Max-Planck-Institut für Mathematik
Bonn

On quasimaps to quadrics

by

M. V. Movshev

Max-Planck-Institut für Mathematik
Preprint Series 2010 (62)

On quasimaps to quadrics

M. V. Movshev

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Mathematics Department
Stony Brook University
Stony Brook, NY 11794-365
USA

MPIM 2010-62

On quasimaps to quadrics

August 4, 2010

Abstract

We put on a rigorous basis results of Aisaka and Aldo Arroyo on curved $\beta\gamma$ systems on a quadric. Drinfeld's quasimaps turned to be indispensable in accurate computation of mini-BRST cohomology.

1 Introduction

A mathematically rigorous derivation of the Hilbert space of string theory in the formalism of pure spinors [4] remains a challenging problem [2]. The difficulty is to give a mathematical meaning a $\beta\gamma$-system with the target the space of pure spinors SO(10)/U(5), more precisely an affine cone over it.The direct attack on the problem would require to develop analysis on the space of loops in the target. The paper [1] proposes to investigate a more simple model where the target is an affine quadric. The space of loops in the quadric is an infinite-dimensional "manifold", defined by an infinite set of equations. The Koszul or mini-BRST(in terminology of [1]) complex, constructed from the set of equations hide all difficulties of analysis in the BRST differential. The advantage of this formulation is that constraint fields of the original problem are replaced by free fields of mini-BRST setup.

More precisely following [1] we consider a space of loops that lie on a quadric in a complex linear space V. It is convenient to choose an orthonormal basis

$e_i, i = 1\dots, n$. Then a loop

$$\lambda : S^1 \to V$$

is completely characterizes by n functions $\lambda^i(z), z \in S^1 \subset \mathbb{C}$:

$$\lambda(z) = \lambda^1(z)e_1 + \cdots + \lambda^n(z)e_n$$

The loop belongs to the quadric if $\sum_{i=1}^n (\lambda^i(z))^2 = 0$ for all z. One can go on and expand λ into Fourier series

$$\lambda(z) = \sum_{k\in\mathbb{Z}} \sum_{i=1}^n \lambda^i[k]e_i z^k$$

From this we obtain an infinite set of equations on Fourier coefficients:

$$f[k] = \sum_{t+s=k} \sum_{i=1}^n \lambda^i[t]\lambda^i[s] = 0 \tag{1}$$

Following the standard in the gauge theory and the commutative algebra construction we define Koszul complex. For every constraint f_k we introduce an odd variable c_k, that carries degree one. In the space of functions in λ_i^k, c_k we define a differential by the formula

$$\sum_k f_k \frac{\partial}{\partial c_k}.$$

One of the assertions that has been used implicitly many times in [1] was that the complex has cohomology only in degree zero. Though the authors attribute this to irreducibility of the constraint, this statement hard to make precise in infinite dimensions because cohomology depend also on the class of functions we are dealing with.

One of the goals of the present work is to make this statement precise.

The key technical novelty is that we use finite-dimensional spaces of Drinfeld's quasimaps to approximate the loop spaces.[1] This means that we work with Fourier series of the form

$$\lambda(z) = \sum_{-N_1\leq k\leq N_2} \sum_{i=1}^n \lambda^i[k]e_i z^k$$

[1] As author learned from [7] this idea was first introduced by Drinfeld in a context of quasimaps to full flag spaces.

Equations $f_k = 0$ define a finite-dimensional algebraic variety, which typically is singular. As a side remark we mention that De Rham and Dolbeault complexes have a limited usefulness in this setup. Never the less we can study algebras of polynomial functions $AQ = AQ^{N_2}_{-N_1}(n)$ in variables $\lambda^i[k]$ subject to relations $f[k]$ and define Koszul complexes $C = C^{N_2}_{-N_1}(n)$ using even $\lambda^i[k]$ $N_1 \leq k \leq N_2$ and odd $c[k]$ $-2N_1 \leq k \leq 2N_2$ as generators. One of our results is

Theorem 1 *The zero cohomology of the complex $C^{N_2}_{-N_1}(n)$ is equal to $AQ^{N_2}_{-N_1}(n)$, $n \geq 3$. All other cohomology vanish.*

In our paper we use ideas of [12] in the version of [20] [19] to compute Poincaré series of AQ. The method lets us not only to compute the series but to prove important Koszul property of algebra AQ. This becomes crucial in the proof acyclicity of resolution proposed by Aisaka and Arroyo.

We believe that direct limit of $Spec(AQ^{N_2}_{-N_1})$ capture most of the pertinent properties of loop spaces for "good" targets. This lets us to avoid difficulties of infinite-dimensional analysis and allows to use methods of commutative algebra and algebraic geometry. We want to emphasize that the method can easily be extended to a more realistic case of pure spinors. We plan to elaborate on this in our following publication.

Few questions were left unanswered:

- Do spaces of Drinfeld's quasimaps have intrinsic physical meaning or it is just a convenient mathematical tool?
- It is known (see e.g. [15]) that $\beta\gamma$-systems might suffer from anomalies. How anomalies appear in the proposed approach? (See a remark in the end of Section 3)
- Depending on the answer of the previous item, what is the construction of the Virasoro action?
- Characterize in geometric terms conical algebraic varieties whose $\beta\gamma$-systems systems satisfy $*$-duality. Give a geometric construction of $*$ duality using the language of quasimaps.

We hope to address these questions in the near future.

The paper is organized as follows. In Section 2 we illustrate our methods of computation of Poincaré series with a simplest example of a quadric. Section 3 is technically central, where we prove straightened law and Koszul property for AQ. In Section 4 compute Poincaré series of AQ. In Section 6 we compute semi-infinite cohomology of Lie algebra LH. This cohomology will serve an approximation to semi-infinite cohomology of algebra $LH^{\infty}_{-\infty}$ discussed in Section 7. The latter is an algebraic counterpart of quantum mini-BRST complex of [1].

The paper is supplemented by two appendices. In the first we discuss exceptional cases of quadrics in 4,3 or 2-D spaces. The basics of theory of Gröbner bases is reviewed in the second appendix.

Acknowledgment

The author would like to thank N.Berkovits, M. Finkelberg, V. Gorbunov, A.Schwarz for stimulating discussions. The author is also thankful to Max-Plank-Institute (Bonn) for excellent recearch environment, where part of this work has been done.

2 Finite-dimensional quadric

Before we delve into algebra of quasimaps let us treat the simplest case of maps of degree zero to a quadric, that is the quadric itself . The algebra of algebraic functions on an affine non-degenerate quadric $A = \bigoplus_{i\geq 0} A_i$ is a quotient $\mathbb{C}[\lambda^i]/(\sum_{i=1}^n \lambda_i^2)$ has Poincaré series

$$A(t) = \sum_{i\geq 0} \dim A_i t^i = \frac{1-t^2}{(1-t)^{\dim(V)}}.$$

Let us see how our method reproduces this result.

2.1 The case of an even-dimensional V

We suppose that $\dim(V) = 2n$, $n \geq 3$ and decompose

$$V = W + W^*$$

into a direct sum of complementary isotropic subspaces. The space W^* is dual to W with the pairing defined by the formula $< g, f >= (g, f)$. We choose a basis $g_1, \dots, g_n$ in W and the dual basis $f^1, \dots, f^n$ in W^*.

The adjoint representation of the complex Lie algebra $\mathfrak{so}_{2n}$ is isomorphic to

$$\Lambda^2(V) \cong \Lambda^2(W^*) + W^* \otimes W + \Lambda^2(W)$$

We identify the space $W^* \otimes W$ with the Lie subalgebra $\mathfrak{gl}_n$ of $\mathfrak{so}_{2n}$. A subalgebra of $\mathfrak{gl}_n$ of diagonal matrices (with respect to the basis $g_1, \dots, g_n$) is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}_{2n}$. It is spanned by $f^i g_i, i = 1 \dots, n$. Denote by ρ the fundamental representation

$$\rho : \mathfrak{so}_{2n} \to \mathrm{End}(V)$$

Simple positive root vectors of $\mathfrak{so}_{2n}$ relative to $\mathfrak{h}$ are $r = g_1 \wedge g_2$ and $r_i = g_{i+1} \otimes f_i$. The operators $R = \rho(r), R_i = \rho(r_i)$

$$Rg_k = 0 \quad Rf^k = \begin{cases} g_2 & \text{if } k = 1, \\ -g_1 & \text{if } k = 2, \\ 0 & \text{if } k \neq 1, 2 \end{cases}$$

$$R_i g_k = \begin{cases} g_{k+1} & \text{if } i = k, \\ 0 & \text{if } i \neq k \end{cases} \qquad R_i f^k = \begin{cases} -f^{k-1} & \text{if } i = k-1, \\ 0 & \text{if } i \neq k-1 \end{cases}$$

$i = 1, \dots, n-1$ are of great importance in our construction.

We define the following Hasse diagram:

$$\begin{tikzcd}
 & & & \langle f^1 \rangle \arrow[dr, "R"] & & & \\
\mathcal{G}(2n) : \langle f^n \rangle \arrow[r, "R_{n-1}"] & \dots \arrow[r, "R_2"] & \langle f^2 \rangle \arrow[ur, "R_1"] \arrow[dr, "R"'] & & \langle g_2 \rangle \arrow[r, "R_2"] & \dots \arrow[r, "R_{n-1}"] & \langle g_n \rangle \\
 & & & \langle g_1 \rangle \arrow[ur, "R_1"'] & & &
\end{tikzcd}$$

formed by weight subspaces of V.

In this diagram the brackets $\langle \rangle$ stand for spans of vectors. One-dimensional linear spaces α, β from the above diagram are connected by an arrow $\alpha \to \beta$ if one of the operators R or R_i transforms linear space α isomorphically to linear space β.

The diagram will be used for describing a basis in the algebra

$$A = \mathbb{C}[g_1, \dots, g_n, f^1, \dots, f^n]/(\sum_{i=1}^{n} g_i f^i)$$

The graph defines a partial order on the set of generators $\mathcal{G} = \mathcal{G}(2n)$ of algebra A ($\alpha \leq \beta$ iff there is an arrow $\alpha \to \beta$). In this poset any two elements $\alpha, \beta \in \mathcal{G}$ have a *unique* supremum and infinum :

$$\delta \leq \alpha, \beta \leq \gamma$$

A poset with this property is called a *lattice*. We denote

$$\gamma \stackrel{\text{def}}{=} \alpha \vee \beta, \quad \delta \stackrel{\text{def}}{=} \alpha \wedge \beta.$$

It is convenient to use uniform notations for generators of A

$$\{e_\alpha | \alpha \in \mathcal{G}\} = \{g_1, \dots, g_n, f^1, \dots, f^n\}$$

Thus A is a quotient of $\mathbb{C}[e_\alpha] \quad \alpha \in \mathcal{G}$

$$0 \to I \to \mathbb{C}[e_\alpha] \xrightarrow{p} A \to 0$$

Any monomial $m = \prod_{i=1}^{k} e_{\alpha_i} \in \mathbb{C}[e_\alpha]$ can be rewritten as $m = \prod_{\alpha \in B \subset \mathcal{G}} e_\alpha^{n_\alpha}$. We call B the support supp(m).

A set $B \subset \mathcal{G}$ is called a path if all elements of B are comparable. The poset $\mathcal{G}$ has precisely two incomparable elements $\langle g_1 \rangle$ and $\langle f^1 \rangle$.

Observe that the only $g_1 f^1$ of the monomials of the defining relation

$$\sum_{i=1}^{n} g_i f^i \tag{2}$$

has support that is not a path. It is useful to rewrite the defining relation in the form

$$g_1 f^1 = -g_2 f^2 - \sum_{i \geq 3} g_i f^i \tag{3}$$

It is equivalent to

$$e_\alpha e_\beta = c_{\alpha\vee\beta,\alpha\wedge\beta} e_{\alpha\vee\beta} e_{\alpha\wedge\beta} + \sum_{\substack{\gamma \geq \alpha\vee\beta \\ \gamma' \leq \alpha\wedge\beta}} c_{\gamma,\gamma'} e_\gamma e_{\gamma'} \tag{4}$$

with coefficients $c_{\gamma,\gamma'}$ equal to -1 or 0.

