PIECEWISE LINEAR STRUCTURES ON TOPOLOGICAL MANIFOLDS

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ABSTRACT. This is a survey paper where we expose the Kirby–Siebenmann results on classification of PL structures on topological manifolds and, in particular, the homotopy equivalence $TOP/PL = K(\mathbb{Z}/2, 3)$ and the Hauptvermutung for manifolds.

Prologue

In his paper [42] Novikov wrote:

> Sullivan's Hauptvermutung theorem was announced first in early 1967. After the careful analysis made by Bill Browder and myself in Princeton, the first version in May 1967 (before publication), his theorem was corrected: a necessary restriction on the 2-torsion of the group $H_3(M)$ was missing. This gap was found and restriction was added. Full proof of this theory has never been written and published. Indeed, nobody knows whether it has been finished or not. Who knows whether it is complete or not? This question is not clarified properly in the literature. Many pieces of this theory were developed by other topologists later. In particular, the final Kirby–Siebenmann classification of topological multidimensional manifolds therefore is not proved yet in the literature.

I do not want to discuss here whether the situation is so dramatic as Novikov wrote. However, it is definitely true that there is no detailed enough and well-ordered exposition of Kirby–Siebenmann classification, such that can be recommended to advanced students which are willing to learn the subject. The fundamental book of Kirby–Siebenmann [28] was written by pioneers and, in a sense, posthaste. It contains all the necessary results, but it is really "Essays", and one must do a lot of work in order to do it readable for general audience.

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The job on hand is an attempt of (or, probably, an approximation to) such an expository paper.

Contents

Introduction

Throughout the paper we use abbreviation PL for "piecewise linear".

Hauptvermutung (main conjecture) is an abbreviation for *die Hauptvermutung der kombinatorischen Topologie* (the main conjecture of combinatorial topology). It seems that the conjecture was first formulated in the papers of Steinitz [54] and Tietze [59] in 1908.

The conjecture claims that the topology of a simplicial complex determines completely its combinatorial structure. In other words, two simplicial complexes are simplicially isomorphic whenever they are homeomorphic. This conjecture was disproved by Milnor [35] in 1961.

However, for manifolds one can state a refined version of the *Hauptvermutung*. A *PL manifold* is defined to be a simplicial complex such that the star of every point (the union of all closed simplices containing the point) is simplicially isomorphic to the n-dimensional ball. (Equivalently, a PL manifold is a manifold with a fixed maximal PL atlas.) There are examples of simplicial complexes which are homeomorphic to topological manifolds but, nevertheless, are not PL manifolds (the double suspensions over Poincaré spheres, see [6]). Moreover, there exists a topological manifold which is homeomorphic to a simplicial complex but do not admit a PL structure, see Example 22.5.

Now, the *Hauptvermutung* for manifolds asks whether any two homeomorphic PL manifolds are PL isomorphic. Furthermore, the related question asks whether every topological manifold is homeomorphic to a PL manifold. Both these questions were solved (negatively) by Kirby and Siebenmann [27, 28]. In fact, Kirby and Siebenmann classified PL structures on high-dimensional topological manifolds. It turned out that a topological manifold can have different PL structures, or not to have any. Below we give a brief description of these results.

Let $BTOP$ and BPL be the classifying spaces for stable topological and PL bundles, respectively. We regard the forgetful map $p : BPL \to BTOP$ as a fibration and denote its fiber by TOP/PL.

Let $f : M \to BTOP$ classify the stable tangent bundle of a topological manifold M. It is clear that every PL structure on M gives us a p-lifting of f and that every two such liftings are fiberwise homotopic. (By the definition, a map $\widehat{f} : M \to BPL$ is a p-lifting of f if $p\widehat{f} = f$.)

It is remarkable that the inverse is also true provided that $\dim M \geq 5$. In greater detail, M admits a PL structure if f admits a p-lifting (the Existence Theorem 6.3), and PL structures on M are in a bijective correspondence with fiberwise homotopy classes of p-liftings of f (the

Classification Theorem 6.2). Kirby and Siebenmann proved these theorems and, moreover, they proved that TOP/PL is the Eilenberg–Mac Lane space $K(\mathbb{Z}/2, 3)$. Thus, there is only one obstruction

$$\varkappa(M) \in H^4(M; \mathbb{Z}/2)$$

to a p-lifting of f, and the set of fiberwise homotopic p-liftings of f (if they exist) is in bijective correspondence with $H^3(M; \mathbb{Z}/2)$. In other words, for every topological manifold M, $\dim M \geq 5$ there is a class $\varkappa(M) \in H^4(M; \mathbb{Z}/2)$ with the following property: M admits a PL structure if and only if $\varkappa(M) = 0$. Furthermore, given a homeomorphism $h : V \to M$ of two PL manifolds, there exists a class

$$\varkappa(h) \in H^3(M; \mathbb{Z}/2)$$

with the following property: $\varkappa(h) = 0$ if and only if h is concordant to a PL isomorphism (or, equivalently, to the identity map 1_M). Finally, every class $a \in H^3(M; \mathbb{Z}/2)$ has the form $a = \varkappa(h)$ for some homeomorphism h. These results give us the complete classification of PL structures on a topological manifold of dimension ≥ 5.

We must explain the following. It can happen that two different PL structures on M yield PL isomorphic PL manifolds (like that two p-liftings $f : M \to BPL$ of f can be non-fiberwise homotopic). Indeed, roughly speaking, a PL structure on a topological manifold M is a concordance class of PL atlases on M (see Section 3 for accurate definitions). However, a PL automorphism of a PL manifold can turn the atlas into a non-concordant to the original one, see Example 22.3. So, in fact, the set of pairwise non-isomorphic PL manifolds which are homeomorphic to a given PL manifold is in a bijective correspondence with the set $H^3(M; \mathbb{Z}/2)/R$ where R is the following equivalence relation: two PL structure are equivalent if the corresponding PL manifolds are PL isomorphic. The *Hauptvermutung* for manifolds claims that $H^3(M; \mathbb{Z}/2)/R$ is one-element. But this is wrong.

Namely, there exists a PL manifold M which is homeomorphic but not PL isomorphic to $\mathbb{RP}^{2n+1}$, see Example 22.1. So, here we have a counterexample to the *Hauptvermutung*.

For completeness of the picture, we mention again that there are topological manifolds which do not admit any PL structure, see Example 22.4.

Comparing the classes of smooth, PL and topological manifolds, we see that there is a big difference between first and second classes, and not so big difference between second and third ones. From the homotopy-theoretical point of view, one can say that the space PL/O

(which classifies smooth structures on PL manifold, see Remark 6.7) has many non-trivial homotopy groups, while TOP/PL is an Eilenberg–Mac Lane space. Geometrically, one can mention that there are many smooth manifolds which are PL isomorphic to S^n but pairwise non-diffeomorphic, while any PL manifold $M^n, n \geq 5$ is PL isomorphic to S^n provided that it is homeomorphic to S^n.

It is worthwhile to go one step deeper and explain the following. Let M^{4k} be a closed connected almost parallelizable manifold (i.e. M becomes parallelizable after deletion of a point). Let σ_k denote the minimal natural number which can be realized as the signature of the manifold M^{4k}. In fact, for every k we have three numbers $\sigma_k^S, \sigma_k^{PL}$ and σ_k^{TOP} while M^{4k} is a smooth, PL or topological manifold, respectively. Milnor and Kervaire [37] proved that

$$\sigma_k^S = c_k(2k-1)!$$

where $c_k \in \mathbb{N}$. On the other hand,

$$\sigma_1^{PL} = 16 \text{ and } \sigma_k^{PL} = 8 \text{ for } k > 1.$$

Finally,

$$\sigma_k^{TOP} = 8 \text{ for all } k.$$

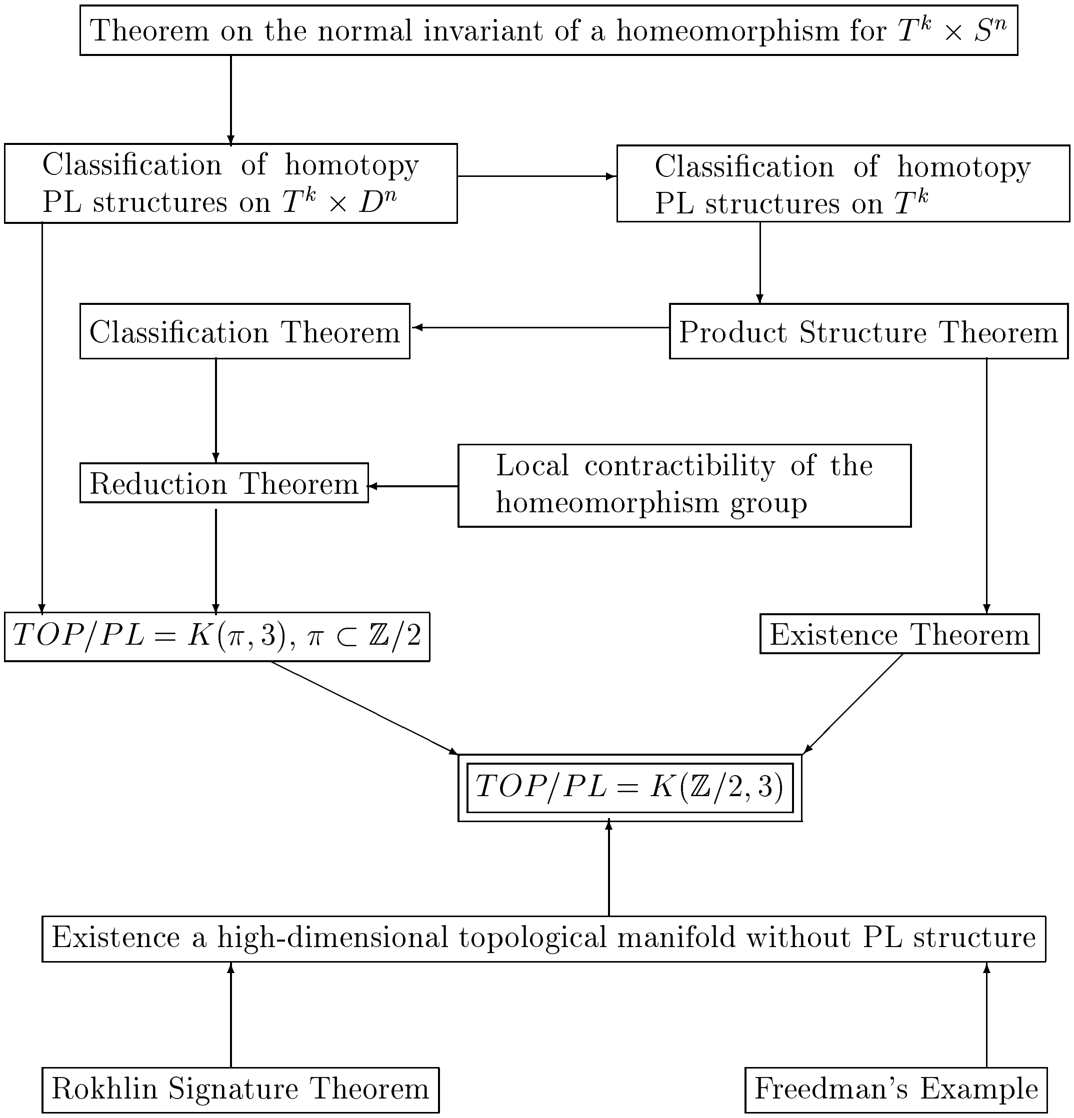
So, here we can see again the big difference between smooth and PL cases. On the other hand, $\sigma_k^{PL} = \sigma_k^{TOP}$ for $k > 1$. Moreover, we will see below that the number

$$2 = 16/8 = \sigma_1^{PL}/\sigma_1^{TOP}$$

yields the group $\mathbb{Z}/2 = \pi_3(TOP/PL)$.

It makes sense to say here about low dimensional manifolds, because of the following remarkable contrast. There is no difference between PL and smooth manifolds in dimension < 7: every PL manifold $V^n, n < 7$ admits a unique smooth structure. However, there are infinitely many smooth manifolds which are homeomorphic to $\mathbb{R}^4$ but pairwise non-diffeomorphic, see [17, 26].

The paper is organized as follows. The first chapter contains the architecture of the proof of the Main Theorem: $TOP/PL \simeq K(\mathbb{Z}/2, 3)$. In fact, we comment the following graph there:


Theorem on the normal invariant of a homeomorphism for $T^k \times S^n$
Classification of homotopy PL structures on $T^k \times D^n$
Classification of homotopy PL structures on T^k
Classification Theorem
Product Structure Theorem
Reduction Theorem
Local contractibility of the homeomorphism group
$TOP/PL = K(\pi, 3)$, $\pi \subset \mathbb{Z}/2$
Existence Theorem
$TOP/PL = K(\mathbb{Z}/2, 3)$
Existence a high-dimensional topological manifold without PL structure
Rokhlin Signature Theorem
Freedman's Example

Namely, we formulate without proofs the boxed claims (and provide the necessary definitions), while we prove all the implications (arrows), i.e., we explain how a claim can be deduced from another one.

The second chapter contains a proof of the Sullivan Theorem on a normal invariant of a homeomorphism for $T^k \times S^n$, and also a proof of the Browder–Novikov Theorem 4.6 about homotopy properties of normal bundles. We need this theorem in order to define the concept of normal invariant.

The third chapter contains several application of the Main theorem and, in particular, the counterexample to the *Hauptvermutung*.

Let me tell something more about the graph. As we have already seen, the classification theory of PL structures on topological manifolds splits into two parts. The first part reduces the original geometric problem to a homotopy one (classification of p-liftings of a map $M \to BTOP$), the second part solves this homotopy problem by proving that $TOP/PL = K(\mathbb{Z}/2, 3)$.

The key result for the first part is the Product Structure Theorem 6.1. Roughly speaking, this theorem establishes a bijection between PL structures on M and $M \times \mathbb{R}$. The Classification Theorem 6.2 and the Existence Theorem 6.3 are the consequences of the Product Structure Theorem.

Passing to the second part, the description of the homotopy type of TOP/PL, we have the following. Because of the Classification Theorem, for $n \geq 5$ there is a bijection between the set $\pi_n(TOP/PL)$ and the set of PL structures on S^n. By the Smale Theorem, every PL manifold $M^n, n \geq 5$, is PL isomorphic to S^n whenever it is homeomorphic to S^n. So, $\pi_n(TOP/PL) = 0$ for $n \geq 5$.

What about $n < 5$? Again, because of the Classification Theorem, the group $\pi_n(TOP/PL)$ is in a bijective correspondence with the set of PL structures on $\mathbb{R}^k \times S^n$ provided that $k + n \geq 5$. However, this set of PL structures is uncontrollable. In order to make the situation more manageable, one can consider the PL structures on the compact manifold $T^k \times S^n$ and then pass to the universal covering. We can't do it directly, but there is a trick (the Reduction Theorem 8.7) which allows us to estimate PL structure on $\mathbb{R}^k \times S^n$ in terms of so-called *homotopy PL structures* on $T^k \times S^n$ (more precisely, we should consider the homotopy PL structures on $T^k \times D^n$ modulo the boundary), see Section 3 for the definitions. Now, using results of Hsiang and Shaneson [23] and Wall [62, 63] about homotopy PL structures on $T^k \times D^n$, one can prove that $\pi_i(TOP/PL) = 0$ for $i \neq 3$ and that $\pi_3(TOP/PL)$

has at most 2 elements. Finally, there exists a high-dimensional topological manifold which does not admit any PL structure. Thus, by the Existence Theorem, the space TOP/PL is not contractible. Therefore $TOP/PL = K(\mathbb{Z}/2, 3)$

It is worthwhile to mention that the proof of the Product Structure Theorem uses the classification of homotopy PL structures on T^k.

Now I say some words about the top box of the above graph. Let F_n be the monoid of pointed homotopy equivalences $S^n \to S^n$, let BF_n be the classifying space for F_n, and let $BF = \lim_{n\to\infty} BF_n$. There is an obvious forgetful map $BPL \to BF$, and we denote by F/PL the homotopy fiber of this map. For every homotopy equivalence of closed PL manifolds $h : V \to M$ Sullivan [56, 57] defined the *normal invariant* of h to be a certain homotopy class $j_F(h) \in [M, F/PL]$, see Section 4. Sullivan proved that, for every *homeomorphism* $h : V \to M$, $j_F(h) = 0$ whenever $H_3(M)$ is 2-torsion free. Moreover, this theorem implies that if, in addition, M is simply-connected then h is homotopic to a PL isomorphism. Thus the *Hauptvermutung* holds for simply-connected manifolds with $H_3(M)$ 2-torsion free.

Definitely, the above formulated Sullivan Theorem on the Normal Invariant of a Homeomorphism is interesting by itself. However, in the paper on hand this theorem plays also an additional important role. Namely, the Sullivan Theorem for $T^k \times S^n$ is a lemma in classifying of homotopy structures on $T^k \times D^n$. For this reason, we first prove the Sullivan Theorem for $T^k \times S^n$, then use it in the proof of the Main Theorem, and then (in Chapter 3) prove the Sullivan Theorem in full generality.

You can also see that the proof of the Main Theorem uses the difficult Freedman's example of a 4-dimensional almost parallelizable topological manifold of signature 8. This example provides the equality $\sigma_1^{TOP} = 8$. Actually, the original proof of the Main Theorem appeared before Freedman's Theorem and therefore did not use the last one. However, as we noticed above, the Freedman results clarify the relations between PL and topological manifolds, and thus they should be incorporated in the exposition of the global picture.

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Notation and conventions

We work mainly with CW-spaces and topological manifolds. However, when we quit these clases by taking products or functional spaces, we equip the last ones with the compactly generated topology, (following Steenrod [53] and McCord [33], see e.g.[48] for the exposition). All maps are supposed to be continuous. All neighbourhoods are supposed to be open.

Given two topological spaces X, Y, we denote by $[X, Y]$ the set of homotopy classes of maps $X \to Y$. We also use the notation $[X, Y]^\bullet$ for the set of pointed homotopy classes of pointed maps $X \to Y$ of pointed spaces.

It is quite standard to denote by $[f]$ the homotopy class of a map f. However, here we usually do not distinguish a map and its homotopy class and use the same symbol, say f for a map as well as for the homotopy class. In this paper this does not lead to any confusion.

We use the term *inessential map* for null-homotopic maps; otherwise we say that a map is called *essential.*

We use the sign $\simeq$ for homotopy of maps or homotopy equivalence of spaces.

We reserve the term *bundle* for locally trivial bundles and the term *fibration* for Hurewicz fibrations. Given a space F, an F*-bundle* is a bundle whose fiber is F, and an F*-fibration* is a fibration whose fibers are homotopy equivalent to F.

Given a bundle or fibration $\xi = \{p : E \to B\}$, we say that B is the *base* of ξ and that E is the *total space* of ξ. Furthermore, given a space X, we set

$$\xi \times X = \{p \times 1 : E \times X \to B \times X\}.$$

Given two bundles $\xi = \{p : E \to B\}$ and $\eta = \{q : Y \to X\}$, a *bundle morphism* $\varphi : \xi \to \eta$ is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{f} & X. \end{array}$$

We say that f is the *base* of the morphism φ or that φ is a *morphism over* f. We also say that g *is a map over* f. If $X = B$ and $f = 1_B$ we say that g is a map over B (and φ is a morphism over B).

Given a map $f : Z \to B$ and a bundle (or fibration) ξ over B, we use the notation $f^*\xi$ for the induced bundle over Z. Recall that there is a canonical bundle morphism $\mathfrak{I}_{f,\xi} : \xi \to \eta$ over f, see [48] (or [16] where it is denoted by $\mathrm{ad}(f)$). Following [16], we call $\mathfrak{I}_{f,\xi}$ the *adjoint morphism of* f, or just the f*-adjoint* morphism. Furthermore, given a bundle morphism $\varphi : \xi \to \eta$ with the base f, there exists a unique bundle morphism $c(\varphi) : \xi \to f^*\eta$ over the base of ξ such that the composition

$$\xi \xrightarrow{c(\varphi)} f^*\eta \xrightarrow{\mathfrak{I}_{f,\eta}} \eta$$

coincides with φ. Following [16], we call $c(\varphi)$ the *correcting morphism.*

Given a subspace A of a space X and a bundle ξ over X, we denote by $\xi|A$ the bundle $i^*\xi$ where $i : A \subset X$ is the inclusion.

Chapter 1. **Architecture**

1. A result from homotopy theory

Recall that an H-space is a space F with a base point f_0 and a multiplication map $\mu : F \times F \to F$ such that f_0 is a homotopy unit, i.e. the maps $f \mapsto \mu(f, f_0)$ and $f \mapsto \mu(f_0, f)$ are homotopic to the identity rel $\{f_0\}$. For details, see [4].

1.1. Definition. (a) Let (F, f_0) be an H-space with the multiplication $\mu : F \times F \to F$. A *principal F-fibration* is an F-fibration $p : E \to B$ equipped with a map $m : E \times F \to E$ such that the following holds:

(i) the diagrams

$$\begin{array}{ccc} E \times F \times F & \xrightarrow{m \times 1} & E \times F \\ {\scriptstyle 1\times\mu}\downarrow & & \downarrow{\scriptstyle m} \\ E \times F & \xrightarrow{m} & E \end{array} \qquad \begin{array}{ccc} E \times F & \xrightarrow{m} & E \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{p} & B \end{array}$$

commute;

(ii) the map

$$E \longrightarrow E, \quad e \mapsto m(e, f_0)$$

is a homotopy equivalence;

(iii) for every $e_0 \in E$, the map

$$F \longrightarrow p^{-1}(p(e_0)), \quad f \mapsto m(e_0, f)$$

is a homotopy equivalence.

