ON INTEGRAL MODELS OF AFFINE TORIC VARIETIES

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1. The goal of this paper is to further investigate the class of (the systems of) Diophantine equations studied by one of us in a recent series of papers, [8-11]. Each of these systems of equations defines an integral model of an affine toric variety. To be precise, given an affine toric variety X defined over $\mathbb{Q}$, these equations define an affine $\mathbb{Z}$-scheme $\mathcal{X}$ whose generic fibre $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to X. After formalising this construction in n°2, we go on to generalise it. In n°3 a class of integral models of an affine toric variety defined over a number field is described; in the last section (n°4) we establish some finiteness properties of these integral models. It seems to be the first attempt to study such models systematically (cf., however, [14]). On the other hand, integral models of algebraic tori have been discussed in the literature, [1], [3], [4], [7], [12], [13], and we comment on the relation of our "standard model" of an algebraic torus to other models, leaving the detailed discussion of this point to a subsequent publication.

In what follows $\mathbb{Q}, \mathbb{Z}$, and $\mathbb{Q}_p$ stand for the field of rational numbers, the ring of rational integers, and the p-adic number field respectively; $G_{m,L}$ denotes the multiplicative group defined over a field L; let $\mathcal{A}(\mathfrak{v})$ be the category of commutative $\mathfrak{v}$-algebras for a given commutative ring $\mathfrak{v}$.

2. Let k be a field of characteristic zero. An algebraic torus T of dimension d defined over k splits over a finite normal extension $K \mid k$, so that $T \times_k K \cong G^d_{m,K}$. Let $\mathfrak{G}$ be the Galois group of $K \mid k$, and let $M = \mathrm{Hom}(T \times_k K, G_m)$ be the $\mathfrak{G}$-module of characters of T. Let $\rho : \mathfrak{G} \to GL(d, \mathbb{Z})$ be the integral representation afforded by M; the dual module $\check{M} = \mathrm{Hom}(M, \mathbb{Z})$ affords the contragredient representation $\check{\rho}$. We fix a $\mathbb{Z}$-basis $\{e_1, \dots, e_d\}$ of $\check{M}$, a basis $\{\omega_1, \dots, \omega_n\}$ of $K \mid k$, introduce a system of independent variables $x = (\dots, x_{ij}, \dots), 0 \le i \le d,\ 1 \le j \le n$, and write

$$gt_i = \sum_{j=1}^{n} x_{ij} g\omega_j,\ t_i^g = \prod_{j=1}^{d} t_j^{r_{ji}(g)},\ 1 \le i \le d, g \in \mathfrak{G},$$

where $\check{\rho}(g)e_i = \sum_{j=1}^{d} e_j r_{ji}(g)$. The torus T can be defined by the following system of equations in $k[x]$:

$$gt = t^g, t_0 \prod_{i=1}^{d} t_i = 1, g \in \mathfrak{G}, \tag{1}$$

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where $t := (t_1, \dots, t_d)$, $t_0 = \sum_{j=1}^{n} x_{oj}\omega_j$. Given a $\mathfrak{G}$-invariant strongly convex rational polyhedral cone $\sigma_{\mathbb{Q}}$ in the $\mathbb{Q}$-vector space $V = M \otimes_{\mathbb{Z}} \mathbb{Q}$, one defines the dual cone

$$\check{\sigma}_{\mathbb{Q}} = \{v \mid v \in \check{V}, (v \mid u) \geq 0 \quad \text{for} \quad u \in \sigma_{\mathbb{Q}}\}$$

in $\check{V} = \check{M} \otimes_{\mathbb{Z}} \mathbb{Q}$, and the semigroup $\check{\sigma} = \check{\sigma}_{\mathbb{Q}} \cap \check{M}$; let $\{u_1, \dots, u_l\}$ be the minimal set of generators of $\check{\sigma}$ (cf. [6]). We write

$$u_i = \sum_{j=1}^{d} e_j c_{ji},\ 1 \leq i \leq \ell;\ e_j = \sum_{i=1}^{\ell} u_i \tilde{c}_{ij},\ 1 \leq j \leq d.$$

