Max-Planck-Institut für Mathematik
Bonn

Quantum KdV hierarchy and quasimodular forms

by

Jan-Willem M. van Ittersum
Giulio Ruzza

Max-Planck-Institut für Mathematik
Preprint Series 2022 (12)

Date of submission: March 18, 2022

Quantum KdV hierarchy and quasimodular forms

by

Jan-Willem M. van Ittersum
Giulio Ruzza

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

IRMP
UC Louvain
Chemin du Cyclotron 2
1348 Louvain-la-Neuve
Belgium

MPIM 22-12

Quantum KdV hierarchy and quasimodular forms

Jan-Willem M. van Ittersum[1] and Giulio Ruzza[2]

[1] *Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany;* ittersum@mpim-bonn.mpg.de
[2] *IRMP, UCLouvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium;* giulio.ruzza@uclouvain.be

Abstract

Dubrovin [10] has shown that the spectrum of the quantization (with respect to the first Poisson structure) of the dispersionless Korteweg–de Vries (KdV) hierarchy is given by shifted symmetric functions; the latter are related by the Bloch–Okounkov Theorem [1] to quasimodular forms on the full modular group. We extend the relation to quasimodular forms to the full quantum KdV hierarchy (and to the more general Intermediate Long Wave hierarchy). These quantum integrable hierarchies have been described by Buryak and Rossi [6] in terms of the Double Ramification cycle in the moduli space of curves. The main tool and conceptual contribution of the paper is a general effective criterion for quasimodularity.

AMS Subject Classification (2020): 05A17, 11F11, 14H70, 37K10
Keywords: partitions, modular forms, double ramification cycles, quantum integrable hierarchies.

1 Introduction

1.1 Differential polynomials and q-series.

Let $\Lambda := \mathbb{Q}[\mathbf{p}]$ be the ring of polynomials with rational coefficients in the variables p_j, for $j \geq 1$, collectively denoted by $\mathbf{p} := (p_1, p_2, p_3, \dots)$. Assigning weight k to p_k we have the grading $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$, where Λ_n consists of polynomials of homogeneous weight n. For any linear operator $G \in \mathrm{End}\,(\Lambda)$ such that[1]

$$\left[G\,, \sum_{k\geq 1} k\, p_k \frac{\partial}{\partial p_k} \right] \;=\; 0, \tag{1.1}$$

i.e., such that G restricts to a linear operator on Λ_n for any $n \geq 0$, we introduce the q-series

$$\{G\}_q \;:=\; \frac{\sum_{n\geq 0} q^n \operatorname{tr}_{\Lambda_n} G}{\sum_{n\geq 0} q^n \dim \Lambda_n}. \tag{1.2}$$

Equivalently, $\{G\}_q = q^{-1/24}\eta(q) \sum_{n\geq 0} q^n \operatorname{tr}_{\Lambda_n} G$, where $\eta(q) = q^{1/24} \prod_{k\geq 1}(1-q^k)$ is the Dedekind eta function.

In certain cases, the q-series (1.2) specializes to the well-studied q-bracket, introduced in [1]; the latter is attached to a function $f : \mathscr{P} \to \mathbb{Q}$ from the set $\mathscr{P}$ of partitions to the rationals and defined by

$$\langle f \rangle_q \;:=\; \frac{\sum_{\lambda\in\mathscr{P}} q^{|\lambda|} f(\lambda)}{\sum_{\lambda\in\mathscr{P}} q^{|\lambda|}} \;=\; q^{-1/24}\eta(q) \sum_{\lambda\in\mathscr{P}} q^{|\lambda|} f(\lambda), \tag{1.3}$$

where $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$ denotes the integer $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a partition of. (Namely, if G is the diagonal operator acting as multiplication by $f(\lambda)$ on $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$, then $\{G\}_q = \langle f \rangle_q$.)

[1] Throughout this paper we denote by $[A, B] := AB - BA$ the commutator of A and B.

In particular, Bloch and Okounkov [1] proved that for the class of *shifted symmetric* functions of partitions, the q-bracket (1.3) is quasimodular of homogeneous weight; this class consists of homogeneous polynomials of certain basic functions $Q_k : \mathscr{P} \to \mathbb{Q}$, for $k \geq 0$, where the weight of Q_k is defined to be k. The basic functions are defined by $Q_0(\lambda) = 1$ and

$$Q_k(\lambda) := \frac{1}{(k-1)!} \sum_{i \geq 1} \left[\left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i + \tfrac{1}{2}\right)^{k-1} \right] + \beta_k \tag{1.4}$$

for $k \geq 1$, where $\beta_k = \left(\frac{1}{2^{k-1}} - 1\right) \frac{B_k}{k!}$. The central characters of the symmetric group are shifted symmetric functions [20] and hence these functions appear in the study of asymptotic properties of partitions [19, 26], as well as in many works in enumerative geometry, e.g., in the Hurwitz/Gromov–Witten theory of an elliptic curve [9, 13, 14, 16, 24, 25], or in the determination of the Siegel–Veech constants of the moduli space of flat surfaces [14, 7, 8].

The appearance of shifted symmetric functions in the study of integrable hierchies is part of an interesting story. As explained in [10], Eliashberg [11] solved the quantization problem for the classical Hopf hierarchy by using ideas coming from *Symplectic Field Theory*. Concretely, he constructed a commuting family of quantum Hamiltonians G_k^{Hopf} ($k \geq -2$), which are operators on Λ depending on a constant[2] c, obtained from differential polynomials by the procedure explained in the next paragraph. In particular, after inserting a *quantization* parameter $\hbar$ in these differential polynomials (see Remark 3.1) in the limit $\hbar \to 0$ they reduce to the Hamiltonian densities of the classical Hopf hierarchy. Rossi [27] showed that the operators G_k^{Hopf}, under the *boson-fermion correspondence* (see, e.g., [23]), are quadratic in fermions, a fact that was exploited by Dubrovin to diagonalize these operators (to then provide applications to the symplectic field theory of the disk). Namely, [10, Theorem 1.4]

$$G_k^{\mathsf{Hopf}} \, s_\lambda(\mathbf{p}) = E_k^{[0]}(\lambda) \, s_\lambda(\mathbf{p}), \qquad \lambda \in \mathscr{P}, \quad k \geq -2, \tag{1.5}$$

where the eigenvalues $E_k^{[0]} : \mathscr{P} \to \mathbb{Q}$ are shifted symmetric functions, given explicitly by

$$E_k^{[0]} = \sum_{j=0}^{k+2} \frac{c^{k+2-j}}{(k+2-j)!} Q_j, \tag{1.6}$$

and the eigenvectors $s_\lambda(\mathbf{p})$ are the Schur functions[3] [22], defined by

$$s_\lambda(\mathbf{p}) := \det\left[h_{\lambda_i - i + j}(\mathbf{p})\right]_{i,j=1}^{\ell(\lambda)}, \qquad \sum_{k \in \mathbb{Z}} y^k h_k(\mathbf{p}) = \exp\Big(\sum\nolimits_{k \geq 1} \frac{p_k}{k} y^k\Big). \tag{1.7}$$

It follows immediately from (1.5) and the Bloch–Okounkov Theorem that

$$\left\{ G_k^{\mathsf{Hopf}} \right\}_q = \sum_{j=0}^{k+2} \frac{c^{k+2-j}}{(k+2-j)!} \langle Q_j \rangle_q \tag{1.8}$$

is a polynomial in c with quasimodular coefficients; note that this expression is of homogeneous weight $k+2$, provided we consider the quasimodular weight as well as we assign weight $+1$ to c.

The motivation of the present work stems from the construction by Buryak and Rossi [6] of a deformation $G_k^{\mathsf{KdV}}(\epsilon)$ of the operators G_k^{Hopf}, depending (polynomially) on an additional parameter ϵ and satisfying $G_k^{\mathsf{KdV}}(0) = G_k^{\mathsf{Hopf}}$. These are the operators of the *quantum Korteweg–de Vries* (KdV) hierarchy[4], as the corresponding densities (again, after properly introducing the parameter $\hbar$, see Remark 3.1) reduce in the limit $\hbar \to 0$ to the Hamiltonian densities of the classical KdV hierarchy. (Incidentally, the construction in op. cit., which we briefly review in Section 3.1, is much more general, and produces a quantum integrable hierarchy attached to any Cohomological Field Theory; the KdV case corresponds to the trivial Cohomological Field Theory.)

[2] This constant is denoted u_0 in [10].

[3] Expressed in terms of the power sum polynomials.

[4] The quantization is with respect to the first Poisson structure.

Our first goal is to study whether the quasimodularity of the Hopf hierarchy, expressed by (1.8), survives under the deformation in ϵ (as anticipated in [28]). We answer this in the affirmative (Theorem 1.1), by providing a general criterion for quasimodularity (Theorem 1.2). Moreover, as a byproduct of the general criterion, we obtain a simplification of the quantum Hamiltonian operators which we expect to be useful in the study and classification of quantum integrable hierarchies of rank 1 [4].

We now move on to a more detailed explanation of our findings.

From differential polynomials to operators. In this paper we shall be concerned with quasi-modular properties of $\{G\}_q$ for operators G on Λ which are obtained out of a polynomial $g \in \mathbb{Q}[\mathbf{u}]$ by the following *quantization* procedure. Here $\mathbb{Q}[\mathbf{u}]$ is the ring of polynomials with rational coefficients in the variables u_j, for $j \geq 0$, collectively denoted by $\mathbf{u} := (u_0, u_1, u_2, \dots)$.

First, for $j \in \mathbb{Z}_{\geq 0}$, we introduce the Fourier series[5] $v_j(x) := \sum_{k\in\mathbb{Z}} (\mathrm{i}k)^j\, \omega_k\, \mathrm{e}^{\mathrm{i}kx}$, where we assign weight k to ω_k. Note that $v_j(x) = \partial_x^j v_0(x)$ for $j \geq 1$. Next, given $g \in \mathbb{Q}[\mathbf{u}]$, define a formal power series (of homogeneous weight 0) in the variables ω_k, $k \in \mathbb{Z}$, by

$$\int_0^{2\pi} g(\mathbf{v}(x)) \,\frac{\mathrm{d}x}{2\pi}, \qquad \mathbf{v}(x) \;:=\; (v_0(x), v_1(x), \dots). \tag{1.9}$$

Write all monomials in this series as a product of ω_k's where all the ω_k's with $k \geq 0$ appear to the left of all ω_k's with $k < 0$ (*normal ordering*); in this expression we finally replace ω_k with the operator $P_k \in \operatorname{End}(\Lambda)$, defined by

$$(P_k f)(\mathbf{p}) \;:=\; \begin{cases} p_k\, f(\mathbf{p}) & k \geq 1, \\ c\, f(\mathbf{p}) & k = 0, \\ -k\, \frac{\partial f(\mathbf{p})}{\partial p_{-k}}, & k \leq -1, \end{cases} \qquad (k \in \mathbb{Z}, f \in \Lambda) \tag{1.10}$$

with $c \in \mathbb{Q}$ an arbitrary constant. The operator on Λ (which also depends on the parameter c) obtained in this way out of $g \in \mathbb{Q}[\mathbf{u}]$ will be denoted $\overline{g} \;=\; \overline{g}(c)$.

It is worth noting that $\overline{g}$ might be complex-valued. More precisely, let $\mathbb{Q}[\mathbf{u}] = \mathbb{Q}[\mathbf{u}]^{\mathsf{even}} \oplus \mathbb{Q}[\mathbf{u}]^{\mathsf{odd}}$, where $\mathbb{Q}[\mathbf{u}]^{\mathsf{even}}$ (respectively, $\mathbb{Q}[\mathbf{u}]^{\mathsf{odd}}$) is the span of monomials which are even (respectively, odd) with respect to the weight operator $\sum_{j\geq 1} j u_j \frac{\partial}{\partial u_j}$; then $\overline{g}$ is purely real for $g \in \mathbb{Q}[\mathbf{u}]^{\mathsf{even}}$, and purely imaginary for $g \in \mathbb{Q}[\mathbf{u}]^{\mathsf{odd}}$.

