_k)_{k=0,1}\bigr).\end{aligned}$$

As $\iota_1 = \bigl(s\mathcal{D} \xhookrightarrow{i} s\mathcal{F}\mathcal{Q} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$ and $j_1 = \bigl(s\mathcal{Q} \xhookrightarrow{\mathrm{in}^{\mathcal{Q}}} s\mathcal{F}\mathcal{Q} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$, the lower triangle in the following diagram commutes:

$$\begin{tikzcd}
sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr) \arrow[dd, "\Phi\cdot\operatorname{restr}_1"'] \arrow[dr, "\Phi"] \arrow[ddrr, "u"] & & \\
 & sA^{\psi u}_{\infty}(\mathcal{E},\mathcal{A})(f,g) \arrow[dl, "\operatorname{restr}_1"'] \arrow[dr, hook, "(\pi\boxtimes 1)M_{01}"] & \\
sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr) & & sA_{\infty}(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f,\pi g) \arrow[ll, "L^i"]
\end{tikzcd} \tag{4.10.1}$$

Thus the whole diagram is commutative and $\Phi\cdot\operatorname{restr}_1 = uL^i$. We have proved in (4.9.3) and (4.5) that for all $(p,p')\in sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}})\bigr)$ and all $k\in\mathbb{Z}_{\geqslant 0}$ we have $i^{\otimes k}((p,p')u_k) = p_k$. Similarly,

$$\mathrm{in}^{\mathcal{Q}}[(p,p')u_1] = (p,p')u'\,\mathrm{pr}_1 = p'_1.$$

Therefore,

$$(p,p')\Phi\operatorname{restr}_1 = \bigl((i^{\otimes k}(p,p')u_k)_{k\geqslant 0},(\mathrm{in}^{\mathcal{Q}\otimes k}(p,p')u_k)_{k=0,1}\bigr) = \bigl((p_k)_{k\geqslant 0},(p'_k)_{k=0,1}\bigr) = (p,p'),$$

and the equation $\Phi\cdot\operatorname{restr}_1 = \mathrm{id}_{sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})((f|_{\mathcal{D}},f|_{\mathcal{Q}}),(g|_{\mathcal{D}},g|_{\mathcal{Q}}))}$ is proven. □

4.11 Lemma. *Denote by v the chain map*

$$v = \mathrm{id} - \mathrm{restr}_1 \cdot \Phi : sA_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA_\infty(\mathcal{E},\mathcal{A})(f,g).$$

Denote by w the chain map

$$w = \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{v} sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xhookrightarrow{(\pi\boxtimes 1)M_{01}} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g)\bigr].$$

There exists a unique homotopy $h : sA_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g)$ of degree -1 such that $w = B_1h + hB_1$,

$$\begin{aligned} h' = \bigl(sA_\infty(\mathcal{E},\mathcal{A})(f,g) &\xrightarrow{h} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g) \\ &\xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{FQ},\mathcal{A})(\pi f,\pi g) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr) = 0, \end{aligned}$$

$$h_k = 0 : sA_\infty(\mathcal{E},\mathcal{A})(f,g) \to \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \mathsf{C}(s\mathcal{FQ}^{\otimes k}(X,Y), s\mathcal{A}(Xf,Yg)), \qquad \textit{for } k>1.$$

Proof. We use Corollary 2.7, setting $P = sA_\infty(\mathcal{E},\mathcal{A})(f,g)$. We have

$$\begin{aligned} w = {}& \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xhookrightarrow{(\pi\boxtimes 1)M_{01}} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g)\bigr] \\ & - \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_1} sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_\mathcal{D},f|_\mathcal{Q}),(g|_\mathcal{D},g|_\mathcal{Q})\bigr) \xrightarrow{u} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g)\bigr] \end{aligned}$$

due to (4.9.4). Due to (2.6.1), w' defined in condition 1 of Corollary 2.7 is

$$\begin{aligned} w' = {}& \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xhookrightarrow{(\pi\boxtimes 1)M_{01}} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g) \\ & \qquad \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{FQ},\mathcal{A})(\pi f,\pi g) \xrightarrow{\mathrm{restr}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] \\ & - \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_1} sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_\mathcal{D},f|_\mathcal{Q}),(g|_\mathcal{D},g|_\mathcal{Q})\bigr) \xrightarrow[\mathrm{restr}_\mathcal{Q}]{u'} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr], \end{aligned}$$

where $j : \mathcal{Q} \hookrightarrow \mathcal{E}$ is the embedding A_1-functor, $j_1 = \bigl(s\mathcal{Q} \hookrightarrow s\mathcal{FQ} \xrightarrow{\pi_1} s\mathcal{E}\bigr)$. We get

$$\begin{aligned} w' = {}& \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{E},\mathcal{A})(f,g) \\ & \qquad \xhookrightarrow{(\pi\boxtimes 1)M_{01}} sA_1(\mathcal{FQ},\mathcal{A})(\pi f,\pi g) \xrightarrow{(\mathrm{in}^\mathcal{Q}\boxtimes 1)M_{01}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] \\ & - \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{((\iota\boxtimes 1)M_{01},(j\boxtimes 1)M_{01})} sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f|_\mathcal{D},f|_\mathcal{Q}),(g|_\mathcal{D},g|_\mathcal{Q})\bigr) \\ & \qquad \xrightarrow{\mathrm{restr}_\mathcal{Q}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] \\ = {}& \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{(j\boxtimes 1)M_{01}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] \\ & - \bigl[sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\mathrm{restr}_{\leqslant 1}} sA_1(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{(j\boxtimes 1)M_{01}} sA_1(\mathcal{Q},\mathcal{A})(jf,jg)\bigr] = 0. \end{aligned}$$

Therefore, $h' = 0$ satisfies $B_1h' + h'B_1 = 0 = w'$. Hence, the unique homotopy h is constructed by Corollary 2.7. □

4.12 Remark. In the case of Lemma 4.11 $h \cdot \mathrm{pr}_k = h_k = 0$ if $k > 1$ or if $k = 0$. Indeed, (2.7.1) together with $h' = 0$ implies that $h_0 = h\,\mathrm{pr}_0 = h'\,\mathrm{pr}_0 = 0$, moreover,

$$rh_1|_{s\mathcal{Q}} = rh'\,\mathrm{pr}_1 = 0 : s\mathcal{Q}(X,Y) = s\mathcal{F}_|\mathcal{Q}(X,Y) \to s\mathcal{A}(Xf,Yg), \tag{4.12.1}$$

where $r \in sA_\infty(\mathcal{E},\mathcal{A})(f,g)$. Therefore, recurrent formula (2.7.2) simplifies here to

$$(-)^r b_k^{\mathcal{FQ}}(rh_1) = rw_k - \sum_{a+1+c=k}^{m,n} (\pi_1^{\otimes a} f_{am} \otimes rh_1 \otimes \pi_1^{\otimes c} g_{cn}) b_{m+1+n}^{\mathcal{A}} :$$
$$(s\mathcal{FQ})^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg). \tag{4.12.2}$$

4.13 Lemma. *The homotopy h constructed in Lemma 4.11 factorizes as*

$$h = \bigl(sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{\eta} sA_\infty(\mathcal{E},\mathcal{A})(f,g) \xrightarrow{(\pi\boxtimes 1)M_{01}} sA_\infty(\mathcal{FQ},\mathcal{A})(\pi f,\pi g)\bigr)$$

for a unique homotopy η of degree -1 such that $v = B_1\eta + \eta B_1$.

Proof. Let us show that h satisfies conditions of Corollary 2.10. The second is obvious. The first is $R_{\mathcal{D}}(rh_1) = 0$ for any $r \in sA_\infty(\mathcal{E},\mathcal{A})(f,g)$, that is,

$$(-)^r(i^{\otimes k} b_k^{\mathcal{FQ}} - b_k^{\mathcal{D}} i)(rh_1) = 0 : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg). \tag{4.13.1}$$

From (4.12.2) we find the formula for $k > 1$

$$(-)^r i^{\otimes k} b_k^{\mathcal{FQ}}(rh_1) = i^{\otimes k}(rw_k) - \sum_{a+1+c=k}^{m,n} (\iota_1^{\otimes a} f_{am} \otimes i(rh_1) \otimes \iota_1^{\otimes c} g_{cn}) b_{m+1+n}^{\mathcal{A}}$$
$$= \iota_1^{\otimes k} r_k - i^{\otimes k}[(\iota r, jr)u_k] - \sum_{a+1+c=k}^{m,n} (\overline{f}_{am} \otimes i(rh_1) \otimes \overline{g}_{cn}) b_{m+1+n}^{\mathcal{A}} : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg).$$

A particular case of (4.12.1) is

$$i(rh_1) = [\mathrm{in}^{\mathcal{D}}(rh')]\,\mathrm{pr}_1 = 0, \tag{4.13.2}$$

due to $h' = 0$, where the A_1-functor $\mathrm{in}^{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{Q}$ is the natural embedding. For our concrete choice of u_k we get

$$(-)^r i^{\otimes k} b_k^{\mathcal{FQ}}(rh_1) = \iota_1^{\otimes k} r_k - i^{\otimes k} \varpi^{\otimes k} \iota_1^{\otimes k} r_k = 0 : s\mathcal{D}^{\otimes k}(X,Y) \to s\mathcal{A}(Xf,Yg),$$

since $\bigl(s\mathcal{D} \xrightarrow{i} s\mathcal{FQ} \xrightarrow{\varpi} s\mathcal{D}\bigr) = \mathrm{id}$. Therefore, $i^{\otimes k} b_k^{\mathcal{FQ}}(rh_1) = 0$ and $b_k^{\mathcal{D}} i(rh_1) = 0$ due to (4.13.2). We conclude that (4.13.1) is satisfied, and by Corollary 2.10 there exists a homotopy

$$\eta : sA_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA_\infty(\mathcal{E},\mathcal{A})(f,g),$$

such that $\deg \eta = -1$, $h = \eta \cdot [(\pi \boxtimes 1)M_{01}]$ and $v = B_1\eta + \eta B_1$. $\square$

Lemmata 4.10 and 4.13 show that the maps restr_1 and Φ given by (4.7.2) and (4.9.4) are homotopy inverse to each other.

The A_∞-functor restr is surjective on objects by Proposition 4.6, and its first component is a homotopy isomorphism. Therefore, it is an A_∞-equivalence by Corollary 1.9, and Theorem 4.7 is proven. $\square$

4.14 Corollary. *The collection of A_∞-functors* (4.7.1) *is natural A_∞-2-equivalence.*

Proof. The restriction A_∞-functors $A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to A^{\psi u}_\infty(\mathcal{D},\mathcal{A})$ and $A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to A_1(\mathcal{Q},\mathcal{A})$ are strict A_∞-2-transformations. By (4.2.1) the restriction A_∞-functor $A^{\psi u}_\infty(\mathcal{E},\mathcal{A}) \to A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ is also a strict A_∞-2-transformation. It is an A_∞-equivalence by Theorem 4.7. $\square$

4.15 Remark. The maps Φ, η constructed in the proof of Theorem 4.7 satisfy

$$\Phi \cdot \eta = 0 : sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \to sA^{\psi u}_\infty(\mathcal{E},\mathcal{A})(f,g).$$

Indeed, $\Phi \cdot \eta$ composed with an embedding,

$$\Phi \cdot \eta \cdot (\pi \boxtimes 1)M_{01} = \Phi \cdot h : sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big) \to sA_\infty(\mathcal{F}\mathcal{Q},\mathcal{A})(\pi f, \pi g),$$

is a degree -1 homotopy such that

$$\begin{aligned} B_1(\Phi h) + (\Phi h)B_1 &= \Phi(B_1 h + h B_1) = \Phi w \\ &= \Phi(\mathrm{id} - \mathrm{restr}_1 \Phi)(\pi \boxtimes 1)M_{01} = (\mathrm{id} - \Phi\, \mathrm{restr}_1)\Phi(\pi \boxtimes 1)M_{01} = 0. \end{aligned}$$

We have $\Phi h \,\mathrm{pr}_k = \Phi \cdot h_k = 0$ for $k > 1$ and

$$(\Phi h)' = \Phi h \,\mathrm{restr}_{\leqslant 1} \mathrm{restr} = \Phi \cdot h' = 0.$$

The 0 homotopy for 0 chain map also has these properties, and by Corollary 2.7 we conclude that $\Phi h = 0$.

4.16 Remark. The equation

$$\eta \cdot \mathrm{restr}_1 = 0 : sA^{\psi u}_\infty(\mathcal{E},\mathcal{A})(f,g) \to sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big)$$

also holds. Indeed, the decomposition $\mathrm{restr}_1 = (\pi \boxtimes 1)M_{01} \cdot L^i$ from diagram (4.10.1) implies that

$$\eta \cdot \mathrm{restr}_1 = \eta \cdot (\pi \boxtimes 1)M_{01} \cdot L^i = h \cdot L^i.$$

For any $r \in sA^{\psi u}_\infty(\mathcal{E},\mathcal{A})(f,g)$ all components of the element

$$rhL^i = \big((i^{\otimes k}(rh_k))_{k \geqslant 0}, (\mathrm{in}^{\mathcal{Q}\otimes k}(rh_k))_{k=0,1}\big) \in sA^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\big((f|_\mathcal{D}, f|_\mathcal{Q}),(g|_\mathcal{D}, g|_\mathcal{Q})\big)$$

vanish except, possibly, those indexed by $k = 1$ by Remark 4.12. However, $i(rh_1) = 0$ by (4.13.2), and, moreover, $\mathrm{in}^\mathcal{Q}(rh_1) = 0$ by (4.12.1), thus, all the components of rhL^i vanish.

4.17 Remark. We have not used in the proof of Theorem 4.7 the assumption of pseudounitality of $\mathcal{D}$ and $\mathcal{A}$. Its assertion holds without this property. The restriction A_∞-functor

$$\mathrm{restr} : A_\infty(\mathcal{E}, \mathcal{A}) \to A_{\infty 1}(\mathcal{D}, \mathcal{Q}; \mathcal{A})$$

is an A_∞-equivalence, surjective on objects. Its first component maps admit a chain splitting. In the particular case $\mathcal{D}(X, Y) = 0$ for all $X, Y \in \operatorname{Ob}\mathcal{Q}$ we get Theorem 2.12 of [LM03]: the A_∞-functor

$$\mathrm{restr} : A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \to A_1(\mathcal{Q}, \mathcal{A})$$

is an A_∞-equivalence.

5. Relatively free A_∞-categories

Hinich [Hin97] defines standard cofibrations of differential graded algebras. This notion is generalized by Drinfeld to semi-free differential graded categories [Dri02]. We give a definition in the spirit of these two definitions in the framework of A_∞-categories.

5.1 Definition. Let $e : \mathcal{C} \to \mathcal{D}$ be a strict A_∞-functor such that $\operatorname{Ob} e$ is an isomorphism, and $e_1 : s\mathcal{C} \to s\mathcal{D}$ is an embedding. The A_∞-category $\mathcal{D}$ is *relatively free* over $\mathcal{C}$, if it can be represented as the union of an increasing sequence of its A_∞-subcategories $\mathcal{D}_j$ and differential graded subquivers $\mathcal{Q}_j$

$$\mathcal{D}_0 \subset \mathcal{Q}_1 \subset \mathcal{D}_1 \subset \mathcal{Q}_2 \subset \mathcal{D}_2 \subset \mathcal{Q}_3 \subset \cdots \subset \mathcal{D} \tag{5.1.1}$$

with the same set of objects $\operatorname{Ob}\mathcal{D}$, such that

1. $s\mathcal{D}_0 = (s\mathcal{C})e_1$;
2. for each $j \geqslant 0$ the embedding of graded quivers $\mathcal{D}_j \hookrightarrow \mathcal{Q}_{j+1}$ admits a splitting map $\mathcal{Q}_{j+1} \longrightarrow \mathcal{D}_j$ of degree 0;
3. for each $j > 0$ the unique strict A_∞-functor $\mathcal{F}\mathcal{Q}_j \to \mathcal{D}_j$ extending the embedding $\mathcal{Q}_j \hookrightarrow \mathcal{D}_j$ factors into the natural projection and an isomorphism

   $$\mathcal{F}\mathcal{Q}_j \longrightarrow \mathcal{F}\mathcal{Q}_j/s^{-1}(R_j) \xrightarrow{\sim} \mathcal{D}_j,$$

   where the system of relations $R_j = R_{\mathcal{D}_{j-1}} \subset s\mathcal{F}\mathcal{Q}_j$ is $R_j = \sum_{n\geqslant 2} \operatorname{Im}(\delta_n)$ for

   $$\delta_n = \big((s\mathcal{D}_{j-1})^{\otimes n} \hookrightarrow (s\mathcal{F}\mathcal{Q}_j)^{\otimes n} \xrightarrow{b_n^{\mathcal{F}\mathcal{Q}_j}} s\mathcal{F}\mathcal{Q}_j\big) - \big((s\mathcal{D}_{j-1})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_{j-1}}} s\mathcal{D}_{j-1} \hookrightarrow s\mathcal{F}\mathcal{Q}_j\big).$$

When all differential graded quivers $\mathcal{N}_j = \operatorname{Coker}(\mathcal{D}_{j-1} \hookrightarrow \mathcal{Q}_j)$ have zero differential and the $\Bbbk$-modules $\mathcal{N}_j^k(X, Y)$ are free for all $j \geqslant 1$, $k \in \mathbb{Z}$, we say that $\mathcal{D}$ is *semi-free* over $\mathcal{C}$ in accordance with terminology of Drinfeld. In fact, if in Definition 5.1 one replaces

A_∞-categories with differential graded categories and adds the above assumption on $\mathcal{N}_j$, then one recovers Definition 13.4 from [Dri02] of semi-free differential graded categories.

The system of relations R_j is the minimal one that ensures that the natural embedding $s\mathcal{D}_{j-1} \hookrightarrow s\mathcal{FQ}_j/(R_j) = s\mathcal{D}_j$ is the first component of a strict A_∞-functor. In semi-free case we may say that $\mathcal{D}_j$ is freely generated by $\mathcal{N}_j$ over $\mathcal{D}_{j-1}$.

5.2. The main construction. Let $\mathcal{C}$ be a unital A_∞-category, and let $\mathcal{B} \subset \mathcal{C}$ be its full subcategory. The unit $\mathbf{i}_0^{\mathcal{C}}$ is abbreviated to $\mathbf{i}_0$.

A vertex of a tree is k-ary if it is adjacent to $k+1$ edges. A unary vertex is a 1-ary one.

Define a *labeled tree* $t = (t; X_0, X_1, \dots, X_n)$ as a non-empty (non-reduced) plane rooted tree t with n leaves, such that unary vertices are not joined by an edge, equipped with a sequence $(X_0, X_1, \dots, X_n)$ of objects of $\mathcal{C}$.

Let e be an edge of t. If i is the smallest number such that i-th leaf is above e, the *domain* of e is defined as $\operatorname{dom}(e) = X_{i-1}$. If k is the biggest number such that k-th leaf is above e, the *codomain* of e is defined as $\operatorname{codom}(e) = X_k$. An *admissible tree* is a labeled tree $(t; X_0, X_1, \dots, X_n)$ such that for each edge e adjacent to a unary vertex $\operatorname{dom}(e) \in \operatorname{Ob}\mathcal{B}$ or $\operatorname{codom}(e) \in \operatorname{Ob}\mathcal{B}$ (or both).

The set of vertices $V(t)$ of a rooted tree t has a canonical ordering: $x \preccurlyeq y$ iff the minimal path connecting the root with y contains x. A $\mathcal{C}$-*admissible tree* is an admissible tree $(t; X_0, X_1, \dots, X_n)$ such that top (maximal with respect to $\preccurlyeq$) internal vertices are unary.

Define a graded quiver $\mathcal{E}$ with the set of objects $\operatorname{Ob}\mathcal{E} = \operatorname{Ob}\mathcal{C}$. The $\mathbb{Z}$-graded $\Bbbk$-module of morphisms between $X, Y \in \operatorname{Ob}\mathcal{E}$ is defined as

$$s\mathcal{E}(X,Y) = \bigoplus_{n\geqslant 1} \bigoplus_{\text{admissible } (t;X_0,X_1,\dots,X_n)}^{X_0=X,\, X_n=Y} s\mathcal{E}(t)(X,Y), \tag{5.2.1}$$
$$s\mathcal{E}(t)(X_0,X_n) = s\mathcal{E}(t) = s\mathcal{C}(X_0,X_1) \otimes \dots \otimes s\mathcal{C}(X_{n-1},X_n)\big[|t|_1 - |t|_>\big],$$

where $|t|_1$ is the number of unary internal vertices of t, and $|t|_>$ is the the number of internal vertices of arity > 1 .

The vertices of arity $k > 1$ are interpreted as k-ary multiplications of degree 1. Unary vertices are interpreted as contracting homotopies H of degree -1. Define an A_∞-structure on $\mathcal{E}$ in which operations b_k, $k > 1$, are given by grafting. So for $k > 1$ the operation b_k is a direct sum of maps

$$b_k = s^{|t_1|} \otimes \dots \otimes s^{|t_{k-1}|} \otimes s^{|t_k|-|t|} : s\mathcal{E}(t_1)(Y_0,Y_1) \otimes \dots \otimes s\mathcal{E}(t_k)(Y_{k-1},Y_k) \to s\mathcal{E}(t)(Y_0,Y_k),$$

where $|t| = |t|_> - |t|_1$ and $t = (t_1 \sqcup \dots \sqcup t_k) \cdot \mathfrak{t}_k$. In particular, $|t| = |t_1| + \dots + |t_k| + 1$.

Let $t = (t; X_0, X_1, \dots, X_n)$ be an admissible tree, whose lowest internal vertex is not unary. In particular, t might be the trivial tree $t = (|; X_0, X_1)$. Assume that $X_0 \in \operatorname{Ob}\mathcal{B}$

or $X_n \in \operatorname{Ob} \mathcal{B}$ (or both). Denote by H the $\Bbbk$-linear map

$$H = s : s\mathcal{E}(t)(X_0, X_n) = s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{n-1}, X_n)\bigl[|t|_1 - |t|_>\bigr] \\ \to s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{n-1}, X_n)\bigl[1 + |t|_1 - |t|_>\bigr] = s\mathcal{E}(t \cdot \mathfrak{t}_1)(X_0, X_n)$$

of degree -1. Here $\mathfrak{t}_1 = (\mid; X_0, X_n)$ is the unary corolla.

The operation b_1 is determined by the given differential $b_1 : s\mathcal{C} \to s\mathcal{C}$ and by the recursive substitutions

$$b_k b_1 := - \sum_{\alpha+p+\beta=k}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta}) b_{\alpha+1+\beta}, \qquad k > 1, \tag{5.2.2}$$

$$H b_1 := 1 - b_1 H, \tag{5.2.3}$$

where H stands for a unary vertex. Identity (5.2.2) is satisfied for $k > 1$ by definition of b_1. We have to prove that $b_1^2 = 0$. Assume that $k > 1$, then

$$\begin{aligned} b_k b_1^2 &= - \sum_{\alpha+p+\beta=k}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta}) b_{\alpha+1+\beta} b_1 \\ &= \sum_{\substack{\alpha+p+\beta=k \\ \gamma+q+\delta=\alpha+1+\beta}}^{\gamma+\delta>0} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta})(1^{\otimes\gamma} \otimes b_q \otimes 1^{\otimes\delta}) b_{\gamma+1+\delta} \\ &= \sum_{\alpha+p+\varepsilon+q+\delta=k} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\varepsilon} \otimes b_q \otimes 1^{\otimes\delta}) b_{\alpha+\varepsilon+\delta+2} \\ &\quad - \sum_{\gamma+q+\eta+p+\beta=k} (1^{\otimes\gamma} \otimes b_q \otimes 1^{\otimes\eta} \otimes b_p \otimes 1^{\otimes\beta}) b_{\gamma+\eta+\beta+2} \\ &\quad + \sum_{\gamma+r+\delta=k}^{\gamma+\delta>0} \Bigl\{ 1^{\otimes\gamma} \otimes \Bigl[\sum_{\kappa+p+\lambda=r} (1^{\otimes\kappa} \otimes b_p \otimes 1^{\otimes\lambda}) b_{\kappa+1+\lambda} \Bigr] \otimes 1^{\otimes\delta} \Bigr\} b_{\gamma+1+\delta} \\ &= \sum_{\gamma+1+\delta=k} (1^{\otimes\gamma} \otimes b_1^2 \otimes 1^{\otimes\delta}) b_k, \end{aligned}$$

because the sum in square brackets vanishes for $r > 1$ by (5.2.2). We also have

$$H b_1^2 = (1 - b_1 H) b_1 = b_1 - b_1(1 - b_1 H) = b_1^2 H.$$

By induction the equation $b_1^2\big|_{s\mathcal{C}} = 0$ implies that $b_1^2 = 0$ on $s\mathcal{E}$. Therefore, $\mathcal{E}$ is an A_∞-category.

It has an ideal $(R_\mathcal{C})_+$, generated by the $\Bbbk$-subquiver $R_\mathcal{C} = \sum_{n \geqslant 2} \operatorname{Im}(\delta_n)$ for

$$\delta_n = \bigl((s\mathcal{C})^{\otimes n} \hookrightarrow (s\mathcal{E})^{\otimes n} \xrightarrow{b_n^\mathcal{E}} s\mathcal{E}\bigr) - \bigl((s\mathcal{C})^{\otimes n} \xrightarrow{b_n^\mathcal{C}} s\mathcal{C} \hookrightarrow s\mathcal{E}\bigr)$$

by application of operations $1^{\otimes\alpha}\otimes H\otimes 1^{\otimes\beta}$, $1^{\otimes\alpha}\otimes b_p\otimes 1^{\otimes\beta}$ for $p \geqslant 2$. By Lemma 3.1 $R_{\mathcal{C}} b_1^{\mathcal{E}} \subset (R_{\mathcal{C}}) \subset (R_{\mathcal{C}})_+$, where $(R_{\mathcal{C}})$ denotes the ideal generated by application of $1^{\otimes\alpha}\otimes b_p\otimes 1^{\otimes\beta}$ $(p \geqslant 2)$ only. Similarly to Proposition 2.1 this implies that $(R_{\mathcal{C}})_+ b_1^{\mathcal{E}} \subset (R_{\mathcal{C}})_+$. Indeed, let $t = (t; X_0, X_1, \dots, X_n)$ be an admissible tree, whose lowest internal vertex is not unary. Assume that $X_0 \in \operatorname{Ob}\mathcal{B}$ or $X_n \in \operatorname{Ob}\mathcal{B}$, so that $t\cdot\mathfrak{t}_1$ is admissible. For an arbitrary $z \in (R_{\mathcal{C}})_+(t)(X_0, X_n)$ there exists $zH \in (R_{\mathcal{C}})_+(t\cdot\mathfrak{t}_1)(X_0, X_n)$, and $zb_1 \in (R_{\mathcal{C}})_+(X_0, X_n)$ by induction. Due to (5.2.3) $zHb_1 = z - zb_1H \in (R_{\mathcal{C}})_+$, which proves the claim. Therefore, the ideal $(R_{\mathcal{C}})_+$ is stable with respect to all A_∞-operations, including b_1.

Denote by $\mathcal{D} = \mathcal{E}/s^{-1}(R_{\mathcal{C}})_+ = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ the quotient A_∞-category. It has a direct sum decomposition similar to that of $\mathcal{E}$

$$s\mathcal{D}(X,Y) = \bigoplus_{n\geqslant 1} \bigoplus_{\mathcal{C}\text{-admissible } (t;X_0,X_1,\dots,X_n)}^{X_0=X,\quad X_n=Y} s\mathcal{D}(t)(X,Y),$$

$$s\mathcal{D}(t)(X_0,X_n) = s\mathcal{E}(t)(X_0,X_n) = s\mathcal{C}(X_0,X_1)\otimes\dots\otimes s\mathcal{C}(X_{n-1},X_n)\bigl[|t|_1 - |t|_>\bigr],$$

with the only difference that the sum is taken over $\mathcal{C}$-admissible trees t. We can view $\mathcal{D}$ as a graded $\Bbbk$-subquiver of $\mathcal{E}$.

The category $\mathcal{C}$ is embedded in $\mathcal{D}$ (via a strict A_∞-functor) as

$$s\mathcal{D}_0(X,Y) = s\mathcal{D}(|)(X,Y) = s\mathcal{C}(X,Y)$$

for the trivial tree $t = (|; X, Y)$. Let us show that $\mathcal{D}$ is relatively free over $\mathcal{C}$.

Let us define for $j \geqslant 0$ the A_∞-subcategories $\mathcal{D}_j$ and differential graded subquivers $\mathcal{Q}_{j+1}$ of $\mathcal{D}$ so that embeddings (5.1.1) hold. Each leaf ℓ and the root of a tree can be connected by the unique minimal path. We say that internal vertices occurring at this path are between the root and the leaf ℓ. Define for $j \geqslant 0$ the A_∞-subcategory $\mathcal{D}_j = \oplus_t \mathcal{D}(t)$ of $\mathcal{D}$, where the summation goes over all

$\mathcal{C}$-admissible trees t with no more than j unary internal vertices between the root and any leaf. (C1)

Define for $j \geqslant 1$ the graded subquiver $\mathcal{N}_j = \oplus_t \mathcal{D}(t)$ of $\mathcal{D}$, where the summation goes over all trees t satisfying (C1) and such that

there exists a leaf ℓ of t with j unary internal vertices between the root and ℓ; the lowest internal vertex (adjacent to the root) is unary. (C2)

One can easily see that for $j \geqslant 1$

$$s\mathcal{Q}_j = s\mathcal{D}_{j-1} \oplus s\mathcal{N}_j$$

is a differential graded subquiver of $\mathcal{D}_j \subset \mathcal{D}$. For example, $\mathcal{D}_0 = \mathcal{D}(|) = \mathcal{C}$, $\mathcal{N}_1 = \mathcal{D}\bigl(\dot{|}\bigr)$, $\mathcal{Q}_1 = \mathcal{D}(|) \oplus \mathcal{D}\bigl(\dot{|}\bigr)$, and $\mathcal{D}_1 = \mathcal{D}(|) \oplus \oplus_t \mathcal{D}(t)$, where t runs over admissible trees with the only unary internal vertex v, such that all other internal vertices lie on the minimal path between the root and v.

The inclusion map of differential graded quivers $i : s\mathcal{Q}_j \hookrightarrow s\mathcal{D}_j$ induces a unique strict A_∞-functor $\hat{\imath} : \mathcal{F}\mathcal{Q}_j \to \mathcal{D}_j$ [LM03, Corollary 2.4].

5.3 Proposition. *The map $\hat\imath_1 : s\mathcal{F}\mathcal{Q}_j \to s\mathcal{D}_j$ is surjective and its kernel is $(R_{\mathcal{D}_{j-1}})$. Thus it induces an isomorphism $\iota_1 : s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \to s\mathcal{D}_j$.*

Proof. The strict A_∞-functor $\hat\imath : \mathcal{F}\mathcal{Q}_j \to \mathcal{D}_j$ is described in [LM03, Section 2.6] as follows. Let t be a reduced labeled tree, with n input leaves, and let $\leqslant$ be a linear order on $\operatorname{Vert}(t)$, such that $x \preccurlyeq y$ implies $x \leqslant y$ for all $x, y \in \operatorname{Vert}(t)$. The choice of $\leqslant$ is equivalent to the choice of decomposition into product of elementary forests (2.0.1). The linearly ordered tree $(t, \leqslant)$ determines the map $b^{\mathcal{F}\mathcal{Q}_j}_{(t,\leqslant)} : (s\mathcal{F}\mathcal{Q}_j)^{\otimes n} \to s\mathcal{F}\mathcal{Q}_j$ given by (2.0.2) and a similar map $b^{\mathcal{D}_j}_{(t,\leqslant)} : (s\mathcal{D}_j)^{\otimes n} \to s\mathcal{D}_j$. In the commutative diagram from [LM03, Section 2.6]

$$\begin{tikzcd}
(s\mathcal{Q}_j)^{\otimes n} \arrow[r, "b^{\mathcal{F}\mathcal{Q}_j}_{(t,\leqslant)}", "\pm s^{-|t|}"'] \arrow[d, "i_1^{\otimes n}"'] & s\mathcal{F}_t\mathcal{Q}_j \arrow[d, "\hat\imath_1"] \\
(s\mathcal{D}_j)^{\otimes n} \arrow[r, "b^{\mathcal{D}_j}_{(t,\leqslant)}"] & s\mathcal{D}_j
\end{tikzcd} \tag{5.3.1}$$

the top map is invertible, so $\hat\imath_1$ is uniquely determined by this diagram.

Being the first component of a strict A_∞-functor $\hat\imath_1$ satisfies, in particular, the equation

$$\begin{aligned}
&\bigl((s\mathcal{D}_{j-1})^{\otimes n} \hookrightarrow (s\mathcal{F}_{|}\mathcal{Q}_j)^{\otimes n} \xrightarrow{b_n^{\mathcal{F}\mathcal{Q}_j}} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\hat\imath_1} s\mathcal{D}_j\bigr) = \bigl((s\mathcal{D}_{j-1})^{\otimes n} \xhookrightarrow{i_1^{\otimes n}} (s\mathcal{D}_j)^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_j}} s\mathcal{D}_j\bigr) \\
&= \bigl((s\mathcal{D}_{j-1})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_{j-1}}} s\mathcal{D}_{j-1} \xhookrightarrow{i_1} s\mathcal{D}_j\bigr) = \bigl((s\mathcal{D}_{j-1})^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_{j-1}}} s\mathcal{D}_{j-1} \hookrightarrow s\mathcal{F}_{|}\mathcal{Q}_j \xrightarrow{\hat\imath_1} s\mathcal{D}_j\bigr).
\end{aligned}$$

It implies $R_{\mathcal{D}_{j-1}}\hat\imath_1 = 0$. Since $\hat\imath$ is strict we have also $(R_{\mathcal{D}_{j-1}})\hat\imath_1 = 0$. Thus there is a strict A_∞-functor ι with the first component $\iota_1 : s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \to s\mathcal{D}_j$, identity on objects.

Let us construct a degree 0 map $\phi : s\mathcal{D}_j \to s\mathcal{F}\mathcal{Q}_j$ for $j \geqslant 1$. Let t be a tree that satisfies (C1). Denote by $\operatorname{UV}(t) \subset \operatorname{Vert}(t)$ the subset of unary internal vertices. Let $\operatorname{minUV}(t)$ be the subset of partially ordered set $(\operatorname{UV}(t), \preccurlyeq)$ consisting of minimal elements. Let $L \subset \operatorname{Leaf}(t)$ be the subset of leaves ℓ such that between ℓ and the root there are no unary vertices. Let $\overline{L}$ be the set of leaf vertices above leaves from L. Using the canonical linear ordering $t_< = (t, \leqslant)$ of $\overline{\operatorname{Vert}}(t)$ [LM03, Section 1.7] we can write the set $\overline{L} \sqcup \operatorname{minUV}(t)$ as $\{u_1 < \cdots < u_k\}$. For any $1 \leqslant p \leqslant k$ denote by t_p the $\mathcal{C}$-admissible subtree of t with

$$\overline{\operatorname{Vert}}(t_p) = \{y \in \overline{\operatorname{Vert}}(t) \mid y \succcurlyeq u_p\} \sqcup \{\text{new root vertex } r_p\}.$$

Edges of t_p are all edges of t above u_p plus a new root edge between u_p and r_p. In particular, if $u_p \in \overline{L}$, then $t_p = |$ is the trivial tree. Denote by t' the reduced labeled tree, which is t with all vertices and edges above $\operatorname{minUV}(t)$ removed. It has precisely k leaves. Thus t is the concatenation of a forest and t':

$$t = (t_1 \sqcup t_2 \sqcup \cdots \sqcup t_k) \cdot t'. \tag{5.3.2}$$

We have

$$\operatorname{Vert}(t) = \operatorname{Vert}(t') \sqcup \bigsqcup_{p=1}^{k} \operatorname{Vert}(t_p), \qquad \operatorname{Leaf}(t) = \bigsqcup_{p=1}^{k} \operatorname{Leaf}(t_p), \qquad \operatorname{Leaf}(t') = \bigsqcup_{p=1}^{k} \operatorname{Out}(t_p).$$

Correspondingly, the labels of the p-th leaf of t' are those of $\mathrm{Out}(t_p)$.

Being simply a shift, the $\Bbbk$-linear map $b_{t'_<}^{\mathcal{D}} : s\mathcal{D}(t_1)\otimes\cdots\otimes s\mathcal{D}(t_k) \to s\mathcal{D}(t)$ is invertible. Therefore, for any element $x \in s\mathcal{D}(t)$ there exists a unique tensor $\sum_i z_1^i \otimes \cdots \otimes z_k^i \in s\mathcal{D}(t_1)\otimes\cdots\otimes s\mathcal{D}(t_k)$ such that $x = \sum_i (z_1^i \otimes \cdots \otimes z_k^i) b_{t'_<}^{\mathcal{D}}$. We have $z_p^i \in s\mathcal{D}(t_p) \subset s\mathcal{Q}_j$, in particular, $z_p^i \in s\mathcal{D}(|) = s\mathcal{C}$, if $u_p \in \overline{L}$. Define

$$x\phi = \sum_i (z_1^i \otimes \cdots \otimes z_k^i) b_{t'_<}^{\mathcal{F}\mathcal{Q}_j} \in s\mathcal{F}_{t'}\mathcal{Q}_j.$$

Commutative diagram (5.3.1) implies that $x\phi\hat{\imath}_1 = \sum_i (z_1^i \otimes \cdots \otimes z_k^i) b_{t'_<}^{\mathcal{D}_j} = x$. Therefore,

$$\big[s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \xrightarrow{\iota_1} s\mathcal{D}_j\big] = \mathrm{id}\,. \tag{5.3.3}$$

Let us prove that $\phi\pi_1$ preserves the operations b_n for $n > 1$. Indeed,

$$b_n^{\mathcal{D}_j}(\phi\pi_1) - (\phi\pi_1)^{\otimes n} b_n = \big[(s\mathcal{D}_j)^{\otimes n} \xrightarrow{b_n^{\mathcal{D}_j}\phi - \phi^{\otimes n} b_n^{\mathcal{F}\mathcal{Q}_j}} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big].$$

Consider trees τ_1, τ_2, $\dots$, τ_n satisfying condition (C1), labeled so that the operation $b_n^{\mathcal{D}_j} : s\mathcal{D}(\tau_1)\otimes\cdots\otimes s\mathcal{D}(\tau_n) \to s\mathcal{D}_j$ makes sense. The quiver $(s\mathcal{D}_j)^{\otimes n}$ is a direct sum of such $s\mathcal{D}(\tau_1)\otimes\cdots\otimes s\mathcal{D}(\tau_n)$. If some of trees τ_p are not trivial, then $b_n^{\mathcal{D}_j}\phi = \phi^{\otimes n} b_n^{\mathcal{F}\mathcal{Q}_j}$, because constructing ϕ for $\tau = (\tau_1 \sqcup \tau_2 \sqcup \cdots \sqcup \tau_n)\cdot \mathfrak{t}_n$ is equivalent to decomposing each τ_p as in (5.3.2), collecting the upper parts, and gluing the lowest parts τ'_p into $\tau' = (\tau'_1 \sqcup \tau'_2 \sqcup \cdots \sqcup \tau'_n)\cdot \mathfrak{t}_n$. If all trees τ_p are trivial, then $\mathcal{D}(\tau_p) = \mathcal{D}(|) = \mathcal{C}$ and

$$(b_n^{\mathcal{C}}\phi - \phi^{\otimes n} b_n^{\mathcal{F}\mathcal{Q}_j})\pi_1 = (b_n^{\mathcal{C}} - b_n^{\mathcal{F}\mathcal{Q}_j})\pi_1 = 0 : (s\mathcal{C})^{\otimes n} \to s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}),$$

due to $R_{\mathcal{C}}\pi_1 \subset R_{\mathcal{D}_{j-1}}\pi_1 = 0$.