(b) A *trivial* principal F-fibration is the fibration $p_2 : X \times F \to F$ with the action $m : E \times F \to E$ of the form

$$m : X \times F \times F \to X \times F, \quad m(x, f_1, f_2) = (x, \mu(f_1, f_2)).$$

It is easy to see that if the fibration η is induced from a principal fibration ξ then η turns into a principal fibration in a canonical way.

1.2. Definition. Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be two principal F-fibrations over the same base B. We say that a map $h : E_1 \to E_2$ is an *F-equivariant map over B* if h is a map over B and the diagram

$$\begin{array}{ccc} E_1 \times F & \xrightarrow{h\times 1} & E_2 \times F \\ {\scriptstyle m_1}\big\downarrow & & \big\downarrow{\scriptstyle m_2} \\ E_1 & \xrightarrow{h} & E_2 \end{array}$$

commutes up to homotopy over B.

Notice that, for every $b \in B$, the map

$$h_b : \pi_1^{-1}(b) \to \pi_2^{-1}(b), \quad h_b(x) = h(x)$$

is a homotopy equivalence.

Now, let $p : E \to B$ be a principal F-fibration, and let $f : X \to B$ be an arbitrary map. Given a p-lifting $g : X \to E$ of f and a map $u : X \to F$, consider the map

$$g_u : X \xrightarrow{\Delta} X \times X \xrightarrow{g\times u} E \times F \xrightarrow{m} E.$$

It is easy to see that the correspondence $(g, u) \mapsto g_u$ yields a well-defined map (action)

$$[\mathrm{Lift}_p\, f] \times [X, F] \to [\mathrm{Lift}_p\, f]. \tag{1.1}$$

In particular, for every p-lifting g of f the correspondence $u \mapsto g_u$ induces a map

$$T_g : [X, F] \to [\mathrm{Lift}_p\, f].$$

1.3. Theorem. *Let $\xi = \{p : E \to B\}$ be a principal F-fibration, and let $f : X \to B$ be a map where X is assumed to be paracompact and locally contractible. If F is an H-space with homotopy inversion, then the above action* (1.1) *is free and transitive provided* $[\mathrm{Lift}_p\, f] \neq \emptyset$. *In particular, for every p-lifting $g : X \to E$ of f the map T_g is a bijection.*

Proof. We start with the following lemma.

1.4. Lemma. *The theorem holds if $X = B$, $f = 1_X$ and ξ is the trivial principal F-fibration.*

Proof. In this case every p-lifting $g : X \to X \times F$ of $f = 1_X$ determines and is completely determined by the map

$$\overline{g} : X : \xrightarrow{g} X \times F \xrightarrow{p_2} F.$$

In other words, we have the bijection $[\mathrm{Lift}_p f] \cong [X, F]$, and under this bijection the action (1.1) turns into the multiplication

$$[X, F] \times [X, F] \to [X, F].$$

Now the result follows since $[X, F]$ is a group. $\diamond$

We finish the proof of the theorem. Consider the induced fibration $f^*\xi = \{q : Y \to X\}$ and notice that there is an $[X, F]$-equivariant bijection

$$[\mathrm{Lift}_p f] \cong [\mathrm{Lift}_q 1_X]. \tag{1.2}$$

Now, suppose that $[\mathrm{Lift}_p f] \neq \emptyset$ and take a p-lifting g of f. Regarding Y as the subset of $X \times E$, define the F-equivariant map

$$h : X \times F \to Y, \quad h(x, a) = (x, g(x)a),\ x \in X, a \in F.$$

It is easy to see that the diagram

$$\begin{tikzcd} X \times F \arrow[r, "h"] \arrow[d, "p_1"'] & Y \arrow[d, "q"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes, i.e. h is a map over X. Since X is a locally contractible paracompact space, and by a theorem of Dold [9], there exists a map $k : Y \to X \times F$ over X which is homotopy inverse over X to h. It is easy to see that k is an equivariant map over X. Indeed, if $m_1 : X \times F \times F \to X \times F$ and $m_2 : Y \times F \to Y$ are the corresponding actions then

$$m_1(k \times 1) \simeq khm_1(k \times 1) \simeq km_2(h \times 1)(k \times 1) =\simeq km_2(hk \times 1) \simeq km_2,$$

where $\simeq$ denotes the homotopy over X.

In particular, there is an $[X, F]$-equivariant bijection

$$[\mathrm{Lift}_q 1_X] \cong [\mathrm{Lift}_{p_1} 1_X]$$

where $p_1 : X \times F \to X$ is the projection. Now we compose this bijection with (1.2) and get $[X, F]$-equivariant bijections

$$[\mathrm{Lift}_p f]] \cong [\mathrm{Lift}_q 1_X] \cong [\mathrm{Lift}_{p_1} 1_X],$$

and the result follows from Lemma 1.4. $\square$

1.5. Example. If $p : E \to B$ is an F-fibration then $\Omega p : \Omega E \to \Omega B$ is a principal ΩF-fibration. Here Ω denotes the loop functor.

2. Preliminaries on bundles and classifying spaces

Here we give a brief recollection on $\mathbb{R}^n$ bundles, spherical fibrations and their classifying spaces. For details, see [48].

We define a *topological* $\mathbb{R}^n$*-bundle* over a space B to be an $\mathbb{R}^n$-bundle $p : E \to B$ equipped with a fixed section $s : B \to E$. Given two topological $\mathbb{R}^n$-bundles $\xi = \{p : E \to B\}$ and $\eta = \{q : Y \to X\}$, we define a *topological morphism* $\varphi : \xi \to \eta$ to be a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{f} & X \end{array} \tag{2.1}$$

where g preserves the sections and yields homeomorphism of fibers. The last one means that, for every $b \in B$, the map

$$g_b : \mathbb{R}^n = p^{-1}(b) \to q^{-1}(f(b) = \mathbb{R}^n, \quad g_b(a) = g(a) \text{ for every } a \in p^{-1}(b)$$

is a homeomorphism. As usual, we say that f is the *base* of the morphism φ.

Topological $\mathbb{R}^n$-bundles can also be regarded as $(TOP_n, \mathbb{R}^n)$-bundles, i.e. $\mathbb{R}^n$-bundles with the structure group TOP_n. Here TOP_n is the topological group of self-homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n, f(0) = 0$. The classifying space $BTOP_n$ of the group TOP_n turns out to be a classifying space for topological $\mathbb{R}^n$-bundles over CW-spaces. This means there exists a universal topological $\mathbb{R}^n$-bundle γ^n_{TOP} over $BTOP_n$ with the following

2.1. Universal Property. For every topological $\mathbb{R}^n$-bundle ξ over a CW-space B, every CW-subspace A of B and every morphism

$$\psi : \xi|A \to \gamma^n_{TOP}$$

of topological $\mathbb{R}^n$-bundles, there exists a morphism $\varphi : \xi \to \gamma^n_{TOP}$ which is an extension of ψ.

In particular, for every topological $\mathbb{R}^n$-bundle ξ over B there exists a morphism $\varphi : \xi \to \gamma^n_{TOP}$ of topological $\mathbb{R}^n$-bundles. We call such φ a *classifying morphism* for ξ. The base $f : B \to BTOP_n$ of φ is called a *classifying map* for ξ. It is clear that ξ is isomorphic over B to $f^*\gamma^n_{TOP}$.

2.2. Proposition. *Let $\varphi_0, \varphi_1 : \xi \to \gamma^n_{TOP}$ be two classifying morphisms for ξ. Then there exists a classifying morphism $\Phi : \xi \times I \to \gamma^n_{TOP}$ such that $\Phi|\xi \times \{i\} = \varphi_i, i = 0, 1$. In particular, a classifying map f for ξ is determined by ξ uniquely up to homotopy.*

Proof. This follows from the universal property 2.1 applied to $\xi \times I$, if we put $A = X \times \{0, 1\}$ where X denotes the base of ξ. $\square$.

A *piecewise linear* (in future PL) $\mathbb{R}^n$*-bundle* is a topological $\mathbb{R}^n$-bundle $\xi = \{p : E \to B\}$ such that E and B are polyhedra and $p : E \to B$ and $s : B \to E$ are PL maps. Furthermore, we require that, for every simplex $\Delta \subset B$, there is a PL homeomorphism $h : p^{-1}(\Delta) \cong \Delta \times \mathbb{R}^n$ with $h(s(\Delta)) = \Delta \times \{0\}$. (For definitions of PL maps, see [24, 47].)

A morphism of PL $\mathbb{R}^n$-bundles is a morphism of topological $\mathbb{R}^n$-bundles where the maps g and f in (2.1) are PL maps.

There exists a universal PL $\mathbb{R}^n$-bundle γ^n_{PL} over a certain space BPL_n. This means that the universal property 2.1 remains valid if we replace γ^n_{TOP} by γ^n_{PL} and "topological $\mathbb{R}^n$ bundle" by "PL $\mathbb{R}^n$-bundle" there. So, BPL_n is a classifying space for PL $\mathbb{R}^n$-bundles.

Notice that BPL_n can also be regarded as the classifying space of a certain group PL_n (which is constructed as the geometric realization of a certain simplicial group), [29, 30].

A *sectioned* S^n*-fibration* is defined to be an S^n-fibration $p : E \to B$ equipped with a section $s : B \to E$. Morphisms of sectioned S^n-fibrations are defined to be diagrams like (2.1) where each map g_b is assumed to be a pointed homotopy equivalence.

There exists a universal sectioned S^n-fibration γ^n_F over a certain space BF_n. This means that the universal property 2.1 remains valid if we replace γ^n_{TOP} by γ^n_F and "topological $\mathbb{R}^n$ bundle" by "sectioned S^n-fibration" there. So, BF_n is a classifying space sectioned S^n-fibrations.

The space BF_n can also be regarded as the classifying space for the monoid F_n of pointed homotopy equivalences $(S^n, *) \to (S^n, *)$. Because of this, we shall use the brief term "$(S^n, *)$-fibration" for sectioned S^n-fibrations. Furthermore, we will also use the term "F_n-morphism" for morphism of sectioned S^n-fibrations. Finally, an F_n-morphism over a space is called an F_n-equivalence.

We need also to recall the space BO_n which classifies n-dimensional vector bundles. The universal vector bundle over BO_n is denoted by γ^n_O. This space is well-known and described in many sources, e.g. [38].

Since γ^n_{PL} can be regarded as the (underlying) topological $\mathbb{R}^n$-bundle, there is a classifying morphism

$$\omega = \omega^{PL}_{TOP}(n) : \gamma^n_{PL} \to \gamma^n_{TOP}. \tag{2.2}$$

We denote by $\alpha = \alpha^{PL}_{TOP}(n) : BPL_n \to BTOP_n$ the base of this morphism.

Similarly, given a topological $\mathbb{R}^n$-bundle $\xi = \{p : E \to B\}$, let $\xi^\bullet$ denotes the S^n-bundle $\xi^\bullet = \{p^\bullet : E^\bullet \to B\}$ where $E^\bullet$ is the fiberwise one-point compactification of E. Notice that the added points ("infinities") give us a certain section of $\xi^\bullet$.

In other words, the TOP_n-action on $\mathbb{R}^n$ extends uniquely to a TOP_n-action on the one-point compactification S^n of $\mathbb{R}^n$, and $\xi^\bullet$ is the (TOP_n, S^n)-bundle associated with ξ. Furthermore, the fixed point ∞ of the TOP_n-action on S^n yields a section of $\xi^\bullet$.

So, $\xi^\bullet$ can be regarded as an $(S^n, *)$-fibration over B. In particular, $(\gamma^n_{TOP})^\bullet$ can be regarded as an $(S^n, *)$-fibration over $BTOP_n$. So, there is a classifying morphism

$$\omega^{TOP}_F(n) : \gamma^n_{TOP} \to \gamma^n_F.$$

We denote by $\alpha^{TOP}_F(n) : BTOP_n \to BF_n$ the base of $\omega^{TOP}_F(n)$.

Finally, we notice that an n-dimensional vector bundle over a polyhedron X has a canonical structure of PL $\mathbb{R}^n$-bundle over X. Similarly to above, this gives us a (forgetful) map

$$\alpha^O_{PL}(n) : BO_n \to BPL_n.$$

So, we have a sequence of forgetful maps

$$BO_n \xrightarrow{\alpha'} BPL_n \xrightarrow{\alpha''} BTOP_n \xrightarrow{\alpha'''} BF_n \tag{2.3}$$

where $\alpha' = \alpha^O_{PL}(n)$, etc.

2.3. Constructions. 1. Given an F-bundle $\xi = \{p : E \to B\}$ and an F'-bundle $\xi' = \{p' : E' \to B'\}$, we define the product $\xi \times \xi'$ to be the $F \times F'$-bundle

$$p \times p' : E \times E' \to B \times B'.$$

2. Given an F-bundle $\xi = \{p : E \to B\}$ with a section $s : B \to E$ and an F'-bundle $\xi' = \{p' : E' \to B'\}$ with a section $s' : B' \to E'$, we define the smash product $\xi \wedge \xi'$ to be the $F \wedge F'$-bundle as follows. The map $p \times p' : E \times E' \to B \times B'$ passes through the quotient map $q : E \times E' \to E \times E'/(E \times s(B') \cup E' \times s(B)$, and we set

$$\xi \wedge \eta = \{\pi : E \times E'/(E \times s(B') \cup E' \times s(B) \to B \times B',$$

where π is the unique map with $p \times p' = \pi q$. Finally, the section s and s' yield an obvious section of π.

3. Given an $\mathbb{R}^m$-bundle ξ and an $\mathbb{R}^n$-bundle η over the same space X, the *Whitney sum* of ξ and η is the $\mathbb{R}^{m+n}$-bundle $\xi \oplus \eta = d^*(\xi \times \eta)$ where $d : X \to X \times X$ is the diagonal.

Notice that if ξ and η are a PL $\mathbb{R}^m$ and PL $\mathbb{R}^n$-bundle, respectively, then $\xi \times \eta$ is a PL $\mathbb{R}^{m+n}$-bundle.

4. Given a sectioned S^m-bundle ξ and sectioned S^n-bundle η over the same space X, we set $\xi \dagger \eta = d^*(\xi \wedge \eta)$.

We denote by $r_n = r_n^{TOP} : BTOP_n \to BTOP_{n+1}$ the map which classifies $\gamma^n_{TOP} \oplus \theta^1_{BTOP_n}$. The maps $r_n^{PL} : BPL_n \to BPL_{n+1}$ and $r_n^O : BO_n \to BO_{n+1}$ are defined in a similar way.

We can also regard the above map $r_n : BTOP_n \to BTOP_{n+1}$ as a map induced by the standard inclusion $TOP_n \subset TOP_{n+1}$. Using this approach, we define $r_n^F : BF_n \to BF_{n+1}$ as the map induced by the standard inclusion $F_n \subset F_{n+1}$, see [31, p. 45].

2.4. Remarks. 1. Regarding $\mathbb{R}^m$ as the bundle over the point, we see that $(\mathbb{R}^m)^\bullet = (S^m)$ and, moreover,

$$(\mathbb{R}^m \times \mathbb{R}^n)^\bullet = S^m \wedge S^n, \text{ i.e. } (\mathbb{R}^m \oplus \mathbb{R}^n)^\bullet = S^m \dagger S^n.$$

Therefore $(\xi \oplus \eta)^\bullet = \xi^\bullet \dagger \eta^\bullet$ for every $\mathbb{R}^m$-bundle ξ and $\mathbb{R}^n$-bundle η.

2. Generally, the smash product of (sectioned) fibrations is not a fibrations. But we apply it to bundles only and so do not have any troubles. On the other hand, there is an operation $\wedge^h$, the *homotopy smash product*, such that $\xi \wedge^h \eta$ is the $(F \wedge G)$-fibration over $X \times Y$ if ξ is an F-fibration over X and η is an G-fibration over Y, see [48]. In particular, one can use it in order to define an analog of Whitney sum for spherical fibrations and then use this one in order to construct the map $BF_n \to BF_{n+1}$.

The spaces $BO_n, BPL_n, BTOP_n$ and BF_n are defined uniquely up to weak homotopy equivalence. However, it is useful for us to work with more or less concrete models of classifying spaces BO_n, etc. In greater detail, we do the following.

Choose classifying spaces $B'F_n$ for $(S^n, *)$-fibrations (i.e., in the weak homotopy type BF_n) and consider the maps $r_n^F : B'F_n \to B'F_{n+1}$ as above. We can assume that every $B'F_n$ is a CW-complex and every r_n is a cellular map. We define BF to be the telescope (homotopy direct limit) of the sequence

$$\cdots \longrightarrow B'F_n \xrightarrow{r_n} B'F_{n+1} \longrightarrow \cdots,$$

see e.g. [48]. Furthermore, we define BF_n to be the telescope of the finite sequence

$$\cdots \longrightarrow B'F_{n-1} \xrightarrow{r_{n-1}} B'F_n.$$

(Notice that $BF_n \simeq B'F_n$.) So, we have the sequence (filtration)

$$\cdots \subset BF_n \subset BF_{n+1} \subset \cdots .$$

Here $BF = \bigcup BF_n$, BF_n is closed in BF and BF has the direct limit topology with respect to the filtration $\{BF_n\}$. Moreover, if $f : K \to BF$ is a map of a compact space K then there exists n such that $f(K) \subset BF_n$.

Now, for every n consider a CW-space $B'TOP_n$ in the weak homotopy type $BTOP_n$ and define $B''TOP$ to be the telescope of the sequence

$$\cdots \longrightarrow B'TOP_n \xrightarrow{r_n} B'TOP_{n+1} \longrightarrow \cdots .$$

Furthermore, we define $B''TOP_n$ to be the telescope of the finite sequence

$$\cdots \longrightarrow B'TOP_{n-1} \xrightarrow{r_{n-1}} B'TOP_n.$$

So, we have the diagram

$$\begin{array}{ccccccc} \cdots \subset & B''TOP_n & \subset & B''TOP_{n+1} & \subset \cdots \subset & B''TOP \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ \cdots \subset & BF_n & \subset & BF_{n+1} & \subset \cdots \subset & BF \end{array}$$

where the map p is induced by maps $\alpha_F^{TOP}(n)$. Now we convert every vertical map to a fibration (using Serre construction). Namely, we set

$$BTOP = \{(x, \omega) \mid x \in B''TOP, \omega \in (BF)^I, p(x) = \omega(0)\}$$

and define $\alpha_F^{TOP} : BTOP \to BF$ by setting $\alpha_F^{TOP}(x, \omega) = \omega(1)$. Finally, we set

$$BTOP_n = \{(x, \omega) \in BTOP \mid x \in B''TOP_n, \omega \in (BF_n)^I \subset (B''TOP)^I\}$$

So, we have the commutative diagram

$$\begin{array}{ccccccc} \cdots \subset & BTOP_n & \subset & BTOP_{n+1} & \subset \cdots \subset & BTOP \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ \cdots \subset & BF_n & \subset & BF_{n+1} & \subset \cdots \subset & BF \end{array}$$

where all the vertical maps are fibrations.

Now it is clear how to proceed and get the diagram

$$
\begin{array}{ccccccc}
\dots\subset & BO_n & \subset & BO_{n+1} & \subset\dots\subset & BO \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha^O_{PL}} \\
\dots\subset & BPL_n & \subset & BPL_{n+1} & \subset\dots\subset & BPL \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha^{PL}_{TOP}} \\
\dots\subset & BTOP_n & \subset & BTOP_{n+1} & \subset\dots\subset & BTOP \\
& {\scriptstyle \alpha^{TOP(n)}_F}\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha^{TOP}_F} \\
\dots\subset & BF_n & \subset & BF_{n+1} & \subset\dots\subset & BF
\end{array}
\tag{2.4}
$$

where all the vertical maps are fibrations and all the filtrations have nice properties. Moreover, each of limit spaces has the direct limit topology with respect to the corresponding filtration, and every compact subspace of, say, BO is contained in some BO_n.

Furthermore, the fiber of α^O_{PL} is denoted by PL/O, the fiber of α^{PL}_{TOP} is denoted by TOP/PL, etc. Similarly, the fiber of the composition, say,

$$\alpha^{PL}_F := \alpha^{TOP}_F \circ \alpha^{PL}_{TOP} : BPL \to BF$$

is denoted by F/PL. In particular, we have a fibration

$$TOP/PL \xrightarrow{a} F/PL \xrightarrow{b} F/TOP. \tag{2.5}$$

Finally, $F/TOP = \bigcup F_n/TOP_n$ where F_n/TOP_n denotes the fiber of the fibration $BTOP_n \to BF_n$, and F/TOP has the direct limit topology with respect to the filtration $\{F_n/TOP_n\}$. The same holds for other "homogeneous spaces" $F/PL, TOP/PL$, etc.

Notice that, because of well-known results of Milnor [34], all these "homogeneous spaces" have the homotopy type of CW-spaces. Furthermore, all the spaces $BO, BPL, BTOP, BF, F/PL, TOP/PL$, etc. are infinite loop spaces and, in particular, H-spaces, see [4].

We mention also the following useful fact.

2.5. Theorem. *Let Z denote on of the symbols O, PL, F. The above described map $BZ_n \to BZ_{n+1}$ induces an isomorphism of homotopy groups in dimensions $\le n-1$ and an epimorphism in dimension n.*

Proof. For $Z = O$ and $Z = F$ it is well known, see e.g [5], for $Z = PL$ it can be found in [20]. $\square$

2.6. Remark. Let G_n denote the topological monoid of homotopy equivalences $S^{n-1} \to S^{n-1}$. Then the classifying space BG_n of G_n classifies S^{n-1}-fibrations. Every $h \in TOP_n$ induces a map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ which, in turn, yields a self-map

$$\pi_h : S^{n-1} \to S^{n-1}, \quad \pi_h(x) = h(x)/||h(x)||.$$

So, we have a map $TOP_n \to G_n$ which, in turn, induces a map

$$BTOP_n \longrightarrow BG_n$$

of classifying spaces. In the language of bundles, this map converts a topological $\mathbb{R}^n$-bundle into a spherical fibration via deletion of the section.