Let us give a few examples:

- when g is in the image of the operator $\partial_x \;:=\; \sum_{j\geq 0} u_{j+1} \dfrac{\partial}{\partial u_j}$, we have $\overline{g} = 0$,
- $\overline{u_0} \;=\; c$,
- $\overline{u_0^2} \;=\; c^2 \;+\; 2\sum_{j\geq 1} j\, p_j\, \dfrac{\partial}{\partial p_j}$,
- $\overline{u_0^3} \;=\; c^3 \;+\; 6c\sum_{j\geq 1} j\, p_j\, \dfrac{\partial}{\partial p_j} \;+\; 6\Delta$, where

$$\Delta \;=\; \frac{1}{2} \sum_{j,k\geq 1} \Big((j+k)\, p_j\, p_k\, \frac{\partial}{\partial p_{j+k}} + j\,k\, p_{j+k}\, \frac{\partial^2}{\partial p_j \partial p_k} \Big) \tag{1.11}$$

  is the *cut-and-join* operator [15].

It is also possible to show (see [28, Lemma 2.3]) that any operator $\overline{g}$ is symmetric, i.e., $(v, \overline{g}w) = (\overline{g}v, w)$ for all $v, w \in \Lambda$ with respect to the standard scalar product $(\;,\;)$ on Λ (see [22]). The latter can be defined by

$$(p_\lambda, p_\mu) \;=\; z_\lambda \delta_{\lambda,\mu}, \tag{1.12}$$

[5] With the convention $0^0 = 1$.

where

$$p_\lambda := p_{\lambda_1} \cdots p_{\lambda_\ell} \tag{1.13}$$

is the monomial basis of Λ indexed by partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$, and $z_\lambda := \prod_{m\geq 1} r_m(\lambda)!\, m^{r_m(\lambda)}$, where $r_m(\lambda) := \#\{i \mid \lambda_i = m\}$.

1.2 Quantum Korteweg–de Vries hierarchy.

A construction by Buryak and Rossi [3, 5, 6], also inspired by previous works in Symplectic Field Theory [11, 12], provides an effective construction of quantum integrable hierarchies associated with an arbitrary Cohomological Field Theory (CohFT). Even though the construction is completely general, in this work we restrict to the case of rank 1 CohFTs only.

The output of this construction, which is briefly reviewed in Section 3.1, is a family of *Hamiltonian densities* $g_k(\mathbf{u};\epsilon) \in \mathbb{Q}[\mathbf{u}]^{\mathsf{even}} \otimes \mathbb{Q}[[\epsilon]]$, for $k \geq -2$, possibly depending on the parameters of the CohFT (more details below). They are determined by either an explicit formula in terms of the Double Ramification cycles in the moduli space of curves, see (3.3), or (more effectively) by a recurrence relation of order one, see (3.5) and (3.6).

One of the main properties of the *Hamiltonian operators* $G_k(\epsilon) := \overline{g_k(\mathbf{u};\epsilon)} \in \operatorname{End}(\Lambda) \otimes \mathbb{Q}[[\epsilon]]$ is that they enjoy the commutativity[6]

$$[G_j(\epsilon), G_k(\epsilon)] = 0, \qquad j, k \geq -2. \tag{1.14}$$

The relevance of this construction to the theory of integrable systems stems from the fact that, after introducing in a suitable way a quantization parameter $\hbar$ (see Remark 3.1), the Hamiltonian densities reduce, in the limit $\hbar \to 0$, to those of *classical* integrable hierarchies, well-studied in the literature. We refer to the aforementioned literature, and in particular to the Introduction of [10], for more details on this point. Since in this work we are mainly interested in the quantum Hamiltonian densities, we opted to simplify the exposition by dropping the parameter $\hbar$ (which can be reinstated at any time by the transformations described in Remark 3.1).

The simplest instance of this construction is provided by the quantum *Korteweg–de Vries* (KdV) hierarchy[7] (associated with the trivial CohFT), a prototypical example of integrable system; in this case the first few densities read

$$\begin{aligned}
g_{-2}^{\mathsf{KdV}}(\mathbf{u};\epsilon) &= 1, \qquad g_{-1}^{\mathsf{KdV}}(\mathbf{u};\epsilon) \;=\; u_0\,, \qquad g_{0}^{\mathsf{KdV}}(\mathbf{u};\epsilon) \;=\; \frac{u_0^2}{2} - \frac{1}{24} + \frac{\epsilon}{24}u_2\,,\\
g_{1}^{\mathsf{KdV}}(\mathbf{u};\epsilon) &= \frac{u_0^3}{6} - \frac{u_0}{24} - \frac{u_2}{24} + \frac{\epsilon}{24}\Big(u_0u_2 - \frac{1}{120}\Big) + \Big(\frac{\epsilon}{24}\Big)^2\frac{u_4}{2},\\
g_{2}^{\mathsf{KdV}}(\mathbf{u};\epsilon) &= \frac{u_0^4}{24} - \frac{u_0^2}{48} - \frac{u_0u_2}{24} + \frac{7}{5760} + \frac{\epsilon}{24}\left(\frac{u_0^2u_2}{2} - \frac{u_4}{30} - \frac{u_2}{24} - \frac{u_0}{120}\right) +\\
&\qquad + \Big(\frac{\epsilon}{24}\Big)^2\left(\frac{7u_2^2}{10} + \frac{u_0u_4}{2} - \frac{1}{210}\right) + \Big(\frac{\epsilon}{24}\Big)^3\frac{u_6}{6}.
\end{aligned} \tag{1.15}$$

(For the recursion determining them see (3.5) and (3.6) below, and for further properties see Appendix A.)

The Hamiltonian operators $G_k^{\mathsf{KdV}}(\epsilon) := \overline{g_k^{\mathsf{KdV}}(\mathbf{u};\epsilon)}$ are polynomials of degree k in ϵ; as already mentioned in Section 1.1, their constant term $G_k^{[0]} := [\epsilon^0]G_k^{\mathsf{KdV}}(\epsilon)$ coincides with the Hamiltonian operators of the Hopf hierarchy, i.e., $G_k^{[0]} = G_k^{\mathsf{Hopf}}$, and the quasimodularity of $\{G_k^{[0]}\}_q$ follows by Dubrovin's result (1.5) along with the Bloch–Okounkov theorem. On the other hand, their leading coefficient $G_k^{[\infty]} := [\epsilon^k]G_k^{\mathsf{KdV}}(\epsilon)$ is given by (see Corollary A.4)

$$G_k^{[\infty]} \;=\; \frac{c^2}{2}\delta_{k,0} + \frac{L_{2k+2}}{(-4)^k(2k+1)!!}, \qquad k \geq 0, \tag{1.16}$$

[6] It is appropriate to remark that the setting of [6] is more general, and the only requirement is the commutation relation $[P_a, P_b] = -a\delta_{a+b,0}$; it is convenient however for our purposes to fix the representation (1.10) (see also [10]).

[7] The quantization is with respect to the *first* Poisson structure of the classical KdV hierarchy.

where the operator L_k is

$$L_k \;:=\; -\frac{B_k}{2k} - \frac{\mathrm{i}^k}{2}\overline{u_0 u_{k-2}} \;=\; -\frac{B_k}{2k} + \sum_{j\geq 1} j^{k-1}\, p_j \frac{\partial}{\partial p_j}. \tag{1.17}$$

Therefore, $G_k^{[\infty]}$ are diagonal on the monomial basis (1.13) of Λ;

$$G_k^{[\infty]}\, p_\lambda \;=\; E_k^{[\infty]}(\lambda)\, p_\lambda, \qquad \lambda \in \mathscr{P}, \quad k \geq -2, \tag{1.18}$$

where $E_k^{[\infty]} : \mathscr{P} \to \mathbb{Q}$ are given in terms of the *moment functions* [32]

$$S_k(\lambda) \;:=\; -\frac{B_k}{2k} + \sum_{i=1}^{\ell(\lambda)} \lambda_i^{k-1}, \qquad k \geq 1, \tag{1.19}$$

by

$$E_k^{[\infty]} \;=\; \frac{c^2}{2}\delta_{k,0} \;+\; \frac{S_{2k+2}}{(-4)^k (2k+1)!!}, \qquad k \geq 0. \tag{1.20}$$

It was observed in [32] that the q-bracket of S_{2k+2} is an Eisenstein series which is quasimodular of weight $2k+2$.

Let us denote by $\widetilde{M}$ the ring of quasimodular forms (with rational coefficients) on the full modular group $\mathrm{SL}_2(\mathbb{Z})$ (see e.g., [31, Section 5.3]), containing, for each even $k \geq 2$ the Eisenstein series

$$\mathbb{G}_k \;=\; -\frac{B_k}{2k} + \sum_{m,r\geq 1} m^{k-1} q^{mr}. \tag{1.21}$$

Then $\widetilde{M} = \bigoplus \widetilde{M}_k$ is a graded ring freely generated over the rationals by the *Eisenstein series* $\mathbb{G}_2$, $\mathbb{G}_4$ and $\mathbb{G}_6$, where the weight of $\mathbb{G}_k$ is defined to be k. Moreover, let $\widetilde{M}[c,\epsilon] := \widetilde{M} \otimes \mathbb{Q}[c,\epsilon] =: \bigoplus_k \widetilde{M}[c,\epsilon]_k$, where we assign weight $+1$ to c and -1 to ϵ.

Theorem 1.1. *For the quantum KdV Hamiltonian operators*

$$G_k^{\mathsf{KdV}}(\epsilon) \;:=\; \overline{g_k^{\mathsf{KdV}}(\mathbf{u};\epsilon)} \in \mathrm{End}\,(\Lambda) \otimes \mathbb{Q}[\epsilon, c], \qquad (k \geq -2) \tag{1.22}$$

we have

$$\left\{G_k^{\mathsf{KdV}}(\epsilon)\right\}_q \;\in\; \widetilde{M}[c,\epsilon]_{k+2}\,. \tag{1.23}$$

The proof is given in Section 3.2, for the more general case of the quantum *Intermediate Long Wave* hierarchy, which is a generalization of the KdV hierarchy [6] (see Theorem 3.8). The key step in the proof is a general criterion (Theorem 1.2 below) for quasimodularity (of homogeneous weight) which applies to operators of the form $\overline{g}$ for $g \in \mathbb{Q}[\mathbf{u}]$.

Explicitly, we have the following expressions in terms of the Eisenstein series (1.21)

$$\begin{aligned}
&\left\{G_{-2}^{\mathsf{KdV}}(\epsilon)\right\}_q \;=\; 1, \qquad \left\{G_{-1}^{\mathsf{KdV}}(\epsilon)\right\}_q \;=\; c, \qquad \left\{G_{0}^{\mathsf{KdV}}(\epsilon)\right\}_q \;=\; \mathbb{G}_2 + \frac{c^2}{2},\\
&\left\{G_{1}^{\mathsf{KdV}}(\epsilon)\right\}_q \;=\; c\,\mathbb{G}_2 + \frac{c^3}{6} \;-\; \frac{\epsilon}{24}(2\mathbb{G}_4),\\
&\left\{G_{2}^{\mathsf{KdV}}(\epsilon)\right\}_q \;=\; \frac{1}{2}\mathbb{G}_2^2 + \frac{1}{12}\mathbb{G}_4 + \frac{c^2}{2}\mathbb{G}_2 + \frac{c^4}{24} \;+\; \frac{\epsilon}{24}(-2c\,\mathbb{G}_4) \;+\; \left(\frac{\epsilon}{24}\right)^2 \frac{12}{5}\mathbb{G}_6\,.
\end{aligned} \tag{1.24}$$

We expect that Theorem 1.1 holds true for all rank 1 quantum Double Ramification integrable hierarchies; namely, the quantum KdV Hamiltonian densities are the special case $s_i = 0$ of a more general hierarchy of Hamiltonian densities $g_k(\mathbf{u};\epsilon,\mathbf{s})$, depending on parameters $\mathbf{s}=(s_1,s_3,s_5,\dots)$ (see, Section 3.1). There exist different possible normalizations for these Hamiltonian densities (see, [4, eq. 5.3]), and by Theorem 1.2 below in all normalizations $\{\overline{g_k}(\epsilon,\mathbf{s})\}_q$ belongs to $\widetilde{M}[c][[\mathbf{s},\epsilon]]$. We expect that there exists a convenient normalization such that $\{\overline{g_k}(\epsilon,\mathbf{s})\}_q$ is quasimodular *of homogeneous weight* (where s_k, for k odd, is assigned weight k).