We claim that

$$\big[s\mathcal{F}\mathcal{Q}_j \xrightarrow{\hat{\imath}_1} s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] = \big[s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]. \tag{5.3.4}$$

First of all, the restriction of this equation to $s\mathcal{Q}_j = s\mathcal{F}_|\mathcal{Q}_j$ holds true:

$$\big[s\mathcal{Q}_j \xrightarrow{i_1} s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] = \big[s\mathcal{Q}_j \hookrightarrow s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]. \tag{5.3.5}$$

Indeed, $s\mathcal{Q}_j = s\mathcal{D}_{j-1} \oplus s\mathcal{N}_j$. On the first summand we get for $x \in s\mathcal{D}_{j-1}(t)$

$$\begin{aligned} x \xmapsto{i_1} x = \sum_i (z_1^i \otimes \cdots \otimes z_k^i) b_{t'_<}^{\mathcal{D}_j} &\xmapsto{\phi} \sum_i (z_1^i \otimes \cdots \otimes z_k^i) b_{t'_<}^{\mathcal{F}\mathcal{Q}_j} \\ \xmapsto{\pi_1} \sum_i (z_1^i \otimes \cdots \otimes z_k^i) b_{t'_<}^{\mathcal{F}\mathcal{Q}_j} + (R_{\mathcal{D}_{j-1}}) &= \sum_i (z_1^i \otimes \cdots \otimes z_k^i) b_{t'_<}^{\mathcal{D}_{j-1}} + (R_{\mathcal{D}_{j-1}}) = x\pi_1 \end{aligned}$$

by Proposition 3.2 because $z^i_p \in s\mathcal{D}_{j-1}$. On the second summand we get for $x \in s\mathcal{N}_j(t)$

$$x \xmapsto{i_1} x \xmapsto{\phi} x \xmapsto{\pi_1} x\pi_1$$

since $t' = |$. Thus, (5.3.5) is verified.

Now we prove (5.3.4) on the generic summand $s\mathcal{F}_\tau\mathcal{Q}_j$ of $s\mathcal{F}\mathcal{Q}_j$, where τ is a reduced labeled tree with n leaves. The first map below is an isomorphism:

$$\begin{aligned}
&\big[(s\mathcal{Q}_j)^{\otimes n} \xrightarrow{b^{\mathcal{F}\mathcal{Q}_j}_{\tau_<}} s\mathcal{F}_\tau\mathcal{Q}_j \xrightarrow{\hat{\imath}_1} s\mathcal{D}_j \xrightarrow{\phi\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] \\
&= \big[(s\mathcal{Q}_j)^{\otimes n} \xhookrightarrow{i_1^{\otimes n}} (s\mathcal{D}_j)^{\otimes n} \xrightarrow{b^{\mathcal{D}_j}_{\tau_<}} s\mathcal{D}_j \xrightarrow{\phi\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] \\
&= \big[(s\mathcal{Q}_j)^{\otimes n} \xhookrightarrow{i_1^{\otimes n}} (s\mathcal{D}_j)^{\otimes n} \xrightarrow{\phi\pi_1} (s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}))^{\otimes n} \xrightarrow{b_{\tau_<}} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] \\
&= \big[(s\mathcal{Q}_j)^{\otimes n} \hookrightarrow (s\mathcal{F}\mathcal{Q}_j)^{\otimes n} \xrightarrow{\pi_1^{\otimes n}} (s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}))^{\otimes n} \xrightarrow{b_{\tau_<}} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] \\
&= \big[(s\mathcal{Q}_j)^{\otimes n} \xrightarrow{b^{\mathcal{F}\mathcal{Q}_j}_{\tau_<}} s\mathcal{F}_\tau\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]
\end{aligned}$$

by (5.3.5) and by the fact that the considered maps $\hat{\imath}_1$, $\phi\pi_1$ and π_1 commute with $b_{\tau_<}$.

Rewriting (5.3.4) in the form

$$\begin{aligned}
\big[s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \xrightarrow{\iota_1} s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]& \\
= \big[s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big]&,
\end{aligned}$$

we find by surjectivity of π_1 that

$$\big[s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}}) \xrightarrow{\iota_1} s\mathcal{D}_j \xrightarrow{\phi} s\mathcal{F}\mathcal{Q}_j \xrightarrow{\pi_1} s\mathcal{F}\mathcal{Q}_j/(R_{\mathcal{D}_{j-1}})\big] = \mathrm{id}\,.$$

Together with (5.3.3) this proves that $\phi\pi_1$ is an inverse to ι_1. $\square$

5.4 Corollary. *The A_∞-category $\mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ is relatively free over $\mathcal{C}$.*

5.5. The first equivalence. Let $\mathcal{A}$ be pseudounital, then the restriction functor

$$A^{\psi u}_{\infty 1}(\mathcal{D}_0, \mathcal{Q}_1; \mathcal{A}) \to A^{\psi u}_\infty(\mathcal{C}, \mathcal{A}), \qquad (f, f') \mapsto f$$

takes values in the full subcategory $A^{\psi u}_\infty(\mathcal{C}, \mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$. Indeed, let (f, f') be an object of $A^{\psi u}_{\infty 1}(\mathcal{D}_0, \mathcal{Q}_1; \mathcal{A})$. For arbitrary objects X, Y of $\mathcal{B}$ we have

$$\begin{aligned}
f_1 &= \big[s\mathcal{C}(X,Y) \hookrightarrow s\mathcal{Q}_1(X,Y) \xrightarrow{f'_1} s\mathcal{A}(Xf, Yf)\big] \\
&= \big[s\mathcal{D}(|)(X,Y) \xrightarrow{b_1 H + H b_1} s\mathcal{Q}_1(X,Y) \xrightarrow{f'_1} s\mathcal{A}(Xf, Yf)\big] \\
&= \big[s\mathcal{D}(|)(X,Y) \xrightarrow{b_1(Hf'_1) + (Hf'_1)b_1} s\mathcal{A}(Xf, Yf)\big],
\end{aligned}$$

where H is the map $H : s\mathcal{D}(|) \xrightarrow{s} s\mathcal{D}(\mathfrak{t}_1) \hookrightarrow s\mathcal{Q}_1$. Hence, the above f_1 is null-homotopic. By Definition 6.4 of [LO02] the A_∞-functor $f\big|_{\mathcal{B}}$ is contractible.

A short exact sequence of chain maps of complexes is *semisplit* (resp. *semisplittable*) if it is split (resp. splittable) as a sequence of degree 0 maps of graded $\Bbbk$-modules.

5.6 Lemma. *Let $0 \to C \xrightarrow{\alpha} A \xrightarrow{\beta} B \to 0$ be a semisplittable exact sequence of complexes of $\Bbbk$-modules. If C is contractible, then this sequence is splittable, and the splitting chain map $\nu : B \to A$ can be chosen so that ν is homotopy inverse to β.*

Proof. Let $\phi : A \to C$ be a map of degree 0, such that $\alpha\phi = 1_C$. Assume that $1_C = Hd + dH$ for a homotopy $H : C \to C$ of degree -1. Then $\psi = (\phi H)d = \phi Hd + d\phi H : A \to C$ is a chain map, and $\alpha\psi = \alpha\phi Hd + d\alpha\phi H = Hd + dH = 1_C$. Denote by $\nu : B \to A$ the unique $\Bbbk$-linear map such that $\nu\psi = 0$, $\nu\beta = 1$. The splitting injection $\nu = \ker\psi$ is a chain map. The sequence looks as follows

$$0 \to C \underset{\psi}{\overset{\alpha}{\rightleftarrows}} A \underset{\nu}{\overset{\beta}{\rightleftarrows}} B \to 0.$$

Since A is a direct sum $C \oplus B$, we have

$$\mathrm{id}_A - \beta\nu = \psi\alpha = (\phi Hd + d\phi H)\alpha = (\phi H\alpha)d + d(\phi H\alpha) = \gamma d + d\gamma,$$

where $\gamma = \bigl(A \xrightarrow{\phi} C \xrightarrow{H} C \xrightarrow{\alpha} A\bigr)$ is a homotopy. □

5.7 Proposition. *Let $\mathcal{A}$ be unital, then the restriction strict A_∞-functor*

$$\mathrm{restr} : A^{\psi u}_{\infty 1}(\mathcal{D}_0, \mathcal{Q}_1; \mathcal{A}) \to A^u_\infty(\mathcal{C}, \mathcal{A})_{\mathrm{mod}\,\mathcal{B}} \tag{5.7.1}$$

is an A_∞-equivalence, surjective on objects. The chain surjections restr_1 *admit a chain splitting.*

Proof. First of all, restr is surjective on objects. Indeed, assume that $f : \mathcal{C} \to \mathcal{A}$ is unital and $\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{f} \mathcal{A}$ is contractible. We have to extend the chain maps $f_1 : s\mathcal{C}(X,Y) \to s\mathcal{A}(Xf,Yf)$ to chain maps $f_1' : s\mathcal{Q}_1(X,Y) \to s\mathcal{A}(Xf,Yf)$. If $X, Y \notin \mathrm{Ob}\,\mathcal{B}$, then $s\mathcal{Q}_1(X,Y) = s\mathcal{C}(X,Y)$ and $f_1' = f_1$. If $X \in \mathrm{Ob}\,\mathcal{B}$ or $Y \in \mathrm{Ob}\,\mathcal{B}$, then $s\mathcal{Q}_1(X,Y) = s\mathcal{C}(X,Y) \oplus s\mathcal{D}(\mathsf{t}_1)(X,Y)$ as a graded quiver, and the complex $\mathcal{A}(Xf,Yf)$ is contractible by Proposition 6.1(C1), (C2) of [LO02]. Let $\chi_{XY} : s\mathcal{A}(Xf,Yf) \to s\mathcal{A}(Xf,Yf)$ be a contracting homotopy for $\mathcal{A}(Xf,Yf)$. Define

$$f_1' = \bigl(s\mathcal{D}(\mathsf{t}_1)(X,Y) \xrightarrow{s^{-1}} s\mathcal{C}(X,Y) \xrightarrow{f_1} s\mathcal{A}(Xf,Yf) \xrightarrow{\chi_{XY}} s\mathcal{A}(Xf,Yf)\bigr).$$

Then $Hf_1' = f_1\chi_{XY}$. Since $H = s : s\mathcal{C} \to s\mathcal{D}(\mathsf{t}_1)$ is invertible, the equation

$$Hf_1'b_1 - Hb_1f_1' = Hf_1'b_1 + b_1Hf_1' - f_1' = f_1\chi_{XY}b_1 + b_1f_1\chi_{XY} - f_1 = 0 : \\ s\mathcal{C}(X,Y) \to s\mathcal{A}(Xf,Yf)$$

implies that f_1' is a chain map.

Let us prove that the restriction chain map

$$\mathrm{restr}_1 : sA^{\psi u}_{\infty 1}(\mathcal{C}, \mathcal{Q}_1; \mathcal{A})\bigl((f,f'),(g,g')\bigr) \to sA_\infty(\mathcal{C},\mathcal{A})(f,g), \qquad (r,r') \mapsto r$$

is homotopy invertible. This map is a product over $n \in \mathbb{Z}_{\geqslant 0}$ of the restriction maps $\rho_n : V_n \to V'_n$ of the graded $\Bbbk$-modules of n-th components (compare (4.3.1) with analogous decomposition of $sA_\infty(\mathcal{C},\mathcal{A})(f,g)$). Clearly, the maps $\rho_n = \mathrm{id}$ for $n=0$ or for $n>1$. On the other hand, for $n=1$

$$\rho_1 = \prod \mathsf{C}(\mathrm{in}^{\mathcal{C}},1) : \prod_{X,Y\in\mathrm{Ob}\,\mathcal{C}} \mathsf{C}(s\mathcal{Q}_1(X,Y), s\mathcal{A}(Xf,Yg)) \to \prod_{X,Y\in\mathrm{Ob}\,\mathcal{C}} \mathsf{C}(s\mathcal{C}(X,Y), s\mathcal{A}(Xf,Yg))$$

is surjective with the kernel $\mathrm{Ker}\,\rho_1 = \prod_{X,Y\in\mathrm{Ob}\,\mathcal{C}} \mathsf{C}(s\mathcal{D}(\mathfrak{t}_1)(X,Y), s\mathcal{A}(Xf,Yg))$, because the sequence $0 \to s\mathcal{C} \to s\mathcal{Q}_1 \to s\mathcal{D}(\mathfrak{t}_1) \to 0$ is semisplit. Since we may write the kernel as

$$\mathrm{Ker}\,\rho_1 = \prod_{\substack{X\in\mathrm{Ob}\,\mathcal{B},Y\in\mathrm{Ob}\,\mathcal{C}\\ \text{or } X\in\mathrm{Ob}\,\mathcal{C},Y\in\mathrm{Ob}\,\mathcal{B}}} \mathsf{C}(s\mathcal{D}(\mathfrak{t}_1)(X,Y), s\mathcal{A}(Xf,Yg)),$$

it is contractible, because contractibility of $f\big|_{\mathcal{B}}, g\big|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ implies contractibility of complexes $\mathcal{A}(Xf,Yg)$ by Proposition 6.1(C1), (C2) of [LO02].

Summing up, the first term of the semisplit exact sequence

$$0 \to \mathrm{Ker}\,\mathrm{restr}_1 \longrightarrow sA_{\infty 1}(\mathcal{C},\mathcal{Q}_1;\mathcal{A})\big((f,f'),(g,g')\big) \xrightarrow{\mathrm{restr}_1} sA_\infty(\mathcal{C},\mathcal{A})(f,g) \to 0$$

is contractible. By Lemma 5.6 this sequence admits a splitting chain map

$$\nu : sA_\infty(\mathcal{C},\mathcal{A})(f,g) \to sA_{\infty 1}(\mathcal{C},\mathcal{Q};\mathcal{A})\big((f,f'),(g,g')\big),$$

and ν is homotopy inverse to restr_1. Applying Corollary 1.9 we conclude that (5.7.1) is an A_∞-equivalence. □

5.8 Corollary. *A_∞-2-transformation* (5.7.1) *is a natural A_∞-2-equivalence.*

5.9 Proposition. *Let $\mathcal{A}$ be a unital A_∞-category, let $\mathcal{D}$ be a pseudounital A_∞-category with distinguished cycles $\iota^{\mathcal{D}}_X$, let $\mathcal{Q}$ be a differential graded quiver and let $\mathcal{N}$ be a graded quiver such that $\mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{Q} = \mathrm{Ob}\,\mathcal{N}$ and $\mathcal{Q} = \mathcal{D}\oplus\mathcal{N}$. Suppose that $\mathcal{N}(X,Y) \neq 0$ implies that $\iota^{\mathcal{D}}_X \in \mathrm{Im}\,b_1$ or $\iota^{\mathcal{D}}_Y \in \mathrm{Im}\,b_1$. Then an arbitrary pseudounital A_∞-functor $f : \mathcal{D} \to \mathcal{A}$ extends to an object $(f,f') \in \mathrm{Ob}\,A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ for some f'.*

Proof. Let $s\mathcal{M} = \mathcal{N}$ be the differential graded quiver and let $\alpha : s\mathcal{M} \to s\mathcal{D}$ be the chain map defined in Section 4.5. There exists a homotopy $\overline{f} : s\mathcal{M}(X,Y) \to s\mathcal{A}(Xf,Yf)$ of degree -1 such that

$$\alpha f_1 = \overline{f} b_1 + d^{\mathcal{M}[1]}\overline{f} : s\mathcal{M}(X,Y) \longrightarrow s\mathcal{A}(Xf,Yf).$$

Indeed, the case of $\mathcal{M}(X,Y) = 0$ being obvious, we may assume that $\iota_X \in \mathrm{Im}\,b_1$ or $\iota_Y \in \mathrm{Im}\,b_1$. Then ${}_{Xf}\mathbf{i}^{\mathcal{A}}_0 \in \iota_X f_1 + \mathrm{Im}\,b_1 \subset \mathrm{Im}\,b_1$ or ${}_{Yf}\mathbf{i}^{\mathcal{A}}_0 \in \iota_Y f_1 + \mathrm{Im}\,b_1 \subset \mathrm{Im}\,b_1$. Since $\mathcal{A}$ is unital, the complex $s\mathcal{A}(Xf,Yf)$ is contractible with some contracting homotopy $\overline{h}$. We may take $\overline{f} = \alpha\overline{h}$.

Define a degree 0 map

$$f_1' = \bigl(s\mathcal{Q}(X,Y) = s\mathcal{D}(X,Y) \oplus s\mathcal{N}(X,Y) \xrightarrow{(f_1,s^{-1}\overline{f})} s\mathcal{A}(Xf,Yf)\bigr).$$

For arbitrary $p \in s\mathcal{D}(X,Y)$, $m \in s\mathcal{M}(X,Y) = \mathcal{N}(X,Y)$ we have

$$\begin{aligned}(p,ms)(f_1'b_1^{\mathcal{A}} - b_1^{\mathcal{Q}}f_1') &= pf_1b_1^{\mathcal{A}} + m\overline{f}b_1^{\mathcal{A}} - (pb_1^{\mathcal{D}} + m\alpha)f_1 + (md^{\mathcal{M}[1]}s)s^{-1}\overline{f} \\ &= p(f_1b_1^{\mathcal{A}} - b_1^{\mathcal{D}}f_1) + m(\overline{f}b_1^{\mathcal{A}} + d^{\mathcal{M}[1]}\overline{f} - \alpha f_1) = 0\end{aligned}$$

by (4.5.1). Therefore, f_1' is a chain map and $f_1'\big|_{s\mathcal{D}} = f_1$. □

5.10 Proposition. *In assumptions of Proposition 5.9 the restriction strict A_∞-functor*

$$\mathrm{restr} : A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A}) \to A^{\psi u}_{\infty}(\mathcal{D},\mathcal{A}), \qquad (x,x') \mapsto x \tag{5.10.1}$$

is an A_∞-equivalence, surjective on objects. The chain surjections restr_1 *admit a chain splitting.*

Proof. Let us prove that for an arbitrary pair of objects (f,f'), (g,g') of $A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})$ the restriction chain map

$$sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f,f'),(g,g')\bigr) \to sA_\infty(\mathcal{D},\mathcal{A})(f,g)$$

is homotopy invertible. This map is a product over $n \in \mathbb{Z}_{\geqslant 0}$ of the restriction maps $\rho_n : V_n \to V_n'$ of the graded $\Bbbk$-modules of n-th components (compare (4.3.1) with analogous decomposition of $sA_\infty(\mathcal{D},\mathcal{A})(f,g)$). Clearly, the maps $\rho_n = \mathrm{id}$ for $n = 0$ or for $n > 1$. On the other hand, for $n = 1$

$$\rho_1 = \prod \mathsf{C}(\mathrm{in}^{\mathcal{D}},1) : \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \mathsf{C}(s\mathcal{Q}(X,Y), s\mathcal{A}(Xf,Yg)) \to \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \mathsf{C}(s\mathcal{D}(X,Y), s\mathcal{A}(Xf,Yg))$$

is surjective with the kernel $\mathrm{Ker}\,\rho_1 = \prod_{X,Y\in\mathrm{Ob}\,\mathcal{D}} \mathsf{C}(s\mathcal{N}(X,Y), s\mathcal{A}(Xf,Yg))$. As in proof of Proposition 5.9 $\mathcal{N}(X,Y) \neq 0$ implies that ${}_{Xf}\mathbf{i}_0^{\mathcal{A}} \in \iota_X f_1 + \mathrm{Im}\, b_1 \subset \mathrm{Im}\, b_1$ or ${}_{Yg}\mathbf{i}_0^{\mathcal{A}} \in \iota_Y g_1 + \mathrm{Im}\, b_1 \subset \mathrm{Im}\, b_1$, hence, $s\mathcal{A}(Xf,Yg)$ is contractible. Therefore, for all objects X, Y of $\mathcal{D}$ the complex $\mathsf{C}(s\mathcal{N}(X,Y), s\mathcal{A}(Xf,Yg))$ is contractible. Thus, $\mathrm{Ker}\,\mathrm{restr}_1 = \mathrm{Ker}\,\rho_1$ is contractible.

Summing up, the first term of the semisplit exact sequence

$$0 \to \mathrm{Ker}\,\mathrm{restr}_1 \longrightarrow sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f,f'),(g,g')\bigr) \xrightarrow{\mathrm{restr}_1} sA_\infty(\mathcal{D},\mathcal{A})(f,g) \to 0$$

is contractible. By Lemma 5.6 this sequence admits a splitting chain map

$$\nu : sA_\infty(\mathcal{D},\mathcal{A})(f,g) \to sA_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A})\bigl((f,f'),(g,g')\bigr),$$

and ν is homotopy inverse to restr_1.

By Proposition 5.9 the surjection

$$\mathrm{Ob}\,A^{\psi u}_{\infty 1}(\mathcal{D},\mathcal{Q};\mathcal{A}) \ni (f,f') \mapsto f \in \mathrm{Ob}\,A^{\psi u}_{\infty}(\mathcal{D},\mathcal{A})$$

admits a splitting $f \mapsto \widetilde{f} = (f,f')$. Applying Corollary 1.9 we conclude that (5.10.1) is an A_∞-equivalence. □

5.11 Corollary. *A_∞-2-transformation* (5.10.1) *is a natural A_∞-2-equivalence.*

An easy converse to Lemma 5.6 is given by

5.12 Lemma. *Let $\beta : A \to B$, $\nu : B \to A$ be chain maps of complexes of $\Bbbk$-modules, such that $\nu\beta = \mathrm{id}_B$. Denote $C = \operatorname{Ker}\nu$, then $A \simeq C \oplus B$. If β is a homotopy isomorphism, then the chain complex C is contractible.*

Proof. Being one-sided inverse to homotopy isomorphism β, the map ν is homotopy inverse to β. Therefore, $\mathrm{id}_A - \beta\nu = hd + dh$ for some homotopy $h : A \to A$ of degree -1. Since C is the image of the idempotent $\mathrm{id}_A - \beta\nu = \mathrm{pr}^C \cdot \mathrm{in}^C$, we have

$$\mathrm{id}_C = \mathrm{in}^C \mathrm{pr}^C \mathrm{in}^C \mathrm{pr}^C = \mathrm{in}^C(\mathrm{id}_A - \beta\nu)\,\mathrm{pr}^C = (\mathrm{in}^C h\,\mathrm{pr}^C)d + d(\mathrm{in}^C h\,\mathrm{pr}^C).$$

Thus, $\mathrm{id}_C = Hd + dH$ for $H = \mathrm{in}^C h\,\mathrm{pr}^C : C \to C$, and C is contractible. □

5.13 Theorem. *Let $\mathcal{A}$ be a unital A_∞-category, and let $\mathcal{D} = \cup_{j\geqslant 0}\mathcal{D}_j = \varinjlim_j \mathcal{D}_j$. Then the restriction strict A_∞-functor*

$$\mathrm{restr} : A_\infty^{\psi u}(\mathcal{D},\mathcal{A}) \to A_\infty^{u}(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$$

is an A_∞-equivalence, 2-natural in $\mathcal{A}$, surjective on objects. The chain surjections restr_1 *admit a chain splitting.*

Proof. All restriction strict A_∞-functors in the sequence

$$\begin{aligned} A_\infty^{u}(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}} \longleftarrow A_{\infty 1}^{\psi u}(\mathcal{D}_0,\mathcal{Q}_1;\mathcal{A}) \longleftarrow A_\infty^{\psi u}(\mathcal{D}_1,\mathcal{A}) \\ \longleftarrow A_{\infty 1}^{\psi u}(\mathcal{D}_1,\mathcal{Q}_2;\mathcal{A}) \longleftarrow A_\infty^{\psi u}(\mathcal{D}_2,\mathcal{A}) \longleftarrow A_{\infty 1}^{\psi u}(\mathcal{D}_2,\mathcal{Q}_3;\mathcal{A}) \longleftarrow \dots \end{aligned} \tag{5.13.1}$$

are A_∞-equivalences (and natural A_∞-2-equivalences). They are surjective on objects. The first components are surjective and admit a chain splitting. For the first functor it follows from Proposition 5.7. For other odd-numbered functors it follows from Proposition 5.10. Indeed, if $X \in \operatorname{Ob}\mathcal{B}$, then ${}_X\mathbf{i}_0^{\mathcal{C}} = {}_X\mathbf{i}_0^{\mathcal{C}} H b_1 \in \operatorname{Im}(b_1 : s\mathcal{D}_1(X,X) \to s\mathcal{D}_1(X,X))$. For even-numbered functors it follows from Theorem 4.7.

Let us show that $A_\infty^{\psi u}(\mathcal{D},\mathcal{A})$ is the inverse limit of (5.13.1) on objects and on morphisms. There are restriction strict A_∞-functors

$$\begin{aligned} &\mathrm{restr} : A_\infty^{\psi u}(\mathcal{D},\mathcal{A}) \to A_\infty^{u}(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}} \subset A_\infty^{u}(\mathcal{C},\mathcal{A}), && f \mapsto f|_{\mathcal{C}}, \\ &\mathrm{restr} : A_\infty^{\psi u}(\mathcal{D},\mathcal{A}) \to A_{\infty 1}^{\psi u}(\mathcal{D}_j,\mathcal{Q}_{j+1};\mathcal{A}), && f \mapsto (f|_{\mathcal{D}_j}, f|_{\mathcal{Q}_{j+1}}), \quad j \geqslant 0, \\ &\mathrm{restr} : A_\infty^{\psi u}(\mathcal{D},\mathcal{A}) \to A_\infty^{\psi u}(\mathcal{D}_j,\mathcal{A}), && f \mapsto f|_{\mathcal{D}_j}, \quad j \geqslant 1. \end{aligned}$$

They agree with the functors restr from (5.13.1) in the sense $\mathrm{restr}\cdot\mathrm{restr} = \mathrm{restr}$.

Since $\mathcal{D} = \cup_{j\geqslant 0}\mathcal{D}_j$, pseudounital A_∞-functors $f : \mathcal{D} \to \mathcal{A}$ are in bijection with sequences $(f^j)_j$ of pseudounital A_∞-functors $f^j : \mathcal{D}_j \to \mathcal{A}$ such that $f^{j+1}\big|_{\mathcal{D}_j} = f^j$. In other words, $\operatorname{Ob} A^{\psi u}_\infty(\mathcal{D},\mathcal{A})$ is the inverse limit of the sequence of surjections

$$\begin{aligned}\operatorname{Ob} A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}} \leftarrow \operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D}_0,\mathcal{Q}_1;\mathcal{A}) \leftarrow \operatorname{Ob} A^{\psi u}_\infty(\mathcal{D}_1,\mathcal{A})\\ \leftarrow \operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D}_1,\mathcal{Q}_2;\mathcal{A}) \leftarrow \operatorname{Ob} A^{\psi u}_\infty(\mathcal{D}_2,\mathcal{A}) \leftarrow \operatorname{Ob} A^{\psi u}_{\infty 1}(\mathcal{D}_2,\mathcal{Q}_3;\mathcal{A}) \leftarrow \dots\end{aligned}$$

In particular, the map

$$\operatorname{Ob}\operatorname{restr} : \operatorname{Ob} A^{\psi u}_\infty(\mathcal{D},\mathcal{A}) \to \operatorname{Ob} A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$$

is surjective.

Let $f, g : \mathcal{D} \to \mathcal{A}$ be pseudounital A_∞-functors. Since $\mathcal{D} = \big(\cup_{j\geqslant 0}\mathcal{D}_j\big)\bigcup\big(\cup_{j\geqslant 1}\mathcal{Q}_j\big)$, A_∞-transformations $p : f \to g : \mathcal{D} \to \mathcal{A}$ are in bijection with sequences

$$(p^0, p'^1, p^1, p'^2, p^2, p'^3, \dots)$$

of A_∞-transformations $p^j : f|_{\mathcal{D}_j} \to g|_{\mathcal{D}_j} : \mathcal{D}_j \to \mathcal{A}$, $j \geqslant 0$, and A_1-transformations $p'^j : f|_{\mathcal{Q}_j} \to g|_{\mathcal{Q}_j} : \mathcal{Q}_j \to \mathcal{A}$, $j \geqslant 1$, such that $p^{j+1}\big|_{\mathcal{D}_j} = p^j$, $p^j\big|_{\mathcal{Q}_j} = p'^j$. In other words, $A_\infty(\mathcal{D},\mathcal{A})(f,g)$ is the inverse limit of the sequence of splittable chain surjections

$$\begin{aligned}A_\infty(\mathcal{C},\mathcal{A})(f|_{\mathcal{C}}, g|_{\mathcal{C}}) \leftarrow A_{\infty 1}(\mathcal{D}_0,\mathcal{Q}_1;\mathcal{A})\big((f|_{\mathcal{D}_0}, f|_{\mathcal{Q}_1}),(g|_{\mathcal{D}_0}, g|_{\mathcal{Q}_1})\big) \leftarrow \dots\\ \leftarrow A_\infty(\mathcal{D}_j,\mathcal{A})(f|_{\mathcal{D}_j}, g|_{\mathcal{D}_j}) \leftarrow A_{\infty 1}(\mathcal{D}_j,\mathcal{Q}_{j+1};\mathcal{A})\big((f|_{\mathcal{D}_j}, f|_{\mathcal{Q}_{j+1}}),(g|_{\mathcal{D}_j}, g|_{\mathcal{Q}_{j+1}})\big) \leftarrow \dots\end{aligned}$$

Since these surjections are splittable, the above sequence is isomorphic to the sequence of natural projections

$$C_0 \leftarrow C_0 \times C_1 \leftarrow C_0 \times C_1 \times C_2 \leftarrow \dots \leftarrow \prod_{m=0}^{n} C_m \leftarrow \prod_{m=0}^{n+1} C_m \leftarrow \dots$$

for some complexes C_m of $\Bbbk$-modules. Its inverse limit is $\prod_{m=0}^\infty C_m \simeq A_\infty(\mathcal{D},\mathcal{A})(f,g)$. By Lemma 5.12 all C_m are contractible for $m > 0$. Therefore, $\prod_{m=1}^\infty C_m$ is contractible. We obtain a split exact sequence

$$0 \to \prod_{m=1}^\infty C_m \longrightarrow A_\infty(\mathcal{D},\mathcal{A})(f,g) \xrightarrow{\beta} A_\infty(\mathcal{C},\mathcal{A})(f|_{\mathcal{C}}, g|_{\mathcal{C}}) \to 0$$

with contractible first term. By Lemma 5.6 $\beta = s \operatorname{restr}_1 s^{-1}$ is a homotopy isomorphism.

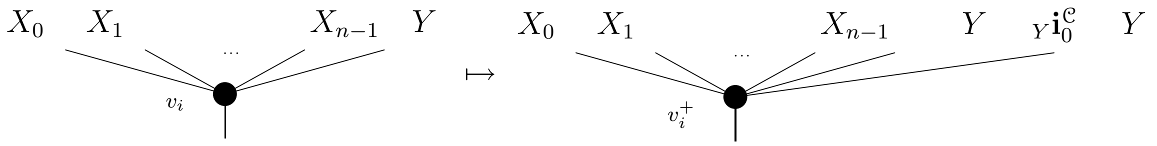
Using Corollary 1.9 we prove the theorem. □

6. Unitality of $\mathcal{D}$

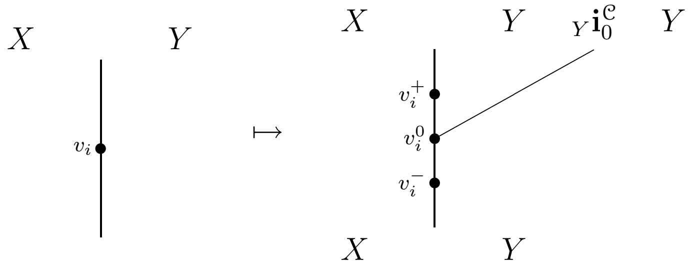
We are going to prove that if $\mathcal{C}$ is strictly unital, then the A_∞-category $\mathcal{D}$ constructed in Section 5.2 is not only pseudounital, but unital with the unit elements ${}_X\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{D}(|)(X,X) \subset s\mathcal{D}(X,X)$. Let us describe $\Bbbk$-linear maps $h : s\mathcal{D}(X,Y) \to s\mathcal{D}(X,Y)$ of degree -1 such that

$$1 - (1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 = b_1 h + h b_1 : s\mathcal{D}(X,Y) \to s\mathcal{D}(X,Y). \tag{6.0.1}$$

The homotopy h is called the *right unit homotopy.* Let t be a $\mathcal{C}$-admissible tree. Let y be its rightmost leaf vertex. Let $v_0 \prec v_1 \prec \dots \prec v_p \prec v_{p+1} = y$ be the directed path connecting the root v_0 with y, $p \geqslant 0$. Vertices v_i and v_{i+1} are connected by an edge for all $0 \leqslant i \leqslant p$. Let t_i^+, $1 \leqslant i \leqslant p$ be the tree t with an extra leaf attached on the right to the vertex v_i if v_i is n-ary, $n > 1$ as in



If v_i is unary, we attach an extra leaf on the right to v_i and add two more unary vertices above and below it as in



The obtained trees t_i^+ are $\mathcal{C}$-admissible.

Let x be a homogeneous element of $s\mathcal{D}(t)(Z,Y)$. Define $x_i^+ = x \otimes_Y \mathbf{i}_0^{\mathcal{C}} \in s\mathcal{D}(t_i^+)(Z,Y) = s\mathcal{D}(t)(Z,Y) \otimes s\mathcal{C}(Y,Y)$. We claim that if $\mathcal{C}$ is strictly unital, then the map

$$h : x \mapsto \sum_{i=1}^{p} \pm x_i^+ \tag{6.0.2}$$

with an appropriate choice of signs satisfies (6.0.1). To describe the signs and to prove the claim we study the set of operations acting in $\mathcal{D}$.

6.1. A multicategory operating in $\mathcal{D}$. Let $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ be the free graded $\Bbbk$-linear (non-symmetric) operad, generated by

— a 0-ary operation $\mathbf{i}_0 \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(0)$ of degree -1,

— a unary operation $H \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(1)$ of degree -1,

— an n-ary operation $b_n \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(n)$ of degree 1 for each $n > 1$.

The construction of free non-symmetric operads uses plane trees instead of abstract trees. Otherwise it is similar to the case of symmetric operads, see e.g. [MSS02]. The operad $\mathscr{A}_\infty$ of operations in A_∞-algebras (e.g. [Mar00]) is a suboperad of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$.

Actually we need a multicategory [Lam69, Lam89] rather than an operad. A multicategory is a many object version of an operad, a (non-symmetric) operad is a one-object multicategory. So we define a graded $\Bbbk$-linear multicategory $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$, whose objects are pairs of objects of $\mathcal{C}$, thus $\operatorname{Ob}\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}} = \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C}$. For $n > 0$ the graded $\Bbbk$-module of morphisms

$$\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_1', X_1), (X_2', X_2), \dots, (X_n', X_n); (Y', Y)\big)$$

is 0 unless $X_1' = Y'$, $X_n = Y$ and $X_i = X_{i+1}'$ for all $1 \leqslant i < n$. For $n = 0$ the graded $\Bbbk$-module of morphisms $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(;(Y', Y))$ is 0 unless $Y' = Y$. The morphisms of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ are freely generated by

— 0-ary operations ${}_X\mathbf{i}_0 \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}(;(X, X))$, $X \in \operatorname{Ob}\mathcal{C}$ of degree $\deg {}_X\mathbf{i}_0 = -1$,

— unary operations $H = H_{X,Y} \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}((X, Y); (X, Y))$, $X, Y \in \operatorname{Ob}\mathcal{C}$, where $X \in \operatorname{Ob}\mathcal{B}$ or $Y \in \operatorname{Ob}\mathcal{B}$, of degree $\deg H = -1$,

— n-ary operations $b_n \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0, X_1), (X_1, X_2), \dots, (X_{n-1}, X_n); (X_0, X_n)\big)$, of degree $\deg b_n = 1$ for X_0, …, $X_n \in \operatorname{Ob}\mathcal{C}$, $n > 1$.

We shall not insist on distinguishing between the operad $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ and its refinement – the multicategory $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$, leaving the choice of context to the reader.

Similarly to the operad $\mathscr{A}_\infty$ [Mar00], governing A_∞-algebras, the multicategory $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ has a differential d – a derivation of degree 1, such that $d^2 = 0$. In general, a derivation d of degree p of a graded $\Bbbk$-linear multicategory $\mathscr{M}$ is a collection of $\Bbbk$-linear endomorphisms d of $\mathscr{M}(Z_1, \dots, Z_n; Z)$ of degree p, such that all compositions (which are of degree 0)

$$\mu_i : \mathscr{M}(Y_1, \dots, Y_k; Z_i) \otimes \mathscr{M}(Z_1, \dots, Z_n; Z) \to \mathscr{M}(Z_1, \dots, Z_{i-1}, Y_1, \dots, Y_k, Z_{i+1}, \dots, Z_n; Z) \tag{6.1.1}$$

satisfy the equation

$$\mu_i d = (1 \otimes d + d \otimes 1)\mu_i$$

with the sign conventions of this article. If $d^2 = 0$, we may say that μ_i are chain maps.

Since $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ is free, its derivations are uniquely determined by their values on generators. In particular, the derivation d of degree 1 is determined by these values:

$$\begin{aligned}
{}_X\mathbf{i}_0 d &= 0, \\
H_{X,Y} d &= 1_{(X,Y)} \qquad \text{(the unit element of } \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X, Y); (X, Y)\big)\text{)}, \\
b_n d &= -\sum_{a+p+c=n}^{p>1,\, a+c>0} (1^{\otimes a} \otimes b_p \otimes 1^{\otimes c}) b_{a+1+c}, \qquad n > 1.
\end{aligned}$$

For $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ we use the notation $(1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})b_{a+1+c}$ referring to the action of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ in A_∞-algebras as a synonym of the usual operadic notation $(b_p \otimes b_{a+1+c})\mu_{a+1} = b_p \circ_{a+1} b_{a+1+c}$. Since the derivation d has odd degree, the $\Bbbk$-linear map d^2 is also a derivation. Its value on all generators is 0 (for b_n it follows from a similar result for $\mathscr{A}_\infty$, see e.g. [Mar00, MSS02]). Therefore, $d^2 = 0$, d is a differential, and $\mathscr{A}_\infty \hookrightarrow \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ is a chain embedding.

The so defined differential d is distinguished by the following property. The action maps

$$\begin{aligned}\alpha : s\mathcal{E}(X_0, X_1) \otimes s\mathcal{E}(X_1, X_2) \otimes \cdots \otimes s\mathcal{E}(X_{n-1}, X_n) \\ \otimes \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0, X_1), (X_1, X_2), \dots, (X_{n-1}, X_n); (X_0, X_n)\big) \to s\mathcal{E}(X_0, X_n) \quad (6.1.2)\end{aligned}$$

are chain maps, where $s\mathcal{E}(\_, \_)$ is equipped with the differential b_1. It suffices to check this on generators of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$. For them the property follows from (5.2.2), (5.2.3) and the equation ${}_X\mathbf{i}_0^{\mathcal{C}} b_1 = 0$.

Let $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ be a submulticategory of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$, generated by H and b_n, $n \geqslant 2$, with the same set of objects $\operatorname{Ob} \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}} = \operatorname{Ob} \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$. It is a differential graded submulticategory without 0-ary operations: $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}(; (X, Y)) = 0$ for all $X, Y \in \operatorname{Ob} \mathcal{C}$. As a $\Bbbk$-linear graded multigraph $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ has the following description. For $n \geqslant 1$

$$\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0, X_1), (X_1, X_2), \dots, (X_{n-1}, X_n); (X_0, X_n)\big) = \bigoplus_{\text{admissible } (t; X_0, X_1, \dots, X_n)} \Bbbk[|t|_1 - |t|_>], \tag{6.1.3}$$

other $\Bbbk$-modules $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_1', X_1), (X_2', X_2), \dots, (X_n', X_n); (Y', Y)\big)$ vanish.

