André-Quillen homology via functor homology

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We obtain André-Quillen homology for commutative algebras using relative homological algebra in the category of functors on finite pointed sets.

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1 Introduction.

Let Γ be the small category of finite pointed sets. For any $n \geq 0$, let $[n]$ be the set $\{0, 1, ..., n\}$ with basepoint 0. We assume that the objects of Γ are the sets $[n]$. A left Γ-module is a covariant functor $\Gamma \to \mathsf{Vect}$ to the category of vector spaces over a field K. For a left Γ-module F we put

$$\pi_0(F) := \mathsf{Coker}(d_0 - d_1 + d_2 : F([2]) \to F([1])),$$

where d_1 is induced by the folding map $[2] \to [1]$, $1, 2 \mapsto 1$ while d_0 and d_2 are induced by the projection maps $[2] \to [1]$ given respectively by $1 \mapsto 1, 2 \mapsto 0$ and $1 \mapsto 0, 2 \mapsto 1$. The category Γ-mod of left Γ-modules is an abelian category with enough projective and injective objects. Therefore one can form the left derived functors of the functor $\pi_0 : \Gamma\text{-}mod \to \mathsf{Vect}$, which we will denote by π_*. Thanks to [5] and [6] we know that $\pi_* F$ is isomorphic to the homotopy of the spectrum corresponding to the Γ-space F according to Segal (see [9] and [1]).

Let A be a commutative algebra over a ground field K and let M be an A-module. There exists a functor $\mathcal{L}(A, M) : \Gamma \to \mathsf{Vect}$, which assigns $M \otimes A^{\otimes n}$ to $[n]$ (see [3] or section 3). Here all tensor products are taken over K. It was proved in [7] that $\pi_*(\mathcal{L}(A, M))$ is isomorphic to a brave new algebra version of André-Quillen homology $\mathsf{H}^{\Gamma}_*(A, M)$ constructed by Alan Robinson and Sarah Whitehouse [10]. The main result of this paper shows that a similar isomorphism exists also for André-Quillen homology if one takes an appropriate relative derived functors of the same functor $\pi_0 : \Gamma\text{-}mod \to \mathsf{Vect}$.

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2 A class of proper exact sequences

Thanks to the Yoneda lemma Γ^n, $n \geq 0$, are projective generators of the category Γ-*mod*. Here

$$\Gamma^n := K[\mathsf{Hom}_\Gamma([n], -)].$$

and $K[S]$ denotes the free vector space generated by a set S. For left Γ-modules F and T one defines the pointwise tensor product $F \otimes T$ to be the left Γ-module given by $(F \otimes T)([n]) = F([n]) \otimes T([n])$. Since $\Gamma^n \otimes \Gamma^m \cong \Gamma^{n+m}$ one sees that the tensor product of two projective left Γ-modules is still projective. We also have $\Gamma^n \cong (\Gamma^1)^{\otimes n}$.

A partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ is a sequence of natural numbers $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. The sum of partition is given by $s(\lambda) := \lambda_1 + \cdots + \lambda_k$, while the group $\Sigma(\lambda)$ is a product of the corresponding symmetric groups

$$\Sigma(\lambda) := \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}.$$

which is identified with the Young subgroup of $\Sigma_{s(\lambda)}$. Let us observe that $\Sigma_n = Aut_\Gamma([n])$ and therefore Σ_n acts on $\Gamma^n \cong (\Gamma^1)^{\otimes n}$. For a partition λ with $s(\lambda) = n$ we let $\Gamma(\lambda)$ be the coinvariants of Γ^n under the action of $\Sigma(\lambda) \subset \Sigma_n$.

For a vector space V we let $S^*(V)$, $\Lambda^*(V)$ and $D^*(V)$ be respectively the symmetric, exterior and divided power algebra generated by V. Let us recall that $S^n(V) = (V^{\otimes n})/\Sigma_n$ is the space of coinvariants of $V^{\otimes n}$ under the action of the symmetric group Σ_n, while $D^n(V) = (V^{\otimes n})^{\Sigma_n}$ is the space of invariants. Moreover for a particiaon $\lambda = (\lambda_1, \cdots, \lambda_k)$ we put

$$S^\lambda := S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_k}.$$

A similar meaning has also Λ^λ and D^λ. It follows from the definition that

$$\Gamma(\lambda) \cong S^\lambda \circ \Gamma^1.$$

In particular $\Gamma(1, \cdots, 1) \cong \Gamma^n$ and $\Gamma(n) \cong S^n \circ \Gamma^1$.

