MATHEMATISCHE ARBEITSTAGUNG 1975

UNIVERSITÄT BONN

Inhalt

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Mathematisches Institut
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Programm der Mathematischen Arbeitstagung 1975 (I)

Samstag, den 21.6.:

17.00 - 18.00 Uhr: M.F. Atiyah: Algebras of operators in Hilbert space and K-theory

Sonntag, den 22.6.:

10.00 - 11.00 Uhr: Moser: Isospectral deformations

12.00 - 13.00 Uhr: Lusztig: Macdonald's conjecture on discrete series of finite Chevalley groups

17.00 - 18.00 Uhr: Kostant: The η-function formula of Macdonald

Montag, den 23.6.:

9.45 - 10.00 Uhr: Festlegung der nächsten Vorträge

10.00 - 11.00 Uhr: MacPherson: Gelfand's formula for the first Pontrjagin class

12.00 - 13.00 Uhr: Ziller: Closed geodesics on homotopy symmetric spaces

17.00 - 18.00 Uhr: Serre: Lower bounds of discriminants of number fields

Die Vorträge finden alle im "Großen Hörsaal" (Wegelerstraße 10) statt.

Erfrischungspausen mit Tee: Sonntag und Montag vormittags von 11.15 - 12.00 Uhr vor dem Großen Hörsaal, nachmittags ab 15.30 Uhr im Diskussionsraum Beringstraße 1. Die Post liegt während der Vormittags-Teepausen aus.

Tischtennis im Keller des Hauses Beringstraße 4.

Den Tagungsbeitrag (≠ DM 10,-) bitte an Frau Gerber bezahlen (vor dem Großen Hörsaal, sonntags und montags vor und nach dem 10-Uhr-Vortrag).

Alle Teilnehmer mögen sich bitte in die Teilnehmerliste eintragen.

Mathematisches Institut
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Programm der Mathematischen Arbeitstagung 1975 (II)

Dienstag, den 24.6.

10.00 - 11.00 Uhr: tom Dieck: Burnside ring of a compact Lie gruop

12.30 - ca. 20.00 Uhr: Dampferfahrt auf dem Rhein nach Andernach
Abfahrt am "Alten Zoll" mit Motorschiff "Verona".

Mittwoch, den 25.6.:

9.45 - 10.00 Uhr: Festlegung der restlichen Vorträge

10.00 - 11.00 Uhr: Jantzen: Modular representations of semi simple groups

12.00 - 13.00 Uhr: Schmid: Blattner's conjecture on the discrete series of semi simple real Lie groups

17.00 - 18.00 Uhr: Varchenko: Newton diagrams of singularities

Die Vorträge finden alle im Großen Hörsaal (Wegelerstraße 10) statt.
Erfrischungspausen mit Tee am Mittwoch, 11.15 Uhr, vor dem Großen Hörsaal und ab 15.30 Uhr im Diskussionsraum der Beringstraße 1.
Die Post liegt während der Vormittags-Teepause aus.
Tischtennis im Keller des Hauses Beringstraße 4.
Die Tagungsgäste, die sich noch nicht in die Teilnehmerliste eingetragen haben, werden gebeten, dies nachzuholen. Sie finden die Liste vor dem Diskussionsraum Beringstraße 1

Mathematisches Institut
der Universität Bonn

Programm der Mathematischen Arbeitstagung 1975 (III)

<u>Donnerstag, den 26.6.:</u>

10.00 - 11.00 Uhr: Kostrikin: Tanaka-Artin's conjecture on the multiplicative group of divisions algebras

12.00 - 13.00 Uhr: Mazur: Rational points on modular curves

17.00 - 18.00 Uhr: Casselman: The $\mathcal{H}$-cohomologiy of representations of semi simple Lie groups

<u>Freitag, den 27.6.:</u>

10.00 - 11.00 Uhr: Parshin: Residues and symbols

12.00 - 13.00 Uhr: Steenbrink: Mixed Hodge structure on vanishing cycles

17.30 - 18.30 Uhr: (Kolloquium) E. Calabi: Nearly flat triangulations of Riemannian manifolds

Die Vorträge finden alle im <u>"Großen Hörsaal"</u> <u>(Wegelerstr. 10)</u> statt.

Erfrischungspausen mit Tee: Donnerstag 11.15 Uhr vor dem Großen Hörsaal und ab 15.30 Uhr im Diskussionsraum der Beringstraße 1; Freitag 11.15 Uhr vor dem Großen Hörsaal und nachmittags um 17.00 Uhr ebenfalls vor dem Großen Hörsaal. Die <u>Post</u> liegt während der Vormittags-Teepausen aus.

<u>Tischtennis</u> im Keller des Hauses Beringstr. 4.

! Die <u>Referenten</u> werden nochmals gebeten, ihre <u>Kurzfassungen</u> möglichst bald !
! bei <u>Herrn Kraft</u> abzugeben, da wir den Tagungsbericht allen Teilnehmern noch !
! vor ihrer Abreise aushändigen möchten. !

Title: M. F ATIYAH

Name of author: Algebras of operators in Hilbert space

Address:

Bibliography: Springer Lecture Notes No. 345
(Proceedings of a conference on Operator Theory)

Let $\mathcal{Q}$ be the algebra of bounded operators on Hilbert space, K the ideal of compact operators and $\mathcal{B} = \mathcal{Q}/K$ the quotient algebra. If $A \in \mathcal{Q}$, and $B = \pi(A) \in \mathcal{B}$, the spectrum of B is also called the essential spectrum of A. Assume that B is normal, then we have the following result of Brown-Douglas-Fillmore characterizing B up to unitary equivalence

Theorem 1. B_1, B_2 normal in $\mathcal{B}$ are unitarily equivalent if and only if

i) $\operatorname{spec} B_1 = \operatorname{spec} B_2$ (= X say)

ii) for each connected component U of $\mathbb{C} - X$ and for $\lambda \in U$ we have

$$\operatorname{index}(A_1 - \lambda) = \operatorname{index}(A_2 - \lambda)$$

Here $\pi(A_i) = B_i$ and, for $\lambda \notin X$, $A_i - \lambda$ is a Fredholm operator and so has an index: it depends only on U and not on choice of $\lambda \in U$.

For a compact subset $X \subset \mathbb{C}$ this theorem can be reformulated as follows. Consider $*$-embeddings $C(X) \to \mathcal{B}$, where $C(X)$ is the C^*-algebra of continuous functions on X. Denote by $\mathrm{Ext}(X)$ the unitary equivalence classes of such embeddings. Then

<u>Theorem 1'</u>. $\mathrm{Ext}(X) \cong H_1(X, \mathbb{Z})$

For this we just observe that

$$H_1(X, \mathbb{Z}) \cong \tilde{H}^0(\mathbb{C} - X; \mathbb{Z})$$

the latter denoting the integer valued continuous functions on $\mathbb{C} - X$ vanishing at ∞.

For more general X, say compact subsets of $\mathbb{C}^N$, Theorem 1' generalizes to

<u>Theorem 2</u>. $\mathrm{Ext}(X) \cong K_1(X)$

where $K_1(X)$ is the homology K-theory functor dual to the usual cohomology K-functor. As definition we can take

$$K_1(X) = \tilde{K}^0(\mathbb{C}^N - X)$$

Elements of this latter group can be viewed as classes of continuous maps into the space $\mathcal{F}$ of Fredholm operators. The map of Theorem 2 can be given explicitly in these terms.

Title: Isospectral Deformations

Name of author: JÜRGEN MOSER

Address: New York University, 251 Mercer Str.
New York, N.Y. 10012

Bibliography:

I) Introduction: In the last ten years a number of partial differential equations have attracted the attention of physicists and mathematicians because of some remarkable properties, such as the Korteweg de Vries equations

(1) $$u_t + uu_x + u_{xxx} = 0,$$

the equation

(2) $$u_{tt} - u_{xx} = \sin u$$

and the nonlinear Schrödinger equation

(3) $$i\psi_t = \psi_{xx} + |\psi|^2\psi.$$

All these equations admit infinitely many conservation laws. Another important property is the simple scattering of wave solutions of (1) which suggests a superposition of solutions for this nonlinear equations. Finally these equations can be viewed as integrable Hamiltonian systems whose integrals (i.e. conservation laws) are in involution.

In this expository lecture we discuss analogous systems in finite dimensions to exhibit clearly the algebraic features in a transparant model.

We consider the system of differential equations

$$\frac{du_k}{dt} = \frac{1}{2}\left(e^{u_{k+1}} - e^{u_{k-1}}\right) \quad \text{for } k=1,2,\dots,2n \tag{4}$$

where e^{u_0}, $e^{u_{2n+1}}$ are to be replaced by zero. This equation was mentioned by M. Henon and was studied by Kac and v. Moerbeke as a discrete analogue of (1). If we introduce a symplectic structure by defining the Poisson brackets

$$\{u_k, u_l\} = \begin{cases} 1 & \text{if } k = l+1 \\ -1 & \text{if } k = l-1 \\ 0 & \text{otherwise} \end{cases}$$

then (4) can be written as a Hamiltonian system

$$\dot{u}_k = \{H, u_k\} \quad \text{with} \quad H = \frac{1}{2}\sum_{j=1}^{2n} e^{u_j}$$

We want to show:

i) This system possesses n integrals which are rational in e^{u_k}.

ii) These integrals are in involution, i.e. their Poisson brackets vanish.

iii) For all solutions $e^{u_k(t)}$ are rational functions of $e^{-\alpha_1 t}, \dots e^{-\alpha_n t}$ with distinct positive $\alpha_1, \dots \alpha_n$.

II Isospectral Deformations.

The argument is based on an entirely different description of the flow, namely as spectrum-preserving deformation of the class of $(2n+1)$ by $(2n+1)$ matrices of the form

$$(5) \qquad L = \begin{pmatrix} 0 & a_1 & & & \\ a_1 & 0 & \ddots & & \bigcirc \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{2n} \\ \bigcirc & & & a_{2n} & 0 \end{pmatrix}, \quad a_j > 0 .$$

We ask for deformations $L(t)$ in the class $\mathcal{L}$ of these matrices such that they remain in the same conjugacy class, i.e. $U(t)^{-1} L(t)\, U(t) = L(0)$ with some orthogonal matrix $U(t)$. This leads to the differential equation

$$(6) \qquad \frac{dL}{dt} = [L, B] = LB - BL$$

with some skew-Hermitean B. For such deformations the eigenvalues are clearly preserved. This approach is due to P. D. Lax who applied it to (1). Flaschka used this method for some finite dimensional problems similar to the present one.

In our case one verifies that the choice of

$$B = \begin{pmatrix} 0 & 0 & a_1 a_2 & & & \\ 0 & 0 & 0 & \ddots & & \bigcirc \\ -a_1 a_2 & 0 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & a_{2n-1} a_{2n} \\ \bigcirc & & \ddots & \ddots & 0 & 0 \\ & & & -a_{2n-1} a_{2n} & 0 & 0 \end{pmatrix}$$

is consistent with deformation in the class $\mathcal{L}$ and the differential equations (6) take the form

$$\dot{a}_k = a_k \left(a_{k+1}^2 - a_{k-1}^2 \right) \qquad (k = 1, \cdots, 2n)$$

where we set $a_0 = 0$, $a_{2n+1} = 0$. With $a_k = \frac{1}{2} e^{\frac{u_k}{2}}$ we identify this equation with (4). Thus it follows that the eigenvalues are integrals of the motion for (4).

These eigenvalues are not independent since the spectrum is symmetric with respect to the origin. Reordering the simple eigenvalues of L one has

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 = \lambda_{n+1} \quad \text{and} \quad \lambda_{2n+2-k} = -\lambda_k$$

and

$$\Delta(\lambda) = \det(\lambda I - L) = \lambda \prod_{k=1}^{n} (\lambda^2 - \lambda_k^2)$$
$$= \lambda \sum_{k=0}^{n} \lambda^{2n-2k} I_k .$$

Then $I_1, I_2, \ldots I_n$ are the desired independent integrals which are rational in $a_k^2 = \frac{1}{4} e^{u_k}$. This proves i). A calculation, or an asymptotic study of the solutions of (4), reveals that $\{I_k, I_l\} = 0$, as claimed in ii).

