On the Index of Elliptic Operators on a Cone

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We consider a special case of Mellin pseudodifferential operators on a cone $X^\wedge = X \times \overline{\mathbb{R}}_+/X \times 0$ where the base X is a smooth compact manifold of dimension n without boundary. The operators have the form

$$(Au)(t) = \frac{1}{2\pi i}\int_\Gamma dz \int_0^\infty \left(\frac{t_1}{t}\right)^z a(t,z)u(t_1)\frac{dt_1}{t_1}. \tag{0.1}$$

Here

$$u(t) \in C_0^\infty(\mathbb{R}_+, C^\infty(X,E)) \tag{0.2}$$

which means that $u(t)$ is a function with compact support on $\mathbb{R}_+$ whose values are sections of a vector bundle E over X. The weight line Γ may be any vertical line $\Gamma_\beta = \{\Re z = \beta\}$ in a complex plane, we assume without loss of generality that Γ coincides with the imaginary axis Γ_0.

The operator-valued Mellin symbol $a(t,z)$ satisfies the following conditions:

1.
$$a(t,z) \in C^\infty(\overline{\mathbb{R}}_+, L^\mu_{cl}(X,\Gamma)), \tag{0.3}$$

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which means that a is a smooth function in $t \in \overline{\mathbb{R}}_+$ whose values are parameter-dependent pseudodifferential operators of order μ on X with a parameter $z = i\tau,\ \tau \in \mathbb{R}$,

2. for $t \in [C, \infty)$ the symbol is independent of t, more precisely

$$a(t,z) = a(\infty, z) \quad t \in [C, \infty),$$

3. for $t \in [0, c]$ with $0 < c < C$ the symbol $a(t,z)$ admits an analytical continuation to some strip $S = \{|\Re z| < \varepsilon\}$ and on each line $\Gamma_\beta = \{\Re z = \beta\}$ is a parameter-dependent operator on X of oder μ, that is satisfies (0.3) uniformly in $|\beta| \le \varepsilon_1 < \varepsilon$.

We use a notation $ML^\mu(X^\wedge)$ for the set of such operators and $ML^\mu_0(X^\wedge)$ if $a(\infty, z) = 0$. Troughout this paper we assume $\mu \le 0$. The operators (0.1) are of great importance for the calculus of pseudodifferential operators on manifolds with conical singularities (see e.g. [1]). Here we restrict ourselves to a model case when a singular manifold is a pure cone and the operator may be written globally in a Mellin form. It is well known that $A \in ML^\mu(X^\wedge)$ is a bounded operator in weighted Sobolev spaces $A : H^{s,(n+1)/2} \to H^{s-\mu,(n+1)/2}$.

We will consider elliptic operators of zero order ($\mu = 0$). The operator $A \in ML^0(X^\wedge)$ is called elliptic if its symbol satisfies the following additional conditions:

1. for $t \in \mathbb{R}_+$ $a(t,z)$ is a parameter-dependent elliptic operator on X with a parameter $z \in \Gamma$,

2. $a(t,z)$ is invertible for for $t \in [0,c]$ and any z in the strip $|\Re z| < \varepsilon$,

3. $a(\infty, z) = 1$ where 1 stands for identity operator.

We prove (section 3) that ellipticity implies Fredholm property in Sobolev spaces and obtain an index formula. A basic observation is that ellipticity conditions imply that the elliptic family $a(t,z)$ paramatrized by $t \in \mathbb{R}_+, z \in \Gamma$ is trivial (that is $a(t,z)$ is invertible) outside a compact set in $\mathbb{R}_+ \times \Gamma$. Thus it defines an index bundle $\operatorname{ind} a \in K_c(\mathbb{R}_+ \times \Gamma)$ where K_c means K-functor with compact supports (see [4]). Its Chern character is represented by a closed differential form with compact support, and we prove the following result

$$\operatorname{ind} A = \int_0^\infty \int_\Gamma \operatorname{ch}(\operatorname{ind} a). \tag{0.4}$$

Another useful form of (0.4) may be obtained in terms of the family $a(t, z)$ and its parametrix $r_0(t, z)$ such that $1 - r_0 a$ and $1 - a r_0$ are trace class operators for any $(t, z) \in \mathbb{R}_+ \times \Gamma$ and $r_0 = a^{-1}$ outside a compact. Then (see e.g. [5]) using the formula for $\mathrm{ch}(\mathrm{ind}\, a)$ in terms of a and r_0 we may write

$$\mathrm{ind}\, A = \frac{1}{2\pi i} \int_{\mathbb{R}_+ \times \Gamma} \mathrm{tr}(dr_0 + r_0 da r_0) \wedge da \tag{0.5}$$

and precisely in this form we prove our result.

The proof follows the scheme developed in [6]. It consists in comparing three expressions:

1. analytical index

$$\mathrm{ind}\, A = \mathrm{tr}(1 - RA) - \mathrm{tr}(1 - AR) \tag{0.6}$$

   where R is a parametrix of A up to a trace class operator,

2. algebraical index

$$\mathrm{ind}\, A = \mathrm{tr}(1 - r \circ a) - \mathrm{tr}(1 - a \circ r) \tag{0.7}$$

   where r is a formal complete symbol of R and $\circ$ means a composition of formal complete symbols,

3. topological index given by (0.6).

The most important step is transition from 1 to 2 or, using the terminology of [6], the theorem on a regularized trace of a product. In contrast to [6] we have not only to watch the order but also to gain weight, which is much more difficult. The second transition from 2 to 3 is based on the machinery developed in [5] and requires no new ideas.

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1 Preliminary Estimates

We will use the following notations. For a function $a = a(t,z)$ we set

$$a^{(k)} = \frac{\partial^k}{\partial z^k} a(t,z),$$

$$a_{(k)} = D^k a(t,z) = \left(-t\frac{\partial}{\partial t}\right)^k a(t,z).$$

For two functions $a(t,z)$ and $b(t,z)$ set

$$a \circ b|_N = \sum_{k=0}^{N-1} \frac{1}{k!} a^{(k)}(t,z) b_{(k)}(t,z). \tag{1.1}$$

For a function $u(t)$, $t \in [0,\infty)$ with values in $C^\infty(X,E)$ we denote by $|u(t)|$ any norm in $C^\infty(X,E)$ at a given t. A similar notation $|a(t,z)|$ for operator-valued functions

$$a(t,z) : C^\infty(X,E) \to C^\infty(X,F)$$

means any operator norm.

Introduce

$$\langle z\rangle^s = (1-z^2)^{s/2} = \exp\left(\frac{s}{2}\ln(1-z^2)\right)$$

for $|\Re z| < \varepsilon < 1$, $s \in \mathbb{C}$, assuming that the branch of $\ln(1-z^2)$ in the strip $|\Re z| < \varepsilon$ is real at $z \in \Gamma_0$. This function is holomorphic in z belonging to the strip $|\Re z| < \varepsilon$ and in s for all $s \in \mathbb{C}$. If s belongs to a horizontal strip $|\Im s| < \delta$, then the inequality holds

$$|\langle z\rangle^s| \le C(1+|z|^2)^{\Re s/2} \tag{1.2}$$

We also need an operator-valued version of the above order reducing function. Let Δ be the Laplace operator on X. Define an order reducing family

$$R^s(z) = (1-\Delta-z^2)^{s/2} \tag{1.3}$$

for $|\Re z| < \varepsilon < 1$ and $|\Im s| < \delta$. A complex power is understood in the sense of elliptic theory (see [7]). This family is holomorphic in z and s belonging to the mentioned strips.

For $u(t) \in C_0^\infty(\mathbb{R}_+)$ its Mellin transform

$$\widehat{u}(z) = \int_0^\infty t^{z-1} u(t) dt$$

is an entire function in z. In virtue of the identity

$$z^k \widehat{u}(z) = \widehat{u}_{(k)}(z) = \int_0^\infty t^{z-1} D^k u(t) dt \tag{1.4}$$

we see that $\widehat{u}(z)$ is rapidly decreasing on any vertical line, that is

$$|\widehat{u}(z)| \leq C |\langle z \rangle^{-k}| \tag{1.5}$$

for any $k \in \mathbb{N}$, $|\Re z| < \varepsilon$.

