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PRINCIPAL MINORS OF GAUSSIAN ORTHOGONAL ENSEMBLE

RENJIE FENG, GANG TIAN, DONGYI WEI, AND DONG YAO

Abstract. In this paper, we prove that the fluctuation of the extreme process of the maxima of all the largest eigenvalues of $m \times m$ principal minors (with fixed m) of the classical Gaussian orthogonal ensemble (GOE) of size $n \times n$ is given by the Gumbel distribution as n tends to infinity. We also derive the joint distribution of such maximal eigenvalue and the corresponding eigenvector in the large n limit, which will imply that these two random variables are asymptotically independent.

1. Introduction

Random matrix theory is a classical topic in probability which has applications to a variety of fields, such as statistics [4], high energy physics [7], wireless communication networks [8], deep neutral network [17], compressed sensing [19] and so forth.

Motivated by high-dimensional statistics and signal processing, the authors in [9] derived the growth order of the maxima of all the largest eigenvalues of the principal minors of the classical random matrices of GOE and Wishart matrices, where the results have applications for the construction of the compressed sensing matrices as in [19].

In this paper, we further study the fluctuation of such maxima for the GOE case. Our main result is that the fluctuation is given by the Gumbel distribution with some Poisson structure involved in the limit, and we also derive the limiting joint distribution of such maxima and its corresponding eigenvector which indicates that these two random variables are asymptotically independent.

The Gaussian orthogonal ensemble (GOE) is the Gaussian measure defined on the space of real symmetric matrices, i.e., $G = (g_{ij})_{1\le i,j\le n}$ is a symmetric matrix whose upper triangular entries are independent real Gaussian variables with the following distribution

$$g_{ij} \overset{\mathrm{d}}{=} \begin{cases} N_{\mathbb{R}}(0,2) \text{ if } i=j; \\ N_{\mathbb{R}}(0,1) \text{ if } i<j. \end{cases}$$

Let $\lambda_1(G) > \lambda_2(G) > \cdots > \lambda_n(G)$ be eigenvalues of GOE, then the distribution of these eigenvalues is invariant under the orthogonal group action and the joint density is

$$\frac{1}{Z_n}\prod_{k=1}^{n} e^{-\frac{1}{4}\lambda_k^2}\prod_{i<j}\left(\lambda_i-\lambda_j\right), \tag{1}$$

where

$$Z_n = 2^{n(n+1)/4}(2\pi)^{n/2}(n!)^{-1}\prod_{j=1}^{n}\frac{\Gamma(1+j/2)}{\Gamma(3/2)} \tag{2}$$

is the partition function. And the limit of the empirical measure of these eigenvalues is given by the classical semicircle law [1].

Let's first introduce some notations in order to present our main results. Given symmetric matrices $G = (g_{ij})_{1\le i\le j\le n}$ sampled from GOE, for $\alpha \subset \{1,\cdots,n\}$ with cardinality $|\alpha| = m \in \mathbb{Z}_+$, we denote $G_\alpha = (g_{ij})_{i,j\in\alpha}$ as the principal minors of G of size $m\times m$, then G_α is also symmetric. Let $\lambda_1(G_\alpha) > \lambda_2(G_\alpha) > \cdots > \lambda_m(G_\alpha)$ be the ordered eigenvalues of G_α. Now we define the extreme process of the maxima of all the largest eigenvalues of the principal minors as

$$T_{m,n} = \max_{\alpha \subset \{1,\cdots,n\}, |\alpha|=m} \lambda_1(G_\alpha).$$

In [9], the authors studied the asymptotic properties of $T_{m,n}$ and proved that under the assumption that m fixed or $m \to +\infty$ with $m = o\Big(\frac{(\ln n)^{1/3}}{\ln \ln n}\Big)$, it holds

$$T_{m,n} - 2\sqrt{m \ln n} \to 0$$

in probability as $n \to +\infty$.

In this paper, we further derive the fluctuation of $T_{m,n}$ when m is fixed as n tends to infinity. Our first result is the following

Theorem 1. *For GOE, we have the following convergence in distribution*

$$T_{m,n}^2 - 4m \ln n - 2(m-2) \ln \ln n \xrightarrow{d} Y$$

as $n \to +\infty$ for fixed m, where the random variable Y has the Gumbel distribution function

$$F_Y(y) = \exp(-c_m e^{-y/4}), \quad y \in \mathbb{R}.$$

Here, the constant $c_m = \frac{(2m)^{(m-2)/2} K_m}{(m-1)! 2^{3/2} \Gamma(1+m/2)}$, where $K_m = \mu(\mathcal{S}_m)$ is the probability of the event

$$\mathcal{S}_m := \Big\{ x \in S^{m-1} : \sum_{j \in \beta} x_j^2 \le \sqrt{k/m},\ \forall\, \beta \subset \{1, \cdots, m\} \text{ with } \forall\, 1 \le |\beta| = k < m \Big\} \tag{3}$$

under the uniform distribution μ on the unit sphere S^{m-1}. In particular, $c_1 = \frac{1}{2\sqrt{\pi}}$ and $c_2 = -\frac{1}{2\sqrt{2}} + \frac{\sqrt{2}}{\pi} \arcsin\left(\frac{1}{2}\right)^{1/4}$.

Our proof for Theorem 1 can imply the joint distribution of the maxima of the largest eigenvalues of principal minors and its corresponding eigenvector. To be more precise, given n, let $v^* \in S^{m-1}$ be the unit eigenvector corresponding to the largest eigenvalue of the principal minor that attains the maxima $T_{m,n}$. By symmetry, $-v^*$ is also the corresponding eigenvector. Now we have the following limit for the joint distribution of $(T_{m,n}, v^*)$.

Theorem 2. *Given any $y \in \mathbb{R}$ and symmetric Borel set $Q \subset S^{m-1}$ such that $-Q = Q$, let $y_m^2 = 4m \ln n + 2(m-2) \ln \ln n + y$, then the joint distribution satisfies*

$$\mathbb{P}(T_{m,n} > y_m, v^* \in Q) \to (1 - F_Y(y)) \nu(Q) \tag{4}$$

as $n \to +\infty$, which implies that $T_{m,n}$ and v^* are asymptotically independent. Here, ν is the uniform distribution on $\mathcal{S}_m$, i.e.,

$$\nu(Q) = \mu(Q \cap \mathcal{S}_m)/\mu(\mathcal{S}_m).$$

Let $a_1, ..., a_n$ be i.i.d. Gaussian random variables $N_{\mathbb{R}}(0, 2)$, for the extreme process

$$M_n := \max\{a_1, ..., a_n\},$$

let

$$a_n = 2\sqrt{\ln n}$$

and

$$b_n = 2\sqrt{\ln n} - \frac{\ln \ln n + \ln(4\pi)}{2\sqrt{\ln n}}.$$

Then for any $y \in \mathbb{R}$, the following classical result holds (Theorem 1.5.3 in [14])

$$\lim_{n \to +\infty} \mathbb{P}\left[a_n \left(M_n - b_n\right) \le y\right] = e^{-e^{-y/2}}.$$

One can check that Theorem 1 for $T^2_{1,n}$ when $m = 1$ is equivalent to this classical result for M_n. In this sense, our result is fundamental which can be considered as a natural generalization of such classical result for the extreme process of the scalar-valued random variables to the matrix-valued random variables (with correlations).

One motivation to study the maxima of the largest eigenvalues of principle minors is from the study of compressed sensing, where one has to recover an input vector f from the corrupted measurements $y = Af + e$. Here, A is a coding matrix and e is an arbitrary and unknown vector of errors. The famous result by Candès-Tao [19] is that if the coding matrix A satisfies the restricted isometry property (Definition 1.1 in [19]), then the input f is the unique solution to some ℓ_1-minimization problem provided that the support S (the number of nonzero entries) of errors e is not too large. Therefore, one of the major goals in compressed sensing is to construct the coding matrix A that satisfies the restricted isometry property. In Section 3 of [19], Candès-Tao proved that the Gaussian random matrices A can satisfy such property with overwhelming probability, and they can derive the estimate about the support S via the probabilistic estimate on the maxima of the largest eigenvalues of principle minors of Wishart matrices $A^T A$. A simple proof based on the concentration measure theory is present in [5]. In this article, we only deal with the GOE case, but it seems that the method can be applied to the Wishart case and it's expected that some Gumbel fluctuation will be observed as well, which can imply better estimates on S. We would like to postpone the Wishart case for further investigate.

Another motivation is that the extreme process of the maxima of the largest eigenvalues of principal minors may provide a model that interpolates between the Gumbel distribution in the Poisson regime and the Tracy-Widom law in the random matrix regime.

It's well-known that for GOE, the largest eigenvalue $T_{n,n}$ when $m = n$ in our setting is asymptotic to $2\sqrt{n}$, and its fluctuation is given by the Tracy-Widom law,

$$F_1(y) = \lim_{n\to+\infty} \mathbb{P}\left((T_{n,n} - 2\sqrt{n})n^{1/6} \le y\right),$$

where $F_1(y)$ can be expressed in term of the Painlevé equation [1].

This together with our main result Theorem 1 indicate that there may exist several transitions from the Gumbel distribution to the Tracy-Widom law when m is increasing with n. There are some other models that have such phenomena. In [13], Johansson studied a family of determinantal processes whose edge behavior interpolates between a Poisson process with density e^{-x} and the Airy kernel point process. This process can be obtained as a scaling limit of a grand canonical version of the random MNS-models [16]. Therefore, it provides a model that the largest eigenvalue has a density transition from the Gumbel distribution to the Tracy-Widom law. Another important model is provided by the (Gaussian) random band matrices. It's also conjectured that there is a transition from the Poisson regime to the random matrix regime while the band width has different critical growth orders, the results in [18] almost confirm this conjecture at the spectral edge of some random band matrices.

Our proof of Theorem 1 is based on Lemma 1, which roughly states that if some random variables are weakly correlated, then the point processes constructed via these random variables have a chance to converge to the Poisson processes. In our case when m is fixed or a very slowly varying function of n, the principal minors of size $m \times m$ are weakly correlated with each other as n large enough, therefore, one can expect that the point process of the largest eigenvalues of these principal minors converges to some Poisson point process, and thus some Gumbel distribution for the extreme process of the maxima of these largest eigenvalues will be observed in the limit. But this is not the case if m is a rapid varying function of n such as $m = \sqrt{n}$, and the arguments in this article will not work, especially, the Poisson limit will not hold any more. Here, we would like to propose some natural questions such as the descriptions of the intermediate phases and the critical growth orders of m corresponding to these phases.

It's worth mentioning that there are many other contexts about the (principal) minors of random matrices, and we list few of them as follows. In [10], Diaconis conjectured that the size of minors of the random matrices sampled from the orthogonal group $O(n)$ with the Haar measure such that the minors can be approximated by independent standard normals is of order $o(\sqrt{n})$, which can be considered as a generalization of the classical Poincaré-Borel Lemma. The conjecture is solved in [12] and we refer to [15] for more details and other relevant results. In [6, 20] and the reference therein, the authors studied the principal minor assignment problems of the determinantal point processes with applications in graph theory and machine learning theory. In statistical physics, the matrix minor process constructed via eigenvalues of minors of random matrices will form an interlacing particle system. For example, the minor process of the Gaussian unitary ensemble is a determinantal point process [11]. One can find some other random matrix minor processes in [2].

Notation. In this paper, c, C and C' stand for positive constants, but their values may change from line to line. For simplicity, the notation $a_n \sim b_n$ means $\lim_{n\to+\infty} a_n/b_n = 1$.

2. Proof of Theorem 1

In this section, we will prove Theorem 1 by assuming some technical lemmas where the proofs of these lemmas are postponed to §3.

The proof of Theorem 1 is based on Lemma 1 with the proof given in [3] by the Stein-Chen method. It provides a criteria to prove the convergence of the total number of occurrences of the point process to the Poisson distribution, and thus it provides a method to derive the distribution for some extreme processes.

Lemma 1. *Let I be an index set, and for $\alpha \in I$, let X_α be a Bernoulli random variable with $p_\alpha = \mathbb{P}(X_\alpha = 1) = 1 - \mathbb{P}(X_\alpha = 0)$. For each $\alpha \in I$, let N_α be a subset of I with $\alpha \in N_\alpha$, that is, $\alpha \in N_\alpha \subset I$. Let*

$$S = \sum_{\alpha\in I} X_\alpha,\ \lambda = \mathbb{E}S = \sum_{\alpha\in I} p_\alpha \in (0, +\infty),$$

let Z be the Poisson random variable with intensity $\mathbb{E}Z = \lambda$, then it holds that

$$\|\mathcal{L}(S) - \mathcal{L}(Z)\| \le 2(b_1 + b_2 + b_3),$$

and the probability of no occurrence has the estimate

$$\begin{aligned}&|\mathbb{P}(S = 0) - e^{-\lambda}|\\ =&|\mathbb{P}(X_\alpha = 0,\ \forall\ \alpha \in I) - e^{-\lambda}|\\ \le&\min(1, \lambda^{-1})(b_1 + b_2 + b_3).\end{aligned}$$

Here, $\|\mathcal{L}(S) - \mathcal{L}(Z)\|$ is the total variation distance between the distributions S and Z, and

$$\begin{aligned}b_1 &= \sum_{\alpha\in I}\sum_{\beta\in N_\alpha} p_\alpha p_\beta,\\ b_2 &= \sum_{\alpha\in I}\sum_{\alpha\neq\beta\in N_\alpha} \mathbb{E}[X_\alpha X_\beta],\\ b_3 &= \sum_{\alpha\in I} \mathbb{E}|\mathbb{E}[X_\alpha|\sigma(X_\beta, \beta \notin N_\alpha)] - p_\alpha|,\end{aligned}$$

and $\sigma(X_\beta, \beta \notin N_\alpha)$ is the σ-algebra generated by $\{X_\beta, \beta \notin N_\alpha\}$. In particular, if X_α is independent of $\{X_\beta, \beta \notin N_\alpha\}$ for each α, then $b_3 = 0$.