Let $\tilde{r}(g) = \tilde{c}r(g)c, \tilde{c} = (\tilde{c}_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq d}, c = (c_{ij})_{1 \leq i \leq d, 1 \leq j \leq \ell}$; clearly, $\check{\rho}(g)u_i = \sum_{j=1}^{\ell} u_j \tilde{r}_{ji}(g), g \in \mathfrak{G}$, $1 \leq i \leq \ell$. In the new variables $y = (\dots, y_{ih}, \dots)$, $1 \leq i \leq \ell$, $1 \leq h \leq n$, equations

$$\sum_{h=1}^{n} y_{ih} g\omega_h = \prod_{j=1}^{\ell} (\sum_{h=1}^{n} y_{jh}\omega_h)^{\tilde{r}_{ji}(g)},\ g \in \mathfrak{G},\ 1 \leq i \leq \ell, \tag{2}$$

define an algebraic variety $X_{T,\sigma}$ over k. It is a toric variety, containing the torus T as a dense open orbit.

Given a subring $\mathfrak{v}$ of the field k, an $\mathfrak{v}$-scheme $\mathcal{X}$ is said to be an *integral model* of $X_{T,\sigma}$ if $\mathcal{X} \times_{\mathfrak{v}} k \cong X_{T,\sigma}$. Suppose now that k is the quotient field of a ring $\mathfrak{v}$ and K is the quotient field of the ring $\mathfrak{O} = \sum_{i=1}^{n} \oplus \omega_i \mathfrak{v}$; let, moreover, $g\omega_i \in \mathfrak{O}$ for $g \in \mathfrak{G}$, $1 \leq i \leq n$. Then equations (2) define a *standard* integral model $\mathcal{X}_0$ of $X_{T;\sigma}$. Clearly,

$$\mathcal{X}_0(A) = \{\alpha \mid \alpha \in B^l,\ g\alpha = \alpha^g \quad \text{for} \quad g \in \mathfrak{G}\}$$

for $A \in \mathcal{A}(\mathfrak{v})$, where $B = A \otimes_{\mathfrak{v}} \mathcal{O}$, and $\alpha_i^g := \prod_{j=1}^{\ell} \alpha_j^{\tilde{r}_{ji}(g)}$, $g \in \mathfrak{G}$, $1 \leq i \leq \ell$. For a standard model $\mathcal{T}_0$ of the k-torus T, defined by (1), this gives

$$\mathcal{T}_0(A) = \{\alpha \mid \alpha \in (B^*)^d,\ g\alpha = \alpha^g \quad \text{for} \quad g \in \mathfrak{G}\} \tag{3}$$

with $\alpha_i^g := \prod_{j=1}^{d} \alpha_j^{r_{ji}(g)}$, $1 \leq i \leq d$, $g \in \mathfrak{G}$, in this case. Note that $\mathcal{T}_0$ is a group $\mathfrak{v}$-scheme since $\mathcal{T}_0(A)$ is an Abelian group for $A \in \mathcal{A}(\mathfrak{v})$. An assiduous reader will note that equations $t^g = gt$, $g \in \mathfrak{G}$, define T as a closed subset in the torus $S = R_{K/k}G_{m,K}^d$, while the condition $\prod_{i=1}^{d} t_i \neq 0$ (equivalent to the equation $t_0 \prod_{i=1}^{d} t_i = 1$) defines S as an open subset in $\mathbb{A}_k^N$, $N = dn$. In other words, equations (1) define a natural embedding of the torus T into the quasi-trivial torus $S = R_{K/k}T_K$. Since S can be regarded as a maximal k-torus of $GL_k(V)$, where V is an N-dimensional linear k-space, one obtains a faithful linear k-representation $\varphi : T \rightarrowtail GL_k(V)$;let $\varphi^* : k[GL(V)] \to k[T]$ be the corresponding homomorphism of the function algebras. The choice of an integer basis of $K \mid k$ induces an integral structure in $GL(V)$, or in other terms, defines a Hopf algebra $H := \mathfrak{v}[GL(V)]$ in $k[GL(V)]$. The group $\mathfrak{v}$-scheme $\mathcal{T}_0 = \operatorname{Spec} \mathfrak{v}[x]/J$, where J denotes the ideal defined by the equations (1), coincides with $\operatorname{Spec} \varphi^*(H)$. This construction is reminiscent of the considerations of V.E. Voskresenskii and T.V. Fomina, [3], [4] (cf. also [15], [7]).