Finally we will consider Mellin transforms of operator-valued symbols $a(t, z) \in ML_0(X^\wedge)$, that is

$$\widehat{a}(\zeta, z) = \int_0^\infty t^{\zeta-1} a(t, z) dt.$$

The integral converges at $\Re\zeta > 0$ so that $\widehat{a}(\zeta, t)$ is a holomorphic function in a half-plane $\Re\zeta > 0$. Similarly to (1.4) we have

$$\widehat{a}(\zeta, t) = \frac{1}{\zeta} \widehat{a}_{(1)}(\zeta, t),$$

where

$$\widehat{a}_{(1)} = \int_0^\infty t^{\zeta-1} Da(t, z) dt$$

is holomorphic in a half-plane $\Re\zeta > -1$ since $Da(t, z)$ vanishes at $t = 0$. Repeating this procedure we may write similarly to (1.4)

$$\widehat{a}(\zeta, z) = \frac{1}{\zeta^k} \widehat{a}_{(k)}(\zeta, z), \tag{1.6}$$

where $\widehat{a}_{(k)}(\zeta, z)$ is holomorphic in $\Re\zeta > -1$. From (1.6) it follows that $\widehat{a}(\zeta, z)$ is rapidly decreasing in ζ on any vertical line Γ_β, so that

$$|\widehat{a}(\zeta, z)| \leq C(z) |\langle \zeta \rangle|^{-k} \tag{1.7}$$

for any $k \in \mathbb{N}$, $\zeta \in \Gamma_\beta$.

For symbols $a(t,z) \in ML_0^\mu(X^\wedge)$ with $\mu \leq 0$ more precise estimates hold

$$||\hat{a}(\zeta, z)|| \leq C|\langle\zeta\rangle|^{-k}|\langle z\rangle|^{\mu}. \tag{1.8}$$

Here $||\cdot||$ means the norm of the operator

$$\hat{a}(\zeta, t) : L^2(X, E) \to L^2(X, E). \tag{1.9}$$

If $\mu < -n/2$ the operator (1.9) belongs to the Hilbert-Schmidt class and its Hilbert-Schmidt norm $||\cdot||_2$ satisfies an equality

$$||\hat{a}(\zeta, z)||_2 \leq C|\langle\zeta\rangle|^{-k}|\langle z\rangle|^{\mu+n/2}. \tag{1.10}$$

Finally, if $\mu < -n$ then the operator (1.9) belongs to the trace class and the trace norm $||\cdot||_1$ satisfies an estimate

$$||\hat{a}(\zeta, z)||_1 \leq C|\langle\zeta\rangle|^{-k}|\langle z\rangle|^{\mu+n}. \tag{1.11}$$

In (1.8)–(1.11) $\zeta \in \Gamma_{\beta_1}$, $\beta_1 \neq 0$, $z \in \Gamma_{\beta_2}$, $|\beta_2| < \varepsilon$ and k may be any integer positive number.

The operator (0.1) corresponding to the operator-valued Mellin symbol $a(t,z)$ will be denoted by $\mathrm{Op}(a)$. With these notations the main theorem of this section is as follows.

Theorem 1 *Let $a(t,z) \in ML_0^\mu(X^\wedge)$, $b(t,z) \in ML_0^\nu(X^\wedge)$ with μ, $\nu \leq 0$. Then for N sufficiently large the operator*

$$C_N = \mathrm{Op}(a)\,\mathrm{Op}(b) - \mathrm{Op}(a \circ b|_N)$$

as an operator in the space $H^{0,(n+1)/2}$ belongs to the trace class.

Proof. Introduce a partition of unity $\rho_0(t), \rho_1(t), \rho_2(t)$ such that

$$\operatorname{supp}\rho_0 \in [0,c],\ \operatorname{supp}\rho_1 \in [c/2, c],\ \operatorname{supp}\rho_2 \in [c/2, \infty),$$
$$\operatorname{supp}\rho_0 \cap \operatorname{supp}\rho_2 = \emptyset$$

and

$$\rho_0(t) \equiv 1,\ t \in [0, c/2],\ \rho_2(t) \equiv 1,\ t \in [c, \infty).$$

Then any operator $\mathrm{Op}(a)$ may be represented as a sum

$$\mathrm{Op}(a) = \sum_{k=0}^{2} \mathrm{Op}(\rho_k a).$$

We emphasize that the symbols $\rho_0(t)a(t,z)$ and $\rho_1(t)a(t,s)$ are holomorphic in z belonging to the strip S.

The operator C_N is then a sum

$$\sum_{i,j=0}^{2} \mathrm{Op}(\rho_i a)\mathrm{Op}(\rho_j b) - \mathrm{Op}((\rho_i a)\circ(\rho_j b)|_N) \tag{1.12}$$

and we consider several cases according to the values of i,j.

Case 1 ($i,j \neq 0$).

In this case the supports of symbols $\rho_i a$ and $\rho_j b$ are separated from $t=0$. The Mellin calculus for such operators on a half- line $t \geq 0$ may be reduced to the usual Fourier calculus of p.d.o. on the whole line $y \in (-\infty,\infty)$ by change of variables $t=e^y$, $z=i\tau$. Indeed, formula (0.1) goes to

$$(Au)(e^y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} e^{i\tau(y_1-y)} a(e^y,i\tau)u(e^{y_1})dy_1,$$

and the o-product of two symbols becomes

$$a(t,z)\circ b(t,z)|_N = \sum_{k=0}^{N-1} \frac{i^k}{k!}\frac{\partial^k}{\partial\tau^k}a(e^y,i\tau)\frac{\partial^k}{\partial y^k}b(e^y,i\tau)$$

which is the usual composition rule for Fourier p.d.o. The symbols

$$\rho_i(e^y)a(e^y,i\tau),\ \rho_j(e^y)b(e^y,i\tau)$$

have compact support in y, so the theorem follows from the usual Fourier calculus of p.d.o. (see e.g. [6]).

Case 2 ($i=2, j=0$).

In this case $(\rho_2 a)\circ(\rho_0 b)|_N = 0$ since the supports of ρ_0 and ρ_2 do not intersect, so we need to prove that the operator

$$\mathrm{Op}(\rho_2 a)\mathrm{Op}(\rho_0 b) = \mathrm{Op}(\rho_2 a)\rho_0\mathrm{Op}(b)$$

belongs to trace class (the so-colled pseudolocality property). The operator $\mathrm{Op}(b)$ is bounded in $H^{0,(n+1)/2}$ since its order $\nu \leq 0$, and thus it is sufficient to prove that $\mathrm{Op}(\rho_2 a)\rho_0$ belongs to trace class. To this end we represent it as a composition

$$H^{0,(n+1)/2} \xrightarrow{C} H^{s,\gamma} \hookrightarrow H^{0,(n+1)/2} \tag{1.13}$$

with some integer $s > (n+1)/2$ and some $\gamma > (n+1)/2$ and show that both operators in (1.13) are Hilbert-Schmidt ones. The second operator is an embedding.

Lemma 2 *For $s > (n+1)/2$, $\delta > 0$ the embedding*

$$H^{s,(n+1)/2+\delta} \hookrightarrow H^{0,(n+1)/2}$$

considered on a subspace of functions $u \in H^{s,(n+1)/2+\delta}$ whose supports belong to the interval $t \in [0,T]$ is a Hilbert-Schmidt operator.

Proof. Let first $u(t) \in C_0^\infty(0,T)$. Then $\widehat{u}(z)T^{-z}$ is an entire function rapidly decreasing in the right half-plane. The norm of $u(t)$ in the Sobolev space $H^{s,(n+1)/2+\delta}$ is equal to an L^2-norm of $R^s(z)\widehat{u}(z)$ on the line $\Gamma_{-\delta}$. Here $R^s(z)$ is an order-reducing family (1.3). The norm of $u(t)$ in $H^{0,(n+1)/2}$ is an L_2-norm of the restriction of $\widehat{u}(z)$ to the line Γ_0. The restrictions of $\widehat{u}(z)$ to $\Gamma_{-\delta}$ and Γ_0 are connected by the Cauchy integral

$$\widehat{u}(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_{-\delta}} \frac{T^{\zeta-z}}{z-\zeta}\widehat{u}(z)dz \tag{1.14}$$

where $\zeta \in \Gamma_0$. Denoting $\widehat{v}(z) = R^s(z)\widehat{u}(z)$ we write (1.14) in the form

$$\widehat{u}(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_{-\delta}} \frac{T^{\zeta-z}}{z-\zeta}R^{-s}(z)\widehat{v}(z)dz.$$

This operator acts between L^2 spaces on lines $\Gamma_{-\delta}$ and Γ_0 and its Hilbert-Schmidt norm is equal to the L^2-norm of its Schwarz kernel

$$K(\zeta,z) = \frac{T^{\zeta-z}}{z-\zeta}R^{-s}(z)$$

which is an operator-valued function. For $s > n/2$ we have an estimate for the Hilbert-Schmidt norm in $L^2(X)$

$$||R^{-s}(z)||_2^2 \leq |\langle z\rangle|^{-2s+n}$$

(cf. (1.10)), so that

$$||K(\zeta,z)||_2^2 \leq \frac{T^{2\delta}}{\delta^2 + (\Im\zeta - \Im z)^2}|\langle z\rangle|^{-2s+n}.$$

Integrating over $\Im\zeta$ and $\Im z$ and using that $2s - n > 1$, we obtain that the L^2-norm of $K(\zeta,z)$ is finite, whence the lemma follows.