One may think of N_α as a 'neighborhood of dependence' for α such that X_α is independent or nearly independent of all of X_β for $\beta \notin N_\alpha$. And Lemma 1 indicates that when b_1, b_2 and b_3 are all small enough, then S which is the total number of occurrences tends to a Poisson distribution.

2.1. **Proof of Theorem 1.** Given any $\ell \times \ell$ symmetric matrix S, we rearrange the eigenvalues of S in descending order $\lambda_1(S) \geq \cdots \geq \lambda_\ell(S)$, and we denote

$$|S|^2 := \operatorname{Tr} S^2 \tag{5}$$

and

$$\lambda_1^*(S) := [(\lambda_1(S)^2 + |S|^2)/2]^{\frac{1}{2}}. \tag{6}$$

For fixed $m \geq 2$, we define the index set

$$I_m = \{\alpha \subset \{1, \cdots, n\}, |\alpha| = m\}$$

and the neighborhood set

$$N_\alpha = \{\beta \in I_m : \alpha \cap \beta \neq \emptyset\} \text{ for } \alpha \in I_m.$$

Throughout the article, for any fixed real number y, we define

$$y_k^2 := 4k \ln n + 2k \ln\ln n,\ y_k > 0,\ 1 \leq k < m$$

and

$$y_m^2 := 4m \ln n + 2(m-2)\ln\ln n + y,\ y_m > 0.$$

For symmetric matrices $G = (g_{ij})_{1\leq i\leq j\leq n}$ sampled from GOE, for $\alpha \in I_m$ where $|\alpha| = m$, we denote $G_\alpha = (g_{ij})_{i,j\in\alpha}$ as the principal minor of size $m \times m$ and we define the event

$$A_\alpha = \Big\{\lambda_1(G_\alpha) > y_m;\quad \lambda_1^*(G_\beta) \leq y_k,\ \forall\ 1 \leq k < m,\ \beta \subset \alpha,\ |\beta| = k\Big\}. \tag{7}$$

Recall the definition of $T_{m,n}$ in §1, we first have

$$\begin{aligned} 0 &\leq \mathbb{P}(\cap_{\alpha\in I_m} A_\alpha^c) - \mathbb{P}(T_{m,n} \leq y_m) \\ &\leq \mathbb{P}(\lambda_1^*(G_\beta) > y_k \text{ for some } \beta \in I_k,\ 1 \leq k < m) \\ &\leq \sum_{k=1}^{m-1} \sum_{\beta\in I_k} \mathbb{P}(\lambda_1^*(G_\beta) > y_k) \\ &= \sum_{k=1}^{m-1} \binom{n}{k} \mathbb{P}(\lambda_1^*(G_{\{1,\cdots,k\}}) > y_k). \end{aligned} \tag{8}$$

Now we need the following lemma and we postpone its proof to the next section.

Lemma 2. *For fixed $k \geq 1$, there exists a constant $C > 0$ (depending on k) so that for all $x > 1$,*

$$\mathbb{P}(|G_{\{1,\cdots,k\}}|^2 > x^2) \leq Cx^{k(k+1)/2-2}e^{-x^2/4}, \tag{9}$$

$$\mathbb{P}(\lambda_1(G_{\{1,\cdots,k\}}) > x) \leq Cx^{k-2}e^{-x^2/4}, \tag{10}$$

$$\mathbb{P}(\lambda_1^*(G_{\{1,\cdots,k\}}) > x) \leq Cx^{k-2}e^{-x^2/4}. \tag{11}$$

Using (11) we have

$$\sum_{k=1}^{m-1} \binom{n}{k} \mathbb{P}(\lambda_1^*(G_{\{1,\cdots,k\}}) > y_k) \leq C \sum_{k=1}^{m-1} n^k y_k^{k-2} e^{-y_k^2/4} \leq C/\ln n. \tag{12}$$

Combining this with (8) we get

$$0 \leq \mathbb{P}(\cap_{\alpha\in I_m} A_\alpha^c) - \mathbb{P}(T_{m,n} \leq y_m) \leq C/\ln n, \tag{13}$$

and thus we have

$$\lim_{n\to+\infty}\mathbb{P}(T_{m,n}\le y_m)=\lim_{n\to+\infty}\mathbb{P}(\cap_{\alpha\in I_m}A_\alpha^c). \tag{14}$$

Therefore, it's enough to derive the limit of $\mathbb{P}(\cap_{\alpha\in I_m}A_\alpha^c)$ to prove Theorem 1. By Lemma 1, we have

$$|\mathbb{P}(\cap_{\alpha\in I_m}A_\alpha^c)-e^{-t_n}|\le b_{n,1}+b_{n,2}, \tag{15}$$

where

$$\begin{aligned}
t_n&=\binom{n}{m}\mathbb{P}(A_{\{1,\cdots,m\}}),\\
b_{n,1}&\le\binom{n}{m}\left(\binom{n}{m}-\binom{n-m}{m}\right)\mathbb{P}(A_{\{1,\cdots,m\}})^2,\\
b_{n,2}&\le\sum_{k=1}^{m-1}\binom{n}{k}\binom{n-k}{m-k}\binom{n-m}{m-k}\mathbb{P}(A_{\{1,\cdots,k,\cdots,m\}}\cap A_{\{1,\cdots,k,m+1,\cdots,2m-k\}}).
\end{aligned}$$

Using (10) we have

$$\mathbb{P}(A_{\{1,\cdots,m\}})\le\mathbb{P}(\lambda_1(G_{\{1,\cdots,m\}})>y_m)\le Cy_m^{m-2}e^{-y_m^2/4}\le Cn^{-m}.$$

And thus we have

$$b_{n,1}\le Cn^{2m-1}\mathbb{P}(A_{\{1,\cdots,m\}})^2\le C'n^{-1},$$

which tends to 0 in the limit.

It remains to find the limit of t_n and show that $b_{n,2}$ tends to 0 in order to complete the proof of Theorem 1.

Let $\alpha=\{1,\cdots,m\},\gamma=\{m-k+1,\cdots,m\},\zeta=\{m-k+1,\cdots,2m-k\}$, then $\alpha\cap\zeta=\gamma$, $|\alpha|=|\zeta|=m$ and $|\gamma|=k$. By rearranging the indices, we have

$$\begin{aligned}
&\mathbb{P}(A_{\{1,\cdots,k,\cdots,m\}}\cap A_{\{1,\cdots,k,m+1,\cdots,2m-k\}})\\
=&\mathbb{P}(A_{\{1,\cdots,m\}}\cap A_{\{m-k+1,\cdots,2m-k\}})\\
\le&\mathbb{P}(\lambda_1(G_\alpha)>y_m,\ \lambda_1(G_\zeta)>y_m,\\
&\lambda_1^*(G_{\alpha\setminus\gamma})\le y_{m-k},\ \lambda_1^*(G_{\zeta\setminus\gamma})\le y_{m-k},\ \lambda_1^*(G_\gamma)\le y_k)\\
=&\mathbb{E}[\mathbb{P}(\lambda_1(G_\alpha)>y_m,\ \lambda_1(G_\zeta)>y_m,\ \lambda_1^*(G_{\alpha\setminus\gamma})\le y_{m-k},\ \lambda_1^*(G_{\zeta\setminus\gamma})\le y_{m-k}|G_\gamma)\\
&\times\mathbf{1}_{\{\lambda_1^*(G_\gamma)\le y_k\}}]\\
=&\mathbb{E}[\mathbb{P}(\lambda_1(G_\alpha)>y_m,\ \lambda_1^*(G_{\alpha\setminus\gamma})\le y_{m-k}|G_\gamma)^2\mathbf{1}_{\{\lambda_1^*(G_\gamma)\le y_k\}}].
\end{aligned}$$

The following lemma will imply that $b_{n,2}$ tends to 0 as $n\to+\infty$.

Lemma 3. *For $\alpha\in I_m,\ \gamma\subset\alpha,\ |\gamma|=k,\ 1\le k<m,\ \beta=\alpha\setminus\gamma,\ x>1,\ \delta,\delta'\in(0,1)$, then there are some constants C and C' (depending on m, δ and δ') such that*

$$\begin{aligned}
&\mathbb{P}(\lambda_1(G_\alpha)>x|G_\beta,G_\gamma)\mathbf{1}_{\{\lambda_1^*(G_\beta)\le(1-\delta)x,\lambda_1^*(G_\gamma)\le(1-\delta')x\}}\\
\le&Cx^{-1}(x/(\lambda_1^*(G_\beta)+1)+x/(\lambda_1^*(G_\gamma)+1))^{k(m-k)-1}e^{-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2}
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\mathbb{P}(\lambda_1(G_\alpha)>x,\ \lambda_1^*(G_\beta)\le(1-\delta)x|G_\gamma)\mathbf{1}_{\{\lambda_1^*(G_\gamma)\le(1-\delta')x\}}\\
\le&C'x^{m-k-2}(x/(\lambda_1^*(G_\gamma)+1))^{k(m-k)-1}e^{((\lambda_1^*(G_\gamma))^2-x^2)/4}.
\end{aligned} \tag{17}$$

By assuming Lemma 3, we have

$$
\begin{aligned}
&\mathbb{E}[\mathbb{P}(\lambda_1(G_\alpha) > y_m,\ \lambda_1^*(G_{\alpha\setminus\gamma}) \le y_{m-k}|G_\gamma)^2\mathbf{1}_{\{\lambda_1^*(G_\gamma)\le y_k\}}] \\
\le& C\mathbb{E}[y_m^{2m-2k-4}(y_m/(\lambda_1^*(G_\gamma)+1))^{2k(m-k)-2}e^{((\lambda_1^*(G_\gamma))^2-y_m^2)/2}\mathbf{1}_{\{\lambda_1^*(G_\gamma)\le y_k\}}] \\
=&: C\mathbb{E}[f(\lambda_1^*(G_\gamma))\mathbf{1}_{\{\lambda_1^*(G_\gamma)\le y_k\}}]
\end{aligned}
\tag{18}
$$

where we define

$$
f(t) = y_m^{2m-2k-4}(y_m/(t+1))^{2k(m-k)-2}e^{(t^2-y_m^2)/2},\ \ t\ge 0. \tag{19}
$$

Integration by parts, we have

$$
\begin{aligned}
\mathbb{E}(f(\lambda_1^*(G_\gamma))\mathbf{1}_{\lambda_1^*(G_\gamma)\le y_k}) = &\int_0^{y_k} f'(t)\mathbb{P}(\lambda_1^*(G_\gamma)>t)dt \\
&- f(y_k)\mathbb{P}(\lambda_1^*(G_\gamma)>y_k) + f(0).
\end{aligned}
\tag{20}
$$

Note that

$$
f'(t) = -(2k(m-k)-2)(t+1)^{-1}f(t) + tf(t) \le tf(t). \tag{21}
$$

Hence, using (11) for $t>1$ and (21), we have

$$
\begin{aligned}
&\int_0^{y_k} f'(t)\mathbb{P}(\lambda_1^*(G_\gamma)>t)dt \\
\le& \int_0^1 tf(t)dt + \int_1^{y_k} f'(t)\mathbb{P}(\lambda_1^*(G_\gamma)>t)dt \\
\le& \max_{0\le t\le 1} f(t) + C\int_1^{y_k} ty_m^{2m-2k-4}(y_m/(t+1))^{2k(m-k)-2}e^{(t^2-y_m^2)/2}t^{k-2}e^{-t^2/4}dt.
\end{aligned}
\tag{22}
$$

Therefore, by (20) we further have

$$
\begin{aligned}
&\mathbb{E}(f(\lambda_1^*(G_\gamma))\mathbf{1}_{\lambda_1^*(G_\gamma)\le y_k}) \le 2\max_{0\le t\le 1} f(t) \\
&+ C\int_1^{y_k} ty_m^{2m-2k-4}(y_m/(t+1))^{2k(m-k)-2}e^{(t^2-y_m^2)/2}t^{k-2}e^{-t^2/4}dt.
\end{aligned}
\tag{23}
$$

We now separate this integration into $1<t<y_k/2$ and $y_k/2<t<y_k$. For $1<t<y_k/2$ the integrand is bounded by

$$
Cy_m^{2m-k-6}y_m^{2k(m-k)-1}e^{y_k^2/16-y_m^2/2},
$$

where we used the fact that $y_k\le y_m$ for n large enough. For $y_k/2<t<y_k$, we can bound the integrand by

$$
Cy_m^{2m-k-6}te^{t^2/4-y_m^2/2},
$$

where we used the fact that $y_m/y_k\le 2\sqrt{m/k}$ as n large enough. Therefore, as n large enough, we have

$$
\begin{aligned}
&\int_1^{y_k} f'(t)\mathbb{P}(\lambda_1^*(G_\beta)>t)dt \\
\le& Cy_m^{2m-k-6}e^{-y_m^2/2}\left(\int_1^{y_k/2} y_m^{2k(m-k)-1}e^{y_k^2/16}dt + \int_{y_k/2}^{y_k} te^{t^2/4}dt\right) \\
\le& Cy_m^{2m-k-6}e^{-y_m^2/2}\left(y_m^{2k(m-k)}e^{y_k^2/16} + e^{y_k^2/4}\right) \\
\le& Cy_m^{2m-k-6}e^{y_k^2/4-y_m^2/2},
\end{aligned}
\tag{24}
$$

where in the last inequality we used the fact that $y_m^{2k(m-k)}e^{-3y_k^2/16}$ can be bounded from above uniformly for all n.