3. Let now k be a number field (that is, a finite extension of $\mathbb{Q}$), and let $\mathfrak{o}$ and $\mathfrak{O}$ denote the rings of integers of k and K respectively. We fix a sequence $\mathfrak{B} = (\mathfrak{B}_1, \dots, \mathfrak{B}_l)$ of fractional ideals of K satisfying the conditions

$$g\mathfrak{B}_i = \prod_{j=1}^{\ell} \mathfrak{B}_j^{\tilde{r}_{ji}(g)} \quad \text{for} \quad g \in \mathfrak{G},\ 1 \le i \le \ell, \tag{4}$$

and define a functor $\mathfrak{F}_{\mathfrak{B}} : \mathcal{A}(\mathfrak{o}) \to \mathrm{Sgr}$, where $\mathcal{A}(\mathfrak{o})$ is the category of commutative $\mathfrak{o}$-algebras, and Sgr is the category of commutative semigroups, as follows. Let $A \in \mathcal{A}(\mathfrak{o})$, and let $B_i = A \otimes_{\mathfrak{o}} \mathfrak{B}_i$, $B = (B_1, \dots, B_\ell)$. By definition,

$$\mathfrak{F}_{\mathfrak{B}}(A) = \{\alpha \mid \alpha \in B, g\alpha = \alpha^g \quad \text{for} \quad g \in \mathfrak{G}\},$$

where $(g\alpha)_i = g\alpha_i$, and $(\alpha^g)_i = \prod_{j=1}^{\ell} \alpha_j^{\tilde{r}_{ji}(g)}$.

Theorem 1. There is an integral model $\mathcal{X}_{\mathfrak{B}}$ of $X_{T,\sigma}$ representing the functor $\mathfrak{F}_{\mathfrak{B}}$ (in other words, $\mathcal{X}_{\mathfrak{B}}$ is an $\mathfrak{o}$-scheme, $\mathcal{X}_{\mathfrak{B}}(A) = \mathfrak{F}_{\mathfrak{B}}(A)$ for $A \in \mathcal{A}(\mathfrak{o})$, and $\mathcal{X}_{\mathfrak{B}} \otimes_{\mathfrak{o}} k \cong X_{T,\sigma}$). The scheme $\mathcal{X}_{\mathfrak{B}}$ can be covered by two open affine subsets.

Proof. It follows from the definition of $X_{T,\sigma}$ that if $\mathcal{X}(A) = \mathfrak{F}_{\mathfrak{B}}(A)$ for $A \in \mathcal{A}(\mathfrak{o})$ then $\mathcal{X} \times_{\mathfrak{o}} k \cong X_{T,\sigma}$. To construct an $\mathfrak{o}$-scheme $\mathcal{X}_{\mathfrak{B}}$ representing the functor $\mathfrak{F}_{\mathfrak{B}}$ we appeal to a theorem of Steinitz's (see, for instance, [5, Theorem 13 on p. 95]), and denote by $\mathrm{St}(\mathfrak{B}_j)$ the Steinitz class of the $\mathfrak{o}$-module $\mathfrak{B}_j$. By definition, $\mathrm{St}(\mathfrak{B}_j) \in H_k$, $1 \le j \le \ell$, where H_k is the class group of k. Since each ideal class contains infinitely many prime ideals, we may choose prime ideals $\mathfrak{p}_{j\nu}$ in such a way that $\mathfrak{p}_{j\nu} \in \mathrm{St}(\mathfrak{B}_j)$ and $\mathfrak{p}_{j\nu} \ne \mathfrak{p}_{i\mu}$ for $(j,\nu) \ne (i,\mu)$, $1 \le i, j \le \ell$, $\{\nu, \mu\} \subseteqq \{1,2\}$. By construction,