A straightforward way to determine $A(t)$ would be to construct a basis m_k^i in A_i, count the number of elements in this basis and write the generating function. Such direct approach technically is not very convenient because the spaces A_i are defined as quotients of graded spaces of polynomial algebra $\mathbb{C}[e_\alpha]$. It is much more transparent to do the counting of dimensions directly in terms of $\mathbb{C}[e_\alpha]$.

One can give the following restatement of the problem. It is to find a subset E in the set of monomials in $\mathbb{C}[e_\alpha]$ such that $p(E)$ is a basis in A.

Not all the monomial $m \in \mathbb{C}[e_\alpha]$ are linearly independent in A because of the relation (2). In particular monomial $g_1 f^1$, whose support $\mathrm{supp}(g_1 f^1)$ is not a path can be represented as a sum of monomials $-g_2 f^2 - \sum_{i\geq 3} g_i f^i$, whose support are paths. This can be generalized to arbitrary monomials: images of monomials whose support is a path (we call them $\mathcal{G}$-monomials) span A.

Proposition 2 *Suppose that* $\dim(V)$ *is* $2n, n \geq 3$*. Let* A *be the algebra of algebraic function on affine non-degenerate quadric in* V^**. Let* $\mathcal{G}(2n)$ *be a lattice define above. Then the image of the set of* $\mathcal{G}$*-monomials in* A *is a basis.*

Proof. If a linear combination of $\mathcal{G}$-monomials belongs to the ideal then it is divisible by the relation (2). But the relation contains a non $\mathcal{G}$-monomial. Thus the original linear combination contains a non $\mathcal{G}$-monomial. ■

Definition 3 *Let* $\mathcal{G}$ *be a lattice,* B *be a graded algebra. Suppose that the first graded component* B_1 *has a basis* $< e_\alpha >$ *labeled by elements of* $\mathcal{G}$*. Suppose that in algebra* B *relations (4) hold. We say that* B *is an algebra with straightened law if the images of* $\mathcal{G}$*-monomials form a basis in* B*.*

Remark 4 *Our definition of algebra with straightened law is a particular case of a Hodge algebra ([10] or [11]).*

We see that A is an algebra with straightened law.

Our results can be used to find a formula for $A(t)$. It is not hard to describe all $\mathcal{G}$-monomials. These are

$$(f^n)^{\alpha_n}\cdots(f^2)^{\alpha_2}(f^1)^{\alpha_1}g_2^{\beta_2}\cdots g_n^{\beta_n}$$

and

$$(f^n)^{\alpha_n}\cdots(f^2)^{\alpha_2}g_1^{\beta_1}g_2^{\beta_2}\cdots g_n^{\beta_n}$$

$\alpha_i, \beta_j \geq 0$

These monomials form sets E_1 and E_2.

Let L be a subset of E. Defines a generating function $L(t)$ as

$$L(t) = \sum_{i\geq 0} \#\{m \in L|\deg(m) = i\}$$

Then

$$\begin{aligned} A(t) &= E_1(t) + E_2(t) - (E_1 \cap E_2)(t) = \\ &= \frac{1}{(1-t)^{2n-1}} + \frac{1}{(1-t)^{2n-1}} - \frac{1}{(1-t)^{2n-2}} = \frac{1-t^2}{(1-t)^{2n}} \end{aligned}$$

2.2 The case of an odd-dimensional V

Let us carry out cursorily a similar analysis of an affine non-degenerate quadric in odd-dimensional space V. Suppose $2n+1 = \dim(V) \geq 1$. We can choose a basis $(f^1, \ldots, f^n, g_1, \ldots, g_n, h)$ in which equation q of the quadric is

$$\sum_{i=1}^{n} f^i g_i + h^2. \tag{5}$$

and

$$\langle f^1, \ldots, f^n\rangle + \langle g_1, \ldots, g_n\rangle + \langle h\rangle = W^* + W + \langle h\rangle$$

The adjoint representation of $\mathfrak{so}_{2n+1}$ is isomorphic to

$$\Lambda^2(V) \cong \Lambda^2(W) + W \otimes W^* + \Lambda^2(W^*) + W \otimes \langle h\rangle + W^* \otimes \langle h\rangle$$

Positive simple root vectors are $r = g_1 h$ and $r_i = g_{i+1} f_i$. Cartan subalgebra is generated by $f^i g_i$.

The corresponding diagram $\mathcal{G}(2n+1)$ constructed with the aid of $R = \rho(r), R_i = \rho(r_i)$ is

$$\mathcal{G}(2n+1) : \langle f^n \rangle \xrightarrow{R_{n-1}} \dots \xrightarrow{R_1} \langle f^1 \rangle \xrightarrow{R} \langle h \rangle \xrightarrow{R} \langle g_1 \rangle \xrightarrow{R_1} \dots \xrightarrow{R_{n-1}} \langle g_n \rangle$$

One of the distinctions of even and odd dimensional cases is that in odd case the Weil group W of $\mathrm{SO}(2n+1)$ does not act transitively on the set of weight vectors f^i, g_j, h. This is because the weight of h is zero, the weights of f^i, g_j are non zero. This makes it impossible for h to belong to the same orbit of the Weil group because the group acts linearly on weights.

A representation of semi-simple group G is called *minuscule* if W acts transitively on the set of weights. Thus the fundamental representation of $\mathfrak{so}_k$ is minuscule only for even k.

To extend a definition of algebras with straightened laws to the case of odd quadric we introduce the following definition.

Definition 5 *We call $\mathcal{T}$ a weakly partly ordered set if it has a (partly defined) binary relation $\leq$ which is antisymmetric, and transitive, i.e., for all a, b, and c in $\mathcal{T}$, we have that:*

- *if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry);*
- *if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).*

The above definition differs from the standard definition of a partly ordered set in the item of reflexivity. For a weak partial order condition $a \leq a$ might fail for some exceptional a.

Definition 6 *We call weakly partly ordered set $\mathcal{T}$ a weak lattice if for any two elements α, β there there are unique supremum $\alpha \vee \beta$, and infinum, $\alpha \wedge \beta$.*

We define a structure of a weak lattice on the set of vertices of the diagram $\mathcal{G}(2n+1)$ using the same prescription as for $\mathcal{G}(2n)$. We only modify relation on $\langle h \rangle$ and make it exceptional. Then the defining relation of the odd quadric (5) can be put into the form (4).

Proposition 7 *Suppose that* $\dim(V)$ *is odd . Let* A *be the algebra of algebraic function on affine non-degenerate quadric in* V^**. Let* $\mathcal{G}(2n+1)$ *be a weak lattice define above. Then the image of the set of* $\mathcal{G}$*-monomials in* A *is a basis.*

Proof. The proof repeats the proof in even-dimensional case. ∎

We are in position to derive a formula for $A(t)$. The set of $\mathcal{G}$-monomials contains elements of the form

$$(f^n)^{\alpha_n} \cdots (f^1)^{\alpha_1} g_1^{\beta_1} \cdots g_n^{\beta_n}$$

and

$$(f^n)^{\alpha_n} \cdots (f^1)^{\alpha_1} g_1^{\beta_1} \cdots g_n^{\beta_n} h$$

$\alpha_i, \beta_j \geq 0$

These monomials constitute nonintersecting set E_1 and E_2.

Then

$$\begin{aligned} A(t) &= E_1(t) + E_2(t) \\ &= \frac{1}{(1-t)^{2n}} + \frac{t}{(1-t)^{2n}} = \frac{1-t^2}{(1-t)^{2n+1}} \end{aligned}$$

3 Drinfelds' quasimaps

The combinatorial approach developed in the last section can be adapted to computation of generating function of Drinfeld's spaces of quasimaps.

Let us review briefly following [7] the basics of theory of quasimaps. Let Σ be an algebraic curve, $X = \mathbf{P}^N$-projective space. In this case points of the space of maps of degree d $\mathrm{Maps}^d(\Sigma, X)$ are classified by the following data:

- A line bundle L on Σ of degree $-d$.
- An embedding of vector bundles $L \to \mathcal{O}_\Sigma^{N+1}$

The reason is that every such embedding defines a one-dimensional subspace in $\mathbb{C}^{N+1}$ for every point $c \in \Sigma$ and thus we get a map $\Sigma \to \mathbf{P}^N$.

Consider, for example, the case when $\Sigma = \mathbf{P}^1$-projectivization of two-dimensional complex symplectic linear space with a basis x, y. In that case L must be isomorphic to the line bundle $\mathcal{O}_{\mathbf{P}^1}(-d)$ (note that such an isomorphism is defined uniquely up to a scalar) and thus $\mathrm{Maps}^d(\mathbf{P}^1, \mathbf{P}^N)$ becomes an open subset in the projectivization of the vector space $\mathrm{Hom}(\mathcal{O}(-d), \mathcal{O}^{N+1}) \cong \mathbb{C}^{(N+1)} \otimes \bigoplus_{i=0}^{d} \langle x^{d-i} y^i \rangle$, i.e. $\mathrm{Maps}^d(\mathbf{P}^1, \mathbf{P}^N)$ is an open subset of $\mathbf{P}^{(N+1)(d+1)-1}$. The reason that it does not coincide with it is that not every non-zero map $\mathcal{O}(-d) \to \mathcal{O}^{N+1}$ gives rise a map $\mathbf{P}^1 \to \mathbf{P}^N$ - we need to consider only those maps which don't vanish in every fiber.

The above example suggests the following compactification of $\mathrm{Maps}^d(\Sigma, \mathbf{P}^N)$. Namely, we define the space of quasi-maps from Σ to X of degree d (denoted by $Q\mathrm{Maps}^d(\Sigma, X)$) to be the scheme classifying the following data:

1. A line bundle L on Σ
2. A non-zero map $\kappa : L \to \mathcal{O}_{\Sigma}^{N+1}$
3. Note that κ defines an honest map $U \to \mathbf{P}^N$ where U is an open subset of Σ. We require that the image of this map lies in X.

For example it is easy to see that if $X = \mathbf{P}^N$ and $\Sigma = \mathbf{P}^1$ then $Q\mathrm{Maps}^d(\Sigma, X) \cong \mathbf{P}(\mathbb{C}^{(N+1)} \otimes \bigoplus_{s=0}^{d} \langle x^{d-s} y^s \rangle)$. Let $z^s = x^s y^{d-s}$. Then the affine cone $CQ\mathrm{Maps}^d(\mathbf{P}^1, \mathbf{P}^N)$ over projective space $Q\mathrm{Maps}^d(\mathbf{P}^1, \mathbf{P}^N)$ is a space of vector-valued polynomials

$$\lambda(z) = \sum_{i=0}^{d} \lambda[s] z^s, \lambda[s] \in \mathbb{C}^{N+1} \tag{6}$$

If a submanifold $X \subset \mathbf{P}^N$ is defined by homogeneous equations $r_i(\lambda) = 0$ then $CQ\mathrm{Maps}^d(\mathbf{P}^1, X)$ is defined by equations

$$r_i[s] = \frac{1}{s!} \frac{d^s r_i(\lambda(z))}{dz^s}|_{z=0} = 0, s \geq 0 \tag{7}$$

on coefficients $\lambda[t]$. This general theory explains SL(2) action on the space $CQ\mathrm{Maps}^d(\mathbf{P}^1, X)$, which is not seen in the description of $CQ\mathrm{Maps}^d(\mathbf{P}^1, X)$ as a variety of polynomial maps $\lambda(z)$.

The spaces of maps of the form

$$\lambda(z) = \sum_{i=-r}^{d-r} \lambda[i] z^i$$

is isomorphic to the space of maps (6). The identification, which we will be using freely, is made by multiplication of $\lambda(z)$ on z^r.