The definition of $f(t)$ in (19) and the fact that $y_m/y_k \sim \sqrt{m/k}$ imply

$$\max_{0\le t\le 1} f(t) \le y_m^{2m-2k-4} y_m^{2k(m-k)-2} e^{(1-y_m^2)/2} \le C y_m^{2m-k-6} e^{y_k^2/4-y_m^2/2} \tag{25}$$

as n large enough. It follows from (18), (23), (24) and (25) that

$$\begin{aligned}&\mathbb{E}[\mathbb{P}(\lambda_1(G_\alpha) > y_m,\ \lambda_1^*(G_{\alpha\setminus\gamma}) \le y_{m-k}|G_\gamma)^2 \mathbf{1}_{\{\lambda_1^*(G_\gamma)\le y_k\}}]\\ \le & C y_m^{2m-k-6} e^{y_k^2/4-y_m^2/2}\end{aligned} \tag{26}$$

as n large enough. Therefore, we have

$$\begin{aligned} b_{n,2} &\le \sum_{k=1}^{m-1} \binom{n}{k}\binom{n-k}{m-k}\binom{n-m}{m-k} \mathbb{P}(A_{\{1,\cdots,k,\cdots,m\}} \cap A_{\{1,\cdots,k,m+1,\cdots,2m-k\}})\\ &\le \sum_{k=1}^{m-1} n^{2m-k} \mathbb{P}(A_{\{1,\cdots,m\}} \cap A_{\{1,\cdots,k,m+1,\cdots,2m-k\}})\\ &\le C\sum_{k=1}^{m-1} n^{2m-k} y_m^{2m-k-6} e^{y_k^2/4-y_m^2/2}\\ &\le C\sum_{k=1}^{m-1} y_m^{2m-k-6} (\ln n)^{k/2-m+2} \le C(\ln n)^{-1},\end{aligned} \tag{27}$$

which tends to 0 as $n \to +\infty$.

The following lemma gives the limit of t_n.

Lemma 4. *We have the limit*

$$\lim_{n\to+\infty} t_n = c_m e^{-y/4}, \tag{28}$$

where

$$c_m = \frac{(2m)^{(m-2)/2} K_m}{(m-1)! 2^{3/2} \Gamma(1+m/2)},$$

where K_m is the constant defined in Theorem 1.

By assuming Lemma 4, by (15) together with the facts that $b_{n,1} \to 0$ and $b_{n,2} \to 0$, we can conclude that

$$\lim_{n\to+\infty} \mathbb{P}(\cap_{\alpha\in I} A_\alpha^c) = \exp(-c_m e^{-y/4}).$$

By (14), this further implies

$$\lim_{n\to+\infty} \mathbb{P}(T_{m,n} \le y_m) = \exp(-c_m e^{-y/4}),$$

which proves Theorem 1.

3. Proofs of lemmas

In this section, we will prove Lemma 2, Lemma 3 and Lemma 4, and thus we complete the proof of Theorem 1.

3.1. **Proof of Lemma 2.**

Proof. We simply have the following estimates. For $s \in \mathbb{R}$, there are some constants $C, C' > 0$ depending on s such that for all $x > 1$ we have

$$\int_x^{+\infty} r^s \exp(-r) dr \leq C x^s \exp(-x) \tag{29}$$

and

$$\int_x^{+\infty} r^s \exp(-r^2/2) dr \leq C' x^{s-1} \exp(-x^2/2). \tag{30}$$

Now let $g_1, \dots, g_\ell$ be ℓ independent $N_{\mathbb{R}}(0,1)$ random variables, then for all $t \geq 1$, we have

$$\mathbb{P}\Big(\sum_{i=1}^{\ell} g_i^2 \geq t\Big) \leq C t^{\ell/2-1} \exp(-t/2), \tag{31}$$

where $C > 0$ only depends on ℓ. The proof of (31) follows if we combine the estimate (29) and the fact that the probability density of the chi-squared distribution $\chi^2(\ell) := \sum_{i=1}^{\ell} g_i^2$ with ℓ degrees of freedom is given by

$$\frac{1}{2^{\ell/2}\Gamma(\ell/2)} x^{\ell/2-1} e^{-x/2}.$$

Lemma 2 holds obviously when $k = 1$ (note that $g_{ii} \overset{\mathrm{d}}{=} N_{\mathbb{R}}(0,2)$), and thus in the followings we consider the case when $k \geq 2$. By the definition of the principal minor, $G_{1,\dots,k} = (g_{ij})_{1 \leq i,j \leq k}$ is also sampled from GOE. Now let $\bar{g}_{ij} = g_{ij}$ if $j \neq i$ and $g_{ij}/\sqrt{2}$ otherwise, then $\bar{g}_{ij}, 1 \leq i \leq j \leq k$ are i.i.d. $N_{\mathbb{R}}(0,1)$ random variables. To prove (9), we note that by definition,

$$|G_{1,\dots,k}|^2 = \mathrm{Tr}(G_{1,\dots,k}^2) = \sum_{i,j=1}^{k} g_{ij}^2 = 2 \sum_{1 \leq i \leq j \leq k} \bar{g}_{ij}^2,$$

therefore, by (29) and (31) we have

$$\mathbb{P}(|G_{1,\dots,k}|^2 > x^2) = \mathbb{P}\Big(\sum_{1 \leq i \leq j \leq k} \bar{g}_{ij}^2 > x^2/2\Big) \leq C x^{k(k+1)/2-2} e^{-x^2/4}.$$

This proves (9). To prove (10), by formula (1), the joint density of eigenvalues $\lambda_1 \geq \cdots \geq \lambda_k$ of $G_{\{1,\cdots,k\}}$ is

$$J_k(\lambda_1, \dots, \lambda_k) = \frac{1}{Z_k} \prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j) \exp\left(-\frac{\sum_{i=1}^{k} \lambda_i^2}{4}\right).$$

Note that if $\lambda_1 > x > 1$, we have $\lambda_1 - \lambda_j \leq (\lambda_1 + 1)(|\lambda_j| + 1) \leq 2\lambda_1(|\lambda_j| + 1)$, and thus we have

$$\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j) \leq C \lambda_1^{k-1} \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (\lambda_i - \lambda_j), \tag{32}$$

which further implies

$$\begin{aligned} &\int_{\lambda_1 > x} J_k(\lambda_1, \dots, \lambda_k) d\lambda_1 \cdots d\lambda_k \leq C \int_x^{\infty} \lambda_1^{k-1} \exp(-\lambda_1^2/4) d\lambda_1 \\ &\times \int_{+\infty > \lambda_2 > \cdots > \lambda_k > -\infty} \prod_{2 \leq i \leq k} (|\lambda_i| + 1) \prod_{2 \leq i < j \leq k} (\lambda_i - \lambda_j) \exp\left(-\frac{\sum_{i=2}^{k} \lambda_i^2}{4}\right) d\lambda_2 \cdots d\lambda_k, \end{aligned}$$

which is bounded from above by $C' x^{k-2} \exp(-x^2/4)$ by (30), thereby proving (10).

Now we prove (11). By definition of $\lambda_1^*(G_{1,\dots,k})$, we have

$$
\begin{aligned}
&\mathbb{P}(\lambda_1^*(G_{1,\dots,k}) > x) \\
=&\mathbb{P}\left(\sum_{i=2}^k \lambda_i^2 + 2\lambda_1^2 > 2x^2\right) \\
\le& \sum_{y=0}^{\lfloor\sqrt{2}x\rfloor+1} \mathbb{P}\left(2\lambda_1^2 > (2x^2-(y+1)^2)_+,\ y^2 \le \sum_{i=2}^k \lambda_i^2 \le (y+1)^2\right) \\
&+\mathbb{P}\left(\sum_{i=2}^k \lambda_i^2 > 2x^2\right) \\
:=&I_1 + I_2.
\end{aligned}
$$

By the fact $\lambda_1 - \lambda_j \le (|\lambda_1|+1)(|\lambda_j|+1)$ for all $2 \le j \le k$, we first have

$$
\prod_{1\le i<j\le k} (\lambda_i - \lambda_j) \le C(|\lambda_1|+1)^{k-1} \prod_{2\le i\le k} (|\lambda_i|+1) \prod_{2\le i<j\le k} (\lambda_i-\lambda_j). \tag{33}
$$

Then by the inequality of arithmetic and geometric means, for $\sum_{i=2}^k \lambda_k^2 \ge 1$, we have

$$
\begin{aligned}
&\prod_{2\le i\le k} (|\lambda_i|+1) \prod_{2\le i<j\le k} (\lambda_i-\lambda_j) \le \prod_{2\le i\le k} (|\lambda_i|+1) \prod_{2\le i<j\le k} (|\lambda_i|+|\lambda_j|) \\
\le& \left(\frac{1+\sum_{i=2}^k |\lambda_i|}{k/2}\right)^{k(k-1)/2} \le \left(\frac{1+\sqrt{k-1}\sqrt{\sum_{i=2}^k \lambda_i^2}}{k/2}\right)^{k(k-1)/2} \le C\left(\sum_{i=2}^k \lambda_i^2\right)^{k(k-1)/4}.
\end{aligned} \tag{34}
$$

Therefore, for $x > 1$, combining (33) and (34), we can bound I_2 as follows

$$
\begin{aligned}
&\int_{\sum_{i=2}^k \lambda_i^2 > 2x^2 > 2} J_k(\lambda_1,\dots,\lambda_k) d\lambda_1 \cdots d\lambda_k \le C \int_0^\infty (|\lambda_1|+1)^{k-1} \exp(-\lambda_1^2/4) d\lambda_1 \\
\times& \int_{\sum_{i=2}^k \lambda_i^2 > 2x^2} \left(\sum_{i=2}^k \lambda_i^2\right)^{k(k-1)/4} \exp\left(-\sum_{i=2}^k \lambda_i^2/4\right) d\lambda_2 \cdots d\lambda_k.
\end{aligned}
$$

The first integral is bounded. Using the polar coordinate to the second integral, and by (30) we have the bound

$$
I_2 \le C \int_{r>\sqrt{2}x} r^{k(k-1)/2} \exp(-r^2/4) r^{k-2} dr \le C x^{k(k+1)/2-3} \exp(-x^2/2), \tag{35}
$$

which can be further bounded by $C' x^{k-2} \exp(-x^2/4)$ for $x > 1$ by choosing C' large enough.

Now we estimate I_1 for $x>1$. For the case $x^2-(y+1)^2/2>1$, by (30) and (32), we have

$$\begin{aligned}
&\mathbb{P}\left(2\lambda_1^2>(2x^2-(y+1)^2)_+,\ y^2\le\sum_{i=2}^k\lambda_i^2\le(y+1)^2\right)\\
\le&C\int_{\lambda_1^2>x^2-(y+1)^2/2>1}|\lambda_1|^{k-1}\exp(-\lambda_1^2/4)d\lambda_1\\
&\times\int_{\lambda_2>\cdots>\lambda_k:(y+1)^2\ge\sum_{i=2}^k\lambda_i^2\ge y^2}\prod_{2\le i\le k}(|\lambda_i|+1)\prod_{2\le i<j\le k}(\lambda_i-\lambda_j)\exp\big(-\sum_{i=2}^k\lambda_i^2/4\big)d\lambda_2\cdots d\lambda_k\\
\le&Cx^{k-2}\exp(-(x^2-(y+1)^2/2)/4)\\
&\times\int_{\mathbb{R}^{k-1}:(y+1)^2\ge\sum_{i=2}^k\lambda_i^2\ge y^2}\prod_{2\le i\le k}(|\lambda_i|+1)\prod_{2\le i<j\le k}(|\lambda_i|+|\lambda_j|)\exp(-\sum_{i=2}^k\lambda_i^2/4)d\lambda_2\cdots d\lambda_k.
\end{aligned}$$

We denote the last integral as

$$A_k(y):=\int_{\mathbb{R}^{k-1}:(y+1)^2\ge\sum_{i=2}^k\lambda_i^2\ge y^2}\prod_{2\le i\le k}(|\lambda_i|+1)\prod_{2\le i<j\le k}(|\lambda_i|+|\lambda_j|)\exp(-\sum_{i=2}^k\lambda_i^2/4)d\lambda_2\cdots d\lambda_k.$$

For the case $x^2-(y+1)^2/2\le1$, by (33) and the arguments as above, we simply have

$$\begin{aligned}
&\mathbb{P}\left(2\lambda_1^2>(2x^2-(y+1)^2)_+,\ y^2\le\sum_{i=2}^k\lambda_i^2\le(y+1)^2\right)\\
\le&C\Big(\int_{\mathbb{R}}(|\lambda_1|+1)^{k-1}\exp(-\lambda_1^2/4)d\lambda_1\Big)A_k(y)\\
\le&CA_k(y)\\
\le&Cx^{k-2}\exp(-(x^2-(y+1)^2/2)/4)A_k(y)
\end{aligned}$$

for $x>1$. Therefore, in both cases, by the polar coordinate, we further have the upper bound,

$$\begin{aligned}
&\mathbb{P}\left(2\lambda_1^2>(2x^2-(y+1)^2)_+,\ y^2\le\sum_{i=2}^k\lambda_i^2\le(y+1)^2\right)\\
\le&Cx^{k-2}e^{-x^2/4}\int_{y+1>r>y}(1+r)^{k-1}r^{(k-1)(k-2)/2}r^{k-2}\exp((r+1)^2/8-r^2/4)dr.
\end{aligned}$$