□

To prove that C in (1.13) is a Hilbert-Schmidt operator we use that $\rho_0\rho_2$ vanishes on the diagonal $t = t_1$. So, writing

$$\left(\frac{t_1}{t}\right)^z = (\ln t_1 - \ln t)^{-N} \frac{\partial^N}{\partial z^N} \left(\frac{t_1}{t}\right)^z$$

for any $N \in \mathbb{N}$ and integrating by parts we represent C as an integral operator

$$(Cu)(t) = \frac{1}{2\pi i} \int_\Gamma dz \int_0^\infty \left(\frac{t_1}{t}\right)^z \frac{\rho_2(t)\rho_0(t_1)a^{(N)}(t,z)}{(\ln t - \ln t_1)^N t_1^{1/2}} v(t_1) dt_1$$

where

$$v(t_1) = t_1^{-1/2} u(t_1) \in L^2(\mathbb{R}_+, L^2(X))$$

if $u \in H^{0,(n+1)/2}$. Inclusion $Cu \in H^{s,\gamma}$ for s integer means that for any $k = 0, 1, \ldots, s$ and for any differential operator P of order $s - k$ on X

$$t^{n/2-\gamma} D^k PCu \in L^2(\mathbb{R}_+, L^2(X)).$$

This operator acting on $v = t^{-1/2}u$ has a Schwarz kernel

$$K(t, t_1) = \int_\Gamma t^{n/2-\gamma} D_t^k \frac{\rho_2(t)\rho_0(t_1)Pa^{(N)}(t,z)}{(\ln t - \ln t_1)^N t_1^{1/2}} \left(\frac{t_1}{t}\right)^z dz \tag{1.15}$$

and we need to estimate the L^2-norm of this kernel, more precisely, we need to prove that

$$\int_0^\infty dt \int_0^\infty dt_1 \|K(t, t_1)\|_2^2 \leq \infty$$

(in fact the integration is over $t \in [c/2, C]$ and over $t_1 \in [0, c]$). We may omit the factor $t^{n/2-\gamma}$ since it is bounded on $[c/2, C]$. Next, since

$$D_t^k \left(\frac{t_1}{t}\right)^z = z^k \left(\frac{t_1}{t}\right)^z$$

and

$$\left|\left(\frac{t_1}{t}\right)^z\right| \leq 1$$

for $z = i\tau$ the Hilbert-Schmidt norm of the integrand may be esimated as

$$C_1 \frac{|z|^k \|Pa^{(N)}(t,z)\|_2}{(1 + |\ln t - \ln t_1|)^N t^{1/2}} \leq \frac{C_2 |\langle z \rangle|^{\mu - N + s + n/2}}{(1 + |\ln t - \ln t_1|)^N t_1^{1/2}}. \tag{1.16}$$

Here we have made use of the fact that $t_1 \neq t$, so that $|\ln t_1 - \ln t| \neq 0$. Moreover,

$$||D_t^k \rho_2 P a^{(N)}(t,z)||_2 \leq C|\langle z\rangle|^{\mu-N+n/2}.$$

For N large enough the integral of (1.16) over z converges and we obtain

$$||K(t,t_1)||_2 \leq \frac{C_3}{(1+|\ln t_1 - \ln t|)^N t_1^{1/2}}.$$

Thus, for L^2-norm of $||K(t,t_1)||$ we obtain an estimate

$$C_4 \int_{c/2}^{C} dt \int_0^c \frac{dt_1}{(1+|\ln t_1 - \ln t|)^N t^1} =$$
$$= C_4 \int_{\ln c/2}^{\ln C} e^y dy \int_{-\infty}^{\ln c} \frac{dy_1}{(1+|y-y_1|)^N} < \infty$$

for $N > 1$. This proves case 2.

Case 3 $(i,j = 0,1)$.

This is the most difficult case. Here we will make use that $\rho_i(t)a(t,z)$ and $\rho_j(t)b(t,z)$ are holomorphic functions in the strip S. To simplify notations, we omit the factors $\rho_i(t), \rho_j(t)$ including them into a and b.

For $u \in C_0^\infty(\mathbb{R}_+)$ we have

$$(Bu)(t) = \operatorname{Op}(b)u = \frac{1}{2\pi i}\int_{\Gamma_1} t^{-z} b(t,z)\hat{u}(z)dz$$

where Γ_1 may be any vertical line with $|\Re\Gamma_1| < \varepsilon$. Then for $\Re\zeta > \Re\Gamma_1$ the Mellin transform of Bu has the form

$$\widehat{Bu}(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_1} \hat{b}(\zeta - z, z)\hat{u}(z)dz.$$

Similarly for $\widehat{ABu}(w)$ we obtain

$$\widehat{ABu}(w) = (2\pi i)^{-2}\int_{\Gamma_2} d\zeta \int_{\Gamma_1} dz \hat{a}(w-\zeta,\zeta)\hat{b}(\zeta - z, z)\hat{u}(z) \qquad (1.17)$$

with $\Re w > \Re\Gamma_2 > \Re\Gamma_1$. The integral converges because of estimates (1.5), (1.8).

Now using the Taylor formula, we write

$$\widehat{a}(w-\zeta,\zeta)=\sum_{k=0}^{N-1}\frac{1}{k!}\widehat{a}^{(k)}(w-\zeta,z)(\zeta-z)^k+T_N\widehat{a}(\zeta-z)^N \tag{1.18}$$

where

$$(T_N\widehat{a})(w-\zeta,z,\zeta)=\int_0^1\widehat{a}^{(N)}(w-\zeta,z+\theta(\zeta-z))\frac{(1-\theta)^{N-1}}{(N-1)!}d\theta. \tag{1.19}$$

The regular terms in (1.18) after substitution into (1.17) and integration over ζ give

$$\frac{1}{2\pi i}\int_{\Gamma_2}\widehat{a}^{(k)}(w-\zeta,z)(\zeta-z)^k\widehat{b}(\zeta-z,z)d\zeta=$$
$$=\frac{1}{2\pi i}\int_{\Gamma_2}\widehat{a}^{(k)}(w-\zeta,z)\widehat{b}_{(k)}(\zeta-z,z)d\zeta=$$
$$=M_{t\to w-z}(a^{(k)}(t,z)b_{(k)}(t,z))$$

resulting in $\mathrm{Op}(a\circ b|_N)u$. So, the operator C_N corresponds to the remainder term in (1.18)

$$\widehat{C_N u}(w)=(2\pi i)^{-2}\int_{\Gamma_2}d\zeta\int_{\Gamma_1}dzT_N\widehat{a}(w-\zeta,\zeta,z)\widehat{b}_{(N)}(\zeta-z,z)\widehat{u}(z). \tag{1.20}$$

Observe that the function $\widehat{b}_{(N)}(\zeta-z,z)$ at $N\geq 1$ is holomorphic in $\Re\zeta>-1$, so the integration line Γ_2 may be shifted arbitrary within the strip $|\Re\zeta|<\varepsilon$. It means that the restriction $\Re\Gamma_1<\Re\Gamma_2<\Re w$ is needed no more, and the only remaining restriction is $\Re\Gamma_2<\Re w$. It will be convenient to take

$$\Re\Gamma_2<\Re w\leq\Re\Gamma_1. \tag{1.21}$$

To prove that (1.20) belongs to the trace class we again represent it as a composition of two Hilbert-Schmidt operators

$$H^{0,(n+1)/2}\overset{C_N}{\to}H^{s,(n+1)/2+\delta}\hookrightarrow H^{0,(n+1)/2}$$

with some $\delta>0$ and some $s>(n+1)/2$ which may be taken as an even integer number. The second operator in this sequence is an embedding. Note that in virtue of lemma 2 the weight gain δ is necessary to have a Hilbert-Schmidt embedding. So, we choose $\Re\Gamma_1=0$, $\Re w=-\delta$, $\Re\Gamma_2<\Re w$ in (1.21).

Lemma 3 *Let $s > (n+1)/2$ be an even integer. Then the operator*

$$C_N : H^{0,(n+1)/2} \to H^{s,(n+1)/2+\delta}$$

is a Hilbert-Schmidt operator provided N is sufficiently large.