Let

$$0 \to T_1 \to T \to T_2 \to 0$$

be an exact sequence of left Γ-modules. It is called a $\mathcal{Y}$-*exact sequence* if for any partition λ with $s(\lambda) = n$ the induced map

$$T([n])^{\Sigma(\lambda)} \to T_2([n])^{\Sigma(\lambda)}$$

is surjective. Here and elsewhere, M^G denotes the subspace of G-fixed elements of a G-module M. For a $\mathcal{Y}$-exact sequence $0 \to T_1 \to T \to T_2 \to 0$ the sequence

$$0 \to T_1([n])^{\Sigma(\lambda)} \to T([n])^{\Sigma(\lambda)} \to T_2([n])^{\Sigma(\lambda)} \to 0$$

is also exact. Following to Section XII.4 of [4] we introduce the related notions. An epimorphism $f : F \to T$ is called $\mathcal{Y}$*-epimorphism* if

$$0 \to \mathsf{Ker}(f) \to F \to T \to 0$$

is a $\mathcal{Y}$-exact sequence. Similarly, a monomorphism $f : F \to T$ is called $\mathcal{Y}$-*monomorphism* if

$$0 \to F \to T \to \mathsf{Coker}(f) \to 0$$

is a $\mathcal{Y}$-exact sequence. A morphism $f : F \to T$ is called $\mathcal{Y}$*-morphism* if $F \to \mathsf{Im}(f)$ is a $\mathcal{Y}$-epimorphism and $\mathsf{Im}(f) \to T$ is a $\mathcal{Y}$-monomorphism. A left Γ-module Z is called $\mathcal{Y}$*-projective* if for any $\mathcal{Y}$-epimorphism $f : F \to T$ and any morphism $g : Z \to T$ there exist a morphism $h : Z \to F$ such that $g = fh$.

Lemma 2.1 i) *If a short exact sequence is isomorphic to a* $\mathcal{Y}$*-exact sequence, then it is also a* $\mathcal{Y}$*-exact sequence.*

ii) *A split short exact sequence is* $\mathcal{Y}$*-exact.*

iii) *A composition of two* $\mathcal{Y}$*-epimorphisms is still a* $\mathcal{Y}$*-epimorphism.*

iv) *If* f *and* g *are two composable epimorphism and* fg *is a* $\mathcal{Y}$*-epimorphism, then* f *is also a* $\mathcal{Y}$*-epimorphism.*

v) *A composition of two* $\mathcal{Y}$*-monomorphisms is still a* $\mathcal{Y}$*-monomorphism.*

vi) *If* f *and* g *are two composable monomorphism and* fg *is a* $\mathcal{Y}$*-monomorphism, then* g *is also a* $\mathcal{Y}$*-monomorphism.*

Proof. The properties i)- iv) are clear. Let $f : B \to C$ and $g : A \to B$ be monomorphisms. One can form the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "1_A"] & B \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "fg"] & C \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & & Y \arrow[r, "1_Y"] \arrow[d] & Y \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Assume f and g are are $\mathcal{Y}$-monomorphisms, then for any partition λ with $s(\lambda) = n$

one has a commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A([n])^{\Sigma(\lambda)} \arrow[r] \arrow[d, "1_A"] & B([n])^{\Sigma(\lambda)} \arrow[r] \arrow[d] & X([n])^{\Sigma(\lambda)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A([n])^{\Sigma(\lambda)} \arrow[r] & C([n])^{\Sigma(\lambda)} \arrow[r, "h"] \arrow[d] & Z([n])^{\Sigma(\lambda)} \arrow[d] & \\
 & & Y([n])^{\Sigma(\lambda)} \arrow[r, "1_Y"] \arrow[d] & Y([n])^{\Sigma(\lambda)} & \\
 & & 0 & &
\end{tikzcd}
$$