By taking the summation, I_1 can be bounded from above by

$$\begin{aligned}
I_1\le&Cx^{k-2}\exp(-x^2/4)\int_0^{\lfloor\sqrt{2}x\rfloor+1}(1+r)^{k-1}r^{(k-1)(k-2)/2}r^{k-2}\exp((r+1)^2/8-r^2/4)dr\\
\le&Cx^{k-2}\exp(-x^2/4)\int_0^{+\infty}\cdots=C'x^{k-2}\exp(-x^2/4).
\end{aligned}$$

This will complete the proof of (11) by the estimates of I_1 and I_2. □

3.2. **Proof of Lemma 3.**

Proof. We first prove (16). We claim that (16) is equivalent to the following statement: for any $\delta>0$, $x>1$, there exists a constant C depending on δ, such that

$$\begin{aligned}
&\mathbb{P}(\lambda_1(G_\alpha)>x|G_\beta,G_\gamma)\mathbf{1}_{\{\lambda_1^*(G_\beta)\le(1-\delta)x,\lambda_1^*(G_\gamma)\le(1-\delta)x\}}\\
\le&Cx^{-1}(x/(\lambda_1^*(G_\beta)+1)+x/(\lambda_1^*(G_\gamma)+1))^{k(m-k)-1}e^{-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2}.
\end{aligned}\tag{36}$$

The implication that (16) $\Rightarrow$ (36) is trivial. We now show that (36) implies (16). For any $\delta, \delta' \in (0,1)$, we define $\bar{\delta} = \min\{\delta, \delta'\}$. (36) implies that there exists a constant $C(\bar{\delta})$ such that

$$\begin{aligned} &\mathbb{P}(\lambda_1(G_\alpha) > x | G_\beta, G_\gamma)\mathbf{1}_{\{\lambda_1^*(G_\beta)\le(1-\bar{\delta})x, \lambda_1^*(G_\gamma)\le(1-\bar{\delta})x\}} \\ \le & Cx^{-1}(x/(\lambda_1^*(G_\beta)+1) + x/(\lambda_1^*(G_\gamma)+1))^{k(m-k)-1}e^{-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2}. \end{aligned} \tag{37}$$

(37) implies (16) since

$$\left\{\lambda_1^*(G_\beta) \le (1-\delta)x, \lambda_1^*(G_\gamma) \le (1-\delta')x\right\} \subset \left\{\lambda_1^*(G_\beta) \le (1-\bar{\delta})x, \lambda_1^*(G_\gamma) \le (1-\bar{\delta})x\right\}.$$

This completes the proof of the equivalence between (16) and (36). We now prove (36). Without loss of generality, we may assume

$$\alpha = \{1, \ldots, m\}, \gamma = \{1, \ldots, k\}, \beta = \{k+1, \ldots, m\}.$$

Since G_β and G_γ are both symmetric matrices sampled from GOE (independently), we can find orthogonal matrices U and U' such that

$$UG_\beta U^t = X, \quad U'G_\gamma {U'}^t = Z,$$

where the diagonal matrices

$$X = \begin{pmatrix} \lambda_1(G_\beta) & & & \\ & \lambda_2(G_\beta) & & \\ & & \ddots & \\ & & & \lambda_\ell(G_\beta) \end{pmatrix}, Z = \begin{pmatrix} \lambda_1(G_\gamma) & & & \\ & \lambda_2(G_\gamma) & & \\ & & \ddots & \\ & & & \lambda_k(G_\gamma) \end{pmatrix},$$

where $\ell := m - k$.

It follows that

$$\begin{pmatrix} U & \\ & U' \end{pmatrix} G_\alpha \begin{pmatrix} U^t & \\ & {U'}^t \end{pmatrix} = \begin{pmatrix} X & V \\ V^t & Z \end{pmatrix},$$

where V is an $\ell \times k$ matrix with i.i.d. $N_{\mathbb{R}}(0,1)$ entries.

Given any $1 \times m$ vector $\mathbf{v}$, we decompose it as

$$\mathbf{v} = (\mathbf{p}, \mathbf{q}), \ \ \mathbf{p} = (p_1, \ldots, p_\ell), \ \ \mathbf{q} = (q_1, \ldots, q_k),$$

then we have

$$\mathbf{v}\begin{pmatrix} X & V \\ V^t & Z \end{pmatrix}\mathbf{v}^t = \sum_{i=1}^{\ell}\lambda_i(G_\beta)p_i^2 + \sum_{j=1}^{k}\lambda_j(G_\gamma)q_j^2 + 2\sum_{i=1}^{\ell}\sum_{j=1}^{k}p_i v_{ij} q_j.$$

For simplicity, we use $\mathbb{P}^*$ to denote the conditional probability (conditional on G_β and G_γ). By Rayleigh quotient, for $x > 1$, we have

$$\begin{aligned} &\mathbb{P}^*(\lambda_1(G_\alpha) > x) \\ =&\mathbb{P}^*\left(\exists(\mathbf{p},\mathbf{q}) \ne \mathbf{0}, \sum_{i=1}^{\ell}\lambda_i(G_\beta)p_i^2 + \sum_{j=1}^{k}\lambda_j(G_\gamma)q_j^2 + 2\sum_{i=1}^{\ell}\sum_{j=1}^{k}p_i v_{ij} q_j \ge x\sum_{i=1}^{\ell}p_i^2 + x\sum_{j=1}^{k}q_j^2\right) \\ =&\mathbb{P}^*\left(\exists(\mathbf{p},\mathbf{q}) \ne \mathbf{0}, 2\sum_{i=1}^{\ell}\sum_{j=1}^{k}p_i v_{ij} q_j \ge \sum_{i=1}^{\ell}(x-\lambda_i(G_\beta))p_i^2 + \sum_{j=1}^{k}(x-\lambda_j(G_\gamma)q_j^2\right). \end{aligned}$$

We define the event

$$\Omega := \left\{\lambda_1^*(G_\beta) \le (1-\delta)x, \lambda_1^*(G_\gamma) \le (1-\delta)x\right\}.$$

For the rest of the proof, all the arguments are restricted on the event Ω and $x > 1$.

On Ω, by the definition $\lambda^*(A)$ for any symmetric matrix A in (6), we simply have $x+\sqrt{2}(1-\delta)x \geq x-\lambda_i(G_\beta) \geq x-\lambda_1^*(G_\beta) \geq \delta x > 0$ and $x+\sqrt{2}(1-\delta)x \geq x-\lambda_j(G_\gamma) \geq x-\lambda_1^*(G_\gamma) \geq \delta x > 0$.

Therefore, on Ω it holds that

$$z_{ij} := (x-\lambda_i(G_\beta))(x-\lambda_j(G_\gamma)) \in [\delta^2 x^2, (\sqrt{2}-\sqrt{2}\delta+1)^2 x^2]. \tag{38}$$

We denote

$$\bar{p}_i = \sqrt{x-\lambda_i(G_\beta)}p_i, \quad \bar{q}_j = \sqrt{x-\lambda_j(G_\gamma)}q_j, \quad \bar{v}_{ij} = \frac{v_{ij}}{\sqrt{z_{ij}}}.$$

Using Cauchy-Schwartz inequality we have

$$\begin{aligned} 2\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{p}_i\bar{v}_{ij}\bar{q}_j &\leq 2\sqrt{\left(\sum_{i=1}^{\ell}\bar{p}_i^2\right)\left(\sum_{i=1}^{\ell}\left(\sum_{j=1}^{k}\bar{v}_{ij}\bar{q}_j\right)^2\right)} \\ &\leq 2\sqrt{\left(\sum_{i=1}^{\ell}\bar{p}_i^2\right)\left(\sum_{j=1}^{k}\bar{q}_j^2\right)\left(\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{v}_{ij}^2\right)} \\ &\leq \sqrt{\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{v}_{ij}^2}\left(\sum_{i=1}^{\ell}\bar{p}_i^2+\sum_{j=1}^{k}\bar{q}_j^2\right). \end{aligned}$$

Therefore, we further have the probabilistic estimate of $\lambda_1(G_\alpha)$ as

$$\begin{aligned} &\mathbb{P}^*(\lambda_1(G_\alpha) > x) \\ =&\mathbb{P}^*\left(\exists(\bar{\mathbf{p}},\bar{\mathbf{q}}) \neq \mathbf{0}, 2\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{p}_i\bar{v}_{ij}\bar{q}_j \geq \sum_{i=1}^{\ell}\bar{p}_i^2+\sum_{j=1}^{k}\bar{q}_j^2\right) \\ \leq&\mathbb{P}^*\left(\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{v}_{ij}^2 \geq 1\right). \end{aligned}$$

If $k=\ell=1$, on the event Ω, by (30) for Gaussian random variable v_{11} we have

$$\begin{aligned} \mathbb{P}^*(\bar{v}_{11}^2 \geq 1) = \mathbb{P}^*(v_{11}^2 > z_{11}) &\leq \frac{C}{\sqrt{z_{11}}}\exp(-z_{11}/2) \\ &\leq \frac{C}{x}\exp(-(x-\lambda_1(G_\beta))(x-\lambda_1(G_\gamma))/2) \\ &\leq \frac{C}{x}\exp(-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2), \end{aligned} \tag{39}$$

where we have used the fact that $z_{11} \geq \delta^2 x^2$. This will imply (36) in the case $m=2$, $k=\ell=1$. Now we consider the case $k \geq 2$ or $\ell \geq 2$. We define

$$\kappa = \begin{cases} \min\left\{\frac{\lambda_1(G_\beta)-\lambda_2(G_\beta)}{x-\lambda_2(G_\beta)}, \frac{\lambda_1(G_\gamma)-\lambda_2(G_\gamma)}{x-\lambda_2(G_\gamma)}\right\} & \text{if } \ell,k \geq 2; \\ \frac{\lambda_1(G_\beta)-\lambda_2(G_\beta)}{x-\lambda_2(G_\beta)} & \text{if } k=1,\ell \geq 2; \\ \frac{\lambda_1(G_\gamma)-\lambda_2(G_\gamma)}{x-\lambda_2(G_\gamma)} & \text{if } \ell=1,k \geq 2. \end{cases}$$

On the event Ω, one can show that

$$0 \leq \kappa \leq \frac{(1-\delta)x-\lambda_2(G_\beta)}{x-\lambda_2(G_\beta)} \leq \frac{(1-\delta)x-(-\sqrt{2}(1-\delta)x)}{x-(-\sqrt{2}(1-\delta)x)} = \frac{(1+\sqrt{2})(1-\delta)}{1+\sqrt{2}(1-\delta)} := c_\delta < 1. \tag{40}$$

Note that

$$
\begin{aligned}
\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{v}_{ij}^2 =&\sum_{i=1}^{\ell}\sum_{j=1}^{k}\frac{v_{ij}^2}{(x-\lambda_i(G_\beta))(x-\lambda_j(G_\gamma))}\\
=&\frac{1}{(x-\lambda_1(G_\beta))(x-\lambda_1(G_\gamma))}\left(v_{11}^2+\sum_{(i,j)\neq(1,1)}\frac{(x-\lambda_1(G_\beta))(x-\lambda_1(G_\gamma))}{(x-\lambda_i(G_\beta))(x-\lambda_j(G_\gamma))}v_{ij}^2\right)\\
\leq&\frac{1}{(x-\lambda_1(G_\beta))(x-\lambda_1(G_\gamma))}\left(v_{11}^2+(1-\kappa)\sum_{(i,j)\neq(1,1)}v_{ij}^2\right),
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
\frac{(x-\lambda_1(G_\beta))(x-\lambda_1(G_\gamma))}{(x-\lambda_i(G_\beta))(x-\lambda_j(G_\gamma))}\leq&\max\left\{\frac{x-\lambda_1(G_\beta)}{x-\lambda_2(G_\beta)},\frac{x-\lambda_1(G_\gamma)}{x-\lambda_2(G_\gamma)}\right\}\\
=&1-\min\left\{\frac{\lambda_1(G_\beta)-\lambda_2(G_\beta)}{x-\lambda_2(G_\beta)},\frac{\lambda_1(G_\gamma)-\lambda_2(G_\gamma)}{x-\lambda_2(G_\gamma)}\right\}
\end{aligned}
$$

for $(i,j)\neq(1,1)$. Hence, by (31) we can conclude the following bounds on the event Ω

$$
\begin{aligned}
&\mathbb{P}^*(\lambda_1(G_\alpha)>x)\\
\leq&\mathbb{P}^*\left(\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{v}_{ij}^2\geq 1\right)\\
\leq&\mathbb{P}^*\left(v_{11}^2+(1-\kappa)\sum_{(i,j)\neq(1,1)}v_{ij}^2\geq z_{11}\right)\\
\leq&\mathbb{P}^*\left(\sum_{i=1}^{\ell}\sum_{j=1}^{k}v_{ij}^2\geq z_{11}\right)\\
\leq&Cz_{11}^{k\ell/2-1}\exp(-z_{11}/2)\\
\leq&Cx^{k\ell-2}\exp(-(x-\lambda_1(G_\beta))(x-\lambda_1(G_\gamma))/2),
\end{aligned}\tag{41}
$$

where in the last step we used (38), and C is a constant depending on δ and m.