We can also consider the space BG by tending n to ∞. In particular, we have the spaces G/PL and G/TOP.

There is an obvious forgetful map $F_n \to G_n$, and it turns out that the induced map $BF \to BG$ (as $n \to \infty$) is a homotopy equivalence. see e.g. [31, Chapter 3]. In particular, $F/PL \simeq G/PL$ and $F/TOP \simeq G/TOP$.

3. Structures on manifolds and bundles

A *PL atlas* on a topological manifold is an atlas such that all the transition maps are PL ones. We define a PL manifold as a topological manifold with a maximal PL atlas. Furthermore, given two PL manifolds M and N, we say that a homeomorphism $H : M \to N$ a PL isomorphism if h is a PL map. (One can prove that in this case h^{-1} is a PL map as well, [24].)

3.1. Definition. (a) We define a ∂_{PL}-manifold to be a topological manifold whose boundary ∂M is a PL manifold. In particular, every closed topological manifold is a ∂_{PL}-manifold. Furthermore, every PL manifold can be canonically regarded as a ∂_{PL}-manifold.

(b) Let M be a ∂_{PL}-manifold. A *PL structuralization* of M is a homeomorphism $h : V \to M$ such that V is a PL manifold and $h : \partial V \to \partial M$ induces a PL isomorphism $\partial V \to \partial M$ of boundaries (or, equivalently, PL isomorphism of corresponding collars). Two PL structuralizations $h_i : V_i \to X, i = 0, 1$ are *concordant* if there exist a PL isomorphism $\varphi : V_0 \to V_1$ and a homeomorphism $H : V_0 \times I \to M \times I$ such that $h_0 = H|V \times \{0\}$ and $H|V_0 \times \{1\} = h_1\varphi$ and, moreover, $H : \partial V_0 \times I \to \partial M \times I$ coincides with $h_0 \times 1_I$. Any concordance class of PL structuralizations is called a *PL structure on M*. We denote by $\mathfrak{T}_{PL}(M)$ the set of all PL structures on X.

(c) If M itself is a PL manifold then $\mathfrak{T}_{PL}(M)$ contains the distinguished element: the concordance class of 1_M. We call it the *trivial element* of $\mathfrak{T}_{PL}(M)$.

3.2. Remarks. 1. Clearly, every PL structuralization of M equips M with a certain PL atlas. Conversely, if we equip M with a certain PL atlas then the identity map can be regarded as a PL structuralization of M.

2. Clearly, if M itself is a PL manifold then the concordance class of any PL isomorphism is the trivial element of $\mathfrak{T}_{PL}(M)$.

3. Recall that two homeomorphism $h_0, h_1 : X \to Y$ are *isotopic* if there exists a homeomorphism $H : X \times I \to Y \times I$ (isotopy) such that $p_2 H : X \times I \to Y \times I \to I$ coincides with $p_2 : X \times I \to I$. Given $A \subset X$, we say that h_0 and H_1 are isotopic $\operatorname{rel} A$ if there exists an isotopy H such that $H(a,t) = h_0(a)$ for every $a \in A$ and every $t \in I$. In particular, if two PL structuralization $h_0, h_1 : V \to M$ are isotopic rel ∂V then they are concordant.

4. Given two PL structuralizations $h_i : V_i \to M, i = 0, 1$, they are not necessarily concordant if V_0 and V_1 are PL isomorphic. We are not able to give examples here, but we do it later, see Remark 3.10(2) and Example 22.3.

3.3. Definition (cf. [5, 48])**.** Given a topological $\mathbb{R}^n$-bundle ξ, a *PL structuralization of* ξ is a morphism $\varphi : \xi \to \gamma^n_{PL}$ of topological $\mathbb{R}^n$-bundles. We say that two PL structuralizations $\varphi_0, \varphi_1 : \xi \to \gamma^n_{PL}$ are *concordant* if there exists a morphism $\Phi : \xi \times 1_I \to \gamma^n_{PL}$ of topological $\mathbb{R}^n$-bundles such that $\Phi|\xi \times 1_{\{i\}} = \varphi_i, i = 0, 1$.

Let $f : X \to BTOP_n$ classify a topological $\mathbb{R}^n$-bundle ξ, and let

$$g : X \to BPL_n$$

be an $\alpha^{PL}_{TOP}(n)$-lifting of f. Then we get a morphism (defined uniquely up to concordance in view of 2.2)

$$\xi \cong f^*\gamma^n_{TOP} = g^*\alpha(n)^*\gamma^n_{TOP} = g^*\gamma^n_{PL} \xrightarrow{\text{classif}} \gamma^N_{PL}, \quad \alpha(n) := \alpha^{PL}_{TOP}(n).$$

Clearly, this morphism $\xi \to \gamma^n_{PL}$ is a PL structuralization of ξ. It is easy to see that in this way we have a correspondence

$$(3.1) \qquad [\mathrm{Lift}_{\alpha(n)} f] \longrightarrow \{\text{PL structures on } \xi\}.$$

3.4. Theorem. *The correspondence* (3.1) *is a bijection.*

Proof. This can be proved similarly to [48, Theorem IV.2.3], cf. also [5, Chapter II]. □

Consider now the map

$$\bar{f} : X \xrightarrow{f} BTOP_n \subset BTOP$$

and the map $\alpha : BPL \to BTOP$ as in (2.4). Then every $\alpha(n)$-lifting of f is the α-lifting of $\bar{f}$. So, we have a correspondence

$$u_\xi : \{\text{PL structures on } \xi\} \longrightarrow [\operatorname{Lift}_{\alpha(n)} f] \longrightarrow [\operatorname{Lift}_\alpha \bar{f}] \tag{3.2}$$

where the first map is the inverse to (3.1). Furthermore, there is a canonical map

$$v_\xi : \{\text{PL structures on } \xi\} \longrightarrow \{\text{PL structures on } \xi \oplus \theta^1\},$$

and these maps respect the maps u_ξ, i.e. $u_{\xi\oplus\theta^1} = v_\xi u_\xi$. So, we have the map

$$\lim_{n\to\infty} \{\text{PL structures on } \xi \oplus \theta^n\} \longrightarrow [\operatorname{Lift}_\alpha \bar{f}] \tag{3.3}$$

where lim means the direct limit of the sequence of sets.

3.5. Proposition. *If X is a finite CW-space then the map* (3.3) *is a bijection.*

Proof. The surjectivity follows since every compact subset of $BTOP$ is contained in some $BTOP_n$. Similarly, every map $X \times I \to BPL$ passes through some BPL_n, and therefore the injectivity holds. □

Furthermore, if ξ itself is a PL bundle then, by Theorem 1.3, there is a bijection

$$[\operatorname{Lift}_\alpha \bar{f}] \cong [X, TOP/PL].$$

Thus, in this case the bijection (3.3) turns into the bijection

$$\lim_{n\to\infty} \{\text{ PL structures on } \xi \oplus \theta^n\} \longrightarrow [X, TOP/PL]. \tag{3.4}$$

3.6. Definition. Let M be a ∂_{PL}-manifold. A *homotopy PL structuralization* of M is a homotopy equivalence $h : V \to X$ such that V is a PL manifold and $h : \partial V \to \partial M$ is a PL isomorphism. Two homotopy PL structuralizations $h_i : V_i \to X, i = 0, 1$ are *equivalent* if there exists a PL isomorphism $\varphi : V_0 \to V_1$ and a homotopy $H : V_0 \times I \to M$ such that $h_0 = H|V \times \{0\}$ and $H|V_0 \times \{1\} = h_1\varphi$ and, moreover, $H|V \times \{t\} : \partial V_0 \to \partial M$ coincides with h_0. Any equivalence class of homotopy PL structuralizations is called a *homotopy PL structure on* X. We denote by $\mathfrak{S}_{PL}(X)$ the set of all homotopy PL structures on X.

If M itself is a PL manifold, we define the *trivial element* of $\mathfrak{S}_{PL}(M)$ as the equivalence class of $1_M : M \to M$.

3.7. Definition. Given an $(S^n, *)$-fibration ξ over X, a *homotopy PL structuralization of* ξ is an $(S^n, *)$-morphism $\varphi : \xi \to (\gamma^n_{PL})^\bullet$. We say that two PL structuralizations $\varphi_0, \varphi_1 : \xi \to (\gamma^n_{PL})^\bullet$ are *equivalent* if there exists a morphism $\Phi : \xi \times 1_I \to (\gamma^n_{PL})^\bullet$ of $(S^n, *)$-fibrations such that $\Phi|\xi \times 1_{\{i\}} = \varphi_i, i = 0, 1$. Every such an equivalence class is called a *homotopy PL structure on* ξ.

Now, similarly to (3.4), for a finite CW-space X we have a bijection

$$\lim_{n\to\infty} \{\text{homotopy PL structures on } \xi \oplus \theta^n\} \longrightarrow [X, F/PL]. \tag{3.5}$$

However, here we can say more.

3.8. Proposition. *The sequence*

$$\{\{\text{homotopy PL structures on } \xi \oplus \theta^n\}\}_{n=1}^\infty$$

stabilizes. In particular, the map

$$\{\text{homotopy PL structures on } \xi \oplus \theta^n\} \to [F/PL]$$

is a bijection if $\dim \xi >> \dim X$

Proof. This follows from 2.5. $\square$

Summarizing, for every PL $\mathbb{R}^N$-bundle ξ we have a commutative diagram

$$\begin{tikzcd}
\{\text{PL structure on } \xi\} \arrow[r] \arrow[d] & {[X, TOP/PL]} \arrow[d, "a_*"'] \\
\{\text{homotopy PL structure on } \xi^\bullet\} \arrow[r] & {[X, F/PL]}
\end{tikzcd}$$

Here the map a in (2.5) induces the map $a_* : [X, TOP/PL] \to [X, F/PL]$. The left vertical arrow converts a morphism of $\mathbb{R}^N$ -bundles into a morphism of $(S^N, *)$-bundles and regards the last one as a morphism of $(S^N, *)$-fibrations.

For a finite CW-space X, the horizontal arrows turn into bijections if we stabilize the picture. i.e. pass to the limit as in (3.4). Furthermore, the bottom arrow is an isomorphism if $N >> \dim X$.

3.9. Remark. Actually, following the proof of the Main Theorem, one can prove that $TOP_m/PL_m = K(\mathbb{Z}/2, 3)$ for $m \geq 5$, see [28, Essay IV, §9]. So, an obvious analog of 2.5 holds for TOP also, and therefore the top map of the above diagram is a bijection for N large enough. But, of course, we are not allowed to use these arguments here.

3.10. Remark. 1. We can also consider *smooth* (=differentiable C^∞) structures on topological manifolds. To do this, we must replace the words "PL" in Definition 3.1 by the word "smooth". The related set of smooth concordance classes is denoted by $\mathfrak{T}_D(M)$.

The set $\mathfrak{S}_D(M)$ is defined in a similar way, we leave it to the reader.

Moreover, recall that every smooth manifold can be canonically converted into a PL manifold (the Cairns–Hirsch–Whitehead Theorem, see e.g [22]). So, we can define the set $\mathfrak{P}_D(M)$ of smooth structures on a PL manifold M. To do this, we must modify definition 3.1 as follows: M is a PL manifold with a compatible smooth boundary, V_i are smooth manifolds, h_i and H are PL isomorphisms.

2. We can now construct an example of two smooth structuralizations $h_i : V \to S^n, i = 1, 2$ which are not concordant. First, notice that there is a bijective correspondence between $\mathfrak{S}_D(S^n)$ and the Kervaire–Milnor group Θ_n of homotopy spheres, [25]. Indeed, Θ_n consists of equivalence classes of oriented homotopy spheres: two oriented homotopy spheres are equivalent if they are orientably diffeomorphic (=h-cobordant). Now, given a homotopy smooth structuralization $h : \Sigma^n \to S^n$, we orient Σ^n so that h has degree 1. In this way we get a well-defined map $u : \mathfrak{S}_D(S^n) \to \Theta_n$. Conversely, given a homotopy sphere Σ^n, consider a homotopy equivalence $h : \Sigma^n \to S^n$ of degree 1. In this way we get a well-defined map $\Theta_n \to \mathfrak{S}_D(S^n)$ which is inverse to u.

Notice that, because of the Smale Theorem, every smooth homotopy sphere $\Sigma^n, n \geq 5$, possesses a smooth function with just two critical points. Thus, $\mathfrak{S}_D(S^n) = \mathfrak{T}_D(S^n) = \mathfrak{P}_D(S^n)$ for $n \geq 5$. Kervaire and Milnor [25] proved that $\Theta_7 = \mathbb{Z}/28$, i.e., because of what we said above, $\mathfrak{S}_D(S^7) = \mathfrak{T}_D(S^7) = \mathfrak{P}_D(S^7)$ consists of 28 elements.

On the other hand, there are only 15 smooth manifolds which are homeomorphic (and PL isomorphic, and homotopy equivalent) to S^7 but mutually non-diffeomorphic. Indeed, if an oriented smooth 7-dimensional manifold Σ is homeomorphic to S^7 then Σ bounds a parallelizable manifold W_Σ, [25]. We equip W an orientation which is compatible with Σ and set

$$a(\Sigma) = \frac{\sigma(W_\Sigma)}{8} \mod 28$$

where $\sigma(W)$ is the signature of W. Kervaire and Milnor [25] proved that the correspondence

$$\Theta_7 \to \mathbb{Z}/28, \quad \Sigma \mapsto a(W_\Sigma)$$

is a well-defined bijection.

However, if $a(\Sigma_1) = -a(\Sigma_2)$ then Σ_1 and Σ_2 are diffeomorphic: namely, Σ_2 is just the Σ_1 with the opposite orientation. So, there are only 15 smooth manifolds which are homeomorphic (and homotopy equivalent, and PL isomorphic) to S^7 but mutually non-diffeomorphic.

In terms of structures, it can be expressed as follows. Let $\rho : S^n \to S^n$ be a diffeomorphism of degree -1. Then the smooth structuralizations $h : \Sigma^7 \to S^7$ and $\rho h : \Sigma^7 \to S^7$ are not concordant, if $a(\Sigma^7) \neq 0, 14$.

For convenience of references, we fix here the following theorem of Smale [50]. Actually, Smale proved it for smooth manifolds, a good proof can also be found in Milnor [36]. However, the proof can be transmitted to the PL case, see Stallings [52, 8.3, Theorem A].

3.11. Theorem. *Let M be a closed PL manifold which is homotopy equivalent to the sphere $S^n, n \geq 5$. Then M is PL isomorphic to S^n.* □

4. From manifolds to bundles

Recall that, for every topological manifold M^n, its tangent bundle τ_M and (stable) normal bundle ν_M are defined. Here τ_M is a topological $\mathbb{R}^n$-bundle, and we can regard ν_N as a topological $\mathbb{R}^N$-bundle with $N >> n$. Furthermore, if M is a PL manifold then τ_M and ν_M turns into PL bundles in a canonical way, see [28, 48].

4.1. Construction. Consider a PL manifold M and a PL structuralization $h : V \to M$. Let $g = h^{-1} : M \to V$. Since g is a homeomorphism, it yields a topological morphism $\lambda : \nu_M \to \nu_V$, and so we have the correcting topological morphism $c(\lambda) : \nu_M \to g^*\nu_V$. Now, the morphism

$$\nu_M \xrightarrow{c(\lambda)} g^*\nu_V \xrightarrow{\text{classif}} \gamma^N_{PL}$$

is a PL structuralization of ν_M. It is easy to see that in this way we have the correspondence

$$j_{TOP} : \mathfrak{T}_{PL}(M) \longrightarrow \{\text{PL structures on } \nu_M\} \longrightarrow [M, TOP/PL]$$

where the last map comes from 3.4. Moreover, it is clear that, in fact, j_{TOP} passes through the map

$$[(M, \partial M), (TOP/PL, *)] \to [M, TOP/PL].$$

So, we can and shall regard j_{TOP} as the map

$$j_{TOP} : \mathfrak{T}_{PL}(M) \longrightarrow [(M, \partial M), (TOP/PL, *)].$$

4.2. Remark. We constructed the map j_{TOP} using PL structuralizations of ν_M. However, we can also construct the map j_{TOP} by considering other bundles related to M. For example, consider the tangent bundle τ_M. The topological morphism

$$\tau_M \longrightarrow g^*\tau_V \xrightarrow{\text{classif}} \gamma^n_{PL}$$

can be regarded as a PL structuralization of τ_M, and so we have another way of constructing of j_{TOP}. Moreover, we can also consider the topological morphism

$$\theta^{N+n} = \tau_M \oplus \nu^N_M \longrightarrow g^*\tau_V \oplus \nu^N_M \xrightarrow{\text{classif}} \gamma^{N+n}_{PL}$$

and regard it as a PL structuralization of θ^{N+n}, etc. One can prove that all these constructions are equivalent.

Now we construct a map $j_F : \mathfrak{S}_{PL}(M) \to [M, F/PL]$, a "homotopy analogue" of j_{TOP}. This construction is more delicate, and we treat only the case of closed manifolds here. So, let M be a connected closed PL manifold.

4.3. Definition. Given an $(S^n, *)$-fibration $\xi = \{E \to B\}$ with a section $s : B \to E$, we define its *Thom space* $T\xi$ as the quotient space $E/s(B)$, Given a topological $\mathbb{R}^N$-bundle η, we define the Thom space $T\eta$ as $T\eta := T(\eta^\bullet)$.

Given a morphism $\varphi : \xi \to \eta$ of $(S^n, *)$-fibrations, we define $T\varphi : T\xi \to T\eta$ to be the unique map such that the diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ T\xi & \xrightarrow{T\varphi} & T\eta \end{array}$$

commutes. Here E' is the total space of η.

4.4. Definition. A pointed space X is called *reducible* if there is a pointed map $f : S^m \to X$ such that $f_* : \widetilde{H}_i(S^m) \to \widetilde{H}_i(X)$ is an isomorphism for $i \geq m$. Every such map f (as well as its homotopy class or its stable homotopy class) is called a *reduction* for X.

We embed M in $\mathbb{R}^{N+n}$, $N >> n$, and let $\nu_M, \dim \nu_M = N$ be a normal bundle of this embedding. Recall that ν_M is a PL bundle $E \to M$ whose total space E is PL isomorphic to a (tubular) neighbourhood U of M in $\mathbb{R}^{N+n}$. We choose such isomorphism and denote it by $\varphi : U \to E$.

One can prove that, for N large enough, the normal bundle always exists, [20, 29, 30].

4.5. Construction–Definition. Let $T\nu_M$ be the Thom space of ν_M. Then there is a unique map

$$\psi : \mathbb{R}^{N+n}/(\mathbb{R}^{N+n} \setminus U) \to T\nu$$

such that $\psi|U = \varphi$. We define the *collapse map* $\iota : S^{N+n} \to T\nu_M$ (the Browder–Novikov map) to be the composition

$$\iota : S^{N+n} \xrightarrow{\text{quotient}} S^{N+n}/(S^{N+n} \setminus U) = \mathbb{R}^{N+n}/(\mathbb{R}^{N+n} \setminus U) \xrightarrow{\psi} T\nu.$$

See [5, II.2.11] or [48, 7.15] for details.

It is well known and easy to see that ι is a reduction for $T\nu$, see Corollary 11.7 below.

It turns out that, for N large enough, the normal bundle of a given embedding $M \to \mathbb{R}^{N+n}$ is unique. For detailed definitions and proofs, see [20, 29, 30]. The uniqueness gives us the following important fact. Let $\nu' = \{E' \to M\}$ be another normal bundle and $\varphi' : U' \to E'$ be another PL isomorphism. Let $\iota : S^{N+n} \to T\nu$ and $\iota' : S^{N+n} \to T\nu'$ be the corresponding Browder–Novikov maps. Then there is a morphism $\nu \to \nu'$ of PL bundles which carries ι to a map homotopic to ι'.

4.6. Theorem. *Consider a PL $\mathbb{R}^N$-bundle η over M such that $T\eta$ is reducible. Let $\alpha \in \pi_{N+n}(T\eta)$ be an arbitrary reduction for $T\eta$. Then there exist an F_N-equivalence $\mu : \nu_M^\bullet \to \eta^\bullet$ such that $(T\mu)_*(\iota) = \alpha$, and such a μ is unique up to fiberwise homotopy over M.*

Proof. We postpone it to the next Chapter, see 11.11. □

4.7. Construction–Definition. Given a homotopy equivalence $h : V \to M$ of closed connected PL manifolds, let ν_V be a normal bundle of a certain embedding $V \subset \mathbb{R}^{N+n}$, and let $u \in \pi_{N+n}(T\nu_V)$ be the homotopy class of a collapsing map $S^{N+n} \to T\nu_V$. Let $g : M \to V$ be homotopy inverse to h and set $\eta = g^*\nu_V$. The adjoint to g morphism

$$\varphi := \mathfrak{I}_{g,\nu_V} : \eta \to \nu_V$$

yields the map $T\varphi : T\eta \to T\nu_V$. It is easy to see that $T\varphi$ is a homotopy equivalence, and so there exists a unique $\alpha \in \pi_{N+n}(T\eta)$ with $(T\varphi)_*(\alpha) = u$. Since u is a reduction for $T\nu_V$, we conclude that α is a reduction for $T\eta$. By Theorem 4.6, we get an F-equivalence $\mu : \nu_M^\bullet \to \eta^\bullet$ with $(T\mu)_*(\iota) = \alpha$. Now, the morphism

$$(\nu_M)^\bullet \xrightarrow{\mu} \eta^\bullet \xrightarrow{\text{classif}} \gamma_F^N$$

is a homotopy PL structuralization of ν_M. Because of the uniqueness of the normal bundle, the concordance class of this structuralization is well defined. So, in this way we have the correspondence

$$j_F : \mathfrak{S}_{PL}(M) \longrightarrow \{\text{homotopy PL structures on } \nu_M\} \cong [M, F/PL]$$

where the last bijection comes from 3.8.