Moreover, as suggested to us by Don Zagier, we expect that the q-series (1.2) associated to arbitrary compositions of the quantum KdV operators give rise to quasimodular forms as well, or, even stronger, that the eigenvalues of the quantum KdV operators are shifted symmetric functions of homogeneous weight. Namely, by [28] there exists a simultaneous basis $r_\lambda(\mathbf{p};\epsilon)\in\Lambda_{|\lambda|}[[\epsilon]]$ of eigenfunctions $E_k(\lambda;\epsilon)$ for $G_k^{\mathsf{KdV}}(\epsilon)$ for all $\lambda\in\mathscr{P}$. Then, we expect that

$$E_k(\lambda;\epsilon)\;\in\;\mathbb{Q}[c,Q_0,Q_1,Q_2,\dots][[\epsilon]]_{k+2}\,, \tag{1.25}$$

where ϵ is assigned weight -1, c weight $+1$, and Q_k weight k. Note $E_k(\lambda;0)=E_k^{[0]}(\lambda)$ is the shifted symmetric function in (1.6) and as a consequence of Theorem 1.1 we have $\langle E_k(\lambda;\epsilon)\rangle_q=\{G_k^{\mathsf{KdV}}(\epsilon)\}_q\in\widetilde{M}[\epsilon,c]_{k+2}$. We have numerical evidence for (1.25) in a few instances, and we hope to return to it in a later publication.

1.3 A criterion for quasimodularity

We will show that the q-series $\{\overline{g}\}_q$ (for $g\in\mathbb{Q}[\mathbf{u}]$) is always a quasimodular form *of mixed weight*. Moreover, we provide a criterion for the quasimodularity of *homogeneous weight* for the q-series $\{\overline{g}\}_q$. To state it, we assign weight $k+1$ to u_k, so that $\mathbb{Q}[\mathbf{u}]$ (as well as its subspaces $\mathbb{Q}[\mathbf{u}]^{\mathsf{even}}$ and $\mathbb{Q}[\mathbf{u}]^{\mathsf{odd}}$ defined at page 3) become graded algebras[8]. Moreover, let $\widetilde{M}[c]:=\widetilde{M}\otimes\mathbb{Q}[c]=:\bigoplus_k\widetilde{M}[c]_k$, be the polynomial ring in c and ϵ over the graded ring of quasimodular forms, graded by the quasimodular weight and by assigning weight $+1$ to c.

Theorem 1.2. *For any $g\in\mathbb{Q}[\mathbf{u}]$ we have $\{\overline{g}(c)\}_q\in\widetilde{M}[c]$, and for any $g\in\mathbb{Q}[\mathbf{u}]^{\mathsf{odd}}$ we even have $\{\overline{g}(c)\}_q=0$. Moreover, let $\mathcal{B}$ be the linear operator on $\mathbb{Q}[\mathbf{u}]$ defined by*

$$\mathcal{B}\;:=\;\exp\Big(-\frac{1}{2}\sum_{i,j\geq 0}(-1)^{\frac{i-j}{2}}\,\frac{B_{i+j+2}}{i+j+2}\,\frac{\partial^2}{\partial u_i\partial u_j}\Big),\qquad B_k=k\text{th Bernoulli number.} \tag{1.26}$$

Then, the mapping $\mathbb{Q}[\mathbf{u}]\to\widetilde{M}[c]$

$$g\mapsto\{\overline{\mathcal{B}g}(c)\}_q \tag{1.27}$$

is a surjective morphism of graded vector spaces.

The proof is given in Section 2 and builds on the previous work [18] of the first author. In the special case $c=0$, the theorem states that $\{\overline{g}(0)\}_q$ is quasimodular of homogeneous weight if $\mathcal{B}^{-1}g$ is homogeneous.

Holomorphic anomaly equation. Just as in [17] we answer the question when $\{\overline{g}(0)\}_q$ is actually modular (rather than quasimodular). Namely, if we, instead, consider c to be a formal variable, the *holomorphic anomaly equation* of $\{\overline{g}\}_q$ (determining the failure of modularity) can be expressed as

$$-\,2\,\mathfrak{d}\,\{\overline{g}\}_q\;=\;\frac{\partial^2}{\partial c^2}\,\{\overline{g}\}_q\,, \tag{1.28}$$

[8]It might be more natural to write u_{k+1} for what is called u_k here, but in order to adhere to standard notation for the Hamiltonian densities introduced in the next section, we do not do so.

where $\mathfrak{d}$ is the unique derivation on quasimodular forms which vanishes on modular forms and for which $\mathfrak{d}(\mathbb{G}_2) = -\frac{1}{2}$, where $\mathbb{G}_2 = -\frac{1}{24} + \sum_{m,r\geq 1} m\, q^{mr}$ is the Eisenstein series of weight 2. Together with the differential operator $q\frac{\partial}{\partial q}$ and the weight operator, this derivation $\mathfrak{d}$ gives an action of $\mathfrak{sl}_2$ on quasimodular forms. Note that (1.28) can equivalently be written as $-2\mathfrak{d}\,\{\overline{g}\}_q = \{\overline{\partial^2 g/\partial u_0^2}\}_q$. Hence, $\{\overline{g}(0)\}_q$ is modular precisely if $\{\overline{\partial^2 g/\partial u_0^2}\}_q = 0$.

Remark 1.3. By (1.6) the holomorphic anomaly equation (1.28) can be explicitly checked in the limit $\epsilon \to 0$, because we know by [32, Theorem 3] that $\mathfrak{d}\,\langle Q_j\rangle_q = -\frac{1}{2}\,\langle Q_{j-2}\rangle_q$ for all $j \geq 2$. In the limit $\epsilon \to \infty$, the holomorphic anomaly equation (1.28) is consistent with (1.20), because, as mentioned earlier, in [32] it is shown that the q-bracket of S_{2k+2} is an Eisenstein series of weight $2k+2$, and every Eisenstein series of weight $2k+2$ with $k > 0$ is actually modular.

Functions on partitions having the same q-bracket. The q-bracket (1.3) is not only a specialization of the q-series (1.2); it is also related to this q-series by the following construction. For any basis $b_\lambda(\mathbf{p})$ of Λ, indexed by partitions $\lambda \in \mathscr{P}$ and satisfying $b_\lambda \in \Lambda_{|\lambda|}$, and for any $G \in \mathrm{End}\,(\Lambda)$ satisfying (1.1) we have the equality $\{G\}_q = \langle f\rangle_q$ where $f(\lambda)$ is defined to be the λth diagonal entry of the matrix representation of G, i.e., $f(\lambda) := (b_\lambda, Gb_\lambda)/(b_\lambda, b_\lambda)$, where the inner product is defined by (1.12). Hence, by studying $\{G\}_q$ for $G \in \mathrm{End}\,(\Lambda)$, we study the q-brackets $\langle f\rangle_q$ for many functions f at the same time.

Incidentally, this reflects the fact that different functions $f : \mathscr{P} \to \mathbb{Q}$ can have the same q-bracket. For example, as observed in [7, Section 13] the moment functions (1.19) and the shifted symmetric functions[9] (in terms of the generators Q_k, defined by (1.4))

$$T_k(\lambda) \;:=\; \frac{(k-2)!}{2}\sum_{i=0}^{k}(-1)^i\, Q_i(\lambda)\, Q_{k-i}(\lambda), \qquad k \geq 2, \tag{1.29}$$

are two instances of functions on partitions for which, in case k is even, the q-bracket equals the holomorphic Eisenstein series (1.21). This particular example can be explained as S_k is the so-called *Möller transform* of [32] of T_k. Now, the Möller transform corresponds to the change of coordinates between the monomial basis p_λ and the Schur basis s_λ of Λ. Indeed, we have[10]

$$S_k = (p_\lambda, L_k p_\lambda)/(p_\lambda, p_\lambda), \qquad T_k = (s_\lambda, L_k s_\lambda)/(s_\lambda, s_\lambda), \tag{1.30}$$

for all $k \geq 1$, where L_k is the operator defined in (1.17).

Outline of the rest of the paper.

In Section 2 we prove Theorem 1.2 and the holomorphic anomaly equation (1.28). In Section 3.1 we review the construction of Double Ramification quantum integrable hierarchies; in Section 3.2 we prove Theorem 3.8 which is a generalization of Theorem 1.1. In Appendix A we give explicit formulas for the limits $\epsilon \to 0, \infty$ of the quantum KdV Hamiltonian densities.

2 Partitions and quasimodular forms

2.1 Proof of Theorem 1.2

Proof of Theorem 1.2. Recall that the $p_\lambda = p_{\lambda_1}\cdots p_{\lambda_r}$ for $\lambda \in \mathscr{P}$ form a basis for Λ. The main observation is that with respect to this basis, for all $\mathbf{k} \in \mathbb{N}^r, \mathbf{l} \in \mathbb{N}^s$ and $\lambda \in \mathscr{P}$ we have

$$[p_\lambda]\, p_{k_1}\cdots p_{k_r}\,\frac{\partial^n}{\partial p_{l_1}\cdots\partial p_{l_s}} p_\lambda \;=\; \begin{cases} \prod_m \binom{r_m(\lambda)}{r_m(\mathbf{l})}\, r_m(\mathbf{l})! & \mathbf{k} \text{ is a permutation of } \mathbf{l},\\ 0 & \text{else,}\end{cases} \tag{2.1}$$

[9]These shifted symmetric functions have a nice interpretation as the moments of the hook-lengths of partitions, see [7, Section 13].

[10]The second equality follows from [7, Section 13] after expanding s_λ in the monomial basis of Λ.

where $[p_\lambda]$ indicates we extract the coefficient of p_λ. Here, $r_m(\lambda) = \#\{i \mid \lambda_i = m\}$ and similarly $r_m(\mathbf{l}) = \#\{i \mid l_i = m\}$.

Given a monomial $u_\mathbf{a} = u_{a_1} \cdots u_{a_n} \in \mathbb{Q}[\mathbf{u}]$, consider

$$\overline{u_\mathbf{a}} \;=\; \sum_{\substack{\mathbf{k}\in\mathbb{Z}^n \\ |\mathbf{k}|=0}} (\mathrm{i}\mathbf{k})^\mathbf{a}\, c^{\#\{j \mid k_j=0\}} \prod_{j \mid k_j>0} p_{k_j} \prod_{j \mid k_j<0} \Bigl(-k_j \frac{\partial}{\partial p_{-k_j}}\Bigr). \tag{2.2}$$

Write $\Pi_2(n,m)$ for the partitions of all $B \subset \{1,2,\ldots,n\}$ in m sets of two elements, i.e., $\pi \in \Pi_2(n,m)$ can be written as $\pi = \{A_1,\ldots,A_m\}$ with $|A_i| = 2$ for all i and $\bigcup A_i = B(\pi)$ for some $B(\pi) \subset \{1,\ldots,n\}$ with $|B(\pi)| = 2m$. Observe that $|\Pi_2(n,m)| = \binom{n}{2m}\cdot(2m-1)!!$. For a set S, write $a_S = \sum_{i\in S} a_i$ and $s(\mathbf{a},S) = \mathrm{i}^{a_S}\sum_{i\in S}(-1)^{a_i}$. We write $|\mathbf{a}|$ for $a_{\{1,\ldots,n\}} = \sum_{i=1}^n a_i$. Also, for $\mathbf{k}\in\mathbb{Z}^m$ write $z_\mathbf{k} = \prod_m r_m(\mathbf{k})!\, m^{r_m(\mathbf{k})}$. Then,

$$\begin{aligned}
[p_\lambda]\,\overline{u_\mathbf{a}}\,p_\lambda \;&=\; [p_\lambda]\sum_{m=0}^{n/2} c^{n-2m} \sum_{\substack{\pi\in\Pi_2(n,m)\\ a_{B(\pi)}=|\mathbf{a}|}} \sum_{k_{A_1},\ldots,k_{A_m}\geq 1} \frac{1}{z_\mathbf{k}} \prod_{A\in\pi}\Bigl(s(\mathbf{a},A)\,k_A^{a_A+2}\,p_{k_A}\Bigr) \prod_{A\in\pi}\frac{\partial}{\partial p_{k_A}} p_\lambda \\
&=\; \sum_{m=0}^{n/2} c^{n-2m} \sum_{\substack{\pi\in\Pi_2(n,m)\\ a_{B(\pi)}=|\mathbf{a}|}} \sum_{k_{A_1},\ldots,k_{A_m}\geq 1} \prod_{A\in\pi} s(\mathbf{a},A)\,k_r^{a_A+1} \prod_m \binom{r_m(\lambda)}{r_m(\mathbf{k})}.
\end{aligned} \tag{2.3}$$

From Lemma 2.1 below, it follows that

$$\{\overline{u_\mathbf{a}}\}_q \;=\; \sum_{m=0}^{n/2} c^{n-2m} \sum_{\substack{\pi\in\Pi_2(n,m)\\ a_{B(\pi)}=|\mathbf{a}|}} \prod_{A\in\pi} s(\mathbf{a},A)\Bigl(\frac{B_k}{2k} + \mathbb{G}_{a_A+2}\Bigr), \tag{2.4}$$

where $\mathbb{G}_k$ for $k\geq 2$ is the holomorphic Eisenstein series (1.21) of weight k.