The embedding of $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ is denoted

$$\iota : \underline{\mathscr{M}} \overset{\text{def}}{=} \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}} \hookrightarrow \mathscr{M} \overset{\text{def}}{=} \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}. \tag{6.1.4}$$

The following general results are valid for an arbitrary embedding $\iota : \underline{\mathscr{M}} \hookrightarrow \mathscr{M}$ of differential graded $\Bbbk$-linear multicategories, such that $\operatorname{Ob} \iota = \mathrm{id}_{\operatorname{Ob} \mathscr{M}}$, and $\underline{\mathscr{M}}(; Z) = 0$ for all $Z \in \operatorname{Ob} \underline{\mathscr{M}} = \operatorname{Ob} \mathscr{M}$.

6.2 Definition. A *right derivation* δ of degree p of the embedding $\iota : \underline{\mathscr{M}} \hookrightarrow \mathscr{M}$ is a collection of $\Bbbk$-linear maps

$$\delta : \underline{\mathscr{M}}(Z_1, \dots, Z_n; Z) \to \mathscr{M}(Z_1, \dots, Z_n; Z)$$

of degree p for all $Z_1, \dots, Z_n, Z \in \operatorname{Ob} \mathscr{M}$, such that compositions μ_i from (6.1.1) satisfy

$$\begin{tikzcd} \underline{\mathscr{M}} \otimes \underline{\mathscr{M}} \arrow[r, "\mu_n"] \arrow[d, "\iota\otimes\delta+\delta\otimes\iota"'] \arrow[dr, phantom, "="] & \underline{\mathscr{M}} \arrow[d, "\delta"] \\ \mathscr{M} \otimes \mathscr{M} \arrow[r, "\mu_n"] & \mathscr{M} \end{tikzcd}, \qquad \begin{tikzcd} \underline{\mathscr{M}} \otimes \underline{\mathscr{M}} \arrow[r, "\mu_i"] \arrow[d, "\iota\otimes\delta"'] \arrow[dr, phantom, "="] & \underline{\mathscr{M}} \arrow[d, "\delta"] \\ \mathscr{M} \otimes \mathscr{M} \arrow[r, "\mu_i"] & \mathscr{M} \end{tikzcd}$$

if $1 \leqslant i < n$.

Clearly, for $n = 0$ the map $\delta : \underline{\mathscr{M}}(;Z) = 0 \to \mathscr{M}(;Z)$ is 0.

An example of a right derivation is given by an *inner right derivation*. Let $\lambda_Z \in \mathscr{M}(Z;Z)$ be a family of morphisms of degree p. For $n > 0$ define

$$\begin{aligned} \mathrm{ad}_\lambda : \underline{\mathscr{M}}(Z_1,\dots,Z_n;Z) &\to \mathscr{M}(Z_1,\dots,Z_n;Z) \\ f &\mapsto (f\otimes\lambda_Z)\mu_1 - (-)^{f\lambda}(\lambda_{Z_n}\otimes f)\mu_n = f\circ_1\lambda_Z - (-)^{fp}\lambda_{Z_n}\circ_n f. \end{aligned}$$

One verifies easily that ad_λ is a right derivation, which we call *inner*.

One can show that if δ is a right derivation of ι, and d is a derivation of $\mathscr{M}$ such that $\underline{\mathscr{M}}d \subset \underline{\mathscr{M}}$, then their commutator $[\delta, d] = \delta d - (-)^{\delta\cdot d} d\delta$ is a right derivation of ι as well. In particular, it applies to a differential d of degree 1. If $\delta = \mathrm{ad}_\lambda$ is inner, then $[\delta, d] = \mathrm{ad}_{\lambda d}$ is also inner.

6.3. Examples of right derivations. Consider embedding (6.1.4) and take the family of morphisms

$$\lambda_{(X,Y)} = (1\otimes{}_Y\mathbf{i}_0)b_2 = {}_Y\mathbf{i}_0\circ_2 b_2 \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X,Y);(X,Y)\big).$$

Since $\lambda_{(X,Y)}d = 0$, we have $[\mathrm{ad}_\lambda, d] = \mathrm{ad}_{\lambda d} = 0$. Thus, $\delta = \mathrm{ad}_{(1\otimes\mathbf{i}_0)b_2}$ commutes with d.

Let us show that $\mathrm{ad}_\lambda = [\eta, d]$ for some right derivation $\eta : \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}} \to \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ of ι. Since $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ is a free multicategory, any such right derivation is uniquely determined by its value on the generators H and b_n, $n > 1$. We define a right derivation η of ι of degree -1 by the following assignment:

$$\begin{aligned} H_{X,Y}\eta &= H_{X,Y}(1\otimes{}_Y\mathbf{i}_0)b_2H_{X,Y}, \\ b_n\eta &= (1^{\otimes n}\otimes{}_{X_n}\mathbf{i}_0)b_{n+1}. \end{aligned} \tag{6.3.1}$$

Any operation $f \in \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ can be presented as

$$f = (g\otimes 1)(1^{\otimes a_1}\otimes e_{p_1})(1^{\otimes a_2}\otimes e_{p_2})\dots(1^{\otimes a_{k-1}}\otimes e_{p_{k-1}})e_{p_k}$$

for some $g \in \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$, where $e_1 = H$ and $e_p = b_p$ if $p > 1$. Then

$$f\eta = \sum_{i=1}^{k}(-)^{k-i}(g\otimes 1)(1^{\otimes a_1}\otimes e_{p_1})\dots(1^{\otimes a_i}\otimes e_{p_i}\eta)\dots(1^{\otimes a_{k-1}}\otimes e_{p_{k-1}})e_{p_k}. \tag{6.3.2}$$

Let us prove that

$$[\eta, d] = \eta d + d\eta = \mathrm{ad}_{(1\otimes\mathbf{i}_0)b_2}\,. \tag{6.3.3}$$

Since the difference of the both sides is a right derivation of ι, it suffices to prove (6.3.3) on generators. First we find that

$$\begin{aligned} H_{X,Y}(\eta d + d\eta) &= [H_{X,Y}(1\otimes{}_Y\mathbf{i}_0)b_2H_{X,Y}]d + 1_{(X,Y)}\eta \\ &= H_{X,Y}(1\otimes{}_Y\mathbf{i}_0)b_2 - (1\otimes{}_Y\mathbf{i}_0)b_2H_{X,Y} = H_{X,Y}\,\mathrm{ad}_{(1\otimes\mathbf{i}_0)b_2}\,. \end{aligned}$$

For $n \geqslant 2$ we have

$$
\begin{aligned}
b_n(\eta d + d\eta) &= (1^{\otimes n} \otimes \mathbf{i}_0)(b_{n+1}d) - \sum_{a+p+c=n}^{p>1,\ c>0} (1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})(b_{a+1+c}\eta) \\
&\quad - \sum_{a+p=n}^{p>1,\ a>0} [(1^{\otimes a} \otimes b_p)b_{a+1}]\eta \\
&= - \sum_{a+p+c=n+1}^{p>1,\ a+c>0} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})b_{a+1+c} \\
&\quad - \sum_{a+p+c=n}^{p>1,\ c>0} (1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})(1^{\otimes a+1+c} \otimes \mathbf{i}_0)b_{a+c+2} \\
&\quad - \sum_{a+p=n}^{p>1,\ a>0} (1^{\otimes a} \otimes b_p)(1^{\otimes a+1} \otimes \mathbf{i}_0)b_{a+2} + \sum_{a+p=n}^{p>1,\ a>0} [1^{\otimes a} \otimes (1^{\otimes p} \otimes \mathbf{i}_0)b_{p+1}]b_{a+1} \\
&= - \sum_{a+p+c=n+1}^{p>1,\ a+c>0} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_p \otimes 1^{\otimes c})b_{a+1+c} \\
&\quad + \sum_{a+p+e=n+1}^{p>1,\ e>1} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_p \otimes 1^{\otimes e})b_{a+1+e} \\
&\quad + \sum_{a+p=n}^{p>1,\ a>0} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_p \otimes 1)b_{a+2} + \sum_{a+q=n+1}^{q>2,\ a>0} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_q)b_{a+1} \\
&= - \sum_{0+p+1=n+1} (1^{\otimes n} \otimes \mathbf{i}_0)(b_p \otimes 1)b_2 - \sum_{a+2+0=n+1} (1^{\otimes n} \otimes \mathbf{i}_0)(1^{\otimes a} \otimes b_2)b_{a+1} \\
&= b_n(1 \otimes \mathbf{i}_0)b_2 - [1^{\otimes n-1} \otimes (1 \otimes \mathbf{i}_0)b_2]b_n \\
&= b_n \operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2} .
\end{aligned}
$$

Thus equation (6.3.3) is verified.

Notice that the graded quiver $s\mathcal{E}$ defined by (5.2.1) is a free $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$-algebra, generated by the graded quiver $s\mathcal{C}$. Indeed, (5.2.1) can be written as

$$
\begin{aligned}
s\mathcal{E}(X,Y) = \bigoplus_{n\geqslant 1} \bigoplus_{X_1,\dots,X_{n-1}\in\operatorname{Ob}\mathcal{C}}^{X_0=X,\ X_n=Y} & s\mathcal{C}(X_0,X_1) \otimes \dots \otimes s\mathcal{C}(X_{n-1},X_n) \otimes \\
& \otimes \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\bigl((X_0,X_1),(X_1,X_2),\dots,(X_{n-1},X_n);(X_0,X_n)\bigr)
\end{aligned}
$$

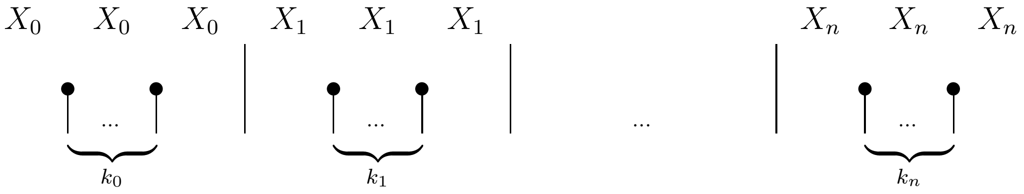
due to (6.1.3). Compare with the usual free algebras over an operad, e.g. [MSS02]. The operations H, b_n for $n \geqslant 2$ act in $s\mathcal{E}$ via multicategory compositions in $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$.

The multigraph $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ is expressed via $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$ as follows:

$$\begin{aligned}
&\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),(X_1,X_2),\dots,(X_{n-1},X_n);(X_0,X_n)\big) \\
&\quad= \bigoplus_{k_0,\dots,k_n\in\mathbb{Z}_{\geqslant 0}} \underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((Y_0,Y_1),(Y_1,Y_2),\dots,(Y_{p-1},Y_p);(X_0,X_n)\big)[k_0+\dots+k_n], \qquad (6.3.4)
\end{aligned}$$

$$(Y_0,Y_1,\dots,Y_p) = (\underbrace{X_0,\dots,X_0}_{k_0+1},\underbrace{X_1,\dots,X_1}_{k_1+1},\dots,\underbrace{X_n,\dots,X_n}_{k_n+1}),$$

where $p = k_0+\dots+k_n+n$. Indeed, 0-ary operations can be performed first. The summand of (6.3.4) corresponds to insertion of k_0 symbols ${}_{X_0}\mathbf{i}_0$, k_1 symbols ${}_{X_1}\mathbf{i}_0$, and so on. In terms of trees such summand is described by concatenation of the forest



with an admissible tree $(t;Y_0,Y_1,\dots,Y_p)$.

The action (6.1.2) of $\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}$ in $s\mathcal{E}$ is described as follows. An element

$$({}_{X_0}\mathbf{i}_0^{\otimes k_0}\otimes 1\otimes {}_{X_1}\mathbf{i}_0^{\otimes k_1}\otimes 1\otimes\dots\otimes 1\otimes {}_{X_n}\mathbf{i}_0^{\otimes k_n})\cdot f \in \mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big),$$

where $f\in\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((Y_0,Y_1),(Y_1,Y_2),\dots,(Y_{p-1},Y_p);(X_0,X_n)\big)$, acts by the map

$$\begin{aligned}
&s\mathcal{E}(X_0,X_1)\otimes\dots\otimes s\mathcal{E}(X_{n-1},X_n) \xrightarrow{{}^{\mathcal{C}}_{X_0}\mathbf{i}_0^{\otimes k_0}\otimes 1\otimes{}^{\mathcal{C}}_{X_1}\mathbf{i}_0^{\otimes k_1}\otimes 1\otimes\dots\otimes 1\otimes{}^{\mathcal{C}}_{X_n}\mathbf{i}_0^{\otimes k_n}} \\
&s\mathcal{C}(X_0,X_0)^{\otimes k_0}\otimes s\mathcal{E}(X_0,X_1)\otimes s\mathcal{C}(X_1,X_1)^{\otimes k_1}\otimes\dots\otimes s\mathcal{E}(X_{n-1},X_n)\otimes s\mathcal{C}(X_n,X_n)^{\otimes k_n}
\end{aligned}$$

followed by the action of f via multiplication in multicategory $\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}$, that is, via grafting of trees.

Define a $\Bbbk$-linear map $\bar{h} : s\mathcal{E}(X,Y)\to s\mathcal{E}(X,Y)$ of degree -1 for all objects X, Y of

$\mathcal{C}$ as follows:

$$\begin{array}{c}
s\mathcal{E}(X,Y) \\
\Big\| \\
\bigoplus\limits_{X_1,\dots,X_{n-1}\in\operatorname{Ob}\mathcal{C}}^{n,\,X_0=X,\,X_n=Y} s\mathcal{C}(X_0,X_1)\otimes\dots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big) \\
\Big\downarrow{\scriptstyle -1^{\otimes n}\otimes\eta} \\
\bigoplus\limits_{X_1,\dots,X_{n-1}\in\operatorname{Ob}\mathcal{C}}^{n,\,X_0=X,\,X_n=Y} s\mathcal{C}(X_0,X_1)\otimes\dots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes\mathscr{A}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big) \\
\Big\downarrow{\scriptstyle \alpha} \\
s\mathcal{E}(X,Y)
\end{array}$$

The concrete choice (6.3.1) of value of η on generators shows that on the summand

$$s\mathcal{C}(X_0,X_1)\otimes\dots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big)$$

the map $\bar{h}$ takes values in

$$s\mathcal{C}(X_0,X_1)\otimes\dots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes s\mathcal{C}(X_n,X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big(\dots,(X_{n-1},X_n),(X_n,X_n);(X_0,X_n)\big)$$

Due to (6.3.2) the explicit formula for $\bar{h}$ has the form (6.0.2) with the concrete choice of signs.

Let us compute the commutator

$$\begin{aligned}
-\bar{h}b_1^{\mathcal{E}}-b_1^{\mathcal{E}}\bar{h} &= (1^{\otimes n}\otimes\eta)\alpha b_1^{\mathcal{E}}+b_1^{\mathcal{E}}(1^{\otimes n}\otimes\eta)\alpha \\
&= (1^{\otimes n}\otimes\eta)\Big(\sum_{a+1+c=n}(1^{\otimes a}\otimes b_1\otimes 1^{\otimes c})\otimes 1+1^{\otimes n}\otimes d\Big)\alpha \\
&\quad+\Big(\sum_{a+1+c=n}(1^{\otimes a}\otimes b_1\otimes 1^{\otimes c})\otimes 1+1^{\otimes n}\otimes d\Big)(1^{\otimes n}\otimes\eta)\alpha \\
&= [1^{\otimes n}\otimes(\eta d+d\eta)]\alpha=[1^{\otimes n}\otimes\operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2}]\alpha:
\end{aligned}$$

$$s\mathcal{C}(X_0,X_1)\otimes\dots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big((X_0,X_1),\dots,(X_{n-1},X_n);(X_0,X_n)\big)\to$$
$$s\mathcal{C}(X_0,X_1)\otimes\dots\otimes s\mathcal{C}(X_{n-1},X_n)\otimes s\mathcal{C}(X_n,X_n)\otimes\underline{\mathscr{A}}_\infty^{\mathcal{C}/\mathcal{B}}\big(\dots,(X_{n-1},X_n),(X_n,X_n);(X_0,X_n)\big).$$

We write an element $z_1\otimes\dots\otimes z_n\otimes f$ of the source, which is a direct summand of $s\mathcal{E}(X_0,X_n)$, in the form $(z_1\otimes\dots\otimes z_n)f$, meaning that $z_i\in s\mathcal{C}(X_{i-1},X_i)$ and f is a composition of expressions $1^{\otimes a}\otimes b_p\otimes 1^{\otimes c}$ and $1^{\otimes a}\otimes H\otimes 1^{\otimes c}$ ending in b_p or H. Then

$$\begin{aligned}
-[(z_1\otimes\dots\otimes z_n)f](\bar{h}b_1^{\mathcal{E}}+b_1^{\mathcal{E}}\bar{h}) &= [(z_1\otimes\dots\otimes z_n)(f.\operatorname{ad}_{(1\otimes\mathbf{i}_0)b_2})]\alpha \\
&= \big\{[(z_1\otimes\dots\otimes z_n)f]\otimes{}_{X_n}\mathbf{i}_0^{\mathcal{C}}\big\}b_2^{\mathcal{E}}-(z_1\otimes\dots\otimes z_n\otimes{}_{X_n}\mathbf{i}_0^{\mathcal{C}})(1^{\otimes n-1}\otimes b_2^{\mathcal{E}})f. \quad (6.3.5)
\end{aligned}$$

6.4 Proposition. *Assume that $\mathcal{C}$ is a unital A_∞-category with the unit elements satisfying equations*

$$(1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 = 1,$$
$$(1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}})b_{n+1} = 0, \qquad \text{if } n > 1.$$

Then there exists a map $h : s\mathcal{D} \to s\mathcal{D}$ such that

$$\bigl(s\mathcal{E} \xrightarrow{\bar{h}} s\mathcal{E} \xrightarrow{\pi} s\mathcal{D}\bigr) = \bigl(s\mathcal{E} \xrightarrow{\pi} s\mathcal{D} \xrightarrow{h} s\mathcal{D}\bigr).$$

The map $h : s\mathcal{D} \to s\mathcal{D}$ is a right unit homotopy for $\mathcal{D}$.

Proof. We have to prove that $(R)_+\bar{h} \subset (R)_+$. The left hand side is the sum of images of maps

$$[(1^{\otimes a} \otimes b_n^{\mathcal{E}} \otimes 1^{\otimes c})f]\eta - (1^{\otimes a} \otimes b_n^{\mathcal{C}} \otimes 1^{\otimes c})(f\eta) : s\mathcal{C}^{\otimes k} \to s\mathcal{E}. \tag{6.4.1}$$

If $c > 0$, this map equals

$$(1^{\otimes a} \otimes b_n^{\mathcal{E}} \otimes 1^{\otimes c})(f\eta) - (1^{\otimes a} \otimes b_n^{\mathcal{C}} \otimes 1^{\otimes c})(f\eta),$$

and the claim holds. If $c = 0$, expression (6.4.1) is

$$\begin{aligned}
(1^{\otimes a} \otimes b_n^{\mathcal{E}})(f\eta) &- (1^{\otimes a} \otimes b_n^{\mathcal{C}})(f\eta) + (-)^f(1^{\otimes a} \otimes (1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}})b_{n+1}^{\mathcal{E}})f \\
&= (1^{\otimes a} \otimes b_n^{\mathcal{E}})(f\eta) - (1^{\otimes a} \otimes b_n^{\mathcal{C}})(f\eta) \\
&\quad + (-)^f(1^{\otimes a+n} \otimes \mathbf{i}_0^{\mathcal{C}})(1^{\otimes a} \otimes b_{n+1}^{\mathcal{E}})f - (-)^f(1^{\otimes a+n} \otimes \mathbf{i}_0^{\mathcal{C}})(1^{\otimes a} \otimes b_{n+1}^{\mathcal{C}})f \\
&\quad + (-)^f(1^{\otimes a} \otimes (1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}})b_{n+1}^{\mathcal{C}})f : s\mathcal{C}^{\otimes k} \to s\mathcal{E}
\end{aligned}$$

and the last summand equals 0. So the claim holds, and h exists.

Property (6.3.5) turns into

$$hb_1^{\mathcal{D}} + b_1^{\mathcal{D}}h = 1 - (1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 : s\mathcal{D}(X,Y) \to s\mathcal{D}(X,Y).$$

Therefore, h is a right unit homotopy for $\mathcal{D}$. □

6.5 Theorem. *Assume that $\mathcal{C}$ is a unital A_∞-category with the unit elements satisfying equations*

$$(1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 = 1, \qquad (\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 = -1,$$
$$(1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}})b_{n+1} = 0, \qquad (\mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes n})b_{n+1} = 0, \qquad \text{if } n > 1.$$

Then the A_∞-category $\mathcal{D}$ is unital.

Proof. Besides constructing $\mathcal{D}$ from the pair $(\mathcal{C}, \mathcal{B})$, we may apply the construction to the pair $(\mathcal{C}^{\mathrm{op}}, \mathcal{B}^{\mathrm{op}})$, and we get an A_∞-category isomorphic to $\mathcal{D}^{\mathrm{op}}$. The opposite A_∞-category $\mathcal{A}^{\mathrm{op}}$ to an A_∞-category $\mathcal{A}$ is the opposite quiver, equipped with operations b_k^{op}, see Definition A.5. In particular, $b_1^{\mathrm{op}} = b_1$ and $(x \otimes \mathbf{i}_0)b_2^{\mathrm{op}} = -x(\mathbf{i}_0 \otimes 1)b_2$. Thus we may use $h_{\mathrm{op}} = h$ for $\mathcal{D}^{\mathrm{op}}$ in place of h' for $\mathcal{D}$. Thus, A_∞-category $\mathcal{D}$ is unital. □

6.6 Corollary. *If $\mathcal{C}$ is strictly unital, then the A_∞-category $\mathcal{D}$ is unital.*

6.7 Corollary. *The A_∞-2-functor $\mathcal{A} \mapsto A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ is unitally representable for an arbitrary unital A_∞-category $\mathcal{C}$.*

Proof. By Corollary B.11 it suffices to prove unital representability for differential graded categories $\widetilde{\mathcal{C}}$ in place of $\mathcal{C}$. In this case the representing unital A_∞-category $\mathcal{D}$ exists by Corollary 6.6. □

7. The example of complexes

Let $\mathcal{A}$ be a $\Bbbk$-linear Abelian category, let $\mathcal{C} = \mathsf{C}(\mathcal{A})$ be the differential graded category of complexes in $\mathcal{A}$, and let $\mathcal{B}$ be its full subcategory of acyclic complexes. Let $\mathcal{D} = \mathsf{Q}(\mathcal{C}|\mathcal{B})$ be the quotient unital A_∞-category. The embedding $e : \mathcal{C} \hookrightarrow \mathcal{D}$ induces a $\Bbbk$-linear functor of homotopy categories

$$H^0e : \operatorname{Ho}\mathcal{C} = H^0(\mathcal{C}, m_1) \to H^0(\mathcal{D}, m_1) = \operatorname{Ho}\mathcal{D},$$

where $m_1 = sb_1s^{-1}$. Morphisms of $\operatorname{Ho}\mathcal{C}$ are homotopy equivalence classes $[q]$ of chain morphisms $q : X \to Y$.

7.1 Proposition. *For a quasi-isomorphism q its image $[qe] \in \operatorname{Ho}\mathcal{D}(X,Y)$ is invertible.*

Proof. If $q : X \to Y$ is a quasi-isomorphism, then $C = \operatorname{Cone} q$ is acyclic. The complex C is the graded object $Y \oplus X[1]$ with the differential given by the matrix

$$d^C = \begin{pmatrix} d^Y & 0 \\ s^{-1}q & d^{X[1]} \end{pmatrix} = \begin{pmatrix} d^Y & 0 \\ s^{-1}q & -s^{-1}d^X s \end{pmatrix}.$$

There is a semisplit exact sequence of chain morphisms $0 \to Y \xrightarrow{n} C \xrightarrow{k} X[1] \to 0$, where $n = \operatorname{in}^Y = (1,0)$, $k = \operatorname{pr}^{X[1]} = \binom{0}{1}$. Thus $n \in \mathcal{C}(Y,C)^0$, $k \in \mathcal{C}(C,X[1])^0$ are cycles, $nm_1 = 0$, $km_1 = 0$. The morphisms $s \in \mathcal{C}(X,X[1])^{-1}$, $s^{-1} \in \mathcal{C}(X[1],X)^1$ are also cycles, because in our conventions $sm_1 = sd^{X[1]} + d^X s = 0$, similarly $s^{-1}m_1 = 0$. Hence, $ks^{-1} \in \mathcal{C}(C,X)^1$ is also a cycle, $(ks^{-1})m_1 = 0$.

Since $C \in \operatorname{Ob}\mathcal{B}$, we have a map

$$\underline{H} = sHs^{-1} : \mathcal{C}(C,C) \longrightarrow \mathcal{D}(\mathfrak{t}_1)(C,C) \hookrightarrow \mathcal{D}(C,C), \qquad f \mapsto fsHs^{-1} = f\underline{H}.$$

It satisfies the equation $\underline{H}m_1 + m_1\underline{H} = e : \mathcal{C}(C,C) \to \mathcal{D}(C,C)$. In particular, there is an element $1_C\underline{H} \in \mathcal{D}(C,C)$. Define an element

$$p = (n \otimes 1_C\underline{H} \otimes ks^{-1})(1 \otimes m_2)m_2 \in \mathcal{D}(Y,X),$$

where $m_2 = (s \otimes s)b_2s^{-1}$. More generally, $m_n = s^{\otimes n}b_ns^{-1}$. We have $\deg p = 0$ and

$$pm_1 = -(n \otimes 1_C \otimes ks^{-1})(1 \otimes m_2)m_2 = nks^{-1} = 0.$$

Let us show that $[p] \in H^0\mathcal{D}(Y,X)$ is inverse to $[q] \in H^0\mathcal{D}(X,Y)$. Define $h = \big(X \xrightarrow{s} X[1] \xrightarrow{\mathrm{in}^{X[1]}} C\big)$, then $h \in \mathcal{C}(X,C)^{-1}$, and

$$hm_1 = hd^C + d^X h = (0,s)\begin{pmatrix} d^Y & 0 \\ s^{-1}q & -s^{-1}d^X s \end{pmatrix} + d^X(0,s) = (q,0) = qn.$$

One can check that

$$\begin{aligned}(h \otimes 1_C\underline{H} \otimes ks^{-1})(1 \otimes m_2)m_2m_1 - (q \otimes n \otimes 1_C\underline{H} \otimes ks^{-1})(1 \otimes 1 \otimes m_2)m_3m_1 \\ = -hks^{-1} + (q \otimes n \otimes 1_C\underline{H} \otimes ks^{-1})(1 \otimes 1 \otimes m_2)(1 \otimes m_2)m_2 = -1_X + (q \otimes p)m_2,\end{aligned}$$

because $hk = s$.

Denote by z the morphism $z = \mathrm{pr}^Y : C \to Y$, then $z \in \mathcal{C}(C,Y)^0$ and

$$zm_1 = zd^Y - d^C z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} d^Y - \begin{pmatrix} d^Y & 0 \\ s^{-1}q & d^{X[1]} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ s^{-1}q \end{pmatrix} = -ks^{-1}q.$$

One can check that

$$\begin{aligned}(n \otimes 1_C\underline{H} \otimes z)(1 \otimes m_2)m_2m_1 - (n \otimes 1_C\underline{H} \otimes ks^{-1} \otimes q)(1 \otimes m_3)m_2m_1 \\ - (n \otimes 1_C\underline{H} \otimes ks^{-1} \otimes q)(1 \otimes m_2 \otimes 1)m_3m_1 \\ = nz - (n \otimes 1_C\underline{H} \otimes ks^{-1} \otimes q)(1 \otimes m_2 \otimes 1)(m_2 \otimes 1)m_2 = 1_Y - (p \otimes q)m_2,\end{aligned}$$

because $nz = 1_Y$. Therefore, the cycles $p \in \mathcal{D}(Y,X)^0$ and $q \in \mathcal{D}(X,Y)^0$ are inverse to each other modulo boundaries. $\square$

7.2 Corollary. *The functor H^0e factors through a functor $D(\mathcal{A}) \to H^0\mathcal{D}$, identity on objects.*

We would like to know, when this functor is an equivalence.

A. The Yoneda Lemma for unital A_∞-categories

A.1. Basic identities in symmetric closed monoidal category of complexes. We want to work out in detail a system of notations suitable for computations in symmetric closed monoidal categories. Actually we need only the category of $\mathbb{Z}$-graded $\Bbbk$-modules with a differential of degree 1. The corresponding system of notations was already used in [Lyu03, LO02].

There exists a $\mathscr{U}'$-small set S of $\mathscr{U}$-small $\Bbbk$-modules such that any $\mathscr{U}$-small $\Bbbk$-module M is isomorphic to some $\Bbbk$-module $N \in S$ (due to presentations $\Bbbk^{(P)} \to \Bbbk^{(Q)} \to M \to 0$). We turn S into a category of $\Bbbk$-modules $\Bbbk$-mod with $\operatorname{Ob}\Bbbk\text{-mod} = S$. Thus $\Bbbk$-mod is an Abelian $\Bbbk$-linear symmetric closed monoidal $\mathscr{U}'$-small $\mathscr{U}$-category. The category $\mathsf{C} = \mathsf{C}(\Bbbk\text{-mod})$ of complexes in $\Bbbk$-mod inherits all these properties from $\Bbbk$-mod, except that the symmetry becomes $c : X \otimes Y \to Y \otimes X$, $x \otimes y \mapsto (-)^{xy} y \otimes x = (-)^{\deg x \cdot \deg y} y \otimes x$. Therefore, we may consider the category of complexes as enriched in itself (a differential graded category), and it is denoted by C in this case. The (inner) hom-object between complexes X and Y is the complex

$$\mathsf{C}(X,Y)^n = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\Bbbk}(X^i, Y^{i+n}), \qquad (f^i)_{i\in\mathbb{Z}} d = (f^i d^{i+\deg f} - (-)^{\deg f} d^i f^{i+1})_{i\in\mathbb{Z}}.$$

The product $X = \prod_{\iota\in I} X_\iota$ in the category of complexes of $\Bbbk$-modules of the family of objects $(X_\iota)_{\iota\in I}$ coincides with the product in the category of $\mathbb{Z}$-graded $\Bbbk$-modules (and differs from the product in the category of $\Bbbk$-modules). It is given by $X^m = \prod_{\iota\in I} X_\iota^m$.

Given a complex Z and an element $a \in \mathsf{C}(X,Y)$, we assign to it elements $1\otimes a \in \mathsf{C}(Z\otimes X, Z\otimes Y)$, $(z\otimes x)(1\otimes a) = z\otimes xa$, and $a\otimes 1 \in \mathsf{C}(X\otimes Z, Y\otimes Z)$, $(x\otimes z)(a\otimes 1) = (-)^{za} xa\otimes z$. Clearly, $(1\otimes a)c = c(a\otimes 1) \in \mathsf{C}(Z\otimes X, Y\otimes Z)$ and $(a\otimes 1)c = c(1\otimes a) \in \mathsf{C}(X\otimes Z, Z\otimes Y)$. If $g \in \mathsf{C}(Z,W)$, then we have $(1\otimes a)(g\otimes 1) = (-)^{ag}(g\otimes 1)(1\otimes a) \in \mathsf{C}(Z\otimes X, W\otimes Y)$ (Koszul's rule).

For any pair of complexes X, $Y \in \operatorname{Ob}\mathsf{C}$ denote by

$$\mathrm{ev}_{X,Y} : X\otimes \mathsf{C}(X,Y) \to Y, \qquad \mathrm{coev}_{X,Y} : Y \to \mathsf{C}(X, X\otimes Y)$$

the canonical evaluation and coevaluation maps respectively. Then the adjunction isomorphisms are explicitly given as follows:

$$\begin{aligned} \mathsf{C}(Y,\mathsf{C}(X,Z)) &\longleftrightarrow \mathsf{C}(X\otimes Y, Z), \\ f &\longmapsto (1_X\otimes f)\,\mathrm{ev}_{X,Z}, \\ \mathrm{coev}_{X,Y}\,\mathsf{C}(X,g) &\longleftarrow\!\shortmid\; g. \end{aligned} \tag{A.1.1}$$

Given a complex Z and an element $a \in \mathsf{C}(X,Y)$, we assign to it the element $\mathsf{C}(1,a) = \mathsf{C}(Z,a) = a\phi$ of $\mathsf{C}(\mathsf{C}(Z,X),\mathsf{C}(Z,Y))$ obtained from the equation

$$m_2^{\mathsf{C}} = \bigl(\mathsf{C}(Z,X)\otimes\mathsf{C}(X,Y) \xrightarrow[\exists!]{1\otimes\phi} \mathsf{C}(Z,X)\otimes\mathsf{C}(\mathsf{C}(Z,X),\mathsf{C}(Z,Y)) \xrightarrow{\mathrm{ev}} \mathsf{C}(Z,Y)\bigr),$$

commutative for a unique chain map ϕ. Despite that the map a is not a chain map we write this element as $a : X \to Y$, and we write $a\phi$ as

$$\mathsf{C}(1,a) : \mathsf{C}(Z,X) \to \mathsf{C}(Z,Y), \qquad (f^i)_{i\in\mathbb{Z}} \mapsto (f^i a^{i+\deg f})_{i\in\mathbb{Z}}.$$

Similarly, given a complex X and an element $g \in \mathsf{C}(W,Z)$, we assign to it the element $\mathsf{C}(g,1) = \mathsf{C}(g,X) = g\psi \in \mathsf{C}(\mathsf{C}(Z,X),\mathsf{C}(W,X))$ obtained from the diagram

$$\begin{tikzcd} \mathsf{C}(Z,X)\otimes\mathsf{C}(W,Z) \arrow[r, "c"] \arrow[d, "\exists!"', "1\otimes\psi"] & \mathsf{C}(W,Z)\otimes\mathsf{C}(Z,X) \arrow[d, "m_2^{\mathsf{C}}"] \\ \mathsf{C}(Z,X)\otimes\mathsf{C}(\mathsf{C}(Z,X),\mathsf{C}(W,X)) \arrow[r, "\mathrm{ev}"] & \mathsf{C}(W,X) \end{tikzcd}$$

commutative for a unique chain map ψ. Although the map $\mathsf{C}(g,1)$ is not a chain map we write it as

$$\mathsf{C}(g,1):\mathsf{C}(Z,X)\to\mathsf{C}(W,X),\qquad (f^i)_{i\in\mathbb{Z}}\mapsto\big((-)^{\deg f\cdot\deg g}g^if^{i+\deg g}\big)_{i\in\mathbb{Z}}.$$

For each pair of homogeneous elements $a\in\mathsf{C}(X,Y)$, $g\in\mathsf{C}(W,Z)$ we have

$$\big(\mathsf{C}(Z,X)\xrightarrow{\mathsf{C}(Z,a)}\mathsf{C}(Z,Y)\xrightarrow{\mathsf{C}(g,Y)}\mathsf{C}(W,Y)\big)=(-)^{ag}\big(\mathsf{C}(Z,X)\xrightarrow{\mathsf{C}(g,X)}\mathsf{C}(W,X)\xrightarrow{\mathsf{C}(W,a)}\mathsf{C}(W,Y)\big).$$

This equation follows from one of the standard identities in symmetric closed monoidal categories [EK66], and can be verified directly. We also have $\mathsf{C}(1,a)\mathsf{C}(1,h)=\mathsf{C}(1,ah)$ and $\mathsf{C}(g,1)\mathsf{C}(e,1)=(-)^{ge}\mathsf{C}(eg,1)$, whenever these maps are defined.

One easily sees that $m_1^{\mathsf{C}}=d:\mathsf{C}(X,Y)\to\mathsf{C}(X,Y)$ coincides with $\mathsf{C}(1,d_Y)-\mathsf{C}(d_X,1)$.

Let $f:A\otimes X\to B$, $g:B\otimes Y\to C$ be two homogeneous $\Bbbk$-linear maps of arbitrary degrees. Then the following holds:

$$\begin{aligned}&\big(X\otimes Y\xrightarrow{\operatorname{coev}_{A,X}\otimes\operatorname{coev}_{B,Y}}\mathsf{C}(A,A\otimes X)\otimes\mathsf{C}(B,B\otimes Y)\xrightarrow{\mathsf{C}(A,f)\otimes\mathsf{C}(B,g)}\mathsf{C}(A,B)\otimes\mathsf{C}(B,C)\\&\xrightarrow{m_2}\mathsf{C}(A,C)\big)=\big(X\otimes Y\xrightarrow{\operatorname{coev}_{A,X\otimes Y}}\mathsf{C}(A,A\otimes X\otimes Y)\xrightarrow{\mathsf{C}(A,f\otimes1)}\mathsf{C}(A,B\otimes Y)\xrightarrow{\mathsf{C}(A,g)}\mathsf{C}(A,C)\big).\end{aligned}\tag{A.1.2}$$

Indeed, $(\operatorname{coev}_{A,X}\otimes\operatorname{coev}_{B,Y})(\mathsf{C}(A,f)\otimes\mathsf{C}(B,g))=(\operatorname{coev}_{A,X}\mathsf{C}(A,f)\otimes\operatorname{coev}_{B,Y}\mathsf{C}(B,g))$, for coev has degree 0. Denote $\bar f=\operatorname{coev}_{A,X}\mathsf{C}(A,f)$, $\bar g=\operatorname{coev}_{B,Y}\mathsf{C}(B,g)$. The morphisms $\bar f$ and $\bar g$ correspond to f and g by adjunction. Further, the morphism m_2 comes by adjunction from the following map:

$$A\otimes\mathsf{C}(A,B)\otimes\mathsf{C}(B,C)\xrightarrow{\operatorname{ev}_{A,B}\otimes1}B\otimes\mathsf{C}(B,C)\xrightarrow{\operatorname{ev}_{B,C}}C,$$

in particular the following diagram commutes:

$$\begin{tikzcd}
A\otimes\mathsf{C}(A,B)\otimes\mathsf{C}(B,C) \arrow[r, "\operatorname{ev}_{A,B}\otimes1"] \arrow[d, "1\otimes m_2"'] & B\otimes\mathsf{C}(B,C) \arrow[d, "\operatorname{ev}_{B,C}"] \\
A\otimes\mathsf{C}(A,C) \arrow[r, "\operatorname{ev}_{A,C}"] & C
\end{tikzcd}$$

Thus we have a commutative diagram

$$\begin{tikzcd}
A\otimes X\otimes Y \arrow[r, "1\otimes\bar f\otimes1"] & A\otimes\mathsf{C}(A,B)\otimes Y \arrow[r, "\operatorname{ev}_{A,B}\otimes1"] \arrow[d, "1\otimes1\otimes\bar g"'] & B\otimes Y \arrow[d, "1\otimes\bar g"] \\
& A\otimes\mathsf{C}(A,B)\otimes\mathsf{C}(B,C) \arrow[r, "\operatorname{ev}_{A,B}\otimes1"] \arrow[d, "1\otimes m_2"'] & B\otimes\mathsf{C}(B,C) \arrow[d, "\operatorname{ev}_{B,C}"] \\
& A\otimes\mathsf{C}(A,C) \arrow[r, "\operatorname{ev}_{A,C}"] & C
\end{tikzcd}$$

The top row composite coincides with $f \otimes 1$ and the right-hand side vertical composite coincides with g (by adjunction). Thus $(1 \otimes \bar{f} \otimes 1)(1 \otimes 1 \otimes \bar{g})(1 \otimes m_2)\operatorname{ev}_{A,C}$ coincides with $(f \otimes 1)g$. But the mentioned morphism comes from $(\bar{f} \otimes \bar{g})m_2$ by adjunction, so that $(\bar{f} \otimes \bar{g})m_2 = \operatorname{coev}_{A,X\otimes Y} \mathsf{C}(A, (f \otimes 1)g)$ (the latter morphism is the image of $(f \otimes 1)g$ under the adjunction), and we are done.

One verifies similarly the following assertion [EK66]: given $f \in \mathsf{C}(A, B)$, then the diagram

$$\begin{tikzcd} X \arrow[r, "\operatorname{coev}_{A,X}"] \arrow[d, "\operatorname{coev}_{B,X}"'] & \mathsf{C}(A, A \otimes X) \arrow[d, "\mathsf{C}(A,f\otimes 1)"] \\ \mathsf{C}(B, B \otimes X) \arrow[r, "\mathsf{C}(f,B\otimes X)"] & \mathsf{C}(A, B \otimes X) \end{tikzcd} \tag{A.1.3}$$

commutes.