The map j_F is called the *normal invariant*, and its value on a homotopy PL structure is called the normal invariant of this structure.

Notice that there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{T}_{PL}(M) & \xrightarrow{j_{TOP}} & [M, TOP/PL] \\ \beta \Big\downarrow & & \Big\downarrow a_* \\ \mathfrak{S}_{PL}(M) & \xrightarrow{j_F} & [M, F/PL] \end{array} \tag{4.1}$$

where β regards a PL structuralization as the homotopy PL structuralization.

5. Homotopy PL structures on $T^k \times D^n$

Below T^k denotes the k-dimensional torus.

5.1. Theorem. *Assume that $k+n \geq 5$. If $x \in \mathfrak{S}_{PL}(T^k \times S^n)$ can be represented by a homeomorphism $M \to T^k \times S^n$ then $j_F(x) = 0$.* □

This is a special case of the Sullivan Normal Invariant Homeomorphism Theorem. We prove 5.1 (in fact, a little bit general result) in the next chapter.

We also prove the Sullivan Theorem in full generality in Chapter 3, section 21. We must do this repetition since the proof in Chapter 3 uses 5.1.

5.2. Construction–Definition. Let $x \in \mathfrak{T}_{PL}(M)$ be represented by a map $h : V \to M$, and let $p : \widetilde{M} \to M$ be a covering. Then we have a commutative diagram

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\widetilde{h}} & \widetilde{M} \\ q \Big\downarrow & & \Big\downarrow p \\ V & \xrightarrow{h} & M \end{array}$$

where q is the induced covering. Since $\widetilde{h}$ is defined uniquely up to deck transformations, the concordance class of $\widetilde{h}$ is well defined. So, we have a well-defined map

$$p^* : \mathfrak{T}_{PL}(M) \to \mathfrak{T}_{PL}(\widetilde{M})$$

where $p^*(x)$ is the concordance class of $\widetilde{h}$. Similarly, one can construct a map

$$p^* : \mathfrak{S}_{PL}(M) \to \mathfrak{S}_{PL}(\widetilde{M}).$$

If p^* is a finite covering, we say that a class $p^*(x) \in \mathfrak{S}_{PL}(\widetilde{M})$ *finitely covers* the class x.

5.3. Theorem. *Let $k + n \geq 5$ Then the following holds:*

(i) *if $n > 3$ then the set $\mathfrak{S}_{PL}(T^k \times D^n)$ consists of precisely one* (*trivial*) *element;*

(ii) *if $n < 3$ then every element from $\mathfrak{S}_{PL}(T^k \times D^n)$ can be finitely covered by the trivial element;*

(iii) *the set $\mathfrak{S}_{PL}(T^k \times D^3)$ contains at most one element which cannot be finitely covered by the trivial element.*

Some words about the proof. First, we mention the proof given by Wall, [62] and [63, Section 15 A]. Wall proved the bijection $w : \mathfrak{S}_{PL}(T^k \times D^n) \to H^{3-n}(T^k)$. Moreover, he also proved that finite coverings respect this bijection, i.e. if $p : T^k \times D^n \to T^k \times D^n$ is a finite covering then there is the commutative diagram

$$\begin{array}{ccc} \mathfrak{S}_{PL}(T^k \times D^n) & \xrightarrow{w} & H^{3-n}(T^k; \mathbb{Z}/2) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ \mathfrak{S}_{PL}(T^k \times D^n) & \xrightarrow{w} & H^{3-n}(T^k; \mathbb{Z}/2)\,. \end{array}$$

Certainly, this result implies all the claims(i)–(iii). Walls proof uses difficult algebraic calculations.

Another proof of the theorem can be found in [23, Theorem C]. Minding the complaint of Novikov concerning Sullivan's results (see Prologue), we must mention that Hsiang and Shaneson [23] use a Sullivan's result. Namely, they consider the so-called surgery exact sequence

$$\xrightarrow{\partial} \mathfrak{S}_{PL}(S^k \times T^n) \xrightarrow{j_F} [S^k \times T^n, F/PL] \longrightarrow \cdots$$

and write (page 42, Section 10):

> By [44], every homomorphism $h : M \to S^k \times T^n, k = n \geq 5$, represents an element in the image of ∂.

Here the item [44] of the citation is our bibliographical item [56]. So, in fact, Hsiang and Shaneson use Theorem 5.1. As I already said, we prove 5.1 in next Chapter and thus fix the proof. □

6. The product structure theorem, or from bundles to manifolds

Let M be an n-dimensional ∂_{PL}-manifold. Then every PL structuralization $h : V \to M$ yields a PL structuralization

$$h \times 1 : V \times \mathbb{R}^k \to M \times \mathbb{R}^k.$$

Thus, we have a well-defined map

$$e : \mathfrak{T}_{PL}(M) \to \mathfrak{T}_{PL}(M \times \mathbb{R}^k).$$

6.1. Theorem (The Product Structure Theorem). *For every $n \geq 5$ and every $k \geq 0$, the map $e : \mathfrak{T}_{PL}(M) \to \mathfrak{T}_{PL}(M \times \mathbb{R}^k)$ is a bijection.*

In particular, if $\mathfrak{T}_{PL}(M \times \mathbb{R}^k) \neq \emptyset$ then $\mathfrak{T}_{PL}(M) \neq \emptyset$.

Concerning the proof. I did not find a proof which is essentially better then the original one. So, I refer the reader to the original source [28]. I want also to mention here that the proof of Theorem 6.1 uses the Theorem 5.3 for $n = 0$. For another approach to the proof of Theorem 6.1, see [13, Remark 5.3]. □

6.2. Corollary (The Classification Theorem). *If $\dim M \geq 5$ and M admits a PL structure, then the map*

$$j_{TOP} : \mathfrak{T}_{PL}(M) \to [(M, \partial M), (TOP/PL, *)]$$

is a bijection.

Proof. We construct a map

$$\sigma : [(M, \partial M), (TOP/PL, *)] \to \mathfrak{T}_{PL}(M) \tag{6.1}$$

which is inverse to j_{TOP}. For simplicity of notations, we consider the case of M closed. Take an element $a \in [M, TOP/PL]$ and, using 3.4, interpret it as a concordance class of a morphism $\varphi : \theta_M^N \to \gamma_{PL}^N$ of topological $\mathbb{R}^N$-bundles (cf. Remark 4.2). The morphism φ yields a correcting isomorphism $\theta_M^N \to b^*\gamma_{PL}^N$ of topological $\mathbb{R}^N$-bundles over M, where $b : M \to BPL$ is the base of the morphism φ. So, we have the commutative diagram

$$\begin{array}{ccc} M \times \mathbb{R}^N & \xrightarrow{h} & W \\ \downarrow & & \downarrow \\ M & = & M \end{array}$$

where h is a fiberwise homeomorphism and $W \to M$ is a PL $\mathbb{R}^N$-bundle $b^*\gamma_{PL}^N$. In particular, W is PL manifold. Regarding $h^{-1}W \to M \times \mathbb{R}^N$ as a PL structuralization of $M \times \mathbb{R}^N$, we conclude that, by the Product

Structure Theorem 6.1, h^{-1} is concordant to a map $g \times 1$ for some PL structuralization $g : V \to M$. We define $\sigma(a) \in \mathfrak{T}_{PL}(M)$ to be the concordance class of g. One can check that σ is a well-defined map which is inverse to j_{TOP}. Cf. [28, Essay IV]. $\square$

6.3. Corollary (The Existence Theorem)**.** *A topological manifold M with* $\dim M \geq 5$ *admits a PL structure if and only if the tangent bundle to M admits a PL structure.*

Proof. Only claim "if" needs a proof. Let $\tau = \{\pi : D \to M\}$ be the tangent bundle of M, and let $\nu = \{r : E \to M\}$ be a stable normal bundle of M, $\dim \nu = N$. Then E is homeomorphic to an open subset of $\mathbb{R}^{N+n}$, and therefore we can (and shall) regard E as a PL manifold. Since τ is a PL bundle, we conclude that $r^*\tau$ is a PL bundle over E. In particular, the total space $M \times \mathbb{R}^{N+n}$ of $r^*\tau$ turns out to be a PL manifold. Now, because of the Product Structure Theorem 6.1, M admits a PL structure. $\square$

Let $f : M \to BTOP$ classify the stable tangent bundle of a closed topological manifold M, $\dim M \geq 5$.

6.4. Corollary. *The following conditions are equivalent:*

(i) *M admits a PL structure;*
(ii) *τ admits a PL structure;*
(iii) *there exists k such that $\tau \oplus \theta^k$ admits a PL structure;*
(iv) *the map f admits an α_{TOP}^{PL}-lifting to BPL.*

Proof. It suffices to prove that (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). The implication (iii) $\Longrightarrow$ (i) can be proved similarly to 6.3. Furthermore, since M is compact, we conclude that $f(M) \subset BTOP_m$ for some m. So, if (iv) holds then f lifts to BPL_m, i.e. $\tau \oplus \theta^{m-k}$ admits a PL structure. $\square$

6.5. Remark. It follows from 1.3, 6.3 and 6.2 that the set $\mathfrak{T}_{PL}(M)$ of PL structures on M is in a bijective correspondence with the set of fiber homotopy classes of α_{TOP}^{PL}-liftings of f. (We leave it to the reader to extend the result on ∂_{PL}-manifolds.)

6.6. Remark. It is well known that j_F is not a bijection in general. The "kernel"' and "cokernel" of j_F can be described in terms of so-called Wall groups, [63]. (For M simply-connected, see also Theorem 13.2.) On the other hand, the bijectivity of j_{TOP} (the Classification Theorem) follows from the Product Structure Theorem. So, informally speaking, kernel and cokernel of j_F play the role of obstructions to splitting of structures. It seems interesting to develop and clarify these naive arguments.

6.7. Remark. Since tangent and normal bundles of smooth manifolds turn out to be vector bundles, one can construct a map

$$k : \mathfrak{P}_D(M) \to [M, PL/O]$$

which is an obvious analog of j_{TOP} (here we assume M to be closed). Moreover, the obvious analog of the Product Structure Theorem (as well as of the Classification and Existence Theorems) holds without any dimensional restriction. In particular, k is a bijection for every smooth manifold, [22].

It is well known (although difficult to prove) that $\pi_i(PL/O) = 0$ for $i \leq 6$. (See [48, IV.4.27(iv)] for the references.) Thus, every PL manifold M of dimension ≤ 7 admits a smooth structure, and this structure is unique if $\dim M \leq 6$.

7. Non-contractibility of TOP/PL

7.1. Theorem (Freedman's Example). *There exists a closed topological* 4*-dimensional manifold F with $w_1(F) = 0 = w_2(F)$ and such that the signature of F is equal to* 8*.*

Here w_i denotes the i-th Stiefel–Whitney class.

Proof. See [14] or the original work [15]. □

7.2. Theorem (Rokhlin Signature Theorem). *Let M be a closed* 4*-dimensional PL manifold with $w_1(M) = 0 = w_2(M)$. Then the signature of M is divisible by* 16*.*

Proof. See [26, 37] or the original work [44]. In fact, Rokhlin proved the result for smooth manifolds, but the proof works for PL manifolds as well. On the other hand, in view of 6.7, there is no difference between smooth and PL manifolds in dimension 4. □

7.3. Corollary. *The topological manifolds F and $F \times T^k, k \geq 1$ do not admit any PL structure.*

Proof. The claim about F follows from 7.2. Suppose that $F \times T^k$ has a PL structure. Then $F \times \mathbb{R}^k$ has a PL structure. So, because of the Product Structure Theorem 6.1, $F \times \mathbb{R}$ has a PL structure. Hence, by 6.7, it possesses a smooth structure. Then the projection $p_2 : F \times \mathbb{R} \to \mathbb{R}$ can be C^0-approximated by a map $f : F \times \mathbb{R} \to \mathbb{R}$ which coincides with p_2 on $F \times (-\infty, 0)$ and is smooth on $F \times (1, \infty)$. Take a regular value $a \in (0, \infty)$ of f (which exists because of the Sard Theorem) and set $W = f^{-1}(a)$. Then W is a smooth manifold (by the Implicit Function Theorem), and it is easy to see that $w_1(W) = 0 =$

$w_2(W)$ (because it holds for both manifolds $\mathbb{R}$ and $F \times \mathbb{R}$). On the other hand, both manifolds F and W cut the "tube" $F \times \mathbb{R}$. So, they are (topologically) bordant, and therefore W has signature 8. But this contradicts the Rokhlin Theorem 7.2.

□

7.4. Corollary. *The space TOP/PL is not contractible.*

Proof. Indeed, suppose that TOP/PL is contractible. Then every map $X \to BTOP$ lifts to BPL, and so, by 6.3, every topological manifold of dimension greater than 4 admits a PL structure. But this contradicts 7.3. □

7.5. Remark. Kirby and Siebenmann [27, 28] constructed the original example of a topological manifold which does not admit a PL structure. Again, the Rokhlin Theorem 7.2 is one of the main ingredients of the proof.

8. Homotopy groups of TOP/PL

Let M be a compact topological manifold equipped with a metric ρ. Then the space $\mathcal{H}$ of homeomorphisms gets a metric d with $d(f,g) = \sup\{x \in M \mid \rho(f(x), g(x))\}$.

8.1. Theorem. *The space $\mathcal{H}$ is locally contractible.*

Proof. See [7, 11]. □

8.2. Corollary. *There exists $\varepsilon > 0$ such that every homeomorphism $h \in \mathcal{H}$ with $d(H, 1_M) < \varepsilon$ is isotopic to 1_M.* □

8.3. Construction. We regard the torus T^k as a commutative Lie group (multiplicative) and equip it with the invariant metric ρ. Consider the map $p_\lambda : T^k \to T^k, p_\lambda(a) = a^\lambda, \lambda \in \mathbb{N}$. Then p_λ is a λ^k-sheeted covering. It is also clear that all the deck transformations of the covering torus are isometries.

8.4. Lemma. *Let $h : T^k \times D^n \to T^k \times D^n$ is a self-homeomorphism which is homotopic $\operatorname{rel} \partial(T^k \times D^n)$ to the identity. Then there exists a commutative diagram*

$$\begin{array}{ccc}
T^k \times D^n & \xrightarrow{\widetilde{h}} & T^k \times D^n \\
{\scriptstyle p_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle p_\lambda} \\
T^k \times D^n & \xrightarrow{h} & T^k \times D^n
\end{array}$$

where the lifting $\widetilde{h}$ of h is isotopic $\operatorname{rel} \partial(T^k \times D^n)$ to the identity.

Proof. (Cf. [28, Essay V].) First, consider the case $n = 0$. Without loss of generality we can assume that $h(e) = e$ where e is the neutral element of T^k. Consider a covering $p_\lambda : T^k \to T^k$ as in 8.3 and take a covering $\widetilde{h} : T^k \to T^k, p_\lambda \widetilde{h} = \widetilde{h} p_\lambda$ of h such that $\widetilde{h}(e) = e$. In order to distinguish the domain and the range of p_λ, we denote the domain of p_λ by $\widetilde{T}$ and the range of p_λ by T. Since all the deck transformations of $\widetilde{T}$ are isometries, we conclude that the diameter of each of (isometric) fundamental domain tends to zero as $\lambda \to \infty$. Furthermore, since h is homotopic to 1_T, we conclude that every point of the lattice $L := p_\lambda^{-1}(e)$ is fixed under $\widetilde{h}$.

Given $\varepsilon > 0$, choose δ such that $\rho(\widetilde{h}(x), \widetilde{h}(y)) < \varepsilon/2$ whenever $\rho(x, y) < \delta$. Furthermore, choose λ so large that the diameter of any closed fundamental domain is less then $\min\{\varepsilon/2, \delta\}$. Now, given $x \in \widetilde{T}$, choose $a \in L$ such that a and x belong to the same closed fundamental domain. Now,

$$\rho(x, \widetilde{h}(x) \leq \rho(x, a) + \rho(a, \widetilde{h}(x)) = \rho(x, a) + \rho(\widetilde{h}(a), \widetilde{h}(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, for every $\varepsilon > 0$ there exists λ such that $d(\widetilde{h}, 1_{\widetilde{T}}) < \varepsilon$. Thus, by 8.2, $\widetilde{h}$ is isotopic to $1_{\widetilde{T}}$ for λ large enough.

The proof for $n > 0$ is similar but a bit more technical. Let $D_\eta \subset D^n$ be the disk centered at 0 and having the radius η. We can always assume that h coincides with identity outside of $T^k \times D_\eta$. Now, asserting as for $n = 0$, take a covering p_λ as above and choose λ and η so small that the diameter of every fundamental domain in $\widetilde{T} \times D_\eta$ is small enough. Then

$$\widetilde{h} : \widetilde{T} \times D_\eta \to \widetilde{T} \times D_\eta$$

is isotopic to the identity (and $\widetilde{h}$ coincides with identity outside $\widetilde{T} \times D_\eta$). This isotopy is not an isotopy rel $\widetilde{T} \times \partial D_\eta$. Nevertheless, we can easily extend it to the whole $\widetilde{T} \times D^n$ so that this extended isotopy is an isotopy rel $\partial(\widetilde{T} \times D^n)$.

If you want formulae, do the following. Given $a = (b, c) \in \widetilde{T} \times D_\eta$, set $|a| = |c|$. Consider an isotopy

$$\varphi : \widetilde{T} \times D_\eta \times I \to \widetilde{T} \times D_\eta \times I, \quad \varphi(a, 0) = a, \; \varphi(a, 1) = \widetilde{h}(a), \quad a \in \widetilde{T} \times D_\eta.$$

Define $\psi : \widetilde{T} \times D_\eta \times I \to \widetilde{T} \times D_\eta = \times I$ by setting

$$\psi(a, t) = \begin{cases} \varphi(a, t) & \text{if } |a| \leq \eta, \\ \varphi(a, \frac{|a|-1}{\eta-1} t) & \text{if } |a| \geq \eta. \end{cases}$$

Then ψ is the desired isotopy rel $\partial(\widetilde{T} \times D^n)$. $\square$

8.5. Corollary. *Let $\beta : \mathfrak{T}_{PL}(T^k \times D^n) \to \mathfrak{S}_{PL}(T^k \times D^n)$ be the forgetful map as in (4.1). If $\beta(x) = \beta(y)$ then there exists a finite covering $p : T^k \times D^n \to T^k \times D^n$ such that $p^*(x) = p^*(y)$.* □

Consider the map

$$\begin{aligned}\psi : \quad & \pi_n(TOP/PL) = [(D^n, \partial D^n), (TOP/PL, *)] \xrightarrow{p_2^*} \\ & [(T^k \times D^n, \partial(T^k \times D^n)), TOP/PL] \xrightarrow{j_{TOP}} \mathfrak{T}_{PL}(T^k \times D^n)\end{aligned}$$

where σ is the map from (6.1).

8.6. Lemma. *The map ψ is injective. Moreover, if $p^*\psi(x) = p^*\psi(y)$ for some finite covering $p : T^k \times D^n \to T^k \times D^n$ then $x = y$.*

In particular, if $p^*\psi(x)$ is the trivial element of $\mathfrak{T}_{PL}(T^k \times D^n)$ then $x = 0$.

Proof. The injectivity of ψ follows from the injectivity of p_2^* and σ. Furthermore, for every finite covering $p : T^k \times D^n \to T^k \times D^n$ we have the commutative diagram

$$\begin{array}{ccc}\pi_n(TOPL/PL) & \xrightarrow{\psi} & \mathfrak{T}_{PL}(T^k \times D^n) \\ \| & & \uparrow{\scriptstyle p^*} \\ \pi_n(TOPL/PL) & \xrightarrow{\psi} & \mathfrak{T}_{PL}(T^k \times D^n)\end{array}$$

Therefore $x = y$ whenever $p^*\psi(x) = p^*\psi(y)$. Finally, if $p^*\psi(x)$ is trivial element then $p^*\psi(x) = p^*\psi(0)$, and thus $x = 0$. □

Consider the map

$$\varphi : \pi_n(TOP/PL) \xrightarrow{\psi} \mathfrak{T}_{PL}((T^k \times D^n) \xrightarrow{\beta} \mathfrak{S}_{PL}((T^k \times D^n)$$

where β is the forgetful map described in (4.1).

8.7. Theorem (The Reduction Theorem)**.** *The map φ is injective. Moreover, if $p^*\varphi(x) = p^*\varphi(y)$ for some finite covering*

$$p : T^k \times D^n \to T^k \times D^n$$

then $x = y$.

In particular, if $p^*\varphi(x)$ is the trivial element of $\mathfrak{T}_{PL}(T^k \times D^n)$ then $x = 0$.

We call it the Reduction Theorem because it *reduces* the calculation of the group $\pi_i(TOP/PL)$ to the calculation of the sets $\mathfrak{S}_{PL}(T^k \times D^n)$.

Proof. If $\varphi(x) = \varphi(y)$ then $\beta\psi(x) = \beta\psi(y)$. Hence, by Corollary 8.5, there exists a finite covering $\pi : T^k \times D^n \to T^k \times D^n$ such that $\pi^*\psi(x) = \pi^*\psi(y)$. So, by Lemma 8.6, $x = y$, i.e. φ is injective.