The main property of the scheme $Q\mathrm{Maps}^d(\Sigma, X)$ is that it possesses a stratification

$$Q\mathrm{Maps}^d(\Sigma, X) = \bigcup_{d'=0}^{d} \mathrm{Maps}^{d'}(\Sigma, X) \times \mathrm{Sym}^{d-d'}(\Sigma)$$

The corresponding strata in $CQ\mathrm{Maps}^d(\Sigma, X)$ correspond maps $\lambda(z)$ that can be factored into $\tilde{\lambda}(z)f(z)$, where a scalar polynomial $f(z)$ has degree $d - d'$. The corresponding point in $\mathrm{Sym}^{d-d'}(\Sigma)$ is the zero divisor of f. Note that $Q\mathrm{Maps}^d(\Sigma, X)$ is typically singular even for a smooth X. Its singularities are located at the described strata.

The Chern class $c_1(\mu)$ of the line bundle μ, that defines an embedding $X \to \mathbf{P}^N$ is an element in $H^2(X)$. The degree d of the map $\lambda : \Sigma \to X$ is the value of the pairing $< c_1(\mu), [\Sigma] >$. In theory $\mathrm{Maps}^d(\Sigma, X)$ might contain components of different dimension. The (virtual) dimension of $\mathrm{Maps}^d(\Sigma, X)$ can be computed with an aid of RiemannRoch formula:

$$\mathrm{vdim} = \dim(X) + < c_1(T_X), [\Sigma] >, < c_1(\mu), [\Sigma] >= d$$

If it is possible to find two $[\Sigma_1], [\Sigma_2]$ such that $< c_1(T_X), [\Sigma_1] > \neq < c_1(T_X), [\Sigma_2] >$ and $< c_1(\mu), [\Sigma_1] >=< c_1(\mu), [\Sigma_2] >= d$, then reducibility of $CQ\mathrm{Maps}^d(\Sigma, X)$ is questionable. If the first Chern class $c_1(T_X)$ of X is very large negative, then the space $Q\mathrm{Maps}^d(\Sigma, X)$ set-theoretically consists of one stratum $X \times \mathrm{Sym}^d(\Sigma)$. We believe that this unusually degenerate structure of the space can be a source of anomalies in $\beta\gamma$-system on the cone CX.

The spaces of quasimaps to compact homogenous spaces have an interpretation of semi-infinite closed Schubert cells of a suitable semi-infinite flag variety of an affine Lie algebra $\hat{\mathfrak{g}}$ (see e.g. [7]). Properties of algebras of homogeneous functions on closed Schubert cells in finite dimensional partial flag spaces of finite dimensional semi-simple $\mathfrak{g}$ were under intensive scrutiny. Its Koszul property was established in [18] [6] [5]. Thus Koszul property of its semi-infinite analogs is expected and has been proved for quasimaps to Grassmannian [20] and to Lagrangian Grassmannian [19].

Returning to the case of quasimaps to a quadric we note that the algebra of functions AQ (defined in the Introduction) carries several gradings. The most coarse is defined as the degree in variables $\lambda^i[k]$. Thus

$$\deg \prod_s (\lambda^{i_s}[k_s])^{\alpha_s} = \sum_i \alpha_s$$

Then AQ can be decomposed into a direct sum

$$AQ = \bigoplus_{i \geq 0} AQ_i, \tag{8}$$

where AQ_i is a linear space of elements of degree i.

Our plan is to compute generating function (Poincaré series)

$$AQ^{N_2}_{-N_1}(t) = \sum_{i \geq 0} \dim AQ^{N_2}_{-N_1,i} t^i$$

using the example of a quadric as a guide. We want to stress that our method has been extracted (with suitable modifications) from works [20] [19].

3.1 Quasi-maps to even-dimensional quadric

Let us consider an algebra of polynomials $\mathbb{C}[f^i[s], g_j[t]]$ with $1 \leq i, j \leq n (n \geq 1)$, $-N_1 \leq s, t \leq N_2, N_1, N_2 \geq 0$. The algebra

$$AQ = AQ^{N_2}_{-N_1}(2n) = \mathbb{C}[f^i[s], g_j[t]]/(r[-2N_1], \ldots, r[2N_2])$$

is a quotient of a polynomial algebra by the ideal generated by relations

$$r[l] = \sum_{s+t=l} \sum_{i=1}^{n} f^i[s] g_i[t] = 0 \quad -2N_1 \leq l \leq 2N_2 \tag{9}$$

To make a connection with the previous section we note that $f^i[s], g_j[t]$ are Taylor coefficients (after multiplication on z^{N_1}) of the coordinate functions of a polynomial map $\lambda(z)$ from $\mathbb{C}^*$ to $V = W + W^*$. The relations $r[l]$ are obtained by prescription (7) from the equation of the quadric.

Dimension of V will be greater or equal to six to the end of the section. The generating space AQ_1 of AQ (8) can be identified with a subspace of $V \otimes \mathbb{C}[z, z^{-1}]$:

$$g_i[t] \to g_i \otimes z^t$$
$$f^i[t] \to f^i \otimes z^t$$

Operators R, R_i and the elements of the Cartan subalgebra from Section 2.1 act on AQ_1. An additional not everywhere defined on AQ_1 operator $\hat{R}$ will be proved useful. The space $V \otimes \mathbb{C}[z, z^{-1}]$ is a representation ρ of the Lie algebra $\mathfrak{so}_{2n} \otimes \mathbb{C}[z, z^{-1}]$. Under identification $\mathfrak{so}_{2n} \otimes \mathbb{C}[z, z^{-1}] \cong \Lambda^2(V) \otimes \mathbb{C}[z, z^{-1}]$ operator $\hat{R}$ coincides with

$$\rho(\hat{r}) = \rho(f^{n-1} \wedge f^n \otimes z). \tag{10}$$

It acts by the formula

$$\hat{R} f^i[t] = 0 \quad \hat{R} g_i[t] = \begin{cases} f^n[t+1] & \text{if } i = n-1, \\ -f^{n-1}[t+1] & \text{if } i = n, \\ 0 & \text{if } i \neq n-1, n \end{cases}$$

A reader familiar with theory of affine Lie algebras will immediately recognize in $r, r_i, \hat{r}$ positive simple root vectors of $\hat{\mathfrak{so}}_{2n}$.

Mimicking considerations of a quadric we can use operators $R, \hat{R}, R_i$ and one-dimensional spaces $\langle g_i[t] \rangle, \langle f^j[t] \rangle$ to define the following affine diagram :

$$
\begin{tikzcd}
 & & \langle f^1[t]\rangle \arrow[dr, "R"] & & \\
\hat{\mathcal{G}}(2n):\cdots \arrow[r, "R_2"] & \langle f^2[t]\rangle \arrow[ur, "R_1"] \arrow[dr, "R"'] & & \langle g_2[t]\rangle \arrow[r, "R_2"] & \cdots \\
 & & \langle g_1[t]\rangle \arrow[ur, "R_1"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \langle f^n[t+1]\rangle \arrow[dr, "R_{n-1}"] & & \\
\cdots \arrow[r, "R_{n-2}"] & \langle g_{n-1}[t]\rangle \arrow[ur, "\hat{R}"] \arrow[dr, "R_{n-1}"'] & & \langle f^{n-1}[t+1]\rangle \arrow[r, "R_{n-2}"] & \cdots t\in\mathbb{Z} \\
 & & \langle g_n[t]\rangle \arrow[ur, "\hat{R}"'] & &
\end{tikzcd}
$$

This diagram defines an infinite poset

$$\hat{\mathcal{G}} = \hat{\mathcal{G}}(2n) = \{e_\alpha[t] | e_\alpha \in \mathcal{G}(2n), t \in \mathbb{Z}\}.$$

A finite sub-poset $\mathcal{G}^{N_2}_{-N_1}$ labels subspaces of the first component of $AQ^{N_2}_{-N_1}$. Alternatively, poset $\mathcal{G}^{N_2}_{-N_1}$ is an interval

$$G^{N_2}_{-N_1} = [\langle f^n[-N_1]\rangle, \langle g_n[N_2]\rangle] \overset{\text{def}}{=} \{\gamma | \langle f^n[-N_1]\rangle \leq \gamma \leq \langle g_n[N_2]\rangle\} \tag{11}$$

The equations (9) can be rewritten in a form similar to (3)

$$\begin{aligned}
&g_1[t]f^1[t] = -g_2[t]f^2[t] - \sum_{3\leq i\leq n} g_i[t]f^i[t] - \sum_{s\neq 0}\sum_{i=1}^{n} g_i[t+s]f^i[t-s] \\
&g_n[t]f^n[t+1] = -g_{n-1}[t]f^{n-1}[t+1] - \sum_{1\leq i\leq n-2} g_i[t]f^i[t+1] - \\
&-\sum_{s\neq 0}\sum_{i=1}^{n} g_i[t+s]f^i[t+1-s]
\end{aligned} \tag{12}$$

Index s in the summations belongs to a maximal subset of integers such that all monomials are elements of $AQ^{N_2}_{-N_1}$

If we label uniformly generators e_α of $AQ^{N_2}_{-N_1}$ by elements of $\mathcal{G}^{N_2}_{-N_1}$, then relations (12) would have a form (4).

Proposition 8 *Suppose $n \geq 3$. Introduce a total order on $\hat{\mathcal{G}}(2n)$, which is a refinement of the partial order. We set $\langle f^1[t]\rangle$ to be grater then $\langle g_1[t]\rangle$ and*

$\langle f^n[t+1]\rangle$ to be grater then $\langle g_n[t]\rangle$. Then the generator (3) of the ideal of relations defines a Gröbner basis of the ideal with respect to the degree-lexicographic order on monomials.

Proof. A short introduction Gröbner bases technique is given in Appendix B. Here we follow notations of that section.

We need to compute appropriate S-polynomials. Note that the leading monomials of relations $\{r[l]\}$ (12) are $g_1[t]f^1[t]$ and $g_n[t]f^n[t+1]$. If $n \geq 3$, then these monomials are relatively prime in the semigroup generated by $g_i[s], f^j[t]$. By Lemma (44) the set of S-polynomials, that we have to compute in order to find $Gr_1(I(r[l]))$, is empty. Thus $Gr(I(r[l])) = \{r[l]\}$.

■

One can prove the following theorem

Theorem 9 *The image of the set of $\mathcal{G}^{N_2}_{-N_1}(2n)$-monomials in $QA^{N_2}_{-N_1}(2n)$ is a basis. Thus $QA^{N_2}_{-N_1}(2n)$ is an algebra with a straightened law.*

Proof. We can interpret results of the previous proposition as follows. The semigroup ideal $T(\{r[l]\})$ coincides with the set of elements divisible by $g_1[t]f^1[t]$, $g_n[t]f^n[t+1]$. These are precisely non-$\mathcal{G}^{N_2}_{-N_1}$-monomials. The complement of non-$\mathcal{G}^{N_2}_{-N_1}(2n)$-monomials in the semigroup of monomials is the set of $\mathcal{G}^{N_2}_{-N_1}(2n)$-monomials. By Theorem 32 item 1 the image of this set in $QA^{N_2}_{-N_1}(2n)$ is a basis.

■

3.1.1 Digression about quadratic and Koszul algebras

Recall, that a graded (not necessarily commutative) algebra $A = \bigoplus_{n\geq 0} A_n$ is a quadratic if $A_0 = \mathbb{C}$, $W = A_1$ generates A and all relations follow from quadratic relations $\sum_{i,j} r^k_{ij} x^i x^j = 0$ where $x^1, \ldots, x^{\dim W}$ is a basis of $W = A_1$. The space of quadratic relations R. is the subspace of $W \otimes W$ spanned by $r^1_{ij}, r^2_{ij}, \ldots)$. Then $A_2 = W \otimes W/R$ and A is a quotient of free algebra (tensor algebra) $\bigoplus_{n\geq 0} W^{\otimes n}$ with respect to the ideal generated by R. The dual quadratic algebra $A^!$ is defined as a quotient $\bigoplus_{n\geq 0} (W^*)^{\otimes n} / I(R^\perp)$. The ideal $I(R^{\perp})$ of the free algebra $\bigoplus_{n\geq 0}(W^*)^{\otimes n}$ is generated by $R^{\perp}\subset W^*\otimes W^*$ (here $R^{\perp}$ stands for the subspace of $W^*\otimes W^*=(W\otimes W)^*$ that is orthogonal to $R\subset W\otimes W$).