Observe that if $\mathbf{a} = (i,j)$ for some $i,j\in\mathbb{Z}$, then

$$s(\mathbf{a},\{1,2\}) \;=\; (-1)^{\frac{i+j}{2}}\bigl((-1)^i + (-1)^j\bigr) \;=\; \begin{cases} 2(-1)^{\frac{i-j}{2}} & i\equiv j \mod 2 \\ 0 & \text{else.}\end{cases} \tag{2.5}$$

Hence, $s(\mathbf{a},A)$ vanishes if a_A+2 is odd. As the Eisenstein series $\mathbb{G}_k$ are (quasi)modular for even k, we have shown that $\{\overline{g}(c)\}_q \in \widetilde{M}[c]$ for all $g\in\mathbb{Q}[\mathbf{u}]$.

Next, assume $u_\mathbf{a}\in\mathbb{Q}[\mathbf{u}]^{\mathsf{odd}}$. Then $|\mathbf{a}|$ is odd. Hence, by (2.4) we see that $\{u_\mathbf{a}\}_q$ is a polynomial in c with quasimodular forms of mixed *odd* weights as its coefficients, i.e., $\{u_\mathbf{a}\}_q = 0$. Therefore, we have shown that $\{\overline{g}(c)\}_q = 0$ for all $g\in\mathbb{Q}[\mathbf{u}]^{\mathsf{odd}}$.

By definition of $\mathcal{B}$ we deduce from (2.4) that

$$\{\overline{\mathcal{B}u_\mathbf{a}}\}_q \;=\; \sum_{m=0}^{n/2} c^{n-2m} \sum_{\substack{\pi\in\Pi_2(n,m)\\ a_{B(\pi)}=|\mathbf{a}|}} \prod_{A\in\pi} s(\mathbf{a},A)\,\mathbb{G}_{a_A+2}\,, \tag{2.6}$$

which is of homogeneous weight $|\mathbf{a}|+n$ (where n is the length of the vector $\mathbf{a}$). Therefore, it follows that $g\mapsto\{\overline{\mathcal{B}g}(c)\}_q$ is a morphism of graded vector spaces. Moreover, as every quasimodular form has a (highly non-unique) respresentation as a polynomial in Eisenstein series, it follows that the q-bracket is surjective on $\widetilde{M}$. □

In order to state Lemma 2.1 needed in the above proof, we define the $\underline{x}$-*bracket*[11] of a function $f:\mathscr{P}\to\mathbb{Q}$ by

$$\langle f\rangle_{\underline{x}} \;=\; \frac{\sum_{\lambda\in\mathscr{P}} f(\lambda)\,x_\lambda}{\sum_{\lambda\in\mathscr{P}} x_\lambda} \qquad (x_\lambda = x_{\lambda_1}x_{\lambda_2}\cdots). \tag{2.7}$$

[11]In [18] this was called the $\underline{u}$-bracket.

Then, one has $\langle f\rangle_q = \langle f\rangle_{(q,q^2,q^3,\dots)}$. Observe that the $\underline{x}$-bracket defines an isomorphism of vector spaces (but not of algebras!)

$$\mathbb{Q}^{\mathscr{P}} \xrightarrow{\sim} \mathbb{Q}[[x_1,x_2,x_3,\dots]], \qquad f \mapsto \langle f\rangle_{\underline{x}}\,. \tag{2.8}$$

Given $f,g\in\mathbb{Q}^{\mathscr{P}}$ we define their *induced product* $f\odot g$ by

$$\langle f\odot g\rangle_{\underline{x}} \;=\; \langle f\rangle_{\underline{x}}\,\langle g\rangle_{\underline{x}}\,, \tag{2.9}$$

where the product of $\langle f\rangle_{\underline{x}}$ and $\langle g\rangle_{\underline{x}}$ is the usual product of power series. In particular, $\langle f\odot g\rangle_q = \langle f\rangle_q\,\langle g\rangle_q$. The following result completes the proof of Theorem 1.2 (after setting $x_k = q^k$ for all k).

Lemma 2.1. *Given $n\geq 0$ and $\mathbf{k}\in(\mathbb{Z}_{>0})^n$, we have*

$$\Big\langle \prod_{m\geq 1}\binom{r_m(\lambda)}{r_m(\mathbf{k})}\Big\rangle_{\underline{x}} \;=\; \prod_{k\in\mathbf{k}}\Big(\sum_{r=1}^{\infty} x_k^r\Big) \quad \textit{or equivalently} \quad \prod_{m\geq1}\binom{r_m(\lambda)}{r_m(\mathbf{k})} \;=\; \bigodot_{k\in\mathbf{k}} r_k(\lambda). \tag{2.10}$$

Proof. See [18, Proposition 7.2.3(ii)]. □

2.2 Holomorphic anomaly equation

A *quasimodular form* is a holomorphic function $f=f(\tau)$ of τ in the complex upper half plane, admitting a Fourier series (q-series, $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$) at infinity and such that for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$

$$(c\tau+d)^{-k} f\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; \sum_{j=0}^{p}\frac{(\mathfrak{d}^j f)(\tau)}{j!}\Big(\frac{1}{2\pi i}\frac{c}{c\tau+d}\Big)^j, \qquad \operatorname{Im}\tau>0, \tag{2.11}$$

where $\mathfrak{d}$ is the derivation defined in the introduction[12]. The transformation (2.11) of the quasimodular forms in Theorem 1.2 is determined by the following holomorphic anomaly equation.

Proposition 2.2. *For all $g\in\mathbb{Q}[\mathbf{u}]$, we have*

$$-\,2\,\mathfrak{d}\,\{\overline{g}(c)\}_q \;=\; \frac{\partial^2}{\partial c^2}\,\{\overline{g}(c)\}_q \;=\; \left\{\overline{\frac{\partial^2 g}{\partial u_0^2}}\right\}_q . \tag{2.12}$$

Proof. Recall that by (2.6) we have

$$\{\overline{\mathcal{B}u_{\mathbf{a}}}\}_q \;=\; \sum_{m=0}^{n/2} c^{n-2m} \sum_{\substack{\pi\in\Pi_2(n,m)\\ a_{B(\pi)}=|\mathbf{a}|}}\;\prod_{A\in\pi} s(\mathbf{a},A)\,\mathbb{G}_{a_A+2}\,. \tag{2.13}$$

For the first equation, it suffices to show that $-2\,\mathfrak{d}$ and $\frac{\partial^2}{\partial c^2}$ agree on $\{\overline{\mathcal{B}u_{\mathbf{a}}}\}_q$. We compute

$$-\,2\,\mathfrak{d}\{\overline{\mathcal{B}u_{\mathbf{a}}}\}_q \;=\; 2\sum_{m=0}^{n/2} c^{n-2m} \sum_{\substack{\pi\in\Pi_2(n,m)\\ a_{B(\pi)}=|\mathbf{a}|}}\;\sum_{\substack{A'\in\pi\\ a_{A'}=0}}\;\prod_{\substack{A\in\pi\\ A\neq A'}} s(\mathbf{a},A)\,\mathbb{G}_{a_A+2}\,, \tag{2.14}$$

where we used that $\mathfrak{d}$ is a derivation which annilates $\mathbb{G}_{a_A+2}$ except if $a_A=0$. Now, let $\pi'\in\Pi_2(n,m-1)$ be the partition after removing A' from π. Note that there are $\binom{n-2m+2}{2}$ partitions π

[12]This is not meant to be a defintion of *a quasimodular form*, as in the definition of $\mathfrak{d}$, and hence also here, we make use of the fact that $\widetilde{M}=M[\mathbb{G}_2]$, where M denotes the space of holomorphic *modular* forms.

yielding π'. Hence, the first equality follows from the computation

$$-2\,\mathfrak{d}\{\overline{\mathcal{B}u_{\mathbf{a}}}\}_q \;=\; 2\sum_{m=0}^{n/2} c^{n-2m}\binom{n-2m+2}{2}\sum_{\substack{\pi'\in\Pi_2(n,m-1)\\ a_{B(\pi)}=|\mathbf{a}|}}\prod_{A\in\pi'} s(\mathbf{a},A)\,\mathbb{G}_{a_A+2} \tag{2.15}$$

$$=\; \frac{\partial^2}{\partial c^2}\sum_{m=1}^{n/2} c^{n-2m+2}\sum_{\substack{\pi'\in\Pi_2(n,m-1)\\ a_{B(\pi)}=|\mathbf{a}|}}\prod_{A\in\pi'} s(\mathbf{a},A)\,\mathbb{G}_{a_A+2} \tag{2.16}$$

$$=\; \frac{\partial^2}{\partial c^2}\{\overline{\mathcal{B}u_{\mathbf{a}}}\}_q\,. \tag{2.17}$$

The second equality follows immediately from the next independent lemma. $\square$

Lemma 2.3. *For any $g\in\mathbb{Q}[\mathbf{u}]$ we have $\frac{\partial\bar{g}}{\partial c} = \overline{\frac{\partial g}{\partial u_0}}$.*

Proof. For any $g(\mathbf{u})$ the operator $\overline{g(u_0-c,u_1,\dots)}$ is independent of c by construction. Hence

$$\bar{g}(c) \;=\; \sum_{s\geq 0}\frac{c^s}{s!}\,\overline{(\partial_{u_0})^s g(u_0-c,u_1,\dots)} \tag{2.18}$$

and so

$$\partial_c\,\bar{g}(c) \;=\; \sum_{s\geq 1}\frac{c^{s-1}}{(s-1)!}\overline{(\partial_{u_0})^s g(u_0-c,u_1,\dots)} \;=\; \sum_{s\geq 0}\frac{c^s}{s!}\overline{(\partial_{u_0})^s(\partial_{u_0}g)(u_0-c,u_1,\dots)} = \overline{\frac{\partial g}{\partial u_0}}, \tag{2.19}$$

as desired. $\square$

3 Applications to quantum integrable hierarchies

3.1 Double Ramification quantum integrable hierarchies

A Cohomological Field Theory (CohFT) [21] consists of

- a finite-dimensional $\mathbb{Q}$-vector space V, equipped with a non-degenerate symmetric two-form $\eta\in\mathrm{Sym}^2(V^*)$ and with a distinguished element $\mathbf{1}\in V$, and
- linear maps $c_{g,n}: V^{\otimes n}\to H^{\mathsf{even}}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$, indexed by $g,n\geq 0$ such that $2g-2+n\geq 0$.

