Effective nonvanishing, effective global generation

Mark Andrea A. de Cataldo*

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Abstract

We prove a multiple-points higher-jets nonvanishing theorem by the use of local Seshadri constants. Applications are given to effectivity problems such as constructing rational and birational maps into Grassmannians, and the global generation of vector bundles.

0 Introduction

Kollár's nonvanishing theorem [10], 3.2 is an instrument to make Kawamata-Shokurov base-point-freeness assertion into an effective one. His result can be applied to a variety of other situations; see [10], §4, [11], §8 and [12], §14.

The basic set-up is as follows. Let $g : X \to S$ be a surjective morphism of proper varieties, where X in nonsingular and complete, M be a nef and g-big line bundle on X, L be a nef and big line bundle on S and $N = K_X + M + mg^*L$ be a line bundle varying with the positive integer m. Kollár proves, under the necessary assumption that the first direct image sheaf $g_*N \neq 0$, that $h^0(X, N) = h^0(S, g_*N) > 0$ and the sections of g_*N generate this sheaf at a general point of S for *every* $m > (1/2)(\dim S^2 + \dim S)$ (this is what makes the result into an "effective" tool).

The proof starts with the choice of a very general point x on S and ends with producing sections of g_*N which generate at x and therefore at a general point.

The purpose of this note is to observe that we can obtain more precise statements by considering the local Seshadri constants of L on S and we can also simplify considerably the proof. See the discussion at the beginning of §2 and Remark 2.3. Our main result is the Effective Nonvanishing Theorem 2.2, a "multiple-points higher-jets" version of [10], Theorem 3.2.

The proof hinges on Demailly's observation that given a nef line bundle $\mathcal{L}$ on X, a big enough local Seshadri constant for $\mathcal{L}$ at a point x can be used together with Kawamata-Viehweg Vanishing Theorem to produce sections of the adjoint line bundle $K_X + \mathcal{L}$ with nice generating properties at x (cf. [3], Proposition 7.10). An effective way to force a big enough local Seshadri constant is Theorem 1.6, which is due Ein-Küchle-Lazarsfeld.

As applications we offer some generalizations (to the case of nef vector bundles) of the results concerning line bundles in [5] and [11]: effective construction of rational and birational maps, and nonvanishing on varieties with big enough algebraic fundamental group. These results in the case of one point follow easily from results of Kollár by considering the tautological line bundle of the projectivization of the vector bundles in question. We present them as an exemplification of the unifying character of Theroem 2.2 and also because they are new in the case of multiple-points and higher-jets.
We also show how the global generation results for line bundles of Anghern-Siu, Demailly, Tsuji and Siu (see [3] and [2] for a bibliography) generalize to vector bundles of the form $K_X^{\otimes a} \otimes E \otimes \det E \otimes L^{\otimes m}$,

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where a and m are appropriate positive integers, E is a nef vector bundle and L is and ample line bundle. We give explicit upper bounds on m which depend only on the dimension of the variety, and not also on the Chern classes of the variety and the bundles in question. However, we do not expect these bounds to be otpimal since they do not match with the line bundle case (*i.e.* assuming that the vector bundle E is the trivial line bundle).

In our paper [2] we prove upper bounds as above for vector bundles E subject to curvature conditions which seem to be the natural differential-geometric analogue of nefness and indeed imply nefness. These bounds match exactly the results in the line bundle case. The methods employed there are analytical.

In the final section we prove, using the language and techniques of algebraic Nadel ideals, a global generating statement for nef vector bundles which indeed matches the result of Anghern and Siu in the line bundle case.

The paper is organized as follows. §1 is preliminary and consists of easy and mostly known facts about local Seshadri constants, and of more elaborate ones, such as Theorem 1.6, which makes Theorem 2.2 into an effective statement. §2 is devoted to the main result, Theorem 2.2. §3 is devoted to the applications discussed above. §4 is devoted to the proof of Corollary 4.6, a major improvment of the results of Theorem 3.7.1 and 3.7.3.

§4 has been written while the author enjoyed the hospitality of the Max-Planck-Institut fúr Mathematik of Bonn.

1 Notation and preliminaries

We mostly employ the notation of [9].

A *variety* is an integral separated scheme of finite type over an uncountable algebraically closed field of characteristic zero k.

We say that *a property holds at a very general point on* X if it holds for every point in the intersection, $\mathfrak{U}$, of some at most countable family of Zariski-open dense subsets of X. Any such set $\mathfrak{U}$ meets any Zariski-open dense subset of X.

The term "point" refers to a closed one.

Vector bundles and associated locally free sheaves are identified. Cartier divisors are at times identified with the associated invertible sheaves and the additive and multiplicative notation are both used, at times simultaneously.

The symbol $B(a,b)$ denotes the usual binomial coefficient.

Let X be a variety, n be its dimension and $Div(X)$ be the group of Cartier divisors on X. A $\mathbb{Q}$-*Cartier divisor* is an element of $Div(X) \otimes \mathbb{Q}$. The linear and numerical equivalence of $\mathbb{Q}$-divisors are denoted by "$\approx$" and "$\equiv$," respectively. A $\mathbb{Q}$-*divisor* is an element in $Z_{n-1}(X) \otimes \mathbb{Q}$, where $Z_{n-1}(X)$ is the free group of Weil divisors on X.

The symbols $\lfloor a \rfloor$ and $\langle a \rangle$ denote the biggest integer less than or equal to a, and $a - \lfloor a \rfloor$, respectively. These symbols are used in conjunction with $\mathbb{Q}$-divisors when these divisors are written as a $\mathbb{Q}$-combination of *distinct* prime divisors.

Given any proper morphism of varieties $\pi : X \to S$, we have the notions of (π-)ample, (π-)nef, (π-)big and (π-)nef and (π-)big for (numerical equivalence classes of) $\mathbb{Q}$-Cartier divisors on X.