The diagram chasing shows that h is an epimorphism and therefore fg is a $\mathcal{Y}$-monomorphism and v) is proved. Assume now fg is a $\mathcal{Y}$-monomorphism. Them we have the following commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A([n])^{\Sigma(\lambda)} \arrow[r] \arrow[d, "1_A"] & B([n])^{\Sigma(\lambda)} \arrow[r, "l"] \arrow[d] & X([n])^{\Sigma(\lambda)} \arrow[d] & \\
0 \arrow[r] & A([n])^{\Sigma(\lambda)} \arrow[r] & C([n])^{\Sigma(\lambda)} \arrow[r] \arrow[d] & Z([n])^{\Sigma(\lambda)} \arrow[r] \arrow[d] & 0 \\
 & & Y([n])^{\Sigma(\lambda)} \arrow[r, "1_Y"] & Y([n])^{\Sigma(\lambda)} &
\end{tikzcd}
$$

The diagram chasing shows that l is an epimorphism and therefore f is a $\mathcal{Y}$-monomorphism and therefore we get vi). $\square$

As an immediate corollary we obtain that the class of all $\mathcal{Y}$-exact sequences is proper in the sense of Mac Lane [4]. We now show that there are enough $\mathcal{Y}$-projective objects.

Lemma 2.2 i) *For any partition λ the left Γ-module $\Gamma(\lambda)$ is a $\mathcal{Y}$-projective object.*

ii) *A morphism $f : F \to T$ of left Γ-modules is $\mathcal{Y}$-epimorphism iff for any partition λ the induced morphism*

$$\mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), F) \to \mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), T)$$

is an epimorphism.

iii) *For any left* Γ*-module* F *there is a* $\mathcal{Y}$*-projective object* Z *and* $\mathcal{Y}$*-epimorphism* $f: Z \to F$.

iv) *Any projective* $\mathcal{Y}$*-module is a direct summand of the sum of objects of the form* $\Gamma(\lambda)$.

v) *The tensor product of two* $\mathcal{Y}$*-projective left* Γ*-modules is still* $\mathcal{Y}$*-projective.*

Proof. Let λ be a partition with $s(\lambda) = n$. By definition $\Gamma(\lambda) = H_0(\Sigma(\lambda), \Gamma^n)$. Hence for any left Γ-module F one has

$$\mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), F) \cong H^0(\Sigma(\lambda), \mathsf{Hom}_{\Gamma-mod}(\Gamma^n, F) \cong F(n)^{\Sigma(\lambda)}.$$

The assertions i) and ii) are immediate consequence of this isomorphisms. To proof iii) we set

$$X(\lambda) := \mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), F).$$

Moreover, for each $x \in X(\lambda)$ we let $f_x : \Gamma(\lambda) \to F$ be the corresponding morphism. Take $Z = \bigoplus_\lambda \bigoplus_{x \in X(\lambda)} \Gamma(\lambda)$. Then the collection f_x, $x \in X(\lambda)$ yields the morphism $f : Z \to F$. We have to show that it is a $\mathcal{Y}$-epimorphism. Let $g : \Gamma(\lambda) \to F$ be a morphism of left Γ-modules. By ii) we need to lift g to Z. By our construction $g \in X(\lambda)$ and therefore the inclusion $\Gamma(\lambda) \to Z$ corresponding to the summand $g \in X(\lambda)$ is an expected lifting and iii) is proved. The proof of iii) shows that one can assume P to be a sum of Γ^λ and iv) follows. To proof the last statement one observes that, for any partitions λ and μ one has

$$\Gamma(\lambda) \otimes \Gamma(\mu) \cong (\Gamma^{s(\lambda)} \otimes \Gamma^{s(\mu)})^{\Sigma(\lambda)\times\Sigma(\mu)} = (\Gamma^{s(\lambda)+s(\mu)})^{\Sigma(\lambda)\times\Sigma(\mu)}$$

and therefore $\Gamma(\lambda) \otimes \Gamma(\mu)$ is $\mathcal{Y}$-projective. □