$$\mathfrak{B}_j = \mathfrak{p}_{j\nu}\omega_{1j}^{(\nu)} \oplus \sum_{h=2}^{n} \oplus \mathfrak{o}_\nu \omega_{hj},\ 1 \le j \le \ell,\ \nu = 1, 2,$$

for some ω_{hj}, $\omega_{1j}^{(\nu)}$ in K. Let $\mathfrak{A}_\nu = \prod_{j=1}^{\ell} \mathfrak{p}_{j\nu}$, $\nu = 1, 2$. Since $(\mathfrak{A}_1, \mathfrak{A}_2) = 1$, there are two integers b_1, b_2 in $\mathfrak{o}$ such that $(b_1, b_2) = 1$ and $\mathfrak{A}_\nu | b_\nu, \nu = 1, 2$. Let $\mathfrak{o}_\nu = \mathfrak{o}[1/b_\nu]$, and let $\mathfrak{o}_{12} = \mathfrak{o}[1/b_1 b_2] (= \mathfrak{o}[1/b_1, 1/b_2])$. Clearly, $\mathfrak{p}_{j\nu}\mathfrak{o}_\nu = \mathfrak{o}_\nu$, and $\mathfrak{p}_{j\nu}\mathfrak{o}_{12} = \mathfrak{o}_{12}$ for $1 \le j \le \ell$, $\nu = 1, 2$. Therefore

$$\mathfrak{B}_j \otimes_{\mathfrak{o}} \mathfrak{o}_\nu = \mathfrak{o}_\nu \omega_{1j}^{(\nu)} \oplus \sum_{h=2}^{n} \oplus \mathfrak{o}_\nu \omega_{hj} \tag{5}$$

is a free $\mathfrak{o}_\nu$-module, and

$$\mathfrak{B}_j \otimes_{\mathfrak{o}} \mathfrak{o}_{12} = \mathfrak{o}_{12} \omega_{1j}^{(\nu)} \oplus \sum_{h=2}^{n} \oplus \mathfrak{o}_{12} \omega_{hj} \quad (\nu = 1, 2) \tag{6}$$

is a free $\mathfrak{o}_{12}$-module. Note that $\mathfrak{p}_{j1} = \alpha_j \mathfrak{p}_{j2}$ and $\omega_{1j}^{(2)} = \alpha_j \omega_{1j}^{(1)}$ for some α_j in k^* since $\mathfrak{p}_{j\nu} \in \mathrm{St}(\mathfrak{B}_j)$, $\nu = 1, 2$, $1 \le j \le \ell$. In view of (4), equations

$$gz^{(\nu)} = (z^{(\nu)})^g,\ g \in \mathfrak{G}, \tag{7}$$

where

$$gz_j^{(\nu)} := y_{j1}^{(\nu)} g\omega_{1j}^{(\nu)} + \sum_{h=2}^{n} y_{jh} g\omega_{hj},\ (z^{(\nu)})_j^g = \prod_{i=1}^{\ell} (z_i^{(\nu)})^{\tilde{r}_{ij}(g)},\ 1 \le j \le \ell,$$

define an ideal $\mathfrak{A}_\nu$ in $\mathfrak{o}_\nu[y^{(\nu)}]$. After clearing the denominators, if necessary, we may (and shall) regard $\mathfrak{A}_\nu$ as an ideal in $\mathfrak{o}[y^{(\nu)}]$. Let $\mathfrak{C}_\nu = \operatorname{Spec} \mathfrak{o}[y^{(\nu)}]/\mathfrak{A}_\nu$, and let $\mathcal{U}_\nu = \{\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{C}_\nu, \mathfrak{p} \nmid b_\nu\}$, $\nu = 1, 2$. By definition, $\mathcal{X}_\mathfrak{B} = W_1 \cup W_2$ with $W_\nu \cong \mathcal{U}_\nu$ and $W_1 \cap W_2 \cong \mathcal{U}_{12}^{(\nu)}$, where $\mathcal{U}_{12}^{(\nu)} = \{\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{C}_\nu, \mathfrak{p} \nmid b_1 b_2\}$, $\nu = 1, 2$, the gluing isomorphism $\varphi : \mathcal{U}_{12}^{(1)} \to \mathcal{U}_{12}^{(2)}$ being given by $\varphi : y_{1j}^{(1)} \mapsto \alpha_j y_{1j}^{(2)}$, $1 \le j \le \ell$. Identifying W_ν with $\operatorname{Spec} \mathfrak{o}_\nu[y^{(\nu)}]/\mathfrak{A}_\nu$, and W_{12} with