Let $\mathcal{A} \in (Div(X) \otimes \mathbb{Q})/ \equiv$ be a numerical class and D be a $\mathbb{Q}$-Cartier divisor on X. By abuse of notation, $\mathcal{A} \equiv D$ means that $A \equiv D$ for one, and thus all, the elements in $\mathcal{A}$. This remark plays a role when we use the canonical divisor class together with $\mathbb{Q}$-Cartier divisors.

In what above, the field $\mathbb{R}$ can replace $\mathbb{Q}$ with minor changes.

The following two vanishing-injectivity theorems are needed for Theorem 2.2.

Theorem 1.1 (Cf. [9], 1.2.3) *Let X be a nonsingular variety and $\pi : X \to S$ be a proper morphism onto a variety S. Assume that N is a Cartier divisor on X and that M and Δ are $\mathbb{Q}$-Cartier divisors*

on X with the following properties: (1) *M is π-nef and π-big,* (2) *the support of Δ is a divisor with normal crossings, and $\lfloor\Delta\rfloor = 0$, and* (3) *$N \equiv M + \Delta$.*
Then $R^i\pi_*\mathcal{O}_X(K_X + N) = 0$ for $i > 0$.

Theorem 1.2 (Cf. [12], 10.13 and 9.17, and [7], 5.12.b) *Let $\pi : X \to S$ and Δ be as above with X projective, D be an effective Cartier divisor on X such that it does not dominate S via π, and L be a nef and big $\mathbb{Q}$-Cartier divisor on S. Let N be a Cartier divisor such that $N \equiv \Delta + \pi^*L$.*
Then the following natural homomorphisms are injective for every $i \geq 0$:

$$H^i(X, K_X + N) \longrightarrow H^i(X, K_X + N + D).$$

Local Seshadri constants. Good references for what follows are [3], §7 and [5].

Definition 1.3 Let X be a complete variety, L be a *nef* $\mathbb{Q}$-Cartier divisor on X, and x be a point on X. The following nonnegative real number is called the *Shesadri constant of L at x*:

$$\epsilon(L, x) = \inf\{\frac{L \cdot C}{mult_x C}\},$$

where the infimum is taken over all integral curves passing through x and $mult_x C$ is the multiplicity of C at x.

Let x be a point in X_{reg}, $b_x : X' \to X$ be the blowing-up of X at x and E be the corresponding exceptional divisor on X'. The $\mathbb{Q}$-Cartier divisor b_x^*L on X' is nef as well. In particular, there is a well-defined nonnegative real number:

$$\epsilon'(L, x) := \sup\{\epsilon' \in \mathbb{Q} |\, b_x^*L - \epsilon' E \;\; is\ nef\}.$$

It is clear that the $\mathbb{R}$-Cartier divisor $b_x^*L - \epsilon'(L, x)E$ is nef, and that the $\mathbb{Q}$-Cartier divisor $b_x^*L - \epsilon' E$ is nef for every rational number ϵ' with the property that $0 \leq \epsilon' \leq \epsilon'(L, x)$.

Fact 1.4 *We have that $\epsilon(L, x) = \epsilon'(L, x)$ for every $x \in X_{reg}$.* This follows from the formula: $(b_x^*L - \epsilon E) \cdot \tilde{C} = L \cdot C - \epsilon\, mult_x C$, where ϵ is any real number, $\tilde{C}$ is any integral curve in X' not contained in E and $C := b_x(\tilde{C})$.

We collect the simple properties of $\epsilon(L, x)$ which, together with Theorem 1.6, are essential in the sequel of the paper.

Lemma 1.5 *Let L be a nef $\mathbb{Q}$-Cartier divisor on a complete variety X and x be a point in X_{reg}. Then*
(1.5.1) *$L^n \geq \epsilon(L, x)^n$;*
(1.5.2) *for every $t \in \mathbb{Q}^+$, $\epsilon(tL, x) = t\, \epsilon(L, x)$;*
(1.5.3) *Let $f : X' \to X$ be a proper and birational morphism and x be a point on X over which f is an isomorphism; then $\epsilon(L, x) = \epsilon(f^*L, f^{-1}\{x\})$;*
(1.5.4) *if L is Cartier, ample and generated by its global sections on X, then $\epsilon(L, x) \geq 1$;*
(1.5.5) *if L is Cartier and the global sections of L generate jets of order s at x,* i.e. *the natural evaluation map $H^0(X, L) \to \mathcal{O}_X(L)/\mathfrak{m}_x^{s+1}\mathcal{O}_X(L)$ is surjective, then $\epsilon(L, x) \geq s$.*

Proof. The first property follows from the fact that since $b_x^*L - \epsilon(L, x)E$ is nef, then its top self-intersection is nonnegative. The second one is an obvious consequence of the bilinearity of the intersection product.
The third property follows from the fact that there is a natural bijection, given by taking strict transforms, between the sets of integral curves on X through x and on X' through $x' := f^{-1}\{x\}$. If C and C' correspond to each other in this bijection, then $L \cdot C = b_x^*L \cdot C'$ and $mult_x C = mult_{x'} C'$ so that the two local Seshadri constants are the same.