If P is a $\mathbb{Z}$-graded $\Bbbk$-module, then $sP = P[1]$ denotes the same $\Bbbk$-module with the grading $(sP)^d = P^{d+1}$. The "identity" map $P \to sP$ of degree -1 is also denoted s. We also define the map $\sigma : P \to P[1]$, $p \mapsto (-)^{\deg p} ps$. The map s commutes with the components of the differential b in an A_∞-category (A_∞-algebra) in the following sense: $s^{\otimes n} b_n = m_n s$. On the other hand, $\sigma^{\otimes n} b_n = (-)^n m_n \sigma$. The main identity $b^2 = 0$ written in components takes the form

$$\sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^k s\mathcal{A} \to s\mathcal{A}. \tag{A.1.4}$$

A.2. A_∞-functor h^X. Let $\mathcal{A}$ be an A_∞-category. Following Fukaya [Fuk02, Definition 7.28] for any object X of $\mathcal{A}$ we define a cocategory homomorphism $h^X : Ts\mathcal{A} \to Ts\mathsf{C}$ as follows. It maps an object Z to the complex $h^X Z = (s\mathcal{A}(X, Z), b_1)$. Actually, $h^X Z$ is some fixed complex of $\Bbbk$-modules from $\operatorname{Ob}\mathsf{C}$, with a fixed isomorphism $h^X Z \xrightarrow{\sim} (s\mathcal{A}(X, Z), b_1)$. These isomorphisms enter implicitly into the structure maps of h^X, however, we shall pretend that they are identity morphisms. The closed monoidal structure of C gives us the right to omit these isomorphisms in all the formulae.

We require h^X to be pointed, that is, $(T^0 s\mathcal{A})h^X \subset T^0 s\mathsf{C}$. Therefore, h^X is completely specified by its components h^X_k for $k \geqslant 1$:

$$\begin{aligned} h^X_k = \Big[& s\mathcal{A}(Z_0, Z_1) \otimes \cdots \otimes s\mathcal{A}(Z_{k-1}, Z_k) \xrightarrow{\operatorname{coev}} \mathsf{C}(h^X Z_0, h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k) \\ & \xrightarrow{\mathsf{C}(1, b_{k+1})} \mathsf{C}(s\mathcal{A}(X, Z_0), s\mathcal{A}(X, Z_k)) \xrightarrow{\sigma} s\mathsf{C}(s\mathcal{A}(X, Z_0), s\mathcal{A}(X, Z_k)) \Big]. \end{aligned} \tag{A.2.1}$$

We claim that h^X is actually an A_∞-functor $h^X : \mathcal{A} \to \mathsf{C}$. A similar statement is already known from Fukaya's work [Fuk02, Proposition 7.18] under slightly more restrictive general assumptions which are not needed for this result. Nevertheless, for the sake of completeness, we give a full proof in our own notations. We have to verify the following equation for an arbitrary $k \geqslant 1$:

$$\sum_{p+t+q=k} (1^{\otimes p} \otimes b_t \otimes 1^{\otimes q}) h^X_{p+1+q} = h^X_k b^{\mathsf{C}}_1 + \sum_{i+j=k} (h^X_i \otimes h^X_j) b^{\mathsf{C}}_2 : \\ h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k \to s\mathsf{C}(h^X Z_0, h^X Z_k).$$

Let us gradually expand each summand. First of all,

$$\begin{aligned} h^X_k b^{\mathsf{C}}_1 &= \operatorname{coev} \mathsf{C}(1, b_{k+1}) \sigma b^{\mathsf{C}}_1 \\ &= - \operatorname{coev} \mathsf{C}(1, b_{k+1}) m^{\mathsf{C}}_1 \sigma \\ &= - \operatorname{coev} \mathsf{C}(1, b_{k+1}) \bigl(\mathsf{C}(1, b_1) - \mathsf{C}(b_1, 1)\bigr) \sigma \\ &= - \operatorname{coev} \mathsf{C}(1, b_{k+1} b_1) \sigma - \operatorname{coev} \mathsf{C}(b_1, 1) \mathsf{C}(1, b_{k+1}) \sigma. \end{aligned}$$

Similarly,

$$\begin{aligned} (h^X_i \otimes h^X_j) b^{\mathsf{C}}_2 &= (\operatorname{coev} \mathsf{C}(1, b_{i+1}) \sigma \otimes \operatorname{coev} \mathsf{C}(1, b_{j+1}) \sigma) b^{\mathsf{C}}_2 \\ &= -(\operatorname{coev} \otimes \operatorname{coev})(\mathsf{C}(1, b_{i+1}) \otimes \mathsf{C}(1, b_{j+1})) m_2 \sigma \\ &= - \operatorname{coev} \mathsf{C}(1, (b_{i+1} \otimes 1^{\otimes j}) b_{j+1}) \sigma. \end{aligned} \tag{A.2.2}$$

The sign is due to the fact σ and b have odd degree. The last line is due to (A.1.2).

Further, given $\alpha : Y \to U$, then the following diagram commutes

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[r, "1 \otimes \alpha \otimes 1"] \arrow[d, "\operatorname{coev}_{A, X \otimes Y \otimes Z}"'] & X \otimes U \otimes Z \arrow[d, "\operatorname{coev}_{A, X \otimes U \otimes Z}"] \\ \mathsf{C}(A, A \otimes X \otimes Y \otimes Z) \arrow[r, "\mathsf{C}(A, 1 \otimes 1 \otimes \alpha \otimes 1)"] & \mathsf{C}(A, A \otimes X \otimes U \otimes Z), \end{tikzcd}$$

for coev is a functor morphism. Applying this to $\alpha = b_t$ we obtain

$$\begin{aligned} (1^{\otimes p} \otimes b_t \otimes 1^{\otimes q}) h^X_{p+1+q} &= (1^{\otimes p} \otimes b_t \otimes 1^{\otimes q}) \operatorname{coev} \mathsf{C}(1, b_{p+2+q}) \sigma \\ &= \operatorname{coev} \mathsf{C}(1, (1^{\otimes p+1} \otimes b_t \otimes 1^{\otimes q}) b_{p+2+q}) \sigma. \end{aligned} \tag{A.2.3}$$

Thus,

$$\sum_{p+t+q=k} (1^{\otimes p} \otimes b_t \otimes 1^{\otimes q}) h^X_{p+1+q} = \operatorname{coev} \mathsf{C}\bigl(1, \sum_{\substack{p+t+q=k+1, \\ p \geqslant 1}} (1^{\otimes p} \otimes b_t \otimes 1^{\otimes q}) b_{p+1+q}\bigr) \sigma,$$

so that

$$\sum_{p+t+q=k} (1^{\otimes p} \otimes b_t \otimes 1^{\otimes q}) h^X_{p+1+q} - h^X_k b^{\mathsf{C}}_1 - \sum_{\substack{i+j=k, \\ i,j \geqslant 1}} (h^X_i \otimes h^X_j) b^{\mathsf{C}}_2 = \operatorname{coev} \mathsf{C}(b_1, 1) \mathsf{C}(1, b_{k+1}) \sigma \\ + \operatorname{coev} \mathsf{C}\bigl(1, \sum_{\substack{p+t+q=k+1, \\ p \geqslant 1}} (1^{\otimes p} \otimes b_t \otimes 1^{\otimes q}) b_{p+1+q} + b_{k+1} b_1 + \sum_{\substack{i+j=k, \\ i,j \geqslant 1}} (b_{i+1} \otimes 1^{\otimes j}) b_{j+1}\bigr) \sigma.$$

Finally, $\operatorname{coev} \mathsf{C}(b_1, 1) = \operatorname{coev} \mathsf{C}(1, b_1 \otimes 1^{\otimes k})$ by (A.1.3), and the sum in the right-hand side of the above formula vanishes by equation (A.1.4).

A.3 Lemma. *Let $\mathcal{A}$ be a unital A_∞-category. Then for each object X of $\mathcal{A}$ the A_∞-functor $h^X : \mathcal{A} \to \mathsf{C}$ is unital.*

Proof. For arbitrary objects Y, Z of $\mathcal{A}$ the map h^X_1 is given by

$$h^X_1 = \bigl[s\mathcal{A}(Y,Z) \xrightarrow{\text{coev}} \mathsf{C}(h^XY, h^XY \otimes h^YZ) \xrightarrow{\mathsf{C}(1,b_2)} \mathsf{C}(h^XY, h^XZ) \xrightarrow{\sigma} s\mathsf{C}(h^XY, h^XZ)\bigr].$$

We need this formula for $Z = Y$. We have to find such $v \in (s\mathsf{C})^{-2}(h^XY, h^XY)$ that

$${}_Y\mathbf{i}^{\mathcal{A}}_0 h^X_1 - {}_{h^XY}\mathbf{i}^{\mathsf{C}}_0 = v b^{\mathsf{C}}_1 : \Bbbk \to (s\mathsf{C})^{-1}(h^XY, h^XY).$$

We have ${}_{h^XY}\mathbf{i}^{\mathsf{C}}_0 = 1_{h^XY} s$. We seek for an element v in the form $v = h's$, where $h' \in \mathsf{C}^{-1}(h^XY, h^XY)$. The required equation is:

$$\bigl[\Bbbk \xrightarrow{{}_Y\mathbf{i}^{\mathcal{A}}_0} (s\mathcal{A})^{-1}(Y,Y) \xrightarrow{\text{coev}} \mathsf{C}^{-1}(h^XY, h^XY \otimes h^YY) \xrightarrow{\mathsf{C}(1,b_2)} \mathsf{C}^0(h^XY, h^XY)\bigr]$$
$$= \bigl[\Bbbk \xrightarrow{1_{h^XY} + h'm^{\mathsf{C}}_1} \mathsf{C}^0(h^XY, h^XY)\bigr].$$

By functoriality of coev we have ${}_Y\mathbf{i}^{\mathcal{A}}_0 \,\text{coev}_{h^XY,h^YY} = \text{coev}_{h^XY,\Bbbk}\, \mathsf{C}(h^XY, 1 \otimes {}_Y\mathbf{i}^{\mathcal{A}}_0)$, and the above equation reduces to

$$\bigl(h^XY = h^XY \otimes \Bbbk \xrightarrow{1 \otimes {}_Y\mathbf{i}^{\mathcal{A}}_0} h^XY \otimes h^YY \xrightarrow{b_2} h^XY\bigr) = \bigl(h^XY \xrightarrow{1 + h'b_1 + b_1h'} h^XY\bigr).$$

The existence of such a homotopy h' immediately follows from the definition of a unital A_∞-category, see [Lyu03, Lemma 7.4]. □

A.4. The opposite A_∞-category. Let $\mathcal{A}$ be a graded $\Bbbk$-quiver. Then its *opposite quiver* $\mathcal{A}^{\text{op}}$ is defined as the quiver with the same class of objects $\operatorname{Ob}\mathcal{A}^{\text{op}} = \operatorname{Ob}\mathcal{A}$, and with graded $\Bbbk$-modules of morphisms $\mathcal{A}^{\text{op}}(X,Y) = \mathcal{A}(Y,X)$.

Let $\gamma : Ts\mathcal{A}^{\text{op}} \to Ts\mathcal{A}$ denote the following cocategory anti-isomorphism:

$$\gamma = (-1)^k \omega^0_c : s\mathcal{A}^{\text{op}}(X_0,X_1) \otimes \dots \otimes s\mathcal{A}^{\text{op}}(X_{k-1},X_k) \to s\mathcal{A}(X_k,X_{k-1}) \otimes \dots \otimes s\mathcal{A}(X_1,X_0), \tag{A.4.1}$$

where the permutation $\omega^0 = \left(\begin{smallmatrix} 1 & 2 & \dots & k-1 & k \\ k & k-1 & \dots & 2 & 1 \end{smallmatrix}\right)$ is the longest element of $\mathfrak{S}_k$, and ω^0_c is the corresponding signed permutation, the action of ω^0 in tensor products via standard symmetry. Clearly, $\gamma\Delta = \Delta(\gamma \otimes \gamma)c = \Delta c(\gamma \otimes \gamma)$, which is the anti-isomorphism property. Notice also that $(\mathcal{A}^{\text{op}})^{\text{op}} = \mathcal{A}$ and $\gamma^2 = \text{id}$.

When $\mathcal{A}$ is an A_∞-category with the codifferential $b : Ts\mathcal{A} \to Ts\mathcal{A}$, then $\gamma b \gamma : Ts\mathcal{A}^{\text{op}} \to Ts\mathcal{A}^{\text{op}}$ is also a codifferential. Indeed,

$$\gamma b \gamma \Delta = \gamma b \Delta c(\gamma \otimes \gamma) = \gamma \Delta (1 \otimes b + b \otimes 1) c(\gamma \otimes \gamma) = \Delta(\gamma \otimes \gamma) c(1 \otimes b + b \otimes 1) c(\gamma \otimes \gamma)$$
$$= \Delta(\gamma \otimes \gamma)(b \otimes 1 + 1 \otimes b)(\gamma \otimes \gamma) = \Delta(\gamma b \gamma \otimes 1 + 1 \otimes \gamma b \gamma).$$

A.5 Definition (cf. Fukaya [Fuk02] Definition 7.8). The *opposite A_∞-category* $\mathcal{A}^{\text{op}}$ to an A_∞-category $\mathcal{A}$ is the opposite quiver, equipped with the codifferential $b^{\text{op}} = \gamma b \gamma : Ts\mathcal{A}^{\text{op}} \to Ts\mathcal{A}^{\text{op}}$.

The components of b^{op} are computed as follows:

$$
\begin{aligned}
b^{\mathrm{op}}_k = (-)^{k+1}\bigl[& s\mathcal{A}^{\mathrm{op}}(X_0,X_1)\otimes\dots\otimes s\mathcal{A}^{\mathrm{op}}(X_{k-1},X_k) \\
& \xrightarrow{\omega^0_c} s\mathcal{A}(X_k,X_{k-1})\otimes\dots\otimes s\mathcal{A}(X_1,X_0) \xrightarrow{b_k} s\mathcal{A}(X_k,X_0) = s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\bigr].
\end{aligned}
$$

The sign $(-1)^k$ in (A.4.1) ensures that the above definition agrees with the definition of the opposite usual category.

A.6. The Yoneda A_∞-functor. Since $\mathcal{A}$ is $\mathscr{U}$-small, and C is a $\mathscr{U}'$-small $\mathscr{U}$-category, $A_\infty(\mathcal{A},\mathsf{C})$ is a $\mathscr{U}'$-small differential graded $\mathscr{U}$-category as well. Indeed, every its set of morphisms $sA_\infty(\mathcal{A},\mathsf{C})(f,g)$ is isomorphic to the product of graded $\Bbbk$-modules

$$
\prod_{k=0}^{\infty}\prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \mathsf{C}\bigl(T^k s\mathcal{A}(X,Y), s\mathsf{C}(Xf,Yg)\bigr),
$$

that is $\mathscr{U}$-small.

Denote by Y a pointed cocategory homomorphism $Ts\mathcal{A}^{\mathrm{op}} \to TsA_\infty(\mathcal{A},\mathsf{C})$ given as follows: on objects $X\mapsto h^X$, the components

$$
Y_n : s\mathcal{A}^{\mathrm{op}}(X_0,X_1)\otimes\dots\otimes s\mathcal{A}^{\mathrm{op}}(X_{n-1},X_n) \to sA_\infty(\mathcal{A},\mathsf{C})(h^{X_0},h^{X_n}) \tag{A.6.1}
$$

are determined by the following formulas. The composition of Y_n with

$$
\mathrm{pr}_k : sA_\infty(\mathcal{A},\mathsf{C})(h^{X_0},h^{X_n}) \to \mathsf{C}(h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k))
$$

(that is, the k-th component of the coderivation $(x_1\otimes\dots\otimes x_n)Y_n$) is given by the formula

$$
\begin{aligned}
Y_{nk} = (-)^n\bigl[& s\mathcal{A}(X_1,X_0)\otimes\dots\otimes s\mathcal{A}(X_n,X_{n-1}) \xrightarrow{\mathrm{coev}} \\
& \mathsf{C}(h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\dots\otimes h^{X_n}X_{n-1}) \\
& \xrightarrow{\mathsf{C}(1,\tau_c b_{k+n+1})} \mathsf{C}(h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, h^{X_n}Z_k) \\
& \xrightarrow{\sim} \mathsf{C}(h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, \mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)) \\
& \xrightarrow{\mathsf{C}(1,\sigma)} \mathsf{C}(h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k))\bigr],
\end{aligned}
$$

where the permutation $\tau\in\mathfrak{S}_{k+n+1}$ is $\tau^{k,n} = \left(\begin{smallmatrix} 0 & 1 & \dots & k & k+1 & \dots & k+n \\ n & 1+n & \dots & k+n & n-1 & \dots & 0 \end{smallmatrix}\right)$.

In other words, the coderivation $(x_1\otimes\dots\otimes x_n)Y_n$ has components $s\mathcal{A}(Z_0,Z_1)\otimes\dots\otimes s\mathcal{A}(Z_{k-1},Z_k) \to s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)$, $(z_1\otimes\dots\otimes z_k)\mapsto(z_1\otimes\dots\otimes z_k\otimes x_1\otimes\dots\otimes x_n)Y'_{nk}$, where

$$
\begin{aligned}
Y'_{nk} = (-)^n\bigl[& s\mathcal{A}(Z_0,Z_1)\otimes\dots\otimes s\mathcal{A}(Z_{k-1},Z_k)\otimes s\mathcal{A}(X_1,X_0)\otimes\dots\otimes s\mathcal{A}(X_n,X_{n-1}) \\
& \xrightarrow{\mathrm{coev}} \mathsf{C}(h^{X_0}Z_0, h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\dots\otimes h^{X_n}X_{n-1}) \\
& \xrightarrow{\mathsf{C}(1,\tau_c b_{k+n+1})} \mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k) \xrightarrow{\sigma} s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\bigr].
\end{aligned} \tag{A.6.2}
$$

We claim that Y is an A_∞-functor. An equivalent statement is already proved by Fukaya [Fuk02, Lemma 9.8]. Since our conventions (and even definitions) differ from Fukaya's, it would take some efforts to establish an equivalence. Instead, we give a proof in our notations.

A.7 Proposition. *The pointed cocategory homomorphism Y is an A_∞-functor $\mathcal{A}^{\mathrm{op}} \to A_\infty(\mathcal{A}, \mathsf{C})$.*

Proof. The equation to prove is the following:

$$\sum_{l>0;i_1+\dots+i_l=n} (Y_{i_1} \otimes Y_{i_2} \otimes \dots \otimes Y_{i_l}) B_l = \sum_{\alpha+p+\beta=n} (1^{\otimes\alpha} \otimes b_p^{\mathrm{op}} \otimes 1^{\otimes\beta}) Y_{\alpha+1+\beta} : \\ s\mathcal{A}^{\mathrm{op}}(X_0, X_1) \otimes \dots \otimes s\mathcal{A}^{\mathrm{op}}(X_{n-1}, X_n) \to sA_\infty(\mathcal{A}, \mathsf{C})(h^{X_0}, h^{X_n}).$$

We compare the k-th components of the both sides:

$$[Y_n B_1]_k + \sum_{p+q=n} [(Y_p \otimes Y_q) B_2]_k = \sum_{\alpha+p+\beta=n} [(1^{\otimes\alpha} \otimes b_p^{\mathrm{op}} \otimes 1^{\otimes\beta}) Y_{\alpha+1+\beta}]_k : \\ s\mathcal{A}(X_1, X_0) \otimes \dots \otimes s\mathcal{A}(X_n, X_{n-1}) \to sA_\infty(\mathcal{A}, \mathsf{C})(h^{X_0}, h^{X_n}) \\ \xrightarrow{\mathrm{pr}_k} \mathsf{C}(h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k, s\mathsf{C}(h^{X_0} Z_0, h^{X_n} Z_k)).$$

Let us denote by $\bar{r}$ an element of $s\mathcal{A}(X_1, X_0) \otimes \dots \otimes s\mathcal{A}(X_n, X_{n-1})$. The terms in the left hand side are expanded to

$$[(\bar{r} Y_n) B_1]_k = (\bar{r} Y_n)_k b_1^{\mathsf{C}} \\ + \sum_{i+j=k}^{i>0} (h_i^{X_0} \otimes (\bar{r} Y_n)_j) b_2^{\mathsf{C}} + \sum_{i+j=k}^{j>0} ((\bar{r} Y_n)_i \otimes h_j^{X_n}) b_2^{\mathsf{C}} - (-)^r \sum_{\alpha+p+\beta=k} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta}) (\bar{r} Y_n)_{\alpha+1+\beta},$$

$$[(Y_p \otimes Y_q) B_2]_k = \sum_{i+j=k}^{p,q>0} (Y'_{pi} \otimes Y'_{qj}) b_2^{\mathsf{C}}.$$

To reduce further the number of terms we introduce the notation $Y'_{0k} = h_k^X$. Indeed, when $n = 0$, formulas (A.6.2) and (A.2.1) coincide. So the left hand side expands to the following expression:

$$[(\bar{r} Y_n) B_1]_k + \sum_{p+q=n} [(Y_p \otimes Y_q) B_2]_k \\ = (\bar{r} Y_n)_k b_1^{\mathsf{C}} + \sum_{i+j=k}^{p,q\geqslant 0} (Y'_{pi} \otimes Y'_{qj}) b_2^{\mathsf{C}} - (-)^r \sum_{\alpha+p+\beta=k} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta}) (\bar{r} Y_n)_{\alpha+1+\beta},$$

Thus the equation to prove is this one:

$$\big[h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}\xrightarrow{Y'_{nk}} s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)$$

$$\xrightarrow{b_1^{\mathsf{C}}} s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\big]+\sum_{\substack{p+q=n\\ i+j=k}}^{\substack{p,q,i,j\geqslant 0\\ p+i>0,q+j>0}}\big[h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}\xrightarrow{1\otimes c\otimes 1}$$

$$h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{i-1}}Z_i\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_p}X_{p-1}\otimes h^{Z_i}Z_{i+1}\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_{p+1}}X_p\otimes\cdots\otimes h^{X_n}X_{n-1}$$

$$\xrightarrow{Y'_{pi}\otimes Y'_{qj}} s\mathsf{C}(h^{X_0}Z_0,h^{X_p}Z_i)\otimes s\mathsf{C}(h^{X_p}Z_i,h^{X_n}Z_k)\xrightarrow{b_2^{\mathsf{C}}} s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\big]$$

$$-\sum_{\alpha+p+\beta=k}\big[h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}\xrightarrow{1^{\otimes\alpha}\otimes b_p\otimes 1^{\otimes\beta+n}}$$

$$h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{\alpha-1}}Z_\alpha\otimes h^{Z_\alpha}Z_{\alpha+p}\otimes h^{Z_{\alpha+p}}Z_{\alpha+p+1}\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}$$

$$\xrightarrow{Y'_{n,\alpha+1+\beta}} s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\big]$$

$$=\sum_{\alpha+p+\beta=n}\big[h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}\xrightarrow{1^{\otimes k+\alpha}\otimes b_p^{\mathrm{op}}\otimes 1^{\otimes\beta}}$$

$$h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_\alpha}X_{\alpha-1}\otimes h^{X_{\alpha+p}}X_\alpha\otimes h^{X_{\alpha+p+1}}X_{\alpha+p}\otimes\cdots\otimes h^{X_n}X_{n-1}$$

$$\xrightarrow{Y'_{\alpha+1+\beta,k}} s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\big].$$

Substituting explicit expression for Y'_{nk} we get the following equation to verify:

$$-(-)^n\big[h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}$$

$$\xrightarrow{\mathrm{coev}} \mathsf{C}(h^{X_0}Z_0,h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1})$$

$$\xrightarrow{\mathsf{C}(1,\tau_c^{n,k}b_{k+n+1})} \mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\xrightarrow{\mathsf{C}(1,b_1)-\mathsf{C}(b_1,1)} \mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\xrightarrow{\sigma} s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\big]$$

$$-\sum_{\substack{p+q=n\\ i+j=k}}^{\substack{p,q,i,j\geqslant 0\\ p+i>0,q+j>0}}(-)^{p+q}\big[h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}\xrightarrow{1^{\otimes i}\otimes c\otimes 1^{\otimes n-p}}$$

$$h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{i-1}}Z_i\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_p}X_{p-1}\otimes h^{Z_i}Z_{i+1}\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_{p+1}}X_p\otimes\cdots\otimes h^{X_n}X_{n-1}$$

$$\xrightarrow{\mathrm{coev}\otimes\mathrm{coev}} \mathsf{C}(h^{X_0}Z_0,h^{X_0}Z_0\otimes h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{i-1}}Z_i\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_p}X_{p-1})$$

$$\otimes\,\mathsf{C}(h^{X_p}Z_i,h^{X_p}Z_i\otimes h^{Z_i}Z_{i+1}\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_{p+1}}X_p\otimes\cdots\otimes h^{X_n}X_{n-1})$$

$$\xrightarrow{\mathsf{C}(1,\tau_c^{i,p}b_{1+i+p})\otimes\mathsf{C}(1,\tau_c^{j,q}b_{1+j+q})} \mathsf{C}(h^{X_0}Z_0,h^{X_p}Z_i)\otimes\mathsf{C}(h^{X_p}Z_i,h^{X_n}Z_k)\xrightarrow{m_2^{\mathsf{C}}} \mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)$$

$$\xrightarrow{\sigma} s\mathsf{C}(h^{X_0}Z_0,h^{X_n}Z_k)\big]$$

$$-(-)^n\sum_{\alpha+p+\beta=k}\big[h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}\xrightarrow{1^{\otimes\alpha}\otimes b_p\otimes 1^{\otimes\beta+n}}$$

$$h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{\alpha-1}}Z_\alpha\otimes h^{Z_\alpha}Z_{\alpha+p}\otimes h^{Z_{\alpha+p}}Z_{\alpha+p+1}\otimes\cdots\otimes h^{Z_{k-1}}Z_k\otimes h^{X_1}X_0\otimes\cdots\otimes h^{X_n}X_{n-1}$$

$$
\xrightarrow{\mathrm{coev}} \mathsf{C}(h^{X_0}Z_0, h^{X_0}Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{\alpha-1}}Z_\alpha \otimes h^{Z_\alpha}Z_{\alpha+p} \otimes h^{Z_{\alpha+p}}Z_{\alpha+p+1} \otimes \dots \otimes h^{Z_{k-1}}Z_k
$$

$$
\otimes h^{X_1}X_0 \otimes \dots \otimes h^{X_n}X_{n-1}) \xrightarrow{\mathsf{C}(1,\tau_c^{\alpha+1+\beta,n}b_{n+\alpha+\beta+2})} \mathsf{C}(h^{X_0}Z_0, h^{X_n}Z_k) \xrightarrow{\sigma} s\mathsf{C}(h^{X_0}Z_0, h^{X_n}Z_k)\big]
$$

$$
= \sum_{\alpha+p+\beta=n} (-1)^{\alpha+\beta+p}\big[h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k \otimes h^{X_1}X_0 \otimes \dots \otimes h^{X_n}X_{n-1} \xrightarrow{1^{\otimes k+\alpha}\otimes(\omega_c^0 b_p)\otimes 1^{\otimes\beta}}
$$

$$
h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k \otimes h^{X_1}X_0 \otimes \dots \otimes h^{X_\alpha}X_{\alpha-1} \otimes h^{X_{\alpha+p}}X_\alpha \otimes h^{X_{\alpha+p+1}}X_{\alpha+p} \otimes \dots \otimes h^{X_n}X_{n-1}
$$

$$
\xrightarrow{\mathrm{coev}} \mathsf{C}(h^{X_0}Z_0, h^{X_0}Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k \otimes h^{X_1}X_0 \otimes \dots \otimes h^{X_\alpha}X_{\alpha-1} \otimes h^{X_{\alpha+p}}X_\alpha \otimes
$$

$$
h^{X_{\alpha+p+1}}X_{\alpha+p} \otimes \dots \otimes h^{X_n}X_{n-1}) \xrightarrow{\mathsf{C}(1,\tau_c^{k,\alpha+1+\beta}b_{k+\alpha+\beta+2})} \mathsf{C}(h^{X_0}Z_0, h^{X_n}Z_k)
$$

$$
\xrightarrow{\sigma} s\mathsf{C}(h^{X_0}Z_0, h^{X_n}Z_k)\big].
$$

Using functoriality of coev, identities (A.1.2) and (A.1.3) we reduce all terms of this equation to the form

$$
\big[h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k \otimes h^{X_1}X_0 \otimes \dots \otimes h^{X_n}X_{n-1}
$$

$$
\xrightarrow{\mathrm{coev}} \mathsf{C}(h^{X_0}Z_0, h^{X_0}Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k \otimes h^{X_1}X_0 \otimes \dots \otimes h^{X_n}X_{n-1})
$$

$$
\longrightarrow \dots \longrightarrow \mathsf{C}(h^{X_0}Z_0, h^{X_n}Z_k) \xrightarrow{\sigma} s\mathsf{C}(h^{X_0}Z_0, h^{X_n}Z_k)\big].
$$

Cancelling the above coev and σ, we obtain an equation between complexes and maps of the form $\mathsf{C}(h^{X_0}Z_0, \_)$. Simplifying it we get an equation between maps $\_ : h^{X_0}Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k \otimes h^{X_1}X_0 \otimes \dots \otimes h^{X_n}X_{n-1} \to h^{X_n}Z_k$, whose corollary is the required identity. Multiplying it with the composite symmetry

$$
\left(\begin{smallmatrix} 1 & \dots & n-1 & n & n+1 & \dots & n+k+1 \\ k+n+1 & \dots & k+3 & k+2 & 1 & \dots & k+1 \end{smallmatrix}\right)_c : h^{X_n}X_{n-1} \otimes \dots \otimes h^{X_1}X_0 \otimes h^{X_0}Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k
$$

$$
\to h^{X_0}Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k \otimes h^{X_1}X_0 \otimes \dots \otimes h^{X_n}X_{n-1}
$$

and with the common sign $(-1)^n$ we get the equation

$$
b_{n+1+k}b_1 + (1^{\otimes n} \otimes b_1 \otimes 1^{\otimes k})b_{n+1+k} + \sum_{\substack{p+q=n \\ i+j=k}}^{\substack{p,q,i,j\geqslant 0 \\ p+i>0,q+j>0}} (1^{\otimes q} \otimes b_{p+1+i} \otimes 1^{\otimes j})b_{q+1+j}
$$

$$
+ \sum_{\alpha+p+\beta=k} (1^{\otimes n+1+\alpha} \otimes b_p \otimes 1^{\otimes\beta})b_{n+1+\alpha+1+\beta} + \sum_{\alpha+p+\beta=n} (1^{\otimes\alpha} \otimes b_p \otimes 1^{\otimes\beta+1+k})b_{\alpha+1+\beta+1+k} = 0.
$$

This is identity (A.1.4), so the proof is finished. □

A.8. The Yoneda embedding. Now our goal is to prove that the Yoneda A_∞-functor Y is an equivalence of $\mathcal{A}^{\mathrm{op}}$ with its image. This is already proven by Fukaya in the case of strictly unital A_∞-category $\mathcal{A}$ [Fuk02, Theorem 9.1]. We are going to extend this result to arbitrary unital A_∞-categories.

A.9 Proposition. *Let $\mathcal{A}$ be a unital A_∞-category. Let X, W be objects of $\mathcal{A}$. Then the map $Y_1 : s\mathcal{A}^{\mathrm{op}}(X,W) \to sA_\infty(\mathcal{A},\mathsf{C})(h^X,h^W)$ is homotopy invertible.*

Proof. The composition of Y_1 with

$$\mathrm{pr}_k : sA_\infty(\mathcal{A},\mathsf{C})(h^X,h^W) \to \mathsf{C}(h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k, s\mathsf{C}(h^X Z_0, h^W Z_k))$$

is given by the formula

$$\begin{aligned} Y_{1k} = -\bigl[& s\mathcal{A}(W,X) \xrightarrow{\mathrm{coev}} \\ & \mathsf{C}(h^X Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k, h^X Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k \otimes h^W X) \\ & \xrightarrow{\mathsf{C}(1,\tau_c^{k,1}b_{k+2})} \mathsf{C}(h^X Z_0 \otimes h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k, h^W Z_k) \\ & \xrightarrow{\sim} \mathsf{C}(h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k, \mathsf{C}(h^X Z_0, h^W Z_k)) \\ & \xrightarrow{\mathsf{C}(1,\sigma)} \mathsf{C}(h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{k-1}}Z_k, s\mathsf{C}(h^X Z_0, h^W Z_k))\bigr], \end{aligned}$$

where the permutation $\tau^{k,1} \in \mathfrak{S}_{k+2}$ is $\tau^{k,1} = \bigl(\begin{smallmatrix} 0 & 1 & \dots & k & k+1 \\ 1 & 2 & \dots & k+1 & 0 \end{smallmatrix}\bigr)$.

Define a degree 0 map $\alpha : sA_\infty(\mathcal{A},\mathsf{C})(h^X,h^W) \to s\mathcal{A}^{\mathrm{op}}(X,W)$ as follows:

$$\begin{aligned} \alpha = \bigl[& sA_\infty(\mathcal{A},\mathsf{C})(h^X,h^W) \xrightarrow{\mathrm{pr}_0} s\mathsf{C}(h^X X, h^W X) \\ & \xrightarrow{\sigma^{-1}} \mathsf{C}(h^X X, h^W X) \xrightarrow{\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \mathsf{C}(\Bbbk, h^W X) = h^W X\bigr]. \end{aligned}$$

It is a chain map. Indeed, pr_0 is a chain map, and so is $\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)$, since

$$\begin{aligned} \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)b_1 &= \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\mathsf{C}(1,b_1) \\ &= -\sigma^{-1}[\mathsf{C}(1,b_1) - \mathsf{C}(b_1,1)]\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1) = b_1^{\mathsf{C}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1). \end{aligned}$$

Let us prove that α is homotopy inverse to Y_1. First of all,

$$Y_{10} = -\bigl[h^W X \xrightarrow{\mathrm{coev}} \mathsf{C}(h^X Z, h^X Z \otimes h^W X) \xrightarrow{\mathsf{C}(1,cb_2)} \mathsf{C}(h^X Z, h^W Z) \xrightarrow{\sigma} s\mathsf{C}(h^X Z, h^W Z)\bigr].$$

Using this expression for $Z = X$ we compute

$$\begin{aligned} Y_1\alpha &= -\bigl[h^W X \xrightarrow{\mathrm{coev}} \mathsf{C}(h^X X, h^X X \otimes h^W X) \xrightarrow{\mathsf{C}(1,cb_2)} \mathsf{C}(h^X X, h^W X) \\ &\qquad \xrightarrow{\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \mathsf{C}(\Bbbk, h^W X) = h^W X\bigr] \\ &= \bigl[h^W X \xrightarrow{\mathrm{coev}} \mathsf{C}(h^X X, h^X X \otimes h^W X) \xrightarrow{\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \mathsf{C}(\Bbbk, h^X X \otimes h^W X) \\ &\qquad \xrightarrow{\mathsf{C}(1,cb_2)} \mathsf{C}(\Bbbk, h^W X) = h^W X\bigr] \\ &= \bigl[h^W X \xrightarrow{\mathrm{coev}} \mathsf{C}(\Bbbk, \Bbbk \otimes h^W X) \xrightarrow{\mathsf{C}(1,{}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)} \mathsf{C}(\Bbbk, h^X X \otimes h^W X) \\ &\qquad \xrightarrow{\mathsf{C}(1,cb_2)} \mathsf{C}(\Bbbk, h^W X) = h^W X\bigr] \\ &= \bigl[h^W X = \Bbbk \otimes h^W X \xrightarrow{{}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1} h^X X \otimes h^W X \xrightarrow{cb_2} h^W X\bigr] \\ &= \bigl[h^W X = h^W X \otimes \Bbbk \xrightarrow{1\otimes{}_X\mathbf{i}_0^{\mathcal{A}}} h^W X \otimes h^X X \xrightarrow{b_2} h^W X\bigr]. \end{aligned}$$

Therefore, the map $Y_1\alpha$ is homotopic to identity map $\mathrm{id} : h^W X \to h^W X$ by [Lyu03, Lemma 7.4]. Now we are going to prove that αY_1 is homotopy invertible.

The graded $\Bbbk$-module $sA_\infty(\mathcal{A}, \mathsf{C})(h^X, h^W)$ is $V = \prod_{n=0}^{\infty} V_n$, where

$$V_n = \prod_{Z_0,\dots,Z_n \in \operatorname{Ob}\mathcal{A}} \mathsf{C}(h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{n-1}}Z_n, s\mathsf{C}(h^X Z_0, h^W Z_n))$$

and all products are taken in the category of graded $\Bbbk$-modules. In other terms, for $d \in \mathbb{Z}$

$$V^d = \prod_{n=0}^{\infty} V_n^d, \qquad V_n^d = \prod_{Z_0,\dots,Z_n \in \operatorname{Ob}\mathcal{A}} \mathsf{C}(h^{Z_0}Z_1 \otimes \dots \otimes h^{Z_{n-1}}Z_n, s\mathsf{C}(h^X Z_0, h^W Z_n))^d.$$

We consider V_n^d as Abelian groups with discrete topology. The Abelian group V^d is equipped with the topology of the product. Thus, its basis of neighborhoods of 0 is given by $\Bbbk$-submodules $\Phi_m^d = 0^{m-1} \times \prod_{n=m}^{\infty} V_n^d$. They form a filtration $V^d = \Phi_0^d \supset \Phi_1^d \supset \Phi_2^d \supset \dots$. We call a $\Bbbk$-linear map $a : V \to V$ of degree p continuous if the induced maps $a^{d,d+p} = a\big|_{V^d} : V^d \to V^{d+p}$ are continuous for all $d \in \mathbb{Z}$. This holds if and only if for any $d \in \mathbb{Z}$ and $m \in \mathbb{N} \overset{\text{def}}{=} \mathbb{Z}_{\geqslant 0}$ there exists an integer $\kappa = \kappa(d,m) \in \mathbb{N}$ such that $(\Phi_\kappa^d)a \subset \Phi_m^{d+p}$. We may assume that

$$m' < m'' \quad \text{implies} \quad \kappa(d,m') \leqslant \kappa(d,m''). \tag{A.9.1}$$

Indeed, a given function $m \mapsto \kappa(d,m)$ can be replaced with the function $m \mapsto \kappa'(d,m) = \min_{n \geqslant m} \kappa(d,n)$ and κ' satisfies condition (A.9.1). Continuous linear maps $a : V \to V$ of degree p are in bijection with families of $\mathbb{N} \times \mathbb{N}$-matrices $(A^{d,d+p})_{d\in\mathbb{Z}}$ of linear maps $A_{nm}^{d,d+p} : V_n^d \to V_m^{d+p}$ with finite number of non-vanishing elements in each column of $A^{d,d+p}$. Indeed, to each continuous map $a^{d,d+p} : V^d \to V^{d+p}$ corresponds the inductive limit over m of $\kappa(d,m) \times m$-matrices of maps $V^d/\Phi_{\kappa(d,m)}^d \to V^{d+p}/\Phi_m^{d+p}$. On the other hand, to each family $(A^{d,d+p})_{d\in\mathbb{Z}}$ of $\mathbb{N}\times\mathbb{N}$-matrices with finite number of non-vanishing elements in each column correspond obvious maps $a^{d,d+p} : V^d \to V^{d+p}$, and they are continuous. Thus, $a = (a^{d,d+p})_{d\in\mathbb{Z}}$ is continuous. A continuous map $a : V \to V$ can be completely recovered from one $\mathbb{N} \times \mathbb{N}$-matrix $(a_{nm})_{n,m\in\mathbb{N}}$ of maps $a_{nm} = (A_{nm}^{d,d+p})_{d\in\mathbb{Z}} : V_n \to V_m$ of degree p. Naturally, not any such matrix determines a continuous map, however, if the number of non-vanishing elements in each column of (a_{nm}) is finite, then this matrix does determine a continuous map.