Now, suppose that $p^*\varphi(x) = p^*\varphi(y)$ for some finite covering $p : T^k \times D^n \to T^k \times D^n$. Then $\beta^*p^*\psi(x) = \beta^*p^*\psi(y)$. Now, by Corollary 8.5, there exists a finite covering

$$q : T^k \times D^n \to T^k \times D^n$$

such that $q^*p^*\psi(x) = q^*p^*\psi(y)$, i.e. $(pq)^*\psi(x) = (pq)^*\psi(y)$. Thus, by Lemma 8.6, $x = y$. □

8.8. Corollary (The Main Theorem)**.** *$\pi_i(TOP/PL) = 0$ for $i \neq 3$. Furthermore, $\pi_3(TOP/PL) = \mathbb{Z}/2$. Thus, $TOP/PL = K(\mathbb{Z}/2.3)$.*

Proof. The equality $\pi_i(TOP/PL) = 0$ for $i \neq 3$ follows from Theorem 5.3(i,ii) and Theorem 8.7. Furthermore, again because of 5.3 and 8.7, we conclude that $\pi_3(TOP/PL)$ has at most two elements. In other words, $TOP/PL = K(\pi, 3)$ where $\pi = \mathbb{Z}/2$ or $\pi = 0$. Finally, by Corollary 7.4, the space TOP/PL is not contractible. Thus, $TOP/PL = K(\mathbb{Z}/2, 3)$. □

8.9. Remark. Notice that, for $i > 5$, the set $\mathfrak{T}_{PL}(S^i)$ consists of just one element by the Smale Theorem 3.11. Because of this, the equality $\pi_i(TOP/PL) = 0, i > 5$ follows from Theorem 6.2. However, for i small we need Theorem 5.3. (Moreover, the proof of Theorem 6.1 uses 5.3 for $n = 0$.)

From now on and till the end of the section we fix a closed topological manifold M and let $f : M \to BTOP$ denote the classifying map for the stable tangent bundle of M.

8.10. Construction–Definition. Let

$$\varkappa \in H^4(BTOP; \pi_3(TOP/PL)) = H^4(BTOP; \mathbb{Z}/2)$$

be the characteristic class of the fibration

$$\zeta := \{\alpha_{TOP}^{PL} : BPL \to BTOP\},$$

(see e.g. [39, 48] for the definition of the characteristic class of the fibration). We define the *Kirby–Siebenmann class* $\varkappa(M) \in H^4(M; \mathbb{Z}/2)$ *of M* by setting

$$\varkappa(M) = f^*\varkappa.$$

Clearly, the class $\varkappa(M)$ can also be described as the characteristic class of the TOP/PL-fibration $f^*\zeta$ over M.

8.11. Corollary. *The manifold M admits a PL structure if and only if $\varkappa(M) = 0$. In particular, if $H^4(M; \mathbb{Z}/2) = 0$ then M admits a PL structure. Furthermore, if M admits a PL structure then the set of all PL structure on M is in a bijective correspondence with $H^3(M; \mathbb{Z}/2)$.*

Proof. By 6.4, M admits a PL structure if and only if f admits an α_{TOP}^{PL}-lifting.

$$\begin{array}{ccc} & & BPL \\ & & \downarrow \\ M & \xrightarrow{f} & BTOP \end{array}$$

By the Main Theorem 8.8, the fiber of the fibration

$$\alpha_{TOP}^{PL} : BPL \longrightarrow BTOP$$

is the Eilenberg–Mac Lane space $K(\mathbb{Z}/2, 3)$. Thus, because of the obstruction theory, f lifts to BPL if and only if $f^*\varkappa = 0$.

Finally, by 6.2 and 8.8, we have the bijections

$$\mathfrak{T}_{PL}(M) \cong [M, TOP/PL] \cong [M, K(\mathbb{Z}/2, 3] = H^3(M; \mathbb{Z}/2)$$

provided $\mathfrak{T}_{PL}(M) \neq \emptyset$. □

9. Do it

We recommend that the reader return to the introduction and look again the graph of our proof of the Main Theorem.

Chapter 2. Tools

Here we prove two important results which we used without proofs in Chapter 1. First, we prove Theorem 4.6. Notice that Browder [5] proved the Theorem for M simply-connected. In fact, his proof works for every orientable M. Here we follow Browder's proof, the only essential modification is that we use the duality theorem 11.6 for arbitrary M while Browder [5] uses it for M simply-connected.

Another goal of this chapter is to prove Theorem 5.1. In fact, we prove here a little bit more general result, Theorem 17.1. The proof uses the Sullivan's result on the homotopy type of F/PL. Notice that Madsen and Milgram [31] gave a detailed proof of those Sullivan result.

10. Stable equivalences of spherical bundles

We denote by $\sigma^k = \sigma^k_X$ the trivial S^k-bundle over X with a fixed section. In another words, $\sigma^k = (\theta^k)^\bullet$.

Given a sectioned spherical bundle ξ over a finite CW-space X, let $\operatorname{aut}\xi$ denote the group of fiberwise homotopy classes of self-equivalences $\xi \to \xi$ over X, where we assume that all homotopies preserve the section.

10.1. Proposition. *There is a natural bijection*

$$\operatorname{aut}\sigma^k = [X, F_k].$$

Proof. Because of the exponential law, every map $X \to F_k$ yields a section-preserving map $X \times S^k \to X \times S^k$ over X, and vice versa. Cf. [5, Prop. I.4.7]. □

Consider the map

$$\mu : F_k \times F_k \to F_{2k}, \quad \mu(a,b) = a \wedge b : S^{2k} = S^k \wedge S^k \to S^k \wedge S^k = S^{2k}$$

where we regard $a, b \in F_k$ as pointed maps $S^k \to S^k$. Let $T : F_k \times F_k \to F_k \times F_k$ be the transpose map, $T(a,b) = (b,a)$.

10.2. Lemma. *The maps $\mu : F_k \times F_k \to F_{2k}$ and $\mu T : F_k \times F_k \to F_{2k}, k > 0$ are homotopic.*

Proof. Consider the map

$$\tau : S^{2k} = S^k \wedge S^k \to S^k \wedge S^k = S^{2k}, \quad \tau(u,v) = (v,u)$$

and notice that, for every $a, b \in F_k$, we have

$$(\mu \circ T)(a,b) = \tau \circ \mu(a,b) \circ \tau.$$

First, consider the case of k odd. Then there is a pointed homotopy h_t between τ and $1_{S^{2k}}$. Now, the pointed homotopy $h_t\circ\mu(a,b)\circ h_t$ is a pointed homotopy between $(\mu\circ T)(a,b)$ and $\mu(a,b)$ which yields a homotopy between μT and μ.

Now consider the case of k even. We regard S^{2k} as $\mathbb{R}^{2k}\cup\infty$ with $\mathbb{R}^{2k}=\{(x_1,\dots,x_{2k})\}$ and define $\tau',\tau'':S^{2k}\to S^{2k}$ by setting

$$\begin{aligned}\tau'(x_1,x_2,x_3,\dots,x_{2k}) &= (x_2,x_1,x_3,\dots,x_{2k}),\\ \tau''(x_1,\dots,x_{2k-2}x_{2k-1}x_{2k}) &= (x_1,\dots,x_{2k-1},x_{2k-2},x_{2k}),\end{aligned}$$

(i.e. τ' permutes the first two coordinates and τ'' permutes the last two coordinates). Since k is even, we conclude that $\tau'\simeq\tau\simeq\tau''$. Furthermore, $\tau''\tau'\simeq 1_{S^{2k}}$. If we fix such pointed homotopies then we get the pointed homotopies

$$\begin{aligned}(\mu\circ T)(a,b) &= \tau\circ\mu(a,b)\tau\simeq\tau''\circ\mu(a,b)\tau'=\tau''\circ(a\wedge b)\tau'\\ &= \tau''\circ(a\wedge 1)\circ(1\wedge b)\tau'=(a\wedge\tau'')\circ(\tau'\wedge b)\\ &= (a\wedge 1)\circ(\tau''\tau')\circ(1\wedge b)\simeq a\wedge b=\mu(a,b)\end{aligned}$$

which yield the homotopy $\mu\circ T\simeq\mu$. □

10.3. Corollary. *Let $\varphi,\psi:\sigma^k\to\sigma^k$ be two automorphisms of σ^k. Then the automorphisms $\varphi\dagger\psi$ and $\psi\dagger\varphi$ of σ^{2k} are fiberwise homotopic.* □

Given two spherical bundles ξ and η over X, consider the bundle $\xi\wedge\eta$ over $X\times X$. We denote by $\Delta:X\to X\times X$ the diagonal and consider the Δ-adjoint bundle morphism

$$J:=\mathfrak{I}_{\Delta,\xi\wedge\eta}:\xi\dagger\eta\to\xi\wedge\eta.$$

10.4. Proposition. *For every automorphism $\varphi:\eta\to\eta$ the diagram*

$$\begin{array}{ccc}\xi\dagger\eta & \xrightarrow{J} & \xi\wedge\eta\\ {\scriptstyle 1\dagger\varphi}\big\downarrow & & \big\downarrow{\scriptstyle 1\wedge\varphi}\\ \xi\dagger\eta & \xrightarrow{J} & \xi\wedge\eta\end{array}$$

commutes □

10.5. Corollary. *The diagram*

$$\begin{array}{ccccc}\xi\dagger\eta\dagger\eta & \xrightarrow{J} & (\xi\dagger\eta)\wedge\eta & \xrightarrow{(1\dagger 1)\wedge\varphi} & (\xi\dagger\eta)\wedge\eta\\ \Big\| & & & & \Big\|\\ \xi\dagger\eta\dagger\eta & \xrightarrow{J} & (\xi\dagger\eta)\wedge\eta & \xrightarrow{(1\dagger\varphi)\wedge 1} & (\xi\dagger\eta)\wedge\eta\end{array}$$

commutes up to homotopy. □

11. Proof of Theorem 4.6

We need some preliminaries on stable duality [51]. Given a pointed map $f : X \to Y$, let $Sf : SX \to SY$ denote the (reduced) suspension over f. So, we have a well-defined map $S : [X, Y]^\bullet \to [SX, SY]^\bullet$.

11.1. Proposition. *Suppose that $\pi_i(Y) = 0$ for $i < n$ and that X is a CW-space with $\dim X < 2n - 1$. Then the map $S : [X, Y]^\bullet \to [SX, SY]^\bullet$ is a bijection.*

Proof. This is the well-known Freudenthal Suspension theorem, see e.g [58] □

Given two pointed spaces X, Y, we define $\{X, Y\}$ to be the direct limit of the sequence

$$[X, Y]^\bullet \xrightarrow{S} [SX, SY]^\bullet \xrightarrow{S} \cdots \longrightarrow [S^n X, S^n Y]^\bullet \xrightarrow{S} \dots .$$

In particular, we have the obvious maps

$$(Y, *)^{(X,*)} \longrightarrow [X, Y]^\bullet \longrightarrow \{X, Y\}.$$

The image of a pointed map $f : X \to Y$ in $\{X, Y\}$ is called the *stable homotopy class of f*. The standard notation for this one is $\{f\}$, but, as usual, in most cases we will not distinguish f, $[f]$ and $\{f\}$.

It is well known that, for $n \geq 2$, the set $[S^n X, S^n Y]^\bullet$ has a natural structure of the abelian group, and the corresponding maps S are homomorphisms, [58]. So, $\{X, Y\}$ turns out to be a group. Furthermore, by Theorem 11.1, if X is a finite CW-space then the map

$$[S^N X, S^N Y]^\bullet \to \{S^N X, S^N Y\}$$

is a bijection for N large enough.

11.2. Definition. A map $f : S^d \to A \wedge A^\perp$ is called *a (stable) d-duality* if the maps

$$u_E : \{A, E\} \to \{S, E \wedge A^\perp\}, \quad u_E(\varphi) = (\varphi \wedge 1_{A^\perp})u$$

and

$$u^E : \{A^\perp, E\} \to \{S, A \wedge E\}, \quad u^E(\varphi) = (1_A \wedge \varphi)u$$

are isomorphisms.

11.3. Proposition. *Let $u : S^d \to A \wedge A^\perp$ be a d-duality between two finite CW-spaces. Then, for every i and π, u yields an isomorphism*

$$H_i(u; \pi) : \widetilde{H}^i(A^\perp; \pi) \to \widetilde{H}_{d-i}(A, \pi).$$

Proof. Recall that

$$H^n(A^\perp;\pi)=[A,k(\pi,n)]^\bullet=[S^NA,K(\pi,N+n)]^\bullet$$

where $K(\pi,i)$ is the Eilenberg–Mac Lane space. Because of Theorem 11.1, the last group coincides with $\{S^NA,K(\pi,N+n)\}$ for N large enough, and therefore

$$H^n(A^\perp;\pi)=\{S^NA,K(\pi,N+n)\}\text{ for }N\text{ large enough.}$$

Furthermore, let $\varepsilon_n:SK(\pi,n)\to K(\pi,n+1)$ be the map which is adjoint to the standard homotopy equivalence $K(\pi,n)\to\Omega K(\pi,n+1)$, see e.g. [58]. Whitehead [64] noticed that

$$\widetilde{H}_n(A;\pi)=\varinjlim\,[S^{N+n},K(\pi,N)\wedge A]^\bullet.$$

Here $\varinjlim$ is the direct limit of the sequence

$$\begin{aligned}[S^{N+n},K(\pi,N)\wedge A]^\bullet &\longrightarrow [S^{N+n+1},SK(\pi,N)\wedge A]^\bullet\\ &\xrightarrow{\varepsilon_*} [S^{N+n+1},K(\pi,N)\wedge A]^\bullet\end{aligned}$$

(see [19, Ch 18] or [48, II.3.24] for greater details). Since ε_n is an n-equivalence, and because of Theorem 11.1, we conclude that

$$\widetilde{H}_n(A;\pi)=[S^{N+n},K(\pi,N)\wedge A]\text{ for }N\text{ large enough.}$$

So, again because of Theorem 11.1,

$$\widetilde{H}_n(A;\pi)=\{S^{N+n},K(\pi,N)\wedge A\}$$

for N large enough.

Now, consider a d-duality $u:S^d\to A\wedge A^\perp$. Fix i and choose N large enough such that

$$\begin{aligned}\widetilde{H}^i(A^\perp;\pi)&=\{S^NA^\perp,K(\pi,N+i)\},\\ \widetilde{H}_{d-i}(A;\pi)&=\{S^{N+d},K(\pi,N+i)\wedge A\}.\end{aligned}$$

By suspending the domain and the range, we get a duality (denoted also by u)

$$u:S^{N+d}\to A\wedge S^NA^\perp.$$

This duality yields the desired isomorphism

$$\begin{aligned}H_i(u;\pi):&=u^{K(\pi,N+i)}:\widetilde{H}^i(A^\perp;\pi)=\{S^NA^\perp,K(\pi,N+i)\}\\ &\to\{S^{N+d},K(\pi,N+i)\wedge A\}=\widetilde{H}_{d-i}(A;\pi)\end{aligned}$$

□

11.4. Definition. Dualizing 4.4, we say that a pointed map $a : A \to S^k$ (or its stable homotopy class $a \in \{A, S^k\}$) is a *coreduction* if the induced map

$$a^* : \widetilde{H}^i(S^k) \to \widetilde{H}^i(A)$$

is an isomorphism for $i \leq k$.

11.5. Proposition. *Let $u : S^d \to A \wedge A^\perp$ be a d-duality between two finite CW-spaces, and let $k \leq d$. A class $\alpha \in \{A^\perp, S^k\}$ is a coreduction if and only if the class $\beta := u^{S^k}\alpha \in \{S^{d-k}, A\}$ is a reduction.*

Proof. Let $H_i(u) : \widetilde{H}^i(A^\perp) \to \widetilde{H}_{d-i}(A)$ be the isomorphism as in 11.3. Notice that the standard homeomorphism $v : S^d \to S^k \wedge S^{d-k}$ is a d-duality. It is easy to see that the diagram

$$\begin{tikzcd}
\widetilde{H}^i(A^\perp) \arrow[r, "H_i(u)"] & \widetilde{H}_{d-i}(A) \\
\widetilde{H}^i(S^k) \arrow[u, "\alpha^*"] \arrow[r, "H_i(v)"] & \widetilde{H}_{d-i}(S^{d-k}) \arrow[u, "\beta_*"']
\end{tikzcd}$$

commutes. In particular, the left vertical arrow is an isomorphism if and only if the right one is. □

Consider a closed connected n-dimensional PL manifold M and embed it in $\mathbb{R}^{N+n+k}$ with N large enough. Let $\iota : S^{N+n+k} \to T\nu^{N+k}$ be a collapse map as in 4.5, and let

$$J : (\nu^{N+k})^\bullet = (\nu^N)^\bullet \dagger \sigma^k \longrightarrow (\nu^N)^\bullet \wedge \sigma^k$$

be the morphism as in 10.4.

11.6. Theorem. *The map*

$$S^{N+n+k} \xrightarrow{\ \iota\ } T\nu^{N+k} \xrightarrow{\ TJ\ } T\nu^N \wedge \sigma^k$$

is an $(N + n + k)$-duality map.

Proof. This is actually proved in [10]. For greater detail, see [48, V.2.3(i)] (where in the proof the reference 2.8(a) must be replaced by 2.8(b)). □

11.7. Corollary. *The collapse map $\iota : S^{N+n} \to T\nu^N$ is a reduction.*

Proof. Recall that $T\sigma^k = (M \times S^k)/M = S^k(M^+)$. Consider a surjective map $e : M^+ \to S^0$ and define $\varepsilon = S^k e : T\sigma^k \to S^k$. Since the map

$$S^k\iota : S^k S^{N+n} = S^{N+n+k} \to T\nu^{N+k} = S^k T\nu$$

can be written as

$$S^{N+n+k} \xrightarrow{S^k\iota} T\nu^{N+k} = T\nu^N \wedge T\sigma^k \xrightarrow{1\wedge\varepsilon} T\nu^N \wedge S^k = S^k T\nu,$$

where the composition of first two maps is the duality from 11.6, we conclude that ι is dual to ε with respect to duality (11.2). Clearly, ε is a coreduction. Thus, the result follows from 11.5. □

For technical reasons, it will be convenient for us to consider the duality

$$S^{N+n+2k} \to T\nu^{N+2k} \xrightarrow{TJ} T\nu^{N+k} \wedge T\sigma^k. \tag{11.1}$$

This duality yields an isomorphism

(11.2)

$$D := u^{S^k} : \{T\sigma^k, S^k\} \to \{S^{N+n+2k}, S^k \wedge T\nu^{N+k}\} = \{S^{N+n+k}, T\nu^{N+k}\}.$$

11.8. Proposition. *For every automorphism $\varphi : \sigma^k \to \sigma^k$ the following diagram commutes up to homotopy:*

$$\begin{array}{ccccccc}
S^{N+n+2k} & \xrightarrow{\iota} & T\nu^{N+2k} & \xrightarrow{TJ} & T\nu^{N+k}\wedge T\sigma^k & \xrightarrow{T(1\dagger\varphi)\wedge 1} & T\nu^{N+k}\wedge T\sigma^k \\
\| & & \| & & & & \| \\
S^{N+n+2k} & \longrightarrow & T\nu^{N+2k} & \xrightarrow{TJ} & T\nu^{N+k}\wedge T\sigma^k & \xrightarrow{1\wedge T\varphi} & T\nu^{N+k}\wedge T\sigma^k
\end{array}$$

Proof. This follows from 10.5 . □

Every automorphism $\varphi : \sigma^k \to \sigma^k$ yields a homotopy equivalence

$$T(1 \dagger \varphi) : T\nu^{N+k} = T(\nu^N \dagger \sigma^k) \longrightarrow T(\nu^N \dagger \sigma^k) = T\nu^{N+k}$$

and hence an isomorphism

$$T(1 \dagger \varphi)_* : \{S^{N+n+k}, T\nu^{N+k}\} \to \{S^{N+n+k}, T\nu^{N+k}\}.$$

So, we have the $\operatorname{aut}\sigma^k$-action

$$\begin{aligned}
a_\nu : \operatorname{aut}\sigma^k \times \{S^{N+n+k}, T\nu^{N+k}\} &\to \{S^{N+n+k}, T\nu^{N+k}\}, \\
a_\nu(\varphi, \alpha) &= T(1 \dagger \varphi)_*(\alpha).
\end{aligned}$$

Similarly, every automorphism φ of σ^k induces a homotopy equivalence $T\sigma^k \to T\sigma^k$, and therefore we have the action

$$a_\sigma : \operatorname{aut}\sigma^k \times \{T\sigma^k, S^k\} \to \{T\sigma^k, S^k\}.$$

11.9. Theorem. *The diagram*

$$\begin{array}{ccc}
\operatorname{aut}\sigma^k \times \{T\sigma^k, S^k\} & \xrightarrow{a_\sigma} & \{T\sigma^k, S^k\} \\
{\scriptstyle 1\times D}\downarrow & & \downarrow{\scriptstyle D} \\
\operatorname{aut}\sigma^k \times \{S^{N+n+k}, T\nu^{N+k}\} & \xrightarrow{a_\nu} & \{S^{N+n+k}, T\nu^{N+k}\}
\end{array}$$

commutes.