If L is ample, Cartier and generated by its global sections on X, then the rational map φ defined by $|L|$ is a finite morphism. Let C be any integral curve on X passing through x. Since C is not contracted by φ, there is an effective divisor D in $|L|$ passing through x but not containing C. It follows that $L \cdot C = D \cdot C \geq mult_x C$. This implies the third property.
Finally, if $s = 0$, then there is nothing left to prove. Assume that $s \geq 1$. Then the global sections of $L \otimes \mathfrak{m}_x^s$ generate $L \otimes \mathfrak{m}_x^s$ at x. Given any integral curve $\tilde{C}$ on X' not contained in E, we find a divisor $D \in |L \otimes \mathfrak{m}_x^s|$ not containing the curve $b_x(\tilde{C})$. It follows that the effective divisor $b_x^*(D) \in |b_x^* L - sE|$ does not contain $\tilde{C}$. In particular, $(b_x^* L - sE) \cdot \tilde{C} \geq 0$. This is enough to establish the last property. $\square$

If X is complete and L is a nef $\mathbb{Q}$-Cartier divisor on X, then Shesadri's criterion of ampleness asserts that L is ample iff $\epsilon(L) := \inf\{\epsilon(L,x) | \, x \in X\} > 0$. An example of R. Miranda's (cf. [3], 7.14) shows that given any positive real number ϵ, there exists a nonsingular rational surface X, a point $x \in X$ and an ample line bundle L on X, such that $\epsilon(L,x) \leq \epsilon$.
In particular, we *cannot* expect to have a statement of the form: *let X be a nonsingular projective variety of dimension n and L be an ample line bundle on X, then $\epsilon(L) \geq C_n$, for some positive constant depending only on n.*

What is known is the following result of Ein, Küchle and Lazarsfeld. The authors prove it for projective varieties, but by Chow's Lemma and Lemma 1.5.3 the statement is true for every complete variety.

Theorem 1.6 (Cf. [5], Theorem 1) *Let L be a nef and big Cartier divisor on a complete variety X of dimension n. Then at a very general point x on X we have:*

$$\epsilon(L,x) \geq \frac{1}{n}.$$

The example that follows shows that Theorem 1.6 cannot hold as stated for an ample and effective integral $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety. As is pointed out in [5], if m is the smallest positive integer such that mL is Cartier, then Theorem 1.6 holds if we replace "$\epsilon(L,x) \geq \frac{1}{n}$" by "$\epsilon(L,x) \geq \frac{1}{nm}$."

Example 1.7 Let $S_m \subset \mathbb{P}^{m+1}$ be the surface which is a cone of vertex v over the rational normal curve of degree m in $\mathbb{P}^m$ and $\mathfrak{l}$ be any line belonging to the ruling of S_m. The Weil divisor $\mathfrak{l}$ is an integral $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor; to be precise it is m-Cartier. The Cartier divisor $m\mathfrak{l}$ is very ample so that, for every $x \in S_m \setminus \{v\}$, we have that $\epsilon(\mathfrak{l},x) \geq \frac{1}{m}$. On the other hand, fix $x \in S_m \setminus \{v\}$ and let C be the line on S_m passing through x. We have $\mathfrak{l} \cdot C = \frac{1}{m}$, so that $\epsilon(\mathfrak{l},x) \leq \frac{1}{m}$. It follows that $\epsilon(\mathfrak{l},x) = \frac{1}{m}$, for every $x \in S_m \setminus \{v\}$.

2 An effective nonvanishing theorem

In this section we prove a nonvanishing theorem very similar to [10], Theorem 3.2. While the statement is clearly inspired by [10], Theorem 3.2, its simpler proof is inspired by [4], Lemma 3.21.

The basic nonvanishing and global generation at a generic point follow easily from [10], Theorem 3.2 (and in fact are slightly weaker than this latter result). The "multiple-points higher-jets" statements do not follow directly from the results in the literature.
Let us point out that Kollár's result implies a version of Theorem 2.2 with x general, p=1 and s=0. However, one can use this result in place of Kollár's in proving the effective base-point-freeness result 1.1 of [10].

The advantages of Theorem 2.2 are at least two.
The former is the simplicity of its proof which consists of basic yoga and one blowing-up procedure. However, we should stress that more often than not this becomes an effective result if used in conjunction with the non-trivial result Theorem 1.6.

The latter is that it is a "multiple-point higher-jets" effective result which, at least in principle, can be applied to prescribed points and can give more than mere nonvanishing. For example, one can use this result to obtain increased lower bounds of log-plurigenera (cf. [10], §4). We shall see some other applications in the following sections.

Remark 2.1 Let $g : Y \to S$ be a proper morphism of varieties with Y nonsingular and Δ a divisor on Y such that $Supp(\Delta)$ has simple normal crossings.
By virtue of generic smoothness, there exists a largest Zariski-open dense subset U of S such that (i) $g_{|g^{-1}(U)} : g^{-1}(U) \to U$ is smooth, (ii) for every point $x \in S$, any irreducible component F of the fiber F_x of g is not contained in $Supp(\Delta)$ and (iii) $Supp(\Delta)$ has simple normal crossings on F.

Theorem 2.2 (**Effective Nonvanishing**) *Let the following data be given.*
(2.2.1) (Y, Δ)*: a log-pair, where* Y *is nonsingular and complete,* $\lfloor \Delta \rfloor = 0$ *and* $Supp(\Delta)$ *has simple normal crossings.*
(2.2.2) N*: a Cartier divisor on* Y*.*
(2.2.3)
- $g : Y \to S$*: a proper and surjective morphism onto a complete variety* S *of positive dimension,*
- $U = U(g, \Delta)$*: the Zariski-open dense set of* S *defined in* Remark 2.1,
- V*: the Zariski-open dense subset of* S *over which the formation of* g_* *for* N *commutes with base extensions.*
(2.2.4)
- p*: a positive integer,*
- $\{s_1, \ldots, s_p\}$*: a* p*-tuple of non-negative integers,*
- $\{x_1, \ldots, x_p\}$*:* p *distinct points in* $U \cap V$*.*
(2.2.5) M*: a* $\mathbb{Q}$*-Cartier divisor on* Y *such that either it is nef and* g*-big, or* X *is projective and* $M \equiv 0$*.*
(2.2.6) $L_1, \ldots, L_p$*:* p $\mathbb{Q}$*-Cartier divisors on* S *such that all* L_j *are nef and big and either*
(a) $\epsilon(L_j, x_j) > \dim S + s_j$, $\forall j = 1, \ldots, p$, *or*
(b) $\epsilon(L_j, x_j) \geq \dim S + s_j$, $\forall j = 1, \ldots, p$ *and* $L_{j_0}^{\dim S} > \epsilon(L_{j_0})^{\dim S}$ *for at least one index* j_0, $1 \leq j_0 \leq p$.
Assume that