Here, $\overline{\mathcal{M}}_{g,n}$ is the Deligne–Mumford moduli space of stable curves of genus g with n marked points and $H^{\mathsf{even}}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ is the even part of its rational cohomology ring. The maps $c_{g,n}$ have to satisfy a number of axioms prescribing their behavior under natural maps between the moduli spaces, i.e., under permutation of marked points, forgetting of marked points, and gluing of curves; for more details see [3, Section 3] and references therein.

In [6] a family of Hamiltonian densities $g_k(\mathbf{u};\epsilon)$ is defined starting from an arbitrary CohFT. We shall consider here only the case of one-dimensional CohFTs, namely $V=\mathbb{Q}\mathbf{1}$, under the additional assumption $\eta(\mathbf{1}\otimes\mathbf{1})=1$; by a slight abuse of notation we shall denote $c_{g,n}$ the value at $\mathbf{1}^{\otimes n}$. A result of Teleman [30] implies that all such CohFTs are given by

$$c_{g,n} \;=\; \exp\Bigl(\sum\nolimits_{j\geq 1} s_{2j-1}\,\mathrm{ch}_{2j-1}(\mathbb{E})\Bigr), \tag{3.1}$$

in terms of parameters s_k, for $k\geq 1$ and odd, where $\mathrm{ch}_k(\mathbb{E})\in H^k(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ are the Chern characters of the Hodge bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{g,n}$. The densities are defined by[13]

$$g_{-2}(\mathbf{u};\epsilon) \;:=\; 1, \qquad g_{-1}(\mathbf{u};\epsilon) \;:=\; u_0\,, \tag{3.2}$$

[13] The density $g_{-2}=1$ is not introduced in [6].

and $g_k(\mathbf{u};\epsilon)$ for $k\geq 0$ is defined in terms of the Fourier series (already used in the introduction) $v_j(x)=\sum_{\ell\in\mathbb{Z}}(\mathrm{i}\ell)^j\,\omega_\ell\,\mathrm{e}^{\mathrm{i}\ell x}$, for $j\geq 0$, by requiring that

$$g_k(\mathbf{v}(x);\epsilon) \;:=\; \sum_{\substack{g,n\geq 0\\ 2g-2+n\geq 0}} \frac{1}{n!} \sum_{(a_1,\dots,a_n)\in\mathbb{Z}^n} \int_{\mathrm{DR}_g(-|a|,a_1,\dots,a_n)} \psi_1^k\,\Lambda_g^\vee(\epsilon)\,c_{g,n+1}\,\omega_{a_1}\dots\omega_{a_n}\,\mathrm{e}^{\mathrm{i}|a|x}, \tag{3.3}$$

where $|a|:=\sum_{i=1}^n a_i$ and $\mathbf{v}(x)\;:=\;(v_0(x),v_1(x),\dots)$; see [5, 6] for more details. Here, we use the following standard notations (see, e.g., [6] and references therein for more details):

- $\mathrm{DR}_g(a_0,a_1,\dots,a_n)\in H_{2g-2+n}(\overline{\mathcal{M}}_{g,n+1},\mathbb{Q})$ is the *Double Ramification cycle* (roughly speaking, defined as the locus in $\overline{\mathcal{M}}_{g,n+1}$ of stable curves with marked points $(C;p_0,\dots,p_n)$ such that $\mathcal{O}_C\simeq\mathcal{O}_C(\sum_{i=0}^n a_ip_i)$),
- $\psi_1\in H^2(\overline{\mathcal{M}}_{g,n+1},\mathbb{Q})$ is the Chern class of the cotangent line bundle at the first marked point, and
- $\Lambda_g^\vee(\epsilon)=1-\epsilon\lambda_1+\dots+(-\epsilon)^g\lambda_g$, where $\lambda_i\in H^{2i}(\overline{\mathcal{M}}_{g,n+1},\mathbb{Q})$ are the Chern classes of the Hodge bundle.

It is worth pointing out that all densities $g_k(\mathbf{u};\epsilon)$ are sums of even monomials with respect to the weight operator $\sum_{j\geq 0}u_j\frac{\partial}{\partial u_j}$, i.e., $g_k(\mathbf{u};\epsilon)\in\mathbb{Q}[\mathbf{u}]^{\mathsf{even}}[\epsilon]$ (see [6, Appendix B]).

Remark 3.1. To simplify the exposition we have omitted the quantization parameter $\hbar$ of [6]; the normalization of op. cit. can be recovered from (3.3) by the transformations

$$\epsilon\mapsto\epsilon(\mathrm{i}\hbar)^{-\frac{1}{2}},\qquad s_k\mapsto s_k(\mathrm{i}\hbar)^{\frac{k}{2}},\qquad u_k\mapsto u_k(\mathrm{i}\hbar)^{-\frac{1}{2}},\qquad g_k\mapsto(\mathrm{i}\hbar)^{\frac{k+2}{2}}g_k\,. \tag{3.4}$$

This follows directly from the dimensional constraints of the integrals over the Double Ramification cycle in (3.3). Moreover, the parameter ϵ in this paper corresponds to ε^2 in [6]. To compare with the normalization of [10], where $\epsilon=0$, we need to replace $\mathrm{i}\hbar$ with $\hbar$.

It is also follows from [6, Theorem 3.5 and Lemma 3.7], combined with Remark 3.1 (see also [28, Section 4]), that the densities $g_k(\mathbf{u};\epsilon)$ in (3.3) can be determined from $g_{-1}(\mathbf{u};\epsilon)=u_0$ by the recursion

$$\frac{\partial g_{k+1}(\mathbf{u};\epsilon)}{\partial u_0}\;=g_k(\mathbf{u};\epsilon), \tag{3.5}$$

$$(k+2+\mathcal{D})\partial_x g_{k+1}(\mathbf{u};\epsilon)\;=[g_k(\mathbf{u};\epsilon),G_1(\epsilon)],\qquad k\geq -1, \tag{3.6}$$

provided one has computed $G_1(\epsilon)=\overline{g_1(\mathbf{u};\epsilon)}$ in advance. Here

$$\partial_x\;:=\;\sum_{i\geq 0}u_{i+1}\frac{\partial}{\partial u_i},\qquad \mathcal{D}\;:=\;\epsilon\frac{\partial}{\partial\epsilon}\;+\;\sum_{i\geq 1}(2i-1)s_{2i-1}\frac{\partial}{\partial s_{2i-1}}, \tag{3.7}$$

and for $f,g\in\mathbb{Q}[\mathbf{u}][[\epsilon]]$ we have [6, Equation 2.2]

$$[f(\mathbf{u};\epsilon),\overline{g}(\epsilon)]\;=\;\sum_{n\geq 1}\frac{(-1)^{n-1}}{n!}\sum_{\substack{r_1,\dots,r_n\geq 0\\ s_1,\dots,s_n\geq 0}}\frac{\partial^n f}{\partial u_{s_1}\cdots\partial u_{s_n}}(-1)^{|\mathbf{r}|}P_{r_1+s_1+1,\dots,r_n+s_n+1}(\partial_x)\,\frac{\partial^n g}{\partial u_{r_1}\cdots\partial u_{r_n}}, \tag{3.8}$$

where $|\mathbf{r}|=\sum_{i=1}^n r_i$ and $P_{\ell_1,\dots,\ell_n}(\xi)$ are polynomials in ξ defined by the sequence of their coefficients;

$$[\xi^j]\,P_{\ell_1,\dots,\ell_n}(\xi)\;=\;\begin{cases}(-1)^{\frac{n-1-j+|\ell|}{2}}\;[\xi^j]\widetilde{P}_{\ell_1,\dots,\ell_n}(\xi) & n-1-j+|\ell|\text{ is even,}\\ 0 & \text{otherwise,}\end{cases} \tag{3.9}$$

and $\widetilde{P}_{\ell_1,\dots,\ell_n}(\xi)$ are polynomials in ξ determined by their values at positive integers ξ;

$$\widetilde{P}_{\ell_1,\dots,\ell_n}(\xi)\;=\;\sum_{\substack{a_1,\dots,a_n\geq 0\\ a_1+\dots+a_n=\xi}}a_1^{\ell_1}\cdots a_n^{\ell_n}. \tag{3.10}$$

Later we shall need the following particular cases.

Lemma 3.2. *We have $P_\ell(\xi) = \xi^\ell$ and*

$$P_{\ell,m}(\xi) = \frac{\ell!\,m!}{(\ell+m+1)!}\xi^{\ell+m+1} + \\ + \sum_{i\geq 0}\frac{B_{2i+2}}{2i+2}\left((-1)^{\ell+i}\binom{\ell}{2i+1-m} + (-1)^{m+i}\binom{m}{2i+1-\ell}\right)\xi^{\ell+m-2i-1}. \quad (3.11)$$

Proof. (See also [18, Lemma 6.1.2].) Only (3.11) needs a proof. We compute the generating series (for ξ integer)

$$\mathbf{P}(u,v;\xi) := \sum_{\ell,m\geq 0}\widetilde{P}_{\ell,m}(\xi)\frac{u^\ell}{\ell!}\frac{v^m}{m!} = \sum_{a=0}^{\xi}\mathrm{e}^{au}\mathrm{e}^{(\xi-a)v} = \mathrm{e}^{\xi v}\sum_{a=0}^{\xi}\mathrm{e}^{a(u-v)} = \frac{\mathrm{e}^{u(\xi+1)}-\mathrm{e}^{v(\xi+1)}}{\mathrm{e}^u-\mathrm{e}^v}. \quad (3.12)$$

The proof is complete by Taylor expanding $\mathbf{P}(u,v;\xi) = \frac{\mathrm{e}^{u\xi}-1}{\mathrm{e}^{u-v}-1} + \frac{\mathrm{e}^{v\xi}-1}{\mathrm{e}^{v-u}-1}$ using $\frac{z}{\mathrm{e}^z-1} = \sum_{j\geq 0}B_j\frac{z^j}{j!}$ and by (3.9). □

Remark 3.3. It is explained in [6, Section 3.5] that the recursion equations (3.5) and (3.6) uniquely determines the g_k's, for all $k\geq -1$, from $g_{-1} = u_0$ and G_1. Indeed, one first uses (3.6) to determine the g_k's for $k\geq -1$ up to a constant depending on $k,\epsilon,\mathbf{s}$ only; indeed, this yet undetermined constant does not affect the right-hand side of (3.6) so that the recursion works. These constants are finally determined by (3.5).

3.2 Quantum Intermediate Long Wave hierarchy

The quntum Intermediate Long Wave hierarchy corresponds to the construction of Buryak and Rossi for the *Hodge* CohFT (see also [2])

$$c_{g,n} = 1 + \mu\lambda_1 + \cdots + \mu^g\lambda_g, \quad (3.13)$$

where μ is a parameter and $\lambda_k \in H^{2k}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ are the Chern classes of the Hodge bundle. In terms of the parameters $\mathbf{s}$ in (3.1) we have $s_{2i-1} = (2i-2)!\,\mu^{2i-1}$, whence (3.7) reduces to

$$\mathcal{D} = \epsilon\frac{\partial}{\partial\epsilon} + \mu\frac{\partial}{\partial\mu}. \quad (3.14)$$

Let us denote $g_k^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu)$ and $G_k^{\mathsf{ILW}}(\epsilon,\mu) := \overline{g_k^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu)}$ the densities and operators for this hierarchy; we know from [6, Lemma 4.2] that[14]

$$G_1^{\mathsf{ILW}}(\epsilon,\mu) = \overline{\frac{u_0^3}{6} - \frac{u_0}{24} + (\epsilon-\mu)\sum_{g\geq 1}(\epsilon\mu)^{g-1}\frac{|B_{2g}|}{2(2g)!}\left(u_0u_{2g} - \frac{|B_{2g+2}|}{2g+2}\right)}. \quad (3.15)$$

We start by making the recursion (3.6) more explicit in this case.