The differential $D \overset{\text{def}}{=} B_1 : V \to V$, $r \mapsto (r)B_1 = rb - (-)^r br$ is continuous and the function κ for it is simply $\kappa(d,m) = m$. Its matrix is given by

$$D = B_1 = \begin{bmatrix} D_{0,0} & D_{0,1} & D_{0,2} & \dots \\ 0 & D_{1,1} & D_{1,2} & \dots \\ 0 & 0 & D_{2,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$D_{k,k} = \mathsf{C}(1, b_1^{\mathsf{C}}) - \mathsf{C}\bigl(\sum_{\alpha+1+\gamma=k} 1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\gamma}, 1\bigr) : V_k \to V_k,$$

$$r_k D_{k,k} = r_k b_1^{\mathsf{C}} - (-)^r \sum_{\alpha+1+\gamma=k} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\gamma}) r_k,$$

(one easily recognizes the differential in the complex V_k),

$$r_k D_{k,k+1} = (r_k \otimes h_1^W) b_2^{\mathsf{C}} + (h_1^X \otimes r_k) b_2^{\mathsf{C}} - (-)^r \sum_{\alpha+\gamma=k-1} (1^{\otimes\alpha} \otimes b_2 \otimes 1^{\otimes\gamma}) r_k.$$

Further we will see that we do not need to compute other components.

Composition of αY_1 with pr_k equals

$$\begin{aligned}
\alpha Y_{1k} = -\bigl[sA_\infty(\mathcal{A}, \mathsf{C})(h^X, h^W) \xrightarrow{\mathrm{pr}_0} s\mathsf{C}(h^X X, h^W X) \xrightarrow{\sigma^{-1}} \mathsf{C}(h^X X, h^W X) \\
\xrightarrow{\mathsf{C}({}_X \mathbf{i}_0^{\mathcal{A}}, 1)} \mathsf{C}(\Bbbk, h^W X) = h^W X \xrightarrow{\mathrm{coev}} \\
\mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k, h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k \otimes h^W X) \\
\xrightarrow{\mathsf{C}(1, \tau_c^{k,1} b_{k+2})} \mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k, h^W Z_k) \\
\xrightarrow{\sim} \mathsf{C}(h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X Z_0, h^W Z_k)) \\
\xrightarrow{\mathsf{C}(1,\sigma)} \mathsf{C}(h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k, s\mathsf{C}(h^X Z_0, h^W Z_k))\bigr].
\end{aligned}$$

Clearly, αY_1 is continuous (take $\kappa(d,m) = 1$). Its $\mathbb{N} \times \mathbb{N}$-matrix has the form

$$\alpha Y_1 = \begin{bmatrix} * & * & * & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The reader might skip the proof of the following lemma at first reading.

A.10 Lemma. *The map $\alpha Y_1 : V \to V$ is homotopic to a continuous map $V \to V$, whose $\mathbb{N} \times \mathbb{N}$-matrix is upper-triangular with the identity maps $\mathrm{id} : V_k \to V_k$ on the diagonal.*

Proof. Define a continuous $\Bbbk$-linear map $H : sA_\infty(\mathcal{A}, \mathsf{C})(h^X, h^W) \to sA_\infty(\mathcal{A}, \mathsf{C})(h^X, h^W)$ of degree -1 by its matrix

$$H = \begin{bmatrix} 0 & 0 & 0 & \dots \\ H_{1,0} & 0 & 0 & \dots \\ 0 & H_{2,1} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

so $\kappa(d,m) = m+1$, where $H_{k+1,k}$ maps the factor indexed by $(X, Z_0, \dots, Z_k)$ to the factor indexed by $(Z_0, \dots, Z_k)$ as follows:

$$
\begin{aligned}
H_{k+1,k} = \bigl[&\mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, s\mathsf{C}(h^X X, h^W Z_k)) \\
&\xrightarrow{\mathsf{C}(1,\sigma^{-1})} \mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X X, h^W Z_k)) \\
&\xrightarrow{\mathsf{C}(1,\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(\Bbbk, h^W Z_k)) \\
&\xrightarrow{\sim} \mathsf{C}(h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X Z_0, h^W Z_k)) \\
&\xrightarrow{\mathsf{C}(1,\sigma)} \mathsf{C}(h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, s\mathsf{C}(h^X Z_0, h^W Z_k))\bigr].
\end{aligned}
$$

Other factors are ignored. Here the adjunction isomorphism is that of (A.1.1).

The composition of continuous maps $V \to V$ is continuous as well. In particular, one finds the matrices of $B_1 H$ and HB_1:

$$
B_1 H = \begin{bmatrix}
D_{0,1}H_{1,0} & D_{0,2}H_{2,1} & D_{0,3}H_{3,2} & \dots \\
D_{1,1}H_{1,0} & D_{1,2}H_{2,1} & D_{1,3}H_{3,2} & \dots \\
0 & D_{2,2}H_{2,1} & D_{2,3}H_{3,2} & \dots \\
0 & 0 & D_{3,3}H_{3,2} & \dots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix},
$$

$$
HB_1 = \begin{bmatrix}
0 & 0 & 0 & \dots \\
H_{1,0}D_{0,0} & H_{1,0}D_{0,1} & H_{1,0}D_{0,2} & \dots \\
0 & H_{2,1}D_{1,1} & H_{2,1}D_{1,2} & \dots \\
0 & 0 & H_{3,2}D_{2,2} & \dots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

We have $D_{k+1,k+1}H_{k+1,k} + H_{k+1,k}D_{k,k} = 0$ for all $k \geqslant 0$. Indeed, conjugating the expanded left hand side with $\mathsf{C}(1,\sigma)$ we come to the following identity

$$
\begin{aligned}
&\bigl[\mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X X, h^W Z_k)) \xrightarrow{-\mathsf{C}(1,m_1^{\mathsf{C}})+\mathsf{C}(\sum_{\alpha+\gamma=k} 1^{\otimes\alpha}\otimes b_1\otimes 1^{\otimes\gamma},1)} \\
&\mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X X, h^W Z_k)) \xrightarrow{\mathsf{C}(1,\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \\
&\mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(\Bbbk, h^W Z_k)) \xrightarrow{\sim} \mathsf{C}(h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X Z_0, h^W Z_k))\bigr] \\
&+ \bigl[\mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X X, h^W Z_k)) \xrightarrow{\mathsf{C}(1,\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \\
&\mathsf{C}(h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(\Bbbk, h^W Z_k)) \xrightarrow{\sim} \mathsf{C}(h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X Z_0, h^W Z_k)) \\
&\xrightarrow{-\mathsf{C}(1,m_1^{\mathsf{C}})+\mathsf{C}(\sum_{\alpha+\gamma=k-1} 1^{\otimes\alpha}\otimes b_1\otimes 1^{\otimes\gamma},1)} \mathsf{C}(h^{Z_0} Z_1 \otimes \cdots \otimes h^{Z_{k-1}} Z_k, \mathsf{C}(h^X Z_0, h^W Z_k))\bigr] = 0.
\end{aligned}
$$

After reducing all terms to the common form, beginning with $\mathsf{C}(1, \mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1))$, all terms cancel each other, so the identity is proven.

Therefore, the chain map $a = \alpha Y_1 + B_1 H + HB_1$ is also represented by an upper-triangular matrix. Its diagonal elements are chain maps $a_{kk} : V_k \to V_k$. We are going to show that they are homotopic to identity maps.

First of all, let us compute the matrix element $a_{00} : V_0 \to V_0 = \prod_{Z\in\mathrm{Ob}\,\mathcal{A}} s\mathsf{C}(h^X Z, h^W Z)$. We have

$$a_{00}\,\mathrm{pr}_Z = (\alpha Y_{10} + B_1 H_{1,0})\,\mathrm{pr}_Z : V_0 \to V_0 \xrightarrow{\mathrm{pr}_Z} s\mathsf{C}(h^X Z, h^W Z).$$

In the expanded form these terms are as follows:

$$\alpha Y_{10}\,\mathrm{pr}_Z = -\bigl[V_0 \xrightarrow{\mathrm{pr}_X} s\mathsf{C}(h^X X, h^W X) \xrightarrow{\sigma^{-1}} \mathsf{C}(h^X X, h^W X) \xrightarrow{\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \mathsf{C}(\Bbbk, h^W X) = h^W X$$
$$\xrightarrow{\mathrm{coev}} \mathsf{C}(h^X Z, h^X Z \otimes h^W X) \xrightarrow{\mathsf{C}(1,cb_2)} \mathsf{C}(h^X Z, h^W Z) \xrightarrow{\sigma} s\mathsf{C}(h^X Z, h^W Z)\bigr],$$

$$B_1 H_{1,0}\,\mathrm{pr}_Z = [(1\otimes h_1^W)b_2 + (h_1^X \otimes 1)b_2]H_{1,0}\,\mathrm{pr}_Z,$$

$$(1\otimes h_1^W)b_2 = \bigl[V_0 \xrightarrow{\mathrm{pr}_X} s\mathsf{C}(h^X X, h^W X) \xrightarrow{\mathrm{coev}} \mathsf{C}(h^X Z, h^X Z \otimes s\mathsf{C}(h^X X, h^W X))$$
$$\xrightarrow{\mathsf{C}(1,c(1\otimes\mathrm{coev}))} \mathsf{C}(h^X Z, s\mathsf{C}(h^X X, h^W X) \otimes \mathsf{C}(h^W X, h^W X \otimes h^X Z))$$
$$\xrightarrow{\mathsf{C}(1,1\otimes\mathsf{C}(1,b_2))} \mathsf{C}(h^X Z, s\mathsf{C}(h^X X, h^W X) \otimes \mathsf{C}(h^W X, h^W Z)) \xrightarrow{\mathsf{C}(1,1\otimes\sigma)}$$
$$\mathsf{C}(h^X Z, s\mathsf{C}(h^X X, h^W X) \otimes s\mathsf{C}(h^W X, h^W Z)) \xrightarrow{\mathsf{C}(1,b_2^{\mathsf{C}})} \mathsf{C}(h^X Z, s\mathsf{C}(h^X X, h^W Z))\bigr],$$

$$(h_1^X\otimes 1)b_2 = \bigl[V_0 \xrightarrow{\mathrm{pr}_Z} s\mathsf{C}(h^X Z, h^W Z) \xrightarrow{\mathrm{coev}} \mathsf{C}(h^X Z, h^X Z \otimes s\mathsf{C}(h^X Z, h^W Z))$$
$$\xrightarrow{\mathsf{C}(1,\mathrm{coev}\otimes 1)} \mathsf{C}(h^X Z, \mathsf{C}(h^X X, h^X X \otimes h^X Z) \otimes s\mathsf{C}(h^X Z, h^W Z))$$
$$\xrightarrow{\mathsf{C}(1,\mathsf{C}(1,b_2)\otimes 1)} \mathsf{C}(h^X Z, \mathsf{C}(h^X X, h^X Z) \otimes s\mathsf{C}(h^X Z, h^W Z)) \xrightarrow{\mathsf{C}(1,\sigma\otimes 1)}$$
$$\mathsf{C}(h^X Z, s\mathsf{C}(h^X X, h^X Z) \otimes s\mathsf{C}(h^X Z, h^W Z)) \xrightarrow{\mathsf{C}(1,b_2^{\mathsf{C}})} \mathsf{C}(h^X Z, s\mathsf{C}(h^X X, h^W Z))\bigr],$$

$$H_{1,0}\,\mathrm{pr}_Z = \bigl[V_1 \xrightarrow{\mathrm{pr}_{X,Z}} \mathsf{C}(h^X Z, s\mathsf{C}(h^X X, h^W Z)) \xrightarrow{\mathsf{C}(1,\sigma^{-1})} \mathsf{C}(h^X Z, \mathsf{C}(h^X X, h^W Z))$$
$$\xrightarrow{\mathsf{C}(1,\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \mathsf{C}(h^X Z, \mathsf{C}(\Bbbk, h^W Z)) = \mathsf{C}(h^X Z, h^W Z) \xrightarrow{\sigma} s\mathsf{C}(h^X Z, h^W Z)\bigr].$$

We claim that in the sum

$$a_{00}\,\mathrm{pr}_Z = \alpha Y_{10}\,\mathrm{pr}_Z + (1\otimes h_1^W)b_2 H_{1,0}\,\mathrm{pr}_Z + (h_1^X\otimes 1)b_2 H_{1,0}\,\mathrm{pr}_Z$$

the first two summands cancel each other, while the last, $(h_1^X\otimes 1)b_2 H_{1,0}$ is homotopic to identity. Indeed, $\alpha Y_{10}\operatorname{pr}_Z + (1\otimes h_1^W)b_2H_{1,0}\operatorname{pr}_Z$ factors through

$$
\begin{aligned}
-\bigl[s\mathsf{C}(h^XX,h^WX) &\xrightarrow{\sigma^{-1}} \mathsf{C}(h^XX,h^WX) \xrightarrow{\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \mathsf{C}(\Bbbk,h^WX) = h^WX \\
&\xrightarrow{\mathrm{coev}} \mathsf{C}(h^XZ,h^XZ\otimes h^WX) \xrightarrow{\mathsf{C}(1,cb_2)} \mathsf{C}(h^XZ,h^WZ)\bigr] \\
+\bigl[s\mathsf{C}(h^XX,h^WX) &\xrightarrow{\mathrm{coev}} \mathsf{C}(h^XZ,h^XZ\otimes s\mathsf{C}(h^XX,h^WX)) \\
&\xrightarrow{\mathsf{C}(1,c(1\otimes\mathrm{coev}))} \mathsf{C}(h^XZ,s\mathsf{C}(h^XX,h^WX)\otimes\mathsf{C}(h^WX,h^WX\otimes h^XZ)) \\
&\xrightarrow{\mathsf{C}(1,1\otimes\mathsf{C}(1,b_2))} \mathsf{C}(h^XZ,s\mathsf{C}(h^XX,h^WX)\otimes\mathsf{C}(h^WX,h^WZ)) \\
&\xrightarrow{\mathsf{C}(1,1\otimes\sigma)} \mathsf{C}(h^XZ,s\mathsf{C}(h^XX,h^WX)\otimes s\mathsf{C}(h^WX,h^WZ)) \xrightarrow{\mathsf{C}(1,b_2^{\mathsf{C}})} \mathsf{C}(h^XZ,s\mathsf{C}(h^XX,h^WZ)) \\
&\xrightarrow{\mathsf{C}(1,\sigma^{-1})} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^WZ)) \xrightarrow{\mathsf{C}(1,\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \mathsf{C}(h^XZ,\mathsf{C}(\Bbbk,h^WZ)) = \mathsf{C}(h^XZ,h^WZ)\bigr].
\end{aligned}
$$

Composing this sum with σ we get

$$
\begin{aligned}
-\bigl[\mathsf{C}(h^XX,h^WX) &\xrightarrow{\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \mathsf{C}(\Bbbk,h^WX) = h^WX \\
&\xrightarrow{\mathrm{coev}} \mathsf{C}(h^XZ,h^XZ\otimes h^WX) \xrightarrow{\mathsf{C}(1,cb_2)} \mathsf{C}(h^XZ,h^WZ)\bigr] \\
-\bigl[\mathsf{C}(h^XX,h^WX) &\xrightarrow{\mathrm{coev}} \mathsf{C}(h^XZ,h^XZ\otimes\mathsf{C}(h^XX,h^WX)) \\
&\xrightarrow{\mathsf{C}(1,c(1\otimes\mathrm{coev}))} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^WX)\otimes\mathsf{C}(h^WX,h^WX\otimes h^XZ)) \\
&\xrightarrow{\mathsf{C}(1,1\otimes\mathsf{C}(1,b_2))} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^WX)\otimes\mathsf{C}(h^WX,h^WZ)) \xrightarrow{\mathsf{C}(1,m_2^{\mathsf{C}})} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^WZ)) \\
&\xrightarrow{\mathsf{C}(1,\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \mathsf{C}(h^XZ,\mathsf{C}(\Bbbk,h^WZ)) = \mathsf{C}(h^XZ,h^WZ)\bigr].
\end{aligned}
$$

This expression vanishes by the properties of closed monoidal category C. Therefore, $\alpha Y_{10}\operatorname{pr}_Z + (1\otimes h_1^W)b_2H_{1,0}\operatorname{pr}_Z$ also vanishes.

Now we prove that $(h_1^X\otimes 1)b_2H_{1,0}$ is homotopic to identity. It maps each factor $s\mathsf{C}(h^XZ,h^WZ)$ into itself via the following map:

$$
\begin{aligned}
\bigl[s\mathsf{C}(h^XZ,h^WZ) &\xrightarrow{\mathrm{coev}} \mathsf{C}(h^XZ,h^XZ\otimes s\mathsf{C}(h^XZ,h^WZ)) \\
&\xrightarrow{\mathsf{C}(1,\mathrm{coev}\otimes 1)} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^XX\otimes h^XZ)\otimes s\mathsf{C}(h^XZ,h^WZ)) \\
&\xrightarrow{\mathsf{C}(1,\mathsf{C}(1,b_2)\otimes 1)} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^XZ)\otimes s\mathsf{C}(h^XZ,h^WZ)) \\
&\xrightarrow{\mathsf{C}(1,\sigma\otimes 1)} \mathsf{C}(h^XZ,s\mathsf{C}(h^XX,h^XZ)\otimes s\mathsf{C}(h^XZ,h^WZ)) \\
&\xrightarrow{\mathsf{C}(1,b_2^{\mathsf{C}})} \mathsf{C}(h^XZ,s\mathsf{C}(h^XX,h^WZ)) \xrightarrow{\mathsf{C}(1,\sigma^{-1})} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^WZ)) \\
&\xrightarrow{\mathsf{C}(1,\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \mathsf{C}(h^XZ,\mathsf{C}(\Bbbk,h^WZ)) = \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma} s\mathsf{C}(h^XZ,h^WZ)\bigr]
\end{aligned}
$$

$$= \big[s\mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma^{-1}} \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\mathrm{coev}} \mathsf{C}(h^XZ,h^XZ\otimes\mathsf{C}(h^XZ,h^WZ))$$
$$\xrightarrow{\mathsf{C}(1,\mathrm{coev}\otimes 1)} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^XX\otimes h^XZ)\otimes\mathsf{C}(h^XZ,h^WZ))$$
$$\xrightarrow{\mathsf{C}(1,\mathsf{C}(1,b_2)\otimes 1)} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^XZ)\otimes\mathsf{C}(h^XZ,h^WZ)) \xrightarrow{\mathsf{C}(1,m_2^{\mathsf{C}})} \mathsf{C}(h^XZ,\mathsf{C}(h^XX,h^WZ))$$
$$\xrightarrow{\mathsf{C}(1,\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \mathsf{C}(h^XZ,\mathsf{C}(\Bbbk,h^WZ)) = \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma} s\mathsf{C}(h^XZ,h^WZ)\big].$$

$$= -\big[s\mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma^{-1}} \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\mathrm{coev}} \mathsf{C}(h^XZ,h^XZ\otimes\mathsf{C}(h^XZ,h^WZ))$$
$$\xrightarrow{\mathsf{C}(1,({}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)b_2\otimes 1)} \mathsf{C}(h^XZ,h^XZ\otimes\mathsf{C}(h^XZ,h^WZ)) \xrightarrow{\mathsf{C}(1,\mathrm{ev})} \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma} s\mathsf{C}(h^XZ,h^WZ)\big].$$

Due to [Lyu03, Lemma 7.4] there exists a homotopy $h'' : h^XZ \to h^XZ$ (a map of degree -1), such that $({}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)b_2 = -1 + h''b_1 + b_1h''$. Therefore, the map considered above equals to

$$\mathrm{id}_{s\mathsf{C}(h^XZ,h^WZ)} - \big[s\mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma^{-1}} \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\mathrm{coev}} \mathsf{C}(h^XZ,h^XZ\otimes\mathsf{C}(h^XZ,h^WZ))$$
$$\xrightarrow{\mathsf{C}(1,(h''b_1+b_1h'')\otimes 1)} \mathsf{C}(h^XZ,h^XZ\otimes\mathsf{C}(h^XZ,h^WZ)) \xrightarrow{\mathsf{C}(1,\mathrm{ev})} \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma} s\mathsf{C}(h^XZ,h^WZ)\big]$$
$$= \big(\mathrm{id}_{s\mathsf{C}(h^XZ,h^WZ)} + b_1^{\mathsf{C}}H'_{00} + H'_{00}b_1^{\mathsf{C}}\big),$$

where

$$H'_{00} = \big[s\mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma^{-1}} \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\mathsf{C}(h'',1)} \mathsf{C}(h^XZ,h^WZ) \xrightarrow{\sigma} s\mathsf{C}(h^XZ,h^WZ)\big].$$

Therefore, $(h_1^X\otimes 1)b_2H_{1,0}$ and a_{00} are homotopic to identity.

Now we are proving that diagonal elements $a_{kk} : V_k \to V_k$ are homotopic to identity maps for $k>0$. Using the explicit formula for adjunction isomorphism we can re-write the formula for H as follows. An element $r_{k+1}\in V_{k+1}$ is mapped to direct product over $Z_0,\dots,Z_k\in\operatorname{Ob}\mathcal{A}$ of

$$r_{k+1}H_{k+1,k} = \mathrm{coev}_{h^XZ_0,*}\,\mathsf{C}(h^XZ_0, r_{k+1}^{X,Z_0,\dots,Z_k}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))\sigma :$$
$$h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k \to s\mathsf{C}(h^XZ_0,h^WZ_k).$$

Here $r_{k+1}^{X,Z_0,\dots,Z_k} : h^XZ_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k \to s\mathsf{C}(h^XX,h^WZ_k)$ is one of the coordinates of r_{k+1}.

Thus, $r_kD_{k,k+1}H_{k+1,k}$ is the sum of three terms (A.10.1a)–(A.10.1c):

$$\mathrm{coev}_{h^XZ_0,*}\,\mathsf{C}(h^XZ_0,(r_k^{X,Z_0,\dots,Z_{k-1}}\otimes h_1^W)b_2^{\mathsf{C}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))\sigma :$$
$$h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k \to s\mathsf{C}(h^XZ_0,h^WZ_k), \quad \text{(A.10.1a)}$$

where $r_k^{X,Z_0,\dots,Z_{k-1}} : h^XZ_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-2}}Z_{k-1} \to s\mathsf{C}(h^XX,h^WZ_{k-1})$. Since $b_2^{\mathsf{C}} = (\sigma\otimes\sigma)^{-1}m_2\sigma = -(\sigma^{-1}\otimes\sigma^{-1})m_2\sigma$, we have

$$(r_k\otimes h_1^W)b_2^{\mathsf{C}}\sigma^{-1} = -(r_k\sigma^{-1}\otimes h_1^W\sigma^{-1})m_2^{\mathsf{C}}.$$

Further, for any $a : X \to A$ the following equation holds:

$$\bigl(\mathsf{C}(A,B)\otimes\mathsf{C}(B,C) \xrightarrow{m_2} \mathsf{C}(A,C) \xrightarrow{\mathsf{C}(a,C)} \mathsf{C}(X,C)\bigr)$$
$$= \bigl(\mathsf{C}(A,B)\otimes\mathsf{C}(B,C) \xrightarrow{\mathsf{C}(a,B)\otimes 1} \mathsf{C}(X,B)\otimes\mathsf{C}(B,C) \xrightarrow{m_2} \mathsf{C}(X,C)\bigr).$$

Thus,

$$(r_k\otimes h_1^W)b_2^{\mathsf{C}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1) = -(r_k\sigma^{-1}\otimes h_1^W\sigma^{-1})m_2^{\mathsf{C}}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)$$
$$= -(r_k\sigma^{-1}\otimes h_1^W\sigma^{-1})(\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1)m_2^{\mathsf{C}} = (r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes h_1^W\sigma^{-1})m_2^{\mathsf{C}}$$

(we have used the fact that $\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)$ has degree -1 and $h_1^W\sigma^{-1}$ has degree 1).

$$\operatorname{coev}_{h^X Z_0,*}\mathsf{C}(h^X Z_0,(h_1^X\otimes r_k^{Z_0,\dots,Z_k})b_2^{\mathsf{C}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))\sigma :$$
$$h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k \to s\mathsf{C}(h^X Z_0, h^W Z_k), \quad \text{(A.10.1b)}$$

where $r_k^{Z_0,\dots,Z_k} : h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k \to s\mathsf{C}(h^X Z_0, h^W Z_k)$. Similarly to above

$$(h_1^X\otimes r_k)b_2^{\mathsf{C}}\sigma^{-1} = (-)^{r+1}(h_1^X\sigma^{-1}\otimes r_k\sigma^{-1})m_2^{\mathsf{C}},$$

so that

$$(h_1^X\otimes r_k)b_2^{\mathsf{C}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1) = (-)^{r+1}(h_1^X\sigma^{-1}\otimes r_k\sigma^{-1})(\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1)m_2^{\mathsf{C}}$$
$$= (h_1^X\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes r_k\sigma^{-1})m_2^{\mathsf{C}}$$

($r_k\sigma^{-1}$ has degree $\deg r+1$ and $\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)$ has degree -1).

For each α, γ, such that $\alpha+\gamma=k-1$

$$\operatorname{coev}_{h^X Z_0,*}\mathsf{C}(h^X Z_0,(1^{\otimes\alpha}\otimes b_2\otimes 1^{\otimes\gamma})r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))\sigma :$$
$$h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k \to s\mathsf{C}(h^X Z_0, h^W Z_k), \quad \text{(A.10.1c)}$$

where r_k means $r_k^{X,Z_0,\dots,Z_{\alpha-1},Z_{\alpha+1},\dots,Z_k} : h^X Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{\alpha-1}}Z_{\alpha+1}\otimes\dots\otimes h^{Z_{k-1}}Z_k \to s\mathsf{C}(h^X X, h^W Z_k)$, and $Z_{-1}=X$.

Thus, $r_k D_{k,k+1}H_{k+1,k} = \operatorname{coev}_{h^X Z_0,*}\mathsf{C}(h^X Z_0,\Sigma_1)\sigma$, where

$$\Sigma_1 = (r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes h_1^W\sigma^{-1})m_2^{\mathsf{C}} + (h_1^X\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes r_k^{Z_0,\dots,Z_k}\sigma^{-1})m_2^{\mathsf{C}}$$
$$-(-)^r\sum_{\alpha+\gamma=k-1}(1^{\otimes\alpha}\otimes b_2\otimes 1^{\otimes\gamma})r_k^{X,Z_0,\dots,Z_{\alpha-1},Z_{\alpha+1},\dots,Z_k}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1) :$$
$$h^X Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k \to h^W Z_k.$$

Similarly, $r_k H_{k,k-1}D_{k-1,k}$ is the sum of three terms (A.10.2a)–(A.10.2c).

$$\bigl(\operatorname{coev}_{h^X Z_0,*}\mathsf{C}(h^X Z_0, r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))\sigma\otimes\operatorname{coev}_{h^W Z_{k-1},h^{Z_{k-1}}Z_k}\mathsf{C}(h^W Z_{k-1},b_2)\sigma\bigr)b_2^{\mathsf{C}}$$
$$= -\bigl(\operatorname{coev}_{h^X Z_0,*}\mathsf{C}(h^X Z_0, r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1))\otimes\operatorname{coev}_{h^W Z_{k-1},h^{Z_{k-1}}Z_k}\mathsf{C}(h^W Z_{k-1},b_2)\bigr)m_2\sigma$$
$$= -\operatorname{coev}_{h^X Z_0,*}\mathsf{C}\bigl(h^X Z_0,(r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1)b_2\bigr)\sigma :$$
$$h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k \to s\mathsf{C}(h^X Z_0, h^W Z_k), \quad \text{(A.10.2a)}$$

where r_k means $r_k^{X,Z_0,\dots,Z_{k-1}} : h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-2}} Z_{k-1} \to s\mathsf{C}(h^X X, h^W Z_{k-1})$. Here we use (A.1.2) (compare with (A.2.2)).

$$\begin{aligned}
&\big(\mathrm{coev}_{h^X Z_0, h^{Z_0} Z_1} \mathsf{C}(h^X Z_0, b_2)\sigma \otimes \mathrm{coev}_{h^X Z_1,*} \mathsf{C}(h^X Z_1, r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1))\sigma\big) b_2^{\mathsf{C}} \\
&\quad = (-)^r \big(\mathrm{coev}_{h^X Z_0, h^{Z_0} Z_1} \mathsf{C}(h^X Z_0, b_2) \otimes \mathrm{coev}_{h^X Z_1,*} \mathsf{C}(h^X Z_1, r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1))\big) m_2^{\mathsf{C}} \sigma \\
&\qquad\qquad = (-)^r \mathrm{coev}_{h^X Z_0,*} \mathsf{C}\big(h^X Z_0, (b_2 \otimes 1^{\otimes k-1}) r_k \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1)\big)\sigma : \\
&\qquad\qquad\qquad\qquad h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k \to s\mathsf{C}(h^X Z_0, h^W Z_k), \quad \text{(A.10.2b)}
\end{aligned}$$

where r_k means the coordinate $r_k^{X,Z_1,\dots,Z_k} : h^X Z_1 \otimes h^{Z_1} Z_2 \otimes \dots \otimes h^{Z_{k-1}} Z_k \to s\mathsf{C}(h^X X, h^W Z_k)$. We have used that $r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1)$ has degree $\deg r$ and formula (A.1.2).

$$\begin{aligned}
&(1^{\otimes\alpha} \otimes b_2 \otimes 1^{\otimes\gamma}) \mathrm{coev}_{h^X Z_0,*} \mathsf{C}\big(h^X Z_0, r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1)\big)\sigma \\
&\qquad = \mathrm{coev}_{h^X Z_0,*} \mathsf{C}(h^X Z_0, 1^{\otimes\alpha+1} \otimes b_2 \otimes 1^{\otimes\gamma}) \mathsf{C}\big(h^X Z_0, r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1)\big)\sigma \\
&\qquad\quad = \mathrm{coev}_{h^X Z_0,*} \mathsf{C}\big(h^X Z_0, (1^{\otimes\alpha+1} \otimes b_2 \otimes 1^{\otimes\gamma}) r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1)\big)\sigma : \\
&\qquad\qquad\qquad\qquad h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k \to s\mathsf{C}(h^X Z_0, h^W Z_k), \quad \text{(A.10.2c)}
\end{aligned}$$

where r_k means the coordinate $r_k^{X,Z_0,\dots,Z_\alpha,Z_{\alpha+2},\dots,Z_k} : h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_\alpha} Z_{\alpha+2} \otimes \dots \otimes h^{Z_{k-1}} Z_k \to s\mathsf{C}(h^X X, h^W Z_k)$. Here we use functoriality of coev (compare with (A.2.3)).

Thus, $r_k H_{k,k-1} D_{k-1,k} = \mathrm{coev}_{h^X Z_0,*} \mathsf{C}(h^X Z_0, \Sigma_2)\sigma$, where

$$\begin{aligned}
\Sigma_2 = &-(r_k^{X,Z_0,\dots,Z_{k-1}} \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1) \otimes 1) b_2 + (-)^r (b_2 \otimes 1^{\otimes k-1}) r_k^{X,Z_1,\dots,Z_k} \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1) \\
&- (-)^{r-1} \sum_{\alpha+\gamma=k-2} (1^{\otimes\alpha+1} \otimes b_2 \otimes 1^{\otimes\gamma}) r_k^{X,Z_0,\dots,Z_\alpha,Z_{\alpha+2},\dots,Z_k} \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1) : \\
&\qquad\qquad\qquad\qquad h^X Z_0 \otimes h^{Z_0} Z_1 \otimes \dots \otimes h^{Z_{k-1}} Z_k \to h^W Z_k.
\end{aligned}$$

The element rH has degree $\deg r - 1$, so the sign $(-)^{r-1}$ arises. Combining this with the expression for $r_k D_{k,k+1} H_{k+1,k}$ we obtain

$$r_k D_{k,k+1} H_{k+1,k} + r_k H_{k,k-1} D_{k-1,k} = \mathrm{coev}_{h^X Z_0,*} \mathsf{C}(h^X Z_0, \Sigma)\sigma,$$

where $\Sigma = \Sigma_1 + \Sigma_2$. We claim that $\Sigma = (h_1^X \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1) \otimes r_k^{Z_0,\dots,Z_k}\sigma^{-1}) m_2$. Indeed, first of all,

$$\begin{aligned}
&(b_2 \otimes 1^{\otimes k-1}) r_k^{X,Z_1,\dots,Z_k} \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1) \\
&\qquad + \sum_{\alpha+\gamma=k-2} (1^{\otimes\alpha+1} \otimes b_2 \otimes 1^{\otimes\gamma}) r_k^{X,Z_0,\dots,Z_\alpha,Z_{\alpha+2},\dots,Z_k} \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1) \\
&\qquad\qquad = \sum_{\alpha+\gamma=k-1} (1^{\otimes\alpha} \otimes b_2 \otimes 1^{\otimes\gamma}) r_k^{X,Z_0,\dots,Z_{\alpha-1},Z_{\alpha+1},\dots,Z_k} \sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}}, 1),
\end{aligned}$$

so that

$$\Sigma = (r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes h_1^W\sigma^{-1})m_2^{\mathsf{C}} + (h_1^X\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes r_k^{Z_0,\dots,Z_k}\sigma^{-1})m_2^{\mathsf{C}}$$
$$- (r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1)b_2 : h^X Z_0\otimes h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k\to h^W Z_k.$$

Further, for all $y\in h^X Z_0$, $z_i\in h^{Z_{i-1}}Z_i$ we have the action of the third summand

$$\begin{aligned}(y\otimes z_1\otimes\dots\otimes z_{k-1}\otimes z_k)&(r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1)b_2\\ &= (-)^{rz_k}((y\otimes z_1\otimes\dots\otimes z_{k-1})r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes z_k)b_2\\ &= (-)^{rz_k+(r+1)+(y+z_1+\dots+z_{k-1})}(({}_X\mathbf{i}_0^{\mathcal{A}})(y\otimes z_1\otimes\dots\otimes z_{k-1})r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\otimes z_k)b_2.\end{aligned}$$

Notice that $r_k\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)$ has degree $\deg r$, and $(y\otimes z_1\otimes\dots\otimes z_{k-1})r_k\sigma^{-1}$ has degree $(r+1)+(y+z_1+\dots+z_{k-1})$. Similarly, the action of the first summand is

$$\begin{aligned}(y\otimes z_1\otimes\dots\otimes z_{k-1}\otimes z_k)&(r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes h_1^W\sigma^{-1})m_2^{\mathsf{C}}\\ &= (-)^{rz_k+(r+1)+(y+z_1+\dots+z_k)}(({}_X\mathbf{i}_0^{\mathcal{A}})(y\otimes z_1\otimes\dots\otimes z_{k-1})r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\otimes z_k h_1^W\sigma^{-1})m_2^{\mathsf{C}},\end{aligned}$$

and since $z_k h_1^W\sigma^{-1} = z_k\operatorname{coev}_{h^W Z_{k-1},h^{Z_{k-1}}Z_k}\mathsf{C}(h^W Z_{k-1},b_2) = (t\mapsto(t\otimes z_k)b_2)$, we obtain

$$\begin{aligned}(y\otimes z_1\otimes\dots\otimes z_{k-1}\otimes z_k)&(r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes h_1^W\sigma^{-1})m_2^{\mathsf{C}}\\ &= (-)^{rz_k+(r+1)+(y+z_1+\dots+z_k)}(({}_X\mathbf{i}_0^{\mathcal{A}})(y\otimes z_1\otimes\dots\otimes z_{k-1})r_k^{X,Z_0,\dots,Z_{k-1}}\sigma^{-1}\otimes z_k)b_2.\end{aligned}$$

We see that the first and the third terms in Σ cancel each other. Hence, only the second summand remains in $\Sigma = (h_1^X\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes r_k^{Z_0,\dots,Z_k}\sigma^{-1})m_2^{\mathsf{C}}$,

$$\begin{aligned}(y\otimes z_1\otimes\dots\otimes z_k)&(h_1^X\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes r_k^{Z_0,\dots,Z_k}\sigma^{-1})m_2^{\mathsf{C}}\\ &= (yh_1^X\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes(z_1\otimes\dots\otimes z_k)r_k^{Z_0,\dots,Z_k}\sigma^{-1})m_2^{\mathsf{C}}.\end{aligned}$$

We have

$$yh_1^X\sigma^{-1} = (y)\operatorname{coev}_{h^X X,h^X Z_0}\mathsf{C}(h^X X,b_2) = (t\mapsto(t\otimes y)b_2)$$

so that $yh_1^X\sigma^{-1}\mathsf{C}({}_X\mathbf{i}_0^{\mathcal{A}},1) = (-)^{y+1}({}_X\mathbf{i}_0^{\mathcal{A}}\otimes y)b_2 = -(y)({}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)b_2 = y - yh''b_1 - yb_1h''$. Then

$$(z_1\otimes\dots\otimes z_k)\operatorname{coev}_{h^X Z_0,*}\mathsf{C}(h^X Z_0,\Sigma) = \big(y\mapsto(y-yh''b_1-yb_1h'')(z_1\otimes\dots\otimes z_k)(r_k^{Z_0,\dots,Z_k}\sigma^{-1})\big)$$

and

$$(z_1\otimes\dots\otimes z_k)\operatorname{coev}_{h^X Z_0,*}\mathsf{C}(h^X Z_0,\Sigma)\sigma = \big(y\mapsto(y-yh''b_1-yb_1h'')(z_1\otimes\dots\otimes z_k)r_k^{Z_0,\dots,Z_k}\big).$$

We see that for $k>0$

$$\begin{aligned}&a_{kk} = D_{k,k+1}H_{k+1,k} + H_{k,k-1}D_{k-1,k} = 1+f :\\ &\mathsf{C}(h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, s\mathsf{C}(h^X Z_0,h^W Z_k))\to\mathsf{C}(h^{Z_0}Z_1\otimes\dots\otimes h^{Z_{k-1}}Z_k, s\mathsf{C}(h^X Z_0,h^W Z_k)),\end{aligned}$$

where $f = -\mathsf{C}(h^{Z_0}Z_1\otimes\cdots\otimes h^{Z_{k-1}}Z_k, \sigma^{-1}\mathsf{C}(h''b_1+b_1h'', 1)\sigma)$. More precisely, $D_{k,k+1}H_{k+1,k}+H_{k,k-1}D_{k-1,k}$ is a diagonal map, whose components are $1+f$. We claim that $f = m_1^{\mathsf{C}}H'_{kk} + H'_{kk}m_1^{\mathsf{C}}$, where $H'_{kk} = \mathsf{C}(h^{Z_0}Z_1 \otimes \cdots \otimes h^{Z_{k-1}}Z_k, \sigma^{-1}\mathsf{C}(h'', 1)\sigma)$. Indeed, $m_1^{\mathsf{C}} = \mathsf{C}(1, b_1^{\mathsf{C}}) - \mathsf{C}(\sum_{\alpha+\gamma=k-1}(1^{\otimes\alpha}\otimes b_1\otimes 1^{\otimes\gamma}), 1)$, so that

$$\begin{aligned}
m_1^{\mathsf{C}}H'_{kk} &= \big(\mathsf{C}(1, b_1^{\mathsf{C}}) - \mathsf{C}(\sum_{\alpha+\gamma=k-1} 1^{\otimes\alpha}\otimes b_1\otimes 1^{\otimes\gamma}, 1)\big)\mathsf{C}\big(1, \sigma^{-1}\mathsf{C}(h'', 1)\sigma\big) \\
&= \mathsf{C}\big(1, b_1^{\mathsf{C}}\sigma^{-1}\mathsf{C}(h'', 1)\sigma\big) + \mathsf{C}\big(1, \sigma^{-1}\mathsf{C}(h'', 1)\sigma\big)\mathsf{C}\big(\sum_{\alpha+\gamma=k-1} 1^{\otimes\alpha}\otimes b_1\otimes 1^{\otimes\gamma}, 1\big), \\
H'_{kk}m_1^{\mathsf{C}} &= \mathsf{C}\big(1, \sigma^{-1}\mathsf{C}(h'', 1)\sigma\big)\big(\mathsf{C}(1, b_1^{\mathsf{C}}) - \mathsf{C}(\sum_{\alpha+\gamma=k-1} 1^{\otimes\alpha}\otimes b_1\otimes 1^{\otimes\gamma}, 1)\big) \\
&= \mathsf{C}\big(1, \sigma^{-1}\mathsf{C}(h'', 1)\sigma b_1^{\mathsf{C}}\big) - \mathsf{C}\big(1, \sigma^{-1}\mathsf{C}(h'', 1)\sigma\big)\mathsf{C}\big(\sum_{\alpha+\gamma=k-1} 1^{\otimes\alpha}\otimes b_1\otimes 1^{\otimes\gamma}, 1\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
H'_{kk}m_1^{\mathsf{C}} + m_1^{\mathsf{C}}H'_{kk} &= \mathsf{C}\big(1, b_1^{\mathsf{C}}\sigma^{-1}\mathsf{C}(h'', 1)\sigma + \sigma^{-1}\mathsf{C}(h'', 1)\sigma b_1^{\mathsf{C}}\big) \\
&= -\mathsf{C}\big(1, \sigma^{-1}(m_1^{\mathsf{C}}\mathsf{C}(h'', 1) + \mathsf{C}(h'', 1)m_1^{\mathsf{C}})\sigma\big).
\end{aligned}$$

Since $m_1^{\mathsf{C}} = \mathsf{C}(1, b_1) - \mathsf{C}(b_1, 1)$, we have

$$\begin{aligned}
&m_1^{\mathsf{C}}\mathsf{C}(h'', 1) + \mathsf{C}(h'', 1)m_1^{\mathsf{C}} \\
&\quad = \mathsf{C}(1, b_1)\mathsf{C}(h'', 1) - \mathsf{C}(b_1, 1)\mathsf{C}(h'', 1) + \mathsf{C}(h'', 1)\mathsf{C}(1, b_1) - \mathsf{C}(h'', 1)\mathsf{C}(b_1, 1) \\
&\quad = \mathsf{C}(h''b_1 + b_1h'', 1),
\end{aligned}$$

so that $H'_{kk}m_1^{\mathsf{C}} + m_1^{\mathsf{C}}H'_{kk} = -\mathsf{C}(1, \sigma^{-1}\mathsf{C}(h''b_1 + b_1h'', 1)\sigma) = f$.