The case of commutative A has its specifics. Quadratic relations of algebra A contains commutators $x^ix^j - x^jx^i$. An easy computation shows that $R^{\perp}$ contains only anti-commutators. We can interpret $A^!$ as a universal enveloping of some Lie algebra L. In the following we will refer to L as to Koszul dual to commutative A.

Let $\mathfrak{g}$ be a super Lie algebra with structure constants $f^{\gamma}_{\alpha\beta}$ in basis b_{α}. We equip algebra of functions on generators c^{α} dual to b_{α}, having the reverse parity an operator d. It acts by the formula

$$d = f^{\gamma}_{\alpha\beta}c^{\alpha}c^{\beta}\frac{\partial}{\partial c^{\gamma}} = f^{\gamma}_{\alpha\beta}c^{\alpha}c^{\beta}b_{\gamma}$$

and satisfies $d^2=0$.

This formula reminiscences BRST differential in the gauge theory. In mathematical literature (see e.g. [8]) the complex $(\mathbb{C}[c^{\alpha}],d)$ is known as Cartan-Chevalley complex of Lie super-algebra $\mathfrak{g}$ and is usually denoted by $C(\mathfrak{g}) = C^{\bullet}(\mathfrak{g})$.

We will give a definition of Koszul algebra that is most suited to our purposes. Other definitions and their equivalence are discussed in [16].

Definition 10 *A quadratic commutative algebra A is called Koszul if the cohomology $H(L)$ of the complex $C(L)$ for Koszul dual L is equal to A.*

Theorem 11 *The algebra $QA^{N_2}_{-N_1}(2n)$ $(n\geq 3)$ is Koszul .*

Proof. Follows from straightened law property of the algebra $QA^{N_2}_{-N_1}(2n)$ (see [13]). ■

3.2 Quasi-maps to odd-dimensional quadric

The relevant algebra of polynomials is $\mathbb{C}[f^i[s],g_j[t],h[k]]$ with $1\leq i,j\leq n (n\geq 1)$, $-N_1\leq s,t,k\leq N_2, N_1,N_2\geq 0$. The algebra

$$AQ = AQ^{N_2}_{-N_1}(2n+1) = \mathbb{C}[f^i[s],g_j[t],h[k]]/(r[-2N_1],\dots,r[2N_2]) \tag{13}$$

is a quotient by the relation ideal

$$r[l] = \sum_{s+t=l} \left(\sum_{i=1}^{n} f^i[s] g_i[t] \right) + h[s]h[t] = 0 \quad -2N_1 \le l \le 2N_2 \tag{14}$$

The element $\hat{r}$ (see equation 10) gets modified as follows

$$\hat{r} = f^{n-1} \wedge f^n \otimes z$$

This enables us to find the corresponding periodic (affine) Hasse diagram ($n \geq 2$)

$$\hat{\mathcal{G}}(2n+1) : \cdots \xrightarrow{R_1} \langle f^1[t] \rangle \xrightarrow{R} \langle h[t] \rangle \xrightarrow{R} \langle g_1[t] \rangle \xrightarrow{R_1} \cdots$$

$$\begin{tikzcd}
 & & \langle f^n[t+1] \rangle \arrow[dr, "R_{n-1}"] & & \\
\cdots \arrow[r, "R_{n-2}"] & \langle g_{n-1}[t] \rangle \arrow[ur, "\hat{R}"] \arrow[dr, "R_{n-1}"'] & & \langle f^{n-1}[t+1] \rangle \arrow[r, "R_{n-2}"] & \cdots t \in \mathbb{Z} \\
 & & \langle g_n[t] \rangle \arrow[ur, "\hat{R}"'] & &
\end{tikzcd}$$

We can construct a weak lattice (see Definition 5) from the above diagram

$$\hat{\mathcal{G}}(2n+1) = \{ e_\alpha[t] | e_\alpha \in \mathcal{G}(2n+1), t \in \mathbb{Z} \}$$

Construction is the same as in the even case, except that non-reflexive elements $h[t]$ are present. We define $\hat{\mathcal{G}}_{-N_1}^{N_2}(2n+1)$ by the formula (11).

Relations (14) can be put in the form

$$\begin{aligned}
&(h[t])^2 = -g_1[t]f^1[t] - \\
&-\sum_{i=2}^{n} g_i[t] f^i[t] \\
&-\sum_{s \neq 0} \left(h[t+s]h[t-s] + \sum_{i=1}^{n} g_i[t+s] f^i[t-s] \right) \\
&g_n[t] f^n[t+1] = -g_{n-1}[t] f^{n-1}[t+1] - \\
&-\sum_{i=1}^{n-2} g_i[t] f^i[t+1] - \\
&-\sum_{s \neq 0} \left(h[t+1+s]h[t-s] + \sum_{i=1}^{n} g_i[t+s] f^i[t+1-s] \right)
\end{aligned} \tag{15}$$

Proposition 12 *Suppose $n \geq 2$. Introduce a total order on $\hat{\mathcal{G}}^{N_2}_{-N_1}(2n+1)$, which is a refinement of the partial order. We set $\langle f^1[t]\rangle < \langle h[t]\rangle < \langle g_1[t]\rangle$ and $\langle f^n[t+1]\rangle$ to be grater then $\langle g_n[t]\rangle$. Then the generators (15) of the ideal of relations defines a Gröbner basis of the ideal with respect to the degree-lexicographic order on monomials.*

Proof. It is similar to the proof of Proposition 9.

We need to compute appropriate S-polynomials. Note that the leading monomials of relations $\{r[l]\}$ (12) are $h[t]^2$ and $g_n[t]f^n[t+1]$. If $n \geq 1$, then these monomials are relatively prime in the semigroup generated by $g_i[s], f^j[t], h[k]$. By Lemma (44) the set of S-polynomials, that we have to compute in order to find $Gr_1(I(r[l]))$, is empty. Thus $Gr(I(r[l])) = \{r[l]\}$.

■

As a corollary we get the following

Theorem 13 *The image of the set of $\hat{\mathcal{G}}^{N_2}_{-N_1}(2n+1)$-monomials in $QA^{N_2}_{-N_1}(2n+1)$ is a basis.*

Theorem 14 *$QA^{N_2}_{-N_1}(2n+1)$ is a Koszul algebra.*

Proof. Formally $QA^{N_2}_{-N_1}(2n+1)$ is not an algebra with straightened law. However the proof of [13] can be easily adapted to our case.

Alternatively the basis of $\hat{\mathcal{G}}^{N_2}_{-N_1}(2n+1)$ monomials is Poincaré-Birkhoff-Witt basis (see [16] for definition). Priddy in [17] have proven that any quadratic PBW-algebra is Koszul. ■

4 Generating functions

Theorem 9 enables us to find various generating function related to algebra $QA^{N_2}_{-N_1}$, besides $QA^{N_2}_{-N_1}(t)$. The group of obvious SO(n)-symmetries of algebra $QA^{N_2}_{-N_1}$ can be extended to SO(n) $\times$ SL(2). The action of SL(2) factor is explained in Section 3.

It makes sense to define a generating function of $\mathrm{SO}(n) \times \mathrm{SL}(2)$-characters $QA(g_1, g_2, t)$. This formal function is completely determined by its restriction $QA(z_1, \ldots, z_{[\frac{n}{2}]}, q, t)$ on the maximal complex torus $\mathbb{C}^{\times [\frac{n}{2}]} \times \mathbb{C}^{\times} \subset \mathrm{SO}(n) \times \mathrm{SL}(2)$.

For simplicity of exposition we let $z_i = 1$. Theorem 9 reduces computation of $QA(q, t)$ to combinatorics. We can associate to any k-tuple

$$m = (e_{\alpha_1}[l_1], \ldots, e_{\alpha_k}[l_k]) \in \bigcup_{k \geq 1} \hat{\mathcal{G}}^{\times k} / \Sigma_k$$

a local weight

$$w(m) = \prod_{s=1}^{k} q^{l_s}$$

and a function $\deg(m) = k$. Let E be some set of m. We define a generating function

$$\hat{E}(t) = \sum_{m \in E} w(m) t^{\deg(m)}.$$

The direct corollary of Theorem 9 is that

$$QA^{N_2}_{-N_1}(q, t) = T^{N_2}_{-N_1}(q, t),$$

where $T^{N_2}_{-N_1}$ is a subset of $\bigcup_{k \geq 1} \hat{\mathcal{G}}^{N_2 \ \times k}_{-N_1} / \Sigma_k$ which consists of all $\hat{\mathcal{G}}^{N_2}_{-N_1}$-elements.

Even-dimensional quadric

We will be interested in the intervals

$$\Delta_l = [f^{n-2}[l], f^{n-1}[l+1]].$$

Then $\bigcup_l \Delta_l = \hat{\mathcal{G}}$. Smaller subintervals will also be used:

$$\Delta'_l = [f^n[l], f^{n-1}[l]]$$

$$\Delta''_l = [f^{n-2}[l], g_n[l]]$$

The set $\hat{\mathcal{G}}^{N_2}_{-N_1}$ is a union

$$\hat{\mathcal{G}}^{N_2}_{-N_1} = \Delta'_{-N_1} \cup \left(\bigcup_{l=-N_1}^{N_2-1} \Delta_t \right) \cup \Delta''_{N_2}$$

The last equality implies that

$$T^{N_2}_{-N_1} = T(\Delta'_{-N_1}) \times \left(\prod_{l=-N_1}^{N_2-1} T(\Delta_l) \right) \times T(\Delta''_{N_2})$$

This allows us to write a formula for generating function

$$T_{-N_1}^{N_2}(q,t) = T(\Delta'_{-N_1})(q,t)T(\Delta''_{N_2})(q,t)\prod_{l=-N_1}^{N_2-1} T(\Delta_l)(q,t)$$

It is not hard to find formulas for local factors:

$$T(\Delta'_l)(q,t) = \frac{1}{(1-q^l t)^2}$$

The formula for $T(\Delta''_l)(q,t)$ and $T([f^{n-2}[l], g_2[l]])(q,t)$ can be obtained along the same lines as for $A(t)$:

$$T(\Delta''_l)(q,t) = \frac{1-q^{2l}t^2}{(1-q^l t)^{2n-2}}$$

$$T([f^{n-2}[l], g_2[l]])(q,t) = \frac{1-q^{2l}t^2}{(1-q^l t)^{2n-(2+n-2)}} = \frac{1-q^{2l}t^2}{(1-q^l t)^n}$$

Finally $\Delta_l = [f^{n-2}[l], g_2[l]] \cup [g_3[l], f^{n-1}[l+1]]$. Using inclusion-exclusion principle we get

$$T([g_3[l], f^{n-1}[l+1]])(q,t) = T([g_3[l], g_{n-2}[l+1]])(q,t)T([g_{n-1}[l], f^{n-1}[l+1]])(q,t) =$$
$$\frac{1}{(1-q^l t)^{n-4}}\left(\frac{1}{(1-q^l t)(1-q^l t)(1-q^{l+1}t)} + \frac{1}{(1-q^l t)(1-q^{l+1}t)(1-q^{l+1}t)} - \frac{1}{(1-q^l t)(1-q^{l+1}t)}\right) =$$
$$= \frac{1}{(1-q^l t)^{n-4}}\frac{1-q^{2l+1}t^2}{(1-q^l t)^2(1-q^{l+1}t)^2}$$

Finally

$$T_{-N_1}^{N_2}(q,t) = \frac{1-q^{2N_2}t^2}{(1-q^{-N_1}t)^2(1-q^{N_2}t)^{2n-2}}\prod_{l=-N_1}^{N_2-1}\frac{(1-q^{2l}t^2)(1-q^{2l+1}t^2)}{(1-q^l t)^n(1-q^l t)^{n-4}(1-q^l t)^2(1-q^{l+1}t)^2} =$$
$$= \frac{\prod_{l=-2N_1}^{2N_2}(1-q^l t^2)}{\prod_{l=-N_1}^{N_2}(1-q^l t)^{2n}}$$

Odd-dimensional quadric

The proof of the formula

$$T_{-N_1}^{N_2}(q,t) = \frac{\prod_{l=-2N_1}^{2N_2}(1-q^l t^2)}{\prod_{l=-N_1}^{N_2}(1-q^l t)^{2n+1}}, n \geq 1$$

is similar to even-dimensional case and is left to the interested reader as an exercise. We have proved a

Proposition 15 *The generating function* $AQ^{N_2}_{-N_1}(n)(q,t)$, $n \geq 5, N_1, N_2 \geq 0$ *is*

$$\frac{\prod_{l=-2N_1}^{2N_2}(1-q^l t^2)}{\prod_{l=-N_1}^{N_2}(1-q^l t)^n}$$

We will treat the remaining $n = 2, 3, 4$ cases in the appendix

5 The mini-BRST resolution

A special interest is the value of $T^{N_2}_{-N_1}(q,t)$ at $q = 1$:

$$T^{N_2}_{-N_1}(1,t) = \frac{(1-t^2)^{2(N_1+N_2)+1}}{(1-t)^{n(N_1+N_2+1)}} \tag{16}$$

We know that algebras $AQ = AQ^{N_2}_{-N_1}(n)$ $n \geq 5, N_1, N_2 \geq 0$ Koszul . The Koszul dual algebra $\left(AQ^{N_2}_{N_1}(n)\right)^!$ (the universal enveloping algebra of a graded Lie algebra L, described in Section 3.1.1) has (see [16]) the generating function

$$1/T^{N_2}_{N_1}(1,-t) = \frac{(1+t)^{n(N_1+N_2+1)}}{(1-t^2)^{2(N_1+N_2)+1}}. \tag{17}$$

For this Lie algebra we introduce a special notation:

$$LH = LH^{N_2}_{-N_1}(n) = \bigoplus_{k\geq 0} LH_k$$

It has finite-dimensional graded components LH_k (we suppress dependence on n, N_1, N_2).