III Inverse Problem

To verify iii) we consider the rational function

(7) $$f(z) = ((zI - L)^{-1} e_1, e_1) = \sum_{k=1}^{2n+1} \frac{r_k^2}{\lambda - \lambda_k} ,$$

e_1 being the vector whose first component is 1 and all others zero. This function has simple poles at the eigenvalues λ_k of L and positive residues r_k^2. Because of the symmetry properties of L one has

$$r_{2n+2-k}^2 = r_k^2 .$$

Moreover, one has $\sum_1^{2n+1} r_j^2 = 1$, and it is natural to introduce the homogeneous coordinates $\rho_1, \rho_2, \ldots \rho_{n+1} > 0$ by

$$r_k = \rho_k \left(\rho_{n+1}^2 + 2 \sum_{j=1}^{n} \rho_j^2 \right)^{-\frac{1}{2}} , \quad k = 1, 2, \ldots, n+1$$

We denote

$$\Sigma = \{ \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 \,;\; \rho_1 > 0, \ldots, \rho_{n+1} > 0 \} / \mathbb{R}_+$$

as the space of spectral data; the ρ_j are related to the so-called normalization constants. Thus we have the map

$$\varphi : \mathcal{L} \rightarrow \Sigma$$

taking the 2n-dimensional space $\mathcal{L}$ of matrices into Σ. The inverse problem consists in studying φ^{-1}. It is ~~remarkable~~ remarkable that φ is bijective and φ^{-1} a rational map: There exist rational functions $R_j(\lambda, \rho)$, homogeneous of degree zero in $\rho = (\rho_1, \ldots, \rho_{n+1})$ such that

$$(8) \qquad a_k^2 = R_k(\lambda, \rho) \qquad k = 1, 2, \ldots, 2n$$

defines the entries of the corresponding matrix (5). To prove this we follow an old device of Stieltjes and express (7) as continued fraction:

$$f(z) = \cfrac{1}{z - \cfrac{a_1^2}{z - \cfrac{a_2^2}{\ddots \; \cfrac{}{z - \cfrac{a_{2n}^2}{z}}}}}.$$

The entries a_k^2 are the desired coefficients of L; because of the rational procedure in determining the entries they depend rationally on λ, ρ.

Viewing (8) as a coordinate transformation we rewrite our system (4) in these variables; one finds

$$\dot{\lambda}_k = 0 \quad ; \quad \dot{\rho}_j = -\lambda_j^2 \rho_j \qquad \begin{array}{l} j = 1,2,\dots n+1 \\ k = 1,2,\dots,n. \end{array}$$

Thus for fixed λ these differential equations are indeed linear, whose solutions are expressed in terms of $e^{-\lambda_j^2 t}$, and via (8) our statement about the solutions $a_k^2 = \frac{1}{4} e^{u_k}$ being a rational function of exponentials is established.

IV Concluding Remarks

This simple example illustrates the interaction of the symplectic structure and the spectral quantities of the matrices in $\mathcal{L}$: The eigenvalues, or their symmetric functions I_k, are in involution. Therefore the manifold $I_k = $ const., i.e. the set of matrices $L \in \mathcal{L}$ conjugate to a fixed one, form a Lagrange-manifold on which the vector fields generated by the I_k form a commuting Lie algebra, making the Lagrange manifold into a group. In our case this group multiplication is given by componentwise multiplication of the ρ_j. Finally we point to the algebraic nature of the problem, which is reflected by the solutions being rational functions of exponentials. In the related but more complicated problem (1) with periodic boundary conditions, where $\mathcal{L}$ is the class of Sturm-Liouville operators $-\left(\frac{d}{dx}\right)^2 + q(x)$, one is led to hyperelliptic functions. This was discovered by S. Novikov 1974, P.D. Lax 1975 and v. Moerbeke and Mc Kean 1975.

We mention another such integrable system belonging to mechanics. Let $x_1 < x_2 < \dots < x_n$ and consider the second order differential equations

$$\ddot{x}_k = - \frac{\partial U}{\partial x_k} , \qquad k = 1, 2, \dots, n$$

where

$$U = \sum_{k<l} (x_k - x_l)^{-2} .$$

This system admits n integrals, rational in x_k and $\dot{x}_k = y_k$, which are in involution. The proof of this fact is again based on isospectral deformation of the class of matrices whose diagonal elements are $y_k = \dot{x}_k$ and the element in the position k, l are $i(x_k - x_l)^{-1}$. The inverse problem for these matrices has not yet been solved.

This report is based on the paper by the author: Three integrable Hamiltonian systems... Advances in Math. 16, 1975 where other references are quoted. Some more recent related references are:

Kac and van Moerbeke, A solution of the periodic Toda problem. Proc. Nat. Acad. Sc. 1975

S. Novikov, Periodic Problem for the K.dV eq., Funct Analys. and Appl. 8 (3), 1974

Lax, Comm. Pure Appl. Math. 28, No. 1. 1975

Mc. Kean + van Moerbeke, On Hill's equation, to appear in Inventiones 1975

Four articles by Kruskal, Flaschka, Newell and the author are contained in Lecture Notes in Physics 38, Springer Verlag 1975

Title: Macdonald's conjecture on discrete series of finite Chevalley groups

Name of author: G. LUSZTIG

Address: Math. Inst., Univ. of Warwick, Coventry, England

Bibliography: P. Deligne, G. Lusztig: Representations of reductive groups over finite fields, to appear.

Let G be a reductive, connected algebraic group defined over a finite field F_q, with Frobenius map F. For $w \in W$ (the Weyl group), let $X(w)$ be the set of Borel subgroups B of G such that B and FB are in relative position w. Let T be a maximal torus and let $B \supset T$ be a Borel subgroup such that $FT = T$ and (B, FB) are in relative position. There is a natural étale covering $\tilde{X}(w) \longrightarrow X(w)$ depending on $(T \subset B)$ with group T^F. This is G^F equivariant. This covering gives rise to locally constant sheaves $\mathcal{F}_\theta$ on $X(w)$, for each character θ of T^F. The group G^F acts naturally in $H^i_c(X(w), \mathcal{F}_\theta)$. The following was proved by Drinfeld for SL_2:

<u>Theorem</u> (joint with Deligne). $H^i_c(X(w), \mathcal{F}_\theta) = 0$, unless $i = l(w)$. If $i = l(w)$, this is an irreducible G^F-module. It is cuspidal $\Leftrightarrow$ T is minisotropic. There is an explicit character formula in terms of Green functions on unipotent elements.

Thus one gets almost all irreducible representations of G^F. To get the other ones one must allow θ to be degenerate.

One has precise results whenever w is a Coxeter element. In this case the eigenspaces of Frobenius give a complete decomposition of $H^*(X(w))$ in irreducible G^F-modules (assuming q large). The eigenvalues of Frobenius are of the form $\varepsilon q^{m/2}$ where ε is a root of 1 of order k, where k is a coefficient of the highest root. There are h distinct eigenvalues, where h is the Coxeter number.

1

Title: On Mac Donald's η-function Formula, the Laplacian and Generalized Exponents

Name of author: Bertram Kostant

Address: MIT , Cambridge,Mass. /USA

Investigations involving the power series $\varphi(x) = \prod_{n=1}^{\infty}(1-x^n)$ have had a long history in mathematics. We will mention a few details. One interest arises since if we write $\varphi(x)^{-1} = \sum_{n=0}^{\infty} p(n)\cdot x^n$ then $n \longmapsto p(n)$ is the classical partition function. The expansion $\varphi(x) = \sum_{n=-\infty}^{+\infty} (-1)^n \, x^{\frac{n^2-3n}{2}}$ is due to Euler. Arising out of his work on Theta functions Jacobi (circa 1828) obtained the expansion :

$$\varphi(x)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1)\, x^{\frac{n(n+1)}{2}} \tag{1}$$

which such a well known combinatorist as Mac Makon has called"the most remarkable formula in all of pure mathematics".(I thank N.Verma for this remark.) The function $\eta(x) = x^{\frac{1}{24}}\cdot\varphi(x)$ is called the Dedekind η-function. (More usually as a function of z in the upper half-plane where we substitute $x = e^{2\pi i z}$. It is then a modular function of z). The expansion of $\eta(x)^{24}$ has $\tau(n)$, the Ramanujan τ-function as coefficient and an expression for this function has been given by F.Dyson. The expansion for $\varphi(x)^{10}$ is due Winquist enabling hin to obtain a simple proof of Ramanujan's result that $p(11m+6) \equiv 0 \mod 11$. The numbers 3, 10 and 24 are the dimensions of the simple compact Lie groups SU(2), Spin (5) and SU(5).

If K is any compact simply connected simple Lie group then I.G. Mac Donald has a remarkable formula for $\eta(x)^{\dim K}$ which generalizes the results stated above.

Let $l = \operatorname{rank} K$ and let $T \subseteq K$ be a maximal torus. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{h}^*$ the pure imaginary dual of $\mathfrak{t}$. The character group $\hat{T}$ of T is isomorphic to $\mathbb{Z}^l$ and may be naturally identified with a lattice $Z \subseteq \mathfrak{h}^* \simeq \mathbb{R}^l$. Let $L(\Sigma) \subset Z$ be the sublattice generated by the roots Σ of (T,K). Also let h be the Coxeter number of K. By definition if W is the Weyl group of (T,K) and $\sigma \in W$ is the Coxeter element then $h = \text{order } \sigma$. For example if $K = SU(n)$ then $W=S_n$ the symmetric group on n-letters and we can take $\sigma = (1,2,\ldots n)$ (the permutation with 1 cycle). Thus $n=h$ for $K=SU(n)$. Next let Σ_+ be a system of positive oots and let (λ, φ) be the bilinear form on $\mathfrak{h}^*$ induced by the Killing form. Let $D = \{\lambda \in Z \mid (\lambda,\varphi) \geq 0 \text{ for all } \varphi \in \Sigma_+\}$ so that by the Cartan-Weyl theory D, as the highest weights, parametrizes $\hat{K}$, the set of equivalence classes of irreducible K-modules. In fact for each $\lambda \in D$ let

$$\pi_\lambda : K \longrightarrow \operatorname{Aut} V_\lambda$$

be an irreducible representation of K with highest weight λ. Also let $\rho = \frac{1}{2}\sum_{\varphi>0} \varphi$ and let for $\nu \in Z$

$$d(\nu) = \frac{\prod_{\varphi>0}(\lambda+\rho,\varphi)}{\prod_{\varphi>0}(\rho,\varphi)}$$

so that by Weyl's formula $d(\lambda) = \dim V_\lambda$ for $\lambda \in D$. However for general $\nu \in Z$, $d(\nu)$ can be zero or negative.

In case K is simply-laced ((φ,φ) is constant for $\varphi\in\Sigma$) then Mac Donald's formula may be written

$$(2)\qquad \eta(x)^{\dim K} = \sum_{\nu\in h\cdot L(\Sigma)} d(\nu)\; x^{(\nu+\rho,\nu+\rho)}$$

This formula for $\eta(x)^{\dim K}$ is in terms of a sum over a lattice. One senses here however that there is an underlying statement in terms of a sum over the irreducible representation of K. This is explicit in Jacobi's formula (1). In fact let $X_1,\dots,X_\lambda$ be an orthonormal base of $\mathfrak{g}$, so that in the universal enveloping algebra of $\mathfrak{g}$ $\sum_{i=1} X_i^2 = z$ is the Casimir element. Thus $\pi_\lambda(z)$ is a scalar operator on V_λ and one knows the scalar $c(\lambda)$ is given by $c(\lambda)=(\lambda+\rho,\lambda+\rho)-(\rho,\rho)$. In (1) the exponents $\frac{n(n+1)}{2}=c(\lambda)$ where π_λ is the $2n+1$ dimensional representation of SU(2). But now in general $\varphi(x)^{\dim K} = \eta(x)^{\dim K}\cdot x^{-\frac{\dim K}{24}}$

However the "stronge formula" of Freudenthal-de Vries assert that $\frac{\dim K}{24} = (\rho,\rho)$. Thus by changing the summations from $h\cdot L(\Sigma)$ to D one can expect

$$\varphi(x)^{\dim K} = \sum_{\lambda\in D} \varepsilon(\lambda)\cdot \dim V_\lambda\; x^{c(\lambda)}$$

where $\varepsilon(\lambda)$ is some, yet to be determined weighting of the representations $\{\pi_\lambda\}=\hat{K}$. In SU(2) by Jacobi's formula the weighting $\varepsilon(\lambda)=0$ if $\dim V_\lambda$ is even and alternating in sign for the odd dimensional representations. But this is exactly how the non-trivial element of the Weyl group operates on the zero weight space of these representation. Gneralizing the usual action of W on $\mathfrak{t}$, for any

$\lambda \in D$ we obtain a representation of W

$$\theta_\lambda : W \longrightarrow \mathrm{Aut}\, V_\lambda^T$$

on the space V_λ^T of T-invariants (to be ignored if $V^T = 0$). This connection of representations of W and K, I think, is interesting. For example if $K=SU(n)$ then there is a natural set λ_i, $i=1,2\ldots, p(n) \in D$ such that θ_{λ_i} is irreducible and runs over all the irreducible representations of $W=S_n$. Now if one looks at a character-table for S_n it is a pluking fact that one always has $\chi(\sigma) \in \{1,0,-1\}$ when $\sigma=(1,2,\ldots,n)$. A generalization of this and also a generalization of Jacobi's formula is given in

Theorem 1. <u>If</u> K <u>is arbitrary then for any</u> $\lambda \in D$ <u>one always has</u> $\mathrm{tr}\,\theta_\lambda(\sigma) = 1,0,-1$ <u>where</u> σ <u>is the Coxeter element. Moreover if</u> K <u>is simply laced then the weighting</u> $\varepsilon(\lambda) = \mathrm{tr}\,\theta_\lambda(\sigma)$. <u>That is</u>

$$\varphi(x)^{\dim K} = \sum_{\lambda \in D} \mathrm{tr}\,\theta_\lambda(\sigma)\cdot \dim V_\lambda \cdot x^{\varepsilon(\lambda)}.$$

We introduce two special kinds of elements in K. If $a \in K$ is regular (the centralizer of a in K is a torus) then the order of $\mathrm{Ad}\,a$ is $\geq h$. We call an element $a \in K$ principal if (1) a is regular and (2) the order of $\mathrm{Ad}\,a = h$. It is then a fact that any two principal elements are conjugate.