Proof. This follows from 11.8 and the definition of D, a_ν and a_σ. $\square$

Because of Theorem 11.1, for k large enough we have

$$\{T\sigma^k, S^k\} = \pi^k(T\sigma^k) \text{ and } \{S^{N+n+k}, T\nu^{N+k}\} = \pi_{N+n+k}(T\nu^{N+k}).$$

Then we can rewrite the diagram from Theorem 11.9 as

$$\begin{array}{ccc} \operatorname{aut}\sigma^k \times \pi^k\left(T\sigma^k\right) & \xrightarrow{a_\sigma} & \pi^k\left(T\sigma^k\right) \\ {\scriptstyle 1\times D}\big\downarrow & & \big\downarrow{\scriptstyle D} \\ \operatorname{aut}\sigma^k \times \pi_{N+n+k}\left(T\nu^{N+k}\right) & \xrightarrow{a_\nu} & \pi_{N+n+k}\left(T\nu^{N+k}\right) \end{array} \tag{11.3}$$

Let $\mathfrak{R} \in \pi_{N+n+k}(T\nu^{N+k})$ be the set of reductions, and let $\mathfrak{C} \in \pi^k(T\sigma^k)$ be the set of coreductions. Then, clearly, $a_\nu(\mathfrak{R}) \subset \mathfrak{R}$ and $a_\sigma(\mathfrak{C}) \subset \mathfrak{C}$. Therefore, in view of Proposition 11.5, the diagram (11.3) yields the commutative diagram

$$\begin{array}{ccc} \operatorname{aut}\sigma^k \times \mathfrak{C} & \xrightarrow{a_\sigma} & \mathfrak{C} \\ {\scriptstyle 1\times D}\big\downarrow & & \big\downarrow{\scriptstyle D} \\ \operatorname{aut}\sigma^k \times \mathfrak{R} & \xrightarrow{a_\nu} & \mathfrak{R} \end{array} \tag{11.4}$$

11.10. Theorem. *For every $\alpha, \beta \in \mathfrak{C}$ there exists an automorphism φ of σ^k such that $a_\sigma(\varphi, \alpha) = \beta$. Moreover, this φ is unique up to fiberwise homotopy. In other words, the action $a_\sigma : \operatorname{aut}\sigma^k \times \mathfrak{C} \to \mathfrak{C}$ is free and transitive.*

Proof. Recall that $T\sigma^k = (M \times S^k)/M$. So, for every $m \in M$, a pointed map $f : T\sigma^k \to S^k$ yields a pointed map $f_m : S^k_m \to S^k$ where S^k_m is the fiber over m. Furthermore, f represents a coreduction if and only if all maps f_m belong to F_k. In other words, every coreduction for $T\sigma^k$ yields a homotopy class $M \to F_k$. Moreover, it is easy to see that, in view of Proposition 10.1, the action a_σ coincides with the map

$$[M, F_k] \times [M, F_k] \to [M, F_k]$$

induced by the product in F_k, and the result follows. $\square$

Since D is an isomorphism, Theorem 11.10 yields the following corollary.

11.11. Corollary. *The action $a_\nu : \operatorname{aut}\sigma^k \times \mathfrak{R} \to \mathfrak{R}$ is free and transitive.* $\square$

Now we can finish the proof of Theorem 4.6. Assuming $\dim \eta = N+k$ to be large enough, we conclude that $\nu^\bullet$ and $\eta^\bullet$ are homotopy equivalent over M, see Atiyah [2, Prop. 3.5]. (Notice that Atiyah works with non-sectioned bundles, but there is no problem to adapt the proof for sectioned ones.) Choose any such F_{N+k}-equivalence $\varphi : \eta^\bullet \to \nu^\bullet$ and consider the induced homotopy equivalence $T\varphi : T\eta \to T\nu$. Clearly, the composition

$$\beta : S^{N+n+k} \xrightarrow{\alpha} T\eta \xrightarrow{T\varphi} T\nu$$

is a reduction. So, by 11.11, there exists an F_{N+k}-equivalence $\lambda : \nu^\bullet \to \eta^\bullet$ over M with $(T\lambda)_*(\beta) = \iota$. Now, we define $\mu : \nu^\bullet \to \eta^\bullet$ to be the fiber homotopy inverse to $\lambda\varphi$. (The existence of an inverse equivalence can be proved following Dold [9], cf. [32]). Clearly, $\mu_*\iota = \alpha$. This proves the existence of the required equivalence μ.

Furthermore, if there exists another equivalence $\mu' : \eta^\bullet \to \nu^\bullet$, then $\mu' \circ \mu^{-1}(\iota) = \iota$, and so μ and μ' are homotopic over M. This proves the uniqueness of μ. Thus, Theorem 4.6 is proved. □

12. Normal maps and F/PL

Throughout the section we fix a closed PL manifold M^n.

12.1. Definition ([5, 31]). A *normal map at M* is a commutative diagram of PL maps

$$\begin{array}{ccc} E & \xrightarrow{\widehat{b}} & E' \\ \downarrow & & \downarrow \\ V & \xrightarrow{b} & M \end{array}$$

where $\nu_V = \{E \to V\}$ is the normal PL $\mathbb{R}^N$-bundle over a closed PL manifold V^n, $\xi = \{E' \to M\}$ is a PL $\mathbb{R}^N$-bundle over M and, finally, $\widehat{b}$ induces a PL isomorphism of fibers and preserves the sections. In other words, a normal map is a bundle morphism $\varphi : \nu_V \to \xi$.

A normal map is called *reducible* if the map

$$S^{N+n} \xrightarrow{\text{collapse}} T\nu_V \xrightarrow{T\varphi} T\xi$$

is a reduction.

Because of the Thom Isomorphism Theorem, a normal map is reducible whenever b is a map of degree 1. (One can prove that ξ is orientable if V and M are.)

12.2. Construction–Definition. Given a map (homotopy class) $f : M \to F/PL$, we represent it by an F_N-morphism $\varphi : \nu_M^\bullet \to (\gamma_{PL}^N)^\bullet$ with N large enough, see 3.5 and/or 3.8. We denote the base of φ by b and set $\xi = b^*\gamma_{PL}^N$. Then the correcting F_N-morphism $\nu_M^\bullet \to \xi^\bullet$ yields a commutative diagram

$$\begin{tikzcd} U^\bullet \arrow[r, "g"] \arrow[d, "q"'] & U'^\bullet \arrow[d, "p"] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{12.1}$$

where $\nu_M = \{q : U \to M\}$, $\xi = \{p : U' \to M\}$ and $U^\bullet, U'^\bullet$ are fiberwise one-point compactifications of U and U', respectively.

We consider M as the zero section of ξ, $M \subset U'$ and deform g to a map $t : U^\bullet \to U'^\bullet$ which is transversal to M. Set $V = t^{-1}(M)$. We can always assume that $V \subset U$. So, we get a morphism of PL $\mathbb{R}^N$-bundles

$$\begin{tikzcd} E \arrow[r, "\widehat{b}"] \arrow[d] & E' \arrow[d] \\ V \arrow[r, "b"] & M \end{tikzcd} \tag{12.2}$$

where $\{E' \to M\}$ is a normal bundle of M in U' and $\{E \to V\}$ is a normal bundle of V in U. Here $b = t|V$. Clearly, $\xi := \{E' \to M\}$ is the bundle of M in U'. Furthermore, since U is a total space of a normal bundle, the normal bundle of U is trivial. Thus, $\{E' \to M\}$ is the normal bundle ν_V of V. In other words, the diagram (12.2) is a normal map at M. We say that the normal map (12.2) is *associated with a map* (*homotopy class*) $f : M \to F/PL$.

Clearly, there any many normal maps associated with a given map $f : M \to F/PL$.

12.3. Construction–Definition. Let

$$\varphi : \nu_V \to \xi \tag{12.3}$$

be a reducible normal map at M and assume that $\dim \nu_V$ is large. Consider a collapse map (homotopy class) $\iota : S^{N+n} \to T\nu_M$ as in 4.5. Since the map

$$\alpha : S^{N+n} \xrightarrow{\text{collapse}} T\nu_V \xrightarrow{T\varphi} T\xi$$

is a reduction, there exists, by Theorem 4.6, a unique F_N-morphism $\mu : \nu_M^\bullet \to \xi^\bullet$ with $\mu_*(\iota) = \alpha$. Now, the morphism

$$\nu_M^\bullet \xrightarrow{\mu_*} \xi^\bullet \xrightarrow{\text{classif}} (\gamma_{PL}^N)^\bullet$$

is a homotopy PL structuralization of ν_M. Thus, it gives us a map $f_\varphi : M \to F/PL$.

12.4. Proposition. *The normal map* (12.3) *is associated with the map* $f_\varphi : M \to F/PL$. □

Recall that a closed manifold is called *almost parallelizable* if it becomes parallelizable after deleting of a point. Notice that every almost parallelizable manifold is orientable (e.g., because its first Stiefel–Whitney class is equal to zero).

12.5. Proposition. *Let V^m be an almost parallelizable PL manifold. Then every map $b : V^m \to S^m$ of degree* 1 *is the base of a reducible normal map at S^m.*

Proof. We regard $S^m = \{(x_1, \dots, x_{m+1}) \bigm| \sum x_i^2 = 1\}$ as the union of two discs, $S^m = D_+ \cup D_-$, where

$$D_+ = \{x \in S^m | x_{m+1} \geq 0\}, \quad D_- = \{x \in S^m | x_{m+1} \leq 0\}.$$

We can always assume (deforming b if necessary) that there is a small closed disk D_0 in V such that $b_+ := b|D_0 : D_0 \to D_+$ is a PL isomorphism. We set $W = V \setminus (\operatorname{Int} D_0)$. Since W is parallelizable, there exists a morphism $\varphi_- : \nu_V|W \to \theta_{D_-}$ of PL bundles such that $b|W : W \to D_-$ is the base of φ. Furthermore, since b_+ is a PL isomorphism, there exists a morphism $\varphi_+ : \nu_V|D_0 \to \theta|D_+$ over b_+ such that φ_+ and φ_- coincide over $b|\partial W : \partial W \to S^{m-1}$. Together φ_+ and φ_- give us a morphism $\varphi : \nu_V \to \xi$ where ξ is a PL bundle over S^m. Clearly, φ is a normal map with the base b, and it is reducible because $\deg b = 1$. □

13. The Sullivan map $s : [M, F/PL] \to P_{\dim M}$

We define the groups P_i by setting

$$P_i = \begin{cases} \mathbb{Z} & \text{if } i = 4k, \\ \mathbb{Z}/2 & \text{if } i = 4k+2, \\ 0 & \text{if } i = 2k+1 \end{cases}$$

where $k \in \mathbb{N}$.

Given a closed connected n-dimensional PL manifold M (which is assumed to be orientable for $n = 4k$), we define a map

$$s : [M, F/PL] \to P_n \tag{13.1}$$

as follows. Given a homotopy class $f : M \to F/PL$, consider a normal map (12.2)

$$\begin{array}{ccc} E & \xrightarrow{\widehat{b}} & E' \\ \downarrow & & \downarrow \\ V & \xrightarrow{b} & M \end{array}$$

associated with f.

For $n = 4k$, let ψ be the symmetric bilinear intersection form on

$$\mathrm{Ker}\{b_* : H_{2k}(Z;\mathbb{Q}) \to H_{2k}(M;\mathbb{Q})\}.$$

We define $s(u) = \frac{\sigma(\psi)}{8}$ where $\sigma(\psi)$ is the signature of ψ. It is well known that $\sigma(\psi)$ is divisible by 8, (see e.g. [5]), and so $s(u) \in \mathbb{Z}$.

Also, it is easy to see that

$$\sigma(u) = \frac{\sigma(Z) - \sigma(M)}{8}$$

where $\sigma(M), \sigma(Z)$ is the signature of the manifold M, Z, respectively.

For $n = 4k + 2$, we define $s(u)$ to be the Kervaire invariant of the normal map $(b, \widehat{b})$. The routine arguments show that s is well-defined, i.e. it does not depend on the choice of the associated normal map. See [5, Ch. III, §4] or [40] for details.

In particular, if b is a homotopy equivalence then $s(u) = 0$.

One can prove that every map s is a homomorphism, where the abelian group structure on $[M, F/PL]$ is given by an H-space structure on F/PL.

Given a map $f : M \to F/PL$, it is useful to introduce the notation $s(M, f) := s([f])$ where $[f]$ is the homotopy class of f.

13.1. Theorem. (i) *The map $s : [S^{4i}, F/PL] \to \mathbb{Z}$ is surjective for all $i > 1$,*

(ii) *The map $s : [S^{4i-2}, F/PL] \to \mathbb{Z}/2$ is surjective for all $i > 0$.*

(iii) *The image of the map $s : [S^4, F/PL] \to \mathbb{Z}$ is the subgroup of index* 2.

Proof. (i) For every $k > 1$ Milnor constructed a parallelizable $4k$-dimensional smooth manifold W^{4k} of signature 8 and such that ∂W is a homotopy sphere, see [5, V.2.9]. Since, by Theorem 3.11, every homotopy sphere of dimension ≥ 5 is PL isomorphic to the standard one, we can form a closed PL manifold

$$V := W \cup_{S^{4k-1}} D^{4k}$$

of the signature 8. Because of Proposition 12.5, there exists a reducible normal map with the base $V^{4k} \to S^{4k}$. Because of Proposition 12.4, this normal map is associated with a certain map (homotopy class) $f : S^{4k} \to F/PL$. Thus, $s(S^{4k}, f) = 1$.

(ii) The proof is similar to that of (i), but we must use (4k+2)-dimensional parallelizable Kervaire manifolds W, $\partial W = S^{4k+1}$ of the Kervaire invariant one, see [5, V.2.11].

(iii) The Kummer algebraic surface [26] gives us an example of 4-dimensional almost parallelizable smooth manifold of the signature 16. So, $\operatorname{Im} s \supset 2\mathbb{Z}$.

Now suppose that there exists $f : S^4 \to F/PL$ with $s(S^4, f) = \mathbb{Z}$. Then there exists a normal map with the base $V^4 \to S^4$ and such that V has signature 8. Since normal bundle of V is induced from a bundle over S^4, we conclude that $w_1(V) = 0 = w_2(V)$. But this contradicts the Rokhlin Theorem 7.2. □

13.2. Theorem. *For every closed simply-connected PL manifold M of dimension ≥ 5, the sequence*

$$0 \longrightarrow \mathfrak{S}_{PL}(M) \xrightarrow{j_F} [M, F/PL] \xrightarrow{s} P_{\dim M}$$

is exact, i.e. j_F is injective and $\operatorname{Im} j_F = s^{-1}(0)$.

Proof. See [5, II.4.10 and II.4.11]. Notice that the map ω in loc. cit is the zero map because, by Theorem 3.11, every homotopy sphere of dimension ≥ 5 is PL isomorphic to the standard sphere. □

13.3. Corollary. *We have $\pi_{4i}(F/PL) = \mathbb{Z}$, $\pi_{4i-2}(F/PL) = \mathbb{Z}/2$, and $\pi_{2i-1}(F/PL) = 0$ for every $i > 0$. Moreover, the map*

$$s : [S^k, F/PL] \to P_k$$

is an isomorphism for $k \neq 4$ and has the form

$$\mathbb{Z} = \pi_4(F/PL) \xrightarrow{s} P_4 = \mathbb{Z}, \quad a \mapsto 2a$$

for $k = 4$.

Proof. First, if $k > 4$ then, because of the Smale Theorem 3.11, $\mathfrak{S}_{PL}(S^k)$ is the one-point set. Now the result follows from 13.2 and 13.1.

If $k \leq 4$ then $\pi_k(PL/O) = 0$, cf. Remark 6.7. So, $\pi_k(F/PL) = \pi_k(F/O)$. Moreover, the forgetful map $\pi_k(BO) \to \pi_k(BF)$ coincides with the Whitehead J-homomorphism. So, we have the long exact sequence

$$\cdots \to \pi_k(F/O) \to \pi_k(BO) \xrightarrow{J} \pi_k(BF) \to \pi_{k-1}(F/O) \to \cdots.$$

For $k \le 5$ all the groups $\pi_k(BO)$ and $\pi_k(BF)$ are known (notice that $\pi_k(BF)$ is the stable homotopy group $\pi_{k+N-1}(S^N)$), and it is also known that J is an epimorphism for $k = 1, 2, 4, 5$, see e.g. [1]. Thus, $\pi_k(F/O) \cong P_i$ for $k \le 4$. The last claim follows from 13.1. □

14. The homotopy type of $F/PL[2]$

As usual, given a space X and an abelian group π, we do not distinguish elements of $H^n(X; \pi)$ and maps (homotopy classes) $X \to K(\pi, n)$. For example, regarding a Steenrod cohomology operation Sq^2 as an element $Sq^2 \in H^{k+2}(K(\mathbb{Z}/2, k); \mathbb{Z}/2)$, we can treat it as a map $Sq^2 : K(\mathbb{Z}/2, k) \to K(\mathbb{Z}/2, k+2)$.

Given a prime p, let $\mathbb{Z}[p]$ be the subring of $\mathbb{Q}$ consisting of all irreducible fractions with denominators relatively prime to p, and let $\mathbb{Z}[1/p]$ be the subgroup of $\mathbb{Q}$ consisting of the fractions $m/p^k, m \in \mathbb{Z}$. Given a simply-connected space X, we denote by $X[p]$ and $X[1/p]$ the $\mathbb{Z}[p]$- and $\mathbb{Z}[1/p]$-localization of X, respectively. Furthermore, we denote by $X[0]$ the $\mathbb{Q}$-localization of X. For the definitions, see [21].

Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}[2] \xrightarrow{2} \mathbb{Z}[2] \xrightarrow{\rho_*} \mathbb{Z}/2 \longrightarrow 0$$

where 2 over the arrow means multiplication by 2 and ρ is the modulo 2 reduction. This exact sequence yields the Bockstein exact sequence

$$\begin{aligned} \cdots \longrightarrow H^n(X; \mathbb{Z}[2]) \xrightarrow{2} H^n(X; \mathbb{Z}[2]) \xrightarrow{\rho} H^n(X; \mathbb{Z}/2) \\ \xrightarrow{\delta} H^{n+1}(X; \mathbb{Z}[2]) \longrightarrow \cdots . \end{aligned} \tag{14.1}$$

Put $X = K(\mathbb{Z}/2, n)$ and consider the fundamental class

$$\iota \in H^n(K(\mathbb{Z}/2, n); \mathbb{Z}/2).$$

Then we have the class $\delta := \delta(\iota) \in H^{n+1}(K(\mathbb{Z}/2, n), \mathbb{Z}[2])$. According to what we said above, we regard δ as a map $\delta : K(\mathbb{Z}/2, n) \to K(\mathbb{Z}[2], n+1)$.

14.1. Proposition (Sullivan [56, 57]). *For every $i > 0$ there are cohomology classes*

$$K_{4i} \in H^{4i}(F/PL; \mathbb{Z}[2]), \; K_{4i-2} \in H^{4i-2}(F/PL; \mathbb{Z}/2)$$

such that

$$s(M^{4i}, f) = \langle f^* K_{4i}, [M] \rangle$$

for every closed oriented PL manifold M, and

$$s(N^{4i-2}, f) = \langle f^* K_{4i-2}, [N]_2 \rangle.$$

for every closed manifold N. Here $[M] \in H^{4i}(M)$ is the orientation of M, $[N]_2 \in H^{4i-2}(N;\mathbb{Z}/2)$ is the modulo 2 fundamental class of N, and $\langle -,-\rangle$ is the Kronecker pairing.

Proof. Let $MSO_*(-)$ denote the oriented bordism theory, see e.g [48]. Recall that if two maps $f : M^{4i} \to F/PL$ and $g : N^{4i} \to F/PL$ are bordant (as oriented singular manifolds) then $s(M,f) = s(N,g)$. Thus, s defines a homomorphism

$$\widetilde{s} : MSO_{4i}(F/PL) \to \mathbb{Z}.$$

It is well known that the Steenrod–Thom map

$$t : MSO_*(-) \otimes \mathbb{Z}[2] \to H_*(-;\mathbb{Z}[2])$$

splits, i.e. there is a natural map $v : H_*(-;\mathbb{Z}[2]) \to MSO_*(-) \otimes \mathbb{Z}[2]$ such that $tv = 1$ (a theorem of Wall [61], see also [55, 48, 3]. In particular, we have a natural homomorphism

$$\widehat{s} : H_{4i}(F/PL;\mathbb{Z}[2]) \xrightarrow{\ v\ } MSO_{4i}(F/PL) \otimes \mathbb{Z}[2] \xrightarrow{\ \widetilde{s}\ } \mathbb{Z}[2].$$

Since the evaluation map

$$\mathrm{ev} : H^*(X;\mathbb{Z}[2]) \to \mathrm{Hom}(H_*(X;\mathbb{Z}[2]),\mathbb{Z}[2]), \quad (\mathrm{ev}(x)(y) = \langle x,y\rangle$$

is surjective for all X, there exists a class $K_{4i} \in H^{4i}(F/PL;\mathbb{Z}[2])$ such that $\mathrm{ev}(K_{4i}) = \widehat{s}$. Now

$$s(M,f) = \widehat{s}(f_*[M]) = \langle K_{4i}, f_*[M]\rangle = \langle f^*K_{4i},[M]\rangle.$$

So, we constructed the desired classes K_{4i}.

The construction of classes K_{4i-2} is similar. Let $MO_*(-)$ denoted the non-oriented bordism theory. Then the map s yields a homomorphism

$$\widetilde{s} : MO_{4i-2}(F/PL) \to \mathbb{Z}/2.$$

Furthermore, there exists a natural map $H_*(-;\mathbb{Z}/2) \to MO_*(-)$ which splits the Steenrod–Thom homomorphism, and so we have a homomorphism

$$\widehat{s} : H_{4i-2}(F/PL;\mathbb{Z}/2) \longrightarrow MO_{4i-2}(F/PL) \otimes \mathbb{Z}[2] \xrightarrow{\ \widetilde{s}\ } \mathbb{Z}/2$$

with $\widehat{s}(f_*([M]_2) = s(M,f)$. Now we can complete the proof similarly to the case of classes K_{4i}. □

We set

$$\Pi := \prod_{i>1} \left(K(\mathbb{Z}[2],4i) \times K(\mathbb{Z}/2,4i-2)\right). \tag{14.2}$$

Together the classes $K_{4i} : F/PL \to K(\mathbb{Z}[2], 4i), i > 1$ and $K_{4i-2} : F/PL \to K(\mathbb{Z}[/2, 4i-2), i > 1$ yield a map

$$K : F/PL \to \Pi.$$

14.2. Lemma. *The map*

$$K[2] : F/PL[2] \to \Pi \tag{14.3}$$

induced an isomorphism of homotopy groups in dimensions ≥ 5.