The formal power series satisfy

$$AQ(-t)^{-1} = \frac{\prod_{k\geq 0}(1+t^{2k+1})^{\dim LH_{2k+1}}}{\prod_{k\geq 1}(1-t^{2k})^{\dim LH_{2k}}}$$

The last identity represents numeric version of Poincaré-Birkhoff-Witt theorem. What is important is that this identity allows unambiguously recover dimensions (and $\mathrm{SO}(n) \times \mathrm{SL}(2)$ characters) of LH_k. Analyzing formula (17) we conclude that L_1 has dimension $n(N_1 + N_2 + 1)$, L_2 has dimension $2(N_1 + N_2) + 1$ and all other graded components vanish.

We will describe Lie algebra $L'H^{N_2}_{-N_1}$. Let H be an odd graded Heisenberg algebra. It is generated by linear space V in degree one. It also contains a central element b in degree two, which satisfies

$$[v_1, v_2] = (v_1, v_2)b \in H_1. \tag{18}$$

The bracket stands for the graded commutator (in our case anti-commutator). We define the loop algebra $\hat{H} = H \otimes \mathbb{C}[z, z^{-1}]$. The algebra $\hat{H}$ contains a subalgebra

$$L'H^{N_2}_{-N_1} = \bigoplus_{l=-N_1}^{N_2} V \otimes z^l + \bigoplus_{l=-2N_1}^{2N_2} \langle b \rangle \otimes z^l$$

Proposition 16 *The universal enveloping* $U(L'H^{N_2}_{-N_1}(n))$ *is isomorphic to* $\left(AQ^{N_2}_{-N_1}(n)\right)^!$

Proof.

The space of generators $LH = LH^{N_2}_{-N_1}(n)$ is dual to the corresponding space of $AQ^{N_2}_{-N_1}(n)$ which is equal to $AQ_1 = \bigoplus_{l=-N_1}^{N_2} V \otimes z^l$. Thus $LH_1 = L'H_1 = \bigoplus_{l=-N_1}^{N_2} V \otimes z^l$ (we identify V with its dual by means of inner product). It is known (see [16]) that the space of elements of degree two LH_2 is dual to the space of relations of $AQ^{N_2}_{-N_1}(n)$ (1). The later can be identified with $\bigoplus_{l=-2N_1}^{2N_2} \langle b \rangle \otimes z^l$.

The Lie algebra LH is generated by LH_1 (this is equivalent to Koszul property of $AQ^{N_2}_{-N_1}(n)$ [16]). The only possible nonzero $\mathrm{SO}(n) \times \mathrm{SL}(2)$-invariant graded commutator

$$\left(\bigoplus_{l=-N_1}^{N_2} V \otimes z^l \right)^{\otimes 2} \to \bigoplus_{l=-2N_1}^{2N_2} \langle b \rangle \otimes z^l$$

is given (up to a constant factor) by the formula (18). ■

Proof of Theorem 1

Let us now put all our ingredients together. The algebra $C = \mathbb{C}[\lambda^i[s]] \otimes \Lambda[c[k]]$ $i = 1, \dots, n, s = -N_1, \dots, N_2, k = -2N_1, \dots, 2N_2$ has two interpretations. In the first it is a truncated version of BRST complex from Section 1. In the second interpretation it is a Cartan-Chevalley complex of Lie algebra $LH^{N_2}_{-N_1}(n)$ $n \geq 5$, whose cohomology is equal to $AQ^{N_2}_{-N_1}$. The quasi-isomorphism

$\zeta : C \to AQ$ is defined on generators by the formula

$$\zeta(\lambda^i[s]) = \lambda^i[s], \zeta(c[s]) = 0.$$

The exceptional cases $n = 3, 4$ are handled in Appendix. ■

As a side remark we note that algebraic variety $Spec(AQ^{N_2}_{-N_1}(n))$ is a complete intersection for the above range of parameters N_1, N_2, n.

6 The Hilbert space

One possible approach to the problem of defining Hilbert space in $\beta\gamma$-system on a quadric [1] utilizes a technique of semi-infinite cohomology. It coincides with quantum mini-BRST complex. Let W be a vector space, decomposed into a direct sum $W = W_+ \oplus W_-$. Any operator $a : W \to W$ has a block-form $a = \left(\begin{smallmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{smallmatrix}\right)$.

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a graded superalgebra. Define linear spaces

$$U_- = \bigoplus_{i \leq 0} \mathfrak{g}_i, U_+ = \bigoplus_{i \geq 1} \mathfrak{g}_i$$

A linear space W_- is commensurable with U_- if projection on U_- has finite dimensional kernel and co-kernel. There are also U_+ commensurable liner spaces.

The following is a slight modification of a definition taken from [21].

Definition 17 *Let $\mathfrak{g}$ be a (super) Lie algebra over the field of complex numbers. We say that $\mathfrak{g}$ is provided with a semi-infinite structure if the following data 1 and 2 are given and condition 3 is satisfied:*

1. *a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ on $\mathfrak{g}$ (that implies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for each i, j), such that $dim \mathfrak{g}_i < \infty$ for each i.*

2. *a 1-cochain β on $\mathfrak{g}$, such that its coboundary $\partial\beta = \mathrm{ad}^*\phi$, ϕ being the 2-cocycle on $\mathfrak{g}$. The two-cocycle $\phi(a, b)$ is defined b the formula*

$$\phi(a, b) = \mathrm{tr}(ad(b)_{-+} ad(a)_{+-} - ad(a)_{-+} ad(b)_{+-})$$

Here tr stands for the (super) trace. Operators $a_{\pm\pm}, b_{\pm\pm}$ are components of adjoint operators computed with respect to decomposition

$$\mathfrak{g} = W_- + W_+,$$

where $W_-(W_+)$ are $U_-(U_+)$ commensurable liner spaces

3. *the cochain β vanishes on $\mathfrak{g}_i$ for all $i \neq 0$*

The Lie algebra $LH = LH^{N_2}_{-N_1}$ is graded by the powers in z. Also it has a U_-, U_+ commensurable polarization

$$W_+ = LH_{\geq 1} = \bigoplus_{l=1}^{N_2} V \otimes z^l + \bigoplus_{l=2}^{2N_2} \langle b \rangle \otimes z^l$$

$$W_- = LH_{\leq 0} = \bigoplus_{l=-N_1}^{0} V \otimes z^l + \bigoplus_{l=-2N_1}^{1} \langle b \rangle \otimes z^l$$

The space $LH + LH^*$ has a canonical symmetric inner product, which enables us to define a Clifford algebra $Cl(LH + LH^*)$. The space $LH + LH^*$ contains an isotropic subspace $LH_{\geq 1} + (LH_{\geq 1})^{\perp} \cong LH_{\geq 1} + (LH_{\leq 0})^*$ (we use the graded dual). We use it to construct the Fock space. The Fock space is the Clifford module $\Lambda^{\frac{\infty}{2}}$ generated by a vacuum vector ω that is annihilated by $LH_{\geq 1} + (LH_{\leq 0})^*$. It is called a module of semi-infinite forms. Let ψ_α be a basis in LH and $\psi^{*\alpha}$ be the dual basis LH^*. Denote by $c^{\gamma}_{\alpha\beta}$ the structure constants of LH :

$$[\psi_\alpha, \psi_\alpha] = c^{\gamma}_{\alpha\beta} \psi_\gamma$$

Proposition 18 *The element $d = c^{\gamma}_{\alpha\beta} \psi^{*\alpha} \psi^{*\beta} \psi_\gamma$ in Clifford algebra satisfies $d^2 = 0$.*

Proof. By general theory (see [21]) the operator d^2, acting in $\Lambda^{\frac{\infty}{2}}$, coincides with the action of operator of multiplication on $\phi(a, b)$, thought of as an element of Clifford algebra. Note that LH is a nilpotent Lie algebra with all double commutators $[a[a', a'']]$ equal to zero. Furthermore $[a, ad(a')_{\pm\pm}(a'')] = 0$. From this we conclude that $ad(a')_{-+} ad(a)_{+-} - ad(a)_{-+} ad(a')_{+-} = 0$ and the cocycle vanishes identically. ■

The operator d defines a differential in $\Lambda^{\frac{\infty}{2}}$. Being a cyclic module over $Cl(LH + LH^*)$ the space $\Lambda^{\frac{\infty}{2}}$ has a $\mathbb{Z}$ grading

$$\Lambda^{\frac{\infty}{2}} = \bigoplus_{i\in\mathbb{Z}} \Lambda^{\frac{\infty}{2}+i}.$$

Creation operators (operators of multiplication on $a \in LH^*$) increase this grading by one, annihilation operators (operators of multiplication on $a' \in LH$) decrease it by one. The space $\Lambda^{\frac{\infty}{2}}$ is isomorphic to the tensor product

$$\Lambda(LH_{\geq 1})^* \otimes \Lambda(LH_{\leq 0}) \otimes \omega$$

of graded exterior powers.

The generating function of euler characters of this complex is

$$\frac{\prod_{l=0}^{2N_2}(1-q^l t^2)}{\prod_{l=0}^{N_2}(1-q^l t)^n} \frac{\prod_{l=1}^{2N_1}(1-q^l t^{-2})}{\prod_{l=1}^{N_1}(1-q^l t^{-1})^n}$$

with an assumption that the grading of vacuum is zero.

The limiting function $N_1, N_2 \to \infty$

$$Z(q,t) = \frac{1-t^2}{(1-t)^n} \prod_{l=1}^{\infty} \frac{(1-q^l t^2)(1-q^l t^{-2})}{(1-q^l t)^n (1-q^l t^{-1})^n}$$

satisfies equations

$$Z(q,t^{-1}) = -(-t)^{n-2} Z(q,t)$$

$$Z(q,qt) = (-1)^n t^{n-4} q^{-1} Z(q,t)$$

which can be established by simple manipulation with the product. Also

$$Z(q,qt^{-1}) = -t^2 q^{-1} Z(q,t),$$

which is a corollary of the previous equations. The same formulas has been found in [1].

One of the results of Voronov [21] is that there is a spectral sequence, that converges to semi-infinite cohomology.

Theorem 19 *For a Lie algebra $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$ with a semi-infinite structure and a $\mathfrak{g}$-module $M \in \mathcal{O}$, there exists a spectral sequence*

$$\{E_r^{p,q}, d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1} | p \leq 1, q \geq 0, r \geq 0\} \tag{19}$$

with the following properties:

1. $E_1^{p,q} = H^q(\mathfrak{n}, \Lambda^{\frac{\infty}{2}+p}(\mathfrak{g}/\mathfrak{n})^* \otimes M)$.