$$N \equiv K_Y + \Delta + M + g^* \sum_{j=1}^{p} L_j.$$

Then the following natural map is surjective

$$H^0(X, N) \simeq H^0(S, g_* N) \longrightarrow\!\!\!\!\!\rightarrow \bigoplus_{j=1}^{p} \frac{g_* N}{\mathfrak{m}_{x_j}^{s_j+1} \cdot g_* N}.$$

In particular, if $g_* N$*, which is torsion-free, is not the zero sheaf, then* $H^0(X, N) \neq \{0\}$*.*

Remark 2.3 The reason for calling this theorem "Effective Nonvanishing" is the last assertion of the theorem and the fact that, for example, if all the L_j were Cartier, then we could make sure, by virtue of Theorem 1.6, that condition (2.2.6) is fulfilled at very general points by taking sufficiently high multiples of the L_j.
Note also that the conclusion of the theorem holds trivially also for $\dim S = 0$, but that in this case (2.1.6) is not meaningful.

Proof. The proof is divided into two cases. The former deals with M nef and g-big. The latter with X projective and $M \equiv 0$. Each case is divided into two sub-cases corresponding to the two numerical assumptions (a) and (b) in (2.2.6).
CASE I: M *is nef and* g*-big.*
First we show that in this case $U = U \cap V$. By virtue of 1.1, we know that $R^i g_* N = 0$ for $i > 0$. By the smoothness of g over U, N is flat over U. By well-known results of Grothendieck (see [8], III.7.7.10) $g_* N$ is locally free on U and the formation of g_* commutes with base extension over U.

In particular, if $Y_{x_j}^{\sigma} := Y \times_S Spec\,(\mathcal{O}_{S,x_j}/\mathfrak{m}_{x_j}^{\sigma})$ is the "σ-thickened fiber" of g at x_j and $N_{x_j}^{\sigma}$ is the pull-back of N to $Y_{x_j}^{\sigma}$ via the natural projection, then the following natural maps are isomorphisms:

$$\frac{g_*N}{\mathfrak{m}_{x_j}^{s_j+1}\cdot g_*N} = g_*N \otimes (\mathcal{O}_{S,x_j}/\mathfrak{m}_{x_j}^{s_j+1}) \longrightarrow H^0(Y_{x_j}^{s_j+1}, N_{x_j}^{s_j+1}).$$

To prove CASE I it is enough to show that the natural map

$$H^0(Y,N) \longrightarrow \bigoplus_{j=1}^{p} H^0(Y_{x_j}^{s_j+1}, N_{x_j}^{s_j+1}), \tag{1}$$

which factors through $g_*N \otimes \mathcal{O}_{S,x_j}/\mathfrak{m}_{x_j}^{s_j+1}$, is surjective.

Consider the following cartesian diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{B} & Y \\ \downarrow g' & & \downarrow g \\ S' & \xrightarrow{b} & S \end{array}$$

where b is the blowing-up of S at all the simple points x_j. Let $F := \coprod F_j$ be the scheme-theoretic-fiber of g corresponding to the union of the points x_j, $j = 1, \ldots, p$. Since g is smooth over U and all the x_j are in U, we see that B coincides with the blowing-up of Y along F. In particular, Y' is a nonsingular variety. Let $E = \sum E_j$ be the exceptional divisor of b and $D = \sum D_j$ the one of B; we have that $D_j = g'^* E_j$, for every $j = 1, \ldots, p$.

The map (1) is surjective iff the natural map $H^1(Y', B^*N - \sum(s_j+1)D_j) \to H^1(Y', B^*N)$ is injective. It is this injectivity that we are going to establish using Theorem 1.1.

Note that $K_{Y'} \approx B^*K_Y + (\dim S - 1)\sum D_j$ and that since no irreducible component of any F_j is contained in any Δ_i and if any such component meets any Δ_i it does so transversally, we have that a) $\Delta' := B^*\Delta = B_*^{-1}\Delta$, *i.e.* the pull-back is the strict transform, b) $\lfloor \Delta' \rfloor = 0$ and c) the support of Δ' has simple normal crossings. The following numerical equality is easily checked:

$$B^*N - \sum (s_j+1)D_j \equiv K_{Y'} + \Delta' + B^*M + B^*g^*\sum L_j - \sum (\dim S + s_j)D_j. \tag{2}$$

SUB-CASE I.A: *Assume that* $\epsilon(L_j, x_j) > \dim S + s_j$, *for every index* j, $1 \le j \le p$.

Since for every index j we have that $\epsilon(L_j, x_j) > \dim S + s_j$, there exists a positive rational number $0 < t < 1$ such that $\epsilon((1-t)L_j, x_j) > \dim S + s_j$ for every j, $1 \le j \le p$. Using the fact that $B^*g^* = g'^*b^*$ we can re-write the r.h.s. of equation (2) as

$$K_{Y'} + \Delta' + B^*(M + tg^*\sum L_j) + g'^*\sum [b^*(1-t)L_j - (\dim S + s_j)E_j]. \tag{3}$$

The last summand is nef by the very definition of $\epsilon((1-t)L_j, x_j)$.

Since M is nef and g-big and $t > 0$, the $\mathbb{Q}$-divisor $M + tg^*\sum L_j$ is nef and big. In particular, $B^*(M + tg^*\sum L_j)$ is nef and big. It follows that the l.h.s. of (2) is a Cartier divisor satisfying the assumptions of Kawamata-Viehweg Vanishing Theorem so that $H^1(Y', B^*N - \sum(s_j+1)D_j) = \{0\}$ and (1) is surjective.