Proof. This follows from Theorem 13.1 and Corollary 13.3. □

14.3. Proposition. *The Postnikov invariant*

$$\varkappa \in H^5(K(\mathbb{Z}/2, 3), \mathbb{Z}[2])$$

of $F/PL[2]$ *is non-zero. Moreover,* $\varkappa = \delta Sq^2 \in H^5(K(\mathbb{Z}/2, 3); \mathbb{Z}[2])$.

Proof. Let $h : \pi_4(F/PL) \to H_4(F/PL)$ be the Hurewicz homomorphism. Suppose that $\varkappa = 0$. Then

$$H_4(F/PL; \mathbb{Z}[2]) = \mathbb{Z}[2] \oplus H_4(K(\mathbb{Z}/2, 2; \mathbb{Z}[2],$$

and therefore the homomorphism

$$\begin{aligned} H^4(F/PL; \mathbb{Z}[2]) &\xrightarrow{\text{ev}} \operatorname{Hom}(H_4(F/PL) \otimes \mathbb{Z}[2], \mathbb{Z}[2]) \\ &\xrightarrow{h^*} \operatorname{Hom}(\pi_4(F/PL) \otimes \mathbb{Z}[2], \mathbb{Z}[2]) \end{aligned}$$

must be surjective. But this contradicts 13.1(ii). Thus, $\varkappa \neq 0$.

Furthermore, since F/PL is an infinite loop space, $\varkappa$ has the form $\Omega^N a$ for some $a \in H^{N+5}(K(\mathbb{Z}/2, N+3), \mathbb{Z}[2])$. But, for general reasons, for N large enough the last group consists of elements of the order 2. Thus, $\varkappa$ has the order 2. It is easy to see that $H^5(K(\mathbb{Z}/2, 3), \mathbb{Z}[2] = \mathbb{Z}/4 = \{x\}$ with $2x = \delta Sq^2$, see [48]. Thus, $\varkappa = \delta Sq^2$. □

14.4. Remark. It is interesting to have a geometrical description of the class $z \in H_4(F/PL)$ with $\langle K_4, z\rangle = 1$. Let η denote the canonical complex line bundle over $\mathbb{CP}^2$. One can prove that 24η is fiberwise homotopy trivial. So, there exist a map $f : \mathbb{CP}^2 \to F/PL$ such that the map

$$\mathbb{CP}^2 \xrightarrow{f} F/PL \longrightarrow BPL$$

classifies 24η. Since $p_1(\eta) = 1$, we conclude that $p_1(24\eta) = 24$, and therefore $L_1(24\eta) = 8$ (here L_1 denotes the first Hirzebruch class), see [38]. Thus, $s(\mathbb{CP}^2, f) = 8/8 = 1$, and therefore $\langle K_4, f_*[\mathbb{C}P^2]\rangle = 1$.

Let Y be the homotopy fiber of the map

$$\delta Sq^2 : K(\mathbb{Z}/2, 2) \to K(\mathbb{Z}[2], 5).$$

In other words, there is a fibration

$$K(\mathbb{Z}[2], 4) \xrightarrow{\ i\ } Y \xrightarrow{\ p\ } K(\mathbb{Z}/2, 2) \tag{14.4}$$

with the characteristic class δSq^2.

Because of Proposition 14.3, the space Y is the Postnikov 4-stage of $F/PL[2]$. In particular, we have a map

$$\psi : F/PL[2] \to Y$$

which induces an isomorphism of homotopy groups in dimension ≤ 4. Together with the map $K[2]$ from 14.3, this map yields a map $\phi : F/PL[2] \to Y \times \Pi$.

14.5. Theorem. *The map*

$$\phi : F/PL[2] \simeq Y \times \Pi \tag{14.5}$$

is a homotopy equivalence.

Proof. This follows from 14.3 and what we said about ψ. □

14.6. Lemma. *Let X be a finite CW-space such that the group $H_*(X)$ is torsion free. Then the group $[X, F/PL[1/2]]$ is torsion free.*

Proof. It suffices to prove that $[X, F/PL[p]]$ is torsion free for every odd prime p. Notice that $F/PL[p]$ is an infinite loop space since F/PL is. So, there exists a connected p-local spectrum E such that

$$\widetilde{E}^0(Y) = [Y, F/PL[p]] = [Y, F/PL] \otimes \mathbb{Z}[p].$$

Moreover, $E^{-i}(\mathrm{pt}) = \pi_i(E) = \pi_i(F/PL) \otimes \mathbb{Z}[p]$, So, because of the isomorphism $\widetilde{E}^0(X) \cong [X, F/PL[p]]$, it suffices to prove that $E^*(X)$ is torsion free. Consider the Atiyah–Hirzebruch spectral sequence for $E^*(X)$. Its initial term is torsion free because $E^*(\mathrm{pt})$ is torsion free. Hence, the spectral sequence degenerates, and thus the group $E^*(X)$ is torsion free. □

14.7. Proposition. *Let X be a finite CW-space such that the group $H_*(X)$ is torsion free. Let $f : X \to F/PL$ be a map such that $f^*K_{4n} = 0$ and $f^*K_{4n+2} = 0$ for every $n \geq 1$. Then f is null-homotopic.*

Proof. Consider the commutative square

$$\begin{array}{ccc} F/PL & \xrightarrow{l_1} & F/PL[2] \\ {\scriptstyle l_2}\downarrow & & \downarrow{\scriptstyle l_3} \\ F/PL[1/2] & \xrightarrow{l_4} & F/PL[0] \end{array}$$

where the horizontal maps are the $\mathbb{Z}[2]$-localizations and the vertical maps are the $\mathbb{Z}[1/2]$-localizations. Because of 14.5, $[X, F/PL]$ is a finitely generated abelian group, and so it suffices to prove that both $l_1 \circ f$ and $l_2 \circ f$ are null-homotopic. First, we remark that $l_2 \circ f$ is null-homotopic whenever $l_1 \circ f$ is. Indeed, since $H_*(X)$ is torsion free, the group $[X, F/PL[1/2]]$ is torsion free by 14.6. Now, if $l_1 \circ f$ is null-homotopic then $l_3 \circ l_1 \circ f$ is null-homotopic, and hence $l_4 \circ l_2 \circ f$ is null-homotopic. Thus, $l_2 \circ f$ is null-homotopic since $[X, F/PL[1/2]]$ is torsion free.

So, it remains to prove that $l_1 \circ f$ is null-homotopic.

Clearly, the equalities $f^* K_{4i} = 0$ and $f^* K_{4i-2} = 0$, $i > 1$, imply that the map

$$X \longrightarrow F/PL \xrightarrow{l_1} F/PL[2] \simeq Y \times \Pi \xrightarrow{p_2} \Pi$$

is null-homotopic. So, it remains to prove that the map

$$g : X \xrightarrow{f} F/PL \xrightarrow{l_1} F/PL[2] \simeq Y \times \Pi \xrightarrow{p_1} Y$$

is null-homotopic.

It is easy to see that $H^4(Y; \mathbb{Z}[2]) = \mathbb{Z}[2]$. Let $u \in H^4(Y; \mathbb{Z}[2])$ be a free generator of the free $\mathbb{Z}[2]$-module $H^4(Y; \mathbb{Z}[2])$. The fibration (14.4) gives us the following diagram with the exact row:

$$\begin{array}{ccccc} H^4(X; \mathbb{Z}[2]) & \xrightarrow{i_*} & [X, Y] & \xrightarrow{p_*} & H^2(X; \mathbb{Z}/2) \\ & & \downarrow{\scriptstyle u_*} & & \\ & & H^4(X; \mathbb{Z}[2]) & & \end{array}$$

Notice that

$$u_* i_* : \mathbb{Z}[2] \to \mathbb{Z}[2]$$

is the multiplication by 2ε where ε is an invertible element of the ring $\mathbb{Z}[2]$. Since $f^* K_2 = 0$, we conclude that $p_*(g) = 0$, and so $g = i_*(a)$ for some $a \in H^4(X; \mathbb{Z}[2])$. Now,

$$0 = u_*(g) = u_*(i_* a) = 2a\varepsilon.$$

But $H^*(X; \mathbb{Z}[2])$ is torsion free, and thus $a = 0$. □

For completeness, we mention also that $F/PL[1/2] \simeq BO[1/2]$. So, there is a Cartesian square (see [31, 57])

$$\begin{array}{ccc} F/PL & \longrightarrow & \Pi \times Y \\ \Big\downarrow & & \Big\downarrow \\ BO[1/2] & \xrightarrow{\mathrm{ph}} & \prod K(\mathbb{Q}, 4i) \end{array}$$

where ph is the Pontryagin character.

15. Splitting theorems

15.1. Definition. Let A^{n+k} and W^{n+k} be two connected PL manifolds (without boundaries), and let M^n be a closed submanifold of A. We say that a map $g : W^{n+k} \to A^{n+k}$ *splits along* M^n if there exists a homotopy

$$g_t : W^{n+k} \to A^{n+k}, \quad t \in I$$

such that:

(i) $g_0 = g$;

(ii) there is a compact subset K of W such that $g_t|W \setminus K = g|W \setminus K$ for every $t \in I$;

(iii) the map g_1 is transversal to M;

(iv) the map $b := (g_1)|g_1^{-1}(M) : g_1^{-1}(M) \to M$ is a homotopy equivalence.

An important special case is when $A^{n+k} = M^n \times B^k$ for some connected manifold B^k. In this case we can regard M as submanifold $M \times \{b_0\}, b_0 \in B$ of A and say that $g : W \to A$ splits along M if it splits along $M \times \{b_0\}$. Clearly, this does not depend on the choice of $\{b_0\}$, i.e. g splits along $M \times \{b_0\}$ if and only if it splits along $M \times \{b\}$ with any other $b \in B$.

Recall that a map f is called *proper* if $f^{-1}(C)$ is compact whenever C is compact. A map $f : X \to Y$ is called a *proper homotopy equivalence* if there exists a map $g : Y \to X$ and the homotopies $F : gf \simeq 1_X$, $G; fg \simeq 1_Y$ such that all the four maps $f, g, F : X \times I \to X$ and $G : Y \times I \to Y$ are proper.

15.2. Theorem. *Let $M^n, n \geq 5$ be a closed connected n-dimensional PL manifold such that $\pi_1(M)$ is a free abelian group. Then every proper homotopy equivalence $h : W^{n+1} \to M^n \times \mathbb{R}$ splits along M^n.*

Proof. Because of the transversality theorem, there is a homotopy $h_t : W \to M \times \mathbb{R}$ which satisfies conditions (i)–(iii) of 15.1. We let $f = h_1$. Because of a crucial theorem of Novikov [41, Theorem 3], there is a sequence of interior surgeries of the inclusion $f^{-1}(M) \subset W$ in W such that the final result of these surgeries is a homotopy equivalence $V \subset W$. (This is the place where we use the fact that $\pi_1(M)$ is free.) Using the Pontryagin–Thom construction, we can realize this sequence of surgeries via a homotopy f_t such that f_t satisfies conditions (i)–(iii) of 15.1 and $f_1^{-1}(M) = V$. □

15.3. Theorem. *Let M^n be a manifold as in* 15.2*. Then every homotopy equivalence $W^{n+1} \to M^n \times S^1$ splits along M^n.*

Proof. This follows from results of Farrell and Hsiang [12, Theorem 2.1] since $\mathrm{Wh}(\mathbb{Z}^m) = 0$ for every m. □

15.4. Corollary. *Let M^n be a manifold as in* 15.2*. Let T^k denote the k-dimensional torus. Then every homotopy equivalence $W^{n+k} \to M^n \times T^k$ splits along M^n.*

Proof. This follows from 15.3 by induction. □

15.5. Theorem. *Let M^n be a manifold as in* 15.2*. Then every homeomorphism $h : W^{n+k} \to M^n \times \mathbb{R}^k$ of a PL manifold W^{n+k} splits along M^k.*

Proof. We use the Novikov's torus trick. The canonical inclusion $T^{k-1} \times \mathbb{R} \subset \mathbb{R}^k$ yields the inclusion $M \times T^{k-1} \times \mathbb{R} \subset M \times \mathbb{R}^k$. We set $W_1 := h^{-1}(M \times T^{k-1} \times R)$. Notice that W_1 is a smooth manifold since it is an open subset of W. Now, set $u = h|W_1 : W_1 \to M \times T^{k-1} \times \mathbb{R}$. Then, by 15.2, u splits along $M \times T^{k-1}$, i.e. there is a homotopy u_t as in 15.1. We set $f := u_1, V := f^{-1}(M \times T^{k-1})$ and $g := f|V$. Because of 15.4, $g : V \to M \times T^{k-1}$ splits along M. Hence, f splits along M, and therefore u splits along M. Let $\overline{u}_t$ be the homotopy which realizes this splitting as in 15.1. Now, we define the homotopy $h_t : W \to M \times \mathbb{R}^n$ by setting $h_t|W_1 := \overline{u}_t|W_1$ and $h_t|W \setminus W_1 := h|W \setminus W_1$. Notice that $\{h_t\}$ is a well-defined and continuous family since the family $\{\overline{u}_t\}$ satisfies 15.1(ii). It is clear that h_t satisfies the conditions (i)—(iii) of 15.1 and that h_1 extends f on the whole W, i.e. h splits along M. □

15.6. Remark. The above used theorems of Novikov and Farrell–Hsiang were originally proved for smooth manifolds, but they are valid for PL manifolds as well, because there is an analog of the Thom Transversality Theorem for PL manifolds, [65].

15.7. Lemma. *Suppose that a map $g : W \to A$ splits along a submanifold M of A. Let $\xi = \{E \to A\}$ be a PL bundle over A, let*

$g^*\xi = \{D \to W\}$, and let $k : g^*\xi \to \xi$ be the g-adjoint bundle morphism. Finally, let $l : D \to E$ be the map of the total spaces induced by k. Then l splits over M. (Here we regard A as the zero section of ξ, and so M turns out to be a submanifold of E).

Proof. Let $G \times I \to A$ be a homotopy which realizes the splitting of g. Recall that $g^*\xi \times I$ is equivalent to $G^*\xi$. Now, the morphism

$$g^*\xi \times I \cong G^*\xi \xrightarrow{\Im_{G,\xi}} \xi$$

gives us the homotopy $D \times I \to E$ which realizes a splitting of l. $\square$

15.8. Lemma. *Let M be a manifold as in* 15.2. *Consider two PL $\mathbb{R}^N$-bundles $\xi = \{U \to M\}$ and $\eta = \{E \to M\}$ over M and a topological morphism $\varphi : \xi \to \eta$ over M of the form*

$$\begin{array}{ccc} U & \xrightarrow{g} & E \\ \downarrow & & \downarrow \\ M & = & M. \end{array}$$

Then there exists k such that the map

$$g \times 1 : U \times \mathbb{R}^k \to E \times \mathbb{R}^k$$

splits along M, where M is regarded as the zero section of η.

Proof. Take a PL $\mathbb{R}^m$-bundle ζ such that $\eta \oplus \zeta = \theta^{N+m}$ and let W be the total space of $\xi \oplus \zeta$. Then the map

$$\varphi \oplus 1 : \xi \oplus \zeta \to \eta \oplus \zeta = \theta^{N+m}$$

yields a map

$$\Phi : W \to M \times \mathbb{R}^{N+m} \tag{15.1}$$

of the total spaces. Because of Theorem 15.5, the map Φ splits along M. Now, considering the morphism

$$\varphi \oplus 1 \oplus 1 : \xi \oplus \zeta \oplus \eta \to \eta \oplus \zeta \oplus \eta$$

and passing to the total spaces, we get a map

$$g \times 1 : U \times \mathbb{R}^{2N+m} \to E \times \mathbb{R}^{2N+m}.$$

In view of Lemma 15.7, this map splits over M because Φ does. So, we can put $k = 2N + m$. $\square$

Now, let $a : TOP/PL \to F/PL$ be a map as in (2.5).

15.9. Theorem. *Let M be as in 15.2. Then the composition*

$$[M, TOP/PL] \xrightarrow{a_*} [M, F/PL] \xrightarrow{s} P_{\dim M}$$

is trivial, i.e., $sa_*(v) = 0$ for every $v \in [M, TOP/PL]$. In other words, $s(M, af) = 0$ for every $f : M \to TOP/PL$.

Proof. In view of 3.4, every element $v \in [M, TOP/PL]$ is a concordance class of a topological morphism

$$\varphi : \nu_M^N \longrightarrow \gamma_{PL}^N$$

of PL $\mathbb{R}^N$-bundles. Passing from the class $v \in [M, TOP/PL]$ to the class $a_*v \in [M, F/PL]$, we must consider the equivalence class of F_N-morphism $\varphi^\bullet : (\nu_M)^\bullet \to (\gamma_{PL}^N)^\bullet$. Now, following the definition of the map $s : [M, F/PL] \to P_{\dim M}$, we get a commutative diagram

$$\begin{array}{ccc} U^\bullet & \xrightarrow{g} & U'^\bullet \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ M & = & M \end{array} \tag{15.2}$$

like (12.1). However, here we know that g is a homeomorphism. Thus, $g(U) = U'$, and so we get the diagram

$$\begin{array}{ccc} U & \xrightarrow{g} & U' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ M & = & M \end{array} \tag{15.3}$$

which is a topological morphism of PL bundles over M. We embed M in U' as the zero section. By the definition of the map s, we conclude that $s(M, a_*v) = 0$ if the map $g : U \to U'$ splits along M (because in this case the associated normal map is a map over a homotopy equivalence). Moreover, since, for every k, the topological morphisms φ and

$$\nu_M \oplus \theta^k \xrightarrow{(\varphi \oplus 1)} \gamma_{PL}^N \oplus \theta^k \longrightarrow \gamma_{PL}^{N+k}$$

represent the same element of $[M, TOP/PL]$, it suffices to prove that there exists k such that the map

$$g \times 1 : U \times \mathbb{R}^k \to U' \times \mathbb{R}^k$$

splits along M. But this follows from Lemma 15.8. $\square$

Now we show that the condition $\dim M \geq 5$ in 15.9 is not necessary.

15.10. Corollary. *Let M be a closed connected PL manifold such that $\pi_1(M)$ is a free abelian group. Then $s(M, af) = 0$ for every map $f : M \to TOP/PL$.*

Proof. Let CP^2 denote the complex projective plane, and let

$$p_1 : M \times CP^2 \to M$$

be the projection on the first factor. Then $s(M \times CP^2, gp_1) = s(M, g)$ for every $g : M \to F/PL$, see [5, Ch. III, §5]. In particular, for every map $f; M \to TOP/PL$ we have

$$s(M, af) = s(M \times \mathbb{CP}^2, (af)p_1) = s(M \times \mathbb{CP}^2, a(fp_1)) = 0$$

where the last equality follows from Theorem 15.9. □

16. Detecting families

Recall the terminology: a singular smooth manifold in a space X is a map $M \to X$ of a smooth manifold.

Given a CW-space X, consider a connected closed smooth singular manifold $\gamma : M \to X$ in X. Then, for every map $f : X \to F/PL$, the invariant $s(M, f\gamma) \in P_{\dim M}$ is defined. Clearly, if f is null-homotopic then $s(M, f\gamma) = 0$.

16.1. Definition. Let $\{\gamma_i : M_i \to X\}_{i \in I}$ be a family of closed connected smooth singular manifolds in X. We say that the family $\{\gamma_i : M_i \to X\}$ is a *detecting family* for X if, for every map $f : X \to F/PL$, the validity of all the equalities $s(M_i, f\gamma_i) = 0, i \in I$ implies that f is null-homotopic.

Notice that F/PL is an H-space, and hence, for every detecting family $\{\gamma_i : M_i \to X\}$, the collection $s(M_i, f\gamma_i)$ determine a map $f : X \to F/PL$ uniquely up to homotopy.

The concept of detecting family is related to Sullivan's "characteristic variety", but it is not precisely the same. If a family $\mathcal{F}$ of singular manifolds in X contains a detecting family, then $\mathcal{F}$ itself is a detecting family. On the contrary, the characteristic variety is in a sense "minimal" detecting family.

16.2. Theorem. *Let X be a connected finite CW-space such that the group $H_*(X)$ is torsion free. Then X possesses a detecting family $\{\gamma_i : M_i \to X\}$ such that each M_i is orientable.*

Proof. Since $H_*(X)$ is torsion free, every homology class in $H_*(X)$ can be realized by a closed connected smooth oriented singular manifold, see e.g. [8, 15.2] or [48, 6.6 and 7.32]. Let $\{\gamma_i : M_i \to X\}$ be a family of smooth oriented closed connected singular manifolds such that the elements $(\gamma_i)_*[M_i]$ generate all the groups $H_{2k}(X)$.

We prove that $\{\gamma_i : M_i \to X\}$ is a detecting family. Consider a map $f : X \to F/PL$ such that $s_i(M_i, f\gamma_i) = 0$ for all i. We must prove that f is null-homotopic.

Because of 14.7, it suffices to prove that $f^*K_{4n} = 0$ and $f^*K_{4n-2} = 0$ for every $n \geq 1$. Furthermore, $H^*(X) = \mathrm{Hom}(H_*(X), \mathbb{Z})$ because $H_*(X)$ is torsion free. So, it suffices to prove that

$$\langle f^*K_{4n}, x\rangle = 0 \text{ for every } x \in H_{4n}(X) \tag{16.1}$$

and

$$\langle f^*K_{4n-2}, x\rangle = 0 \text{ for every } x \in H_{4n-2}(X; \mathbb{Z}/2). \tag{16.2}$$

First, we prove (16.1). Since the classes $(\gamma_i)_*[M_i], \dim M_i = 4n$ generates the group $H_{4n}(X)$, it suffices to prove that

$$\langle f^*K_{4n},\, (\gamma_i)_*[M_i]\rangle = 0 \text{ whenever } \dim M_i = 4n.$$

But, because of 14.1, for every $4n$-dimensional M_i we have

$$0 = s(M_i, f\gamma_i) = \langle (f\gamma_i)^*K_{4n},\, [M_i]\rangle = \langle f^*K_{4n},\, (\gamma_i)_*[M_i]\rangle\,.$$

This completes the proof of the equality (16.1).