Proof. The assumption that s is an even integer serves only to simplify the proof. Using representation (1.20) for C_N we are to estimate the Hilbert-Schmidt norm of the operator between L^2-spaces on the lines Γ_0 and $\Gamma_{-\delta}$ with the Schwarz kernel

$$K(w,z) = \int_{\Gamma_2} R^s(w) T_N \hat{a}(w-\zeta,\zeta,z)\hat{b}_{(N)}(\zeta - z, z)d\zeta.$$

The Hilbert-Schmidt norm of the integrand may be estimated as

$$\begin{aligned} &||R^s(w)T_N\hat{a}(w-\zeta,\zeta,z)||_2\,||\hat{b}_{(N)}(\zeta-z,z)|| \leq \\ &\leq \frac{C_p}{|\langle \zeta - z\rangle|^p}||R^s(w)T_N\hat{a}(w-\zeta,\zeta,z)||_2. \end{aligned}$$

Here we have used (1.8) for $\hat{b}_{(N)}$ with p arbitrary large. In virtue of (1.19) our next step is to estimate the norm

$$||R^s(w)\hat{a}^{(N)}(w-\zeta, z+\theta(\zeta - z))||_2. \tag{1.22}$$

Denoting $z+\theta(\zeta-z)$ by η and putting $s = 2\sigma$, $\sigma \in \mathbb{N}$, we may rewrite $R^s(w)$ using the binomial formula

$$R^s(w) = (1-\Delta-\eta^2+\eta^2-w^2)^\sigma = \sum_{k+l+m=\sigma} C_{klm}R^{2k}(\eta)\eta^{2l}w^{2m}$$

with some constant coefficients C_{klm}. Finally, using (1.10) for the operator $R^{2k}(\eta)a^{(N)}(w-\zeta,\eta)$ of order $\mu - N + 2k$, we obtain

$$||R^{2k}(\eta)a^{(N)}(w-\zeta,\eta)||_2 \leq \frac{C_q}{|\langle w-\zeta\rangle|^q}|\langle\eta\rangle|^{\mu-N+2k+\frac{n}{2}},$$

provided $\mu - N + 2k + \frac{n}{2} < 0$, q may be any positive number. It implies the following rough estimate of (1.22)

$$||R^s(w)a^{(N)}(w-\zeta,\eta)||_2 \leq \frac{C}{|\langle w-q\rangle|^q}|\langle\eta\rangle|^{\mu_1}|\langle w\rangle|^s$$

where $\mu_1 \leq \mu - N + s + n/2$ is supposed to be negative.

Next, writing w as a sum

$$w = (w - \zeta) + (\zeta - z) + z$$

and applying binomial formula, we get

$$|\langle w \rangle|^s \leq C |\langle w - \zeta \rangle|^s |\langle \zeta - z \rangle|^s |\langle z \rangle|^s.$$

These rough estimates result in the following estimate for the kernel $K(w, z)$

$$||K(w,z)||_2 \leq C \int_0^1 d\theta \int_{\Gamma_2} \frac{|\langle z \rangle|^s |d\zeta|}{|\langle w - \zeta \rangle|^{q_1} |\langle \zeta - z \rangle|^{p_1} |\langle z + \theta(\zeta - z) \rangle|^{\mu_2}} \quad (1.23)$$

where q_1, p_1 are arbitrary positive numbers and μ_2 is negative and may be made larger in magnitude at the expense of N. The needed Hilbert-Schmidt norm is

$$\int_{\Gamma_0} \int_{\Gamma_{-\delta}} ||K(w,z)||_2^2 |dz||dw|. \quad (1.24)$$

After integration over $w \in \Gamma_{-\delta}$ we get the following integral which estimates (1.24)

$$C \int_0^1 d\theta \int_{\Gamma_0} |dz| \int_{\Gamma_2} |d\zeta| \frac{|\langle z \rangle|^{2s}}{|\langle \zeta - z \rangle|^{2p_1} |\langle z + \theta(\zeta - z) \rangle|^{2\mu_2}}.$$

First consider the region $|\zeta - z| < |z|/2$. Then

$$\frac{1}{|\langle \zeta - z \rangle|^{2p_1}} \leq 1$$

$$\frac{1}{|\langle z + \theta(\zeta - z) \rangle|^{2\mu_2}} \leq \frac{C}{|\langle z \rangle|^{2\mu_2}}$$

and

$$\int_{\Gamma_0} |dz| \int_{|\zeta - z| < \frac{|z|}{2}} \frac{|d\zeta|}{|\langle z \rangle|^{2\mu_2 - 2s}} \leq \int_{\Gamma_0} \frac{|z|}{|\langle z \rangle|^{2\mu_2 - 2s}} |dz|$$

which is convergent for μ_2 large enough. Now, for $|\zeta - z| \geq |z|/2$ we estimate

$$\frac{1}{|\langle z + \theta(\zeta - z) \rangle|^{2\mu_2}} \leq 1$$

and

$$\int_{\Gamma_0} |dz| \int_{|z|/2}^{\infty} |\langle z\rangle|^{2s} \frac{d|\eta|}{|\eta|^{2p_1}} = \frac{1}{2p_1+1}\int_{\Gamma_0} \frac{|\langle z\rangle|^{2s}}{|z|^{2p_1+1}} \leq \infty.$$

So, the Hilbert-Schmidt norm of C_N is finite.

□

To prove the theorem, it remains to observe that the functions of the type $C_N u$ have support in a finite interval $t \in [0,T]$ since $a \circ b|_N$ vanish at t large enough independent of z. Thus, lemma 2 may be applied implying that C_N belongs to the trace class.

Case 4 $(i = 0, j = 2)$.

Here we have a pseudolocallity property similar to case 2, but the proof runs slightly different. Again we have that $(\rho_0 a) \circ (\rho_2 b) = 0$, so we need to prove that $\mathrm{Op}(\rho_0 a)\rho_2 \mathrm{Op}(b)$ belongs to the trace class. Since $b(t,z)$ has compact support in t, we may assume that $\rho_2(t)$ is also compactly supported. Then since $\mathrm{Op}(b)$ is bounded in $H^{0,(n+1)1/2}$, we again need to prove that $\mathrm{Op}(\rho_0 a)\rho_2$ belongs to the trace class. But multiplication operator $\rho_2(t)$ may be regarded as Mellin p.d.o. with a holomorphic symbol, so we are in assumptions of case 3. This completes the proof of the theorem.

□

2 A Regularized Trace of a Product

For operators $A = \mathrm{Op}(a) \in ML_0^{\mu}(X^\wedge)$ and $B = \mathrm{Op}(b) \in ML_0^{\nu}(X^\wedge)$ with $\mu, \nu \leq 0$ define a regularized trace of a product by

$$\mathrm{Tr}_N AB = \mathrm{Tr}(AB - \mathrm{Op}(a \circ b|_N)).$$

By theorem 1 this trace exists provided N is sufficiently large.

Theorem 4 *The regularized trace of a product does not depend on the order, that is*

$$\mathrm{Tr}_N AB = \mathrm{Tr}_N BA \tag{2.1}$$

Proof. There are several cases corresponding to those listed in the proof of theorem 1.

Case 1 $(i, j \neq 0)$.

The assertion reduces to the theorem on a regularized trace of a product of Fourier p.d.o. [6].

Case 2 ($i = 2, j = 0$) or case 4 ($i = 0, j = 2$).

For $A = \mathrm{Op}(\rho_i a), B = \mathrm{Op}(\rho_j b)$ we have by theorem 1 that AB and BA belong to the trace class. Then their traces are equal by Lidskij's theorem.

Case 3 ($i, j = 0, 1$).

Using (1.20) with

$$\Re\Gamma_2 < \Re w = \Re\Gamma_1 = 0$$

(cf. (1.21)) we get

$$\mathrm{Tr}_N AB = \mathrm{Tr} C_N = (2\pi i)^{-2} \int_{\Gamma_2} d\zeta \int_{\Gamma_1} dz \ \mathrm{tr} T_N \widehat{a}(z-\zeta, \zeta, z) \widehat{b}_{(N)}(\zeta - z, z) \tag{2.2}$$

Recall that Tr denotes the trace of operators on the cone $X^\wedge$ while tr is the trace of operators on the base X. The function

$$T_N \widehat{a}(z-\zeta, \zeta, z) = \frac{1}{z-\zeta} T_N \widehat{a}_{(1)}(z-\zeta, \zeta, z)$$

has a pole of the first order at $\zeta = z$. So, we may shift the lines Γ_1, Γ_2 within the strips $|\Re z| < \varepsilon, |\Re\zeta| < \varepsilon$, provided $|\Re\Gamma_2|$ remains less then $|\Re\Gamma_1|$. Moreover, we may shift Γ_2 crossing Γ_1, but then we must take into account the residue at $\zeta = z$. It is equal to

$$-\frac{1}{2\pi i} \int_{\Gamma_1} \mathrm{tr}\, T_N \widehat{a}_{(1)}(0, z, z) \widehat{b}_{(N)}(0, z) dz.$$

By (1.19) we have

$$T_N \widehat{a}_{(1)}(0, z, z) = \frac{1}{N!} \widehat{a}_{(1)}^{(N)}(0, z),$$

so that the residue is equal to

$$-\frac{1}{2\pi i N!} \int_{\Gamma_1} \mathrm{tr}\, \widehat{a}_{(1)}^{(N)}(0, z) \widehat{b}_{(N)}(0, z) dz.$$

But for $N > 1$

$$\widehat{b}_{(N)}(0, z) = \int_0^\infty -\frac{\partial}{\partial t} D^{N-1} b(t, z) dt = -D^{N-1} b(t, z)|_0^\infty = 0.$$

Thus, for $N > 1$ integral (2.2) *does not depend on a position of the lines* Γ_1, Γ_2 within the strip $|\Re z| < \varepsilon, |\Re\zeta| < \varepsilon$. A similar assertion is true for the integral

$$\mathrm{Tr}_N BA = (2\pi i)^{-2} \int_{\Gamma_2} d\zeta \int_{\Gamma_1} dz \;\, \mathrm{tr}\, T_N \widehat{b}_{(1)}(z-\zeta, \zeta, z)\widehat{a}_{(N)}(\zeta - z, z). \quad (2.3)$$

To prove that (2.2) and (2.3) are equal, we consider families

$$a_s(t,z) = a(t,z)R^{-s}(z)$$

$$b_s(t,z) = b(t,z)R^{-s}(z)$$

where s is a complex parameter ranging in the half-strip $\Re s \geq 0, \;\; |\Im s| \leq \varepsilon$.