SUB-CASE I.B: *Assume that* $\epsilon(L_j, x_j) \ge \dim S + s_j$, $\forall j$, $1 \le j \le p$ *and that* $L_{j_0}^{\dim S} > \epsilon(L_{j_0})^{\dim S}$ *for at least one index* j_0, $1 \le j_0 \le p$.

Using the fact that $B^*g^* = g'^*b^*$ and isolating the index j_0 we write the r.h.s. of (2) as

$$K_{Y'} + \Delta' + B^*M + \sum_{j \ne j_0} g'^*\left[b^*L_j - (\dim S + s_j)E_j\right] + g'^*\left[b^*L_{j_0} - (\dim S + s_{j_0})E_{j_0}\right]. \tag{4}$$

Since M is nef and g-big and $\sum_{j\neq j_0} g'^*(b^*L_j-(\dim S+s_j)E_j)$ is nef we see that $B^*M+\sum_{j\neq j_0} g'^*(b^*L_j-(\dim S+s_j)E_j)$ is nef and g'-big. Since $L_{j_0}^{\dim S} > \epsilon(L_{j_0}, x_{j_0})^{\dim S}$, we see, as in the proof of Lemma 1.5.1, that $(b^*L_{j_0}-(\dim S+s_{j_0})E_{j_0})$ is nef and big. It follows that $B^*M+\sum_{j\neq j_0} g'^*(b^*L_j-(\dim S+s_j)E_j)+g'^*(b^*L_{j_0}-(\dim S+s_{j_0})E_{j_0})$ is nef and big and we conclude as in SUB-CASE I.A.

CASE II: *X is projective, $M\equiv 0$ and the points x_j are in $U\cap V$.*
We by-pass the first paragraph in the proof of CASE I. We proceed *verbatim* as in that case until we hit again (2). We delete M. We can again divide the analysis into two separate sub-cases. We do so and obtain that in the two distinct sub-cases the l.h.s. of (2) is numerically equivalent to the r.h.s. of (3) and (4), respectively and, in both cases, we are in the position to apply Theorem 1.2 to the morphism $g': Y'\to S'$ and infer the desired injectivity statement. □

3 Applications

The local Seshadri constant can be linked, via Kawamata-Viehweg Vanishing Theorem to the production of sections for the adjoint to nef and big line bundles. This observation is due to Demailly; see [3], Proposition 7.10 and [5], Proposition 1.3. In this section we apply Theorem 2.2 to nef vector bundles. Actually, a factor $\det E$ appears and is necessary in our proof. We ignore if it is necessary for the truth of the various statements that follow. First we fix some notation.

Let E be a rank r vector bundle on a nonsingular complete variety X. We denote by $\mathbb{P}_X(E)$ the projectivized bundle of hyperplanes, by $\pi:\mathbb{P}_X(E)\to X$ the natural morphism and by ξ or ξ_E the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(E)}(1)$. We say that E is nef if ξ is nef.

Let p be any positive integer. We say that *the global sections of E generate jets of order $s_1,\dots,s_p\in\mathbb{N}$ at p distinct points $\{x_1,\dots,x_p\}$ of X* if the following natural map is surjective:

$$H^0(X,E)\longrightarrow \bigoplus_{\imath=1}^{p} E_{x_i}\otimes\mathcal{O}_X/\mathfrak{m}_{x_i}^{s_i+1}.$$

We say that *the global sections of E separate p distinct points $\{x_1,\dots,x_p\}$ of X* if the above holds with all $s_i=0$. The case $p=1$ is equivalent to E being generated by its global sections (*generated*, for short) at the point in question.

Rational maps to Grassmannians. Let $V:=H^0(X,E)$ and $h^0:=h^0(X,E):=\dim_k H^0(X,E)$. Consider the Grassmannian $G:=G(r,h^0)$ of r-dimensional quotients of V, the universal quotient bundle $\mathfrak{Q}$ of G and the determinant of $\mathfrak{Q}$, $\mathfrak{q}$.
As soon as E is generated at some point of X, we get a rational map $\varphi: X-->G$ assigning to each point $y\in X$ where E is generated the quotient $E_y\otimes k(y)$.
If E is generated at every point of X, then $f:=\varphi$ is a morphism and $E\simeq f^*\mathfrak{Q}$.
It is clear that:
- V separates arbitrary pairs of points of X iff f is bijective birational onto its image;
- If V separates every pair of points of X and generates jets of order 1 at every point of X, then f is a closed embedding (the converse maybe false if $r>1$).

In the three propositions that follow we generalize to the case of higher rank results in [5]. The analogues to these facts involving arbitrary p and $\{s_1,\dots,s_p\}$ are clear, and left to the reader. We give the reference to the analogous results for line bundles, but we prove only the first of the three propositions to illustrate the method.