We now consider the case when one of the following two conditions holds.

Condition 1: $\lambda_1^*(G_\beta)\leq 1$ or $\lambda_1^*(G_\gamma)\leq 1$. Under this condition, since $\lambda_1(G_\beta)\leq\lambda_1^*(G_\beta)$ and $\lambda_1(G_\gamma)\leq\lambda_1^*(G_\gamma)$, we have

$$
\mathbb{P}^*\left(\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{v}_{i,j}^2\geq 1\right)\leq Cx^{k\ell-2}\exp(-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2),
$$

which further yields the bound (36) by the assumption $\lambda_1^*(G_\beta)\leq 1$ or $\lambda_1^*(G_\gamma)\leq 1$.

Condition 2: $\max\{|\lambda_i(G_\beta)|,2\leq i\leq\ell\}>|\lambda_1(G_\beta)|/2$ or $\max\{|\lambda_i(G_\gamma)|,2\leq i\leq k\}>|\lambda_1(G_\gamma)|/2$. If condition 1 fails (i.e., $\lambda_1^*(G_\beta)>1$ and $\lambda_1^*(G_\gamma)>1$) and condition 2 holds (say, $\max\{|\lambda_i(G_\beta)|,2\leq i\leq\ell\}>|\lambda_1(G_\beta)|/2$), then we have

$$
\lambda_1^*(G_\beta)-\lambda_1(G_\beta)=\frac{\sum_{i=2}^{\ell}\lambda_i^2(G_\beta)}{2(\lambda_1^*(G_\beta)+\lambda_1(G_\beta))}\geq\frac{C(\lambda_1^*(G_\beta))^2}{4\lambda_1^*(G_\beta)}=C'\lambda_1^*(G_\beta).
$$

In this case, on Ω we have

$$(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma)) \le (x-\lambda_1(G_\beta))(x-\lambda_1^*(G_\gamma)) - (\lambda_1^*(G_\beta)-\lambda_1(G_\beta))\delta x$$
$$\le (x-\lambda_1(G_\beta))(x-\lambda_1(G_\gamma)) - C'\lambda_1^*(G_\beta)\delta x \tag{42}$$
$$\le (x-\lambda_1(G_\beta))(x-\lambda_1(G_\gamma)) - C'\delta x.$$

Using (41), we have

$$\mathbb{P}^*(\lambda_1(G_\alpha) > x) \le Cx^{k\ell-2}\exp(-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2)\exp(-C'\delta x).$$

This can be further bounded from above by

$$Cx^{-1}\Big(\frac{x}{1+\lambda_1^*(G_\beta)}+\frac{x}{1+\lambda_1^*(G_\gamma)}\Big)^{k\ell-1}\exp(-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2),$$

this is because on Ω it holds

$$\exp(-C'\delta x) \le C\Big(\frac{1}{1+(1-\delta)x}\Big)^{k\ell-1} \le C\Big(\frac{1}{1+\lambda_1^*(G_\beta)}+\frac{1}{1+\lambda_1^*(G_\gamma)}\Big)^{k\ell-1}$$

for $x>1$ by choosing C large enough. Note that the constant $C>0$ only depends on m and δ, and does not depend on (the conditional) $\lambda_1^*(G_\beta)$ and $\lambda_1^*(G_\gamma)$.

As a summary, we have verified (36) when either of the two conditions is satisfied. Now we assume that both conditions fail so that $\lambda_1^*(G_\beta)>1$, $\lambda_1^*(G_\gamma)>1$, $\max\{|\lambda_i(G_\beta)|, 2\le i\le \ell\} \le |\lambda_1(G_\beta)|/2$ and $\max\{|\lambda_i(G_\gamma)|, 2\le i\le k\} \le |\lambda_1(G_\gamma)|/2$. Then there exists a constant $c>0$ such that

$$\lambda_1(G_\beta)-\lambda_2(G_\beta) \ge c\lambda_1^*(G_\beta) \ge c \text{ and } \lambda_1(G_\gamma)-\lambda_2(G_\gamma) \ge c\lambda_1^*(G_\gamma) \ge c.$$

Therefore, by definition of κ, (40) and the assumptions $\lambda_1^*(G_\beta)>1$, $\lambda_1^*(G_\gamma)>1$, we have

$$1 > c_\delta \ge \kappa \ge c\min\Big\{\frac{\lambda_1^*(G_\beta)}{x-\lambda_2(G_\beta)}, \frac{\lambda_1^*(G_\gamma)}{x-\lambda_2(G_\gamma)}\Big\} \ge \frac{c}{2}\min\Big\{\frac{\lambda_1^*(G_\beta)+1}{x-\lambda_2(G_\beta)}, \frac{\lambda_1^*(G_\gamma)+1}{x-\lambda_2(G_\gamma)}\Big\}. \tag{43}$$

By the fact that $0<\delta x \le x-\lambda_2(G_\beta), x-\lambda_2(G_\gamma) \le x+\sqrt{2}(1-\delta)x$ on Ω, we further have

$$c_\delta \ge \kappa \ge C_\delta \min\Big\{\frac{\lambda_1^*(G_\beta)+1}{x}, \frac{\lambda_1^*(G_\gamma)+1}{x}\Big\}, \tag{44}$$

which implies

$$\frac{1}{\kappa} \le Cx\left(\frac{1}{\lambda_1^*(G_\beta)+1}+\frac{1}{\lambda_1^*(G_\gamma)+1}\right) \tag{45}$$

for some constant $C>0$ that depends on m and δ. Note that the probability density function $p(t)$ for v_{11}^2 is $\exp(-t/2)/\sqrt{2\pi t}$, recall (41), we have

$$\begin{aligned}
&\mathbb{P}^*\left(\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar v_{ij}^2 \ge 1\right)\\
\le&\mathbb{P}^*\left(v_{11}^2+(1-\kappa)\sum_{(i,j)\ne(1,1)}v_{ij}^2 > z_{11}\right)\\
=&\int_0^{z_{11}}\frac{\exp(-(z_{11}-t)/2)}{\sqrt{2\pi(z_{11}-t)}}\mathbb{P}^*\left((1-\kappa)\sum_{(i,j)\ne(1,1)}v_{ij}^2>t\right)dt+\mathbb{P}^*(v_{11}^2>z_{11})\\
\le&C\exp(-z_{11}/2)\left(\frac{1}{\sqrt{z_{11}}}+\int_0^{z_{11}}\frac{\exp(t/2)}{\sqrt{z_{11}-t}}\mathbb{P}^*\left((1-\kappa)\sum_{(i,j)\ne(1,1)}v_{ij}^2>t\right)dt\right),
\end{aligned}$$

where we have used $\mathbb{P}^*(v_{11}^2 > z_{11}) \le \frac{Ce^{-z_{11}/2}}{\sqrt{z_{11}}}$ by (31). By (31) again, we have

$$\begin{aligned}\mathbb{P}^*\left((1-\kappa)\sum_{(i,j)\neq(1,1)} v_{i,j}^2 > t\right) &\le C\left(\frac{t}{1-\kappa}\right)^{(k\ell-1)/2-1}\exp\left(-\frac{t}{2(1-\kappa)}\right)\\ &\le Ct^{(k\ell-1)/2-1}\exp\left(-\frac{t}{2(1-\kappa)}\right),\end{aligned}$$

where in the last inequality we used the fact that $1-\kappa \ge 1-c_\delta > 0$ by (40). It follows that

$$\begin{aligned}&\mathbb{P}^*\left(\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{v}_{ij}^2 \ge 1\right)\\ \le& C\exp(-z_{11}/2)\left(\frac{1}{\sqrt{z_{11}}} + \int_0^{z_{11}}\frac{t^{(k\ell-3)/2}}{\sqrt{z_{11}-t}}\exp\left(\frac{t}{2}-\frac{t}{2(1-\kappa)}\right)dt\right)\\ \le& C'\exp(-z_{11}/2)\left(\frac{1}{\sqrt{z_{11}}} + \frac{1}{\sqrt{z_{11}}}\int_0^{z_{11}/2} t^{(k\ell-3)/2}\exp\left(-\frac{\kappa t}{2(1-\kappa)}\right)dt\right.\\ &\left. + z_{11}^{(k\ell-3)/2}\exp\left(-\frac{\kappa z_{11}}{4(1-\kappa)}\right)\int_{z_{11}/2}^{z_{11}}\frac{1}{\sqrt{z_{11}-t}}dt\right)\\ \le& C'\frac{\exp(-z_{11}/2)}{\sqrt{z_{11}}}\Big(1+\left(\frac{1}{\kappa}\right)^{(k\ell-1)/2}\int_0^\infty s^{(k\ell-3)/2}\exp(-s)ds\\ &+ z_{11}^{(k\ell-1)/2}\exp\Big(-\frac{\kappa z_{11}}{4}\Big)\Big).\end{aligned}$$

Note that the integration of $s^{(k\ell-3)/2}\exp(-s)$ converges since $k\ell \ge 2$, and thus the second term can be bounded from above by $c(1/\kappa)^{(k\ell-1)/2}$. For the third term, the global maxima of the function $x^{(k\ell-1)/2}e^{-\kappa x/4}$ is obtained at the point $x = c'/\kappa$, here c' depends on k, ℓ, and thus the third term can be bounded from above by $C(1/\kappa)^{(k\ell-1)/2}$, where C depends on δ and m. Therefore, on Ω we have

$$\begin{aligned}&\mathbb{P}^*\left(\sum_{i=1}^{\ell}\sum_{j=1}^{k}\bar{v}_{ij}^2 \ge 1\right)\\ \le& C'\frac{\exp(-z_{11}/2)}{\sqrt{z_{11}}}\Big(1 + c(1/\kappa)^{(k\ell-1)/2} + C(1/\kappa)^{(k\ell-1)/2}\Big)\\ \le& C'\frac{\exp(-z_{11}/2)}{\sqrt{z_{11}}}\Big((1/\kappa)^{(k\ell-1)/2} + c(1/\kappa)^{(k\ell-1)/2} + C(1/\kappa)^{(k\ell-1)/2}\Big) \text{ [since } 1 < 1/\kappa]\\ =& C\frac{\exp(-z_{11}/2)}{\sqrt{z_{11}}}(1/\kappa)^{(k\ell-1)/2}\\ \le& C\frac{\exp(-z_{11}/2)}{\sqrt{z_{11}}}(1/\kappa)^{k\ell-1}\\ \le& Cx^{-1}\left(\frac{x}{\lambda_1^*(G_\beta)+1} + \frac{x}{\lambda_1^*(G_\gamma)+1}\right)^{k\ell-1}\exp(-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2)) \text{ [by (38), (45)]}.\end{aligned}$$

This proves (36), and thus we complete the proof of (16).

We now prove (17) using (11) and (16). By convexity of the function $s \to s^{k(m-k)-1}$ for $s > 0$, we have

$$
\begin{aligned}
&x^{-1}(x/(\lambda_1^*(G_\beta)+1) + x/(\lambda_1^*(G_\gamma)+1))^{k(m-k)-1} e^{-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2} \\
\leq & Cx^{-1}(x/(\lambda_1^*(G_\beta)+1))^{k(m-k)-1} e^{-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2} \\
&+ Cx^{-1}(x/(\lambda_1^*(G_\gamma)+1))^{k(m-k)-1} e^{-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2} \\
:= & I_3 + I_4.
\end{aligned} \tag{46}
$$

Let $\bar{\mathbb{E}}$ be the conditional expectation with respect to G_γ and $\ell := m-k$ as before, and we define

$$f(t) = x^{-1}(x/(t+1))^{k\ell-1} \exp(-(x-t)(x-\lambda_1^*(G_\gamma))/2)$$

and

$$h(t) = x^{-1}(x/(\lambda_1^*(G_\gamma)+1))^{k\ell-1} \exp(-(x-t)(x-\lambda_1^*(G_\gamma))/2).$$

We define the event

$$\Omega' = \{\lambda_1^*(G_\gamma) \leq (1-\delta')x\}.$$

Using (16) and (46), we have

$$
\begin{aligned}
&\mathbb{P}(\lambda_1(G_\alpha) > x,\ \lambda_1^*(G_\beta) \leq (1-\delta)x | G_\gamma) \mathbf{1}_{\{\lambda_1^*(G_\gamma) \leq (1-\delta')x\}} \\
=&\mathbb{E}[\mathbb{E}\left(\mathbf{1}_{\{\lambda_1(G_\alpha)>x,\ \lambda_1^*(G_\beta)<(1-\delta)x\}} | G_\beta, G_\gamma\right) | G_\gamma] \mathbf{1}_{\{\lambda_1^*(G_\gamma) \leq (1-\delta')x\}} \\
=&\mathbb{E}[\mathbb{E}\left(\mathbf{1}_{\{\lambda_1(G_\alpha)>x\}} | G_\beta, G_\gamma\right) \mathbf{1}_{\{\lambda_1^*(G_\beta)<(1-\delta)x\}} \mathbf{1}_{\{\lambda_1^*(G_\gamma) \leq (1-\delta')x\}} | G_\gamma] \mathbf{1}_{\{\lambda_1^*(G_\gamma) \leq (1-\delta')x\}} \\
\leq & C\bar{\mathbb{E}}\Big(x^{-1}(x/(\lambda_1^*(G_\beta)+1) + x/(\lambda_1^*(G_\gamma)+1))^{k\ell-1} \\
&\times \exp\left(-(x-\lambda_1^*(G_\beta))(x-\lambda_1^*(G_\gamma))/2\right) \mathbf{1}_{\{\lambda_1^*(G_\beta) \leq (1-\delta)x\}}\Big) \mathbf{1}_{\Omega'} \\
\leq & C\bar{\mathbb{E}}(I_3 \mathbf{1}_{\{\lambda_1^*(G_\beta) \leq (1-\delta)x\}}) \mathbf{1}_{\Omega'} + C\bar{\mathbb{E}}(I_4 \mathbf{1}_{\{\lambda_1^*(G_\beta) \leq (1-\delta)x\}}) \mathbf{1}_{\Omega'} \\
\leq & C\Big(\bar{\mathbb{E}}(f(\lambda_1^*(G_\beta)) \mathbf{1}_{\{\lambda_1^*(G_\beta)<(1-\delta)x\}}) \mathbf{1}_{\Omega'} + \bar{\mathbb{E}}(h(\lambda_1^*(G_\beta)) \mathbf{1}_{\{\lambda_1^*(G_\beta)<(1-\delta)x\}}) \mathbf{1}_{\Omega'}\Big).
\end{aligned} \tag{47}
$$