Lemma 3.4. *The operator*

$$\mathcal{R}^{\mathsf{ILW}} : g(\mathbf{u};\epsilon,\mu) \mapsto [g(\mathbf{u};\epsilon,\mu), G_1^{\mathsf{ILW}}(\epsilon,\mu)] \quad (3.16)$$

can be spelled out as

$$\mathcal{R}^{\mathsf{ILW}} = \mathcal{R}_1^{\mathsf{ILW}} + \mathcal{R}_2^{\mathsf{ILW}}, \quad (3.17)$$

where

$$\mathcal{R}_1^{\mathsf{ILW}} = \sum_{i\geq 0}\partial_x^{i+1}\left(\frac{u_0^2}{2} + (\epsilon-\mu)\sum_{g\geq 1}(\epsilon\mu)^{g-1}\frac{|B_{2g}|}{(2g)!}u_{2g}\right)\frac{\partial}{\partial u_i} - \frac{1}{2}\sum_{i,j\geq 0}\frac{(i+1)!(j+1)!}{(i+j+3)!}u_{i+j+3}\frac{\partial^2}{\partial u_i\partial u_j}, \quad (3.18)$$

$$\mathcal{R}_2^{\mathsf{ILW}} = \sum_{i,j,l\geq 0}\frac{B_{2l+2}}{2l+2}\left((-1)^{i+l}\binom{i+1}{2l-j} + (-1)^{j+l}\binom{j+1}{2l-i}\right)u_{i+j+1-2l}\frac{\partial^2}{\partial u_i\partial u_j}. \quad (3.19)$$

[14] In loc. cit. the formula is given up to constant terms.

Proof. It follows directly from (3.8) and Lemma 3.2. □

Introduce the *reduced* ILW densities

$$\widetilde{g}_k^{\mathsf{ILW}} := \mathcal{B}^{-1} g_k^{\mathsf{ILW}}, \tag{3.20}$$

where $\mathcal{B}$ is the the operator given in (1.26). Remarkably, they satisfy a similar but slightly simpler recursion.

Lemma 3.5. *The reduced ILW densities $\widetilde{g}_k^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu)$ are uniquely determined from $\widetilde{g}_{-2}^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu) = 1$ and $\widetilde{g}_{-1}^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu) = u_0$ by the recursion*

$$\frac{\partial \widetilde{g}_{k+1}^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu)}{\partial u_0} = \widetilde{g}_k^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu), \tag{3.21}$$

$$(k+2+\mathcal{D})\partial_x \widetilde{g}_{k+1}^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu) = \mathcal{R}_1^{\mathsf{ILW}} \widetilde{g}_k^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu), \qquad k \geq -1, \tag{3.22}$$

where $\mathcal{R}_1^{\mathsf{ILW}}$ is given in (3.18).

Proof. For the sake of clarity, let us drop the superscript ILW and the dependence on $\mathbf{u},\epsilon,\mu$ in this proof. The equation (3.21) follows from the chain of equalities

$$\frac{\partial}{\partial u_0}\widetilde{g}_{k+1} = \mathcal{B}^{-1}\frac{\partial}{\partial u_0} g_{k+1} = \mathcal{B}^{-1} g_k = \widetilde{g}_k, \tag{3.23}$$

where we use $[\mathcal{B}, \partial_{u_0}] = 0$. Next, for any power series $\Phi(\xi_0,\xi_1,\dots)$ we have $[\Phi(\partial_{u_0},\partial_{u_1},\dots), u_j] = (\partial_{\xi_j}\Phi)(\partial_{u_0},\partial_{u_1},\dots)$. In particular when $\Phi(\xi_0,\xi_1,\cdots) = \exp(\pm\frac{1}{2}\sum_{i,j\geq 0}\nu_{i,j}\xi_i\xi_j)$, with

$$\nu_{i,j} := (-1)^{\frac{i-j}{2}} \frac{B_{i+j+2}}{i+j+2}, \tag{3.24}$$

we get

$$[\mathcal{B}^{\pm 1}, u_j] = \mp \sum_{i\geq 0} \nu_{i,j} \frac{\partial}{\partial u_i} \mathcal{B}^{\pm 1}. \tag{3.25}$$

We claim that $[\mathcal{B}, (k+2+\mathcal{D})\partial_x] = 0$; indeed, by (3.25) we have

$$[\mathcal{B}, \partial_x] = -\sum_{i,j\geq 0} \nu_{i,j+1} \frac{\partial^2}{\partial u_i \partial u_j} \mathcal{B} = 0, \tag{3.26}$$

because $\nu_{i,j+1} = -\nu_{j,i+1}$, and $[\mathcal{B},\mathcal{D}] = 0$. Therefore,

$$(k+2+\mathcal{D})\partial_x \widetilde{g}_{k+1} = \mathcal{B}^{-1}(k+2+\mathcal{D})\partial_x g_{k+1} = \mathcal{B}^{-1}\mathcal{R} g_k = \mathcal{B}^{-1}\mathcal{R}\mathcal{B}\widetilde{g}_k \tag{3.27}$$

The proof is complete once we show the identity $\mathcal{B}^{-1}\mathcal{R}\mathcal{B} = \mathcal{R}_1$, or, equivalently,

$$[\mathcal{B}, \mathcal{R}_1] = \mathcal{R}_2 \mathcal{B}. \tag{3.28}$$

The operator $\mathcal{R}_1$ consists of three parts, see (3.18); namely, $\mathcal{R}_1 = \mathcal{R}_1^{(a)} + \mathcal{R}_1^{(b)} + \mathcal{R}_1^{(c)}$ with

$$\mathcal{R}_1^{(a)} = \sum_{\ell\geq 0}(\epsilon-\mu)\sum_{g\geq 1}(\epsilon\mu)^{g-1}\frac{|B_{2g}|}{(2g)!} u_{\ell+2g+1}\frac{\partial}{\partial u_\ell}, \tag{3.29}$$

$$\mathcal{R}_1^{(b)} = \frac{1}{2}\sum_{\ell\geq 0}\sum_{k=0}^{\ell+1}\binom{\ell+1}{k} u_k u_{\ell+1-k}\frac{\partial}{\partial u_\ell}, \tag{3.30}$$

$$\mathcal{R}_1^{(c)} = -\frac{1}{2}\sum_{\ell,m\geq 0}\frac{(\ell+1)!(m+1)!}{(\ell+m+3)!} u_{\ell+m+3}\frac{\partial^2}{\partial u_\ell \partial u_m}. \tag{3.31}$$

We compute separately each contribution, using (3.25).

(a) We have

$$[\mathcal{B},\mathcal{R}_1^{(a)}] \;=\; (\epsilon-\mu)\sum_{\ell\geq 0}\sum_{g\geq 1}(\epsilon\mu)^{g-1}\frac{|B_{2g}|}{(2g)!}[\mathcal{B},u_{\ell+2g+1}]\frac{\partial}{\partial u_\ell}$$
$$\;=\; (\epsilon-\mu)\sum_{i,\ell\geq 0}\sum_{g\geq 1}(\epsilon\mu)^{g-1}\frac{|B_{2g}|}{(2g)!}\nu_{i,\ell+2g+1}\frac{\partial^2}{\partial u_i\partial u_\ell}\mathcal{B} \;=\; 0, \tag{3.32}$$

because $\nu_{i,\ell+k}=-\nu_{\ell,i+k}$ for any odd k.

(b) We first compute

$$\begin{aligned}
[\mathcal{B},u_{j_1}u_{j_2}] \;&=\; u_{j_1}[\mathcal{B},u_{j_2}] \;+\; [\mathcal{B},u_{j_1}]u_{j_2}\\
&=\; -\sum_{i\geq 0}\Big(\nu_{i,j_2}u_{j_1}\frac{\partial}{\partial u_i}\mathcal{B}+\nu_{i,j_1}\frac{\partial}{\partial u_i}\mathcal{B}u_{j_2}\Big)\\
&=\; -\sum_{i\geq 0}\Big(\nu_{i,j_2}u_{j_1}\frac{\partial}{\partial u_i}+\nu_{i,j_1}\frac{\partial}{\partial u_i}u_{j_2}\Big)\mathcal{B} \;-\; \sum_{i\geq 0}\nu_{i,j_1}\frac{\partial}{\partial u_i}[\mathcal{B},u_{j_2}]\\
&=\; -\Bigg(\sum_{i\geq 0}\Big(\nu_{i,j_2}u_{j_1}\frac{\partial}{\partial u_i}+\nu_{i,j_1}u_{j_2}\frac{\partial}{\partial u_i}\Big)+\nu_{j_1,j_2}-\sum_{i_1,i_2\geq 0}\nu_{i_1,j_1}\nu_{i_2,j_2}\frac{\partial^2}{\partial u_{i_1}\partial u_{i_2}}\Bigg)\mathcal{B}.
\end{aligned} \tag{3.33}$$

Therefore,

$$\begin{aligned}
[\mathcal{B},\mathcal{R}_1^{(b)}]\mathcal{B}^{-1} \;&=\frac{1}{2}\sum_{\ell\geq 0}\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}[\mathcal{B},u_ku_{\ell+1-k}]\mathcal{B}^{-1}\frac{\partial}{\partial u_\ell}\\
&=\; -\frac{1}{2}\sum_{i,\ell\geq 0}\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}\Big(\nu_{i,\ell+1-k}u_k+\nu_{i,k}u_{\ell+1-k}\Big)\frac{\partial^2}{\partial u_i\partial u_\ell}+\\
&\quad -\frac{1}{2}\sum_{\ell\geq 0}\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}\nu_{k,\ell+1-k}\frac{\partial}{\partial u_\ell}+\\
&\quad +\frac{1}{2}\sum_{i_1,i_2,\ell\geq 0}\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}\nu_{i_1,k}\nu_{i_2,\ell+1-k}\frac{\partial^3}{\partial u_{i_1}\partial u_{i_2}\partial u_\ell}.
\end{aligned} \tag{3.34}$$

Since

$$\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}\nu_{k,\ell+1-k} \;=\; (-1)^{\frac{\ell+1}{2}}\frac{B_{\ell+3}}{\ell+3}\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}(-1)^k \;=\; 0, \qquad \ell\geq 0, \tag{3.35}$$

the second sum in (3.34) vanishes for all $\ell\geq 0$.

(c) Finally, we have

$$[\mathcal{B},\mathcal{R}_1^{(c)}] \;=\; \frac{1}{2}\sum_{i,\ell,m\geq 0}\frac{(\ell+1)!(m+1)!}{(\ell+m+3)!}\nu_{i,\ell+m+3}\frac{\partial^3}{\partial u_i\,\partial u_\ell\,\partial u_m}\mathcal{B}. \tag{3.36}$$

Combining these three computations we obtain, denoting for convenience $b_k := B_k/k$,

$$\begin{aligned}
[\mathcal{B},\mathcal{R}_1]\mathcal{B}^{-1} \;&=\; -\frac{1}{2}\sum_{i,\ell\geq 0}\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}\Big((-1)^{\frac{i+k-\ell-1}{2}}b_{i+k+\ell+3}u_k+(-1)^{\frac{i-k}{2}}b_{i+k+2}u_{\ell+1-k}\Big)\frac{\partial^2}{\partial u_i\partial u_\ell}+\\
&\quad +\frac{1}{2}\sum_{i_1,i_2,\ell\geq 0}\sum_{k=0}^{\ell+1}\binom{\ell+1}{k}(-1)^{\frac{i_1+i_2-\ell-1}{2}}b_{i_1+k+2}b_{i_2+\ell+3-k}\frac{\partial^3}{\partial u_{i_1}\partial u_{i_2}\partial u_\ell}+\\
&\quad +\frac{1}{2}\sum_{i,\ell,m\geq 0}\frac{(\ell+1)!(m+1)!}{(\ell+m+3)!}(-1)^{\frac{\ell+m+3-i}{2}}b_{i+\ell+m+5}\frac{\partial^3}{\partial u_i\partial u_\ell\partial u_m}.
\end{aligned} \tag{3.37}$$

It is easy to check that the first line on the right-hand side of (3.37) equals $\mathcal{R}_2$ (see (3.19)). To complete the proof we need to show that the last two lines in (3.37) cancel each other; to this end we first recall the following theorem by Skoruppa.