Summing up, we have proved that

$$a = \alpha Y_1 + B_1H + HB_1 = 1 + B_1H' + H'B_1 + N,$$

where $H' : sA_\infty(\mathcal{A}, \mathsf{C})(h^X, h^W) \to sA_\infty(\mathcal{A}, \mathsf{C})(h^X, h^W)$ is a continuous $\Bbbk$-linear map of degree -1 determined by a diagonal matrix with the matrix elements

$$H'_{kk} = \mathsf{C}(h^{Z_0}Z_1 \otimes \cdots \otimes h^{Z_{k-1}}Z_k, \sigma^{-1}\mathsf{C}(h'', 1)\sigma) : V_k \to V_k,$$

$H'_{kl} = 0$ for $k \neq l$, and the matrix of the remainder N is strictly upper-triangular: $N_{kl} = 0$ for all $k \geqslant l$. □

The continuous map of degree 0

$$\alpha Y_1 + B_1(H - H') + (H - H')B_1 = 1 + N : V \to V,$$

obtained in Lemma A.10, is invertible (its inverse is determined by the upper-triangular matrix $\sum_{i=0}^{\infty}(-N)^i$, which is well-defined). Therefore, αY_1 is homotopy invertible. We have proved earlier that $Y_1\alpha$ is homotopic to identity. Viewing α, Y_1 as morphisms of homotopy category $H^0(\mathsf{C})$, we see that both of them are homotopy invertible. Hence, they are homotopy inverse to each other. □

Let us define a full subcategory $\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})$ of the $\mathscr{U}'$-small differential graded $\mathscr{U}$-category $A^u_\infty(\mathcal{A},\mathsf{C})$ as follows. Its objects are all A_∞-functors $h^X:\mathcal{A}\to\mathsf{C}$ for $X\in\operatorname{Ob}\mathcal{A}$. They are unital by Lemma A.3. The differential graded category $\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})$ is $\mathscr{U}$-small. Thus, the Yoneda A_∞-functor $Y:\mathcal{A}^{\mathrm{op}}\to A_\infty(\mathcal{A},\mathsf{C})$ takes values in the $\mathscr{U}$-small subcategory $\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})$.

A.11 Theorem. *Let $\mathcal{A}$ be a unital A_∞-category. Then the restricted Yoneda A_∞-functor $Y:\mathcal{A}^{\mathrm{op}}\to\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})$ is an equivalence.*

Proof. Let us verify the assumptions of Corollary 1.9. Proposition A.9 shows that the maps Y_1 are homotopy invertible. The map $\operatorname{Ob}Y:\operatorname{Ob}\mathcal{A}\to\operatorname{Ob}\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})$, $X\mapsto h^X$ is surjective (in our conventions it may happen that $X\neq Y$, but $h^X=h^Y$). Thus, assumptions of Corollary 1.9 are satisfied. We deduce by it that Y is an equivalence. □

A.12 Corollary. *Each $\mathscr{U}$-small unital A_∞-category $\mathcal{A}$ is A_∞-equivalent to a $\mathscr{U}$-small differential graded category* $\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})^{\mathrm{op}}$.

A.13 Remark. We may use the surjective map $\operatorname{Ob}Y:\operatorname{Ob}\mathcal{A}\to\operatorname{Ob}\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})$ to transfer the differential graded category structure of $\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})$ to $\operatorname{Ob}\mathcal{A}$. This new $\mathscr{U}$-small differential graded category is denoted $\widetilde{\mathrm{Rep}}A^u_\infty(\mathcal{A},\mathsf{C})$. Thus, its set of objects is $\operatorname{Ob}\mathcal{A}$, the sets of morphisms are

$$\widetilde{\mathrm{Rep}}A^u_\infty(\mathcal{A},\mathsf{C})(X,Y)=A_\infty(\mathcal{A},\mathsf{C})(h^X,h^Y),$$

and the operations are those of $A_\infty(\mathcal{A},\mathsf{C})$. It is equivalent to $\mathrm{Rep}A^u_\infty(\mathcal{A},\mathsf{C})$ by surjectivity of $\operatorname{Ob}Y$. The Yoneda A_∞-functor can be presented as an A_∞-equivalence $\widetilde{Y}:\mathcal{A}^{\mathrm{op}}\to\widetilde{\mathrm{Rep}}A^u_\infty(\mathcal{A},\mathsf{C})$, identity on objects, whose components $\widetilde{Y}_n=Y_n$ are given by (A.6.1). A quasi-inverse to $\widetilde{Y}$ equivalence $\widetilde{\mathrm{Rep}}A^u_\infty(\mathcal{A},\mathsf{C})\to\mathcal{A}^{\mathrm{op}}$ can be chosen so that it induces the identity map on objects as well by Corollary 1.9.

B. Strict A_∞-2-functor.

The goal of this section is to show that the problem of representing the A_∞-2-functor $\mathcal{A}\mapsto A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ for a pair $(\mathcal{C},\mathcal{B})$ of a unital A_∞-category $\mathcal{C}$ and its full subcategory $\mathcal{B}$ reduces to the case of differential graded $\mathcal{C}$.

B.1. An A_∞-functor. For arbitrary A_∞-categories $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ the left hand side of the equation

$$\begin{aligned}&\big[TsA_\infty(\mathcal{Y},\mathcal{Z})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\xrightarrow{c}TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA_\infty(\mathcal{Y},\mathcal{Z})\xrightarrow{M}TsA_\infty(\mathcal{X},\mathcal{Z})\big]\\&=\big[TsA_\infty(\mathcal{Y},\mathcal{Z})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\xrightarrow{1\boxtimes A_\infty(\_,\mathcal{Z})}TsA_\infty(\mathcal{Y},\mathcal{Z})\boxtimes TsA_\infty(A_\infty(\mathcal{Y},\mathcal{Z}),A_\infty(\mathcal{X},\mathcal{Z}))\\&\qquad\qquad\xrightarrow{\alpha}TsA_\infty(\mathcal{X},\mathcal{Z})\big]\qquad\text{(B.1.1)}\end{aligned}$$

is an A_∞-functor. Therefore, by Proposition 5.5 of [Lyu03] there exists a unique A_∞-functor

$$A_\infty(\_,\mathcal{Z}) : A_\infty(\mathcal{X},\mathcal{Y}) \to A_\infty(A_\infty(\mathcal{Y},\mathcal{Z}), A_\infty(\mathcal{X},\mathcal{Z}))$$

in the right hand side, which makes equation (B.1.1) hold true. The proof of Proposition 3.4 of [Lyu03] contains a recipe for finding the components of $A_\infty(\_,\mathcal{Z})$. Namely, the equation

$$(p \boxtimes 1)M = [p.A_\infty(\_,\mathcal{Z})]\theta \tag{B.1.2}$$

has to hold for all $p \in TsA_\infty(\mathcal{X},\mathcal{Y})$. In particular,

$$f.A_\infty(\_,\mathcal{Z}) = (f \boxtimes 1)M = (f \boxtimes 1_\mathcal{Z})M : A_\infty(\mathcal{Y},\mathcal{Z}) \to A_\infty(\mathcal{X},\mathcal{Z}) \quad \text{for } f \in \operatorname{Ob} A_\infty(\mathcal{X},\mathcal{Y}),$$
$$r.A_\infty(\_,\mathcal{Z})_1 = (r \boxtimes 1)M = (r \boxtimes 1_\mathcal{Z})M : (f \boxtimes 1)M \to (g \boxtimes 1)M \text{ for } r \in sA_\infty(\mathcal{X},\mathcal{Y})(f,g).$$

Other components of $A_\infty(\_,\mathcal{Z})$ are obtained from the recurrent relation, which is equation (B.1.2) written for $p = p^1 \otimes \dots \otimes p^n$:

$$\begin{aligned}(p^1 \otimes \dots \otimes p^n)A_\infty(\_,\mathcal{Z})_n &= (p^1 \otimes \dots \otimes p^n \boxtimes 1)M \\ &- \sum_{i_1+\dots+i_l=n}^{l>1} [(p^1 \otimes \dots \otimes p^n).(A_\infty(\_,\mathcal{Z})_{i_1} \otimes A_\infty(\_,\mathcal{Z})_{i_2} \otimes \dots \otimes A_\infty(\_,\mathcal{Z})_{i_l})]\theta.\end{aligned} \tag{B.1.3}$$

In particular, for $r \otimes t \in T^2sA_\infty(\mathcal{X},\mathcal{Y})$ we get

$$(r \otimes t)A_\infty(\_,\mathcal{Z})_2 = (r \otimes t \boxtimes 1)M - [(r \boxtimes 1)M \otimes (t \boxtimes 1)M]\theta.$$

Given $g^0 \xrightarrow{p^1} g^1 \xrightarrow{p^2} \dots g^{n-1} \xrightarrow{p^n} g^n$ we find from (B.1.3) the components of the A_∞-transformation

$$(p^1 \otimes \dots \otimes p^n)A_\infty(\_,\mathcal{Z})_n \in sA_\infty(A_\infty(\mathcal{Y},\mathcal{Z}), A_\infty(\mathcal{X},\mathcal{Z}))((g^0 \boxtimes 1)M, (g^n \boxtimes 1)M)$$

in the form

$$[(p^1 \otimes \dots \otimes p^n)A_\infty(\_,\mathcal{Z})_n]_m = (p^1 \otimes \dots \otimes p^n \boxtimes 1)M_{nm}.$$

So they vanish for $m > 1$.

If $\mathcal{Z}$ is unital, then the A_∞-functor $A_\infty(\_,\mathcal{Z})$ takes values in the subcategory

$$A^u_\infty(A_\infty(\mathcal{Y},\mathcal{Z}), A_\infty(\mathcal{X},\mathcal{Z})),$$

because the A_∞-functors $(f \boxtimes 1_\mathcal{Z})M$ commute with $(1 \boxtimes \mathbf{i}^\mathcal{Z})M$, so they are unital.

B.2 Proposition. *For arbitrary A_∞-categories $\mathcal{X}$, $\mathcal{Y}$ and unital A_∞-categories $\mathcal{C}$, $\mathcal{D}$ we* *have*

$$
\begin{aligned}
&\bigl[TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{c} TsA_\infty(\mathcal{X},\mathcal{Y}) \boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{A_\infty(\_,\mathcal{C})\boxtimes A_\infty(\mathcal{X},\_)} \\
&\qquad TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{C})) \boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D})) \\
&\qquad\qquad \xrightarrow{M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D}))\bigr] \\
&= \bigl[TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{A_\infty(\mathcal{Y},\_)\boxtimes A_\infty(\_,\mathcal{D})} \\
&\qquad TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{Y},\mathcal{D})) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}),A_\infty(\mathcal{X},\mathcal{D})) \\
&\qquad\qquad \xrightarrow{M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D}))\bigr].
\end{aligned} \tag{B.2.1}
$$

The same statement holds true if one removes the unitality superscript u, and do not assume $\mathcal{C}$, $\mathcal{D}$ unital. The same equation holds true if all four A_∞-categories $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{C}$, $\mathcal{D}$ are unital and all A_∞-categories $A_\infty(,)$ are replaced with their subcategories $A^u_\infty(,)$.

Proof. Due to Proposition 3.4 of [Lyu03] equation (B.2.1) is equivalent to the following one:

$$
\begin{aligned}
&\bigl[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{1\boxtimes c} TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \\
&\xrightarrow{1\boxtimes A_\infty(\_,\mathcal{C})\boxtimes A_\infty(\mathcal{X},\_)} TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{C}))\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D})) \\
&\qquad\qquad \xrightarrow{1\boxtimes M} TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\bigr] \\
&= \bigl[TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{1\boxtimes A_\infty(\mathcal{Y},\_)\boxtimes A_\infty(\_,\mathcal{D})} \\
&\qquad TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{Y},\mathcal{D})) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}),A_\infty(\mathcal{X},\mathcal{D})) \\
&\qquad\qquad \xrightarrow{1\boxtimes M} TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\bigr].
\end{aligned}
$$

Using the definition of M [Lyu03, diagram (4.0.1)] we transform this equation into

$$
\begin{aligned}
&\bigl[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{1\boxtimes c} TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \\
&\xrightarrow{1\boxtimes A_\infty(\_,\mathcal{C})\boxtimes A_\infty(\mathcal{X},\_)} TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{C}))\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D})) \\
&\qquad\qquad \xrightarrow{\alpha\boxtimes 1} TsA_\infty(\mathcal{X},\mathcal{C}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\bigr] \\
&= \bigl[TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D}) \boxtimes TsA_\infty(\mathcal{X},\mathcal{Y}) \xrightarrow{1\boxtimes A_\infty(\mathcal{Y},\_)\boxtimes A_\infty(\_,\mathcal{D})} \\
&\qquad TsA_\infty(\mathcal{Y},\mathcal{C}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}),A_\infty(\mathcal{Y},\mathcal{D})) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}),A_\infty(\mathcal{X},\mathcal{D})) \\
&\qquad\qquad \xrightarrow{\alpha\boxtimes 1} TsA_\infty(\mathcal{Y},\mathcal{D}) \boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}),A_\infty(\mathcal{X},\mathcal{D})) \xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\bigr].
\end{aligned}
$$

Using the definitions of $A_\infty(\_,\mathcal{C})$ and $A_\infty(\mathcal{Y},\_)$ [Lyu03, (6.1.2)] we rewrite this equation as follows:

$$\begin{aligned}&\big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\xrightarrow{c_{(123)}} TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\\
&\xrightarrow{M\boxtimes A_\infty(\mathcal{X},_)} TsA_\infty(\mathcal{X},\mathcal{C})\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}),A_\infty(\mathcal{X},\mathcal{D}))\xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\big]\\
&=\big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\xrightarrow{M\boxtimes A_\infty(_,\mathcal{D})}\\
&\qquad TsA_\infty(\mathcal{Y},\mathcal{D})\boxtimes TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}),A_\infty(\mathcal{X},\mathcal{D}))\xrightarrow{\alpha} TsA_\infty(\mathcal{X},\mathcal{D})\big].\end{aligned}$$

Now we use definitions of $A_\infty(\mathcal{X},\_)$ and $A_\infty(\_,\mathcal{D})$ to get an equivalent form of the required equation:

$$\begin{aligned}&\big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\xrightarrow{c_{(123)}} TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\\
&\xrightarrow{M\boxtimes 1} TsA_\infty(\mathcal{X},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\xrightarrow{M} TsA_\infty(\mathcal{X},\mathcal{D})\big]\\
&=\big[TsA_\infty(\mathcal{Y},\mathcal{C})\boxtimes TsA^u_\infty(\mathcal{C},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\xrightarrow{M\boxtimes 1} TsA_\infty(\mathcal{Y},\mathcal{D})\boxtimes TsA_\infty(\mathcal{X},\mathcal{Y})\\
&\qquad\xrightarrow{c} TsA_\infty(\mathcal{X},\mathcal{Y})\boxtimes TsA_\infty(\mathcal{Y},\mathcal{D})\xrightarrow{M} TsA_\infty(\mathcal{X},\mathcal{D})\big].\end{aligned}$$

This equation holds true due to associativity of M, since M has degree 0.

Other statements are similar or follow from the already proven one. ☐

B.3. An A_∞-2-functor. Let A_∞-category $\mathcal{A}$ be unital. The A_∞-category $\mathcal{E}$ is pseudounital with distinguished elements equal to the unit elements of $\mathcal{C}$.

Strict A_∞-2-functors are defined in [LM03, Definition 3.1]. There is a strict A_∞-2-functor F, given by the following data:

1. the mapping of objects $F:\operatorname{Ob}A^u_\infty\to\operatorname{Ob}A^u_\infty$, $\mathcal{A}\mapsto F\mathcal{A}=A_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$ ($F\mathcal{A}$ is a full subcategory of the unital A_∞-category $A_\infty(\mathcal{C},\mathcal{A})$, hence it is unital as well);

2. the strict unital A_∞-functor $F=F_{\mathcal{A}_1,\mathcal{A}_2}:A^u_\infty(\mathcal{A}_1,\mathcal{A}_2)\to A^u_\infty(F\mathcal{A}_1,F\mathcal{A}_2)$ for each pair of unital A_∞-categories $\mathcal{A}_1$, $\mathcal{A}_2$ given as follows:

$$\operatorname{Ob}F:g\mapsto gF=(1\boxtimes g)M|_{F\mathcal{A}_1},$$

   where $(1\boxtimes g)M:A_\infty(\mathcal{C},\mathcal{A}_1)\to A_\infty(\mathcal{C},\mathcal{A}_2)$. Indeed, if $\mathcal{B}\hookrightarrow\mathcal{C}\xrightarrow{f}\mathcal{A}_1$ is a contractible A_∞-functor, then so is $\mathcal{B}\hookrightarrow\mathcal{C}\xrightarrow{fg}\mathcal{A}_2$. Actually, if ${}_X\mathbf{i}_0f_1=w_Xb_1$ for some $w_X\in(s\mathcal{A}_1)^{-2}(Xf,Xf)$, $X\in\operatorname{Ob}\mathcal{B}$, then ${}_X\mathbf{i}_0f_1g_1=w_Xb_1g_1=(w_Xg_1)b_1$, where $w_Xg_1\in(s\mathcal{A}_2)^{-2}(Xfg,Xfg)$. Furthermore,

$$\begin{aligned}&F_1:A^u_\infty(\mathcal{A}_1,\mathcal{A}_2)(g,h)\to A^u_\infty(F\mathcal{A}_1,F\mathcal{A}_2)(gF,hF),\\
&(r:g\to h:\mathcal{A}_1\to\mathcal{A}_2)\mapsto rF_1=(1\boxtimes r)M|_{F\mathcal{A}_1},\end{aligned}$$

   or more precisely,

$$\Big[sF\mathcal{A}_1(f_0,f_1)\otimes\cdots\otimes sF\mathcal{A}_1(f_{n-1},f_n)\xrightarrow{[rF_1]_n} sF\mathcal{A}_2(f_0g,f_nh)\Big]$$
$$=\Big[sA_\infty(\mathcal{C},\mathcal{A}_1)(f_0,f_1)\otimes\cdots\otimes sA_\infty(\mathcal{C},\mathcal{A}_1)(f_{n-1},f_n)\xrightarrow{(1\boxtimes r)M_{n1}} sA_\infty(\mathcal{C},\mathcal{A}_2)(f_0g,f_nh)\Big].$$

The necessary equations for F_1 are consequences of those for $A_\infty(\mathcal{C},\_)$ [Lyu03, Proposition 6.2]. Clearly, F is a unital A_∞-functor. Let us check that the following diagram commutes:

$$\begin{tikzcd}
TsA^u_\infty(\mathcal{A}_0,\mathcal{A}_1)\boxtimes TsA^u_\infty(\mathcal{A}_1,\mathcal{A}_2) \arrow[r, "M"] \arrow[d, "F_{\mathcal{A}_0,\mathcal{A}_1}\otimes F_{\mathcal{A}_1,\mathcal{A}_2}"'] \arrow[dr, phantom, "="] & TsA^u_\infty(\mathcal{A}_0,\mathcal{A}_2) \arrow[d, "F_{\mathcal{A}_0,\mathcal{A}_2}"] \\
TsA^u_\infty(F\mathcal{A}_0,F\mathcal{A}_1)\boxtimes TsA^u_\infty(F\mathcal{A}_1,F\mathcal{A}_2) \arrow[r, "M"] & TsA^u_\infty(F\mathcal{A}_0,F\mathcal{A}_2).
\end{tikzcd}$$

It follows from a similar diagram for $A_\infty(\mathcal{C},\_)$ in place of F (equation (3.3.1) of [LM03]). The commutativity is clear on objects; since both sides of the required identity are cocategory homomorphisms it suffices to show that

$$(F_{\mathcal{A}_0,\mathcal{A}_1}\boxtimes F_{\mathcal{A}_1,\mathcal{A}_2})M\,\mathrm{pr}_1 = MF_{\mathcal{A}_0,\mathcal{A}_1}\,\mathrm{pr}_1 :$$
$$TsA^u_\infty(\mathcal{A}_0,\mathcal{A}_1)\boxtimes TsA^u_\infty(\mathcal{A}_1,\mathcal{A}_2)\to sA^u_\infty(F\mathcal{A}_0,F\mathcal{A}_2).$$

Since $F_{-,-}$ are strict A_∞-functors, we must show that for any non-negative integers n, m

$$((F_{\mathcal{A}_0,\mathcal{A}_1})_1^{\otimes n}\boxtimes(F_{\mathcal{A}_1,\mathcal{A}_2})_1^{\otimes m})M_{nm} = M_{nm}(F_{\mathcal{A}_0,\mathcal{A}_2})_1 :$$
$$T^nsA^u_\infty(\mathcal{A}_0,\mathcal{A}_1)\boxtimes T^msA^u_\infty(\mathcal{A}_1,\mathcal{A}_2)\to sA^u_\infty(F\mathcal{A}_0,F\mathcal{A}_2).$$

Since M_{nm} vanishes whenever $m>1$, we restrict our attention to $m=0$ and $m=1$. These cases are similar and we will give verification in the case $m=1$. Given diagrams of A_∞-functors and A_∞-transformations

$$f^0\xrightarrow{r^1} f^1\longrightarrow\dots\xrightarrow{r^n} f^n:\mathcal{A}_0\to\mathcal{A}_1,\qquad g^0\xrightarrow{t^1} g^1:\mathcal{A}_1\to\mathcal{A}_2,$$

we must show that $(1\boxtimes(r^1\otimes\cdots\otimes r^n\boxtimes t^1)M)M|_{F\mathcal{A}_0} = ((1\boxtimes r^1)M|_{F\mathcal{A}_0}\otimes\cdots\otimes(1\boxtimes r^n)M|_{F\mathcal{A}_0}\boxtimes(1\boxtimes t^1)M|_{F\mathcal{A}_1})M$. This is a particular case of equation (3.3.1) of [LM03] since it coincides with a similar equation for $A_\infty(\mathcal{C},\_)$ in place of F.

Let $\mathcal{D}$ be the A_∞-category defined in Section 5.2. We claim that $\mathrm{restr}:A_\infty(\mathcal{D},\mathcal{A})\to F\mathcal{A}$ is an A_∞-2-transformation as defined in [LM03, Definition 3.2]. The strict A_∞-functor restr is induced by the inclusion $\iota:\mathcal{C}\hookrightarrow\mathcal{D}$:

$$f\mapsto(\mathcal{C}\hookrightarrow\mathcal{D}\xrightarrow{f}\mathcal{A})=f|_{\mathcal{C}},$$
$$\mathrm{restr}_1:A_\infty(\mathcal{D},\mathcal{A})(f,g)\to F\mathcal{A}(f|_{\mathcal{C}},g|_{\mathcal{C}})=A_\infty(\mathcal{C},\mathcal{A})(f|_{\mathcal{C}},g|_{\mathcal{C}}).$$

The restriction $f|_{\mathcal{B}}$ is a contractible A_∞-functor, for ${}_X\mathbf{i}_0f_1=\varepsilon_Xb_1f_1=(\varepsilon_Xf_1)b_1$ for $X\in\mathcal{B}$. Let us check that the following diagram of A_∞-functors commutes:

$$\begin{tikzcd}
A^u_\infty(\mathcal{A}_1,\mathcal{A}_2) \arrow[r, "A_\infty(\mathcal{D},-)"] \arrow[d, "F"'] \arrow[dr, phantom, "="] & A^u_\infty(A_\infty(\mathcal{D},\mathcal{A}_1),A_\infty(\mathcal{D},\mathcal{A}_2)) \arrow[d, "(1\boxtimes\mathrm{restr}_{\mathcal{A}_2})M"] \\
A^u_\infty(F\mathcal{A}_1,F\mathcal{A}_2) \arrow[r, "(\mathrm{restr}_{\mathcal{A}_1}\boxtimes1)M"] & A^u_\infty(A_\infty(\mathcal{D},\mathcal{A}_1),F\mathcal{A}_2)
\end{tikzcd}$$

All functors in the diagram above are strict (the proof is given in [LM03, Section 3.4]). We must verify the equation

$$A_\infty(\mathcal{D},-)(1\boxtimes \mathrm{restr}_{\mathcal{A}_2})M = F(\mathrm{restr}_{\mathcal{A}_1}\boxtimes 1)M.$$

On objects: given a unital A_∞-functor $g:\mathcal{A}_1\to\mathcal{A}_2$, we are going to check that

$$[(1\boxtimes g)M]_n\cdot \mathrm{restr}_1 = \mathrm{restr}_1^{\otimes n}\cdot[(1\boxtimes g)M]_n$$

for any $n\geqslant 1$. Indeed, for any n-tuple of composable A_∞-transformations

$$f^0 \xrightarrow{r^1} f^1 \longrightarrow \dots \xrightarrow{r^n} f^n:\mathcal{D}\to\mathcal{A}_1$$

we have

$$\begin{aligned}
\{(r^1\otimes\dots\otimes r^n)[(1\boxtimes g)M]_n\}_k\big|_{\mathcal{C}} &= [(r^1\otimes\dots\otimes r^n\boxtimes g)M_{n0}]_k\big|_{\mathcal{C}}\\
&= \Big\{\sum_l (r^1\otimes\dots\otimes r^n)\theta_{kl}g_l\Big\}\Big|_{\mathcal{C}}\\
&= \sum_l (r^1|_{\mathcal{C}}\otimes\dots\otimes r^n|_{\mathcal{C}})\theta_{kl}g_l|_{\mathcal{C}}\\
&= \{(r^1|_{\mathcal{C}}\otimes\dots\otimes r^n|_{\mathcal{C}})[(1\boxtimes g|_{\mathcal{C}})M]_n\}_k.
\end{aligned}$$

The coincidence of A_∞-transformations $((1\boxtimes t)M)\cdot\mathrm{restr}_{\mathcal{D}} = \mathrm{restr}_{\mathcal{C}}\cdot((1\boxtimes t)M)$ follows similarly from the computation:

$$\begin{aligned}
\{(r^1\otimes\dots\otimes r^n)[(1\boxtimes t)M]_n\}_k\big|_{\mathcal{C}} &= [(r^1\otimes\dots\otimes r^n\boxtimes t)M_{n1}]_k\big|_{\mathcal{C}}\\
&= \Big\{\sum_l (r^1\otimes\dots\otimes r^n)\theta_{kl}t_l\Big\}\Big|_{\mathcal{C}}\\
&= \sum_l (r^1|_{\mathcal{C}}\otimes\dots\otimes r^n|_{\mathcal{C}})\theta_{kl}t_l|_{\mathcal{C}}\\
&= \{(r^1|_{\mathcal{C}}\otimes\dots\otimes r^n|_{\mathcal{C}})[(1\boxtimes t|_{\mathcal{C}})M]_n\}_k.
\end{aligned}$$

We are more interested in the following A_∞-2-subfunctor of F, denoted $G:\mathrm{Ob}\,A_\infty^u\to\mathrm{Ob}\,A_\infty^u$, $\mathcal{A}\mapsto G\mathcal{A}=A_\infty^u(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}\subset F\mathcal{A}$. All the structure data of G are restrictions of those of F. Hence, the conclusion that $\mathrm{restr}:A_\infty^{\psi u}(\mathcal{D},\mathcal{A})\to G\mathcal{A}$ is an A_∞-2-natural transformation remains valid. We prove in Theorem 5.13 that this A_∞-functor is an equivalence. Therefore, this restriction A_∞-functor is a 2-natural A_∞-equivalence. If $\mathcal{C}$ is strictly unital, then $\mathcal{D}=\mathsf{Q}(\mathcal{C}|\mathcal{B})$ is unital by Theorem 6.5. Whenever $\mathcal{D}$ is unital, we say that $\mathcal{D}$ *unitally* represents G.

We are going to discuss how an A_∞-2-functor represented by an A_∞-category $\mathcal{X}$ depends on $\mathcal{X}$.

B.4 Proposition. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an A_∞-functor. Then*

$$\lambda_{\mathcal{A}} = (f \boxtimes 1)M : A_\infty(\mathcal{Y}, \mathcal{A}) \to A_\infty(\mathcal{X}, \mathcal{A}), \qquad g \mapsto fg$$

is a strict A_∞-2-transformation between two A_∞-2-functors of $\mathcal{A} \in \operatorname{Ob} A^u_\infty$. If $f : \mathcal{X} \to \mathcal{Y}$ is a unital A_∞-functor, then

$$(f \boxtimes 1)M : A^u_\infty(\mathcal{Y}, \mathcal{A}) \to A^u_\infty(\mathcal{X}, \mathcal{A})$$

is also a strict A_∞-2-transformation.

Proof. The A_∞-functor $(f \boxtimes 1)M$ strictly commutes with the unit transformations $(1 \boxtimes \mathbf{i}^{\mathcal{A}})M$ in $A_\infty(\mathcal{Y}, \mathcal{A})$ and $A_\infty(\mathcal{X}, \mathcal{A})$:

$$(f \boxtimes 1_{\mathcal{A}})M \cdot (1_{\mathcal{X}} \boxtimes \mathbf{i}^{\mathcal{A}})M = (1_{\mathcal{Y}} \boxtimes \mathbf{i}^{\mathcal{A}})M \cdot (f \boxtimes 1_{\mathcal{A}})M$$

due to associativity of M. Therefore, $(f \boxtimes 1)M$ is unital.

We have to prove that the diagram of A_∞-functors

$$\begin{tikzcd}
A^u_\infty(\mathcal{C}, \mathcal{D}) \arrow[r, "A_\infty(\mathcal{Y},_)"] \arrow[d, "A_\infty(\mathcal{X},_)"'] & A^u_\infty(A_\infty(\mathcal{Y}, \mathcal{C}), A_\infty(\mathcal{Y}, \mathcal{D})) \arrow[d, "(1\boxtimes\lambda_{\mathcal{D}})M"] \\
A^u_\infty(A_\infty(\mathcal{X}, \mathcal{C}), A_\infty(\mathcal{X}, \mathcal{D})) \arrow[r, "(\lambda_{\mathcal{C}}\boxtimes 1)M"] & A^u_\infty(A_\infty(\mathcal{Y}, \mathcal{C}), A_\infty(\mathcal{X}, \mathcal{D}))
\end{tikzcd} \tag{B.4.1}$$

commutes. All four A_∞-functors in this diagram are strict. So it suffices to check commutativity on objects and for the first components.

If $h : \mathcal{C} \to \mathcal{D}$ is a unital A_∞-functor, then

$$h.A_\infty(\mathcal{Y}, \_)(1 \boxtimes \lambda_{\mathcal{D}})M = (1 \boxtimes h)M \cdot (f \boxtimes 1)M = (f \boxtimes 1 \boxtimes h)(1 \boxtimes M)M$$

equals to

$$h.A_\infty(\mathcal{X}, \_)(\lambda_{\mathcal{C}} \boxtimes 1)M = (f \boxtimes 1)M \cdot (1 \boxtimes h)M = (f \boxtimes 1 \boxtimes h)(M \boxtimes 1)M$$

due to associativity of M.

If $t \in sA^u_\infty(\mathcal{C}, \mathcal{D})(g, h)$, then it is mapped by the first components of A_∞-functors in diagram (B.4.1) to

$$t.A_\infty(\mathcal{Y}, \_)_1(1 \boxtimes \lambda_{\mathcal{D}})M_{10} = (1 \boxtimes t)M \cdot (f \boxtimes 1)M = (f \boxtimes 1 \boxtimes t)(1 \boxtimes M)M$$

and

$$t.A_\infty(\mathcal{X}, \_)_1(\lambda_{\mathcal{C}} \boxtimes 1)M_{01} = (f \boxtimes 1)M \cdot (1 \boxtimes t)M = (f \boxtimes 1 \boxtimes t)(M \boxtimes 1)M.$$

These expressions are equal due to associativity of M.

The case of unital f and A_∞-2-subfunctors $A^u_\infty(\mathcal{Y}, \mathcal{A})$, $A^u_\infty(\mathcal{X}, \mathcal{A})$ follows from the general case. □

With two strict A_∞-2-transformations λ, μ in $F \xrightarrow{\lambda} G \xrightarrow{\mu} H : A^u_\infty \to A^u_\infty$ is associated the third A_∞-2-transformation $\lambda\mu : F \to H : A^u_\infty \to A^u_\infty$ – their composition, specified by the family of unital A_∞-functors

$$(\lambda\mu)_{\mathcal{C}} = \lambda_{\mathcal{C}}\mu_{\mathcal{C}} : F\mathcal{C} \to H\mathcal{C}, \qquad \mathcal{C} \in \operatorname{Ob} A^u_\infty.$$

In order to verify that $\lambda\mu$ is indeed a strict A_∞-2-transformation, we have to check equation (3.2.1) of [LM03]:

$$F \cdot (1 \boxtimes (\lambda\mu)_{\mathcal{D}})M = H \cdot ((\lambda\mu)_{\mathcal{C}} \boxtimes 1)M : A^u_\infty(\mathcal{C},\mathcal{D}) \to A^u_\infty(F\mathcal{C}, H\mathcal{D}).$$

We do it as follows:

$$\begin{aligned}
&\big[A^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{F} A^u_\infty(F\mathcal{C},F\mathcal{D}) \xrightarrow[(1\boxtimes\lambda_{\mathcal{D}})M\cdot(1\boxtimes\mu_{\mathcal{D}})M]{(1\boxtimes\lambda_{\mathcal{D}}\mu_{\mathcal{D}})M} A^u_\infty(F\mathcal{C},H\mathcal{D})\big] \\
= &\big[A^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{G} A^u_\infty(G\mathcal{C},G\mathcal{D}) \xrightarrow[(1\boxtimes\mu_{\mathcal{D}})M\cdot(\lambda_{\mathcal{C}}\boxtimes1)M]{(\lambda_{\mathcal{C}}\boxtimes1)M\cdot(1\boxtimes\mu_{\mathcal{D}})M} A^u_\infty(F\mathcal{C},H\mathcal{D})\big] \\
= &\big[A^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{H} A^u_\infty(H\mathcal{C},H\mathcal{D}) \xrightarrow[(\lambda_{\mathcal{C}}\mu_{\mathcal{C}}\boxtimes1)M]{(\mu_{\mathcal{C}}\boxtimes1)M\cdot(\lambda_{\mathcal{C}}\boxtimes1)M} A^u_\infty(F\mathcal{C},H\mathcal{D})\big].
\end{aligned}$$

B.5 Definition. A strict *modification* $m : \lambda \to \mu : F \to G : A^u_\infty \to A^u_\infty$ of strict A_∞-2-transformations λ, μ is

1. a family of A_∞-transformations $m_{\mathcal{C}} : \lambda_{\mathcal{C}} \to \mu_{\mathcal{C}} : F\mathcal{C} \to G\mathcal{C}$ for $\mathcal{C} \in \operatorname{Ob} A^u_\infty$ such that
2. for any pair of unital A_∞-categories $\mathcal{C}$, $\mathcal{D}$ the A_∞-transformations

$$(F_{\mathcal{C},\mathcal{D}}\boxtimes\lambda_{\mathcal{D}})\cdot M = (\lambda_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})\cdot M \xrightarrow{(m_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})\cdot M} (\mu_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})\cdot M : A^u_\infty(\mathcal{C},\mathcal{D}) \to A^u_\infty(F\mathcal{C},G\mathcal{D}),$$

$$(F_{\mathcal{C},\mathcal{D}}\boxtimes\lambda_{\mathcal{D}})\cdot M \xrightarrow{(F_{\mathcal{C},\mathcal{D}}\boxtimes m_{\mathcal{D}})\cdot M} (F_{\mathcal{C},\mathcal{D}}\boxtimes\mu_{\mathcal{D}})\cdot M = (\mu_{\mathcal{C}}\boxtimes G_{\mathcal{C},\mathcal{D}})\cdot M : A^u_\infty(\mathcal{C},\mathcal{D}) \to A^u_\infty(F\mathcal{C},G\mathcal{D})$$

are equal, in short,

$$G_{\mathcal{C},\mathcal{D}}(m_{\mathcal{C}} \boxtimes 1)M = F_{\mathcal{C},\mathcal{D}}(1 \boxtimes m_{\mathcal{D}})M : F_{\mathcal{C},\mathcal{D}}(1 \boxtimes \lambda_{\mathcal{D}})M \to G_{\mathcal{C},\mathcal{D}}(\mu_{\mathcal{C}} \boxtimes 1)M.$$

A modification m is *natural* if all A_∞-transformations $m_{\mathcal{C}}$ are natural.

B.6. Example of strict modification. We claim that for an arbitrary A_∞-transformation $r : f \to g : \mathcal{X} \to \mathcal{Y}$ the family of A_∞-transformations

$$m_{\mathcal{C}} = (r \boxtimes 1)M : \lambda_{\mathcal{C}} = (f \boxtimes 1)M \to \mu_{\mathcal{C}} = (g \boxtimes 1)M : F\mathcal{C} = A_\infty(\mathcal{Y},\mathcal{C}) \to G\mathcal{C} = A_\infty(\mathcal{X},\mathcal{C})$$

is a strict modification. In order to prove it, we notice first of all that $(r\boxtimes1)M$ is a $((f\boxtimes 1)M,(g\boxtimes1)M)$-coderivation for an arbitrary $\mathcal{C}$, since M is a cocategory homomorphism.

When we want to indicate the category $\mathcal{C}$, we write this coderivation as $(r \boxtimes 1_{\mathcal{C}})M \in sA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{C}))((f \boxtimes 1_{\mathcal{C}})M, (g \boxtimes 1_{\mathcal{C}})M)$. The equation to verify is

$$\begin{aligned}
&\big[TsA^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{(r\boxtimes 1_{\mathcal{C}})M\boxtimes A_\infty(\mathcal{X},_)} T^1sA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{C}))\boxtimes TsA^u_\infty(A_\infty(\mathcal{X},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D})) \\
&\qquad \xrightarrow{M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D}))\big] \\
&= \big[TsA^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{A_\infty(\mathcal{Y},_)\boxtimes(r\boxtimes 1_{\mathcal{D}})M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{Y},\mathcal{D}))\boxtimes T^1sA^u_\infty(A_\infty(\mathcal{Y},\mathcal{D}), A_\infty(\mathcal{X},\mathcal{D})) \\
&\qquad \xrightarrow{M} TsA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{D}))\big].
\end{aligned}$$

It is an immediate consequence of equation (B.2.1) restricted to the element $r \in T^1sA_\infty(\mathcal{X},\mathcal{Y})$.