Let A_s, B_s be the corresponding operators on the cone. The estimates of theorem 1 show that $\mathrm{Tr}_N A_s B_s$ and $\mathrm{Tr}_N B_s A_s$ are holomorphic functions for s belonging to the half-strip. Thus, it is sufficient to prove the equality

$$\mathrm{Tr}_N A_s B_s = \; \mathrm{Tr}_N B_s A_s$$

for $\Re s$ sufficiently large. To put it differently, we need to verify (2.1) for operators A, B of sufficiently large negative orders. To this end we write

$$\frac{T_N \widehat{a}_{(1)}(z-\zeta,\zeta,z)}{(z-\zeta)} = -\frac{\widehat{a}_{(1)}(z-\zeta,\zeta) - \sum_{k=0}^{N-1} \widehat{a}_{(1)}^{(k)}(z-\zeta,z)\frac{(\zeta-z)^k}{k!}}{(\zeta-z)^{N+1}}$$

and then

$$\mathrm{Tr}_N\, AB = -(2\pi i)^{-2} \int_{\Gamma_2} d\zeta \int_{\Gamma_1} dz \frac{\mathrm{tr}\, \widehat{a}_{(1)}(z-\zeta,\zeta)\widehat{b}_{(1)}(\zeta-z,z)}{(\zeta-z)^2} +$$
$$+ \sum_{k=0}^{N-1} (2\pi i)^{-2} \int_{\Gamma_2} d\zeta \int_{\Gamma_1} dz \frac{\mathrm{tr}\, \widehat{a}_{(1)}^{(k)}(z-\zeta,z)(\zeta-z)^k \widehat{b}_{(1)}(\zeta-z,z)}{(\zeta-z)^2 k!}. \quad (2.4)$$

Each summand in (2.4) makes sense if a and b have large negative orders and $\Re\Gamma_2 < \Re\Gamma_1$ are fixed. Interchanging z and ζ in the first integral and using that $\mathrm{tr}\, \widehat{a}_{(1)}\widehat{b}_{(1)} = \; \mathrm{tr}\, \widehat{b}_{(1)}\widehat{a}_{(1)}$ for trace class operators, we obtain

$$-\,(2\pi i)^{-2} \int_{\Gamma_1} d\zeta \int_{\Gamma_2} dz \frac{\mathrm{tr}\, \widehat{b}_{(1)}(z-\zeta,\zeta)\widehat{a}_{(1)}(\zeta-z,z)}{(\zeta-z)^2} \quad (2.5)$$

Now, the remaining summands may be transformed as follows

$$\int_{\Gamma_{-\delta}} d\eta \int_{\Gamma_1} dz \frac{\operatorname{tr} \hat{a}_{(1)}^{(k)}(-\eta, z)\eta^k \hat{b}_{(1)}(\eta, z)}{\eta^2} =$$
$$= (-1)^k \int_{\Gamma_\delta} d\eta \int_{\Gamma_1} dz \frac{\operatorname{tr} \hat{a}_{(1)}^{(k)}(\eta, z)\eta^k \hat{b}_{(1)}(-\eta, z)}{\eta^2}.$$

Here we have changed η by $-\eta$. Now, integrating by parts with respect to z and permuting $\hat{a}_{(1)}$ and $\hat{b}_{(1)}$ under the trace sign, we obtain

$$\int_{\Gamma_\delta} d\eta \int_{\Gamma_1} dz \frac{\operatorname{tr} \hat{b}_{(1)}^{(k)}(-\eta, z)\eta^k \hat{a}_{(1)}(\eta, z)}{\eta^2} =$$
$$= \int_{\Gamma_1} d\zeta \int_{\Gamma_2} dz \frac{\operatorname{tr} \hat{b}_{(1)}^{(k)}(z-\zeta, z)(\zeta - z)^k \hat{a}_{(1)}(\zeta - z, z)}{(\zeta - z)^2}. \tag{2.6}$$

Here we have shifted the lines of integrations keeping $\Re\Gamma_2 < \Re\Gamma_1$. Now, summing (2.5) and (2.6) for $k = 0, 1, \ldots, N-1$, we get

$$\operatorname{Tr}_N AB = (2\pi i)^{-2} \int_{\Gamma_1} d\zeta \int_{\Gamma_2} dz \frac{\operatorname{tr} T_N \hat{b}_{(1)}(z-\zeta, \zeta, z)\hat{a}_{(N)}(\zeta - z, z)}{\zeta - z}.$$

This expression coincides with the corresponding expression (2.2) for $\operatorname{Tr}_N BA$ except that the lines Γ_1 and Γ_2 are interchanged. But, as we have seen, we may interchange Γ_1 and Γ_2 not affecting the value of the integral.

This completes the proof.

□

3 An Algebraical Index

First we introduce an algebra of formal Mellin symbols on $\mathbb{R}_+$, define elliptic symbols and introduce an algebraical index of elliptic elements. Then constructing a parametrix and applying the theorem on a regularized trace of a product, we prove that the analytical and algebraical index coincide.

A formal symbol is a formal power series

$$a(t, z) = \sum_{k=0}^{\infty} h^k a_k(t, z)$$

where coefficients $a_k(t,z) \in C^\infty(\mathbb{R}_+, L^{\mu-k}(X,\Gamma_0))$, $\mu \leq 0$ satisfy the following conditions:

1. $$\frac{\partial^m}{\partial z^m} a_k(t,z) \in C^\infty(\mathbb{R}_+, L^{\mu-k-m}(X,\Gamma_0)),$$

2. for $0 < t < c$ or $t > C$ $a_0(t,z)$ does not depend on t while $a_k(t,z) = 0$ for $k > 0$.

The powers of a formal parameter h serve for ordering the series terms. Define a product $\circ$ of two symbols by

$$a \circ b = \sum_{k,l,m=0}^{\infty} h^{k+l+m} \frac{1}{m!} \frac{\partial^m}{\partial z^m} a_k(t,z) D^m b_l(t,z).$$

It is easy to check that the symbols form an associative algebra with a unit $a(t,z) \equiv 1$ consisting of the leading term only. We denote this algebra by $\mathcal{A}$. Introduce a *trace ideal* $\mathcal{J}$ consisting of symbols with $\mu < -(n+1)$ and with all the functions $a_k(t,z)$ vanishing at $t \in [0,c]$ and $t \geq C$. A *trace* for $a \in \mathcal{J}$ is defined by

$$\operatorname{Tr} a = \sum_{k=0}^{\infty} h^{k-1} \frac{1}{2\pi i} \int_{\Gamma_0} dz \int_0^\infty \operatorname{tr} a_k(t,z) \frac{dt}{t}.$$

This is a formal series with constant coefficients and the exponents of h ranging from -1 to $+\infty$. Using integrations by parts, one can check that

$$\operatorname{Tr} a \circ b = \operatorname{Tr} b \circ a$$

if one of the symbols belongs to $\mathcal{J}$.

A symbol $a \in \mathcal{A}$ is called elliptic if there exists a symbol r such that $1 - r \circ a$ and $1 - a \circ r$ belong to $\mathcal{J}$. In particular for leading terms a_0 and r_0 we obtain

$$1 - r_0 a_0 \in \mathcal{J}, \quad 1 - a_0 r_0 \in \mathcal{J}. \tag{3.1}$$

Such symbol r is called a (formal) *parametrix of a*. The following construction is well-known.

Lemma 5 *Let there exist a function $r_0(t,z)$ satisfying* (3.1). *Then for N large enough the symbol*

$$r = r_0 \circ \sum_{k=0}^{N}(1 - a \circ r_0)^k = \sum_{k=0}^{N}(1 - r_0 \circ a)^k \circ r_0 \tag{3.2}$$

(the powers are understood with respect to the product $\circ$) is a parametrix of a.

Proof. By direct calculation we have

$$1 - r \circ a = (1 - r_0 \circ a)^{\circ(N+1)}, \tag{3.3}$$

$$1 - a \circ r = (1 - a \circ r_0)^{\circ(N+1)} \tag{3.4}$$

where exponent $\circ(N+1)$ means the $(N+1)$-th power with respect to $\circ$-product. Clearly, these symols belong to $\mathcal{J}$.