Theorem 3.6 ([29]). *Suppose that the symmetric homogeneous bivariate polynomial $H(x,y) = \sum_{\nu=0}^{n} h_\nu x^\nu y^{n-\nu}$ of even positive degree n satisfies $H(x,y) = H(y, y-x)$. Then*

$$\sum_{\substack{0<\nu<n \\ \nu \text{ odd}}} h_\nu \, \mathbb{G}_{\nu+1} \, \mathbb{G}_{n+1-\nu} \;=\; h \, \mathbb{G}_{n+2} \;-\; \frac{h_1}{n} \, q \frac{\mathrm{d}}{\mathrm{d}q} \mathbb{G}_n, \tag{3.38}$$

where $h := -\frac{1}{2}\int_0^1 H(1,y)\,\mathrm{d}y$ and $\mathbb{G}_n$ is the Eisenstein series (1.21).

Since $\mathbb{G}_n$ has a Fourier series in the upper half plane whose constant term is $-b_n/2$ we get

$$\sum_{\substack{0<\nu<n \\ \nu \text{ odd}}} h_\nu \, b_{\nu+1} \, b_{n+1-\nu} \;=\; -2\, h \, b_{n+2}, \tag{3.39}$$

with the notations of Theorem 3.6. We need the following specialization: given positive integers a_1, a_2, a_3 with $a_1 + a_2 + a_3$ even, the polynomial

$$\begin{aligned} H(x,y) \;&:=\; \sum_{\pi \in S_3} x^{a_{\pi(1)}} (-y)^{a_{\pi(2)}} (y-x)^{a_{\pi(3)}} \\ &=\; \sum_{\pi \in S_3} \sum_{k=0}^{a_{\pi(3)}} \binom{a_{\pi(3)}}{k} (-1)^{a_{\pi(2)}+k} x^{a_{\pi(1)}+k} y^{a_{\pi(2)}+a_{\pi(3)}-k} \end{aligned} \tag{3.40}$$

satisfies the condition of Theorem 3.6, because

$$\begin{aligned} H(y, y-x) \;&=\; \sum_{\pi \in S_3} y^{a_{\pi(1)}} (x-y)^{a_{\pi(2)}} (-x)^{a_{\pi(3)}} \\ &=\; (-1)^{a_1+a_2+a_3} \sum_{\pi \in S_3} (-y)^{a_{\pi(1)}} (y-x)^{a_{\pi(2)}} (x)^{a_{\pi(3)}} \;=\; H(x,y). \end{aligned} \tag{3.41}$$

Hence, by (3.39)

$$\begin{aligned} &\sum_{\pi \in S_3} \sum_{k=0}^{a_{\pi(3)}} \binom{a_{\pi(3)}}{k} (-1)^{a_{\pi(2)}+k} b_{a_{\pi(1)}+k+1} \, b_{a_{\pi(2)}+a_{\pi(3)}-k+1} \;= \\ &\qquad\qquad b_{a_1+a_2+a_3+2} \sum_{\pi \in S_3} (-1)^{a_{\pi(1)}} \frac{a_{\pi(2)}! \, a_{\pi(3)}!}{(a_{\pi(2)} + a_{\pi(3)} + 1)!}. \end{aligned} \tag{3.42}$$

Note that $a_{\pi(2)} + a_{\pi(3)} - k + 1$ is even in the left-hand side of the last identity; therefore, multiplying both sides by $(-1)^{\frac{a_1+a_2+a_3}{2}}$, we may write it as

$$\begin{aligned} &\sum_{\pi \in S_3} \sum_{k=0}^{a_{\pi(3)}} \binom{a_{\pi(3)}}{k} (-1)^{\frac{a_{\pi(1)}+a_{\pi(2)}-a_{\pi(3)}}{2}} b_{a_{\pi(1)}+k+1} \, b_{a_{\pi(2)}+a_{\pi(3)}-k+1} \\ &\qquad\qquad = -b_{a_1+a_2+a_3+2} \sum_{\pi \in S_3} (-1)^{\frac{a_{\pi(2)}+a_{\pi(3)}-a_{\pi(1)}}{2}} \frac{a_{\pi(2)}! \, a_{\pi(3)}!}{(a_{\pi(2)} + a_{\pi(3)} + 1)!}. \end{aligned} \tag{3.43}$$

It is clear by this identity that the two cubic operators in the $\partial/\partial u_i$'s, which appear in the second and third line of (3.37), cancel each other. ☐

Corollary 3.7. *For all $k \geq -2$, the reduced density $\widetilde{g}_k^{\mathsf{ILW}}(\mathbf{u};\epsilon)$ is homogeneous of weight $k+2$ if we assign weight $i+1$ to u_i, $+1$ to c, and -1 to ϵ and μ.*

Proof. Let $\mathcal{W} := \sum_{i\geq 0}(i+1)u_i\frac{\partial}{\partial u_i} - \epsilon\frac{\partial}{\partial\epsilon} - \mu\frac{\partial}{\partial\mu}$ the grading operator with respect to the weights of the statement; we need to prove that $\mathcal{W}\widetilde{g}_k^{\mathsf{ILW}} = (k+2)\widetilde{g}_k^{\mathsf{ILW}}$ for $k\geq -1$ (the case $k=-2$ being trivial). It is straightforward to verify the commutation relations

$$\left[\mathcal{W}, \frac{\partial}{\partial u_0}\right] = -\frac{\partial}{\partial u_0}, \qquad [\mathcal{W}, (k+2+\mathcal{D})\partial_x] = (k+2+\mathcal{D})\partial_x, \qquad [\mathcal{W}, \mathcal{R}_1^{\mathsf{ILW}}] = 2\mathcal{R}_1^{\mathsf{ILW}}, \tag{3.44}$$

where $\mathcal{D}$ is given in (3.14). It follows from these relations and (3.21)-(3.22) that the polynomials $f_k := \mathcal{W}\widetilde{g}_k^{\mathsf{ILW}}$ satisfy the recursion

$$\begin{aligned} \frac{\partial}{\partial u_0} f_{k+1} &= f_k + \widetilde{g}_k^{\mathsf{ILW}}, & (3.45)\\ (k+2+\mathcal{D})\partial_x f_{k+1} &= \mathcal{R}_1^{\mathsf{ILW}}(f_k + \widetilde{g}_k^{\mathsf{ILW}}), \qquad k\geq -1, & (3.46)\end{aligned}$$

with initial condition $f_{-1}=u_0$. This recursion uniquely determines (by an argument parallel to that in Remark 3.3) all f_k's, for $k\geq -1$. On the other hand, this recursion is satisfied by $f_k = (k+2)\widetilde{g}_k$, and the proof is complete. □

Let $\widetilde{M}[c,\epsilon,\mu] := \widetilde{M}\otimes\mathbb{Q}[c,\epsilon,\mu] =: \bigoplus_k \widetilde{M}[c,\epsilon,\mu]_{k+2}$, graded by the quasimodular weight and by assigning weight $+1$ to c and -1 to ϵ and μ. The central result of this section now follows directly from Theorem 1.2 and Corollary 3.7.

Theorem 3.8. *For all $k\geq -2$, we have*

$$\left\{G_k^{\mathsf{ILW}}(\epsilon,\mu)\right\}_q \in \widetilde{M}[c,\epsilon,\mu]_{k+2}. \tag{3.47}$$

The quantum KdV hierarchy mentioned in Section 1 corresponds to the special case $\mu=0$ of the ILW hierarchy;

$$g_k^{\mathsf{KdV}}(\mathbf{u};\epsilon) = g_k^{\mathsf{ILW}}(\mathbf{u};\epsilon,\mu)\Big|_{\mu=0}, \qquad G_k^{\mathsf{KdV}}(\epsilon) = G_k^{\mathsf{ILW}}(\epsilon,\mu)\Big|_{\mu=0}. \tag{3.48}$$

Therefore Theorem 1.1 is a direct corollary of Theorem 3.8.

Acknowledgments

We are indebted to Boris Dubrovin, Di Yang and Don Zagier for valuable conversations and suggestions. The work of GR is supported by the Fonds de la Recherche Scientifique-FNRS under EOS project O013018F.

A Closed formulas for the quantum KdV hierarchy for $\epsilon\to 0$ and $\epsilon\to\infty$

A.1 Dispersionless limit $\epsilon\to 0$.

An explicit generating function for the Hamiltonian densities in the case $\epsilon=0$ is due to Eliashberg, see [11] and [6, Proposition 4.1]. Namely, it is known that

$$\sum_{k\geq 2} y^{k+2} g_k^{\mathsf{KdV}}(\mathbf{u};\epsilon=0) = \frac{\exp(y\,S(\mathrm{i}y\partial_x)u_0)}{S(y)}, \qquad S(y) := \frac{\sinh(y/2)}{y/2} = \sum_{k\geq 0}\frac{y^{2k}}{4^k(2k+1)!}, \tag{A.1}$$

where ∂_x is defined by (3.7). We observe here the simplification of this formula when we consider the reduced densities instead.

Proposition A.1. *Denoting $\widetilde{g}_k^{\mathsf{KdV}} := \mathcal{B}^{-1} g_k^{\mathsf{KdV}}$, we have*

$$\sum_{k\geq 2} y^{k+2}\, \widetilde{g}_k^{\mathsf{KdV}}(\mathbf{u};\epsilon=0) \;=\; \exp(y\, S(\mathrm{i}y\partial_x)u_0). \tag{A.2}$$

Remark A.2. By assigning degree -1 to y we see directly that the right-hand side in (A.2) is of homogeneous weight zero (provided u_k has weight $k+1$), hence we recover that $\widetilde{g}_k^{\mathsf{KdV}}(\mathbf{u};\epsilon=0)$ has degree $k+2$. Actually, since by the definition of the operator $\mathcal{B}$ we have $g_k^{\mathsf{KdV}} = \widetilde{g}_k^{\mathsf{KdV}} +$ lower degree terms, it is not difficult to derive (A.2) from (A.1).

A.2 Limit $\epsilon \to \infty$.

We also provide formulas for the (sub)leading terms in $g_k^{\mathsf{KdV}}(\mathbf{u};\epsilon)$ as $\epsilon \to \infty$.

Proposition A.3. *For all $k \geq -1$, the density $g_k^{\mathsf{KdV}}(\mathbf{u};\epsilon)$ is a polynomial in ϵ of degree $k+1$, whose leading and subleading terms are given by*

$$[\epsilon^{k+1}]\, g_k^{\mathsf{KdV}}(\mathbf{u};\epsilon) \;=\; \frac{u_{2k+2}}{24^{k+1}(k+1)!}, \qquad k\geq -1, \tag{A.3}$$

$$[\epsilon^{k}]\, g_k^{\mathsf{KdV}}(\mathbf{u};\epsilon) \;=\; -\frac{1}{(-4)^k(2k+1)!!}\frac{B_{2k+2}}{4k+4} + \frac{1}{24^k}\sum_{j=0}^{k} d(j,k)u_j u_{2k-j}\,, \qquad k\geq 0, \tag{A.4}$$

where the coefficients $d(j,k)$ (for $j,k>0$) are given by the generating series

$$\sum_{j,k\geq 0}(2k+1)!!\, d(j,k)\, y^j x^k \;=\; \frac{1}{2\sqrt{1-2x(1+y)^2}\,(1-2x(1-y+y^2))} \tag{A.5}$$

and in particular satisfy $d(j,k)=0$ for $j>2k$ and $d(j,k)=d(2k-j,k)$ for $j,k\geq 0$ with $j\leq 2k$.