B.7 Lemma. *If $m : \lambda \to \mu : F \to G : A^u_\infty \to A^u_\infty$ is a strict modification, then so is $mB_1 : \lambda \to \mu : F \to G : A^u_\infty \to A^u_\infty$, where $(mB_1)_{\mathcal{C}} = m_{\mathcal{C}}B_1 : \lambda_{\mathcal{C}} \to \mu_{\mathcal{C}} : F\mathcal{C} \to G\mathcal{C}$ for all $\mathcal{C} \in \operatorname{Ob} A^u_\infty$. The $\Bbbk$-linear map*

$$sA_\infty(\mathcal{X},\mathcal{Y})(f,g) \ni r \mapsto (r \boxtimes 1)M \in sA^u_\infty(A_\infty(\mathcal{Y},\mathcal{C}), A_\infty(\mathcal{X},\mathcal{C}))((f \boxtimes 1)M, (g \boxtimes 1)M)$$

is a chain map.

Proof. Let us prove that the family $m_{\mathcal{C}}B_1$ constitutes a strict modification. The identity $(1 \boxtimes B + B \boxtimes 1)M = MB$ implies that

$$\begin{aligned}
&(F_{\mathcal{C},\mathcal{D}} \boxtimes m_{\mathcal{D}}.B_1)M = (F_{\mathcal{C},\mathcal{D}} \boxtimes m_{\mathcal{D}})MB - (-)^m B(F_{\mathcal{C},\mathcal{D}} \boxtimes m_{\mathcal{D}})M = [(F_{\mathcal{C},\mathcal{D}} \boxtimes m_{\mathcal{D}})M].B_1 \\
&= [(m_{\mathcal{C}} \boxtimes G_{\mathcal{C},\mathcal{D}})M].B_1 = (m_{\mathcal{C}} \boxtimes G_{\mathcal{C},\mathcal{D}})MB - (-)^m B(m_{\mathcal{C}} \boxtimes G_{\mathcal{C},\mathcal{D}})M = (m_{\mathcal{C}}.B_1 \boxtimes G_{\mathcal{C},\mathcal{D}})M.
\end{aligned}$$

Here we use the fact that $m_{\mathcal{D}}.B = m_{\mathcal{D}}.B_1$ due to $m_{\mathcal{D}} \in T^1sA_\infty(F\mathcal{D}, G\mathcal{D})(\lambda_{\mathcal{D}}, \mu_{\mathcal{D}})$, and, similarly, $m_{\mathcal{C}}.B = m_{\mathcal{C}}.B_1$. The equation

$$[(r \boxtimes 1)M].B_1 = (r \boxtimes 1)MB - (-)^r B(r \boxtimes 1)M = (r.B \boxtimes 1)M = (r.B_1 \boxtimes 1)M \qquad \text{(B.7.1)}$$

proves that $r \mapsto (r \boxtimes 1)M$ is a chain map. $\square$

B.8 Lemma. *Let m, n in $\lambda \xrightarrow{m} \mu \xrightarrow{n} \nu : F \to G : A^u_\infty \to A^u_\infty$ be strict modifications. Then*

$$\begin{aligned}
&F_{\mathcal{C},\mathcal{D}}[1 \boxtimes (m_{\mathcal{D}} \otimes n_{\mathcal{D}})B_2]M = G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})B_2 \boxtimes 1]M \\
&\quad + G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})(1 \otimes B_1 + B_1 \otimes 1)A_\infty(_, G\mathcal{D})_2] - G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}} \otimes n_{\mathcal{C}})A_\infty(_, G\mathcal{D})_2B_1]
\end{aligned}$$

Proof. Since $A_\infty(\_, G\mathcal{D})$ is an A_∞-functor we have

$$\begin{aligned}
&[(m_{\mathcal{C}} \boxtimes 1)M \otimes (n_{\mathcal{C}} \boxtimes 1)M]B_2 + (m_{\mathcal{C}} \otimes n_{\mathcal{C}}).A_\infty(_, G\mathcal{D})_2B_1 \\
&= [m_{\mathcal{C}}.A_\infty(_, G\mathcal{D})_1 \otimes n_{\mathcal{C}}.A_\infty(_, G\mathcal{D})_1]B_2 + (m_{\mathcal{C}} \otimes n_{\mathcal{C}}).A_\infty(_, G\mathcal{D})_2B_1 \\
&= (m_{\mathcal{C}} \otimes n_{\mathcal{C}})B_2.A_\infty(_, G\mathcal{D})_1 + (m_{\mathcal{C}} \otimes n_{\mathcal{C}})(1 \otimes B_1 + B_1 \otimes 1)A_\infty(_, G\mathcal{D})_2 \\
&= [(m_{\mathcal{C}} \otimes n_{\mathcal{C}})B_2 \boxtimes 1]M + (m_{\mathcal{C}} \otimes n_{\mathcal{C}})(1 \otimes B_1 + B_1 \otimes 1)A_\infty(_, G\mathcal{D})_2. \qquad \text{(B.8.1)}
\end{aligned}$$

Using this identity we find

$$\begin{aligned}
&F_{\mathcal{C},\mathcal{D}}[1\boxtimes(m_{\mathcal{D}}\otimes n_{\mathcal{D}})B_2]M = F_{\mathcal{C},\mathcal{D}}[(1\boxtimes m_{\mathcal{D}})M\otimes(1\boxtimes n_{\mathcal{D}})M]B_2\\
&\quad= [F_{\mathcal{C},\mathcal{D}}(1\boxtimes m_{\mathcal{D}})M\otimes F_{\mathcal{C},\mathcal{D}}(1\boxtimes n_{\mathcal{D}})M]B_2 = [G_{\mathcal{C},\mathcal{D}}(m_{\mathcal{C}}\boxtimes 1)M\otimes G_{\mathcal{C},\mathcal{D}}(n_{\mathcal{C}}\boxtimes 1)M]B_2\\
&\quad= G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}}\boxtimes 1)M\otimes(n_{\mathcal{C}}\boxtimes 1)M]B_2 = G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}}\otimes n_{\mathcal{C}})B_2\boxtimes 1]M\\
&\quad\ + G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}}\otimes n_{\mathcal{C}})(1\otimes B_1+B_1\otimes 1)A_\infty(_,G\mathcal{D})_2] - G_{\mathcal{C},\mathcal{D}}[(m_{\mathcal{C}}\otimes n_{\mathcal{C}})A_\infty(_,G\mathcal{D})_2B_1],
\end{aligned}$$

so the lemma is proven. □

B.9 Proposition. *If unital A_∞-categories $\mathcal{X}$, $\mathcal{Y}$ are equivalent, then A_∞-2-functors $\mathcal{A}\mapsto A^u_\infty(\mathcal{X},\mathcal{A})$ and $\mathcal{A}\mapsto A^u_\infty(\mathcal{Y},\mathcal{A})$ are naturally A_∞-2-equivalent.*

Proof. Let $\phi:\mathcal{X}\to\mathcal{Y}$, $\psi:\mathcal{Y}\to\mathcal{X}$ be A_∞-equivalences, quasi-inverse to each other. Then there are natural A_∞-transformations

$$r:\phi\psi\to\mathrm{id}_{\mathcal{X}}:\mathcal{X}\to\mathcal{X},\qquad p:\mathrm{id}_{\mathcal{X}}\to\phi\psi:\mathcal{X}\to\mathcal{X},$$

inverse to each other, that is,

$$(r\otimes p)B_2\equiv\phi\psi\mathbf{i}^{\mathcal{X}},\qquad (p\otimes r)B_2\equiv\mathbf{i}^{\mathcal{X}}.$$

The A_∞-transformations

$$(r\boxtimes 1)M:(\psi\boxtimes 1)M(\phi\boxtimes 1)M\to\mathrm{id}:A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A}), \tag{B.9.1}$$
$$(p\boxtimes 1)M:\mathrm{id}\to(\psi\boxtimes 1)M(\phi\boxtimes 1)M:A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A}) \tag{B.9.2}$$

are also natural by (B.7.1). We are going to prove that these A_∞-transformations are inverse to each other.

The natural A_∞-transformation

$$(1\boxtimes\mathbf{i}^{\mathcal{A}})M:\mathrm{id}\to\mathrm{id}:A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A})$$

is a unit transformation of A_∞-category $A^u_\infty(\mathcal{X},\mathcal{A})$ by Proposition 7.7 of [Lyu03]. Another unit transformation is given by

$$(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M:\mathrm{id}\to\mathrm{id}:A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A}). \tag{B.9.3}$$

Indeed, values of 0-th components of the both natural transformations on an object f of $A^u_\infty(\mathcal{X},\mathcal{A})$ differ by a boundary since

$${}_f[(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M]_0 = (\mathbf{i}^{\mathcal{X}}\boxtimes f)M_{10} = \mathbf{i}^{\mathcal{X}}f\equiv f\mathbf{i}^{\mathcal{A}} = (f\boxtimes\mathbf{i}^{\mathcal{A}})M_{01} = {}_f[(1\boxtimes\mathbf{i}^{\mathcal{A}})M]_0.$$

Therefore, $[1\otimes(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M_{10}]B_2$ and $[(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M_{10}\otimes 1]B_2$ are homotopy invertible.

Furthermore, by (B.8.1)

$$[(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M\otimes(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M]B_2\equiv[(\mathbf{i}^{\mathcal{X}}\otimes\mathbf{i}^{\mathcal{X}})B_2\boxtimes 1]M\equiv(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M$$

due to Lemma B.7. Therefore, homotopy idempotent (B.9.3) is a unit transformation of $A^u_\infty(\mathcal{X},\mathcal{A})$ by [Lyu03, Definition 7.6]. Since unit transformation is unique up to equivalence by [Lyu03, Corollary 7.10] we have

$$(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M \equiv (1\boxtimes \mathbf{i}^{\mathcal{A}})M : \mathrm{id}\to\mathrm{id} : A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A}).$$

Composing natural A_∞-transformations $(r\boxtimes 1)M$ and $(p\boxtimes 1)M$ given by (B.9.1) and (B.9.2) we get

$$\begin{aligned}[]
[(r\boxtimes 1)M\otimes(p\boxtimes 1)M]B_2 &\equiv [(r\otimes p)B_2\boxtimes 1]M \equiv (\phi\psi\mathbf{i}^{\mathcal{X}}\boxtimes 1)M\\
&\equiv (\mathbf{i}^{\mathcal{X}}\phi\psi\boxtimes 1)M = (\psi\boxtimes 1)M(\phi\boxtimes 1)M(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M,
\end{aligned}$$

$$[(p\boxtimes 1)M\otimes(r\boxtimes 1)M]B_2 \equiv [(p\otimes r)B_2\boxtimes 1]M \equiv (\mathbf{i}^{\mathcal{X}}\boxtimes 1)M$$

by (B.8.1) and Lemma B.7. Since $(\mathbf{i}^{\mathcal{X}}\boxtimes 1)M$ is a unit transformation, the A_∞-transformations $(r\boxtimes 1)M$ and $(p\boxtimes 1)M$ are inverse to each other.

The obtained statement together with one more statement in which ϕ and ψ exchange their places implies that A_∞-functors

$$\begin{aligned}
A^u_\infty(\phi,\mathcal{A}) &= (\phi\boxtimes 1)M : A^u_\infty(\mathcal{Y},\mathcal{A})\to A^u_\infty(\mathcal{X},\mathcal{A}),\\
A^u_\infty(\psi,\mathcal{A}) &= (\psi\boxtimes 1)M : A^u_\infty(\mathcal{X},\mathcal{A})\to A^u_\infty(\mathcal{Y},\mathcal{A})
\end{aligned}$$

are quasi-inverse to each other.

They form strict A_∞-2-transformations by Proposition B.4. Therefore, $A^u_\infty(\phi,\mathcal{A})$ and $A^u_\infty(\psi,\mathcal{A})$ are natural A_∞-2-equivalences. □

Given a pair $(\mathcal{C},\mathcal{B})$ consisting of a unital A_∞-category $\mathcal{C}$ and its full subcategory $\mathcal{B}$, we shall construct another pair $(\widetilde{\mathcal{C}},\widetilde{\mathcal{B}})$ consisting of a differential graded category $\widetilde{\mathcal{C}}$ and its full subcategory $\widetilde{\mathcal{B}}$ as follows. Set $\widetilde{\mathcal{C}}$ to be the differential graded category $\widetilde{\mathrm{Rep}A^u_\infty}(\mathcal{C}^{\mathrm{op}},\mathsf{C})$ of A_∞-functors, represented by objects of $\mathcal{C}$, see Remark A.13. Thus, $\operatorname{Ob}\widetilde{\mathcal{C}} = \operatorname{Ob}\mathcal{C}$. The category $\widetilde{\mathcal{C}}$ is equivalent to $\mathcal{C}$, the Yoneda A_∞-equivalence $\widetilde{Y}:\mathcal{C}\to\widetilde{\mathcal{C}}$ and its quasi-inverse $\Psi:\widetilde{\mathcal{C}}\to\mathcal{C}$ are identity on objects by Remark A.13. We take $\widetilde{\mathcal{B}}$ to be the full subcategory of $\widetilde{\mathcal{C}}$ with the set of objects $\operatorname{Ob}\widetilde{\mathcal{B}} = \operatorname{Ob}\mathcal{B}$. Therefore, the A_∞-functors $\widetilde{Y}$ and Ψ can be restricted to quasi-inverse to each other A_∞-equivalences $Y' = \widetilde{Y}\big|_{\mathcal{B}} : \mathcal{B}\to\widetilde{\mathcal{B}}$ and $\Psi' = \Psi\big|_{\widetilde{\mathcal{B}}} : \widetilde{\mathcal{B}}\to\mathcal{B}$.

B.10 Corollary. *Let $\mathcal{A}$ be a unital A_∞-category. The A_∞-functors*

$$\begin{aligned}
(\Psi\boxtimes 1)M &: A^u_\infty(\mathcal{C},\mathcal{A})\to A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A}), &\qquad f&\mapsto \Psi f,\\
(\widetilde{Y}\boxtimes 1)M &: A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})\to A^u_\infty(\mathcal{C},\mathcal{A}), &\qquad g&\mapsto \widetilde{Y}g,
\end{aligned}$$

are quasi-inverse to each other A_∞-equivalences. The first maps objects of $A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}$ to objects of $A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\mathrm{mod}\,\widetilde{\mathcal{B}}}$, the second does vice versa. Therefore, their restrictions determine quasi-inverse to each other A_∞-equivalences

$$\begin{aligned}
(\Psi\boxtimes 1)M &: A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}\to A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\mathrm{mod}\,\widetilde{\mathcal{B}}},\\
(\widetilde{Y}\boxtimes 1)M &: A^u_\infty(\widetilde{\mathcal{C}},\mathcal{A})_{\mathrm{mod}\,\widetilde{\mathcal{B}}}\to A^u_\infty(\mathcal{C},\mathcal{A})_{\mathrm{mod}\,\mathcal{B}}.
\end{aligned}$$

Proof. Let $f : \mathcal{C} \to \mathcal{A}$ be a unital A_∞-functor such that $f\big|_{\mathcal{B}}$ is contractible. Then for each object X of $\mathcal{B}$ the complex $(s\mathcal{A}(Xf, Xf), b_1)$ is contractible [LO02, Proposition 6.1]. Equivalently we may say that for each object Z of $\widetilde{\mathcal{B}}$ the complex $(s\mathcal{A}(Z\Psi f, Z\Psi f), b_1)$ is contractible, as the following commutative diagram shows:

$$\begin{tikzcd}
\widetilde{\mathcal{B}} \arrow[r, hook] \arrow[d, "\Psi'"'] & \widetilde{\mathcal{C}} \arrow[r, "\Psi f"] \arrow[d, "\Psi"'] & \mathcal{A} \arrow[d, equal] \\
\mathcal{B} \arrow[r, hook] & \mathcal{C} \arrow[r, "f"] & \mathcal{A}
\end{tikzcd}$$

This implies contractibility of the A_∞-functor $\Psi f\big|_{\widetilde{\mathcal{B}}}$. Similarly, if $g \in A^u_\infty(\widetilde{\mathcal{C}}, \mathcal{A})_{\operatorname{mod}\widetilde{\mathcal{B}}}$, then $\widetilde{Y} g \in A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod}\mathcal{B}}$. □

B.11 Corollary. *The A_∞-2-functors $\mathcal{A} \mapsto A^u_\infty(\mathcal{C}, \mathcal{A})_{\operatorname{mod}\mathcal{B}}$ and $\mathcal{A} \mapsto A^u_\infty(\widetilde{\mathcal{C}}, \mathcal{A})_{\operatorname{mod}\widetilde{\mathcal{B}}}$ are naturally A_∞-2-equivalent. Therefore, if one of them is representable, then so is the other.*

C. The Yoneda Lemma for 2-categories

C.1. 2-categories. We recall the definitions of strict 2-categories and associated weak notions originating in [Bén67]. We use the form and the notation of [Lyu99].

C.2 Definition. A (strict) *2-category* $\mathfrak{A}$ consists of

1. a set of objects $\operatorname{Ob}\mathfrak{A}$;
2. for any pair of objects $X, Y \in \operatorname{Ob}\mathfrak{A}$ a category $\mathfrak{A}(X, Y)$;
3. (a) for any object $X \in \operatorname{Ob}\mathfrak{A}$ an object 1_X of $\mathfrak{A}(X, X)$;
   (b) for any triple of objects $X, Y, Z \in \operatorname{Ob}\mathfrak{A}$ a functor
   $$\bullet : \mathfrak{A}(X, Y) \times \mathfrak{A}(Y, Z) \to \mathfrak{A}(X, Z), \qquad (F, G) \mapsto FG = F{\bullet}G = G \circ F;$$
   such that the following functors are equal
4. $F{\bullet}1 = F = 1{\bullet}F$, $F(GH) = (FG)H$.

The 2-category of ($\mathscr{U}$-small) categories is denoted $\mathcal{C}at$.

C.3 Definition. A *weak 2-functor* (a homomorphism in [Bén67]) between 2-categories $\mathfrak{A}$ and $\mathfrak{C}$ consists of

1. a function $F : \operatorname{Ob}\mathfrak{A} \to \operatorname{Ob}\mathfrak{C}$;
2. a functor $F = F_{X,Y} : \mathfrak{A}(X, Y) \to \mathfrak{C}(FX, FY)$ for each pair of objects $X, Y \in \operatorname{Ob}\mathfrak{A}$;

3. (a) an isomorphism $\phi_0 : 1_{FX} \to F1_X$;

   (b) an invertible (natural) transformation

$$\begin{tikzcd}
\mathfrak{A}(X,Y) \times \mathfrak{A}(Y,Z) \arrow[r, "\bullet"] \arrow[d, "F_{X,Y}\times F_{Y,Z}"'] & \mathfrak{A}(X,Z) \arrow[d, "F_{X,Z}"] \\
\mathfrak{C}(FX,FY) \times \mathfrak{C}(FY,FZ) \arrow[r, "\bullet"] \arrow[ur, Rightarrow, "\phi_2"] & \mathfrak{C}(FX,FZ)
\end{tikzcd}$$

   for each triple $X, Y, Z \in \operatorname{Ob}\mathfrak{A}$;

such that

4. (a) for any object $M \in \mathfrak{A}(X,Y)$ the composites

$$FM = FM\bullet 1_{FY} \xrightarrow{FM\bullet\phi_0} FM\bullet F1_Y \xrightarrow{\phi_2} F(M\bullet 1_Y) = FM \qquad \text{(C.3.1)}$$

$$FM = 1_{FX}\bullet FM \xrightarrow{\phi_0\bullet FM} F1_X\bullet FM \xrightarrow{\phi_2} F(1_X\bullet M) = FM \qquad \text{(C.3.2)}$$

   are identity morphisms in $\mathfrak{C}(FM,FM)$;

   (b) For any objects $W, X, Y, Z \in \operatorname{Ob}\mathfrak{A}$ and any object

$$(K, L, M) \in \mathfrak{A}(W,X) \times \mathfrak{A}(X,Y) \times \mathfrak{A}(Y,Z)$$

   there is an equation

$$\begin{aligned}
&\big(FK\bullet(FL\bullet FM) \xrightarrow{FK\bullet\phi_2} FK\bullet F(L\bullet M) \xrightarrow{\phi_2} F(K\bullet(L\bullet M))\big) \\
&\qquad = \big((FK\bullet FL)\bullet FM \xrightarrow{\phi_2\bullet FM} F(K\bullet L)\bullet FM \xrightarrow{\phi_2} F((K\bullet L)\bullet M)\big).
\end{aligned}$$

If ϕ_2 and ϕ_0 are identity isomorphisms, F is called a *strict 2-functor*.

C.4 Definition. A *weak 2-transformation* (pseudo-natural transformation [Gra74]) $\lambda : (F, \phi_2, \phi_0) \to (G, \psi_2, \psi_0) : \mathfrak{A} \to \mathfrak{C}$ is

1. a family of 1-morphisms $\lambda_X : FX \to GX$, $X \in \operatorname{Ob}\mathfrak{A}$;

2. for any 1-morphism $f : X \to Y$ in $\mathfrak{A}$ a 2-isomorphism in $\mathfrak{C}$

$$\lambda_f : Ff\bullet\lambda_Y \xrightarrow{\sim} \lambda_X\bullet Gf : FX \to GY,$$

   which is an isomorphism of functors

$$\lambda_{\_} : F - \bullet\lambda_Y \to \lambda_X\bullet G- : \mathfrak{A}(X,Y) \to \mathfrak{C}(FX,GY),$$

that is, for any 2-morphism $\xi : f \to g : X \to Y$

$$
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\lambda_X"'] & FY \arrow[d, "\lambda_Y"] \arrow[dl, Rightarrow, "\lambda_f"] \\
GX \arrow[r, shift left=2, "Gf"] \arrow[r, shift right=2, "Gg"'] \arrow[r, phantom, "\Downarrow G\xi"] & GY
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, shift left=2, "Ff"] \arrow[r, shift right=2, "Fg"'] \arrow[r, phantom, "\Downarrow F\xi"] \arrow[d, "\lambda_X"'] & FY \arrow[d, "\lambda_Y"] \arrow[dl, Rightarrow, "\lambda_g"'] \\
GX \arrow[r, "Gg"'] & GY
\end{tikzcd},
$$

such that

3. (a) for any object $X \in \operatorname{Ob}\mathfrak{A}$

$$
\begin{tikzcd}
FX \arrow[r, shift left=2, "1_{FX}"] \arrow[r, shift right=2, "F1_X"'] \arrow[r, phantom, "\Downarrow \phi_0"] \arrow[d, "\lambda_X"'] & FX \arrow[d, "\lambda_X"] \arrow[dl, Rightarrow, "\lambda_{1_X}"'] \\
GX \arrow[r, "G1_X"'] & GX
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, "1_{FX}"] \arrow[d, "\lambda_X"'] \arrow[dr, phantom, "="] & FX \arrow[d, "\lambda_X"] \\
GX \arrow[r, shift left=2, "1_{GX}"] \arrow[r, shift right=2, "G1_X"'] \arrow[r, phantom, "\Downarrow \psi_0"] & GX
\end{tikzcd}
;
$$

   (b) for any pair of composable 1-morphisms $f, g \in \mathfrak{A}_1$

$$
\begin{tikzcd}
 & FY \arrow[dr, "Fg"] \arrow[d, Rightarrow, "\phi_2"] & \\
FX \arrow[ur, "Ff"] \arrow[rr, "F(fg)"'] \arrow[d, "\lambda_X"'] & {} & FZ \arrow[d, "\lambda_Z"] \arrow[dll, Rightarrow, "\lambda_{fg}"] \\
GX \arrow[rr, "G(fg)"'] & & GZ
\end{tikzcd}
=
\begin{tikzcd}
 & FY \arrow[dr, "Fg"] \arrow[d, "\lambda_Y"] \arrow[ddl, Rightarrow, "\lambda_f"'] & \\
FX \arrow[ur, "Ff"] \arrow[d, "\lambda_X"'] & GY \arrow[dr, "Gg"] \arrow[d, Rightarrow, "\psi_2"] & FZ \arrow[l, Rightarrow, "\lambda_g"'] \arrow[d, "\lambda_Z"] \\
GX \arrow[ur, "Gf"] \arrow[rr, "G(fg)"'] & {} & GZ
\end{tikzcd}
.
$$

If λ_f are identity isomorphisms, λ is called a *strict 2-transformation.* A weak 2-transformation $\lambda = (\lambda_X)$ for which λ_X are equivalences is called a *2-natural equivalence.*

C.5 Definition. A *modification* $m : \lambda \to \mu : F \to G : \mathfrak{A} \to \mathfrak{C}$ is

1. a family of 2-morphisms $m_X : \lambda_X \to \mu_X$, $X \in \operatorname{Ob}\mathfrak{A}$
   such that

2. for any 1-morphism $f : X \to Y$ in $\mathfrak{A}$

$$
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, shift right=3, "\mu_X"'] \arrow[d, shift left=3, "\lambda_X"] \arrow[d, phantom, "\overset{m_X}{\Leftarrow}"] & FY \arrow[d, "\lambda_Y"] \arrow[dl, Rightarrow, "\lambda_f"'] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\mu_X"'] & FY \arrow[d, shift right=3, "\mu_Y"'] \arrow[d, shift left=3, "\lambda_Y"] \arrow[d, phantom, "\overset{m_Y}{\Leftarrow}"] \arrow[dl, Rightarrow, "\mu_f"'] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}
$$

C.6 Proposition (Invertibility of 2-natural equivalences). *Let $\lambda : F \to G : \mathfrak{A} \to \mathfrak{C}$ be a 2-natural equivalence. Then there exist a weak 2-transformation $\mu : G \to F : \mathfrak{A} \to \mathfrak{C}$ and invertible modifications $\varepsilon : \lambda\mu \to 1_F : F \to F : \mathfrak{A} \to \mathfrak{C}$ and $\eta : 1_G \to \mu\lambda : G \to G : \mathfrak{A} \to \mathfrak{C}$. Thus, μ is quasi-inverse to λ.*

Proof. Since $\lambda_X : FX \to GX$ is an equivalence for every $X \in \operatorname{Ob}\mathfrak{A}$, we obtain: for every $X \in \operatorname{Ob}\mathfrak{A}$ there exist a 1-morphism $\mu_X : GX \to FX$ and invertible 2-morphisms $\varepsilon_X : \lambda_X\mu_X \to 1_{FX} : FX \to FX$, $\beta_X : 1_{GX} \to \mu_X\lambda_X : GX \to GX$.

C.7 Lemma. *There exist such invertible 2-morphisms $\eta_X : 1_{GX} \to \mu_X\lambda_X : GX \to GX$ that the following equations hold true:*

$$\big(\mu_X \xrightarrow{\eta_X\bullet\mu_X} \mu_X\lambda_X\mu_X \xrightarrow{\mu_X\bullet\varepsilon_X} \mu_X\big) = 1_{\mu_X}, \tag{C.7.1}$$

$$\big(\lambda_X \xrightarrow{\lambda_X\bullet\eta_X} \lambda_X\mu_X\lambda_X \xrightarrow{\varepsilon_X\bullet\lambda_X} \lambda_X\big) = 1_{\lambda_X}. \tag{C.7.2}$$

Proof. Consider the following functors:

$$\begin{aligned} \mathfrak{C}(GX,GX) &\longrightarrow \mathfrak{C}(GX,FX), & \mathfrak{C}(GX,GX) &\longrightarrow \mathfrak{C}(FX,GX), \qquad &(\text{C.7.3})\\ f &\longmapsto f\mu_X, & f &\longmapsto \lambda_X f,\\ \phi : f \to g &\longmapsto \phi\bullet\mu_X : f\mu_X \to g\mu_X, & \phi : f \to g &\longmapsto \lambda_X\bullet\phi : \lambda_X f \to \lambda_X g, &(\text{C.7.4})\\ \mathfrak{C}(GX,FX) &\longrightarrow \mathfrak{C}(GX,GX), & \mathfrak{C}(FX,GX) &\longrightarrow \mathfrak{C}(GX,GX),\\ h &\longmapsto h\lambda_X, & h &\longmapsto \mu_X h,\\ \chi : h \to k &\longmapsto \chi\bullet\lambda_X : h\lambda_X \to k\lambda_X, & \chi : h \to k &\longmapsto \mu_X\bullet\chi : \mu_X h \to \mu_X k. \end{aligned}$$

These functors are faithful. Let us prove it for the first one. Indeed, if $\phi\bullet\mu_X = \psi\bullet\mu_X : f\mu_X \to g\mu_X$, then $\phi\bullet\mu_X\bullet\lambda_X = \psi\bullet\mu_X\bullet\lambda_X : f\mu_X\lambda_X \to g\mu_X\lambda_X$ and $(f\bullet\beta_X)(\phi\bullet\mu_X\bullet\lambda_X) = (f\bullet\beta_X)(\psi\bullet\mu_X\bullet\lambda_X) : f \to g\mu_X\lambda_X$, i.e., $\phi\bullet(g\beta_X) = \psi\bullet(g\beta_X) : f \to g\mu_X\lambda_X$, hence $\phi = \psi$, because β_X is invertible. Similarly or by symmetry the other 3 functors are also faithful.

Functors (C.7.3) are full. Let us prove it for the first one. Given $\psi : f\mu_X \to g\mu_X$, we set $\phi = (f\bullet\beta)(\psi\bullet\lambda_X)(g\bullet\beta_X^{-1}) : f \to g$. Then $(f\bullet\beta_X)(\phi\bullet\mu_X\bullet\lambda_X)(g\bullet\beta_X^{-1}) = (f\bullet\beta_X)(f\bullet\beta_X^{-1})\phi = \phi = (f\bullet\beta_X)(\psi\bullet\lambda_X)(g\bullet\beta_X^{-1})$. This yields $\phi\bullet\mu_X\bullet\lambda_X = \psi\bullet\lambda_X$. Since multiplication with λ_X is a faithful functor, we obtain $\phi\bullet\mu_X = \psi$. Notice that if ψ is a 2-isomorphism, then the composition of 2-isomorphisms ϕ is a 2-isomorphism as well. Similarly we prove that the second functor of (C.7.3) is full.

Now let us take $f = 1_{GX}$, $g = \mu_X\lambda_X$. By the first bijection of (C.7.4) we find a 2-isomorphism $\eta_X : 1_{GX} \to \mu_X\lambda_X$ corresponding to the 2-isomorphism $\psi = \mu_X\bullet\varepsilon_X^{-1} : \mu_X \to \mu_X\lambda_X\mu_X$. Then $\eta_X\bullet\mu_X = \mu_X\bullet\varepsilon_X^{-1}$, that is, $(\eta_X\bullet\mu_X)(\mu_X\bullet\varepsilon_X) = 1_{\mu_X}$. By the second bijection of (C.7.4) we find a 2-isomorphism $\gamma_X : 1_{GX} \to \mu_X\lambda_X$ corresponding to the 2-isomorphism $\xi = \varepsilon_X^{-1}\bullet\lambda_X : \lambda_X \to \lambda_X\mu_X\lambda_X$. Then $\lambda_X\bullet\gamma_X = \varepsilon_X^{-1}\bullet\lambda_X$, that is, $(\lambda_X\bullet\gamma_X)(\varepsilon_X\bullet\lambda_X) = 1_{\lambda_X}$. We have

$$\begin{aligned} \gamma = \big(1_{GX} \xrightarrow{\gamma} \mu\lambda \xrightarrow{1_\mu\lambda} \mu\lambda\big) &= \big(1_{GX} \xrightarrow{\gamma} \mu\lambda \xrightarrow{\eta\mu\lambda} \mu\lambda\mu\lambda \xrightarrow{\mu\varepsilon\lambda} \mu\lambda\big)\\ &= \big(1_{GX} \xrightarrow{\eta} \mu\lambda \xrightarrow{\mu\lambda\gamma} \mu\lambda\mu\lambda \xrightarrow{\mu\varepsilon\lambda} \mu\lambda\big) = \big(1_{GX} \xrightarrow{\eta} \mu\lambda \xrightarrow{\mu 1_\lambda} \mu\lambda\big) = \eta. \end{aligned}$$

Therefore, the 2-isomorphism η fulfills both equations (C.7.1) and (C.7.2). □

For any 1-morphism $f : X \to Y$ in $\mathfrak{A}$ we define a 2-isomorphism μ_f as the pasting of

$$\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] & FX \arrow[r, "Ff"] \arrow[d, "\lambda_X"] \arrow[dl, phantom, "\eta_X"] & FY \arrow[d, "\lambda_Y"'] \arrow[dr, "1_{FY}"] \arrow[dl, phantom, "\lambda_f^{-1}"] & \\
& GX \arrow[r, "Gf"'] & GY \arrow[r, "\mu_Y"'] \arrow[ur, phantom, "\varepsilon_Y"] & FY
\end{tikzcd} .$$

Since λ_f determine an isomorphism of functors

$$\lambda_- : F - \bullet \lambda_Y \to \lambda_X \bullet G- : \mathfrak{A}(X, Y) \to \mathfrak{C}(FX, GY),$$

for any 2-morphism $\xi : f \to g : X \to Y$ equation 2 of Definition C.4 holds. It implies

$$\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] & FX \arrow[r, bend left, "Fg"] \arrow[r, bend right, "Ff"'] \arrow[r, phantom, "\Uparrow F\xi"] \arrow[d, "\lambda_X"] \arrow[dl, phantom, "\eta_X"] & FY \arrow[d, "\lambda_Y"'] \arrow[dr, "1_{FX}"] \arrow[dl, phantom, "\lambda_f^{-1}"] & \\
& GX \arrow[r, "Gf"'] & GY \arrow[r, "\mu_Y"'] \arrow[ur, phantom, "\varepsilon_Y"] & FY
\end{tikzcd}
=
\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] & FX \arrow[r, "Fg"] \arrow[d, "\lambda_X"] \arrow[dl, phantom, "\eta_X"] & FY \arrow[d, "\lambda_Y"'] \arrow[dr, "1_{FX}"] \arrow[dl, phantom, "\lambda_g^{-1}"] & \\
& GX \arrow[r, bend left, "Gg"] \arrow[r, bend right, "Gf"'] \arrow[r, phantom, "\Uparrow G\xi"] & GY \arrow[r, "\mu_Y"'] \arrow[ur, phantom, "\varepsilon_Y"] & FY
\end{tikzcd} .$$

This shows that the collection of 2-isomorphisms μ_f determines an isomorphism of functors

$$\mu_- : G - \bullet \mu_Y \to \mu_X \bullet F- : \mathfrak{A}(X, Y) \to \mathfrak{C}(GX, FY).$$

Let us check conditions 3(a),(b) of Definition C.4 for μ. For 3(a) we have

$$\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] & FX \arrow[r, "F1_X"] \arrow[d, "\lambda_X"] \arrow[dl, phantom, "\eta_X"] & FX \arrow[d, "\lambda_X"'] \arrow[dr, "1_{FX}"] \arrow[dl, phantom, "\lambda_{1_X}^{-1}"] & \\
& GX \arrow[r, bend left, "G1_X"] \arrow[r, bend right, "1_{GX}"'] \arrow[r, phantom, "\Uparrow\psi_0"] & GX \arrow[r, "\mu_X"'] \arrow[ur, phantom, "\varepsilon_X"] & FX
\end{tikzcd}
=
\begin{tikzcd}
GX \arrow[r, "\mu_X"] \arrow[dr, "1_{GX}"'] & FX \arrow[r, bend left, "F1_X"] \arrow[r, bend right, "1_{FX}"'] \arrow[r, phantom, "\Uparrow\phi_0"] \arrow[d, "\lambda_X"] \arrow[dl, phantom, "\eta_X"] \arrow[dr, phantom, "="] & FX \arrow[d, "\lambda_X"'] \arrow[dr, "1_{FX}"] & \\
& GX \arrow[r, "1_{GX}"'] & GX \arrow[r, "\mu_X"'] \arrow[ur, phantom, "\varepsilon_X"] & FX
\end{tikzcd}$$

and the required equation follows from (C.7.1).

Equation (C.7.2) and the corresponding property for λ imply

$$\begin{tikzcd}
& & GY \arrow[dr, "Gg"] \arrow[d, phantom, "\Downarrow\psi_2"] & & \\
& GX \arrow[ur, "Gf"] \arrow[rr, "G(fg)"'] \arrow[drr, phantom, "\lambda_{fg}^{-1}"] & {} & GZ \arrow[r, "\mu_Z"] \arrow[dr, phantom, "\varepsilon_Z"] & FZ \\
GX \arrow[ur, "1_{GX}"] \arrow[r, "\mu_X"'] \arrow[ur, phantom, "\eta_X", shift right=3] & FX \arrow[u, "\lambda_X"'] \arrow[rr, "F(fg)"'] & & FZ \arrow[u, "\lambda_Z"] \arrow[ur, "1_{FZ}"'] &
\end{tikzcd}$$

$$
=\quad
\begin{tikzcd}
 & & GY \arrow[r, "1_{GY}"] \arrow[dr, "\mu_Y" description] \arrow[d, phantom, "\Downarrow\varepsilon_Y"] & GY \arrow[dr, "Gg"] \arrow[d, phantom, "\lambda_g^{-1}"] & & \\
 & GX \arrow[ur, "Gf"] \arrow[r, Rightarrow, "\lambda_f^{-1}"] & FY \arrow[u, "\lambda_Y"] \arrow[r, "1_{FY}"'] \arrow[ur, phantom, "\Downarrow\eta_Y"] \arrow[d, phantom, "\Downarrow\phi_2"] & FY \arrow[u, "\lambda_Y"'] \arrow[dr, "Fg"'] & GZ \arrow[r, "\mu_Z"] \arrow[d, phantom, "\varepsilon_Z"] & FZ \\
GX \arrow[ur, "1_{GX}"] \arrow[r, "\mu_X"'] \arrow[ur, phantom, "\eta_X"] & FX \arrow[u, "\lambda_X"] \arrow[ur, "Ff"'] \arrow[rrr, "F(fg)"'] & & & FZ \arrow[u, "\lambda_Z"] \arrow[ur, "1_{FZ}"'] &
\end{tikzcd}
$$

and the assertion 3(b) for μ follows.

Let us verify that collections $\varepsilon = (\varepsilon_X)$ and $\eta = (\eta_X)$ determine modifications. Equation (C.7.2) implies that

$$
\begin{tikzcd}
FX \arrow[rr, "Ff"] \arrow[dr, "\lambda_X"'] \arrow[drr, "\lambda_X"] \arrow[dd, "1_{FX}"'] & & FY \arrow[dr, "\lambda_Y"] \arrow[d, phantom, "\lambda_f\Downarrow"] & \\
\Leftarrow\varepsilon_X & GX \arrow[r, "1_{GX}"] \arrow[dl, "\mu_X"] \arrow[dl, phantom, "\eta_X"] & GX \arrow[r, "Gf"] \arrow[d, phantom, "\lambda_f^{-1}\Downarrow"] & GY \arrow[d, "\mu_Y"] \arrow[d, phantom, "\varepsilon_Y"] \\
FX \arrow[urr, "\lambda_X"'] \arrow[rr, "Ff"'] & & FY \arrow[ur, "\lambda_Y"] \arrow[r, "1_{FY}"'] & FY
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[dd, "1_{FX}"'] & FY \arrow[dr, "\lambda_Y"] \arrow[dd, "1_{FY}"] & \\
= & & GY \arrow[dl, "\mu_Y"] \arrow[l, phantom, "\Longleftarrow \varepsilon_Y"] \\
FX \arrow[r, "Ff"'] & FY &
\end{tikzcd}
\quad .
$$

This means that $\varepsilon : \lambda\mu \to 1_F : F \to F : \mathfrak{A} \to \mathfrak{C}$ is a modification.