□

We define the algebraical index of an elliptic symbol a by

$$\operatorname{ind} a = \operatorname{Tr}(1 - r \circ a) - \operatorname{Tr}(1 - a \circ r). \tag{3.5}$$

By definition it is a formal series in h with constant coefficients. It turns out, however, that all the coefficients vanish except a constant term, so we can treat it as a number. Moreover, the index does not depend on the choice of a parametrix. All these properties are standard consequences of the *stability* of the index.

Lemma 6 *Let $a(\lambda)$ be a family of elliptic symbols, $r(\lambda)$ a family of parametrices. Then*

$$\operatorname{Tr}(1 - r(\lambda) \circ a(\lambda)) - \operatorname{Tr}(1 - a(\lambda) \circ r(\lambda))$$

is independent of λ.

Proof. We have

$$\begin{aligned}(1 - r \circ a)' &= (1 - r \circ a)' \circ (1 - r \circ a) + (1 - r \circ a)' \circ r \circ a = \\ &= \{(1 - r \circ a) \circ r \circ a\}' - (1 - r \circ a) \circ (r \circ a)' - (r \circ a)' \circ (1 - r \circ a)\end{aligned}$$

where prime means derivation by λ. Thus,

$$\operatorname{Tr}(1 - r \circ a)' = \frac{d}{d\lambda} \operatorname{Tr}(1 - r \circ a) \circ r \circ a - 2 \operatorname{Tr}(1 - r \circ a) \circ (r' \circ a + r \circ a').$$

Similarly,

$$\operatorname{Tr}(1 - a \circ r)' = \frac{d}{d\lambda} \operatorname{Tr}(1 - a \circ r) \circ a \circ r - 2 \operatorname{Tr}(1 - a \circ r) \circ (a' \circ r + a \circ r').$$

But

$$\begin{aligned} \operatorname{Tr}(1 - a \circ r) \circ a \circ r &= \operatorname{Tr} r \circ (1 - a \circ r) \circ a = \operatorname{Tr}(1 - r \circ a) \circ r \circ a, \\ \operatorname{Tr}(1 - a \circ r) \circ a' \circ r &= \operatorname{Tr} r \circ (1 - a \circ r) \circ a' = \operatorname{Tr}(1 - r \circ a) \circ r \circ a', \\ \operatorname{Tr}(1 - a \circ r) \circ a \circ r' &= \operatorname{Tr} a \circ (1 - r \circ a) \circ r' = \operatorname{Tr}(1 - r \circ a) \circ r' \circ a. \end{aligned}$$

So, both expressions $\operatorname{Tr}(1 - r \circ a)'$, $\operatorname{Tr}(1 - a \circ r)'$ coincide.

□

In particular, given two parametrices r_1 and r_2 of the same elliptic symbol a, we consider a linear homotopy $r(\lambda) = (1 - \lambda) r_1 + \lambda r_2$ which gives a family of parametrices. Then, lemma 6 implies that index does not depend on the choice of a parametrix.

Now, for a real $\lambda > 0$ define a homomorphism $H_\lambda : \mathcal{A} \to \mathcal{A}$ by

$$H_\lambda a = \sum_{k=0}^{\infty} \lambda^k h^k a_k(t, \lambda z).$$

It is straightforward to check that H_λ is in fact a homomorphism:

$$H_\lambda(a \circ b) = (H_\lambda a) \circ (H_\lambda b).$$

If $a \in \mathcal{J}$, then

$$\operatorname{Tr} H_\lambda a = H_\lambda \operatorname{Tr} a \tag{3.6}$$

where H_λ acts on formal series with constant coefficients replacing h by λh. Equality (3.6) follows by change of variables in the integral

$$\operatorname{Tr} H_\lambda a = \sum_{k=0}^{\infty} \lambda^k h^{k-1} \frac{1}{2\pi i} \int_{\Gamma_0} dz \int_0^\infty \operatorname{tr} a_k(t, \lambda z) \frac{dt}{t} =$$
$$= \sum_{k=0}^{\infty} \lambda^{k-1} h^{k-1} \frac{1}{2\pi i} \int_{\Gamma_0} dz \int_0^\infty \operatorname{tr} a_k(t, z) \frac{dt}{t}.$$

Lemma 7 *The formal series* ind *consists of the constant term only.*

Proof. For $\lambda > 0$ consider a family $a(\lambda) = H_\lambda a$ of elliptic symbols. Then $r(\lambda) = H_\lambda r$ is a family of parametrices since

$$1 - H_\lambda r \circ H_\lambda a = H_\lambda(1 - r \circ a) \in \mathcal{J}.$$

So,

$$\operatorname{ind} a(\lambda) = \operatorname{Tr} H_\lambda(1 - r \circ a) - \operatorname{Tr} H_\lambda(1 - a \circ r) = H_\lambda \operatorname{ind} a.$$

On the other hand, $\operatorname{ind} a(\lambda)$ does not depend on λ by lemma 6.

□

We are going to compare analytical and algebraical indices. Given an elliptic operator

$$A = \operatorname{Op}(a(t,z)),$$

we may treat its symbol $a(t,z)$ as a formal one consisting of the leading term only. The ellipticity conditions listed in the introduction imply that there exists an $r_0(t,z)$ such that $1 - r_0 a$ and $1 - a r_0$ belong to $\mathcal{J}$. Indeed, for $0 < t < c$ and $t > C$ a^{-1} exists by definition. For $t \in [c, C]$ $a(t,z)$ is parameter-dependent elliptic where $z \in \Gamma_0$ is considered as a parameter. In particular it implies that $a(t,z)$ is also invertible at $|z| > M$ for sufficiently large M. As for $t \in [c, C]$, $z \in [-M, M]$ there exists a parametrix $b(t,z)$ since $a(t,z)$ is elliptic at any t, z. Now, using a cut-off function $\varphi(t,z)$ which is equal to 1 in a rectangle $t \in [c, C]$, $|z| \leq M$ and vanishes outside a compact in $\mathbb{R}_+ \times \Gamma_0$ define

$$r_0(t,z) = (1 - \varphi)a^{-1}(t,z) + \varphi b(t,z).$$

This function may serve as a leading term of a formal parametrix given by lemma 5. Thus, the algebraical index is defined. To compute the analytical index of A, we need an operator parametrix R inverting A up to the trace class operators. Then the analytical index of A is given by the formula

$$\operatorname{ind} A = \operatorname{Tr}(1 - RA) - \operatorname{Tr}(1 - AR). \tag{3.7}$$

Introduce a notation

$$r|_N = \sum_{k=0}^{N-1} r_k$$

for a formal symbol

$$r = \sum_{k=0}^{\infty} h^k r_k.$$

Then the following theorem holds.

Theorem 8 *Let r be a formal parametrix (3.2) of the elliptic symbol a. Then for N large enough the operator $R = \mathrm{Op}(r|_N)$ is an operator parametrix of $A = \mathrm{Op}\,(a)$ and*

$$\operatorname{ind} A = \mathrm{Tr}(1 - RA) - \mathrm{Tr}(1 - AR) = \mathrm{Tr}(1 - r \circ a) - \mathrm{Tr}(1 - a \circ r). \quad (3.8)$$

Proof. Denoting $r|_N$ by b and taking M sufficiently large, we get

$$1 - \mathrm{Op}(b)\mathrm{Op}(a) = \mathrm{Op}((1 - b \circ a)|_M) + \{\mathrm{Op}(b)\mathrm{Op}(a) - \mathrm{Op}(b \circ a|_M)\}$$
$$1 - \mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}((1 - a \circ b)|_M) + \{\mathrm{Op}(a)\mathrm{Op}(b) - \mathrm{Op}(a \circ b|_M)\}.$$

By theorems 1 and 4 the operators in curly brackets belong to the trace class and their traces are equal. Taking $M \geq N$ and writing

$$b = r|_N = \sum_{k=0}^{N-1} r_k,$$

$$b \circ a|_M = \sum_{l=0}^{M-1} \sum_{k=0}^{N-1} \frac{1}{l!} \frac{\partial^l}{\partial z^l} r_k D^l a,$$

$$r \circ a|_N = \sum_{0 \leq k+l < N} \frac{1}{l!} \frac{\partial^l}{\partial z^l} r_k D^l a,$$

we see that $\mathrm{Op}((b \circ a)|_M) - \mathrm{Op}((r \circ a)|_N)$ is a finite sum of terms

$$\mathrm{Op}\left(\frac{\partial^l}{\partial z^l} r_k D^l a\right)$$

with $k + l \geq N, \quad k < N, \quad l < M$. If $N > n + 1$, this operator belongs to the trace class since its order is less than $-(n + 1)$ and its symbol vanishes at $t \in [0, c]$ and $t \in [C, \infty)$. The same is true for $\mathrm{Op}(a \circ b|_M) - \mathrm{Op}(a \circ r|_N)$, which is the sum of

$$\mathrm{Op}\left(\frac{\partial^l a}{\partial z^l} D^l r_k\right)$$

with $k+l \geq N, \quad k < N, \quad l < M$. The traces of such operators are equal because

$$\int_{\Gamma_0} dz \int_0^\infty \operatorname{tr} \frac{\partial^l}{\partial z^l} r_k D^l a \frac{dt}{t} = \int_{\Gamma_0} dz \int_0^\infty \operatorname{tr} \frac{\partial^l a}{\partial z^l} D^l r_k \frac{dt}{t},$$

as one can see integrating by parts. Thus,

$$\operatorname{Tr}(1-RA) - \operatorname{Tr}(1-AR) = \operatorname{Tr}\,(1 - r \circ a)|_N - \operatorname{Tr}\,(1 - a \circ r)|_N$$

which is precisely the algebraical index.