2. *If* $\mathfrak{n}$ *is an ideal, then* $E_2^{p,q} = H^{\frac{\infty}{2}+p}(\mathfrak{g}/\mathfrak{n}, H^q(\mathfrak{n}, M)) = H_{-p}(\mathfrak{g}/\mathfrak{n}, H^q(\mathfrak{n}, M) \otimes \Lambda^{\frac{\infty}{2}+0}(\mathfrak{g}/\mathfrak{n})^*)$*. As usual* $H_i(\mathfrak{g})$ *stands for Lie algebra homology .*

3. $E_\infty^{p,q} = \mathrm{gr}^p H^{\frac{\infty}{2}+p+q}(\mathfrak{g}, M)$.

4. *the differentials* d_r*, induce a sequence of epimorphisms* $E_r^{p,q} \to E_r^{p,q} \to \cdots$ *for* r *large enough, so that* $\lim\limits_{\to} E_r^{p,q} = E_\infty^{p,q}$ *.*

The theorem admits an obvious modification for super-algebras. The one dimensional $LH_{\geq 1}$ module $\Lambda^{\frac{\infty}{2}+0}(\mathfrak{g}/\mathfrak{n})^* = \Lambda^{\frac{\infty}{2}+0}(LH/LH_{\geq 1})^*$ (with respect to adjoint action) is trivial. Hence

$$\Lambda^{\frac{\infty}{2}-p}(LH/LH_{\geq 1})^* \cong \Lambda^p(LH/LH_{\geq 1})$$

The sequence adapted to our needs has a form:

$$H^i(LH_{\geq 1}, \Lambda^j(LH/LH_{\geq 1})) \Rightarrow H^{\frac{\infty}{2}+i-j}(LH, \mathbb{C})$$

There is no convergence issues with this sequence because of the grading induced by $\mathbb{C}^*$ action $z \to az$ on the loop parameter. The sequence breaks down into a direct sum of finite-dimensional complexes according to this grading, for which convergence issues are vacuous.

By construction the space $\Lambda^{\frac{\infty}{2}}(LH)$ is a module over the Koszul complex

$$C = C(LH) = \Lambda(LH)^*.$$

Here we interpret it as a complex of cohomological chains. We already know (Theorem 1) that the homomorphism

$$C = C(LH) \xrightarrow{\zeta} AQ_{-N_1}^{N_2}$$

is a quasi-isomorphism. There is a restriction homomorphism $AQ_{-N_1}^{N_2} \to AQ_1^{N_2}$.We define a complex

$$\Lambda^{\frac{\infty}{2}}(LH)_{red} \stackrel{\mathrm{def}}{=} \Lambda^{\frac{\infty}{2}}(LH) \underset{C(LH)}{\otimes} AQ_1^{N_2}$$

and a homomorphism of $C(LH)$-modules

$$red : \Lambda^{\frac{\infty}{2}}(LH) \to \Lambda^{\frac{\infty}{2}}(LH)_{red},$$

defined by the formula $a \to a \otimes 1$

Proposition 20 *The map red is a quasi-isomorphism.*

Proof. Filtration on $\Lambda^{\frac{\infty}{2}}(LH)$ that defines spectral sequence 19 (see [21] for details) induces a filtration on $\Lambda^{\frac{\infty}{2}}(LH)_{red}$ The map red defines a map of spectral sequences. The map of E_1 terms

$$red : H(LH_{\geq 1}, \Lambda^j(LH/LH_{\geq 1})) \to H(\Lambda^j(LH/LH_{\geq 1}) \otimes AQ_1^{N_2})$$

It is a quasi-isomorphism because $U(LH_{\geq 1}) \cong U(LH_0^{N_2-1})$ is a Koszul algebra. The cohomology of any graded $LH_{\geq 1}$ module M (which in our case is $\Lambda^j(LH/LH_{\geq 1})$) can be computed with the complex $M \otimes U(LH_{\geq 1})^! \cong M \otimes AQ_1^{N_2}$ (see [16] for details). Since E_1 terms of spectral sequences coincide we conclude that red is a quasi-isomorphism. ∎

The space $\left(AQ^0_{-N_1}\right)^* \otimes AQ_1^{N_2}$ will be used to compute cohomology $\Lambda^{\frac{\infty}{2}}(LH)$. We denote this space by $AQ^{\frac{\infty}{2}} = (AQ^{N_2}_{-N_1})^{\frac{\infty}{2}}$. The reader should keep in mind that $\left(AQ^0_{-N_1}\right)^* \otimes AQ_1^{N_2}$ is a module over $AQ^0_{-N_1} \otimes AQ_1^{N_2}$, and operators of multiplication on $\lambda^i[s] \otimes \lambda^i[t]$, $-N_1 \leq s \leq 0, 1 \leq t \leq N_2$ are defined.

Proposition 21 *The cohomology of $\Lambda^{\frac{\infty}{2}}(LH)$ coincide with the cohomology of a two-step complex*

$$AQ^{\frac{\infty}{2}} \xrightarrow{d} AQ^{\frac{\infty}{2}} \tag{20}$$

The map d is defined by the formula

$$d(a) = \sum_{s+t=1, s\leq 0, t\geq 1} \lambda^i[s] \otimes \lambda^i[t]a, a \in AQ^{\frac{\infty}{2}} \tag{21}$$

The indices s, t are in the range defined above.

Proof.

The complex $\Lambda^{\frac{\infty}{2}}(LH)_{red}$ as a linear space is isomorphic to

$$AQ_1^{N_2} \otimes \Lambda(LH_{\leq 0}) \otimes \langle \omega \rangle \cong AQ_1^{N_2} \otimes \Lambda(LH^0_{-N_1}) \otimes (\langle b[1]\omega \rangle + \langle \omega \rangle)$$

In the last isomorphism we used decomposition $LH_{\leq 0} = LH^0_{-N_1} + \langle b[1] \rangle$. Introduce a notation for the graded space

$$\langle b[1]\omega \rangle + \langle \omega \rangle = B^{\frac{\infty}{2}-1} + B^{\frac{\infty}{2}+0}$$

We denote temporally $A = AQ_1^{N_2}$. Recall that A is graded . A larger graded space $B = \bigoplus B^{\frac{\infty}{2}+i}$ is defined as a sum

$$B^{\frac{\infty}{2}+i} = A_i \otimes B^{\frac{\infty}{2}+0} + A_{i+1} \otimes B^{\frac{\infty}{2}-1}$$

Filtration of the graded algebra A

$$F^k = \bigoplus_{k \geq i} A_i$$

defines a filtration on $\Lambda^{\frac{\infty}{2}}(LH)_{red}$. The corresponding spectral sequence has E_1 term :

$$E_1^{i,j} = H_{-i}(LH^0_{-N_1}, \mathbb{C}) \otimes B^{\frac{\infty}{2}+i} \to H^{\frac{\infty}{2}+i+j}(LH)_{red}$$

Homology $H_i(LH^0_{-N_1}, \mathbb{C})$ is dual to cohomology , which we know how to compute. We conclude that $H_\bullet(LH^0_{-N_1}, \mathbb{C}) = \left(AQ^0_{-N_1}\right)^*$. The E_2-term coincides with (20) with the following identifications: element a in the domain of d gets identified with

$$ab[1]\omega$$

in some sub-quotient of $\Lambda^{\frac{\infty}{2}}(LH)$. Likewise the image of d is an element

$$\sum_{s+t=1, s \leq 0, t \geq 1} \lambda^i[s] \otimes \lambda^i[t] a\omega$$

The spectral sequence degenerates in E_3 term because of the two-step nilpotency of the algebra LH.

■

7 Limits $N_1, N_2 \to \infty$

Our goal is to define limits of $\Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1})$ and study its properties. Only in these limits Virasoro and affine Lie algebra actions emerge . Taking these limits is not straightforward and we shall discuss this presently.

We start with a remark that there are embeddings

$$LH^{N_2}_{-N_1} \subset LH^{N_2'}_{-N_1'}, N_1 \leq N_1', N_2 \leq N_2'$$

It is well known that inclusions define maps in cohomology and homology complexes:

$$C^*(LH^{N_2'}_{-N_1'}) \to C^*(LH^{N_2}_{-N_1})$$
$$C_*(LH^{N_2}_{-N_1}) \to C_*(LH^{N_2'}_{-N_1'})$$

Semi-infinite cohomology is not a functor of the Lie algebra because it shares properties of both homology and cohomology (see [21]). It has a weaker property:

Proposition 22 *There are morphisms of complexes of* $C^*(LH^{N_2}_{-N_1'})$ *and* $C^*(LH^{N_2'}_{-N_1})$ *modules:*

$$\Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1'}) \to \Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1})$$
$$\Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1}) \to \Lambda^{\frac{\infty}{2}}(LH^{N_2'}_{-N_1})$$

$N_1 \leq N_1', N_2 \leq N_2'$

Proof. The first map follows from the isomorphism of differential graded $C(LH^{N_2}_{-N_1'})$-modules

$$\Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1}) \cong C^*(LH^{N_2}_{-N_1}) \underset{C^*(LH^{N_2}_{-N_1'})}{\otimes} \Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1'})$$

The second from

$$\Lambda^{\frac{\infty}{2}}(LH^{N_2'}_{-N_1}) \cong C_*(LH^{N_2'}_{-N_1}) \underset{C^*(LH^{N_2'}_{-N_1})}{\otimes} \Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1})$$

■

We define $\Lambda^{\frac{\infty}{2}}(LH^{\infty}_{-\infty})$ to be a double limit

$$\Lambda^{\frac{\infty}{2}}(LH^{\infty}_{-\infty}) = \lim_{\overleftarrow{N_1}} \lim_{\overrightarrow{N_2}} \Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1})$$

The space $PQ^{\frac{\infty}{2}}$ it the double limit

$$PQ^{\frac{\infty}{2}} = \lim_{\overleftarrow{N_1}} \lim_{\overrightarrow{N_2}} AQ^{0}_{-N_1} \otimes AQ^{N_2 *}_{1}$$

Proposition 23 *The cohomology of $\Lambda^{\frac{\infty}{2}}(LH^{\infty}_{-\infty})$ coincide with cohomology of a two-term complex*

$$PQ^{\frac{\infty}{2}} \xrightarrow{d} PQ^{\frac{\infty}{2}} \tag{22}$$

with the differential as in (21).

Proof. First of all the operator d is well defined because $\lambda^i[t]$ acts trivially on $AQ^{N_2*}_1$ with $t > N_2$.

Secondly, The complexes $\Lambda^{\frac{\infty}{2}}(LH^{\infty}_{-\infty})$ and (22) decompose into a direct product of eigenspaces of the dilation operator. The eigenspaces are finite-dimensional complexes. The spaces of these complexes stabilize for large N_1 and N_2. The cohomology can be computed using appropriate eigenspace of $\Lambda^{\frac{\infty}{2}}(LH^{N_2}_{-N_1})$ and by Proposition 21 coincide with cohomology of the corresponding eigenspaces of (20). The later eigenspaces stabilize for $N_1' \geq N_1, N_2' \geq N_2$. ■

There is a natural isomorphism $AQ^{N_2}_{-N_1} \cong AQ^{N_1}_{-N_2}$ defined by the formula $\lambda(z) \to \lambda(1/z)$ (compare with [1]). It enables us to define an isomorphism

$$\begin{gathered} \left(AQ^{0*}_{-N_1} \otimes AQ^{N_2*}_1\right)^* \cong AQ^{0}_{-N_1} \otimes AQ^{N_2\ *}_1 \cong \\ AQ^{-1}_{-N_1-1} \otimes AQ^{N_2-1\ *}_0 \cong AQ^{0\ *}_{-N_2+1} \otimes AQ^{N_1+1}_1 \end{gathered} \tag{23}$$

We also get a limiting isomorphism of cohomological degree one

$$PQ^{\frac{\infty}{2}} \cong PQ^{\frac{\infty}{2}*}$$

We denote the pairing between linear spaces of the complex (22) by $< \cdot, \cdot >$.