3 Definition of André-Quillen homology and the functor $\mathcal{L}$

The definition André-Quillen homology is based on the framework of homotopical algebra [8] and it is given as follows. We let $C_*(V_*)$ be the chain complex associated to a simplicial vector space V_*. Let A be a commutative algebra over a ground field K and let M be an A-module. *A simplicial resolution of* A is an augmented simplicial object $P_* \to A$ in the category of commutative algebras, which is a weak equivalence (in other words $C_*(P_*) \to A$ is a weak equivalence). A simplicial resolution is called *free* if P_n is a polynomial algebra over K for all $n \geq 0$. Any commutative algebra posses a free simplicial resolution which is unique up to homotopy. Then the André-Quillen homology is defined by

$$\mathsf{D}_*(A, M) := H_*(C_*(\Omega^1_{P_*} \otimes_{P_*} M)),$$

where Ω^1 is the Kähler 1-differentials and $P_* \to A$ is a free simplicial resolution. In the dimension 0 we have $\mathsf{D}_0(A, M) \cong \Omega^1_A \otimes_A M$.

As we mentioned the functor $\mathcal{L}(A, M) : \Gamma \to \mathsf{Vect}$ is given on objects by $[n] \mapsto M \otimes A^{\otimes n}$. For a pointed map $f : [n] \to [m]$, the action of f on $\mathcal{L}(A, M)$ is given by

$$f_*(a_0 \otimes \cdots \otimes a_n) := b_0 \otimes \cdots \otimes b_m,$$

where $b_j = \prod_{f(i)=j} a_i, \; j = 0, \cdots, n.$

Example 3.1 Let $M = A = K[t]$. In this case one has an isomorphism

$$\mathcal{L}(K[t], K[t]) \cong S^* \circ \Gamma^1.$$

To see this isomorphism, one observes that Γ^1 assigns the free vector space on a set $[n]$ to $[n]$ and therefore both functors in the question assign the ring $K[t_0, \cdots, t_n]$ to $[n]$. An important consequence of this isomorphism is the fact that the functor $\mathcal{L}(K[t], K[t])$ is $\mathcal{Y}$-projective.

Lemma 3.2 *For any commutative algebra A and any A-module M, one has a natural isomorphism $\pi_0(\mathcal{L}(A, M)) \cong \Omega^1_A \otimes_A M$.*

Proof. This is a consequence of Proposition 1.15 and Proposition 2.2 of [5]. $\square$

Lemma 3.3 i) *Let A be a commutative algebras and let*

$$0 \to M_1 \to M \to M_2 \to 0$$

be a short exact sequence of A-modules. Then

$$0 \to \mathcal{L}(A, M_1) \to \mathcal{L}(A, M) \to \mathcal{L}(A, M_2) \to 0$$

is a $\mathcal{Y}$-exact sequence.

ii) *Let $f : B \to A$ be a surjective homomorphism of commutative algebras, then for any A-module M the induced morphism of left Γ-modules*

$$\mathcal{L}(B, M) \to \mathcal{L}(A, M)$$

is a $\mathcal{Y}$-epimorphism.

Proof. One observes that for any partition λ with $s(\lambda) = n$ one has

$$(\mathcal{L}(A, M)([n]))^{\Sigma(\lambda)} = (M \otimes A^{\otimes n})^{\Sigma(\lambda)} \cong M \otimes D^\lambda(A).$$

Since we are over field the tensor product is exact and we obtain i). By the same reason f has a linear section, which yields also a linear section of $D^\lambda(B) \to D^\lambda(A)$, because D^λ is a functor defined on the category of vector spaces . $\square$