Proposition 3.1 (Cf. [5], 1.3 and 4.4) *Let X be a nonsingular complete variety of dimension n. Let E be a rank r nef vector bundle on X, L be a nef and big $\mathbb{Q}$-Cartier divisor on X, Δ' be a $\mathbb{Q}$-Cartier divisor on X such that $\lfloor\Delta'\rfloor=0$ and Supp(Δ') has simple normal crossings, and N' be a Cartier divisor on X such that $N'\equiv L+\Delta'$.*

Let s be a nonnegative integer and x be a point of $X \setminus Supp(\Delta)$.
Assume that either $\epsilon(L,x) > n+s$, or $\epsilon(L,x) \geq n+s$ and $L^n > \epsilon(L,x)^n$.
Then $H^0(X, K_X \otimes E \otimes \det E \otimes N')$ generates s-jets at x and the rational map φ as above is defined. Moreover,

$$h^0(X, K_X \otimes E \otimes \det E \otimes N') \geq rB(n+s,s).$$

In particular, if $\mathcal{L}$ is a nef and big Cartier divisor on X, then

$$H^0(X, K_X \otimes E \otimes \det E \otimes \mathcal{L}^{\otimes m}) \geq rB(n+s,s), \quad \forall\, m \geq n^2+ns.$$

Proof. Set $Y := \mathbb{P}_X(E)$, $S := X$, $g := \pi$, $\Delta := g^*\Delta'$, $M := (r+1)\xi$, $N := K_Y + (r+1)\xi + g^*N'$, $p = 1$, $s_1 = s$. Note that M is nef and g-big and that $g_*N = K_X \otimes E \otimes \det E \otimes N'$.
Apply Theorem 2.2. The only issue is whether $x \in U$; this is why the point x is assumed to be outside of $Supp(\Delta)$.
The lower bound on h^0 stems from the surjection given by Theorem 2.2 and the fact that

$$\dim_k \mathcal{O}_{X,x}/\mathfrak{m}_x^{s+1} = B(n+s,n).$$

The statement about $\mathcal{L}$ is a special case after Theorem 1.6: there exists $x \in X$ such that $\epsilon(\mathcal{L},x) \geq 1/n$. If $m \geq n^2+ns$, then $\epsilon(m\mathcal{L},x) \geq n+s$ and equality holds iff $\epsilon(\mathcal{L},x) = 1/n$ and $m = n^2+ns$; in this case the inequality $\mathcal{L}^n \geq 1 > \epsilon(\mathcal{L},x)^n$ is automatic. ☐

Proposition 3.2 (Cf. [5], 4.5) *Let X, n, E, L, Δ' and N' be as above. Assume that either $n \geq 2$ and $\epsilon(L,x) \geq 2n$ for every x very general, or that $n = 1$ and $\deg N' \geq 3$.*
Then the rational map φ associated with $H^0(X, K_X \otimes E \otimes \det E \otimes N')$ is defined and is birational onto its image.
In particular, if $\mathcal{L}$ is a nef and big Cartier divisor on X, then the rational map φ associated with $H^0(X, K_X \otimes E \otimes \det E \otimes \mathcal{L}^{\otimes m})$ is defined and birational onto its image for every $m \geq 2n^2$.

Proposition 3.3 (Cf. [5], 4.6) *Let X be a complete variety of dimension n with only terminal singularities and of global index i such that K_X is nef and big, (i.e. X is normal, $\mathbb{Q}$-Gorenstein and a minimal variety of general type, and i is the smallest positive integer such that the Weil divisor class iK_X is a Cartier divisor class), and E be a nef vector bundle on X.*
Then the rational map associated with $H^0(X, \mathcal{O}_X(miK_X) \otimes E \otimes \det E)$ is defined and is birational onto its image for every $m \geq 2n^2+1$.

The following follows from results in [11], §8. As is already pointed out in [5], a generically large algebraic fundamental group on the base variety S can be used to produce section by increasing the local Seshadri constants on finite étale covers of S. The reader can consult [11] for the relevant definitions.

Proposition 3.4 (Cf. [11], 8.4) *Let X be a normal and complete variety, N' be an integral big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, and E be a nef vector bundle on X.*
Assume that X has generically large algebraic fundamental group.
Then $h^0(X, \mathcal{O}_X(K_X+N') \otimes E \otimes \det E) > 0$.

Sketch of proof. By the proof of [11], Corollary 8.4 and by the first part of the proof of [11], Theorem 8.3 we are reduced to the case in which X is nonsingular and $N' \equiv L + \Delta$, where L and Δ' are $\mathbb{Q}$-Cartier divisors, L is nef and big, $\lfloor \Delta' \rfloor = 0$ and $Supp(\Delta')$ has simple normal crossings.
Pick a point $x \in X$ such that $\epsilon(L,x) > 0$. By [11], Lemma 8.2 there is a finite étale map of varieties $m : X'' \to X$ and a point $x'' \in X''$ such that $\epsilon(m^*L, x'') \geq n$.
Denote $\deg m$ by d, m^*L by L'', $m^*\Delta'$ by Δ'', m^*N' by N'' and m^*E by E''.
Apply Proposition 3.1 to X'', L'', Δ'', N'', E'' and $s = 0$. We get $h^0(K_{X''} \otimes E'' \otimes \det E'' \otimes N'') > 0$.

Kawamata-Vieheweg Vanishing Theorem applied to the nef and big $\mathbb{Q}$-divisor $(r+1)\xi_{E''} + \pi''^{*}L''$ gives, via Leray spectral sequence, $h^i(X'', K_{X''} \otimes E'' \otimes \det E'' \otimes N'') = 0$, for every $i > 0$. The analogous statement holds on X.
The above vanishing and the multiplicative behavior of Euler-Poincaré characteristics of coherent sheaves under finite étale maps of nonsingular proper varieties gives:

$$h^0(X, K_X \otimes E \otimes \det E \otimes N') = \chi(X, -) = \frac{1}{d}\chi(X'', -'') = \frac{1}{d}h^0(X'', K_{X''} \otimes E'' \otimes \det E'' \otimes N'') > 0.$$

$\square$

Let us point out a consequence of [11], 8.3 as a corollary to the result above. Recall that the integers $I^i_{lm} := h^i(X, S^l(\Omega^1_X) \otimes K_X^{\otimes m})$ are birational invariants of a nonsingular and complete variety X for every $m, l \geq 0$ and that they are independent of the more standard invariants like the plurigenera or the cohomology groups of the sheaves Ω^p_X; for some facts about these invariants and some references see [14]. The assumptions of the "sample" corollary that follows are fulfilled, for example, by projective varieties whose universal covering space is the unit ball in $\mathbb{C}^n$.