Next an element $a \in K$ is called principal of type ρ if it is conjugate to the (regular) element $\exp 2x_\rho$ where $x_\rho \in \mathfrak{t}$ is that element such that $\langle \beta, x_\rho\rangle = 2\pi i(\beta,\rho)$ for any $\beta \in \Sigma$.

Remark 1. If K is simply laced then the two notions are the same. That is $a \in K$ is principal if and only if it is principal of type ρ.

Elements of the type introduced have remarkable character values.

Theorem 2. If $a \in K$ is principal then for any $\lambda \in D$ one has $\chi_\lambda(a) = 1, -1$ or 0, where χ_λ is the character of π_λ. In fact $\chi_\lambda(a) = \operatorname{tr} \theta_\lambda(\sigma)$ where σ is the Coxeter element.

Theorem 2. asserts that we may subsitute $\chi_\lambda(a)$ for $\operatorname{tr}\theta_\lambda(\sigma)$ in the formula (Theorem 1) for $\varphi(x)^{\dim K}$. But more than that is true. If we use a principal element of type ρ instead we do not have to assume that K is simply laced in the formula for $\varphi(x)^{\dim K}$.

Theorem 3. Let $a \in K$ be a principal element of type ρ. Then $\chi_\lambda(a) = 1, -1$ or 0 for any $\lambda \in D$. Furthermore one has

$$\varphi(x)^{\dim K} = \sum_{\lambda \in D} \chi_\lambda(a) \cdot \dim V_\lambda \cdot x^{c(\lambda)}.$$

If M is a compact Riemannian manifold and Δ is the Laplace Bertrami operator on M there is a considerable amount of mathematical activity concerned with the question as to whether the complex valued function $s \longmapsto \operatorname{tr}(\Delta + c)^{-s}$ defined on the half plane and extended meromorphically satisfies a functional equation. Here c is a constant ≥ 0. Consider the question when $M = K$. A natural choice for c from many points of view is (ρ, ρ). But now if $\operatorname{Re} s > \frac{\dim K}{2}$ then $(\Delta + (\rho, \rho))^{-s}$ is given by convolution by a function on K which we may identify with the operator $(\Delta + (\rho,\rho))^{-s}$. But then $\operatorname{tr}(\Delta + (\rho,\rho))^{-s} = (\Delta + (\rho,\rho))^{-s}(e)$

where $e \in K$ is the identity. Our next statement is that if a principal element of type ρ is substituted for e then indeed the resulting function of s is holomorphic and satisfies a functional equation. In fact, since $\eta(e^{2\pi i z})$ is modular in z with zero constant term then the same is true for $\eta^{\dim K}$ and hence the Mellin transform $M(\eta^{\dim K})(s)$ is holomorphic and satisfies a functional equation. But now by the Peter-Weyl theorem one can sum the right side of the formula in Theorem 3 when $x^{c(\lambda)}$ is replaced by $(c(\lambda)+(\rho,\rho))^{-s}$.

Thus one has

Theorem 4: <u>Let</u> $a \in K$ <u>be a principal element of type</u> ρ <u>then</u>

$$M(\eta^{\dim K})(s) = (\Delta + (\rho,\rho))^{-s}(a)$$

<u>so that the function</u> $s \longmapsto (\Delta + (\rho,\rho))^{-s}(a)$ <u>is everywhere holomorphic and satisfies a function equation</u> (K <u>here is arbitrary</u>).

If K is simply-laced there is another statement one can make which in effect applies to K/T rather than K. If the Poincaré polynomial of K is written $p_K(t) = \prod_{i=1}^{l}(1 + t^{2m_i+1})$, $m_1 \leq m_2 \leq \ldots \leq m_l$ then the integers m_i are called the exponents of K. More generally if $\lambda \in D$ and $l(\lambda) = \dim V_\lambda^T$ there is a naturally associated sequence of integer $m_1(\lambda) \leq \ldots \leq m_{l(\lambda)}(\lambda)$ which are called the generalized exponents. The terminology is justified in that

$m_i(\psi) = m_i$ where $\psi \in D$ is the highest root of Σ. We recall the definition. Let S denote the ring of complex-valued polynomial functions on $\mathfrak{k}$ and let S^i denote the homogeneous component of S of degree i. The adjoint action of K on $\mathfrak{h}$ induces an action of K on S and let $J = S^K$ be the ring of polynomial invariants. On the other hand let $H \subseteq S$ denote the set of harmonic polynomials in the generalized sense i.e. all $f \in S$ such that $\partial_n f = 0$ where ∂_n is a constant efficient differential operator without constant term which commutes with the action of K. One knows that $S = J \otimes H$ and that H is a K-submodul with finite multiplicities. In fact the multiplicity of π_λ is $l(\lambda)$. Indeed if $H_\lambda \subseteq H$ for $\lambda \in D$ is the primary π_λ- component then the $m_\iota(\lambda)$ are defined in that we can write

$$H_\lambda = \bigoplus_{\iota=1}^{\ell(\lambda)} H_\lambda^\iota$$

where $H_\lambda^\iota \subseteq S^{m_\iota(\lambda)}$ and H_λ^ι is K-irreduzible.

Now let $w = e^{2\pi i/h}$. By results of Coxeter, Coleman and Steinberg one knows that the eigenvalues of $\theta_\psi(\sigma)$ are w^{m_ι} where $\sigma \in W$ is the Coxeter element. More generally we have proved that the eigenvalues of $\theta_\lambda(\sigma)$ are $w^{m_\iota(\lambda)}$ $\iota = 1, \ldots, l(\lambda)$ for any $\lambda \in D$. Thus one has for any $\lambda \in D$

$$\text{(6)} \qquad \operatorname{tr} \theta_\lambda(\sigma) = w^{m_1(\lambda)} + \ldots + w^{m_{\ell(\lambda)}(\lambda)} = \begin{cases} 1 \\ -1 \\ 0 \end{cases}$$

Now let Ω be the operator on H defined so that Ω is multiplications by w^i on $H^i = H \cap S^i$. Now let $\Box$ be the operator on H defined by the restriction of the

Casimir operator to H (via the adjoint action). Thus corresponds to the Laplace-Beltrami operator on K/T (using the Killing form to define the metric) since H as a K-module is isomorphic to the space of K-finite functions on K/T. If we substitute (6) in Theorem 1, replace φ by η and recall that $\bigoplus_\lambda H_\lambda = H$ then one has (convergence is easy for Re $s > \dim K/2$)

Theorem 5: Assume K is simply connected and let $M(\eta^{\dim K})(s)$ be the Mellin transform of the Dedekind η-function to the dim K power. Then

$$M(\eta^{\dim K})(s) = \operatorname{tr}_\Omega(\Box + (\rho,\rho))^{-s}$$

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Title: Gelfand's formula for the first Pontrjagin class

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A great deal of effort has gone into finding a formula for a cycle representing the i^{th} Pontrjagin class $P_i \in H_{n-4i}(M)$ of an oriented polyhedral n-manifold M. Such a formula would express the idea that the polyhedron has a "curvature" whose Pontrjagin polynomial is a "current" concentrated on the $n-4i$ skeleton.

The conceivable forms of such a formula vary widely as to the degree of explicitness and as to the ammount of input information required. The beautiful formula of Thom that originally established the existance of polyhedral Pontryagin classes required only the polyhedral structure but was not explicit: the cycle was determined only up to homology. At the other extreme, the ideal would be a purely local formula: this would give the coefficient of an $n-4i$ simplex σ in the cycle only in terms of the simplicial decomposition of the star of σ. It is unknown whether a purely local formula is even possible.

Gabrielov, Gelfand, and Lossik have given for the first Pontryagin class the first entirely explicit formula, but they need as input information two pieces of data beyond the polyhedral structure: configuration data and hypercomplex data. Since these data are very complicated we only suggest their nature.

Configuration data is roughly the choice for each $p \in M$ of a flattening of a neighborhood of p which varies continuously with p. The existance of configiuration data is equivalent to the smoothability of M. Therefore Gelfand's formula cannot even be stated for an arbitrary PL manifold let alone a rational homology manifold, for which Thom defined Pontryagin classes, or a pseudomanifold with even dimension strata, for which Goresky and MacPherson have defined L classes.

Hypercomplex data is roughly a subdecomposition of the dual cell complex with only certain types of cells, called hypersimplices, occuring. This always exists but is not unique.

The formula for Gelfand's P_1 cycle assigns to each codimension four simplex σ a sum of terms: one for each hypersimplex in the subdivision of the dual cell of σ and one for each hypersimplex in the dual of each codimension three simplex σ' with σ as a face. The terms for the hypersimplices in the dual of σ are 0 or $+\frac{1}{48}$ or $-\frac{1}{48}$ depending on certain orientations in the flattening of σ given by the configuration data. The terms for the hypersimplices in the dual to σ' are $+\frac{1}{48}$ or $-\frac{1}{48}$ for each degeneration of a certain type in the flattening as you travel from the center of σ' to the center of σ.

Title: Closed Geodesics on Homotopy Symetric Spaces

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Bibliography:

It is a good conjecture that there exist ∞ many closed geodesics on any compact manifold. For most manifolds this is proven now by the combination of the following two theorems:

Here $\Lambda(M) = \{ c: S^1 \to M$ continous $\}$ with compact open topology.

Theorem (Gromoll-Meyer 1969): M compact and $b_i(\Lambda(M),Q)$ unbounded, then there exist ∞ many closed geodesics for any metric on M.

Theorem (Sullivan-Vigue 1975): M compact, simply connected. Then $b_i(\Lambda(M),Q)$ unbounded $\Leftrightarrow$ $H^*(M,Q)$ generated by more than one element.

The theorem of Gromoll-Meyer has been generalized by Klingenberg to hold for $b_i(\Lambda(M),Z_2)$ unbounded, but the methods of Sullivan are naturally restricted to the rationals.

For thr symetric spaces of rank 1 this does not matter, since $b_i(\Lambda(M),K)$ is bounded for any field K, but there are symetric spaces which have $H^*(M,Q)$ generated by one element but have more Z_2 torsion:

$M = SU(3)/SO(3)$, $SO(n+2)/SO(2)\times SO(n)$ n odd , $G_2/SO(4)$

These are actually all such spaces.

One cannot apply the above theorems to these spaces and it is therefore of interest to prove the following theorem:

Theorem : M compact symetric with rank(M) ≥ 2 (or any space homotopy equivalent to such a one). Then $b_i(\Lambda(M),Z_2)$ is unbounded and therefore there exist ∞ many closed geodesics in any metric on such a space.