Note that G_β and G_γ are independent, therefore, $\lambda_1^*(G_\beta)$ and $\lambda_1^*(G_\gamma)$ are independent as well. Hence for any $t > 0$, $\bar{\mathbb{P}}(\lambda_1^*(G_\beta) > t)$ is equal to unconditional probability $\mathbb{P}(\lambda_1^*(G_\beta) > t)$. Then using integration by parts, we have

$$
\begin{aligned}
\bar{\mathbb{E}}(f(\lambda_1^*(G_\beta)) \mathbf{1}_{\{\lambda_1^*(G_\beta)<(1-\delta)x\}}) = &\int_0^{(1-\delta)x} f'(t) \mathbb{P}(\lambda_1^*(G_\beta) > t) dt \\
&- f((1-\delta)x) \mathbb{P}(\lambda_1^*(G_\beta) > (1-\delta)x) + f(0).
\end{aligned} \tag{48}
$$

On the event $\Omega' = \{\lambda_1^*(G_\gamma) \leq (1-\delta')x\}$, one has

$$f'(t) \leq \frac{1}{2}(x/(t+1))^{k\ell-1} \exp(-(x-t)(x-\lambda_1^*(G_\gamma))/2).$$

Therefore, we have

$$
\begin{aligned}
&\int_0^1 f'(t) \mathbb{P}(\lambda_1^*(G_\beta) > t) dt \leq \frac{1}{2} x^{k\ell-1} \exp(-(x-1)(x-\lambda_1^*(G_\gamma))/2) \\
&= \frac{1}{2} x^{k\ell-1} e^{1/4} \exp((\lambda_1^*(G_\gamma)^2 - x^2)/4) \exp\left(-\frac{((x-\lambda_1^*(G_\gamma))-1)^2}{4}\right).
\end{aligned} \tag{49}
$$

On the event Ω', we have $x - \lambda_1^*(G_\gamma) \geq \delta' x$. Therefore, for C large enough, one has

$$\exp\left(-\frac{((x-\lambda_1^*(G_\gamma))-1)^2}{4}\right) \leq C \exp(-\delta'^2 x^2/8). \tag{50}$$

It further follows from (49) and (50) that for $x > 1$, there exists C large enough such that

$$\int_0^1 f'(t)\mathbb{P}(\lambda_1^*(G_\beta) > t)dt \le Cx^{\ell-2}\exp(((\lambda_1^*(G_\gamma))^2 - x^2)/4). \tag{51}$$

If $(1-\delta)x \le 1$, then we trivially have

$$\int_0^{(1-\delta)x} f'(t)\mathbb{P}(\lambda_1^*(G_\beta) > t)dt \le \int_0^1 \cdots \le Cx^{\ell-2}\exp(((\lambda_1^*(G_\gamma))^2 - x^2)/4). \tag{52}$$

Now we consider the case when $(1-\delta)x \ge 1$. Recall (11) in Lemma 2, we have

$$\mathbb{P}(\lambda_1^*(G_\beta) > t) \le Ct^{\ell-2}\exp(-t^2/4), \ \ t \ge 1, \tag{53}$$

this implies that

$$\begin{aligned}\int_1^{(1-\delta)x} f'(t)\mathbb{P}(\lambda_1^*(G_\beta) > t)dt &\le Cx^{\ell-2}\exp(((\lambda_1^*(G_\gamma))^2 - x^2)/4)\\ &\times \int_1^{(1-\delta)x} (x/(t+1))^{k\ell-1}\exp(-(t-(x-\lambda_1^*(G_\gamma)))^2/4)dt,\end{aligned} \tag{54}$$

where we have used the identity

$$\begin{aligned}&-(x-t)(x-\lambda_1^*(G_\gamma))/2 - t^2/4\\ =&-\frac{(t-(x-\lambda_1^*(G_\gamma)))^2}{4} + \frac{(x-\lambda_1^*(G_\gamma))^2}{4} - \frac{x(x-\lambda_1^*(G_\gamma))}{2}\\ =&-\frac{(t-(x-\lambda_1^*(G_\gamma)))^2}{4} + \frac{(\lambda_1^*(G_\gamma))^2 - x^2}{4}.\end{aligned}$$

On Ω', we have,

$$\begin{aligned}&\int_1^{(1-\delta)x} (x/(t+1))^{k\ell-1}\exp(-(t-(x-\lambda_1^*(G_\gamma)))^2/4)dt\\ \le&\int_0^{\min\{\delta' x/2,(1-\delta)x\}} x^{k\ell-1}\exp(-(\delta')^2x^2/16)dt\\ &+\int_{\min\{\delta' x/2,(1-\delta)x\}}^{(1-\delta)x} \Big(\frac{2}{\delta'} + \frac{1}{1-\delta}\Big)^{k\ell-1}\exp(-(t-(x-\lambda_1^*(G_\gamma)))^2/4)dt\\ \le& Cx^{k\ell}\exp(-(\delta')^2x^2/16) + C \le C'.\end{aligned} \tag{55}$$

Combining (51), (54) and (55), for $(1-\delta)x > 1$, we have

$$\int_0^{(1-\delta)x} f'(t)\mathbb{P}(\lambda_1^*(G_\beta) > t)dt = \int_0^1 \cdots + \int_1^{(1-\delta)x} \cdots \le Cx^{\ell-2}\exp(((\lambda_1^*(G_\gamma))^2 - x^2)/4). \tag{56}$$

By (52), the above estimate is actually true for all $x > 1$ on Ω'. We also have

$$f(0) = x^{k\ell-2}\exp(-x(x-\lambda_1^*(G_\gamma))/2) \le Cx^{\ell-2}\exp(((\lambda_1^*(G_\gamma))^2 - x^2)/4) \tag{57}$$

for C large enough, this is because on Ω' we have

$$-x(x-\lambda_1^*(G_\gamma))/2 - ((\lambda_1^*(G_\gamma))^2 - x^2)/4 = -(x-\lambda_1^*(G_\gamma))^2/4 \le -\delta'^2x^2.$$

Combining (48), (56) and (57), on Ω' we get

$$\bar{\mathbb{E}}(f(\lambda_1^*(G_\beta))\mathbf{1}_{\{\lambda_1^*(G_\beta)<(1-\delta)x\}}) \le C'x^{\ell-2}\exp(((\lambda_1^*(G_\gamma))^2 - x^2)/4). \tag{58}$$

On Ω', for $x > 1$, it holds that $x/(\lambda_1^*(G_\gamma)+1) \ge x/((1-\delta')x+1) \ge 1/(2-\delta')$, therefore, we further have the upper bound,

$$\bar{\mathbb{E}}(f(\lambda_1^*(G_\beta))\mathbf{1}_{\{\lambda_1^*(G_\beta)<(1-\delta)x\}}) \le C'x^{\ell-2}(x/(\lambda_1^*(G_\gamma)+1))^{k\ell-1}\exp(((\lambda_1^*(G_\gamma))^2 - x^2)/4). \tag{59}$$

We can similarly control the conditional expectation of $h(\lambda_1^*(G_\beta))\mathbf{1}_{\{\lambda_1^*(G_\beta)\le(1-\delta)x\}}$. Analogously to (48), we have

$$\bar{\mathbb{E}}(h(\lambda_1^*(G_\beta))\mathbf{1}_{\{\lambda_1^*(G_\beta)<(1-\delta)x\}}) = \int_0^{(1-\delta)x} h'(t)\mathbb{P}(\lambda_1^*(G_\beta)>t)dt - h((1-\delta)x)\mathbb{P}(\lambda_1^*(G_\beta)>(1-\delta)x)+h(0). \tag{60}$$

On the event $\Omega' = \{\lambda_1^*(G_\gamma)\le(1-\delta')x\}$, one has

$$h'(t)\le\frac{1}{2}(x/(\lambda_1^*(G_\gamma)+1))^{k\ell-1}\exp(-(x-t)(x-\lambda_1^*(G_\gamma))/2).$$

We basically repeat the proofs of (56) and (57) to get

$$\bar{\mathbb{E}}(h(\lambda_1^*(G_\beta))\mathbf{1}_{\{\lambda_1^*(G_\beta)<(1-\delta)x\}}) \le Cx^{\ell-2}(x/(\lambda_1^*(G_\gamma)+1))^{k\ell-1}\exp(((\lambda_1^*(G_\gamma))^2-x^2)/4). \tag{61}$$

Then (17) follows from (47), (59) and (61) (recall that $\ell=m-k$). □

3.3. **Proof of Lemma 4.**

Proof. By the definition of A_α in (7) with $\alpha=\{1,2,...,m\}$, we have

$$\begin{aligned}&\mathbb{P}(A_{\{1,2,\dots,m\}})\\ =&\mathbb{P}(\lambda_1(G_\alpha)>y_m;\quad \lambda_1^*(G_\beta)\le y_k,\ \forall\ 1\le k<m,\ \beta\subset\alpha,\ |\beta|=k)\\ =&\mathbb{P}(\lambda_1(G_\alpha)>y_m)\mathbb{P}(\lambda_1^*(G_\beta)\le y_k,\ \forall\ 1\le k<m,\ \beta\subset\alpha,\ |\beta|=k|\lambda_1(G_\alpha)>y_m).\end{aligned} \tag{62}$$

Lemma 4 follows from the following two limits,

$$\lim_{n\to+\infty}\mathbb{P}(\lambda_1(G_\alpha)>y_m)/[m2^{-(1+m)/2}/\Gamma(1+m/2)\cdot y_m^{m-2}e^{-y_m^2/4}]=1 \tag{63}$$

and

$$\lim_{n\to+\infty}\mathbb{P}(\lambda_1^*(G_\beta)\le y_k,\ \forall\ 1\le k<m,\ \beta\subset\alpha,\ |\beta|=k|\lambda_1(G_\alpha)>y_m)=K_m, \tag{64}$$

where K_m is the constant defined in Theorem 1. Indeed, (62), (63), (64) and the definition of y_m imply that

$$\begin{aligned}\lim_{n\to+\infty}t_n &= \lim_{n\to+\infty}\binom{n}{m}\mathbb{P}(A_{\{1,\cdots,m\}})\\ &= \lim_{n\to+\infty}n^m/m!\Big(m2^{-(1+m)/2}/\Gamma(1+m/2)\cdot y_m^{m-2}e^{-y_m^2/4}\Big)K_m\\ &= \lim_{n\to+\infty}\frac{y_m^{m-2}(\ln n)^{-(m-2)/2}K_m}{(m-1)!2^{\frac{m+1}{2}}\Gamma(1+m/2)}e^{-y/4}\\ &= \frac{(2m)^{(m-2)/2}K_m}{(m-1)!2^{3/2}\Gamma(1+m/2)}e^{-y/4},\end{aligned}$$

as desired.