□

4 A Topological Index

Following [6], we introduce one more algebra which permits to simplify significantly various calculations with noncommutative differential forms. We will use a real variable τ instead of $z = i\tau$. An element a of our new algebra $\widehat{\mathcal{A}}$ is an operator-valued nonhomogeneous differential form of even degrees on the half-plane $\mathbb{R}^2_+$. So,

$$a = a_0(t,\tau) + a_1(t,\tau) d\tau \wedge dt \tag{4.1}$$

where a_0 and a_1 are pseudodifferential operators on X of nonpositive orders. A product $\hat{\circ}$ of two elements $a, b \in \widehat{\mathcal{A}}$ is defined by

$$a \hat{\circ} b = a \wedge b + \frac{i}{2} da \wedge db. \tag{4.2}$$

One immediately checks that this product is associative.

Any function $a(t,\tau)$ may be considered as an element of $\widehat{\mathcal{A}}$ consisting of 0-component only. So, for functions a, b we have three products:

- ab is the usual point-wise operator product of functions,
- $$a \circ b = ab + iht \frac{\partial a}{\partial \tau} \frac{\partial b}{\partial t} + \ldots \tag{4.3}$$

  is a product in $\mathcal{A}$ as formal symbols,

•

$$a\hat{\circ}b = ab + \frac{i}{2}da \wedge db \tag{4.4}$$

is a product in $\hat{\mathcal{A}}$.

We may also consider the powers of a function a with respect to any of these products using notations $a^k, a^{\circ k}, a^{\hat{\circ} k}$ to distinguish the three possibilities. One can verify a simple rule to pass from $\circ$-product to $\hat{\circ}$-product of functions: keep the terms linear in h, then alternate derivations $\partial/\partial\tau, \partial/\partial t$ and then write $d\tau \wedge dt$ instead of ht. This rule is valid for any number of functions $a_1 \circ a_2 \circ \ldots \circ a_k$ and $a_1 \hat{\circ} a_2 \hat{\circ} \ldots \hat{\circ} a_k$.

Similarly to $\mathcal{J}$ we introduce a *trace ideal* $\widehat{\mathcal{J}} \subset \hat{\mathcal{A}}$. It consists of forms (4.1) where a_0, a_1 are operators of order $\mu < -(n+1)$ with regard for a parameter $\tau \in \mathbb{R}$, vanishing at $t \in [0, c]$ and $t \in [C, \infty)$. For $a \in \widehat{\mathcal{J}}$ define a *trace*

$$\operatorname{Tr} a = \frac{1}{2\pi} \int_{\mathbb{R}^2_+} \operatorname{tr} a = \frac{1}{2\pi} \int_{\mathbb{R}^2_+} \operatorname{tr} a_1(t, \tau) d\tau dt$$

(the orientation of $\mathbb{R}^2_+$ is given by the form $d\tau \wedge dt$). The trace property

$$\operatorname{Tr} a\hat{\circ}b = \operatorname{Tr} b\hat{\circ}a$$

is obviously satisfied if a or b belongs to $\mathcal{J}$.

With this definitions we have the following topological index formula.

Theorem 9 *For any* $N \geq 1$

$$\operatorname{ind} A = \operatorname{Tr}(1 - r_0 \hat{\circ} a)^{\hat{\circ}(N+1)} - \operatorname{Tr}(1 - a \hat{\circ} r_0)^{\hat{\circ}(N+1)} \tag{4.5}$$

where r_0 *is the leading term of the parametrix of* a.

Proof. We start with the algebraical index formula (3.8) taking

$$r = r_0 \circ \sum_{k=0}^{N} (1 - a \circ r_0)^k = \sum_{k=0}^{N} (1 - r_0 \circ a)^k \circ r_0.$$

with N large enough. Then

$$1 - r \circ a = (1 - r_0 \circ a)^{\circ(N+1)}$$

$$1 - a \circ r = (1 - a \circ r_0)^{\circ(N+1)},$$

so

$$\operatorname{ind} A = \operatorname{Tr}(1 - r_0 \circ a)^{\circ(N+1)} - \operatorname{Tr}(1 - a \circ r_0)^{\circ(N+1)}. \tag{4.6}$$

According to lemma 7 we need to extract a constant term in (4.6). It means that we may calculate $(1 - r_0 \circ a)^{\circ(N+1)}$ keeping the terms linear in h. Thus,

$$1 - r_0 \circ a = 1 - r_0 a - iht\frac{\partial r_0}{\partial \tau}\frac{\partial a}{\partial t} + \dots$$

where dots mean higher-degree terms in h. Using induction one easily obtains a formula

$$\begin{aligned}(1 - r_0 \circ a)^{\circ(N+1)} &\sim (1 - r_0 a)^{N+1} - \\ -iht &\left\{\sum_{k=0}^{N}(1 - r_0 a)^k \frac{\partial r_0}{\partial \tau}\frac{\partial a}{\partial t}(1 - r_0 a)^{N-k} - \sum_{k+p+q=N-1}(1 - r_0 a)^k \times \right. \\ &\left. \times \frac{\partial(1 - r_0 a)}{\partial \tau}(1 - r_0 a)^p \frac{\partial(1 - r_0 a)}{\partial t}(1 - r_0 a)^q\right\}\end{aligned} \tag{4.7}$$

where $\sim$ means that the linear terms coincide. The second sum may be written as

$$\sum_{k=0}^{N}\frac{\partial(1 - r_0 a)^k}{\partial \tau}\frac{\partial(1 - r_0 a)}{\partial t}(1 - r_0 a)^{N-k} \tag{4.8}$$

or

$$\sum_{k=0}^{N}(1 - r_0 a)^k \frac{\partial(1 - r_0 a)}{\partial \tau}\frac{\partial(1 - r_0 a)^{N-k}}{\partial t}. \tag{4.9}$$

Using "integration by parts" transform (4.9) to the form

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_{k=0}^{N}(1 - r_0 a)^k \frac{\partial(1 - r_0 a)}{\partial \tau}(1 - r_0 a)^{N-k} - \\ &-\sum_{k=0}^{N}\frac{\partial(1 - r_0 a)^k}{\partial t}\frac{\partial(1 - r_0 a)}{\partial \tau}(1 - r_0 a)^{N-k} - \\ &-\sum_{k=0}^{N}(1 - r_0 a)^k \frac{\partial^2(1 - r_0 a)}{\partial \tau \partial t}(1 - r_0 a)^{N-k}.\end{aligned} \tag{4.10}$$

If $N \geq 1$ all the written terms belong to the trace ideal $\mathcal{J}$ since they contain a factor $1 - r_0 a \in \mathcal{J}$ or its derivatives.