The prove is of geometric nature, it uses Morse Theory for the energy function on the free loop space $\Lambda(M)$. One has to prove that the critical points, which are the closed geodesics, form non degenerate critical submanifolds in $\Lambda(M)$ and then one applies Bott's idea of K-cycles for the loop space with fixed end points to show that the relative homology classes coming from Morse Theory can actually be completed to non vanishing cycles of $\Lambda(M)$. Using the structure theory of globally symetric spaces, in particular the lattice generated by the star vectors of the roots , one can then show, that the Z_2 betti numbers of the free loop space $\Lambda(M)$ are unbounded.

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Title: Lower bounds for discriminants of number fields

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Let K be an algebraic number field of degree n, with r_1 real places and $2r_2$ complex ones, so that $n = r_1 + 2r_2$. Let $D = D_K$ denote the absolute value of the discriminant of K. Using geometry of numbers, Minkowski [1] proved in 1891 that

$$
\begin{aligned}
D^{1/n} &\geqslant (\pi/4)^{2r_2/n}\, n^2 (n!)^{-2/n} \\
&\geqslant (e^2)^{r_1/n} (\pi e^2/4)^{2r_2/n} + o(1) \qquad (n \to \infty) \\
&\geqslant (7.389\ldots)^{r_1/n} (5.803\ldots)^{2r_2/n} + o(1).
\end{aligned}
$$

This estimate was later improved by Rogers [5] and Mulholland [2] to :

$$D^{1/n} \geqslant (32.561\ldots)^{r_1/n} (15.775\ldots)^{2r_2/n} + o(1).$$

They also used geometry of numbers.

A completely different method has been introduced recently by H.Stark [6] . This method is based on the functional equation of the zeta function $\zeta_K(s)$ of K. Stark remarks that one has

$$\log D = r_1(\log\pi - \psi(s/2)) + 2r_2(\log 2\pi - \psi(s)) + 2Z(s) + 2\sum{}^{*}\mathrm{Re}(1/(s-\rho)) - 2/s - 2/(s-1) ,$$

where $Z(s) = -\zeta_K'(s)/\zeta_K(s)$, $\psi(s) = \Gamma'(s)/\Gamma(s)$, and ρ runs through the non trivial zeros of ζ_K (with ρ and $\overline{\rho}$ collected together in order to have an absolutely convergent summation). Taking $s = 1 + n^{-1/2}$, say, gives

$$\frac{1}{n}\log D \geqslant a_1 r_1/n + 2a_2 r_2/n + o(1) ,$$

with $a_1 = \log\pi - \psi(1/2)$, $a_2 = \log 2\pi - \psi(1)$, hence

$$D^{1/n} \geqslant (A_1)^{r_1/n}(A_2)^{2r_2/n} + o(1)$$

with $A_1 = 22.38\ldots$, $A_2 = 11.19\ldots$.

An improved version of Stark's method is given by Odlyzko [3], [4], who obtains

$A_1 = 50.66\ldots$, $A_2 = 19.96\ldots$ [3]

$A_1 = 55$, $A_2 = 21$ [4]

and under the assumption of the generalized Riemann conjecture :

$$A_1 = 136 , \quad A_2 = 34.5 .$$

Those improved lower bounds are of interest in connection with :

a) class field towers ,

b) construction of Galois representations of low conductors (esp. those related to modular forms).

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Title: The Burnside ring of a compact Lie group

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Notation:

G compact Lie group

$H < G$ closed subgroup

X G-space

X^H = H-fixed point set $= \{x \in X \mid Hx = x\}$

(H) conjugacy class of H

G_x isotropy group at $x \in X$

$\chi(Y)$ Euler characteristic of Y

Question 1. What are the relations among the $\chi(X^H)$, all $H < G$?

Since $(H) = (K)$ implies $\chi(X^H) = \chi(X^K)$ we can put it

Question 1'. Which functions from the set of conjugacy classes to $\mathbb{Z}$ have the form $(H) \mapsto \chi(X^H)$?

We first construct the ring of all such functions

$$A(G) \underset{\text{Def}}{=} \text{Set of compact differentiable } G\text{-manifolds} / \sim$$

where

$$M \sim N \iff \text{for all } H < G \quad \chi(M^H) = \chi(N^H)$$

$M + N$ disjoint union induces $+$ in $A(G)$

$M \times N$ cartesian product $\cdot$ in $A(G)$

$A(G)$ becomes a commutative ring with 1.

Definition. $A(G)$ Burnside ring of G.

$$\varphi_H : A(G) \longrightarrow \mathbb{Z} \ : \ M \mapsto \chi(M^H)$$

is a ring homomorphism. By definition of $A(G)$ their product $\varphi = (\varphi_H) : A(G) \longrightarrow \prod_{(H)} \mathbb{Z}$

is an injective ring homomorphism.

Theorem 1 $A(G)$ additively free abelian group on G/H, (H) all conj. classes with NH/H finite (NH normaliser of H).

Proof. (a) $X_{(H)} := \{x \in X \mid (G_x) = (H)\}$

$$X = \cup X_{(H)} \quad \text{disjoint union}$$

$$X = \sum_{(H)} \chi_c(X_{(H)}/G)\, G/H \,, \quad \text{equality in } A(G);$$

χ_c Euler characteristic using homology with compact support.

(b) $\mathrm{Aut}_G(G/H) = NH/H$ acts freely on G/H hence on all G/H^K. $\dim NH/H > 0 \Rightarrow S^1 \subset NH/H$ $\Rightarrow$ free S^1-action on $G/H^K \Rightarrow \chi(G/H^K) = 0 \Rightarrow$ $G/H = 0$ in $A(G)$. ∎

Corollary. G finite, then $A(G) \cong$ classical Burnside ring of finite G-sets.

Theorem 2 Let $G/H \times G/K = \sum_{(L)} n_L\, G/L$ in $A(G)$. Then $n_L \geq 0$, [1], [5].

Proof. $n_L = \chi_c((G/H \times G/K)_{(L)}/G)$. The set inside $\chi_c(\cdot\cdot)$ is finite because G/H^K consists of finitely many NK/K orbits. ∎

$$\begin{array}{ccc} A(G) & \xrightarrow{\varphi_H} & \mathbb{Z} \\ \cup & & \cup \\ q(H,p) := \varphi_H^{-1}(p) & \longrightarrow & (p) \end{array} \qquad p \text{ prime or } 0$$

$q(H,p)$ is prime ideal of $A(G)$.

Theorem 3. Let $q \in A(G)$ be a prime ideal. Then there exists a unique (H) with NH/H finite, $G/H \notin q$ with

$$q = q(H,p).$$

Remark. $G/H \notin q(H,p) \iff \varphi_H(G/H) = \chi(G/H^H) = \chi(NH/H) = |NH/H| \not\equiv 0 \mod p$.

Proof of Th. 3. Dress [2a] for G finite.

Remark. Let $0 \to H \to K \to P \to 0$ be exact, P extension of torus by finite p-group. Then $\chi(X^K) = \chi((X^H)^{K/H}) \equiv \chi(X^H) \mod p$
$\Rightarrow q(H,p) = q(K,p)$.

Examples. $G = SO(3)$, then subgroup with NH/H finite are:
G, T = max torus, NT, Icosahedral, Octahetral, Tetrahedral, D_m Dihedral $(m \geq 2)$.
In general: $(G/T)^2 = |W|\, G/T$, W Weyl group
$(G/H)^2 = |NH/H|\, G/H + \sum a_L\, G/L$, $L \subsetneq H$.
If $NH/H = 1$, does one get idempotent elements? Yes. E.g.
$e_1 = I + \text{Tetra} + D_5 + D_4 + D_3$
$e_2 = NT + O + \text{smaller}$
$1 - e_1 - e_2$ are indecomposable orthogonal idempotents of $A\,SO(3)$.

%.

$$\begin{array}{ccc}
\operatorname{Spec} A(G) & \xrightarrow[\pi]{} & \operatorname{Spec} \mathbb{Z} \\
\cup & & \cup \\
\pi^{-1}(p) & & (p) \\
\shortparallel & & \\
\end{array}$$

$\operatorname{Spec}(A(G) \otimes \mathbb{Z}_p)$ — totally disconneted, compact Hausdorff, $\operatorname{Spec}(A(G) \otimes \mathbb{Q})$ = Set of homo $A(G) \to \mathbb{Z}$ = Set of (H), NH/H finite.
S = Set of closed subgroup of G with Hausdorff metric is compact Hausdorff
$S_0 \subset S$, S_0 = Set of H with NH/H finite

closed subspace. Φ = Set of (H), NH/H finite quotient of S_0.

Theorem 4. $\operatorname{Spec}(AG \otimes \mathbb{Q})$ homeo Φ. [2c]

Topology on Φ uniquely determined by

(i) totally disconnected compact Hausdorff

(ii) for all $a \in A(G)$ $\Phi \to \mathbb{Z} : a \mapsto \varphi(a)$ is continuous (i.e locally constant).

This gives

$\alpha : A(G) \longrightarrow C(\Phi, \mathbb{Z})$ = ring of continuous functions $\Phi \to \mathbb{Z}$

$a \mapsto (\varphi \mapsto \varphi(a))$. α is injective. α is the inclusion of $A(G)$ into the integral closure of $A(G)$ in its total quotient ring.

Theorem 5. (a) There exists a natural number n s.th. $n\, C(\Phi, \mathbb{Z}) \subset \alpha A(G)$. (b) The smallest such $n = n_G$ is l.c.m $\{ |NH/H| \;\; (H) \in \Phi \}$.

Examples G finite, the $n_G = |G|$.

$n_{SU(3)} = 6$.

Corollary Cokernel $\alpha \underset{\text{Def.}}{=\!=}$ "Relations between the $\chi(X^H)$". Relations only n_G-torsion.

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Question 2. V finite dim G-module, $V^c = V \cup \infty$.

Homotopy classes $[V^c, W^c]$ of maps $V^c \to W^c$?

Stabilisation : $V^c \xrightarrow[f]{} W^c \mapsto$ (suspension)

$\mathrm{id}_U \oplus f : (U \oplus V)^c \longrightarrow (U \oplus W)^c$.

$$\varinjlim \, [(U \oplus V)^c, (U \oplus W)^c] =: \omega(V^c, W^c)$$

Similarly $\omega(X, Y)$ $(= \omega^0_G(X; Y)$ more systematic notation)

$\omega(X; S^0) = \omega^0_G(X)$. $\omega^0_G(S^0) = \omega^0_G$.

[4] for G finite

[8], [9] for unstable homotopy sets.

Theorem 6. $A(G) \cong \omega_G^0$.

A map $A(G) \to \omega_G^0$ is given by asigning to X the fixed point index [Dold 3] of $X \to$ Point.

other definitions of $A(G)$

i) compact euclian neighbourhood retracts (equiv.) X (= ENR) instead of manifolds

ii) $U \subset X$ open, X ENR, $f: U \to X$ G-map with compact fixed point set $Fix(f)$

$f \underset{\text{eq}}{\sim} g \iff$ Lefschetz number $(f^H) = L(g^H)$

iii) [Dold 3] $f_0 \sim f_1 \iff \exists$

$$\begin{tikzcd} \tilde{U} \arrow[r, "F"] \arrow[dr] & Y \arrow[d] \\ & X \times [0,1] \end{tikzcd}$$

which restricts over $X \times \{i\}$ to f_i.

Theorem 7. a) The "edge homomorphism"

$\omega_G^0(S) \to H^0(S; \underset{\sim}{\omega_G^0})$ induces homeo of prime ideal spectra.

b) $\operatorname{Spec} \omega_G^0(S) = \varprojlim \operatorname{Spec} \omega_G^0(G/H)$,

the limit taken over the category of homogeneous spaces over S, $G/H \to S$.

(Result analogous to [Quillen, 6].

Let t_*^G equivariant homology theory, e.g. $t_*^G(X) = \omega_*^G(Y; X)$. $t_*^G(X)$ is module over $\omega_*^G = A(G)$. Let $q \subset A(G)$ prime ideal.

Question 3. What is localization ~~of~~ $t_*^G(X)$ at q, $t_*^G(X)_q$?

Suppose $q = q(H, p)$. Put

$F_0 H = \{K \mid q(H,p) = q(K,p)\}$

$F'H = \{K \mid K \lneqq L,\ L \in F_0 H\}$

$F(H) = F'(H) \cup F_0(H)$.

Let for a family F of subgps closed under conjugates and subgroups EF be the universal G-space with isotropy in F, e.g. the classifying space of the category of G/K, $K \in F$, and G-maps.