Proof. It is straightforward from the definition of $\mathcal{B}$ and the identity

$$\sum_{j=1}^{2k}(-1)^j\, d(j,k) \;=\; \frac{1}{(2k+1)!!}\,[x^k]\frac{1}{2(1-6x)} \;=\; \frac{6^k}{2(2k+1)!!} \tag{A.6}$$

to check that the statement is equivalent to the fact that the reduced density $\widetilde{g}_k^{\mathsf{KdV}} := \mathcal{B}^{-1} g_k^{\mathsf{KdV}}$ is a polynomial in ϵ of degree $k+1$ for $k\geq -1$ whose leading and subleading terms are given by

$$[\epsilon^{k+1}]\, \widetilde{g}_k^{\mathsf{KdV}}(\mathbf{u};\epsilon) \;=\; \frac{u_{2k+2}}{24^{k+1}(k+1)!}, \qquad k\geq -1, \tag{A.7}$$

$$[\epsilon^{k}]\, \widetilde{g}_k^{\mathsf{KdV}}(\mathbf{u};\epsilon) \;=\; \frac{1}{24^k}\sum_{j=0}^{k} d(j,k)u_j u_{2k-j}\,, \qquad k\geq 0. \tag{A.8}$$

Therefore, it suffices to show (A.7) and (A.8). The $\widetilde{g}_k$'s are determined by (3.21) and (3.22) with $\mu=0$, namely,

$$\Big(k+2+\epsilon\frac{\partial}{\partial\epsilon}\Big)\partial_x \widetilde{g}_{k+1}^{\mathsf{KdV}} \;=\; (\mathcal{R}_0+\epsilon\,\mathcal{R}_1)\widetilde{g}_k^{\mathsf{KdV}}, \qquad \frac{\partial \widetilde{g}_{k+1}^{\mathsf{KdV}}}{\partial u_0} \;=\; \widetilde{g}_k^{\mathsf{KdV}}, \qquad k\geq -1, \tag{A.9}$$

with initial condition $\widetilde{g}_{-1} = u_0$, where

$$\mathcal{R}_0 \;:=\frac{1}{2}\sum_{i,j\geq 0}\left(\frac{(i+1)!}{(i+1-j)!j!}u_{i+1-j}u_j\frac{\partial}{\partial u_i} - \frac{(i+1)!(j+1)!}{(i+j+3)!}u_{i+j+3}\frac{\partial^2}{\partial u_i\partial u_j}\right), \tag{A.10}$$

$$\mathcal{R}_1 \;:=\frac{1}{12}\sum_{i\geq 0}u_{i+3}\frac{\partial}{\partial u_i}. \tag{A.11}$$

It follows that $\widetilde{g}_k$ is a polynomial of degree at most $k+1$ in ϵ. Moreover, the leading term satisfies the recursion

$$(2k+4)\,\partial_x\big([\epsilon^{k+2}]\widetilde{g}^{\mathsf{KdV}}_{k+1}\big) \;=\; \mathcal{R}_1\big([\epsilon^{k+1}]\widetilde{g}^{\mathsf{KdV}}_k\big), \qquad k \geq -1, \tag{A.12}$$

with $[\epsilon^0]\,\widetilde{g}^{\mathsf{KdV}}_{-1} = u_0$. Hence, for $k \geq -1$ it follows that $[\epsilon^{k+1}]\,\widetilde{g}^{\mathsf{KdV}}_k(\mathbf{u};\epsilon) = \frac{u_{2k+2}}{24^{k+1}(k+1)!} + c_k$ for some constants c_k depending on k only; by Corollary 3.7, $[\epsilon^{k+1}]\,\widetilde{g}^{\mathsf{KdV}}_k(\mathbf{u};\epsilon)$ must be of homogeneous weight $2k+3$, hence $c_k = 0$. Then the subleading term is determined by the recursion

$$(2k+3)\,\partial_x\big([\epsilon^{k+1}]\,\widetilde{g}^{\mathsf{KdV}}_{k+1}\big) \;=\; \frac{\mathcal{R}_0(u_{2k+2})}{24^{k+1}(k+1)!} + \mathcal{R}_1\big([\epsilon^{k}]\,\widetilde{g}^{\mathsf{KdV}}_k\big), \qquad k \geq 0 \tag{A.13}$$

with $[\epsilon^0]\widetilde{g}^{\mathsf{KdV}}_0 = u_0^2/2$. Therefore, $[\epsilon^k]\,\widetilde{g}^{\mathsf{KdV}}_k = 24^{-k}\sum_{j=0}^{2k} d(j,k)u_j u_{2k-j}$ for some coefficients $d(j,k)$; again, this is in principle only true up to a constant depending on k only, however Corollary 3.7 again implies that this constant vanishes. Here the coefficients are assumed to satisfy $d(j,k) = d(2k-j,k)$ and, as a consequence of (A.13), are subject to the recurrence

$$(2k+3)(d(j,k+1)+d(j-1,k+1)) = \frac{1}{2(k+1)!}\binom{2k+3}{j} + 2(d(j-3,k)+d(j,k)), \quad j,k \geq 0, \tag{A.14}$$

where $d(j,k) = 0$ for $j < 0$ or $j > 2k$, with initial condition $d(0,0) = 1/2$. Therefore

$$(2k+3)(1+y)\,\Delta_{k+1}(y) \;=\; \frac{(1+y)^{2k+3}}{2(k+1)!} + 2(1+y^3)\,\Delta_k(y), \qquad \Delta_k(y) \;:=\; \sum_{j=0}^{2k} d(j,k)\,y^j. \tag{A.15}$$

Dividing by $(1+y)$, multiplying by $x^{k+1}(2k+1)!!$, and summing over $k \geq 0$ we obtain

$$D(x,y) - \frac{1}{2} \;=\; \frac{1}{2}\left(\frac{1}{\sqrt{1-2x(y+1)^2}} - 1\right) \;+\; 2x(1-y+y^2)D(x,y), \tag{A.16}$$

where $D(x,y) := \sum_{k\geq 0}(2k+1)!!\,\Delta_k(y)\,x^k$, and (A.8) follows. □

In particular, we obtain the following immediate consequence, whose proof is omitted.

Corollary A.4. *$G^{\mathsf{KdV}}_k(\epsilon)$ is a polynomial in ϵ of degree k for $k \geq 0$ whose leading coefficient is*

$$[\epsilon^k]\,G^{\mathsf{KdV}}_k(\epsilon) \;=\; \frac{c^2}{2}\delta_{k,0} \;+\; \frac{1}{(-4)^k(2k+1)!!}\left(-\frac{B_{2k+2}}{4k+4} + \sum_{j\geq 1} j^{2k+1}p_j\frac{\partial}{\partial p_j}\right), \qquad k \geq 0. \tag{A.17}$$

References

[1] S. Bloch & A. Okounkov. "The character of the infinite wedge representation". *Adv. Math.* 149 (2000), no. 1, 1–60.

[2] G. Bonelli, A. Sciarappa, A. Tanzini, & P. Vasko. "Six-dimensional supersymmetric gauge theories, quantum cohomology of instanton moduli spaces and gl(N) Quantum Intermediate Long Wave Hydrodynamics". *J. High Energ. Phys.* 2014 (2014), no. 141, 29 pp.

[3] A. Buryak. "Double ramification cycles and integrable hierarchies". *Comm. Math. Phys.* 336 (2015), no. 3, 1085–1107.

[4] A. Buryak, B. Dubrovin, J. Guéré, & P. Rossi. "Integrable systems of double ramification type". *Int. Math. Res. Not. IMRN* 2020, no. 24, 10381–10446.

[5] A. Buryak & P. Rossi. "Recursion relations for double ramification hierarchies". *Comm. Math. Phys.* 342 (2016), no. 2, 533–568.

[6] A. Buryak & P. Rossi. "Double ramification cycles and quantum integrable systems". *Lett. Math. Phys.* 106 (2016), no. 3, 289–317.

[7] D. Chen, M. Möller, & D. Zagier, "Quasimodularity and large genus limits of Siegel–Veech constants," *J. Amer. Math. Soc.* 31 (2018), no. 4, 1059–1163.

[8] D. Chen, M. Möller, A. Sauvaget, & D. Zagier, "Masur–Veech volumes and intersection theory on moduli spaces of Abelian differentials," *Invent. Math.* 222 (2020), no. 1, 283–373.

[9] R. Dijkgraaf, "Mirror symmetry and elliptic curves," *The moduli space of curves (Texel Island, 1994)*, 149–163, *Progr. Math.*, 129, Birkhäuser Boston, Boston, MA, 1995.

[10] B. Dubrovin. "Symplectic field theory of a disk, quantum integrable systems, and Schur polynomials". *Ann. Henri Poincaré* 17 (2016), no. 7, 1595–1613.

[11] Y. Eliashberg. "Symplectic field theory and its applications". *International Congress of Mathematicians. Vol. I*, 217–246, *Eur. Math. Soc., Zürich*, 2007.

[12] Y. Eliashberg, A. Givental, & H. Hofer. "Introduction to symplectic field theory". *Geom. Funct. Anal.* 2000, Special Volume, Part II, 560–673.

[13] P. Engel, "Hurwitz theory of elliptic orbifolds, I," *Geom. Topol.* 25 (2021), no. 1, 229–274.

[14] A. Eskin & A. Okounkov, "Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials," *Invent. Math.* 145 (2001), no. 1, 59–103.

[15] I. P. Goulden. "A differential operator for symmetric functions and the combinatorics of multiplying transpositions". Trans. Amer. Math. Soc. 344 (1994), no. 1, 421–440.

[16] M. A. Hahn, J.-W. M. van Ittersum, & F. Leid, "Triply mixed coverings of arbitrary base curves: Quasimodularity, quantum curves and recursions," *ArXiv e-prints:1901.03598* (2019), to appear in Ann. Henri Poincaré D, 41 pp.

[17] J.-W. M. van Ittersum. "When is the Bloch–Okounkov q-bracket modular?". *Ramanujan J.* 52 (2020), no. 3, 669–682.

[18] J.-W. M. van Ittersum. "A symmetric Bloch–Okounkov theorem". *Res. Math. Sci.* 8 (2021), no. 2, Paper No. 19, 42 pp.

[19] V. Ivanov & G. Olshanski, "Kerov's central limit theorem for the Plancherel measure on Young diagrams," *Symmetric functions 2001: surveys of developments and perspectives*, 93–151, vol. 74 of *NATO Sci. Ser. II Math. Phys. Chem.*, Kluwer Acad. Publ., Dordrecht, 2002.

[20] S. Kerov & G. Olshanski, "Polynomial functions on the set of Young diagrams.," *C. R. Acad. Sci., Paris, Sér. I*, 319 (1994), no. 2, 121–126.

[21] M. Kontsevich & Yu. Manin. "Gromov–Witten classes, quantum cohomology, and enumerative geometry". *Comm. Math. Phys.* 164 (1994), no. 3, 525–562.

[22] I. G. Macdonald. "Symmetric functions and Hall polynomials". Oxford Classic Texts in the Physical Sciences. *The Clarendon Press, Oxford University Press, New York*, 2015.

[23] T. Miwa, M. Jimbo, & E. Date. "Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras". Cambridge Tracts in Mathematics, 135. *Cambridge University Press, Cambridge,* 2000.

[24] H. Ochiai, "Counting functions for branched covers of elliptic curves and quasi-modular forms," Representation theory of vertex operator algebras and related topics (Japanese) (Kyoto, 2000), No. 1218 (2001), 153–167, 2001.

[25] A. Okounkov & R. Pandharipande, "Gromov-Witten theory, Hurwitz theory, and completed cycles," *Ann. of Math. (2)*, 163 (2006), no. 2, 517–560.

[26] R. Ríos-Zertuche. "An introduction to the half-infinite wedge." *Mexican mathematicians abroad: recent contributions*, 197–237, Contemp. Math., 657, Aportaciones Mat., *Amer. Math. Soc., Providence, RI*, 2016.

[27] P. Rossi. "Gromov-Witten invariants of target curves via symplectic field theory". *J. Geom. Phys.* 58 (2008), no. 8, 931–941.

[28] G. Ruzza & D. Yang. "On the spectral problem of the quantum KdV hierarchy". *J. Phys. A* 54 (2021), no. 37, Paper No. 374001, 27 pp.

[29] N.-P. Skoruppa. "A quick combinatorial proof of Eisenstein series identities". *J. Number Theory* 43 (1993), no. 1, 68–73.

[30] C. Teleman. "The structure of 2D semi-simple field theories". *Invent. Math.* 188 (2012), no. 3, 525–588.

[31] D. Zagier. "Elliptic modular forms and their applications". *The 1-2-3 of modular forms*, 1–103, Universitext, *Springer, Berlin,* 2008.

[32] D. Zagier. "Partitions, quasimodular forms, and the Bloch–Okounkov theorem". *Ramanujan J.* 41 (2016), no. 1-3, 345–368.