Corollary 3.5 (Cf. [11], 8.5) *Let X be a nonsingular complete variety with K_X nef and big, Ω^1_X nef and generically large algebraic fundamental group.*
Then $I^0_{1m} \geq 0$ for every $l \geq 0$ and $m \geq 3$.

We now observe that the global generation results of Anghern-Siu, Demailly, Tsuji and Siu can be used to deduce analogous statements for vector bundles of the form $K_X^{\otimes a} \otimes E \otimes \det E \otimes L^{\otimes m}$, where E and L are a nef vector bundle and an ample line bundle on X, respectively. The idea is simple: once the sections of a line bundle of the form $\mathcal{L} := K_X + mL$ generate the s jets at every point, the local Seshadri constant is at least s at every point by virtue of Lemma 1.5.5. We then use Proposition 3.1. However, this idea is applied here in a rather primitive way; we expect these results to be far from otpimal.
We shall give statements concerning $p = 1, 2$ and low values for the jets. In the same way one can prove statements concerning more points and higher jets. We omit the details.

For ease of reference we collect the line bundle results in the literature in the following result. First some additional notation. Let n and p be positive integers and $\{s_1, \ldots, s_p\}$ be a p-tuple of nonnegative integers. Let us define the following integers:

$$m_1(n,p) := \frac{1}{2}(n^2 + 2pn - n + 2),$$

$$m_2(n, p; s_1, \ldots, s_p) = 2n \sum_{i=1}^{p} B(3n + 2s_i - 3, n) + 2pn + 1.$$

Theorem 3.6 *Let X be a nonsingular projective variety of dimension n, and L be an ample Cartier divisor on X.*

(3.6.1) (Cf. [15]) *if $m \geq m_2(n, p; s_1, \ldots, s_p)$, then the global sections of $2K_X + mL$ generate simultaneous jets of order $s_1, \ldots, s_p \in \mathbb{N}$ at arbitrary p distinct points of X;*

(3.6.2) (Cf. [1]) *If $m \geq m_1(n,p)$, then the global sections of $K_X + mL$ separate arbitrary p distinct points of X;*

Theorem 3.7 *Let X, n and L be as above. Let E be a nef vector bundle on X. Then the vector bundles $K_X^{\otimes a} \otimes E \otimes \det E \otimes L^{\otimes m}$:*
(3.7.1) *are generated by their global sections and the associated morphism to a Grassmannian $f : X \to G$ is finite, for $a = 2$ and for every $m \geq (1/2)(m_2(n, 1; 2n) + 1)$;*
(3.7.2) *have global sections which separate arbitrary pairs of points, 1-jets at an arbitrary point, and f is a closed embedding, for $a = 2$ and for every $m \geq (1/2)(m_2(n, 1; 4n) + 1)$;*

(3.7.3) *are generated by their global sections and f is finite, for $a = n+1$ and for every $m \geq nm_1(n,1)$;*
(3.7.4) *have global sections which separate arbitrary pairs of points, 1-jets at an arbitrary point, and f is a closed embedding, for $a = 2n+1$ and for every $m \geq 2nm_1(n,1)$.*

Proof. Let us observe that all the vector bundles in question are ample. One sees this easily by observing that $K_X + (n+1)L$ is always nef (Fujita) and that "nef $\otimes$ ample = ample." As soon as f is defined, these bundles are pull-backs under f so that they can be ample only if f is finite.

Let $L' := K_X + (1/2)(m_2(n,1;2n)+1)L$. By virtue of Theorem 3.6.1, the global sections of $2L'$ generate $2n$ jets at every point $x \in X$. By virtue of Lemma 1.5.5, $\epsilon(L',x) \geq n$ for every $x \in X$. We can apply Proposition 3.1 which assumptions are readily verified. This proves (3.7.1).
The proof of (3.7.2) is similar. We observe that we need $\epsilon(L',x) \geq 2n$ to separate points and $\epsilon(L',x) \geq n+1$ to separate 1-jets. We then use Proposition 3.2 in the former case and Proposition 3.1 with $s=1$ in the latter.
(3.7.3) and (3.7.4) are proved similarly using Theorem 3.6.2 and Lemma 1.5.4. $\square$

4 Better bounds for global generation

In this section we greatly improve upon Theorem 3.7.1 and 3.7.3. The method is similar to the one of the previous section. However, it does not use Seshadri constants. It needs a similar, local positivity result which allows one to apply the same techniques used before, and based on Theorem 2.2, to produce sections. Once the local positivity at one point has been established, the technique employed in 2.2 emerges in all its simplicity.

Let us recall, for the readers's convenience few basic facts about the algebraic counterparts to Nadel Ideals. The reference is [6].

Let X be a nonsingular projective variety and D be an effective $\mathbb{Q}$-divisor. Let $f : X' \to X$ be an embedded resolution for (X,D). The integral divisor $K_{X'/X} - f^*\lfloor D \rfloor$ can be written as $P - N$, where P and N are integral divisors without common components and P is f-exceptional.

The *multiplier ideal* $\mathcal{I}(D)$ associated with (X,D) is, by definition,

$$\mathcal{I}(D) := f_*\mathcal{O}_{X'}(P-N) = f_*\mathcal{O}_{X'}(-N) \subseteq \mathcal{O}_X.$$

One checks that this ideal sheaf is independent of the resolution chosen and that $\mathcal{O}_{X'}(P-N)$ has trivial higher direct images. As a consequence, we get the following vanishing result.

Proposition 4.1 (Cf. [6], 1.4) *Let X be a nonsingular projective variety, L be a line bundle on X and D be an effective $\mathbb{Q}$-divisor on X. Assume that $L - D$ is nef and big.*
Then $H^j(X, K_X \otimes L \otimes \mathcal{I}(D)) = \{0\}$, for every $j > 0$.