Theorem 8. The following groups are naturally isomorphic

$$t_*^G(X)_q \cong t_*^G((EFH, EF'H)\times X)_q \cong t_*^G((EFH, EF'H)\times X)_{(p)}$$

$$\cong$$

$$t_*^H((EFH, EF'H)\times X)_q^{\mathrm{inv}} \cong t_*^H(EFH, EFH')\times X)_{(p)}$$

if $q(H,p) = q$ and H is the defining group as in Theorem 3.

$t_*^G(X)_{(p)}$ is essentially the direct sum of the $t_*^G(X)_{q(H,p)}$. (Technically, if there are an infinite number of $q(H,p)$ one has to take continuity into account; "sections of the structure sheaf")

Title: Modular Representations of Semi-simple Groups

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Denote by G an almost simple, connected algebraic group over an algebraically closed field k, by T a maximal torus of G, by R the root system, by R^+ a positive system of roots, by W the Weyl group and by (,) a scalar product on $X\otimes\mathbb{R}$, where X is the group of characters of T. Let be $X^+=\{\lambda\in X \mid (\lambda,\alpha)\geq 0$ for all $\alpha\in R^+\}$ and $\rho=\frac{1}{2}\sum_{\alpha\in R^+}\alpha$. If V is a (rational, as always) G-module and $\mu\in X$, let V^μ be the μ weight space $\{x\in V / hx=\mu(h)x$ for all $h\in T\}$; it is $V=\coprod_{\mu\in X} V^\mu$. Let $\mathbb{Z}[X]$ be the group ring of X and $e(\lambda)$ with $\lambda\in X$ be the canonical basis of it. For all $\lambda\in X$ set $\chi(\lambda)=\sum_{w\in W}\det(w)e(w(\lambda+\rho))/\sum_{w\in W}\det(w)e(w(\rho))$. For a G-module V denote by ch(V) the formal character $\sum_{\mu\in X}\dim(V^\mu)e(\mu)\in\mathbb{Z}[X]$ of V.

For $\lambda\in X^+$ let $V(\lambda)$ be a complex irreducible representation of the corresponding complex group with the highest weight λ; choose $v\in V(\lambda)^\lambda$, $v\neq 0$. There is a unique smallest (resp. greatest) "admissible lattice" $V(\lambda)_{\mathbb{Z}}$ (resp. $V(\lambda)_{\mathbb{Z}}^{\max}$) such that $V(\lambda)_{\mathbb{Z}}^\lambda=\mathbb{Z}v$ (resp. $V(\lambda)_{\mathbb{Z}}^{\max,\lambda}=\mathbb{Z}v$). Then $V(\lambda)_k=V(\lambda)_{\mathbb{Z}}\otimes k$ is a G-module and has a unique simple quotient $L_k(\lambda)$.

<u>Theorem</u> (Chevalley) <u>The different</u> $L_k(\lambda)$ <u>are not isomorphic; each simple G-module is isomorphic to one</u> $L_k(\lambda)$.

For all $\lambda\in X^+$ we have $\chi(\lambda)=ch(V(\lambda)_k)$ (H. Weyl), therefore there are $a_{\lambda,\lambda'}\in\mathbb{N}$ such that $\chi(\lambda)=\sum_{\lambda'\in X^+}a_{\lambda,\lambda'}ch(L_k(\lambda'))$. Let W_p be the group generated by W and the translations by $p\alpha$ with $\alpha\in R$.

<u>Theorem: If</u> a) (Humphreys) $p>h$=<u>Coxeter number of</u> R <u>or if</u> b) (Carter & Lusztig) <u>R is of type</u> A_n, <u>then</u> $a_{\lambda,\lambda'}\neq 0$ <u>implies that there is</u> $w\in W_p$ <u>with</u> $\lambda'+\rho=w(\lambda+\rho)$

<u>Theorem: If type(R)</u>$\neq E_n, F_4$ <u>and</u> $p>h$, <u>if</u> $a_{\lambda,\lambda'}\neq 0$, <u>then there are reflections</u> $s_1,s_2,\ldots,s_n$ <u>in</u> W_p <u>such that</u>

$$\lambda+\rho>s_1(\lambda+\rho)>(s_2s_1)(\lambda+\rho)>\ldots>(s_n\ldots s_2s_1)(\lambda+\rho)=\lambda'+\rho.$$

(We define $\lambda\geq\mu$ by $\lambda-\mu\in\mathbb{N}R^+$)

Let C be a chamber with respect to the reflection group W_p i.e. there are integers n_α $(\alpha\in R^+)$ such that

$C=\{ x\in X\otimes\mathbb{R} \mid n_\alpha p<2(x,\alpha)/(\alpha,\alpha)<(n_\alpha+1)p$ for all $\alpha\in R^+\}$;

we then set

$\hat{C}=\{x\in X\otimes\mathbb{R} \mid n_\alpha p<2(x,\alpha)/(\alpha,\alpha)<(n_\alpha+1)p$ for all $\alpha\in R^+\}$.

Theorem: If $\lambda\in X^+,\lambda+\rho\in C,\mu+\rho\in X\cap\bar{C}$ and $ch(L_k(\lambda))=$ $\sum_{w\in W_p} b_{\lambda,w}\chi(w(\lambda+\rho)-\rho)$, then $\sum_{w\in W_p} b_{\lambda,w}\chi(w(\mu+\rho)-\rho)=ch(L_k(\mu))$ for $\mu+\rho\in\hat{C}$; if $\mu+\rho\notin\hat{C}$ the sum is 0.

Theorem: Assume $\lambda,\mu\in X^+,\lambda+\rho\in C,s\in W_p$ a reflection such that $\mu+\rho=s(\lambda+\rho)<\lambda+\rho$, assume there are not reflections $s_1,\ldots,s_n$ in W_p $(n>1)$ such that $\lambda+\rho>s_1(\lambda+\rho)>\ldots>(s_n\ldots s_1)(\lambda+\rho)=\mu+\rho$; then: a) (Carter & Lusztig) If type$(R)=A_n \Rightarrow 0\neq Hom_G(V(\mu)_k,V(\lambda)_k)$. b) If type$(R)\neq E_n,F_4$, then the multiplicity of $L_k(\mu)$ in $V(\lambda)_k$ is 1.

For all $\lambda\in X^+,\mu\in X$ let be $D_\lambda(\mu)=[V(\lambda)^{\max,\mu}_{\mathbb{Z}}:V(\lambda)^{\mu}_{\mathbb{Z}}]$. Let ν_p be the p-adic valuation: $\nu_p(p^n)=n$ and $\nu_p(D_\lambda)=\sum_\mu \nu_p(D_\lambda(\mu))e(\mu)$ in $\mathbb{Z}[X]^W$; there are $c_{\lambda,\mu}$ such that $\nu_p(D_\lambda)=\sum_{\mu\in X^+}c_{\lambda,\mu}ch(L_k(\mu))$. If the $a_{\lambda,\mu}$ are as above, then:

Proposition: For $\lambda\neq\mu$ we have $a_{\lambda,\mu}\leq c_{\lambda,\mu}$ and $a_{\lambda,\mu}\neq 0$ if and only if $c_{\lambda,\mu}\neq 0$.

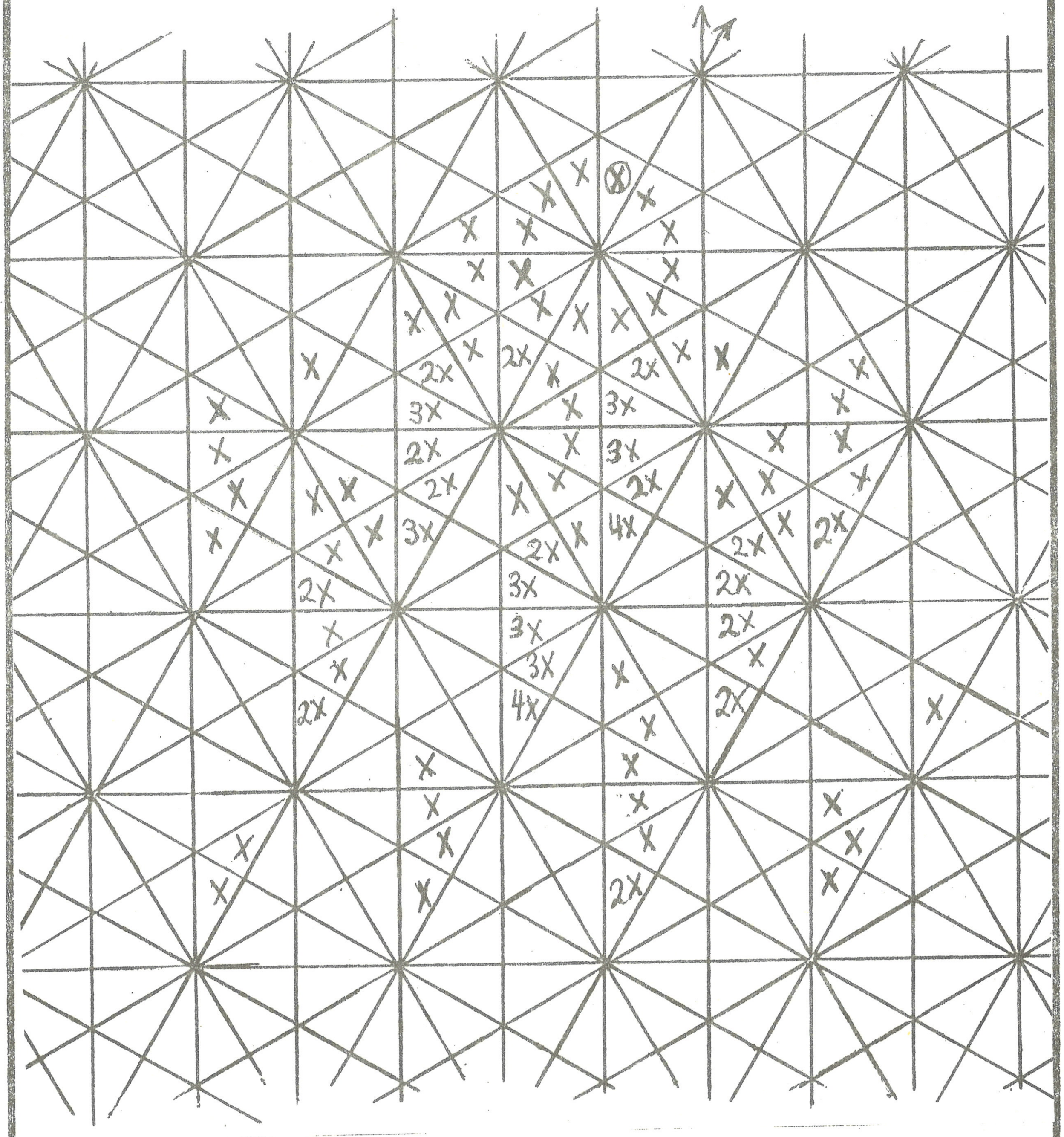
Theorem: If $p>h$ and if type$(R)\neq E_n,F_4$ then:

$$\nu_p(D_\lambda)=\sum_{\alpha\in R^+}\ \sum_{0<r<2(\lambda+\rho,\alpha)/(\alpha,\alpha)}\nu_p\left(\frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)}-r\right)\chi(\lambda-r\alpha).$$

The $ch(L_k(\lambda))$ have been known for a long time for type A_1, for type A_2 they have been determined by B. Braden (1967). These methods give them for types A_3,B_2 and G_2 ($p\neq 5$ here).

Example: Let R be of type G_2; the picture on the following page gives the system of hyperplanes for W_p; let C be the chamber indicated by $\otimes$ and assume $\lambda\in X^+,\lambda+\rho\in C$; for all $w\in W_p$ there is written $n_w\chi$ inside the chamber $w(C)$; hereby we leave away 0χ and write χ instead of 1χ. Then we have:

$$\mathrm{ch}(V(\lambda)_k)=\sum_{w\in W_p} n_w \,\mathrm{ch}(L_k(w(\lambda+\rho)-\rho))$$ provided all $w(\lambda+\rho)-\rho$ with $n_w\neq 0$ are in X^+ and all $2(\lambda+\rho,\alpha)/(\alpha,\alpha)$ are "sufficiently" smaller than p^2 (all $\alpha\in R^+$).



type G_2

Title: Blattner's conjecture on the discrete series

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Let G be a connected, semisimple Lie group. For reasons of convenience, assume that G has a simply connected complexification $G^{\mathbb{C}}$.

An irreducible, unitary representation of G is said to be *square-integrable* if it can be realized on an invariant subspace of $L^2(G)$. The *discrete series* is the set of isomorphism classes of such irreducible, square-integrable representations.