Also equation (C.7.2) implies that

$$
\begin{tikzcd}
 & GX \arrow[r, "Gf"] \arrow[dl, "\mu_X"'] \arrow[dd, "1_{GX}"] & GY \arrow[dd, "1_{GY}"] \\
FX \arrow[dr, "\lambda_X"'] \arrow[r, phantom, "\overset{\eta_X}{\Longleftarrow}"] & \arrow[r, phantom, "="] & \\
 & GX \arrow[r, "Gf"'] & GY
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
GX \arrow[r, "1_{GX}"] \arrow[d, "\mu_X"'] \arrow[d, phantom, "\eta_X"] & GX \arrow[rr, "Gf"] \arrow[d, phantom, "\Downarrow\lambda_f^{-1}"] & & GY \arrow[dl, "\mu_Y"] \arrow[dd, "1_{GX}"] \\
FX \arrow[ur, "\lambda_X"] \arrow[r, "Ff"'] \arrow[dr, "\lambda_X"'] & FY \arrow[urr, "\lambda_Y"] \arrow[r, "1_{FY}"'] \arrow[drr, "\lambda_Y"'] \arrow[d, phantom, "\Downarrow\lambda_f"] & FY \arrow[dr, "\lambda_Y"] \arrow[u, phantom, "\varepsilon_Y"] \arrow[r, phantom, "\overset{\eta_Y}{\Longleftarrow}"] \arrow[d, phantom, "="] & \\
 & GX \arrow[rr, "Gf"'] & & GY
\end{tikzcd}
$$

This means that $\eta : 1_G \to \mu\lambda : G \to G : \mathfrak{A} \to \mathfrak{C}$ is a modification.

Therefore, μ is a weak 2-transformation quasi-inverse to λ. □

C.8 Remark. Clearly, if $\lambda : F \to G : \mathfrak{A} \to \mathfrak{C}$ and $\mu : G \to F : \mathfrak{A} \to \mathfrak{C}$ are weak 2-transformations, quasi-inverse to each other, then both are 2-natural equivalences.

C.9. The 2-functor $\mathfrak{A}(A, \_)$. We fix a strict 2-category $\mathfrak{A}$ till the end of this section. An arbitrary object $A \in \operatorname{Ob}\mathfrak{A}$ gives rise to a strict 2-functor $\mathfrak{A}(A, \_) : \mathfrak{A} \to \mathcal{C}at$. It is specified by the following data:

1. the function $\operatorname{Ob}\mathfrak{A} \to \operatorname{Ob}\mathcal{C}at$, $X \mapsto \mathfrak{A}(A, X)$;

2. the functor $\mathfrak{A}(A,\_)_{XY} : \mathfrak{A}(X,Y) \to \mathfrak{Cat}(\mathfrak{A}(A,X),\mathfrak{A}(A,Y))$ for each pair of objects $X, Y \in \operatorname{Ob}\mathfrak{A}$.

The functor $\mathfrak{A}(A,\_)_{XY}$ is given as follows. For any 1-morphism $f : X \to Y$ the functor $\mathfrak{A}(A,f) : \mathfrak{A}(A,X) \to \mathfrak{A}(A,Y)$ is given by the following formulas:

$$
\begin{gathered}
(\phi : A \to X) \mapsto (\phi f : A \to Y), \\
(\pi : \phi \to \psi : A \to X) \mapsto (\pi_\bullet f : \phi f \to \psi f : A \to Y).
\end{gathered}
$$

For any 2-morphism $\alpha : f \to g : X \to Y$ the natural transformation $\mathfrak{A}(A,\alpha) : \mathfrak{A}(A,f) \to \mathfrak{A}(A,g) : \mathfrak{A}(A,X) \to \mathfrak{A}(A,Y)$ is given explicitly by its components:

$$
(\phi)\mathfrak{A}(A,\alpha) = \phi_\bullet\alpha : (\phi)\mathfrak{A}(A,f) = \phi f \to \phi g = (\phi)\mathfrak{A}(A,g), \qquad \phi \in \mathfrak{A}(A,X).
$$

We want to generalize the classical Yoneda Lemma to 2-categories. If we were using strict 2-functors and strict 2-transformations, we would view 2-categories as *cat*-categories, where *cat* is the category of categories. The Yoneda Lemma for enriched categories [EK66, Theorem I.8.6] would be sufficient. However, we need weak 2-transformations, so we prove a direct generalization in the weak setup.

Let $[\mathfrak{A},\mathfrak{Cat}]$ denote the strict 2-category of weak 2-functors $\mathfrak{A} \to \mathfrak{Cat}$, their weak 2-transformations and their modifications, see e.g. [Lyu99, Appendix A.1.5].

A weak 2-transformation $\lambda : \mathfrak{A}(A,\_) \to G : \mathfrak{A} \to \mathfrak{Cat}$ involves, in particular, a functor $\lambda_A : \mathfrak{A}(A,A) \to GA$. Evaluating it on the object $1_A \in \operatorname{Ob}\mathfrak{A}(A,A)$ we get an object $(1_A)\lambda_A \in \operatorname{Ob} GA$. A modification $m : \lambda \to \mu : \mathfrak{A}(A,\_) \to G : \mathfrak{A} \to \mathfrak{Cat}$ involves, in particular, a natural transformation $m_A : \lambda_A \to \mu_A : \mathfrak{A}(A,A) \to GA$. Evaluating it on the object $1_A \in \operatorname{Ob}\mathfrak{A}(A,A)$ we get a morphism $(1_A)m_A : (1_A)\lambda_A \to (1_A)\mu_A$ of GA.

C.10 Proposition. *Let $G : \mathfrak{A} \to \mathfrak{Cat}$ be a weak 2-functor. Let A be an object of $\mathfrak{A}$. Then the functor*

$$
\begin{aligned}
\operatorname{ev}_{1_A} : [\mathfrak{A},\mathfrak{Cat}](\mathfrak{A}(A,\_),G) &\longrightarrow GA, \\
\lambda &\longmapsto (1_A)\lambda_A, \\
m : \lambda \to \mu &\longmapsto (1_A)m_A : (1_A)\lambda_A \to (1_A)\mu_A,
\end{aligned}
$$

is full and faithful.

Proof. Clearly, the assignment ev_{1_A} gives a functor. Let us show that it is faithful. Let two modifications $m, n : \lambda \to \mu : \mathfrak{A}(A,\_) \to G : \mathfrak{A} \to \mathfrak{Cat}$ be given such that $(1_A)m_A = (1_A)n_A$. The modification m is a family of natural transformations $(m_C)_{C\in\operatorname{Ob}\mathfrak{A}}$ satisfying the equation

$$
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, shift right=4, "\mu_B"'] \arrow[d, shift left=4, "\lambda_B"] \arrow[d, phantom, "\overset{m_B}{\Leftarrow}"] & \mathfrak{A}(A,C) \arrow[d, "\lambda_C"] \arrow[dl, Rightarrow, "\lambda_f"'] \\
GB \arrow[r, "Gf"'] & GC
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, "\mu_B"'] & \mathfrak{A}(A,C) \arrow[d, shift right=4, "\mu_C"'] \arrow[d, shift left=4, "\lambda_C"] \arrow[d, phantom, "\overset{m_C}{\Leftarrow}"] \arrow[dl, Rightarrow, "\mu_f"'] \\
GB \arrow[r, "Gf"'] & GC
\end{tikzcd}
\tag{C.10.1}
$$

for an arbitrary 1-morphism $f : B \to C$ of $\mathfrak{A}$. In particular, it holds for $B = A$. Restrict this equation to the object 1_A of $\mathfrak{A}(A, A)$. Then it gives for an arbitrary 1-morphism $f : A \to C$ the equation

$$\begin{aligned}\big[(f)\lambda_C \xrightarrow{(1_A)\lambda_f} ((1_A)\lambda_A)(Gf) \xrightarrow{((1_A)m_A)(Gf)} ((1_A)\mu_A)(Gf)\big]& \\ = \big[(f)\lambda_C \xrightarrow{(f)m_C} (f)\mu_C \xrightarrow{(1_A)\mu_f} ((1_A)\mu_A)(Gf)\big]&.\end{aligned}$$

Therefore, the value of m_C on an arbitrary object f of $\mathfrak{A}(A, C)$ is completely determined by the morphism $(1_A)m_A$:

$$(f)m_C = \big[(f)\lambda_C \xrightarrow{(1_A)\lambda_f} ((1_A)\lambda_A)(Gf) \xrightarrow{((1_A)m_A)(Gf)} ((1_A)\mu_A)(Gf) \xrightarrow{(1_A)\mu_f^{-1}} (f)\mu_C\big].$$

Thus, $m = n$ and ev_{1_A} is faithful.

Let us prove that ev_{1_A} is full. Let $\lambda, \mu : \mathfrak{A}(A, \_) \to G : \mathfrak{A} \to \mathcal{C}at$ be weak 2-transformations. Let $\phi : (1_A)\lambda_A \to (1_A)\mu_A$ be a morphism of GA. We claim that there is a modification $m : \lambda \to \mu$ such that $(1_A)m_A = \phi$. The value of m_C on an arbitrary 1-morphism $f : A \to C$ can be only

$$(f)m_C = \big[(f)\lambda_C \xrightarrow{(1_A)\lambda_f} ((1_A)\lambda_A)(Gf) \xrightarrow{(\phi)(Gf)} ((1_A)\mu_A)(Gf) \xrightarrow{(1_A)\mu_f^{-1}} (f)\mu_C\big], \tag{C.10.2}$$

as we have seen. Let us verify that, indeed, this formula determines a modification.

First of all, each m_C is a natural transformation. Indeed, for each 2-morphism $\xi : f \to g : A \to C$ of $\mathfrak{A}$ the following diagram commutes:

$$\begin{tikzcd}
(f)\lambda_C \arrow[r, "(1_A)\lambda_f"] \arrow[d, "(\xi)\lambda_C"'] & ((1_A)\lambda_A)(Gf) \arrow[r, "(\phi)(Gf)"] \arrow[d, "((1_A)\lambda_A)(G\xi)"'] & ((1_A)\mu_A)(Gf) \arrow[r, "(1_A)\mu_f^{-1}"] \arrow[d, "((1_A)\mu_A)(G\xi)"] & (f)\mu_C \arrow[d, "(\xi)\mu_C"] \\
(g)\lambda_C \arrow[r, "(1_A)\lambda_g"] & ((1_A)\lambda_A)(Gg) \arrow[r, "(\phi)(Gg)"] & ((1_A)\mu_A)(Gg) \arrow[r, "(1_A)\mu_g^{-1}"] & (g)\mu_C
\end{tikzcd} \tag{C.10.3}$$

The central square commutes because $G\xi : Gf \to Gg$ is a natural transformation. Condition 2 for λ from Definition C.4 implies, in particular, equation

$$\begin{tikzcd}
\mathfrak{A}(A,A) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda_A"'] & \mathfrak{A}(A,C) \arrow[d, "\lambda_C"] \arrow[dl, Rightarrow, "\lambda_f"] \\
GA \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\xi"] & GC
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,A) \arrow[r, bend left, "\mathfrak{A}(A,f)"] \arrow[r, bend right, "\mathfrak{A}(A,g)"'] \arrow[r, phantom, "\Downarrow\mathfrak{A}(A,\xi)"] \arrow[d, "\lambda_A"'] & \mathfrak{A}(A,C) \arrow[d, "\lambda_C"] \arrow[dl, Rightarrow, "\lambda_g"'] \\
GA \arrow[r, "Gg"'] & GC
\end{tikzcd}.$$

Restricting this equation to the object 1_A of $\mathfrak{A}(A, A)$ we get an equation, which expresses precisely commutativity of the left square of diagram (C.10.3). The right square of (C.10.3) commutes by the same reasoning applied to μ instead of λ. Thus, m_C is a natural transformation.

Secondly, we have to prove equation (C.10.1) for the family (m_C) and for an arbitrary 1-morphism $f : B \to C$ of $\mathfrak{A}$. On an arbitrary object $g : A \to B$ of $\mathfrak{A}(A, B)$ this equation reads:

$$\big[(gf)\lambda_C \xrightarrow{(g)\lambda_f} ((g)\lambda_B)(Gf) \xrightarrow{((g)m_B)(Gf)} ((g)\mu_B)(Gf)\big]$$
$$= \big[(gf)\lambda_C \xrightarrow{(gf)m_C} (gf)\mu_C \xrightarrow{(g)\mu_f} ((g)\mu_B)(Gf)\big].$$

Substituting definition (C.10.2) of m_C we get an equation, which expresses commutativity of the exterior of the following diagram:

$$\begin{tikzcd}
((g)\lambda_B)(Gf) \arrow[r, "((1_A)\lambda_g)(Gf)"] & ((1_A)\lambda_A)(Gg)(Gf) \arrow[r, "(\phi)(Gg)(Gf)"] \arrow[d, "((1_A)\lambda_A)(g,f)\psi_2"'] & ((1_A)\mu_A)(Gg)(Gf) \arrow[r, "((1_A)\mu_g^{-1})(Gf)"] \arrow[d, "((1_A)\mu_A)(g,f)\psi_2"] & ((g)\mu_B)(Gf) \\
(gf)\lambda_C \arrow[u, "(g)\lambda_f"] \arrow[r, "(1_A)\lambda_{gf}"] & ((1_A)\lambda_A)(G(gf)) \arrow[r, "(\phi)(G(gf))"] & ((1_A)\mu_A)(G(gf)) \arrow[r, "(1_A)\mu_{gf}^{-1}"] & (gf)\mu_C \arrow[u, "(g)\mu_f"']
\end{tikzcd} \tag{C.10.4}$$

The middle square commutes, because $(g, f)\psi_2 : (Gg)(Gf) \to G(gf)$ is a morphism of functors. Property 3(b) of Definition C.4 for λ implies, in particular, the equation

$$\begin{tikzcd}
& \mathfrak{A}(A,B) \arrow[dr, "\mathfrak{A}(A,f)"] \arrow[d, Rightarrow, "="] & \\
\mathfrak{A}(A,A) \arrow[ur, "\mathfrak{A}(A,g)"] \arrow[rr, "\mathfrak{A}(A,gf)"'] \arrow[d, "\lambda_A"'] & & \mathfrak{A}(A,C) \arrow[d, "\lambda_C"] \arrow[dll, Rightarrow, "\lambda_{gf}"] \\
GA \arrow[rr, "G(gf)"'] & & GC
\end{tikzcd}
=
\begin{tikzcd}
& \mathfrak{A}(A,B) \arrow[dr, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda_B"] \arrow[ddl, Rightarrow, "\lambda_g"] & \\
\mathfrak{A}(A,A) \arrow[ur, "\mathfrak{A}(A,g)"] \arrow[d, "\lambda_A"'] & GB \arrow[dr, "Gf"] \arrow[d, phantom, "\Downarrow\psi_2"] & \mathfrak{A}(A,C) \arrow[l, Rightarrow, "\lambda_f"'] \arrow[d, "\lambda_C"] \\
GA \arrow[ur, "Gg"] \arrow[rr, "G(gf)"'] & & GC
\end{tikzcd} .$$

Restricting this equation to the object 1_A of $\mathfrak{A}(A, A)$, we will get precisely the left square of diagram (C.10.4), therefore, it commutes. The right square of (C.10.4) commutes by the same reasoning applied to μ instead of λ. Therefore, m is a modification, and ev_{1_A} is full. □

C.11 Proposition. *For each object x of GA there is a weak 2-transformation*

$$\lambda^x = \lambda^{A,x} = {}^G\lambda^{A,x} : \mathfrak{A}(A, \_) \to (G, \psi_2, \psi_0) : \mathfrak{A} \to \mathcal{C}at,$$

specified by the family of functors λ^x_C, $C \in \operatorname{Ob}\mathfrak{A}$:

$$\begin{aligned}
\lambda^x_C : \mathfrak{A}(A, C) &\longrightarrow GC, \\
f : A \to C &\longmapsto (x)(Gf), \\
\xi : f \to g : A \to C &\longmapsto (x)(G\xi) : (x)(Gf) \to (x)(Gg),
\end{aligned}$$

and by the family of invertible natural transformations

$$\lambda^x_f : \mathfrak{A}(A,f)_\bullet\lambda^x_C \to \lambda^x_B{}_\bullet Gf : \mathfrak{A}(A,B) \to GC,$$

$f \in \operatorname{Ob}\mathfrak{A}(B,C)$, *which map an object* $g \in \operatorname{Ob}\mathfrak{A}(A,B)$ *to the isomorphism of* GC*:*

$$(g)\lambda^x_f \overset{\text{def}}{=} (x)(g,f)\psi_2^{-1} : (gf)\lambda^x_C = (x)(G(gf)) \to (x)(Gg)(Gf) = (g)\lambda^x_B(Gf). \qquad \text{(C.11.1)}$$

For each morphism $u : x \to y$ *of* GA *there is a modification*

$$\lambda^u = \lambda^{A,u} = {}^G\lambda^{A,u} : \lambda^x \to \lambda^y : \mathfrak{A}(A,\_) \to (G,\psi_2,\psi_0) : \mathfrak{A} \to \mathfrak{C}at,$$

specified by the family of natural transformations λ^u_C, $C \in \operatorname{Ob}\mathfrak{A}$*:*

$$\lambda^u_C : \lambda^x_C \to \lambda^y_C : \mathfrak{A}(A,C) \to GC,$$
$$\lambda^u_C : (f : A \to C) \longmapsto \big((f)\lambda^u_C = (u)(Gf) : (f)\lambda^x_C = (x)(Gf) \to (y)(Gf) = (f)\lambda^y_C\big). \qquad \text{(C.11.2)}$$

The correspondence

$$\begin{aligned} \Lambda : GA &\longrightarrow [\mathfrak{A},\mathfrak{C}at](\mathfrak{A}(A,\_),G),\\ x &\longmapsto \lambda^x,\\ u : x \to y &\longmapsto \lambda^u : \lambda^x \to \lambda^y, \end{aligned}$$

is a functor.

Proof. As $G : \mathfrak{A}(A,C) \to \mathfrak{C}at(GA,GC)$ is a functor, $G1_f = 1_{Gf}$ for the unit 2-morphism $1_f : f \to f : A \to C$ of $\mathfrak{A}$, and for each pair of composable 2-morphisms $f \xrightarrow{\xi} g \xrightarrow{\chi} h : A \to C$ of $\mathfrak{A}$ we have

$$G(\xi\chi) = \big(Gf \xrightarrow{G\xi} Gg \xrightarrow{G\chi} Gh\big).$$

Evaluating these equations on x we get $(1_f)\lambda^x_C = (x)1_{Gf} = 1_{(x)(Gf)}$ and $(\xi\chi)\lambda^x_C = (\xi\lambda^x_C)(\chi\lambda^x_C)$, thus, λ^x_C is a functor.

We claim that λ^x_f given by (C.11.1) is a natural transformation. Indeed, naturality of ψ_2, expressed by

$$\begin{tikzcd} G(gf) \arrow[d, "G(\xi_\bullet f)"'] & (Gg)(Gf) \arrow[l, "(g,f)\psi_2"'] \arrow[d, "G\xi_\bullet Gf"] \arrow[dl, phantom, "="] \\ G(hf) & (Gh)(Gf) \arrow[l, "(h,f)\psi_2"'] \end{tikzcd}$$

implies commutativity of

$$\begin{tikzcd} (x)(G(gf)) \arrow[r, "(x)(g,f)\psi_2^{-1}"] \arrow[d, "(x)(G(\xi_\bullet f))"'] \arrow[dr, phantom, "="] & (x)(Gg)(Gf) \arrow[d, "(x)(G\xi)(Gf)"] \\ (x)(G(hf)) \arrow[r, "(x)(h,f)\psi_2^{-1}"] & (x)(Gh)(Gf) \end{tikzcd}$$

which is nothing else, but naturality of λ^x_f:

$$\begin{tikzcd}
(g)\mathfrak{A}(A,f)\lambda^x_C \arrow[r, "(g)\lambda^x_f"] \arrow[d, "(\xi)\mathfrak{A}(A,f)\lambda^x_C"'] \arrow[dr, phantom, "="] & (g)\lambda^x_B(Gf) \arrow[d, "(\xi)\lambda^x_B(Gf)"] \\
(h)\mathfrak{A}(A,f)\lambda^x_C \arrow[r, "(h)\lambda^x_f"] & (h)\lambda^x_B(Gf)
\end{tikzcd}$$

We claim that

$$\lambda^x_{\_} : \mathfrak{A}(A,\_)\lambda^x_C \to \lambda^x_B \bullet G- : \mathfrak{A}(A,B) \to GC$$

is a morphism of functors. That is, for each 2-morphism $\xi : f \to g : B \to C$ of $\mathfrak{A}$ the following equation holds:

$$\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda^x_B"'] & \mathfrak{A}(A,C) \arrow[d, "\lambda^x_C"] \arrow[dl, Rightarrow, "\lambda^x_f"] \\
GB \arrow[r, shift left=2, "Gf"] \arrow[r, shift right=2, "Gg"'] \arrow[r, phantom, "\Downarrow G\xi"] & GC
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, shift left=2, "\mathfrak{A}(A,f)"] \arrow[r, shift right=2, "\mathfrak{A}(A,g)"'] \arrow[r, phantom, "\Downarrow \mathfrak{A}(A,\xi)"] \arrow[d, "\lambda^x_B"'] & \mathfrak{A}(A,C) \arrow[d, "\lambda^x_C"] \arrow[dl, Rightarrow, "\lambda^x_g"'] \\
GB \arrow[r, "Gg"'] & GC
\end{tikzcd} . \tag{C.11.3}$$

Indeed, for each 1-morphism $h : A \to B$ of $\mathfrak{A}$ we have

$$\begin{tikzcd}
(x)(G(hf)) \arrow[r, "(x)(h{,}f)\psi_2^{-1}"] \arrow[d, "(x)(G(h\bullet\xi))"'] \arrow[dr, phantom, "="] & (x)(Gh)(Gf) \arrow[d, "(x)(Gh)(G\xi)"] \\
(x)(G(hg)) \arrow[r, "(x)(h{,}g)\psi_2^{-1}"] & (x)(Gh)(Gg)
\end{tikzcd}$$

by naturality of ψ_2. Rewriting this equation in the form

$$\begin{tikzcd}
(hf)\lambda^x_C \arrow[r, "(h)\lambda^x_f"] \arrow[d, "(h\bullet\xi)\lambda^x_C"'] \arrow[dr, phantom, "="] & (h)\lambda^x_B(Gf) \arrow[d, "((h)\lambda^x_B)(G\xi)"] \\
(hg)\lambda^x_C \arrow[r, "(h)\lambda^x_g"] & (h)\lambda^x_B(Gg)
\end{tikzcd}$$

we deduce that (C.11.3) holds on h. Therefore, condition 2 of Definition C.4 is satisfied.

Let us verify condition 3(a) of Definition C.4, that is, equation

$$\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "1_{\mathfrak{A}(A,B)}", "\mathfrak{A}(A,1_B)"'] \arrow[d, "\lambda^x_B"'] & \mathfrak{A}(A,B) \arrow[d, "\lambda^x_B"] \arrow[dl, Rightarrow, "\lambda^x_{1_B}"] \\
GB \arrow[r, "G1_B"'] & GB
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "1_{\mathfrak{A}(A,B)}"] \arrow[d, "\lambda^x_B"'] \arrow[dr, phantom, "="] & \mathfrak{A}(A,B) \arrow[d, "\lambda^x_B"] \\
GB \arrow[r, shift left=2, "1_{GB}"] \arrow[r, shift right=2, "G1_B"'] \arrow[r, phantom, "\Downarrow \psi_0"] & GB
\end{tikzcd} .$$

On an object $f : A \to B$ of $\mathfrak{A}(A, B)$ it reads:

$$(f)\lambda^x_{1_B} = ((f)\lambda^x_B)\psi_0 : (f)\lambda^x_B \to (f)\lambda^x_B(G1_B). \tag{C.11.4}$$

It follows from condition (C.3.1) for G,

$$\big[Gf \xrightarrow{Gf\bullet\psi_0} (Gf)(G1_B) \xrightarrow{(f,1_B)\psi_2} Gf\big] = 1_{Gf} : Gf \to Gf : GA \to GB,$$

which, evaluated on $x \in \operatorname{Ob} GA$, can be written as

$$(x)(f, 1_B)\psi_2^{-1} = ((x)(Gf))\psi_0 : (x)(Gf) \to (x)(Gf)(G1_B).$$

This is precisely (C.11.4).

Let us verify condition 3(b) of Definition C.4, that is, equation

$$\begin{tikzcd}
 & \mathfrak{A}(A,C) \arrow[dr, "\mathfrak{A}(A,g)"] \arrow[d, Rightarrow, no head, "="] & \\
\mathfrak{A}(A,B) \arrow[ur, "\mathfrak{A}(A,f)"] \arrow[rr, "\mathfrak{A}(A,fg)"'] \arrow[d, "\lambda^x_B"'] & {} & \mathfrak{A}(A,D) \arrow[d, "\lambda_D"] \arrow[dll, Rightarrow, "\lambda^x_{fg}"] \\
GB \arrow[rr, "G(fg)"'] & & GD
\end{tikzcd}
=
\begin{tikzcd}
 & \mathfrak{A}(A,C) \arrow[dr, "\mathfrak{A}(A,g)"] \arrow[d, "\lambda^x_C"] \arrow[dl, Rightarrow, "\lambda^x_f"'] & \\
\mathfrak{A}(A,B) \arrow[ur, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda^x_B"'] & GC \arrow[dr, "Gg"] \arrow[d, Rightarrow, "\psi_2"] & \mathfrak{A}(A,D) \arrow[l, Rightarrow, "\lambda^x_g"'] \arrow[d, "\lambda^x_D"] \\
GB \arrow[ur, "Gf"] \arrow[r, "G(fg)"'] & {} \arrow[r] & GD
\end{tikzcd} \tag{C.11.5}$$

for arbitrary pair of composable 1-morphisms $B \xrightarrow{f} C \xrightarrow{g} D$ of $\mathfrak{A}$. We have to check this equation on an arbitrary 1-morphism $h : A \to B$. Condition 4(b) of Definition C.3 for G is the equation

$$\begin{tikzcd}
(Gh)(Gf)(Gg) \arrow[r, "Gh\bullet(f,g)\psi_2"] \arrow[d, "(h,f)\psi_2\bullet Gg"'] \arrow[dr, phantom, "="] & (Gh)\bullet G(fg) \arrow[d, "(h,fg)\psi_2"] \\
G(hf).Gg \arrow[r, "(hf,g)\psi_2"'] & G(hfg)
\end{tikzcd}$$

Evaluating it on x we get the equation

$$\begin{aligned}(x)(h, fg)\psi_2^{-1} = \big[&(x)(G(hfg)) \xrightarrow{(x)(hf,g)\psi_2^{-1}} (x)(G(hf))(Gg) \\ &\xrightarrow{((x)(h,f)\psi_2^{-1})(Gg)} (x)(Gh)(Gf)(Gg) \xrightarrow{((x)(Gh))(f,g)\psi_2} (x)(Gh)(G(fg))\big],\end{aligned}$$

which can be rewritten as

$$\begin{aligned}(h)\lambda^x_{fg} = \big[&(hfg)\lambda^x_D \xrightarrow{(hf)\lambda^x_g} (hf)\lambda^x_C(Gg) \\ &\xrightarrow{((h)\lambda^x_f)(Gg)} (h)\lambda^x_B(Gf)(Gg) \xrightarrow{((h)\lambda^x_B)(f,g)\psi_2} (h)\lambda^x_B(G(fg))\big].\end{aligned}$$

And this is precisely (C.11.5), evaluated on $h : A \to B$.

Therefore, all conditions of Definition C.4 are satisfied, and λ^x is a weak 2-transformation.

Let us show that correspondence (C.11.2) defines a natural transformation. Indeed, for each 2-morphism $\xi : f \to g : A \to C$ of $\mathfrak{A}$ the diagram

$$\begin{tikzcd}
(f)\lambda^x_C = (x)(Gf) \arrow[r, "(f)\lambda^u_C", "(u)(Gf)"'] \arrow[d, "(\xi)\lambda^x_C = "', "(x)(G\xi)"] & (y)(Gf) = (f)\lambda^y_C \arrow[d, "(y)(G\xi)"', "= (\xi)\lambda^y_C"] \\
(g)\lambda^x_C = (x)(Gg) \arrow[r, "(u)(Gg)", "(g)\lambda^u_C"'] & (y)(Gg) = (g)\lambda^y_C
\end{tikzcd}$$

commutes due to $G\xi : Gf \to Gg : GA \to GC$ being a natural transformation.

We claim that property 2 of Definition C.5 holds for λ^u. For an arbitrary 1-morphism $f : B \to C$ of $\mathfrak{A}$ we have to prove the equation

$$\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, shift right=3ex, "\lambda^y_B"'] \arrow[d, shift left=3ex, "\lambda^x_B"] \arrow[d, phantom, "\overset{\lambda^u_B}{\Leftarrow}"] & \mathfrak{A}(A,C) \arrow[d, "\lambda^x_C"] \arrow[dl, Rightarrow, "\lambda^x_f"'] \\
GB \arrow[r, "Gf"] & GC
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[r, "\mathfrak{A}(A,f)"] \arrow[d, "\lambda^y_B"'] & \mathfrak{A}(A,C) \arrow[d, shift right=3ex, "\lambda^y_C"'] \arrow[d, shift left=3ex, "\lambda^x_C"] \arrow[d, phantom, "\overset{\lambda^u_C}{\Leftarrow}"] \arrow[dl, Rightarrow, "\lambda^y_f"'] \\
GB \arrow[r, "Gf"] & GC
\end{tikzcd} .$$

On the object $g : A \to B$ of $\mathfrak{A}(A,B)$ this equation reads

$$\begin{tikzcd}
(gf)\lambda^x_C = (x)(G(gf)) \arrow[r, "(g)\lambda^x_f", "(x)(g,f)\psi_2^{-1}"'] \arrow[d, "(gf)\lambda^u_C = "', "(u)(G(gf))"] \arrow[dr, phantom, "="] & (x)(Gg)(Gf) = (g)\lambda^x_B(Gf) \arrow[d, "(u)(Gg)(Gf)"', "= (g)\lambda^u_B(Gf)"] \\
(gf)\lambda^y_C = (y)(G(gf)) \arrow[r, "(y)(g,f)\psi_2^{-1}", "(g)\lambda^y_f"'] & (y)(Gg)(Gf) = (g)\lambda^y_B(Gf)
\end{tikzcd}$$

It holds due to $(g,f)\psi_2^{-1} : G(gf) \to (Gg)(Gf) : GA \to GC$ being a natural transformation. Therefore, λ^u is a modification.

The unit morphism $1_x : x \to x$ of GA goes to the identity transformation

$$\lambda^{1_x}_C : (f : A \to C) \longmapsto \bigl((1_x)(Gf) = 1_{(x)(Gf)} : (x)(Gf) \to (x)(Gf)\bigr),$$

because Gf is a functor. For a pair of composable morphisms $x \xrightarrow{u} y \xrightarrow{v} z$ of GA we have $\lambda^{uv}_C = \bigl(\lambda^x_C \xrightarrow{\lambda^u_C} \lambda^y_C \xrightarrow{\lambda^v_C} \lambda^z_C\bigr)$, since $(uv)(Gf) = (u)(Gf) \cdot (v)(Gf)$ due to Gf being a functor. Therefore, Λ is a functor. $\square$

C.12 Proposition. *Functors*

$$\mathrm{ev}_{1_A} : [\mathfrak{A}, \mathcal{C}at](\mathfrak{A}(A,\_), G) \to GA \quad \textit{and} \quad \Lambda : GA \to [\mathfrak{A}, \mathcal{C}at](\mathfrak{A}(A,\_), G)$$

are equivalences, quasi-inverse to each other.

Proof. We have

$$\big[GA \xrightarrow{\Lambda} [\mathfrak{A}, \mathcal{C}at](\mathfrak{A}(A,\_), G) \xrightarrow{\mathrm{ev}_{1_A}} GA\big] = G1_A.$$

Indeed, for any object x of GA

$$(x)\Lambda\,\mathrm{ev}_{1_A} = (\lambda^x)\,\mathrm{ev}_{1_A} = (1_A)\lambda^x_A = (x)(G1_A),$$

for any morphism $u : x \to y$ of GA

$$(u)\Lambda\,\mathrm{ev}_{1_A} = (\lambda^u)\,\mathrm{ev}_{1_A} = (1_A)\lambda^u_A = (u)(G1_A).$$

An isomorphism of functors $\psi_0 : 1_{GA} \to G1_A$ implies that an arbitrary object x of GA is isomorphic to $(x)(G1_A) = ((x)\Lambda)\,\mathrm{ev}_{1_A}$. Thus, ev_{1_A} is essentially surjective on objects. By Proposition C.10 ev_{1_A} is an equivalence. Therefore, Λ is isomorphic to a functor quasi-inverse to ev_{1_A}. Hence, Λ itself is an equivalence quasi-inverse to ev_{1_A}. □

C.13. Example of strict 2-functor $G = \mathfrak{A}(B,\_)$. Applying Proposition C.11 to the strict 2-functor $G = \mathfrak{A}(B,\_) : \mathfrak{A} \to \mathcal{C}at$, we get the following. An arbitrary 1-morphism $f : B \to A$ gives rise to the strict 2-transformation $f^* = {}^{\mathfrak{A}(B,-)}\lambda^{A,f} : \mathfrak{A}(A,\_) \to \mathfrak{A}(B,\_)$. It is specified by the family of functors f^*_C, $C \in \mathrm{Ob}\,\mathfrak{A}$:

$$\begin{aligned} f^*_C = \mathfrak{A}(f,C) : \mathfrak{A}(A,C) &\longrightarrow \mathfrak{A}(B,C) \\ (\phi : A \to C) &\longmapsto (f)\mathfrak{A}(B,\phi) = f\phi : B \to C, \\ (\pi : \phi \to \psi : A \to C) &\longmapsto (f)\mathfrak{A}(B,\pi) = f_\bullet\pi : f\phi \to f\psi : B \to C. \end{aligned}$$

An arbitrary 2-morphism $\alpha : f \to g : B \to A$ gives rise to the modification $\alpha^* = {}^{\mathfrak{A}(B,-)}\lambda^{A,\alpha} : f^* \to g^* : \mathfrak{A}(A,\_) \to \mathfrak{A}(B,\_)$ given by the family of natural transformations $\alpha^*_C : f^*_C \to g^*_C : \mathfrak{A}(A,C) \to \mathfrak{A}(B,C)$, $C \in \mathrm{Ob}\,\mathfrak{A}$. The transformation α^*_C is specified by its components:

$$\alpha^*_C = \mathfrak{A}(\alpha,C) : (\phi : A \to C) \longmapsto (\alpha)\mathfrak{A}(B,\phi) = \alpha_\bullet\phi : (\phi)f^*_C = f\phi \to g\phi = (\phi)g^*_C.$$

By Proposition C.11 the correspondence $f \mapsto f^*$, $\alpha \mapsto \alpha^*$ determines a functor $Y_{AB} : \mathfrak{A}^{\mathrm{op}}(A,B) = \mathfrak{A}(B,A) \to [\mathfrak{A}, \mathcal{C}at](\mathfrak{A}(A,\_), \mathfrak{A}(B,\_))$. One easily verifies that in fact we have a strict 2-functor $Y : \mathfrak{A}^{\mathrm{op}} \to [\mathfrak{A}, \mathcal{C}at]$.

C.14 Corollary. *Y is a local equivalence, i.e., for each pair of objects $A, B \in \mathrm{Ob}\,\mathfrak{A}^{\mathrm{op}}$ the functor Y_{AB} is an equivalence.*

C.15 Proposition. *Let $G : \mathfrak{A} \to \mathcal{C}at$ be a weak 2-functor. Then the following conditions are equivalent:*

1. *there exists an object A of $\mathfrak{A}$ and a 2-natural equivalence $\lambda : \mathfrak{A}(A,\_) \to G$;*

2. *there exists an object A of $\mathfrak{A}$ and an object x of GA such that the weak 2-transformation $\lambda^x : \mathfrak{A}(A,\_) \to G$ is a 2-natural equivalence.*

Proof. Clearly, the second property implies the first one. Assume that condition 1) holds. By Proposition C.12 the weak 2-transformation λ is isomorphic to λ^x for some $x \in \operatorname{Ob} GA$. By Proposition C.6 λ is a quasi-invertible 1-morphism of $[\mathfrak{A}, \mathfrak{C}at]$, hence, so is λ^x. By Remark C.8 condition 2) holds. $\square$

C.16 Definition. A weak 2-functor $G : \mathfrak{A} \to \mathfrak{C}at$ is *representable* if it satisfies equivalent conditions of Proposition C.15. A pair (A, x) consisting of an object A of $\mathfrak{A}$ and an object x of GA is said to *represent* G, if $\lambda^x = \lambda^{A,x} = {}^G\lambda^{A,x} : \mathfrak{A}(A,\_) \to G$ is a 2-natural equivalence.

C.17. Uniqueness of the representing pair. Let two pairs (A, x) and (B, y) represent G. Then there is a quasi-inverse to $\lambda^{B,y} : \mathfrak{A}(B,\_) \to G$ weak 2-transformation $\lambda^{B,y-} : G \to \mathfrak{A}(B,\_)$. Define a 2-natural equivalence $\mu = \lambda^{A,x} \bullet \lambda^{B,y-} : \mathfrak{A}(A,\_) \to \mathfrak{A}(B,\_)$. It is isomorphic to the 2-transformation ${}^{\mathfrak{A}(B,-)}\lambda^{A,f}$ for some $f \in \operatorname{Ob} \mathfrak{A}(B, A)$. There is an invertible modification m:

$$\begin{tikzcd}
\mathfrak{A}(A,_) \arrow[rr, "{}^G\lambda^{A,x}"] \arrow[dr, "{}^{\mathfrak{A}(B,-)}\lambda^{A,f}"'] & \arrow[d, phantom, "\Downarrow m"] & G \\
& \mathfrak{A}(B,_) \arrow[ur, "{}^G\lambda^{B,y}"'] &
\end{tikzcd}$$

Then $(1_A)m_A : (x)(G1_A) \to (y)(Gf)$ is an isomorphism of GA. Therefore,

$$x = (x)(1_{GA}) \xrightarrow{(x)\psi_0} (x)(G1_A) \xrightarrow{(1_A)m_A} (y)(Gf)$$

is an isomorphism of GA. By symmetry we get a 1-morphism $g : A \to B$ of $\mathfrak{A}$ and an isomorphism $y \xrightarrow{\sim} (x)(Gg)$ of GB. By construction the strict 2-transformations

$$\begin{aligned}
{}^{\mathfrak{A}(B,-)}\lambda^{A,f} &: \mathfrak{A}(A,_) \to \mathfrak{A}(B,_), \\
{}^{\mathfrak{A}(A,-)}\lambda^{B,g} &: \mathfrak{A}(B,_) \to \mathfrak{A}(A,_)
\end{aligned}$$

are quasi-inverse to each other. In particular,

$$\begin{aligned}
1_A \simeq ((1_A)^{\mathfrak{A}(B,-)}\lambda_A^{A,f})^{\mathfrak{A}(A,-)}\lambda_A^{B,g} &= ((f)\mathfrak{A}(B, 1_A))^{\mathfrak{A}(A,-)}\lambda_A^{B,g} \\
&= (f)^{\mathfrak{A}(A,-)}\lambda_A^{B,g} = (g)\mathfrak{A}(A, f) = gf,
\end{aligned}$$

and by symmetry $1_B \simeq fg$. Therefore, 1-morphisms f and g are quasi-inverse to each other.

Summing up, a pair $(A \in \operatorname{Ob} \mathfrak{A}, x \in \operatorname{Ob} GA)$ representing a weak 2-functor $G : \mathfrak{A} \to \mathfrak{C}at$ is unique up to equivalence f of the first objects, such that Gf preserves the second object up to an isomorphism.

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