Let us now write down a constant term of the trace of (4.7). We represent the second sum in (4.7) as a half-sum of expression (4.8) and (4.10). We may drop the first sum in (4.10) since complete derivatives vanish under integration and permute cyclically the factors under trace sign. Finally we obtain

$$\frac{1}{2\pi i}\int_{\mathbb{R}^2_+}(N+1)\mathrm{tr}(1-r_0a)^N\left(\frac{\partial r_0}{\partial\tau}\frac{\partial a}{\partial t}-\frac{1}{2}\frac{\partial^2(r_0a)}{\partial\tau\partial t}\right)-$$
$$-\frac{1}{2}\sum_{k=0}^{N}\left\{\frac{\partial(1-r_0a)^k}{\partial\tau}\frac{\partial(1-r_0a)}{\partial t}-\frac{\partial(1-r_0a)^k}{\partial t}\frac{\partial(1-r_0a)}{\partial\tau}\right\}\times$$
$$\times(1-r_0a)^{N-k}d\tau dt.$$

Next,

$$\frac{\partial r_0}{\partial\tau}\frac{\partial a}{\partial t}-\frac{1}{2}\frac{\partial^2(r_0a)}{\partial\tau\partial t}=\frac{1}{2}\frac{\partial r_0}{\partial\tau}\frac{\partial a}{\partial t}-\frac{1}{2}\frac{\partial r_0}{\partial t}\frac{\partial a}{\partial\tau}-\frac{1}{2}\frac{\partial^2 r}{\partial t\partial\tau}a-\frac{1}{2}r\frac{\partial^2 a}{\partial t\partial\tau}$$

and

$$\left(\frac{\partial r_0}{\partial\tau}\frac{\partial a}{\partial t}-\frac{\partial r_0}{\partial t}\frac{\partial a}{\partial\tau}\right)d\tau\wedge dt=dr_0\wedge da,$$
$$\left\{\frac{\partial(1-r_0a)^k}{\partial\tau}\frac{\partial(1-r_0a)}{\partial t}-\frac{\partial(1-r_0a)^k}{\partial t}\frac{\partial(1-r_0a)}{\partial\tau}\right\}d\tau\wedge dt=$$
$$=d(1-r_0a)^k\wedge d(1-r_0a).$$

Thus, for the constant term of $\mathrm{Tr}(1-r_0\circ a)^{\circ(N+1)}$ we get an expression

$$\frac{1}{2\pi i}\int_{\mathbb{R}^2_+}\left\{\frac{(N+1)}{2}\mathrm{tr}(1-r_0a)^N dr_0\wedge da-\right.$$
$$\left.\frac{1}{2}\sum_{k=0}^{N}\mathrm{tr}\, d(1-r_0a)^k\wedge d(1-r_0a)(1-r_0a)^{N-k}\right\}-$$
$$-\frac{1}{2\pi i}\frac{N+1}{2}\int_{\mathbb{R}^2_+}\mathrm{tr}(1-r_0a)^N\left(\frac{\partial^2 r_0}{\partial\tau\partial t}a+r_0\frac{\partial^2 a_0}{\partial\tau\partial t}\right)d\tau dt. \qquad (4.11)$$

A similar expression may be written for the constant term of $\mathrm{Tr}(1-a\circ r_0)^{\circ(N+1)}$ by interchanging a and r_0. Note that

$$\mathrm{tr}(1-ar_0)^N\frac{\partial^2 a}{\partial\tau\partial t}r_0=\mathrm{tr}\, r_0(1-ar_0)^N\frac{\partial^2 a}{\partial\tau\partial t}=$$
$$=\mathrm{tr}(1-r_0a)^N r_0\frac{\partial^2 a}{\partial\tau\partial t}.$$

It implies that the last integral in (4.11) does not change under permutation of a and r_0. Thus, taking the difference of (4.11) and the corresponding expression obtained by interchanging a and r_0, we find

$$\begin{aligned}\operatorname{ind} A = \frac{1}{2\pi i}\int_{\mathbb{R}^2_+} \Big\{ & \frac{(N+1)}{2}\operatorname{tr}(1-r_0a)^N dr_0\wedge da - \\ & -\frac{1}{2}\sum_{k=0}^{N}\operatorname{tr} d(1-r_0a)^k\wedge d(1-r_0a)(1-r_0a)^{N-k} - \\ & -\frac{N+1}{2}\operatorname{tr}(1-ar_0)^N da\wedge dr_0 + \\ & +\frac{1}{2}\sum_{k=0}^{N}\operatorname{tr} d(1-ar_0)^k\wedge d(1-ar_0)(1-ar_0)^{N-k}\Big\}. \end{aligned} \tag{4.12}$$

Taking into account the rule for passing from $\circ$-product to $\hat{\circ}$-product one easily recognizes formula (4.5) in (4.12). In particular, we obtain that the right hand side of (4.12) or (4.5) is independent of N provided N is large enough. Using $\hat{\circ}$-product, we prove now that (4.5) is valid for $N \geq 1$. Indeed,

$$\begin{aligned} & \operatorname{Tr}(1-r_0\hat{\circ}a)^{\hat{\circ}(N+1)} - \operatorname{Tr}(1-a\hat{\circ}r_0)^{\hat{\circ}(N+1)} = \\ & = \operatorname{Tr}(1-r_0\hat{\circ}a)^{\hat{\circ}N} - \operatorname{Tr}(1-a\hat{\circ}r_0)^{\hat{\circ}N} - \\ & -\operatorname{Tr}(1-r_0\hat{\circ}a)^{\hat{\circ}N}\hat{\circ}r_0\hat{\circ}a + \operatorname{Tr}(1-a\hat{\circ}r_0)^{\hat{\circ}N}\hat{\circ}a\hat{\circ}r_0. \end{aligned}$$

But by the associativity of the $\hat{\circ}$-product

$$\operatorname{Tr}(1-r_0\hat{\circ}a)^{\hat{\circ}N}\hat{\circ}r_0\hat{\circ}a = \operatorname{Tr} r_0\hat{\circ}(1-a\hat{\circ}r_0)^{\hat{\circ}N}\hat{\circ}a = \operatorname{Tr}(1-a\hat{\circ}r_0)^{\hat{\circ}N}\hat{\circ}a\hat{\circ}r_0.$$

Here we have used that $(1-a\hat{\circ}r_0)^{\hat{\circ}N} \in \widehat{\mathcal{J}}$ for $N \geq 1$, so a cyclic permutation of factors under a trace sign is possible.

□

For $N = 1$ (4.12) becomes

$$\operatorname{ind} A = \frac{1}{2\pi i}\int_{\mathbb{R}^2_+} \operatorname{tr}(1-r_0a)dr_0\wedge da - \operatorname{tr}(1-ar_0)da\wedge dr_0$$

since

$$\operatorname{tr} d(1-r_0a)\wedge d(1-r_0a) = \operatorname{tr} d(1-ar_0)\wedge d(1-ar_0) = 0.$$

Integrating by parts in the first summand, we get

$$-\int_{\mathbb{R}^2_+} \operatorname{tr}\,(d(1-r_0a))r_0da = \int_{\mathbb{R}^2_+} \operatorname{tr}(dr_0ar_0 \wedge da + r_0dar_0 \wedge da).$$

The second summand may be transformed as follows

$$-\operatorname{tr}(1-ar_0)da \wedge dr = \operatorname{tr} dr(1-ar_0) \wedge da.$$

So, (4.12) at $N = 1$ gives formula (0.5).

5 Example

Consider the simplest example of a singular integral operator on a half–line (a cone over a point)

$$(Au)(x) = \alpha(x)u(x) + \frac{\beta(x)}{\pi i}\int_0^\infty \frac{u(y)}{y-x}dy \tag{5.1}$$

where $\beta(x)$ and $1-\alpha(x)$ belong to $C_0^\infty(\overline{R}_+)$. For $x > 0$ this operator has a principal (Fourier) symbol

$$a_0(x,\xi) = \alpha(x) + \beta(x)\operatorname{sgn}\xi$$

and we assume as usual that $a_0(x, \pm 1) \neq 0$.

Let us pass to the Mellin representation. Taking

$$u(y) = \frac{1}{2\pi i}\int_\Gamma y^{-z}\hat{u}(z)dz$$

with $0 < \Re\Gamma < 1$, substituting it into (5.1) and calculating the singular integral

$$\frac{1}{\pi i}\int_0^\infty \frac{y^{-z}}{y-x}dy = x^{-z}\frac{1+e^{2\pi i z}}{1-e^{2\pi i z}}$$

we represent (5.1) in the form

$$(Au)(x) = \frac{1}{2\pi i}\int_\Gamma x^{-z}a(x,z)\hat{u}(z)dz \tag{5.2}$$

where the Mellin symbol is

$$a(x,z) = \alpha(x) + \beta(x)\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}}. \tag{5.3}$$

The ellipticity conditions for the operator (5.2) besides $\alpha(x)\pm\beta(x) \neq 0$ mean that the line $\Gamma = \Gamma_\sigma$ does not contain the poles of $a(t,z)$ as well as the zeros of $a(0,z)$, so that $a^{-1}(0,z)$ is holomorphic in a strip containing Γ_σ. Now, since $rda \wedge rda = 0$ for scalar functions we obtain by the Stokes formula

$$\mathrm{ind}A = \frac{1}{2\pi i}\int_{\Gamma\times\mathbb{R}_+} dr \wedge da = \frac{1}{2\pi i}\int_{\partial(\Gamma\times\mathbb{R}_+)} a^{-1}da$$

which is equal to the variation of $\arg a$ along the "boundary" $\partial(\Gamma\times\mathbb{R}_+)$. The latter consists of the line $z \in \Gamma$, $x = 0$ and two half-lines $z = \sigma\pm i\infty$, $x \in \mathbb{R}_+$. This gives a final formula

$$\mathrm{ind}A = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} a^{-1}(0,z)a'(0,z)dz + \frac{1}{2\pi}\Delta\arg\frac{\alpha(x)+\beta(x)}{\alpha(x)-\beta(x)}$$

where Δ means variation along the positive half-axis $x > 0$. It is precisely formula (0.3) from [2].

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