$$\operatorname{Spec} \mathfrak{o}_{12}[y^{(1)}]/\mathfrak{A}_1 \cong \operatorname{Spec} \mathfrak{o}_{12}[y^{(2)}]/\mathfrak{A}_2,$$

and bearing in mind the relations (5), (6), and $(b_1, b_2) = 1$, one can easily check that the $\mathfrak{o}$-scheme $\mathcal{X}_\mathfrak{B}$ represents the functor $\mathcal{F}_\mathfrak{B}$, as required.

Corollary 1.
There is an $\mathfrak{o}$-scheme $\mathcal{T}_0$ satisfying relation (3).

Proof. Take $\sigma = \{0\}$, then $\check{\sigma}_\mathbb{Q} = \mathbb{Q}^d$, $\ell = 2d$, and we may let $u_i = e_i$, $u_{i+d} = -e_i$ for $1 \le i \le d$, $\mathfrak{B}_j = \mathcal{O}$ for $1 \le j \le d$. In this case, $\tilde{r}_{ij}(g) = r_{ij}(g)$ and $\tilde{r}_{ij+d}(g) = -r_{ij}(g)$ for $1 \le i, j \le d$, $\tilde{r}_{ij}(g) = 0$ for $i > d$, $g \in \mathfrak{G}$. Therefore $(\alpha^g)_i = \prod_{j=1}^d \alpha_j^{r_{ji}(g)}$, $\alpha_i \alpha_{i+d} = 1$ for $1 \le i \le d$, $g \in \mathfrak{G}$, and the assertion follows from the definition of the functor $\mathfrak{F}_\mathfrak{B}$ and the theorem.

The $\mathfrak{o}$-group scheme $\mathcal{T}_0$ is called the *standard* integral model of the k-torus T; if the extension $K \mid k$ has an integral basis, $\mathcal{T}_0$ coincides with the affine $\mathfrak{o}$-scheme defined in n° 2.

Remark 1. An $\mathfrak{o}$-scheme $\mathcal{X}_\mathfrak{B}$ may be regarded as a $\mathcal{T}_0$-toric variety in the following sense. There is an $\mathfrak{o}$-scheme morphism $\varphi_\mathfrak{B} : \mathcal{T}_0 \times \mathcal{X}_\mathfrak{B} \to \mathcal{X}_\mathfrak{B}$ such that the diagram

$$\begin{array}{ccc} \mathcal{T}_0 \times \mathcal{X}_\mathfrak{B} & \xrightarrow{\varphi_\mathfrak{B}} & \mathcal{X}_\mathfrak{B} \\ \downarrow & & \downarrow \\ T \times X_{T,\sigma} & \xrightarrow{\psi} & X_{T,\sigma} \end{array}$$

commutes (here the vertical arrows are induced by the base change $\mathfrak{o} \to \mathfrak{o} \otimes_\mathfrak{o} k (= k)$). Thus the action of the $\mathfrak{o}$-scheme $\mathcal{T}_0$ on any of the $\mathfrak{o}$-schemes $\mathcal{X}_\mathfrak{B}$, defined by $\varphi_\mathfrak{B}$, extends the natural action of the k-torus T on the k-variety $X_{T,\sigma}$ (defined by ψ). By definition, the morphism $\varphi_\mathfrak{B}$ induces the group $\mathcal{T}_0(A)$ action on the set $\mathcal{X}_\mathfrak{B}(A)$ for $A \in \mathcal{A}(\mathfrak{o})$.