We first prove (63). By formula (1) for G_α where $|\alpha|=m$, we have

$$\begin{aligned}&\mathbb{P}(\lambda_1(G_\alpha)>y_m)\\ =&\frac{1}{Z_m}\int_{\lambda_1>\cdots>\lambda_m;\lambda_1>y_m}e^{-\sum_{i=1}^m\lambda_i^2/4}\prod_{1\le i<j\le m}|\lambda_i-\lambda_j|d\lambda_1\cdots d\lambda_m\\ =&\frac{1}{Z_m}\int_{\lambda_1>y_m}e^{-\lambda_1^2/4}g(\lambda_1)d\lambda_1,\end{aligned} \tag{65}$$

where we denote

$$g(\lambda_1) = \int_{\lambda_1 > \cdots > \lambda_m} e^{-\sum_{i=2}^{m} \lambda_i^2/4} \prod_{1 \le i < j \le m} |\lambda_i - \lambda_j| d\lambda_2 \cdots d\lambda_m.$$

We claim that

$$\lim_{\lambda_1 \to +\infty} g(\lambda_1)/(\lambda_1^{m-1} Z_{m-1}) = 1. \tag{66}$$

Dividing into two cases $\lambda_m > -\sqrt{\lambda_1}$ and $\lambda_m < -\sqrt{\lambda_1}$, $g(\lambda_1)$ is bounded from above by

$$\begin{aligned} &\int_{\lambda_2 > \cdots > \lambda_m > -\sqrt{\lambda_1}} (\lambda_1 + \sqrt{\lambda_1})^{m-1} \prod_{2 \le i < j \le m} (\lambda_i - \lambda_j) e^{-\sum_{i=2}^m \lambda_i^2/4} d\lambda_2 \cdots d\lambda_m \\ &+ \int_{\lambda_2 > \cdots > \lambda_m, \lambda_m < -\sqrt{\lambda_1}} \prod_{1 \le i < j \le m} (\lambda_i - \lambda_j) e^{-\sum_{i=2}^m \lambda_i^2/4} d\lambda_2 \cdots d\lambda_m \\ &:= I_5 + I_6. \end{aligned} \tag{67}$$

Note that I_5 is further bounded from above by $(\lambda_1 + \sqrt{\lambda_1})^{m-1} Z_{m-1}$. For I_6, we note that

$$\prod_{1 \le i < j \le m} (\lambda_i - \lambda_j) \le \prod_{1 \le i < j \le m} (|\lambda_i| + 1)(|\lambda_j| + 1) \le \prod_{i=1}^{m} (|\lambda_i| + 1)^{m-1}.$$

Therefore, we can bound I_6 from above by

$$\begin{aligned} &(|\lambda_1| + 1)^{m-1} \int_{\lambda_m < -\sqrt{\lambda_1}} (|\lambda_m| + 1)^{m-1} \exp(-\lambda_m^2/4) d\lambda_m \\ &\times \left(\int_{\mathbb{R}} (|\lambda_2| + 1)^{m-1} \exp(-\lambda_2^2/4) d\lambda_2 \right)^{m-2}. \end{aligned} \tag{68}$$

For λ_1 large enough, (68) can be further bounded from above by (using (30))

$$C \lambda_1^{m-1} \sqrt{\lambda_1}^{m-2} \exp(-\lambda_1/4) \le C' \lambda_1^{m-2}.$$

Combining the estimates for I_5 and I_6 we get

$$g(\lambda_1) \le Z_{m-1} (\lambda_1 + \sqrt{\lambda_1})^{m-1} + C \lambda_1^{m-2}$$

for λ_1 large enough. It follows that

$$\limsup_{\lambda_1 \to +\infty} \frac{g(\lambda_1)}{\lambda_1^{m-1}} \le Z_{m-1}.$$

For the lower bound, for all $\lambda_1 > 1$ we have

$$g(\lambda_1) \ge (\lambda_1 - \sqrt{\lambda_1})^{m-1} \int_{\sqrt{\lambda_1} \ge \lambda_2 > \cdots > \lambda_m} e^{-\sum_{i=2}^{m} \lambda_i^2/4} \prod_{2 \le i < j \le m} |\lambda_i - \lambda_j| d\lambda_2 \cdots d\lambda_m.$$

It follows that

$$\liminf_{\lambda_1 \to +\infty} \frac{g(\lambda_1)}{\lambda_1^{m-1}} \ge \liminf_{\lambda_1 \to +\infty} \int_{\sqrt{\lambda_1} \ge \lambda_2 > \cdots > \lambda_m} e^{-\sum_{i=2}^{m} \lambda_i^2/4} \prod_{2 \le i < j \le m} |\lambda_i - \lambda_j| d\lambda_2 \cdots d\lambda_m = Z_{m-1}.$$

Then (66) follows from the upper and lower bounds.

Therefore, as $n \to +\infty$, i.e., $y_m \to +\infty$, we have

$$\int_{\lambda_1 > y_m} e^{-\lambda_1^2/4} g(\lambda_1) d\lambda_1 \sim Z_{m-1} \int_{\lambda_1 > y_m} e^{-\lambda_1^2/4} \lambda_1^{m-1} d\lambda_1. \tag{69}$$

Standard results for the upper incomplete Gamma function imply

$$\begin{aligned}\int_{\lambda_1>y_m} e^{-\lambda_1^2/4}\lambda_1^{m-1}d\lambda_1 &= \int_{s\geq y_m^2/4} e^{-s}2^{m-1}s^{(m-2)/2}ds \\ &\sim 2^{m-1}e^{-y_m^2/4}(y_m^2/4)^{(m-2)/2} \\ &= 2e^{-y_m^2/4}y_m^{m-2}.\end{aligned} \tag{70}$$

Combining (65), (69) and (70) we get

$$\begin{aligned}&\mathbb{P}(\lambda_1(G_\alpha)>y_m) \\ \sim&\frac{Z_{m-1}}{Z_m}\int_{\lambda_1>y_m} e^{-\lambda_1^2/4}\lambda_1^{m-1}d\lambda_1 \\ \sim&(Z_{m-1}/Z_m)2y_m^{m-2}e^{-y_m^2/4} \\ =&m(2\pi)^{-1/2}\Gamma(3/2)/\Gamma(1+m/2)\cdot 2^{1-m/2}y_m^{m-2}e^{-y_m^2/4} \\ =&m2^{-(1+m)/2}/\Gamma(1+m/2)\cdot y_m^{m-2}e^{-y_m^2/4}.\end{aligned} \tag{71}$$

This completes the proof of (63).

Now we prove (64). We define the following two auxiliary events:

$$B_\alpha := \Big\{y_m<\lambda_1(G_\alpha)<y_m+1, \lambda_2(G_\alpha)<\sqrt{\log\log n}, \lambda_m(G_\alpha)>-\sqrt{\log\log n}\Big\}$$

and

$$D_\alpha := \Big\{y_m<\lambda_1(G_\alpha)<y_m+1\Big\}.$$

Then it holds that

$$B_\alpha\subset D_\alpha\subset\Big\{\lambda_1(G_\alpha)>y_m\Big\}.$$

We first show that

$$\lim_{n\to+\infty}\mathbb{P}(B_\alpha|\lambda_1(G_\alpha)>y_m)=1. \tag{72}$$

Note that (72) follows if we can prove the following two limits,

$$\lim_{n\to+\infty}\mathbb{P}(D_\alpha|\lambda_1(G_\alpha)>y_m)=1 \tag{73}$$

and

$$\lim_{n\to+\infty}\mathbb{P}(B_\alpha|D_\alpha)=1. \tag{74}$$

By (63), we find that

$$\lim_{n\to+\infty}\frac{\mathbb{P}(\lambda_1(G_\alpha)>y_m+1)}{\mathbb{P}(\lambda_1(G_\alpha)>y_m)}=0,$$

which implies (73). We now prove (74) by finding a sequence of numbers $\epsilon_n\to 0$ such that for n large enough it holds that

$$\frac{\mathbb{P}(\lambda_2(G_\alpha)>\sqrt{\log\log n}\text{ or }\lambda_m(G_\alpha)<-\sqrt{\log\log n}|D_\alpha)}{\mathbb{P}(\lambda_2(G_\alpha)<1,\lambda_m(G_\alpha)>0|D_\alpha)}\leq\epsilon_n. \tag{75}$$

Indeed, (75) implies that

$$\mathbb{P}(\lambda_2(G_\alpha)>\sqrt{\log\log n}\text{ or }\lambda_m(G_\alpha)<-\sqrt{\log\log n}|D_\alpha)\leq\frac{\epsilon_n}{\epsilon_n+1}, \tag{76}$$

which converges to 0 as $n \to +\infty$, and thus this yields (74). To prove (75), we divide the set $\Big\{\lambda_2(G_\alpha) > \sqrt{\log\log n}$ or $\lambda_m(G_\alpha) < -\sqrt{\log\log n}\Big\}$ as the union of two disjoint subsets $S_1 \cup S_2$ as follows. On the subset $S_1 := \Big\{\lambda_m(G_\alpha) \le -\sqrt{\log\log n}\Big\}$, if we condition on D_α, we have

$$\begin{aligned}
&\prod_{1\le i<j\le m}(\lambda_i-\lambda_j)\\
\le& C(\lambda_1+|\lambda_m|)^{m-1}(|\lambda_2|+|\lambda_m|)^{(m-1)(m-2)/2}\\
\le& C\lambda_1^{m-1}|\lambda_m|^{m-1}(|\lambda_2|+1)^{(m-1)(m-2)/2}|\lambda_m|^{(m-1)(m-2)/2}\\
=& C\lambda_1^{m-1}|\lambda_m|^{(m-1)m/2}(|\lambda_2|+1)^{(m-1)(m-2)/2}.
\end{aligned}\tag{77}$$

On the subset $S_2 := \Big\{\lambda_2(G_\alpha) > \sqrt{\log\log n}, \lambda_m(G_\alpha) > -\sqrt{\log\log n}\Big\}$, if we condition on D_α where $\lambda_1(G_\alpha) \sim 2\sqrt{m\log n}$, then we easily have the upper bound

$$\prod_{1\le i<j\le m}(\lambda_i-\lambda_j) \le C\lambda_1^{m-1}\lambda_2^{(m-1)(m-2)/2}.$$

It follows that the left hand side of (75) can be bounded from above by the summation of

$$\frac{C[\int_{y_m}^{y_m+1}\lambda_1^{m-1}e^{-\lambda_1^2/4}d\lambda_1]\int_{\lambda_m<-\sqrt{\log\log n}}|\lambda_m|^{(m-1)m/2}(|\lambda_2|+1)^{(m-1)(m-2)/2}e^{-\sum_{i=2}^m\lambda_i^2/4}d\lambda_2\cdots d\lambda_m}{[\int_{y_m}^{y_m+1}(\lambda_1-1)^{m-1}e^{-\lambda_1^2/4}d\lambda_1]\int_{1>\lambda_2>\cdots>\lambda_m>0}\prod_{2\le i<j\le m}(\lambda_i-\lambda_j)e^{-\sum_{i=2}^m\lambda_i^2/4}d\lambda_2\cdots d\lambda_m}$$

and

$$\frac{C[\int_{y_m}^{y_m+1}\lambda_1^{m-1}e^{-\lambda_1^2/4}d\lambda_1]\int_{\lambda_2>\sqrt{\log\log n},\lambda_m>-\sqrt{\log\log n}}\lambda_2^{(m-2)(m-1)/2}e^{-\sum_{i=2}^m\lambda_i^2/4}d\lambda_2\cdots d\lambda_m}{[\int_{y_m}^{y_m+1}(\lambda_1-1)^{m-1}e^{-\lambda_1^2/4}d\lambda_1]\int_{1>\lambda_2>\cdots>\lambda_m>0}\prod_{2\le i<j\le m}(\lambda_i-\lambda_j)e^{-\sum_{i=2}^m\lambda_i^2/4}d\lambda_2\cdots d\lambda_m}.$$

The above summation can be further bounded from above by

$$\begin{aligned}
&C\Big(\int_{\lambda_m<-\sqrt{\log\log n}}|\lambda_m|^{(m-1)m/2}\exp(-\lambda_m^2/4)d\lambda_m\\
&+\int_{\lambda_2>\sqrt{\log\log n}}\lambda_2^{(m-1)(m-2)/2}\exp(-\lambda_2^2/4)d\lambda_2\Big)\\
&:=\epsilon_n
\end{aligned}$$

as n large enough. Clearly it holds that $\epsilon_n \to 0$ since $\sqrt{\log\log n} \to +\infty$. This completes the proof of (75) and thus the proof of (72).

Now we are ready to prove (64). We define the event

$$H_\alpha = \Big\{\lambda_1^*(G_\beta) \le y_k,\ \forall\ 1\le k<m,\ \beta\subset\alpha,\ |\beta|=k\Big\}.$$

Using (72), we see that (64) is equivalent to

$$\lim_{n\to+\infty}\mathbb{P}(H_\alpha|B_\alpha) = K_m.\tag{78}$$

To prove (78) we need to use the fact that

$$G_\alpha \overset{\mathrm{d}}{=} U^T\operatorname{diag}(\lambda_1,\cdots,\lambda_m)U$$

where $|\alpha| = m$ and

$$U \overset{\mathrm{d}}{=} \mathcal{U}(O(m))$$

is sampled from the uniform measure on the orthogonal group $O(m)$ which is independent of $(\lambda_1, \cdots, \lambda_m)$ and

$$(\lambda_1, \cdots, \lambda_m) \stackrel{\mathrm{d}}{=} (\lambda_1(G_\alpha), \cdots, \lambda_m(G_\alpha)).$$

Given any

$$X := U^T \operatorname{diag}(\lambda_1, \cdots, \lambda_m) U, \ U \in O(m), \ \lambda_1 \geq \cdots \geq \lambda_m,$$

by definition it holds

$$\lambda_1(X) = \lambda_1, \ \lambda_1^*(X)^2 = \lambda_1^2 + \sum_{k=2}^m \lambda_k^2/2.$$

Let $\lambda_1^-(X) := \sqrt{\sum_{k=2}^m \lambda_k^2} = \sqrt{|X|^2 - \lambda_1^2(X)}$, $X_1 := U^T \operatorname{diag}(\lambda_1, 0, \cdots, 0) U$, $X_2 := X - X_1$, then we have $|X_2| (= \sqrt{\mathrm{Tr}(X_2^2)}) = \lambda_1^-(X)$. For $\beta \subset \alpha = \{1, \cdots, m\}$ we have

$$|\lambda_1^*(X_\beta) - \lambda_1^*((X_1)_\beta)| \leq |(X_2)_\beta| \leq |X_2| = \lambda_1^-(X). \tag{79}$$

Here, by the Lipschitz continuity of eigenvalues for symmetric matrices (see Corollary A.6 in [1]), we can further derive the fact that

$$|\lambda_1^*(A) - \lambda_1^*(B)| \leq |A - B|.$$

Since $(X_1)_\beta$ is a matrix of rank at most 1, we have

$$\lambda_1^*((X_1)_\beta) = (\mathrm{Tr}((X_1)_\beta^2))^{1/2} = |\lambda_1| \sum_{k \in \beta} u_k^2, \tag{80}$$

here $\mathbf{u} = (u_1, \cdots, u_m)$ is the first row of U. Note that $\mathbf{u}$ has the uniform distribution on the unit sphere S^{m-1}. By (79) and (80), we have

$$\left| \lambda_1^*(X_\beta) - |\lambda_1| \sum_{k \in \beta} u_k^2 \right| \leq \lambda_1^-(X). \tag{81}$$

Now we replace X by G_α. On the event B_α we have

$$\lambda_1^-(G_\alpha) \leq \sqrt{m \log\log n} \text{ and } \quad y_m < \lambda_1 < y_m + 1,$$

which together with (81) imply

$$y_m \sum_{k \in \beta} u_k^2 - \sqrt{m \log\log n} \leq \lambda_1^*(G_\beta) \leq (y_m + 1) \sum_{k \in \beta} u_k^2 + \sqrt{m \log\log n} \tag{82}$$

for all $\beta \subset \alpha$.