Choose a maximal compact subgroup $K \subset G$, and a maximal torus $H \subset K$. Assume that $\mathrm{rk}\, K = \mathrm{rk}\, G$ (i.e. H is a compact Cartan subgroup of G). Exactly in this situation, G has a non-empty discrete series (Harish-Chandra). Via exponentiation, $\hat{H} = \mathrm{Hom}(H, \mathbb{C}^*)$ becomes isomorphic to a lattice Λ in $i\mathfrak{h}^*$ ($\mathfrak{h}$ = Lie algebra of H, $\mathfrak{h}^*$ = dual space of $\mathfrak{h}$). Let W = Weyl group of H in G, $W_{\mathbb{C}}$ = Weyl group of $H^{\mathbb{C}}$ in $G^{\mathbb{C}}$. Then $W \subset W_{\mathbb{C}}$, and both Weyl groups operate faithfully on H, $i\mathfrak{h}^*$, and Λ. Fix a Weyl chamber C for W in $i\mathfrak{h}^*$, and enumerate those Weyl chambers for $W_{\mathbb{C}}$, which lie in C, as $C_1, \ldots, C_N$ (note: all Weyl chambers are to be open; $\overline{C}$ will denote the closure of C).

Harish-Chandra has constructed a natural bijection between the discrete series and $\bigcup_{j=1}^{N} (\Lambda \cap C_j)$. The precise definition of this bijection involves a certain description of the characters of the discrete series.

Fix an element λ of one of the $C_j \cap \Lambda$, and let π_λ be a discrete series representation corresponding to λ. For any $\mu \in \Lambda \cap \overline{C}$ ($\cong$ set of isomorphism classes of irreducible K-modules), set $n_\lambda(\mu)$ = multiplicity with which the irreducible K-module of highest weight μ occurs in $\pi_\lambda|_K$. Blattner has conjectured a formula for the integers $n_\lambda(\mu)$, which is formally analogous to Kostant's formula for the multiplicity of a weight in the finite-dimensional case.

<u>Theorem</u> (joint with H. Hecht) Blattner's formula for the $n_\lambda(\mu)$ is correct.

According to the multiplicity formula, there is a distinguished $\mu(\lambda) \in \Lambda \cap \overline{C}$, such that
a) the irreducible K-module of highest weight $\mu(\lambda)$ occurs in $\pi_\lambda|_K$, and
b) for every $\mu \in \Lambda \cap \overline{C}$, which is lower than

$\mu(\lambda)$ (in an appropriate sense), the irreducible K-module of highest weight μ does not occur in $\pi_\lambda|_K$.

<u>Theorem</u> Among all (not necessarily unitary) irreducible representations of G, π_λ is uniquely characterized, up to infinitesimal equivalence, by the conditions a) and b).

The preceeding statement is a formal analogue of the "theorem of the highest weight" in the representation theory of compact Lie groups. As a corollary to the proof of this theorem, one obtains the following result, which was proven independently by G. Zuckerman:

<u>Corollary</u> Let π be a representation of G with a finite decomposition series. Suppose that each irreducible sub-quotient is infinitesimally equivalent to a discrete series representation. Then π is completely reducible.

In addition to the discrete series representation π_λ, consider a finite-dimensional representation σ of G. Then $\pi_\lambda \otimes \sigma$ necessarily has a finite decomposition series. The next statement comes up as a by-product of the proof of Blattner's conjecture:

<u>Theorem</u> (joint with H. Hecht) Every irreducible

sub-quotient of $\pi_\lambda \otimes \sigma$ is infinitesimally equivalent to a discrete series representation $\Longleftrightarrow$

(*) for every noncompact root β which has a positive inner product with λ, and for every weight ν of σ, $(\lambda+\nu, \beta) > 0$.

<u>Corollary</u> In the situation (*), $\pi_\lambda \otimes \sigma$ is a direct sum of discrete series representations (up to infinitesimal equivalence).

An irreducible, unitary representation π of G is said to be <u>integrable</u> if there exist non-zero vectors u, v, such that the function $g \mapsto (\pi(g)u, v)$ lies in $L^1(G)$. Every integrable representation is necessarily square-integrable. Trombi and Varadarajan have shown: if π_λ is integrable, then the condition (*) holds, for the finite-dimensional, irreducible representation of highest weight ρ (ρ = one half of the sum of the positive roots). The last corollary easily implies that this condition is not only necessary, but also sufficient for the integrability of π_λ. With more care, one should be able to use infinitesimal arguments of this sort to derive much finer information about the asymptotic behavior of the matrix coefficients of discrete series representations.

Title: Newton's diagrams of singularities

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Let N be the set of nonnegative integes, R_+ be the set of nonnegative real numbers and R be the set of all real numbers. Let K be a subset of N^k, $\Gamma_+(K)$ be the convex hull of the set $\left(\bigcup_{n\in K}(n+R_+^k)\right)$ in R_+^k. The polygon $\Gamma(K)$ which is the union of all compact facets of $\Gamma_+(K)$ is called Newton's diagram of K. Let $f=$
$\sum_{n\in N^k} a_n x^n$, $a_n \in \mathbb{C}$, and $\operatorname{supp} f = \{n \in N^k : a_n \neq 0, n \neq 0\}$
The diagram $\Gamma(\operatorname{supp} f)$ is denoted by $\Gamma(f)$ and called Newton's diagram of f. Let $\Gamma_-(K)$ be the union of all segments connecting the origin and the points of $\Gamma(K)$. Let F_K be the set of all functions analytic in the origin for which $\Gamma(f)=\Gamma(K)$, and for any $f(x)=\sum_{n\in N^k} a_n x^n$ in F_K let f_K be the polynomial $\sum_{n\in\Gamma(K)} a_n x^n$. For any compact polygon S in R_+^k let $V(S)=k!\,V_k-(k-1)!\,V_{k-1}+\ldots+(-1)^{k-1}1!\,V_1+(-1)^k$, where V_k – k-dim volume of S, V_q, $q<k$, – the sum of the q-dim volumes of intersections of S with all q-dim coordinate planes. Let $\mu(f)=\mathbb{C}[[x_1,\ldots,x_k]]/\left(\frac{\partial f}{\partial x}\right)$ be Milnor's number of series f.

<u>Теорема</u> (Kushnirenko) Suppose that $K \cap (\text{any coord. axes}) \neq \emptyset$
1. $\mu(f) \geq V(\Gamma_-(K))$, 2. $\exists$ a proper algebraic subset A of $\{f_K \mid f \in F_K\}$ such that for $f \in F_K$ with $f_K \notin A$ $\mu(f) = V(\Gamma_-(K))$.

Let $K\subset N^k$ – finite and $\Delta(K)$ be the convex hull of K. $\forall$ polynomial $f(x)=\sum a_n x^n$ let $\Delta(f)$ be the convex hull of $\{n\in N^k \mid a_n\neq 0\}$
Let $\Delta_1,\Delta_2\subset R^k$ be convex polygons and $\alpha_1,\alpha_2\in R^+$. Then $\alpha_1\Delta_1+\alpha_2\Delta_2 \underset{def}{=} \{\alpha_1 v_1+\alpha_2 v_2\in R^k \mid v_1\in\Delta, v_2\in\Delta_2\}$. The unique polylinear symmetric function $V(\Delta_1,\dots,\Delta_k)$ on k-tuples of convex polygons in R^k with condition $V(\Delta,\dots,\Delta)=V(\Delta)$ – ordinary volume is called the mixed volume of Minkowski.

<u>Th.</u> (D. Bernshtein) Let $K_1,\dots,K_k\subset N^k$ – finite $\Rightarrow \exists$ a proper algebraic $A\subset\{$set of all k-tuples of polynomials $(f_1(x),\dots,f_k(x))$ $\mid \Delta(f_i)=\Delta(K_i)\}$ such that if $(f_1,\dots,f_k)$ with $\Delta(f_i)=\Delta(K_i)$ is not in A then $\#\{x\in \mathbb{C}^k \mid f_1(x)=\dots=f_k(x)=0,\ x_1,\dots,x_k\neq 0\} = k!\,V(\Delta_1,\dots,\Delta_k)$.

<u>Th.</u> Let $K\subset N^k$ and $(t_0,\dots,t_0)=\{x\in\mathbb{R}^k \mid x_1=\dots=x_k\}\cap\{$boundary of $\Gamma_+(K)\}$. Let $\beta(K)$ be the number $1/t_0 \Rightarrow \exists$ a proper algebraic $A\subset\{f_K \mid f\in F_K\}$ such that for any $f\in F_K$ having singularity in the origin and $f_K\notin A$ the following is true. $\forall$ sufficiently small neighborhood V of the origin in R^k we have 1. $\forall \varphi\in C^\infty(R^k,R)$ vanishing outside V and $\forall\varepsilon>0$
$I(f,\varphi,\lambda)=\int_{R^k} e^{i\lambda f(x)}\varphi(x)\,dx<\lambda^{-\beta(K)+\varepsilon}$ when $\lambda\to+\infty$.
2. if $\beta(K)<1$ then $\exists\varphi\in C^\infty(R^k,R)$ vanishing outside V such that $\forall\varepsilon>0$ $I(f,\varphi,\lambda)>\lambda^{-\beta(K)-\varepsilon}$ when $\lambda\to+\infty$.

Let f be a real function analytic in the origin. Let $f_+(x)=\begin{cases} f(x) & f(x)>0\\ 0 & f(x)\le 0\end{cases}$, $\varphi\in C^\infty(R^k,R)$ with a sufficiently small support $\lambda\in\mathbb{C}$. Then $\int_{R^k} f_+^\lambda(x)\varphi(x)\,dx$ is meromorphic function of λ and it's poles belong to a finite number of arithmetic progressions

<u>Th.</u> Let $K\subset N^k \Rightarrow \exists$ proper algebraic $A\subset\{f_K \mid f\in F_K\}$ such that if $f\in F_K$, $f_K\notin A$ then the arithmetic progressions containing the poles of $\int_{R^k} f_+^\lambda(x)\varphi(x)\,dx$ can be calculated from $\Gamma(K)$.

1

Title: Tannaka-Artin's conjecture on the multiplicative group of a divison algebra (by V.P. Platonov)

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Izv. Akad. Nauk SSSR Ser. Mat. (to appear)

Let A be a simple finitedimensional associative algebra with a center F, $A^{(1)}$ be the commutator group of a multiplicative group A^* and $S(A) = \{a \in A^* \mid Nrd_{A/F}(a) = 1\}$ where $Nrd_{A/F} : A \to F$ is a reduced norm on A. It is clearly that $A^{(1)} \subseteq S(A)$.

Tannaka and Artin (1943) posed the question: is $S(A) = A^{(1)}$?

We have $A = L(n, D)$, D is a division algebra with a center F, so $S(A) = SL(n, D) = A^{(1)} \iff S(D) = D^{(1)}$.

The T.-A. question was formulated by H. Bass also in his book on algebraic K-theory in the form: $SK_1(A) = 1$?

If one takes into account a well known Dieudonne's results, then one has the implications:

Positive solution of the T.-A. problem $\iff$ either (i) $PSL(n, D)$, $n>1$, is a simple group or (ii) $SL(n,D)$ $n>1$, is generated by unipotent elements (the special case of Kneser-Tits problem (1963) on F-isotropic simple connected algebraic group G).

Some progress was achived in a direction of a positiv solution:

1) Nakayma-Matsusima (1943): for p-adic number field F;
2) S. Wang (1950): for an algebraic number field F;
3) V.P. Platonov - V.T. Janchevski (1974): for any local compact field and for any functional (in one indeterm.) with a finite constant field.

But in general (and as a rule) the answer to the T.-A. question is turn out to be negative. This unexpected result has been achived quite recently by V.P. Platonov (Minsk, USSR). He delivered a lecture about it on the meeting of Moscow mathematical society (April, 1975) and published two short notes [1, 2]. The details will appear in [3].

The aim of this talk is to formulate the result by Platonov and to give some details.

As it was remarked we can take A to be a division algebra. Then $[A:F]=p^{2m}$ because A is a tensor product of division algebras with relatively prime indices. $S(A)=A^{(1)}$ if $m=1$ (Kodama). It is of especial interest that for $m=2$ an inequality $S(A)\neq A^{(1)}$ holds.