The following lemma is an easy consequence of the definitions and is a functorial property of these ideals.

Lemma 4.2 *Let $\pi : P \to X$ be a smooth morphism of nonsingular projective varieties and D be an effective $\mathbb{Q}$-divisor on X. Then $\pi^*\mathcal{I}(D) = \mathcal{I}(\pi^*D)$.*

Proof. Consider the following cartesian diagram

$$\begin{array}{ccc} P' & \xrightarrow{f'} & P \\ \downarrow \pi' & \square & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

where $f : X' \to X$ is an embedded resolution of singularities of the log-pair (X,D). Since π is smooth, $f' : P' \to P$ is a resolution of (P, π^*D).

We have $\mathcal{I}(\pi^*D) = f'_*(K_{P'/P} - \lfloor f'^*(\pi^*D)\rfloor) = f'_*(\pi'^*K_{X'/X} - \lfloor \pi'^* f^*D\rfloor) = f'_*(\pi'^*K_{X'/X} - \pi'^*\lfloor f^*D\rfloor) = f'_*(\pi'^*(K_{X'/X} - \lfloor f^*D\rfloor)) = \pi^*(f_*(K_{X'/X} - \lfloor f^*D\rfloor)) = \pi^*\mathcal{I}(D)$, where: the second equality holds because the formation of the sheaf of relative differentials $\Omega^1_{*/*}$ commutes with base change and the relative cononical sheaf is, in the simple case under scrutiny, the determinant of $\Omega^1_{*/*}$; the third equality holds because π' is smooth; the fifth stems from the fact that cohomology commutes with the flat base extension π. □

The following result is a $\mathbb{Q}$-divisors reformulation of the result of Anghern-Siu. The result is due to Kollár [13]. The formulation given below in terms of algebraic multiplier ideals is due to Ein [6].

Theorem 4.3 *Let X be a nonsingular projective variety of dimension n and L be an ample line bundle on X such that*

$$L^d \cdot Z > B(n+1,2)^d$$

for every d-dimensional integral cycle Z on X. Then, for every point $x \in X$ there exists an effective $\mathbb{Q}$-divisor D such that $D \equiv \lambda L$ for some positive rational number $0 < \lambda < 1$ and x is in the support of an isolated component of $V(\mathcal{I}(D))$.

Remark 4.4 A similar statement holds if we consider several distinct points.

Theorem 4.5 *Let $\pi : P \to X$ be a smooth morphism with connected fibers of nonsingular projective varieties, n be the dimension of X, M be a nef and π-big line bundle on P, $\mathcal{L}$ be an ample line bundle on X such that*

$$\mathcal{L}^d \cdot Z > B(n+1,2)^d$$

for every d-dimensional integral cycle Z on X.
Then, the vector bundle $\pi_*(K_P + M) \otimes \mathcal{L}$ is generated by its global sections.
In particular, if L is any ample line bundle on X, then we can choose, $\mathcal{L} := B(n+1,2)\,L$.

Proof. Let $x \in X$ be an arbitrary point. Let D be a $\mathbb{Q}$-divisor as in Theorem 4.3. Since $\mathcal{L} - D$ is ample and M is nef and π-big, the $\mathbb{Q}$-divisor $M + \pi^*(\mathcal{L} - D)$ is nef and big on P. The smoothness of π implies, by virtue of Lemma 4.2, that $\pi^*\mathcal{I}(D) = \mathcal{I}(\pi^*D)$. It follows that $H^1(P, (K_P + M + \pi^*\mathcal{L}) \otimes \pi^*\mathcal{I}(D)) = H^1(P, (K_P + M + \pi^*\mathcal{L}) \otimes \mathcal{I}(\pi^*D)) = \{0\}$, the second equality stemming from Ein's version of Nadel Vanishing Theorem Proposition 4.1.

Since $V(\mathcal{I}(D))$ has isolated support at x, we conclude that, if we denote by F_x the fiber of π over x:

$$H^0(P, K_P + M + \pi^*\mathcal{L}) \twoheadrightarrow H^0(F_x, (K_P + M + \pi^*\mathcal{L}) \otimes \mathcal{O}_{F_x}).$$

The result follows by the natural identification between the map given above and the map

$$H^0(X, \pi_*(K_P + M) \otimes \mathcal{L}) \to \pi_*(K_P + M) \otimes \mathcal{L} \otimes \mathcal{O}_X/\mathfrak{m}_x,$$

which holds because $R^1\pi_*(K_P + M) = 0$ is the zero sheaf by relative vanishing. □

Corollary 4.6 *Let X be a nonsingular projective variety of dimension n, E be a nef vector bundle on X, and $\mathcal{L}$ be an ample line bundle on X. Assume that*

$$\mathcal{L}^d \cdot Z > B(n+1,2)^d$$

for every d-dimensional integral cycle Z on X. Then $K_X \otimes E \otimes \det E \otimes \mathcal{L}$ is generated by its global sections at every point of X.
In particular, if L is any ample line bundle on X, then we can choose $\mathcal{L} = B(n+1,2)\,L$.

Proof. Set $P := \mathbb{P}(E)$, $\pi :=$ the natural projection onto X, $M := (r+1)\xi_E$, where r is the rank of E, and apply Theorem 4.5. □

Remark 4.7 A similar statement hold for the simultaneous generation at several points; see Remark 4.4. The same is true for Theorem 4.5.

Remark 4.8 The paper [2] contains similar results without the factor $\det E$. However, the assumption E nef is there replaced by a stronger curvature condition on E, and the methods are purely analytic

Remark 4.9 We do not know if similar statements hold without the factor $\det E$.

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Author's address: Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, 53225 Bonn, Germany. e-mail: markan@mpim-bonn.mpg.de