4 Relative derived functors

By Lemma 2.2 the class of $\mathcal{Y}$-exact sequences has enough projective objects. Thanks to [4] this allows us to construct the relative derived functors. Let us recall that an augmented chain complex $X_* \to F$ is called a *$\mathcal{Y}$-resolution* of F if it is exact (that is $H_i(X_*) = 0$, for $i > 0$ and $H_0(X_*) \cong F$) and all boundary maps $X_{n+1} \to X_n$ are $\mathcal{Y}$-morphisms, $n \geq 0$. It follows from Lemma 2.2 that $X_* \to F$ is a $\mathcal{Y}$-resolution iff for any partition λ the augmented complex

$$\mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), X_*) \to \mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), F)$$

is exact. A $\mathcal{Y}$-resolution $Z_* \to F$ is called *$\mathcal{Y}$-projective resolution* if for all $n \geq 0$ the left Γ-nodule Z_n is a $\mathcal{Y}$-projective object. We define $\pi_*^{\mathcal{Y}}(F)$ using relative derived functors of the functor $\pi_0 : \Gamma\text{-}mod \to \mathsf{Vect}$. In other words we put

$$\pi_n^{\mathcal{Y}}(F) := H_n(\pi_0(Z_*)), \ n \geq 0,$$

where $Z_* \to F$ is a $\mathcal{Y}$-projective resolution. By [4] this gives the well-defined functors $\pi_n^{\mathcal{Y}} : \Gamma\text{-}mod \to \mathsf{Vect}, \ \ n \geq 0$.

Lemma 4.1 *If K is a field of chracteristic zero, then $\pi_*(F) \cong \pi_*^{\mathcal{Y}}(F)$.*

Proof. In this case all exact sequences are $\mathcal{Y}$-exact, because for any finite group G, the functor $M \mapsto M^G$ is exact. $\square$

Lemma 4.2 *For left Γ-modules F, T one has an isomorphism*

$$\pi_*^{\mathcal{Y}}(F \otimes T) \cong \pi_*^{\mathcal{Y}}(F) \otimes T([0]) \oplus F([0]) \otimes \pi_*^{Y}(T).$$

Proof. The result in the dimension 0 is known (see Lemma 4.2 of [5]). Let $Z_* \to F$ and $R_* \to T$ be $\mathcal{Y}$-projective resolutions. By Lemma 2.2 $Z_* \otimes R_* \to F \otimes T$ is aslo $\mathcal{Y}$-projective resolution. Thus

$$\pi_*^{\mathcal{Y}}(F \otimes T) = H_*(\pi_0(Z_* \otimes R_*)) \cong$$

$$H_*(\pi_0^{\mathcal{Y}}(Z_*) \otimes R_*([0]) \oplus Z_*([0]) \otimes \pi_0^{Y}(R_*)) \cong$$

$$\pi_*^{\mathcal{Y}}(F) \otimes T([0]) \oplus F([0]) \otimes \pi_*^{Y}(T),$$

where the last isomorphism follows from the Eilenberg-Zilber theorem and Künneth theorem. $\square$

Lemma 4.3 *Let $\epsilon : X_* \to A$ be a simplicial resolution in the category of commutative algebras and let M be an A-module. Then the associated chain complex of the simplicial Γ-module $C_*(\mathcal{L}(X_*, M)) \to \mathcal{L}(A, M)$ is a $\mathcal{Y}$ $-$resolution.*

Proof. Since ϵ is weak equivalence of simplicial algebras it is a homotopy equivalence in the category of simplicial vector spaces. Thus $M\otimes D^\lambda(X_*)\to M\otimes D^\lambda(A_*)$ is also a homotopy equivalence, for any particion λ. It follows that

$$\mathcal{L}(X_*,M)([n])^{\Sigma(\lambda)}\to\mathcal{L}(A,M)([n])^{\Sigma(\lambda)}$$

is also a homotopy equivalence of simplicial vector spaces. $\square$

The following is our main result.

Theorem 4.4 *For any commutative ring A and any A-module M, there is a canonical isomorphism*

$$\mathsf{D}_i(A,M)\cong\pi_i^{\mathcal{Y}}(\mathcal{L}(A,M)),\ \ i\geq 0$$

between the André-Quillen homology and relative derived functors of π_0 applied on the functor $\mathcal{L}(A,M)$.