Some notation. Let p, r be prime numbers $p\neq r$, $\mathbb{Q}$ - rational field, $\mathbb{Q}_p$ be p-adic numbers field; $F(x,y)$ - rational functions field, $F\langle x,y\rangle$ be a formal power series field; $T(a,b)$ be a quaternion algebra over F with parametres $a,b\in F$.

<u>The main theorem</u> (V.P. Platonov). Let p be a prime number with $\left(\frac{p}{3}\right)=-1$; $A(p,3)=T(x,p)\otimes_{Q(x,y)} T(y,3)$; $\tilde{A}(p,3)=A(p,3)\otimes_{Q(x,y)} Q\langle x,y\rangle$; $\tilde{\tilde{A}}(p,3)=A(p,3)\otimes_{Q_3(x,y)} Q_3\langle x,y\rangle$.

Then we have $S\tilde{\tilde{A}}(p,3)\neq\tilde{\tilde{A}}^{(1)}(p,3)$ and $SA(p,3)\not\subset\tilde{\tilde{A}}^{(1)}(p,3)$ for $p=2$ or for $p\equiv 1 \pmod 8$. Also $S\tilde{A}(2,3)\neq\tilde{A}^{(1)}(2,3)$.

<u>Some remarks</u>. Platonov proved a much more general result. He gave some method for the calculating of $SK_1(A)=S(A)/A^{(1)}$ too. In particular he showed that $SK_1(A)$ may be any elementary abelian p-group at last. The all results are extended for $x_1, x_2, \dots x_n$, $n\geq 2$ instead of x, y. The division algebra A with an almost any indice can be constructed for which $S(A)\neq A^{(1)}$. At last the conditions on the field and on primes are taken only for clearity.

But we restrict our attention on the above special case.

<u>Some details</u>

(I) $\tilde{A}(p,r)$ and $\tilde{\tilde{A}}(p,r)$, $\left(\frac{p}{r}\right)=-1$, are division algebras.

Let us denote: O_D be the ring of integral elements in a div alg. D; $\Omega(p,r)=Q(\sqrt{p},\sqrt{r})$; $\tilde{\Omega}(p,r)=Q_r(\sqrt{p},\sqrt{r})$.

(2). $\left(\frac{p}{r}\right)=-1,\ r\neq 2,\ \Rightarrow O_{\tilde{\tilde\Omega}(p,r)} = O_r + O_r\sqrt{p} + O_r\sqrt{r} + O_r\sqrt{pr}$

(3) $N_{\tilde\Omega(2,3)/Q(\sqrt3)}(\delta_0) = 2+\sqrt3,\quad \delta_0 = 1+\sqrt3+\frac12\sqrt2+\frac12\sqrt6.$

(4) $N_{\tilde{\tilde\Omega}(2,3)/Q_3(\sqrt3)}(\tilde{\tilde\Omega}(2,3)) \supset Q_3.$

(5) $\left(\frac{p}{3}\right)=-1$ & $t\in Q_3^{*} \Longrightarrow (3+\sqrt3)t \notin N_{\tilde{\tilde\Omega}(p,3)/Q_3(\sqrt3)}(\tilde{\tilde\Omega}(p,3))$

$2+\sqrt3 = \epsilon\cdot\sigma(\epsilon)^{-1},\quad \epsilon=(3+\sqrt3)t,\quad \sigma: \alpha+\beta\sqrt3\mapsto\alpha-\beta\sqrt3.$

(6) $\left(\frac{p}{3}\right)=-1 \Longrightarrow (2+\sqrt3)\in N_{\tilde\Omega(p,3)/Q(\sqrt3)}(\tilde\Omega(p,3))$

(7) Let $\tilde{\mathfrak a}(p,r)$ be a prime ideal of $O_{\tilde A(p,r)}$, Then we have

$$O_{\tilde A(p,3)}/\tilde{\mathfrak a}(p,3) \cong T(x,p)\otimes_{Q\langle x\rangle} Q(\sqrt3)\langle x\rangle$$

(8) Let $\tilde D(p,3) = T(x,p)\otimes_{Q(x)} Q(\sqrt3)\langle x\rangle$, $\tilde{\tilde D}(p,3) = T(x,p)\otimes_{Q_3(x)} Q_3(\sqrt3)\langle x\rangle$ $\left(\frac{p}{3}\right)=-1$

Then we have:

$O_{\tilde D(p,3)}/\mathfrak a_{\tilde D(p,3)} \cong \tilde\Omega(p,3)$; $O_{\tilde{\tilde D}(p,3)}/\mathfrak a_{\tilde{\tilde D}(p,3)} \cong \tilde{\tilde\Omega}(p,3)$,

(9) Let us keep the notation of Pr. (9).

$\left(\frac{p}{3}\right)=-1 \Rightarrow (2+\sqrt3)\in Nrd_{\tilde{\tilde D}(p,3)/Q_3(\sqrt3)\langle x\rangle}(\tilde{\tilde D}(p,3))$

(10) $t\in(Q_3\langle x\rangle)^{*} \Rightarrow (3+\sqrt3)t \notin Nrd_{\tilde{\tilde D}(p,3)/Q_3(\sqrt3)\langle x\rangle}(\tilde{\tilde D}(p,3))$

We can put in general $t\in Q_3[[x]]$. Notation: $\bar t$ is a class of t mod $\mathfrak a_{\tilde{\tilde D}(p,3)}$.

(11) Let $\pi=\sqrt{y}$. Then π generates an involutiv inner automorphism of $\tilde A(p,3)$ and $\tilde{\tilde A}(p,3)$.; π induces an aut. $\tilde\varphi$ (corresp. $\tilde{\tilde\varphi}$) of $O_{\tilde A(p,3)}/\tilde{\mathfrak a}$ and $O_{\tilde{\tilde A}(p,3)}/\tilde{\tilde{\mathfrak a}}$; Moreover $\tilde\varphi$, $\tilde{\tilde\varphi}$ are not indentity on the center.

(12) $w\in\tilde A^{(1)}$ or $\tilde{\tilde A}^{(1)} \Longrightarrow Nrd_{\tilde D(p,3)/Q(\sqrt3)\langle x\rangle}(\bar w) = \theta\tilde\varphi(\theta)^{-1}$

where $\theta\in Nrd_{\tilde D(p,3)/Q(\sqrt3)\langle x\rangle}(\tilde D(p,3))$.

(The same statement holds for $\tilde{\tilde D}$).

(13) Let $\delta_0\in S\tilde{\tilde A}(p,3)$, if $\left(\frac{p}{3}\right)=-1$, and

$$Nrd_{\tilde{\tilde D}(p,3)/Q_3(\sqrt3)\langle x\rangle}(\delta) = 2+\sqrt3$$

Then $\delta\notin\tilde{\tilde A}^{(1)}(p,3)$; $p=2\Rightarrow\delta_0\notin\tilde{\tilde A}^{(1)}(2,3)$ (see (2)).

(14) Conclusion: compare (5), (12) and (13).

Title: Rational points on modular curves

Name of author: B. Mazur

Address: Harvard University

Bibliography: [1] B. Mazur, J.-P. Serre, Points rationnels des courbes modulaires $X_0(N)$ séminaire Bourbaki no. 469 June 1975

[2] B. Mazur, Modular curves and the Eisenstein ideal (in preparation)

Let N be a positive integer. Let $X_1(N)$ and $X_0(N)$ denote the modular curves over Q as in [1] . The arithmetic structure of these curves is related to the arithmetic of elliptic curves in two ways (one of which is conjectural):

A. To a given pair (E,e_N) consisting in an elliptic curve E and a point of order N in E, defined over a field K containing Q we may associate a noncuspidal point $j(E,e_N)$ in $X_1(N)$ rational over K. For $N \geq 3$, this establishes a bijection between the K-isomorphism classes of such pairs and noncuspidal rational points of $X_1(N)$ over K.

To a given pair (E,C_N) consisting in an elliptic curve E and a cyclic subgroup of order N in E, defined over a field K containing Q we may associate a noncuspidal point $j(E,C_N)$ in $X_0(N)$ rational over K. This establishes a bijection between the K-rational noncuspidal points of $X_0(N)$ and equivalence classes of pairs (E,C_N) over K, where two pairs are equivalent if they are isomorphic over some extension field of K.

B. Conjecture of Weil Any elliptic curve E over Q may be obtained as a quotient (over Q) of J, the jacobian of $X_0(N)$.

Suppose that N is a <u>prime number</u>. The <u>Hecke algebra</u> $\underline{T}$ is the subring of the ring of endomorphisms of J generated by the Hecke operators T_l for prime numbers $l \neq N$, and by the "canonical involution" w. These facts concerning $\underline{T}$ are known. $\underline{T}$ is a commutative algebra, which as a Z-module is free of rank $g = \dim J$. $\underline{T}\otimes Q$ is isomorphic to a product of totally real number fields $\prod_j k_j$. There is are one:one correspondences

$$\begin{array}{ccc}
\left\{\begin{array}{l}\mathbb{C}\text{-simple}\\ \text{abelian variety}\\ \text{factors of } J\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{l}Q\text{-simple}\\ \text{abelian variety}\\ \text{factors of } J\end{array}\right\} \\
\updownarrow & & \updownarrow \\
\left\{\begin{array}{l}\text{fields } k_j \text{ occurring}\\ \text{in the product de-}\\ \text{composition of } \underline{T}\otimes Q\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{l}\text{irreducible components}\\ \text{of Spec } \underline{T}\end{array}\right\}.
\end{array} \qquad \ast$$

Since N is a prime number, there are two cusps $0, \infty$ in $X_0(N)$ and these are rational points. The linear equivalence class of their difference gives an element $c = Cl(\,0-\infty\,)$ in $J(Q)$. It is a theorem of Ogg that c is a point of order precisely $n = \mathrm{num}\left(\frac{N-1}{12}\right)$ where num means numerator. One can check that $n > 1$ if and only if $g > 0$. The action of the Hecke operators on c is easily computed: $T_l \cdot c = (1+l)\cdot c$; $w.c = -c$.

By the <u>Eisenstein ideal</u> I in $\underline{T}$ one means the ideal generated by the elements $1+l-T_l$ for all prime numbers $l \neq N$, and by $1+w$. Since $I\cdot c = 0$, and c is nontrivial if $g > 0$, one sees that $I \neq \underline{T}$ whenever $g > 0$.

By the <u>Eisenstein quotient</u> $\tilde{J}$ one means a certain abelian variety over Q, which is a quotient of J, and, up to isogeny, can be taken to be the product of those simple abelian variety factors of J which correspond (under $\ast$) to irreducible components of Spec $\underline{T}$ which meet the support of the Eisenstein ideal I.

Three recent theorems were discussed, in the lecture.

THEOREM 1. $\tilde{J}(Q)$ is finite.

From this result one easily concludes

THEOREM 2. If $g > 0$, then $X_0(N)$ has only a finite number of rational points.

The third theorem has to do with $J_+ = (1+w).J$ which may be identified with the jacobian of the quotient curve $X_0^+(N) = X_0(N)/w$.

THEOREM 3. If the genus of $X_0^+(N)$ is greater than 0, then $J_+(Q)$ contains a point of infinite order.

Calculations for low values of N were discussed. In particular, $X_0(163)$ has precisely one noncuspidal rational point. More generally, using the above methods, one has a complete determination of the set of rational points of $X_0(N)$ for all prime numbers $N < 250$, with five exceptions: $N = 53, 113, 137, 151,$ and 227. This has has been carried out with the help of A. Ogg, and A. Brumer and K. Kramer.

In contrast, I know of no value of N for which $X_0^+(N)$ has been shown to have only a finite number of rational points.

The $\mathfrak{n}$-cohomology of $\mathfrak{g}$-modules

W. Casselman

References: 1. F. Aribaud, Bull. Math. Soc. France 95 (1967)
2. P. Cartier, Annals of Math. 74 (1961)
3. B. Kostant, " "

Scott Osborne and I recently (found) a generalization and a simpler proof of an old result of Kostant.

Let

$$\mathfrak{g} = \text{semi-simple algebra} / \mathbb{C}$$
$$\mathfrak{p} = \mathfrak{m} + \mathfrak{n} = \text{a parabolic subalgebra}$$

One has the Harish-Chandra homomorphism $\sigma : Z(\mathfrak{g}) \longrightarrow Z(\mathfrak{m})$, where $Z(\mathfrak{g})$ etc. = center of ~~enveloping~~ enveloping algebra of $\mathfrak{g}$ etc: it is characterized by the property that for any $X \in Z(\mathfrak{g})$, $X - \sigma(X) \in U(\mathfrak{g})\mathfrak{n}$.