By (82) and the fact that the first row $\mathbf{u}$ of the orthogonal group are independent of the eigenvalues $\lambda_i(G_\alpha), 1 \leq i \leq m$, we have

$$\begin{aligned} &\mathbb{P}\left(\sum_{k \in \beta} u_k^2 \leq \frac{y_k - \sqrt{m \log\log n}}{y_m + 1}, \forall \beta \subset \alpha, |\beta| = k \right) \\ \leq &\mathbb{P}(H_\alpha | B_\alpha) \\ \leq &\mathbb{P}\left(\sum_{k \in \beta} u_k^2 \leq \frac{y_k + \sqrt{m \log\log n}}{y_m}, \forall \beta \subset \alpha, |\beta| = k \right). \end{aligned} \tag{83}$$

This together with the facts that $y_k \sim 2\sqrt{k \log n}$ and $y_m \sim 2\sqrt{m \log n}$ as $n \to +\infty$ imply

$$\lim_{n \to +\infty} \mathbb{P}(H_\alpha | B_\alpha) = \mathbb{P}\left(\sum_{k \in \beta} u_k^2 \leq \sqrt{k/m}, \forall \beta \in \alpha, |\beta| = k \right),$$

which is exactly the definition of K_m. This proves (78), and hence (64). Therefore, we complete the proof of Lemma 4. $\square$

4. Proof of Theorem 2

Now we prove Theorem 2. As before, for $G = (g_{ij})_{1\leq i\leq j\leq n}$ sampled from GOE, we denote $G_\alpha = (g_{ij})_{i,j\in\alpha}$ as the principal minor of size $m \times m$ for $\alpha \subset \{1, ..., n\}$ with $|\alpha| = m$ and I_m is the collection of all such α. Let $v_1(G_\alpha)$ be the eigenvector of the largest eigenvalue of G_α and $v^* \in S^{m-1}$ be the eigenvector of the largest eigenvalue of the principal sub-matrix that attains the maximal eigenvalue $T_{m,n}$, i.e., we have

$$\alpha^* := \mathrm{argmax}_{\alpha\in I_m}\lambda_1(G_\alpha)$$

and

$$v^* = v_1(G_{\alpha^*}). \tag{84}$$

As in §3.3, we recall the definition of the event

$$H_\alpha := \{\lambda_1^*(G_\beta) \leq y_k, \, \forall 1 \leq k < m, \, \beta \subset \alpha, |\beta| = k\}. \tag{85}$$

We define the random variable $\hat{\alpha}$ as follows. If the event $\cup_{\alpha\in I_m} H_\alpha$ holds, then we set

$$\hat{\alpha} := \mathrm{argmax}_{\alpha\in I_m, \; H_\alpha \text{ holds}}\lambda_1(G_\alpha).$$

Otherwise, we set $\hat{\alpha}$ to be $\{1, \dots, m\}$. We now set

$$\hat{v} := \lambda_1(G_{\hat{\alpha}}). \tag{86}$$

In other words, $\hat{v}$ is the eigenvector of the largest eigenvalue of the principal sub-matrix G_α that achieves the maximal eigenvalue under the constraint that H_α is true. By (8) and (12), we have

$$\lim_{n\to+\infty} \mathbb{P}(\cap_{\alpha\in I_m} H_\alpha) = 1.$$

On the event $\cap_{\alpha\in I_m} H_\alpha$ we clearly have $\hat{\alpha} = \alpha^*$ and $\hat{v} = v^*$ since the constraint for $\hat{\alpha}$ doesn't have any effect. Recall the definition of A_α in (7) where $A_\alpha = H_\alpha \cap \{\lambda_1(G_\alpha) > y_m\}$, on $\cap_{\alpha\in I_m} H_\alpha$, the two events $\{T_{m,n} \geq y_m\}$ and $\cup_{\alpha\in I_m} A_\alpha$ coincide. In other words, for symmetric Q, we have

$$(\cap_{\alpha\in I_m} H_\alpha) \cap \{T_{G,m,n} \geq y_m, v^* \in Q\} = (\cap_{\alpha\in I_m} H_\alpha) \cap \{\cup_{\alpha\in I_m} A_\alpha, \hat{v} \in Q\}. \tag{87}$$

It follows that

$$\begin{aligned}
&|\mathbb{P}(T_{m,n} \geq y_m, v^* \in Q) - \mathbb{P}(\cup_{\alpha\in I_m} A_\alpha, \hat{v} \in Q)| \\
=&|\mathbb{P}\left((\cap_{\alpha\in I_m} H_\alpha)^c \cap \{T_{m,n} \geq y_m, v^* \in Q\}\right) - \mathbb{P}\left((\cap_{\alpha\in I_m} H_\alpha)^c \cap (\cup_{\alpha\in I_m} A_\alpha, \hat{v} \in Q)\right)| \\
\leq&2\mathbb{P}\left((\cap_{\alpha\in I_m} H_\alpha)^c\right),
\end{aligned} \tag{88}$$

which converges to 0 as $n \to +\infty$. Hence, (4) is equivalent to the limit

$$\mathbb{P}(\cup_{\alpha\in I_m} A_\alpha, \hat{v} \in Q) \to (1 - F_Y(y))\nu(Q). \tag{89}$$

All of the rest is to prove this convergence. Now we define four quantities

$$\begin{aligned}
k_1(\alpha) &= \mathbb{P}(A_\alpha, v_1(G_\alpha) \in Q; \forall \alpha' \neq \alpha, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha)), \\
k_2(\alpha) &= \mathbb{P}(A_\alpha, v_1(G_\alpha) \in Q; \forall \alpha' \cap \alpha = \emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha)), \\
k_3(\alpha) &= \mathbb{P}(A_\alpha; \forall \alpha' \neq \alpha, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha)), \\
k_4(\alpha) &= \mathbb{P}(A_\alpha; \forall \alpha' \cap \alpha = \emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'}) < \lambda_1(G_\alpha)).
\end{aligned} \tag{90}$$

We note that

$$
\begin{aligned}
&\mathbb{P}\left(\cup_{\alpha\in I_m} A_\alpha, \hat{v}\in Q\right)\\
=&\sum_{\alpha\in I_m}\mathbb{P}(A_\alpha, v_1(G_\alpha)\in Q; \forall\alpha'\neq\alpha, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'})<\lambda_1(G_\alpha))\\
=&\sum_{\alpha\in I_m} k_1(\alpha).
\end{aligned}
\tag{91}
$$

In fact $k_1(\alpha),\ldots,k_4(\alpha)$ don't depend on the specific choice of α. Recall the definition of $b_{n,2}$ and (27), we have

$$
\lim_{n\to+\infty}\sum_{\alpha,\alpha'\in I_m;\alpha\cap\alpha'\neq\emptyset}\mathbb{P}(A_\alpha\cap A_{\alpha'})=\lim_{n\to+\infty} b_{n,2}=0. \tag{92}
$$

By the definition of $k_1(\alpha)$ and $k_2(\alpha)$ together with the union bound we have

$$
\sum_{\alpha\in I_m}|k_1(\alpha)-k_2(\alpha)|\le\sum_{\alpha\in I_m}\mathbb{P}\left(A_\alpha\cap\left(\cup_{\alpha'\cap\alpha\neq\emptyset}A_{\alpha'}\right)\right)\le\sum_{\alpha,\alpha'\in I_m;\alpha\cap\alpha'\neq\emptyset}\mathbb{P}(A_\alpha\cap A_{\alpha'}). \tag{93}
$$

Combining (91), (92) and (93) we see that

$$
\lim_{n\to+\infty}\left|\mathbb{P}(\cup_{\alpha\in I_m}A_\alpha,\hat{v}\in Q)-\sum_{\alpha\in I_m}k_2(\alpha)\right|=0. \tag{94}
$$

We can similarly show that

$$
\lim_{n\to+\infty}\left|\mathbb{P}(\cup_{\alpha\in I_m}A_\alpha)-\sum_{\alpha\in I_m}k_4(\alpha)\right|\le\lim_{n\to+\infty}\sum_{\alpha\in I_m}|k_3(\alpha)-k_4(\alpha)|=0. \tag{95}
$$

Recall that we have already shown

$$
\mathbb{P}(\cup_{\alpha\in I_m}A_\alpha)\to 1-F_Y(y) \tag{96}
$$

in the proof of Theorem 1 as $n\to+\infty$. Hence, combining (95) and (96) we have

$$
\lim_{n\to+\infty}\sum_{\alpha\in I_m}k_4(\alpha)=1-F_Y(y). \tag{97}
$$

The advantage of introducing $k_2(\alpha)$ is that, we have the conditional probability

$$
\begin{aligned}
&\mathbb{P}(v_1(G_\alpha)\in Q|A_\alpha;\forall\alpha'\cap\alpha=\emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'})<\lambda_1(G_\alpha))\\
=&\mathbb{P}(v_1(G_\alpha)\in Q|A_\alpha),
\end{aligned}
\tag{98}
$$

since G_α is independent of $\{G_{\alpha'}:\alpha'\cap\alpha=\emptyset\}$. Consequently, we have

$$
\begin{aligned}
k_2(\alpha)=&\mathbb{P}(A_\alpha;\forall\alpha'\cap\alpha=\emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'})<\lambda_1(G_\alpha))\\
&\times\mathbb{P}(v_1(G_\alpha)\in Q|A_\alpha;\forall\alpha'\cap\alpha=\emptyset, A_{\alpha'} \text{ fails or } \lambda_1(G_{\alpha'})<\lambda_1(G_\alpha))\\
=&\mathbb{P}(v_1(G_\alpha)\in Q|A_\alpha)k_4(\alpha).
\end{aligned}
\tag{99}
$$

Clearly the term $\mathbb{P}(v_1(G_\alpha)\in Q|A_\alpha)$ is the same for all $\alpha\in I_m$. Let $\alpha_1=\{1,\ldots,m\}$. Then we have

$$
\sum_{\alpha\in I_m}k_2(\alpha)=\sum_{\alpha\in I_m}\mathbb{P}(v_1(G_\alpha)\in Q|A_\alpha)k_4(\alpha)=\mathbb{P}(v_1(G_{\alpha_1})\in Q|A_{\alpha_1})\sum_{\alpha\in I_m}k_4(\alpha). \tag{100}
$$

Let $\mathbf{u}$ be sampled from the uniform distribution on the unit sphere S^{m-1}. Inspecting the proof of (64) in §3.3, especially (72) and (83), we have

$$
\lim_{n\to+\infty}\mathbb{P}(H_\alpha, v_1(G_\alpha)\in Q|\lambda_1(G_\alpha)>y_m)=\mathbb{P}(\mathbf{u}\in\mathcal{S}_m\cap Q), \tag{101}
$$

where $\mathcal{S}_m$ has been defined in (3).

By the fact $A_\alpha = H_\alpha \cap \{\lambda_1(G_\alpha) > y_m\}$, we have

$$\begin{aligned}
&\lim_{n\to+\infty} \mathbb{P}(v_1(G_{\alpha_1}) \in Q | A_{\alpha_1}) \\
=& \lim_{n\to+\infty} \frac{\mathbb{P}(H_\alpha, v_1(G_\alpha) \in Q | \lambda_1(G_\alpha) > y_m)}{\mathbb{P}(H_\alpha | \lambda_1(G_\alpha) > y_m)} \\
=& \frac{\mathbb{P}(\mathbf{u} \in \mathcal{S}_m \cap Q)}{\mathbb{P}(\mathbf{u} \in \mathcal{S}_m)} \\
=& \nu(Q),
\end{aligned} \tag{102}$$

where ν is the uniform distribution on the set $\mathcal{S}_m$.

Combining (94), (97), (100) and (102) we get

$$\begin{aligned}
&\lim_{n\to+\infty} \mathbb{P}\left(\cup_{\alpha\in I_m} A_\alpha, \hat{v} \in Q\right) \\
=& \lim_{n\to+\infty} \sum_{\alpha\in I_m} k_2(\alpha) \\
=& \lim_{n\to+\infty} \mathbb{P}(v_1(G_{\alpha_1}) \in Q | A_{\alpha_1}) \sum_{\alpha\in I_m} k_4(\alpha) \\
=& \nu(Q)(1 - F_Y(y)).
\end{aligned} \tag{103}$$

This proves (89), and thus the proof of Theorem 2.

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