4. Let

$$I(\sigma) = \{\mathfrak{B} \mid \mathfrak{B} \in I(K)^\ell,\ g\mathfrak{B} = \mathfrak{B}^g \quad \text{for} \quad g \in \mathfrak{G}\},$$

where $(\mathfrak{B}^g)_j = \prod_{i=1}^\ell \mathfrak{B}_i^{r_{ij}(g)}$, $1 \le j \le \ell$, and $I(K)$ stands for the group of fractional ideals of K. In this section we prove that the class of integral models

$$\mathcal{D}(T, \sigma) = \{\mathcal{X}_\mathfrak{B} \mid \mathfrak{B} \in I(\sigma)\}$$

of the toric variety $X_{T,\sigma}$, constructed in n° 3, contains only finitely many pairwise non-isomorphic $\mathfrak{o}$-schemes. To state our result, let

$$I_{pr}(\sigma) = \{(\alpha) \mid \alpha \in K^{*\ell},\ g\alpha = \alpha^g \quad \text{for} \quad \mathfrak{G}\},$$

where $(\alpha^g)_j = \prod_{i=1}^{\ell} \alpha_i^{\tilde{r}_{ij}(g)}$, and $(\alpha) := ((\alpha_1), \dots, (\alpha_\ell))$ stands for a sequence of principal ideals in $I(K)^\ell$; let $H(\sigma) = I(\sigma)/I_{pr}(\sigma)$.

Proposition 1. The group $H(\sigma)$ is finite. Moreover, if $\mathfrak{B}^{-1}\mathfrak{B}' \in I_{pr}(\sigma)$ then $\mathcal{X}_{\mathfrak{B}} \cong \mathcal{X}_{\mathfrak{B}'}$ (as $\mathfrak{o}$-schemes).

Proof. Let $\mathfrak{B}' = (\alpha)\mathfrak{B}$ with $(\alpha) \in I_{pr}(\sigma)$. Then the schemes $\mathcal{X}_{\mathfrak{B}}$ and $\mathcal{X}_{\mathfrak{B}'}$ are (locally) defined by the same system of equations (7) if one chooses $\{\alpha_j\omega_{1j}^{(\nu)},\ \alpha_j\omega_{hj} \mid 2 \le h \le n\}$ to serve as an $\mathfrak{o}_\nu$-basis of $\mathfrak{B}_j$ in (5), (6). Let $J(\sigma) = I(\sigma) \cap I_{pr}(K)^\ell$, where $I_{pr}(K) = \{(\alpha) \mid \alpha \in K^*\}$ denotes the group of principal ideals of K. Since $I(\sigma)/J(\sigma)$ is finite, being isomorphic to a subgroup of the finite group $H(K)^\ell$ (here $H(K)$ denotes the ideal class group of K), it suffices to prove that $J(\sigma)/I_{pr}(\sigma)$ is a finite group. Let $(\alpha) \in J(\sigma)$; then $g\alpha = \epsilon(\alpha, g)\alpha^g$ with $\epsilon(\alpha, g) \in (\mathfrak{O}^*)^\ell$. Clearly, $(\alpha) \in I_{pr}(\sigma)$ if and only if

$$\epsilon(\alpha, g) = \frac{g\lambda(\alpha)}{\lambda(\alpha)^g} \quad \text{with} \quad \lambda(\alpha) \in (\mathfrak{O}^*)^\ell. \tag{8}$$

On the other hand, it follows from the identity

$$\epsilon(\alpha, g_1 g_2) = g_1\epsilon(\alpha, g_2)\epsilon(\alpha, g_1)^{g_2}$$

that $\epsilon(\alpha, g)^n$ satisfies relation (8) with $\lambda(\alpha) = \prod_{g\in\mathfrak{G}} g^{-1}\epsilon(\alpha, g)$. Since the group $(\mathfrak{O}^*)^\ell$ is finitely generated, it follows that the group $J(\sigma)/I_{pr}(\sigma)$ is indeed finite. This proves the proposition.