Passing to the case $n = 4k + 2$, notice that the group $H_{2k}(X; \mathbb{Z}/2)$ is generated by the elements $(\gamma_i)_*[M_i]_2, \dim M_i = 2k$, since $H_*(X)$ is torsion free. Now the proof can be completed similarly to the case $n = 4k$. □

17. A special case of the theorem on the normal invariant of a homeomorphism

17.1. Theorem. *Let M be closed connected PL manifold such that each of the group $H_i(M)$ and $\pi_1(M)$ is a free abelian group. Then $j_F(x) = 0$ whenever $x \in \mathfrak{S}_{PL}$ can be represented by a homeomorphism $h : V \to M$.*

Proof. The maps j_{TOP} and j_F from section 3 can be included in the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{T}_{PL}(M) & \xrightarrow{j_{TOP}} & [M, TOP/PL] \\ \Big\downarrow & & \Big\downarrow{\scriptstyle a_*} \\ \mathfrak{S}_{PL}(M) & \xrightarrow{j_F} & [M, F/PL] \end{array} \tag{17.1}$$

where the left arrow is the obvious forgetful map and a_* is induced by a as in (2.5).

Suppose that x can be represented by a homeomorphism $h : V \to M$. Consider a map $f : M \to TOP/PL$ such that $j_{TOP}(h)$ is homotopy class of f. Then, clearly, the class $j_F(x) \in [M, F/PL]$ is represented by the map

$$M \xrightarrow{f} TOP/PL \xrightarrow{a} G/PL.$$

By 16.2, M possesses a detecting family $\{\gamma_i : M_i \to M\}$. We can assume (performing oriented surgeries of M_i if necessary) that $\pi_1(M_i)$ is a subgroup of $\pi_1(M)$, and so $\pi_1(M_i)$ is a free abelian group. Hence, by 15.9 and 15.10, $s(M_i, af\gamma_i) = 0$ for every i. But $\{\gamma_i : M_i \to M\}$ is a detecting family, and therefore af is null-homotopic. Thus, $j_F(x)) = 0$. □

Chapter 3. Applications

18. Topological invariance of rational Pontryagin classes

18.1. Lemma. *The homotopy groups $\pi_i(PL/O)$ are finite. Thus, the space $PL/O[0]$ is contractible.*

Proof. See [48, IV.4.27(iv)]. □

Recall that $H^*(BO;\mathbb{Q}) = \mathbb{Q}[p_1,\dots,p_i,\dots]$ where $p_i, \dim p_i = 4i$ is the universal rational Pontryagin class, see e.g. [38]. (In fact, p_i is the image of the integral Pontryagin class $p_i \in H^*(BO)$.)

18.2. Theorem. *The forgetful map $\alpha = \alpha_{PL}^{O} : BO \to BPL$ induces an isomorphism*

$$\alpha^* : H^*(BPL;\mathbb{Q}) \to H^*(BO;\mathbb{Q}).$$

Proof. It suffices to prove that $\alpha[0] : BO[0] \to BPL[0]$ is a homotopy equivalence. But this holds because the homotopy fiber of $\alpha[0]$ is the contractible space $PL/O[0]$. □

It follows from 18.2 that $H^*(BPL;\mathbb{Q}) == \mathbb{Q}[p_1',\dots,p_i',\dots]$ where p_i' are the cohomology classes determined by the condition $\alpha^*(p_i') = p_i$. Now, given a PL manifold M, we define its rational Pontryagin classes $p_i'(M) \in H^{4i}(M;\mathbb{Q})$ by setting

$$p_i'(M) = t^* p_i'$$

where $t : M \to BPL$ classifies the stable tangent bundle of M. Clearly, if we regard a smooth manifold as a PL manifold then $p_i(N) = p_i'(N)$.

18.3. Corollary (PL invariance of Pontryagin classes, [60, 45])**.** *Let $f : M_1 \to M_2$ be a PL isomorphism of smooth manifolds. Then $f^*p_i(M_2) = p_i(M_1)$.* □

18.4. Theorem. *The forgetful map $\alpha = \alpha_{TOP}^{PL} : BPL \to BTOP$ induces an isomorphism*

$$\alpha^* : H^*(BTOP;\mathbb{Q}) \to H^*(BPL;\mathbb{Q}).$$

Proof. This can be deduced from Theorem 8.8 just in the same manner as we deduced Theorem 18.2 from Theorem 18.1. □

Now we introduce the universal classes $p_i'' \in H^{4i}(BTOP;\mathbb{Q})$ by the equality

$$(\alpha_{TOP}^{PL})^* p_i' = p_i''.$$

Furthermore, given a topological manifold M, we set

$$p_i''(M) = t^*p_i''$$

(where t classifies the stable tangent bundle of M) and get the following corollary.

18.5. Corollary (topological invariance of Pontryagin classes, [41]). *Let $f : M_1 \to M_2$ be a homeomorphism of smooth manifolds. Then $f^*p_i(M_2) = p_i(M_1)$.* □

19. The space F/TOP

It turns out to be that, in view of the Product Structure Theorem, the Transversality Theorem holds for topological manifolds and bundles. I am not able to discuss it here, see [48, IV.7.18] for the references.

Since we have the topological transversality, we can define the maps

$$s' : [M, F/TOP] \to P_{\dim M}$$

which are obvious analogs of maps s defined in (13.1). However, here we allow M to be a topological manifold. The following proposition demonstrates the main difference between F/PL and F/TOP.

19.1. Proposition. *The map $s' : \pi_4(F/TOP) \to \mathbb{Z}$ is a surjection.*

Proof. Notice that the Freedman manifold from Theorem 7.1 is almost parallelizable and has signature 8. Now the proof can be completed just as 13.1(i). □

Recall that in (2.5) we described a fibration

$$TOP/PL \xrightarrow{a} F/PL \xrightarrow{b} F/TOP.$$

19.2. Theorem. *For $i \neq 4$ the map $b : F/PL \to F/TOP$ induces an isomorphism*

$$b_* : \pi_i(F/PL) \to \pi_i(F/TOP).$$

The map

$$b_* : \mathbb{Z} = \pi_i(F/PL) \to \mathbb{Z} = \pi_i(F/TOP)$$

is the multiplication by 2.

Proof. Recall that $TOP/PL = K(\mathbb{Z}/2, 3)$. So, the homotopy exact sequence of the fibration

$$TOP/PL \xrightarrow{a} F/PL \xrightarrow{b} F/TOP$$

yields an isomorphism $b_* : \pi_i(F/TOP) \cong \pi_i(F/PL)$ for $i \neq 4$. Furthermore, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r, equal] & \pi_4(TOP/PL) \arrow[d, "a_*"'] & \\
\mathbb{Z} \arrow[r, equal] & \pi_4(F/PL) \arrow[r, "s"] \arrow[d, "b_*"'] & \mathbb{Z} \arrow[d, equal] \\
& \pi_4(F/TOP) \arrow[r, "s'"] \arrow[d] & \mathbb{Z} \\
& \pi_3(TOP/PL) \arrow[r, equal] \arrow[d] & \mathbb{Z}/2 \\
& \pi_3(F/PL) \arrow[r, equal] & 0
\end{tikzcd}$$

where the middle line is a short exact sequence. So, $\pi_4(F/TOP) = \mathbb{Z}$ or $\pi_4(FTOP) = \mathbb{Z} \oplus \mathbb{Z}/2$. By 13.1, $\operatorname{Im} s$ is the subgroup $2\mathbb{Z}$ of $\mathbb{Z}$, while s' is a surjection by 19.2. Thus, $\pi_4(F/TOP) = \mathbb{Z}$ and b_* is the multiplication by 2. □

Now, following 14.1, we can introduce the classes

$$K'_{4i} \in H^{4i}(F/TOP, \mathbb{Z}[2]) \text{ and } K'_{4i-2} \in H^{4i-2}(F/TOP, \mathbb{Z}/2)$$

such that

$$s'(M^{4i}, f) = \langle f^*K'_{4i}, [M]\rangle \text{ and } s'(N^{4i-2}, f) = \langle f^*K'_{4i-2}, [N]_2\rangle.$$

However, here M and N are assumed to be topological (i,e, not necessarily PL) manifolds.

Together these classes yield the map

$$K' : F/TOP \longrightarrow \prod_{i>0} \left(K(\mathbb{Z}[2], 4i) \times K(\mathbb{Z}/2, 4i-2)\right).$$

19.3. Theorem. *The map*

$$K'[2] : K' : F/TOP[2] \longrightarrow \prod_{i>0} (K(\mathbb{Z}[2], 4i) \times K(\mathbb{Z}/2, 4i-2))$$

is a homotopy equivalence.

Proof. Together 13.1 and 19.1 imply that the homomorphisms $s' : \pi_{2i}(F/TOP) \to P_{2i}$ are surjective. Now, in view of 19.2, all the homomorphisms s' are isomorphisms, and the result follows. □

So, the only difference between F/PL and F/TOP is that $F/TOP[2]$ has trivial Postnikov invariants while $F/PL[2]$ has just one non-trivial Postnikov invariant $\delta Sq^2 \in H^5(K(\mathbb{Z}/2,2);\mathbb{Z}[2])$.

20. The map $a : TOP/PL \to F/PL$

20.1. Proposition. *The map $a : TOP/PL \to F/PL$ is essential.*

Proof. For general reasons, the fibration

$$TOP/PL \xrightarrow{a} F/PL \longrightarrow F/TOP$$

yields a fibration

$$\Omega(F/TOP) \xrightarrow{u} TOP/PL \xrightarrow{a} F/PL.$$

If a is inessential then there exists a map $v : TOP/PL \to \Omega(F/TOP)$ with $uv \simeq 1$. But this is impossible because $\pi_3(TOP/PL) = \mathbb{Z}/2$ while $\pi_3(\Omega(F/TOP)) = \pi_4(F/TOP) = \mathbb{Z}$.

Take an arbitrary map $f : X \to TOP/PL$. Let $\ell : F/PL \to F/PL[2]$ denote the localization map.

20.2. Proposition. *The following three conditions are equivalent:*

(i) *the map*

$$X \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL$$

is essential;

(ii) *the map*

$$X \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL \xrightarrow{\ell} F/PL[2]$$

is essential;

(iii) *the map*

$$X \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL \xrightarrow{\ell} F/PL[2] \xrightarrow{\text{projection}} Y$$

is essential.

Proof. It suffices to prove that (i) $\Rightarrow$ (ii) $\Rightarrow$(iii). To prove the first implication, recall that a map $u : X \to F/PL$ is inessential if both localized maps

$$X \xrightarrow{u} F/PL \to F/PL[2], \quad X \xrightarrow{u} F/PL \to F/PL[1/2]$$

are inessential. Now, (i) $\Rightarrow$ (ii) holds since $TOP/PL[1/2]$ is contractible.

To prove the second implication, notice that a map $v : X \to F/PL[2]$ is inessential if both maps (we use notation as in 14.5)

$$X \xrightarrow{v} F/PL[2] \xrightarrow{K} \Pi, \quad X \xrightarrow{v} F/PL[2] \longrightarrow Y$$

are inessential. So, it suffices to prove that the map

$$X \xrightarrow{\ell a f} F/PL[2] \longrightarrow \Pi$$

is inessential. This holds, in turn, because the map $TOP/PL \to F/PL \to F/TOP$ is inessential and the diagram

$$\begin{array}{ccccc} F/PL[2] & \xrightarrow{K[2]} & \Pi & = & \Pi \\ \downarrow{\scriptstyle b[2]} & & & & \| \\ F/TOP[2] & \xrightarrow{K'[2]} & \prod_{i>0}(K(\mathbb{Z}/2, 4i-2) \times K(\mathbb{Z}[2], 4i)) & \xrightarrow{\text{proj}} & \Pi \end{array}$$

commutes. □

Consider the fibration

$$K(\mathbb{Z}[2], 4) \xrightarrow{i} Y \longrightarrow K(\mathbb{Z}/2.2)$$

as in (14.4).

20.3. Lemma. *For every space X, the homomorphism*

$$H^4(X; \mathbb{Z}[2]) = [X, K(\mathbb{Z}[2], 4)] \xrightarrow{i_*} [X, Y]$$

is injective. Moreover, i_* is an isomorphism if $H^2(X; \mathbb{Z}/2) = 0$.

Proof. The fibration (14.4) yields the exact sequence (see e.g. [39])

(20.1) $$H^1(X; \mathbb{Z}/2) \xrightarrow{\delta Sq^2} H^4(X; \mathbb{Z}[2]) \xrightarrow{i_*} [X, Y] \to H^2(X; \mathbb{Z}/2)$$

where $\delta Sq^2(x) \equiv 0$ (because $\delta Sq^2(x) = 0$ whenever $\deg x = 1$). □

Let $g : TOP/PL \to Y$ be the composition

$$TOP/PL \xrightarrow{a} F/PL \xrightarrow{\ell} F/PL[2] \xrightarrow{\text{proj}} Y.$$

Notice that g is essential because of 20.1 and 20.2.

20.4. Corollary. *The map*

$$TOP/PL = K(\mathbb{Z}/2, 3) \xrightarrow{\delta} K(\mathbb{Z}[2], 4) \xrightarrow{i} Y$$

is homotopic to g, i.e. $g \simeq i\delta$.

Proof. Since the sequence (20.1) is exact, the set $[K(\mathbb{Z}/2, 3), Y]$ has precisely two elements. Since both maps g and $i \circ \delta$ are essential (the last one because of Lemma 20.3), we conclude that $g \simeq i\delta$. □

20.5. Theorem. *Given a map $f : X \to TOP/PL$, the map*

$$X \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL$$

is essential if and only if the map

$$X \xrightarrow{f} TOP/PL = K(\mathbb{Z}/2, 3) \xrightarrow{\delta} K(\mathbb{Z}[2], 5)$$

is essential.

Proof. af is essential $\overset{20.2}{\Longleftrightarrow}$ gf is essential $\overset{20.4}{\Longleftrightarrow}$ $i\delta f$ is essential $\overset{20.3}{\Longleftrightarrow}$ δf is essential. □

21. The theorem on the normal invariant of a homeomorphism

21.1. Lemma. *Let X be a finite CW-space such that $H_n(X)$ is 2-torsion free. Then the homomorphism*

$$\delta : H^n(X; \mathbb{Z}/2) \to H^{n+1}(X; \mathbb{Z}[2])$$

is zero.

Proof. Because of the exactness of the sequence (14.1)

$$H^n(X; \mathbb{Z}/2) \xrightarrow{\delta} H^{n+1}(X; \mathbb{Z}[2]) \xrightarrow{2} H^{n+1}(X; \mathbb{Z}[2]),$$

it suffices to prove that $H^{n+1}(X; \mathbb{Z}[2])$ is 2-torsion free. Since $H_n(X)$ is 2-torsion free, we conclude that $\operatorname{Ext}(H_n(X), \mathbb{Z}[2]) = 0$. Thus,

$$\begin{aligned} H^{n+1}(X; \mathbb{Z}[2]) &= \operatorname{Hom}(H_{n+1}(X; \mathbb{Z}[2]) \oplus \operatorname{Ext}(H_n(X); \mathbb{Z}[2]) \\ &= \operatorname{Hom}(H_{n+1}(X; \mathbb{Z}[2]), \end{aligned}$$

and the result follows. □

21.2. Theorem. *Let M be a closed PL manifold such that $H_3(M)$ is 2-torsion free. Then the normal invariant of any homeomorphism $h : V \to M$ is trivial.*

Proof. Since h is a homeomorphism, the normal invariant $j_F(h)$ turns out to be the homotopy class of a map

$$M \xrightarrow{f} TOP/PL \xrightarrow{a} F/PL$$

where the homotopy class of f is $j_{TOP}(h)$. Because of 20.2 and 20.3, it suffices to prove that the map

$$M \xrightarrow{f} TOP/PL = K(\mathbb{Z}/2, 3) \xrightarrow{\delta} K(\mathbb{Z}[2], 4)$$

is inessential. But this follows from Lemma 21.1. □

21.3. Corollary. *Let $M, \dim M \geq 5$ be a closed simply-connected PL manifold such that $H_3(M)$ is 2-torsion free. Then every homeomorphism $h : V \to M$ is homotopic to a PL isomorphism. In particular, V and M are PL isomorphic.*

Proof. This follows from 13.2 and 21.2. □

21.4. Remark. Rourke [46] suggested another proof of 21.2, using the technique of simplicial sets.

22. A counterexample to the Hauptvermutung, and other examples

22.1. Example. *Two manifolds which are homeomorphic but not PL isomorphic.*

Let $\mathbb{RP}^n$ denote the real projective space of dimension n.

22.2. Lemma. *For every homotopy equivalence $h : \mathbb{RP}^5 \to \mathbb{RP}^5$ we have $j_F(h) = 0$.*

Proof. We triangulate $\mathbb{RP}^4$ and take the induced triangulation of the covering space S^4. Take the corresponding triangulation of the suspension $SS^4 = S^5$. Let $r : S^5 \to S^5$ be the reflection with respect to the equator S^4. Since r is an antipodal map, it yields a map $f : \mathbb{RP}^5 \to \mathbb{RP}^5$. Clearly, f is a map of degree -1.

It follows from the obstruction theory that every homotopy equivalence $\mathbb{RP}^5 \to \mathbb{RP}^5$ is homotopic either to f or to the identity map. (For the homotopy classification of maps $\mathbb{RP}^n \to \mathbb{RP}^n$, see [18].) Since f is a PL isomorphism, the lemma follows. □

Recall that $j_{TOP} : \mathfrak{T}_{PL}(\mathbb{RP}^5) \to [\mathbb{RP}^5, TOP/PL]$ is a bijection. Consider a homeomorphism $k : M \to \mathbb{RP}^5$ such that

$$j_{TOP}(k) \neq 0 \in [\mathbb{RP}^5, TOP/PL] = H^3(\mathbb{RP}^5; \mathbb{Z}/2) = \mathbb{Z}/2.$$

Notice that

$$\delta : \mathbb{Z}/2 = H^3(\mathbb{RP}^5; \mathbb{Z}/2) \to H^4(\mathbb{RP}^5; \mathbb{Z}/2) = \mathbb{Z}/2$$

is an isomorphism, and hence $\delta(j_{TOP}(k)) \neq 0$. So, by Theorem 20.5, $a_* j_{TOP}(k) \neq 0$. In view of commutativity of the diagram (17.1), $j_F(k) = a_* j_{TOP}(k)$, i.e. $j_F(k) \neq 0$. Thus, in view of Lemma 22.2, M is not PL isomorphic to $\mathbb{RP}^5$.

22.3. Example. *A homeomorphism $h : S^3 \times S^n \to S^3 \times S^n, n > 3$ which is homotopic to a PL isomorphism but is not concordant to a PL isomorphism.*

Take an arbitrary homeomorphism $f : V \to S^3 \times S^n$. Then $j_F(f)$ is trivial by Theorem 21.2. Thus, by 13.2, f is homotopic to a PL isomorphism. In particular, V is PL isomorphic to $S^3 \times S^n$.

Now, we refine the situation and take a homeomorphism $h : S^3 \times S^n \to S^3 \times S^n$ such that

$$j_{TOP}(h) \neq 0 \in \mathfrak{T}_{PL}(S^3 \times S^n) = H^3(S^3 \times S^n; \mathbb{Z}/2) = \mathbb{Z}/2.$$

So, h is not concordant to the identity map, and therefore h is not concordant to a PL isomorphism. But, as we have already seen, h is homotopic to a PL isomorphism.

Notice that the maps h and the identity map have the same domain while they are not concordant. So, this example serves also the Remark 3.2(3).

22.4. Example. *A topological manifold which does not admit any PL structure.*

Already constructed $F \times T^n$, see 7.3.

22.5. Example. *A topological manifold which is homeomorphic to a polyhedron but does not admit any PL structure.*

Let M be a closed topological 7-dimensional manifold, and let

$$\varkappa(M) \in H^4(M; \mathbb{Z}/2)$$

be the Kirby–Siebenmann invariant described in 8.10. Let

$$\delta : H^4(M; \mathbb{Z}/2) \to H^5(M; \mathbb{Z})$$

be the Bockstein homomorphism. Scharlemann [49] proved that if $\delta\varkappa(M) = 0$ and if a quadruple suspension over a certain 3-dimensional homology sphere is homeomorphic to S^7, then M admits a simplicial triangulation. Cannon [6] proved that a double suspension over every 3-dimensional homology sphere is homeomorphic to S^5. So, if $\varkappa(M) \neq 0$ while $\delta\varkappa(M) = 0$ then M is homeomorphic to a polyhedron but does not admits a PL structure.

Take $M = F \times T^3$. Then $\varkappa(M) \neq 0$ by 7.3. On the other hand, $\delta\varkappa(M) = 0$ because $F \times T^3$ is torsion free, while $2\delta(x) = 0$ for every x.

Epilogue

Certainly, it is useful (or even necessary) to write a book about this subject. The paper on hand looks as a reasonable point to start. The contents of the book is more or less clear (we can follow the graph from the introduction). From my (maybe, personal) point of view, the accurate treatment of PL manifolds and their normal bundles, and of the Product Structure Theorem – is most difficult thing in this business (while surgery exact sequence, fake tori, splitting theorems, etc. are more or less well exposed in the literature). I appreciate any comments and would be glad to collaborate with anybody who will be interested in it.

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