Proof. Thanks to Lemma 3.2 the result is true for $i=0$. First consider the case, when $M=A=K[t]$. In this case André-Quillen homology vanishes in positive dimensions by definition. On the other hand $\mathcal{L}(K[t],K[t])$ is $\mathcal{Y}$-projective thanks to Example 3.1 and therefore $\pi_i^{\mathcal{Y}}(\mathcal{L}(A,M))$ vanishes for all $i>0$. One can use Lemma 4.2 to conclude that $\pi_i^{\mathcal{Y}}(\mathcal{L}(A,A))$ vanishes for all $i>0$ provided A is a polynomial algebra. For the next step, we proof that the result is true if A is a polynomial algebra and M is any A-module. We have to prove that $\pi_i^{\mathcal{Y}}(\mathcal{L}(A,M))$ also vanishes for $i>0$. We already proved this fact if $M=A$. By additivity the functor $\pi_i^{\mathcal{Y}}(\mathcal{L}(A,-))$ vanishes on free A-modules. By Lemma 3.3 the functor $\pi_*^{\mathcal{Y}}(\mathcal{L}(A,-))$ assignes the long exact sequence to a short exact sequence of A-modules. Therefore we can consider such an exact sequence associated to a short exact sequence of A-modules

$$0\to N\to F\to M\to 0$$

with free F. Since the result is true if $i=0$ one obtains by induction on i, that $\pi_i^{\mathcal{Y}}(\mathcal{L}(A,M))=0$ provided $i>0$. Now consider the general case. Let $P_*\to A$ be a free simplicial resolution in the category of commutative algebras. Then we have

$$\Omega^1_{P_*}\otimes_{P_*}M\cong\pi_0^{\mathcal{Y}}(\mathcal{L}(P_*,M))$$

Thanks to Lemma 4.3 $C_*(\mathcal{L}(P_*,M))\to\mathcal{L}(A,M)$ is a $\mathcal{Y}$-resolution consisting with $\pi_*^{\mathcal{Y}}$-acyclic objects and the result follows. $\square$

The main theorem together with the main result of [7] yields:

Corollary 4.5 *If $Char(K)=0$, then for any commutative algebra A and any A-module M one has a natural isomorphism*

$$\mathsf{D}_*(A,M)\cong\mathsf{H}^\Gamma_*(A,M).$$

This fact was also proved in [10] based on the combinatorical and homotopical analysis of the space of fully grown trees.

Remarks. i) We let $t : \Gamma^{op} \to \mathsf{Vect}$ be the functor which assignes the vector space of all maps $f : [n] \to K$, $f(0) = 0$ to $[n]$. Then $t \otimes_\Gamma F \cong \pi_0(F)$ (see Proposition 2.2 [5]). Hence $\pi_*^{\mathcal{Y}}$ van be also defined as the relative derived functors of the functor $t \otimes_\Gamma (-) : \Gamma\text{-}nod \to \mathsf{Vect}$. More generally one can take any functor $T : \Gamma^{op} \to \mathsf{Vect}$ and define $\mathsf{Tor}_*^{\mathcal{Y}}(T, F)$ as the value of the relative derived functors (with respect of $\mathcal{Y}$-exact sequences) of the functor $T \otimes_\Gamma (-) : \Gamma\text{-}nod \to \mathsf{Vect}$. Then our result claims that

$$\mathsf{D}_*(A, M) \cong \mathsf{Tor}_*^{\mathcal{Y}}(t, \mathcal{L}(A, M)).$$

Based on the Proposition 1.15 of [5] the argument given above shows that

$$\mathsf{D}_*^{\{n\}}(A, M) \cong \mathsf{Tor}_*^{\mathcal{Y}}(\Lambda^n \circ t, \mathcal{L}(A, M)),$$

where $\mathsf{D}_*^{\{n\}}(A, M)$ are defined using Kähler n-differentials:

$$\mathsf{D}_*^{\{n\}}(A, M) := H_*(C_*(\Omega^n_{P_*} \otimes_{P_*} M))$$

and for $n = 1$ one recovers the main theorem.

ii) All results remains true if K is any commutative ring and A and M are projective as K-mudule.

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