Lemma 24 *The map d (22) satisfies $< d(a), b > = < a, d(b) >$.*

Proof. The key moment is that a shift by one in indices in (23) agrees with the rule $s+t=1$ in (21). ∎

As a corollary we get the following Proposition.

Proposition 25 *The cohomology of the complex* $\Lambda^{\frac{\infty}{2}}(LH^{\infty}_{-\infty})$ *has a non-degenerate Poincaré duality pairing.*

Remark 26 *It is still remains unclear how to compute generating functions of individual cohomology groups of (20) or (22).*

Appendix

A Exceptional cases

A.1 $n=4$

The case of two-dimensional projective quadric ($n=4$) requires additional notations. The Lie algebra of the symmetry group of the quadric is non-simple $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$. Choose generators $a=a_+, b=b_+$ of nilpotent subalgebras in left and right copies of $\mathfrak{sl}_2$. Elements a_-, b_- are such that $(a_+, [a_+,a_-], a_-), (b_+, [b_+,b_-], b_-)$ form $\mathfrak{sl}_2$-triples.[2]Define

$$\hat{a} = a_- \otimes z \in \mathfrak{sl}_2 \otimes \mathbb{C}[z, z^{-1}],$$

$$\hat{b} = b_- \otimes z \in \mathfrak{sl}_2 \otimes \mathbb{C}[z, z^{-1}].$$

It is useful to identify four-dimensional fundamental representation of $\mathfrak{so}_4$ with the tensor product $V = W_l \otimes W_r$, where W_l, W_r are spinorial representations of left and right copies of $\mathfrak{sl}_2$. We choose bases (ξ_1, ξ_2) and (η_1, η_2) of W_l, W_r that consist of weight vectors. In this bases

$$a(\xi_1) = \xi_2, b(\eta_2) = \eta_1$$

Linear spaces W_l, W_r are equipped $\mathfrak{so}_4$-invariant skew-symmetric dot-products $< \cdot, \cdot >$, $[\cdot, \cdot]$ normalized by conditions $<\xi_1, \xi_2>=1$, $[\eta_1, \eta_2]=1$. We set

$$f^2 = \xi_1 \otimes \eta_2, f^1 = \xi_2 \otimes \eta_2$$

[2]Elements (e,h,f) define an $\mathfrak{sl}_2$-triple if $[h,e]=2e, [h,f]=-2f, [e,f]=h$.

$$g_1 = \xi_1 \otimes \eta_1, g_2 = \xi_2 \otimes \eta_1$$

The invariant dot-product on $V = W_l \otimes W_r$ is equal to $< \cdot, \cdot > \otimes [\cdot, \cdot]$. The Hasse diagram $\hat{\mathcal{G}}(2)$ is

$$\begin{array}{ccccccccccc}
\cdots \to & \langle f^2[t] \rangle & \xrightarrow{a} & \langle f^1[t] \rangle & \xrightarrow{\hat{a}} & \langle f^2[t+1] \rangle & \xrightarrow{a} & \langle f^1[t+1] \rangle & \to \dots \\
 & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & \\
\cdots \to & \langle g_1[t] \rangle & \xrightarrow{a} & \langle g_2[t] \rangle & \xrightarrow{\hat{a}} & \langle g_1[t+1] \rangle & \xrightarrow{a} & \langle g_2[t+1] \rangle & \to \dots
\end{array}$$

Additional arrows that did not fit to the diagram are

$$\langle g_2[t] \rangle \xrightarrow{\hat{b}} \langle f^1[t+1] \rangle$$

$$\langle g_1[t] \rangle \xrightarrow{\hat{b}} \langle f^2[t+1] \rangle$$

This diagram *does not* define a lattice because, e.g. supremum of $g_1[t], f^1[t]$ consists of two elements: $g_2[t]$ and $f^2[t+1]$.

The diagram $\mathcal{G}_{-N_1}^{N_2}$ coincides with $[\langle f^n[-N_1] \rangle, \langle g_n[N_2] \rangle]$.

$$g_1[t] f^1[t] = -g_2[t] f^2[t] - g_2[t-1] f^2[t+1] - \left(g_1[t-1] f^1[t+1] + \sum_{s \geq 2} \sum_{i=1}^{2} g_i[t+s] f^i[t-s] \right)$$

$$g_2[t] f^2[t+1] = -g_1[t+1] f^1[t] - g_2[t] f^2[t+1] - \left(g_1[t] f^1[t+1] + \sum_{s \geq 2} \sum_{i=1}^{2} g_i[t+s] f^i[t+1-s] \right) \tag{24}$$

Proposition 27 *Suppose $n = 4$. We refine the partial order on the poset $\hat{\mathcal{G}}(4)$ to the total order as follows:*

$$\cdots < \langle f^2[t] \rangle < \langle g_1[t] \rangle < \langle f^1[t] \rangle < \langle g_2[t] \rangle < \langle f^2[t+1] \rangle < \cdots$$

Then the generators (24) of the ideal of relations defines a Gröbner basis of the ideal with respect to the degree-lexicographic order on monomials.

Proof. The proof is the same as in the general case. ■

As a corollary we get that $AQ_{-N_1}^{N_2}(4)$ is a Hodge algebra([10], [11]).

Proposition 28 *The generating function $AQ_{-N_1}^{N_2}(4)(q, t)$, , $N_1, N_2 \geq 0$ is*

$$\frac{\prod_{l=-2N_1}^{2N_2} (1 - q^l t^2)}{\prod_{l=-N_1}^{N_2} (1 - q^l t)^4}$$

Proof. Exercise. ■

A.2 $\quad n = 3$

The fundamental representation V of $\mathfrak{so}_3 \cong \mathfrak{sl}_2$ coincides with the adjoint . We choose R to be a commutator with g, $\hat{R}$ to be commutator with $f \otimes z$. The periodic affine diagram is given below.

$\hat{\mathcal{G}}(3)$:

$$\begin{tikzcd}
& \langle f[t+1]\rangle \arrow[dr, "R"] & \\
\cdots \langle h[t]\rangle \arrow[ur, "\hat{R}"] \arrow[dr, "R"'] & & \langle h[t+1]\rangle \cdots t \in \mathbb{Z} \\
& \langle g[t]\rangle \arrow[ur, "\hat{R}"'] &
\end{tikzcd}$$

We construct a weakly partly ordered set using the standard prescription with a proviso that $\langle h[t]\rangle$ are not reflexive vertices. The attentive reader will immediately notice that $\hat{\mathcal{G}}(3)$ is not a weak lattice because

$$\sup\{\langle h[t]\rangle, \langle h[t]\rangle\} = \{\langle g[t]\rangle, \langle f[t+1]\rangle\}$$

Proposition 29 *Suppose $n = 3$. We define a total order on the set of generators of $AQ^{N_2}_{-N_1}(3)$ (see equation 13 for definition) as follows:*

$$\cdots < \langle g[t-1]\rangle < \langle f[t]\rangle < \langle h[t]\rangle < \langle g[t]\rangle < \langle f[t+1]\rangle < \cdots$$

Then the generators (25) of the ideal of relations defines a Gröbner basis of the ideal with respect to the degree-lexicographic order on monomials.

$$\begin{aligned}
(h[t])^2 &= -g[t]f[t] - g[t-1]f[t+1] - \sum_{s\neq 0} h[t+s]h[t-s] - \sum_{s\neq 0,-1} g[t+s]f[t-s] \\
g[t]f[t+1] &= -h[t]h[t+1] - \sum_{s\neq 0} \left(h[t+s]h[t+1-s] + g[t+s]f[t+1-s]\right)
\end{aligned} \tag{25}$$

In the above formulas only variables $f[t], g[s], h[k]$ are present, that satisfy $-N_1 \leq t, s, k \leq N_2$.

Proof. The proof is no different then the proof of the general case. ■

Proposition 30 *The generating function* $AQ_{-N_1}^{N_2}(3)(q,t)$, $N_1, N_2 \geq 0$ *is*

$$\frac{\prod_{l=-2N_1}^{2N_2}(1-q^l t^2)}{\prod_{l=-N_1}^{N_2}(1-q^l t)^3}$$

Proof. Exercise. ■

A.3 $n = 2$

Remark 31 Suppose $n = 2$. We define a total order on the set of generators of $AQ_{-N_1}^{N_2}(2)$ as follows:

$$\cdots < \langle f[t-1] \rangle < \langle g[t-1] \rangle < \langle f[t] \rangle < \langle g[t] \rangle < \langle f[t+1] \rangle < \cdots$$

Then the generators (26) of the ideal of relations *do not* defines a Gröbner basis of the ideal with respect to the degree-lexicographic order on monomials.

$$\begin{aligned} g[t]f[t] &= -\sum_{s\neq 0} g[t+s]f[t-s] \\ g[t]f[t+1] &= -\sum_{s\neq 0} g[t+s]f[t+1-s] \end{aligned} \tag{26}$$

In the above formulas only variables $f[t], g[s]$ are present, that satisfy $-N_1 \leq t, s \leq N_2$.

The reason is that S-polynomials computed from the pair $g[t]f[t]$ and $g[t]f[t+1]$ are nontrivial. Nontrivial also are reduction $A(S(g[t]f[t], g[t]f[t+1]))$.

Here are results of computation of generating functions $AQ_0^N(2)(1,t)$ that utilize *Maple* package *Gröbner*:

$$\begin{aligned} AQ_0^0(2)(1,t) &= \frac{t+1}{1-t} \\ AQ_0^1(2)(1,t) &= \frac{t^4-2t^3+2t+1}{(1-t)^2} \\ AQ_0^2(2)(1,t) &= \frac{-t^7-3t^6+11t^5-5t^4-5t^3+t^2+3t+1}{(1-t)^3} \\ AQ_0^3(2)(1,t) &= \frac{t^{10}+4t^9+3t^8-48t^7+56t^6-14t^4-8t^3+3t^2+4t+1}{(1-t)^4} \end{aligned} \tag{27}$$

Algebra $AQ_0^3(2)$ is not Koszul because some coefficients of $AQ_0^3(2)(1,-t)^{-1}$ are negative.

We also would like to bring readers attention to the fact that numerators of the above rational functions *are not* palindromic polynomials. This contrasts with palindromic property of numerators of (16) corresponding to a quadric in $n \geq 3$ dimensional space.

B Gröbner bases

A powerful technique of commutative Gröbner bases significantly simplifies computation of Poincaré series of graded algebras. We will review following [14] this technique in the present section.

Let $\mathbb{N}$ be the set of non-negative integers, $\mathbf{T}$ be a commutative semigroup generated by $a_1, \ldots, a_n$, whose elements are $t = a_1^{e_1} \cdots a_n^{e_n}, e_i \in \mathbb{N}$. We define $\deg[t] = e_1 + \cdots + e_n$. We choose the following total degree-lexicographic well order on $\mathbf{T}$:

$$t_1 = a_1^{e_1} \cdots a_n^{e_n} < a_1^{f_1} \cdots a_n^{f_n} = t_2$$

if and only if

$$\deg(t_1) < \deg(t_2) \text{ or } \deg(t_1) = \deg(t_2) \text{ and there is } j : i < j, e_i = f_i, e_j < f_j$$

The order is compatible with the semi-group operation in a sense that for $s, t_1, t_2 \in \mathbf{T}$, $t_1 < t_2$ implies $st_1 < st_2$.

The semigroup ring $\mathbb{C}[\mathbf{T}]$ is nothing else but a polynomial algebra $\mathbb{C}[a_1, \ldots, a_n]$. Each element $f \in \mathbb{C}[\mathbf{T}]$ has a unique ordered representation

$$f = \sum_{i=1}^{s} c_i t_i, c_i \in \mathbb{C}^*, t_i \in \mathbf{T}, t_1 > t_2 \cdots > t_s \tag{28}$$

With every nonzero element $f \in \mathbb{C}[\mathbf{T}]$ we can associate $T(f) = t_1$ - the maximal term of f and $lc(f) = c_1$ - the leading coefficient of f.

If $I \subset \mathbb{C}[\mathbf{T}]$ is an ideal , the set

$$T(I) = \{T(f) \in \mathbf{T} | f \in I\}$$

is a semigroup ideal. Introduce a set

$$E(I) = \mathbf{T} \backslash T(I)$$

If the ideal is graded, the generating function $E(I)[t]$ coincides with Poincaré series of $A = \mathbb{C}[\mathbf{T}]/I$.