Corollary 2. Suppose that σ contains an element v with $(u \mid v) > 0$ for each u in $\check{\sigma}$, and let $I_0(K)$ be the semigroup of integral ideals in $K, I_0(\sigma) = I(\sigma) \cap I_0(K)^\ell$. There is a finite subset S of $I_0(\sigma)$ such that any scheme $\mathcal{X}$ in $D(T, \sigma)$ is isomorphic to a scheme $\mathcal{X}_{\mathfrak{B}}$ with $\mathfrak{B} \in S$.

Proof. Let $v' = \sum_{g\in\mathfrak{G}} gv$; clearly, $(u_i \mid v') > 0$ for $1 \le i \le \ell$ and $gv' = v'$ for $g \in \mathfrak{G}$. Let $\mathcal{X}_{\mathfrak{B}} \in \mathcal{D}(T, \sigma)$; choose a natural number n such that $n\mathfrak{B} \in I_0(K)^\ell$, and let $\alpha = (n^{m_1}, \dots, n^{m_e})$ with $m_i = (u_i \mid v')$. Let $\mathfrak{B}' = (\alpha)\mathfrak{B}$, then $\mathfrak{B}' \in I_0(\sigma)$ and $\mathcal{X}_{\mathfrak{B}} \cong \mathcal{X}_{\mathfrak{B}'}$ since $g\alpha = \alpha^g = \alpha$ for $g \in \mathfrak{G}$.

Remark 2. We do not know whether

$$\mathcal{X}_{\mathfrak{B}} \cong \mathcal{X}_{\mathfrak{B}'} \Rightarrow \mathfrak{B}^{-1}\mathfrak{B}' \in I_{pr}(\sigma),$$

but suspect it is not the case.

Remark 3. If $\sigma = \{0\}$ then $X_{T,\sigma} \cong T$, and therefore $\mathcal{X}_{\mathfrak{B}}$ is an $\mathfrak{o}$-integral model of the k-scheme T for $\mathfrak{B} \in I(\sigma)$. However, $\mathcal{X}_{\mathfrak{B}}$ is not an $\mathfrak{o}$-group scheme unless $\mathfrak{B} \in I_{pr}(\sigma)$, in which case $\mathcal{X}_{\mathfrak{B}} \cong \mathcal{T}_0$.

As it has been noted at the end of n° 2, equations (1) define a closed embedding $\varphi : T \hookrightarrow S$, where $S = R_{K/k}T_K, , T_K \cong G_{m,K}^d$. Let $\mathcal{S}$ be the Néron-Raynaud model of S (cf. [1, Ch. 10]). The connected component $\mathcal{S}^0$ of the identity element in $\mathcal{S}$ is isomorphic to $R_{\mathfrak{O}/\mathfrak{o}}G^d_{m,\mathfrak{O}}$ [12, Lemma 3.1]; therefore $\mathcal{T}_0$ coincides with the schematic closure of T in $\mathcal{S}$ and, moreover, $\mathcal{T}_0 \subseteq \mathcal{S}^0$ (in view of (3)). If the extension $K \mid k$ is tame, one may deduce from the results of Edixhoven, [2], that $\mathcal{T}_0$ is a smooth $\mathfrak{o}$-group scheme, and therefore in this case it coincides with the connected component of the identity element in the Néron-Raynaud model of the torus T, by Proposition 10.1.4 in [1]. In general, however, $\mathcal{T}_0$ may fail to be a smooth $\mathfrak{o}$-scheme. Let, for example, $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{d})$, $d = 3(\bmod 4)$, and let $T = R^1_{K/k}G_m$ be the norm-torus, defined as the kernel of the norm $N : R_{K/k}G_{m,K} \to G_{m,k}$. Then

$$\mathcal{T}_0 = \operatorname{Spec} \mathbb{Z}[x,y]/(x^2 - dy^2 - 1),$$

and the fibre of $\mathcal{T}_0$ at the prime 2 is non-reduced, because $f^2 = 0$ in $F_2[x,y]/(x^2 - dy^2 - 1)$ for $f = x + dy + 1$ (in this case the Néron-Raynaud model can be obtained from $\mathcal{T}_0$ by blowing up two closed points, [2, Example 4.3]).

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