The following theorem holds:

Theorem 32 *1. The algebra $\mathbb{C}[\mathbf{T}]$ is a direct sum of I and the span of $E(I)$:*

$$\mathbb{C}[\mathbf{T}] = I \oplus \langle E(I) \rangle. \tag{29}$$

2. *There is a $\mathbb{C}$-vector space isomorphism $\mathbb{C}[\mathbf{T}]/I$ and $\langle E(I) \rangle$*

3. *Let $can : \mathbb{C}[\mathbf{T}] \to \langle E(I) \rangle$ be projection on the second summand in (29). The element $can(f, I)$ is the canonical form of f with respect to ideal I and the degree-lexicographic order. For each $f \in \mathbb{C}[\mathbf{T}]$ there is $g = can(f, I) \in \langle E(I) \rangle$ such that $f - g \in I$.*

Moreover

1. *$can(f, I) = can(g, I)$ if and only if $f - g \in I$*

2. *$can(f, I) = 0$ if and only if $f \in I$*

Suppose we know all about the semi-group ideal $T(I)$. Then the canonical form $can(f, I)$ can be effectively computed. The iterative procedure that does this computation can be described as follows.

Procedure 33 *Given a polynomial f we traverse through the set $\{t_i\}$ (28) starting from the greatest element, descend down the order and looking for an element $t = t_i \in T(I)$. As*

$$t = \sum_{g_k < t} c_k g_k \mathrm{mod}\ I,$$

we can replace t in the sum (28) by a linear combination of monomials of lower order, still keeping the image of f in $\mathbb{C}[\mathbf{T}]/I$ unchanged. After collecting similar terms we repeat the procedure. Eventually this procedure terminates.

The reader should note that the described modifications simplify the structure of f, because some monomials become smaller relative to the order. In this sense $can(f, I)$ is the smallest representative in the class f mod I. We shall refer to substitution

$$t \xrightarrow{r} \sum_{g_k < t} c_k g_k$$

from Procedure 33 as to reduction r_t.

In case $\mathbf{T} = \mathbb{N}$ we have $\mathbb{C}[\mathbf{T}] = \mathbb{C}[t]$; $\mathbb{C}[t]$ is a principal ideal domain , i.e. for any ideal I there is $g \in \mathbb{C}[t]$ such that

$$I = (g).$$

The above procedure becomes the Euclidian division algorithm of polynomials in one variable:

$$f = sg + r = sg + can(f, I(g))$$

One can think about this procedure as a multidimensional generalization of the Euclidian algorithm (see also [9]).

Remembering that each semigroup ideal in an ordered semigroup has a unique irredundant basis we obtain:

Proposition 34 *[14] If $I \subset \mathbb{C}[\mathbf{T}]$ is an ideal, there is a unique set $Gr(I) \subset I$ such that:*

1. *$G(I) = \{T(g) | g \in G\}$ is an irreducible basis of $T(I)$.*
2. *$lc(g) = 1$ for each $g \in Gr$*
3. *$g = T(g) - can(T(g), I)$ for each $g \in Gr$*

Gr is called the reduced Gröbner basis of I.

Example 35 *Let I be an ideal in $\mathbb{C}[a_1, \ldots, a_n]$ generated by the quadric*

$$a_1^2 + \cdots + a_n^2.$$

Then the semigroup ideal $T(I)$ is generated by a_n^2. This enables us to compute, using the previous proposition as an aid, the Poincaré series of the ideal

$$I[t] = E(I)[t] = \frac{t^2}{(1-t)^n}$$

This example explains that the problem of computing Poincaré of an ideal can be reduced to a combinatorial problem of Poincaré series of semigroup ideals.

Let $\{f_i\}$ a basis of I,i.e. a minimal set of generators of I. Then

$$T(\{f_i\}) \subset G(I)$$

It is not true however that two sets should coincide.

Example 36 *Let $f_1 = x^3 - 2xy$ and $f_2 = x^2y - 2y^2 + x$ Then*

$$x(x^2y - 2y^2 + x) - y(x^3 - 2xy) = x^2$$

so that $x^2 \in I(f_1, f_2)$. Thus, $x^2 \in T(I)$. However x^2 is not divisible by $T(f_1) = x^3$ or $T(f_2) = x^2y$ So $T(I(f_1, f_2))$ is not generated by $T(f_1), T(f_2)$

The procedure of finding $cal(f, I)$ relies on reductions r_t. In general the structure of $r_t, t \in G(I)$ could be quite irregular. It simplifies if

$$G(I) = T(\{f_i\}). \tag{30}$$

We normalize f_i by the condition that

$$lc(f_i) = 1. \tag{31}$$

Let $q_i \in \mathbf{T}$ be equal to $T(f_i)$.

Procedure (33) is modified as follows

Procedure 37 *Given a polynomial f we traverse through the set $\{t_i\}$ (28) starting from the greatest element, descend down the order and look for an element $t = t_i \in T(I)$. Under assumption (30) this means that t is divisible by one of $q_j = q$:*

$$t = sq. \tag{32}$$

This lets us to apply reduction r_{sq}, which finishes an iteration of the algorithm. The algorithm terminates after a finite number of steps.

Equation (33) tells us that monomial t contains q as a sub-monomial. We may try to find $can(f, I)$ by repeatedly lowering the order of monomials in f by applying reductions to q_i-divisible monomials.

Theorem 38 *[14] Under condition (30) algorithm 37 applied to f gives $con(f, I)$.*

The nontrivial part of this theorem is that $con(f, I)$ is unambiguously defined. The issue is that some monomial t in f satisfies

$$t = qs = q's' \text{ with } q' \neq q'$$

This means that we can apply to distinct maximal sequences of reduction $r_{q_s} \cdots r_{q_1}$, $r_{q'_s} \cdots r_{q'_1}$ to the element f. The theorem asserts that

$$r_{q_s} \cdots r_{q_1} f = r_{q'_s} \cdots r_{q'_1} f = con(f, I).$$

The procedure described in (37) can be formally used even when condition (30) is not satisfied. In this case however

$$r_{q_s} \cdots r_{q_1} f \neq r_{q'_s} \cdots r_{q'_1} f \neq con(f, I).$$

This can be rectified if we work with the set $G(I)$. It is convenient to enumerate elements of G by positive integers.

Procedure 39 *Given a polynomial f we traverse through the set $\{t_i\}$ (28) starting from the greatest element, descend down the order and look for an element*

$$t = sq, q \in G(I), s \in \mathbf{T} \tag{33}$$

If factorization $t = sq_i = s'q_j$ is not unique we choose q with the greater index. This lets us to apply reduction r_{sq}, which finishes an iteration of the algorithm. The next iteration is applied to $r_{sq}f$. The procedure terminates after a finite number of steps and converges to $A(f, G)$.

Theorem 40 *[14] The procedure 39 applied to f gives $A(f, G) = con(f, I)$. In particular $A(f, G)$ does not depend on how we enumerated elements of G.*

This theorem explains importance of sets $G(I)$ and $Gr(I)$. The next construction known under the name of Buchberger algorithm effectively computes $Gr(I)$.

We still work under assumption (31). With the notation $l.c.m(a,b)$ for the least common multiple for $a, b \in \mathbf{T}$ we define

$$T(f,g) = l.c.m(f,g)$$

Definition 41 *The polynomial*

$$S(f,g) = \frac{T(f,g)}{T(g)} g - \frac{T(f,g)}{T(f)} f$$

is called S-polynomial of f, g.

Observe that $T(\frac{T(f,g)}{T(g)} g) = T(\frac{T(f,g)}{T(f)} f) = T(f,g)$, which means that in the difference some cancelations should occur.

The method of computation of $A(f,G)$ can be trivially extended to any set $F \supset \{f_i\}$. In this case we can not claim that $A(f,F) = con(f,I)$. The function $A(f,F)$ will be used for computation of Gröbner basis $Gr(I)$.

Algorithm 42 *Computation of $Gr(I)$*

As zero approximation $Gr_0(I)$ to $Gr(I)$ we take a set $\{f_i\}$.

We compute $Gr_i(I)$ as follows. Suppose $Gr_{i-1}(I) = \{f_i | i = 1, \dots n_{i-1}\}$. We compute all possible S-polynomials $S(f,f')$, $f, f' \in Gr_{i-1}(I)$ and then its reductions $A(S(f,f'), Gr_{i-1}(I))$. Let $f_i, i = n_{i-1}+1, \dots, n_i$ be the set of nonzero reductions. Then $Gr_i(I) = \{f_i | i = 1, \dots n_i\}$.

Proposition 43 *After finitely many steps the above algorithm terminates $Gr_{i-1}(I) = Gr_i(I) = Gr(I)$. Its result is a Gröbner basis of I.*

Lemma 44 *[14] If $T(f_i, f_j) = T(f_i)T(f_j)$ then $A(S(f_i, f_j), \{f_i\}) = 0$.*

This lemma could be a useful, because it eliminates computation of $A(S(f_i, f_j), \{f_i\})$ for a significant set of S-polynomials.

References

[1] Y. Aisaka, E. Aldo Arroyo *Hilbert space of curved $\beta\gamma$ systems on quadric cones* JHEP 0808:052,2008, arXiv:0806.0586v1 [hep-th]

[2] Y. Aisaka, E. Aldo Arroyo, N. Berkovits, N. Nekrasov *Pure Spinor Partition Function and the Massive Superstring Spectrum* arXiv:0806.0584v1 [hep-th]

[3] G. Bergman *The diamond lemma for rings theory.* Advances in Math. 29, N. 2, p. 178218, 1978.

[4] N. Berkovits *Super-Poincaré Covariant Quantization of the Superstring* JHEP 0004:018,2000, arXiv:hep-th/0001035v2

[5] R. Bezrukavnikov *Koszul Property and Frobenius Splitting of Schubert Varieties* arXiv:alg-geom/9502021v1

[6] R. Bögvad *Some homogeneous coordinate rings that are Koszul algebras* arXiv:alg-geom/9501011v2

[7] A. Braverman *Spaces of quasi-maps into the flag varieties and their applications* Proc. ICM 2006, Madrid, Zurich: Eur. Math. Soc., 2006

[8] C. Chevalley. *Theory of Lie groups I.* Princeton University Press, Princeton, NJ, 1933.

[9] D. Cox, J. Little, D. O'Shea *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra* Springer, 2007

[10] C. De Concini , D. Eisenbud, C. Procesi *Hodge Algebras.* Astrisque 91 (1982)

[11] T. Hibi *Every affine graded ring has a Hodge algebra structure.* Rend. Sem. Mat. Univ. Polytech. Torino 44 (1986), 278-286

[12] W.V.D. Hodge *Some enumerative results in the theory of forms*, Proc. Camb. Phil. Soc. 39 (1943),2230.

[13] G. R. Kempf *Some wonderful rings in algebraic geometry* Journal of Algebra, Vol 134, Issue 1,October 1990, Pages 222-224

[14] T. Mora *An introduction to commutative and non-commutative Gröbner bases*

[15] N. Nekrasov *Lectures on curved beta-gamma systems, pure spinors, and anomalies* arXiv:hep-th/0511008v1

[16] A. Polishchuk , L. Positselski *Quadratic algebras* Univ. Lecture Ser., vol. 37, Amer. Math. Soc., Providence, RI, 2005

[17] S. Priddy *Koszul resolutions.* Trans. Amer. Math. Soc. 152, p. 3960, 1970.

[18] M.S. Ravi *Coordinate rings of G/P are Koszul* Journal of Algebra Volume 177, Issue 2, 1995, Pages 367-371

[19] J. Ruffo*Quasimaps, straightening laws, and quantum cohomology for the Lagrangian Grassmannian* arXiv:0806.0834v1 [math.AG]

[20] F. Sottile, B. Sturmfels, *A sagbi basis for the quantum Grassmannian,* J. Pure and Appl. Alg. 158 (2001), 347366.

[21] A. A. Voronov *Semi-infinite homological algebra* Inventiones Mathematicae, Vol 113, N. 1, 1993