If W = a $\mathfrak{g}$-module, then by restriction it is a $\mathfrak{p}$-module, so that $H^*(\mathfrak{n}, W)$ is an $\mathfrak{m}$-module, hence an $U(\mathfrak{m})$-module.

Theorem. <u>If</u> $X \in Z(\mathfrak{g})$ <u>annihilates</u> W, <u>then</u> $\sigma(X)$ <u>annihilates</u> $H^*(\mathfrak{n}, W)$.

If W is irreducible, for example, then one knows there exists a homomorphism $\Theta_W : Z(\mathfrak{g}) \to \mathbb{C}$ such that $X \cdot w = \Theta_W(X) \cdot w$ (Θ_W = infinitesimal character), and thus W is annihilated by the maximal ideal in $Z(\mathfrak{g})$ ~~$= \{X - \Theta$~~ generated by $\{X - \Theta_W(X) \cdot I\}$, and in this case, because one knows that ~~is~~ $Z(\mathfrak{m})$ is a finite ring over $\sigma(Z(\mathfrak{g}))$, we deduce that there are only a finite number of possibilities for the infinitesimal characters of the ~~$\mathfrak{m}$-module~~ factors of the $\mathfrak{m}$-module $H^*(\mathfrak{n}, W)$.

When W is irreducible and finite-dimensional, ~~this~~ and X is $C - \Theta_W(C)$ (C = Casimir generator of $\mathcal{U}(\mathfrak{g})$), this result is a lemma due to Aribaud. In this case, we can deduce in a very elementary fashion the result of Kostant's:

Corollary: If W = irreducible and finite-dimensional with highest weight λ, and $\mathfrak{y}$ = a Borel subalgebra, then $H^n(\mathfrak{n}, W) \cong \bigoplus_{l(w)=n} w(\lambda+\rho)-\rho$, where for $w \in W$, $l(w)$ = length of w, and ρ = one-half the sum of the positive roots.

(Kostant also has a result for other $\mathfrak{y}$, a bit more complicated, ~~which~~ which also follows from Aribaud's Lemma.) (An elementary argument due to Cartier plays a role here.)

The proof, as it has to be for such a general result, is trivial: let one be given a resolution of V by injective $\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})X = \mathcal{U}_X$-modules:

$$0 \to V \to V^0 \to V^1 \to \cdots$$

and apply this:

Lemma. Any ~~$\mathcal{U}_X$-injective module~~ injective $\mathcal{U}_X$-module is $\mathfrak{n}$-acyclic.

The cohomology ~~of V is~~ $H^*(\mathfrak{n}, V)$ is therefore the cohomology of the complex

$$0 \to (V^0)^n \to (V^1)^n \to \cdots$$

so that the theorem reduces to the case of H^0, which is trivial.

Details will appear in the Compositio Mathematica

Title: Residues and symbols

Name of author: A. N. Parshin

Address: Steklov Mathematical Institute, Moscow 333, USSR

Bibliography:

1) A. Parshin, Class-fields and algebraic K-theory, УМН, 30(1975), №1

2) A. Parshin, Residues and duality on algebraic surfaces, УМН (to appear)

Let X be a smooth proper surface over k, $K=k(X)$ and $\omega \in \Omega^2_{K/k}$. Consider pairs (m, p) where m is a point of X, p is an irreducible curve on X and also it is fixed some non-singular branch of p in m. If u is a local parameter defining the branch of p in the neighbourhood of m and $\mathrm{res}_u\, \omega$ is a coefficient in the decomposition of ω by the degrees of u, then we can restrict form $\mathrm{res}_u\, \omega$ of degree 1 on the branch of p u then take a usual residue on p in the point m. It can be proved that the resulting number $\mathrm{res}_{m,p}\, \omega$ depends only on m, p (a branch of p) and ω and has the following properties:

1) $\sum_{m \in p} \mathrm{res}_{m,p}\, \omega = 0 \quad \forall p \subset X$

2) $\sum_{p \ni m} \mathrm{res}_{m,p}\, \omega = 0 \quad \forall m \in X$

Now denote by ν_p u ν_m the valuations corresponding to m and p (first one is a valuation on the field K and the second is on the field of functions on p). If $f, g, h \in K^*$ and $()|_p$ is a restriction of () on p then put

$$(f,g,h)_{m,p} = (-1)^{mnl + mqk + mql + pnk + pnl + pqk} \times$$

$$\times f^{nl-kq} g^{kp-ml} h^{mq-np} |_p (m)$$

where $m = \nu_p(f)$, $n = \nu_p(g)$, $k = \nu_p(h)$ and $p = \nu_m(fu^{-m}|_p)$, $q = \nu_m(gt^{-n}|_p)$, $l = \nu_m(hu^{-k}|_p)$.

It is true that this symbol depends only on f, g, h, m and p, is a threelinear skewsymmetric form на K^* and

1) $(f, 1-f, h)_{m,p} = 1$

2) $\prod_{m\in p} (f,g,h)_{m,p} = 1 \quad \forall p \subset X$

3) $\prod_{p \ni m} (f,g,h)_{m,p} = 1 \quad \forall m \in X$

All these objects are the generalizations of the well known constructions for the curves and report is devoted to the application to such things as duality and class field theory on X.

Title: *Mixed Hodge structure on vanishing cycles.*

Name of author: Joseph Steenbrink.

Address: IHES / University of Amsterdam.

Bibliography:
[1] P. Deligne: Théorie de Hodge III. Publ. Math. IHES 43.
[2] W. Schmid: Variation of Hodge Structure. The Singularities of the period mapping. Inv. Math. 22 (1970).
[3] J. Steenbrink: Limits of Hodge Structures. IHES 1975.

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(1) We consider maps $f: Y \to S$ where S is the unit disk in the complex plane, Y is a closed analytic subset of $\mathbb{P}^r(\mathbb{C}) \times S$ for some $r > 0$, f is surjective with connected fibers and has an isolated singular point x_0 with $f(x_0) = 0$ and f smooth of relative dimension n in every point $y \neq x_0$.
Topologically one obtains the singular fiber Y_0 from a smooth fiber Y_s $(s \neq 0)$ by contraction of the intersection F of Y_s with a small ball centered at x_0. This F is diffeomorphic with the Milnor fiber of the map germ $f: (Y, x_0) \to (S, 0)$.

The exact sequence of relative cohomology gives (*):

$$0 \to H^n(Y_0) \to H^n(Y_s) \to \tilde{H}^n(F) \to H^{n+1}(Y_0) \to H^{n+1}(Y_s) \to 0$$

All terms of this sequence carry a mixed Hodge structure such, that all maps preserve the Hodge structure. These mixed Hodge structures and there connections with monodromy and intersection form are our interest.

(2) If M is a complex variety, a mixed Hodge structure on $H^k(M)$ means an increasing filtration W on $H^k(M, \mathbb{C})$, called the weight filtration and defined already over $\mathbb{Q}$, together with a decreasing filtration F on $H^k(M, \mathbb{C})$ such that the following holds:

$$\begin{cases} Gr_r^W H^k(M,\mathbb{C}) = \bigoplus_{p+q=r} F^{p,q} H^k(M,\mathbb{C}) & \text{for all } r\,; \\ \overline{F^{p,q} H^k(M,\mathbb{C})} = F^{q,p} H^k(M,\mathbb{C}) & \text{for } p+q=r. \end{cases}$$

Here $Gr_r^W = W_r/W_{r-1}$ and $F^{p,q}H^k(M,\mathbb{C})$ is the intersection of $F^p \cap W_{p+q}/F^p \cap W_{p+q-1}$ and the complex conjugate of $F^{q+1} \cap W_{p+q}/F^{q+1} \cap W_{p+q-1}$ in $Gr^W_{p+q} H^k(M,\mathbb{C})$ with respect to the rational structure $Gr^W_{p+q} H^k(M,\mathbb{Q})$.

Deligne [1] has shown that every algebraic variety M over $\mathbb{C}$ carries a canonical functorial mixed Hodge structure. Moreover if M is smooth, then $Gr_r^W H^k(M) = 0$ if $r < k$ and if M is complete, then $Gr_r^W H^k(M) = 0$ if $r > k$.

Morphisms of Hodge structures (preserving W and F and the rational structure) are strict : taking $Gr_F^p Gr_q^W$ preserves exactness of sequences.

---

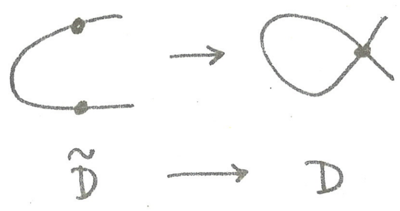
(3) The MHS on $H^*(Y_0)$ is obtained as follows. Let $\tilde{Y}_0 \to Y_0$ be a resolution of singularities for Y_0 with exceptional divisor $C = C_1 \cup \dots \cup C_m$. For $r \geq 0$ denote $\tilde{C}^{(r)} = \coprod_{i_1 < \dots < i_r} C_{i_1} \cap \dots \cap C_{i_r}$, $\tilde{C}^{(0)} = \tilde{Y}_0$. Then one has a spectral sequence :

$$(1) \quad E_1^{p,q} = H^q(\tilde{C}^{(p)}, \mathbb{Q}) \Rightarrow H^{p+q}(Y_0, \mathbb{Q})$$

with $E_2^{pq} = E_\infty^{pq} = Gr_q^W H^{p+q}(Y_0,\mathbb{Q})$ and d_1^{pq} is an alternating sum of restriction maps.

<u>Example</u> : Let $D \subset \mathbb{P}^2$ be given by $y^2 = x^3 - x^2$.

Then one gets



$\tilde{D} \longrightarrow D$

$0 \to H^0(\tilde{D}) \to H^0(2\text{pts}) \to Gr_0^W H^1(D) \to 0$

so $Gr_0^W H^1(D) = \mathbb{Q}$.

Also $Gr_1^W H^1(D) = 0 = H^1(\tilde{D})$.

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(1) If we give $H^*(Y_s)$ its canonical Hodge structure, then it will depend on the choice of $s \in S$ and the natural map $H^n(Y_0) \hookrightarrow H^n(Y_s)$ cannot be a morphism of mixed Hodge structures unless $H^n(Y_0)$ is purely of weight n. Hence we use for $H^n(Y_s)$ the limit mixed Hodge structure in the sense of Schmid [2] and the author [3].

Let $\rho : X \to Y$ be a resolution of singularities for the map f; denoting $g = f \circ \rho$, we have $g^{-1}(0) = E_0 \cup \dots \cup E_m$ a divisor with normal crossings on X which is smooth, E_0 is the strict transform of Y_0 and hence $E_0 \to Y_0$ is a resolution, and $E_1 \cup \dots \cup E_m = \rho^{-1}(x_0)$.
Denote e_i = multiplicity of E_i and $e = \mathrm{lcm}(e_0, \dots, e_m)$.
Let $\tilde S$ be another copy of the disk and $\sigma : \tilde S \to S$ with $\sigma(s) = s^e$. We denote $\tilde X$ the normalization of $X \times_S \tilde S$ and $\pi : \tilde X \to X$, $\tilde g : \tilde X \to \tilde S$ the induced maps.
Denote $D_i = \pi^{-1}(E_i)$, $i = 0, \dots, m$. Then $\tilde g^{-1}(0) = D_0 \cup \dots \cup D_m$ is reduced.

$$\begin{tikzcd}
D_i \arrow[r] \arrow[d, "\pi"] & \tilde X \arrow[rr, "\tilde g"] \arrow[d, "\pi"] & & \tilde S \arrow[d, "\sigma"] \\
E_i \arrow[r] & X \arrow[r, "\rho"'] \arrow[rr, bend left, "g"] & Y \arrow[r, "f"'] & S
\end{tikzcd}$$

Define $\tilde D^{(r)}$, $\tilde E^{(r)}$ $(r \geq 1)$ as before. Then there is a spectral sequence

$$(2) \quad E_1^{-r,q+r} = \bigoplus_{\substack{k \geq 0 \\ k \geq -r}} H^{q-r-2k}(\tilde D^{(2k+r+1)})(-r-k) \Rightarrow H^q(Y_s)$$

with $E_2^{-r,q+r} = E_\infty^{-r,q+r} = Gr^W_{q+r} H^q(Y_s)$. The maps d_1 are made up from restriction maps and their dual maps i.e. Gysin maps.

In general the $\tilde D^{(r)}$ are not smooth. However they are rational homology manifolds, hence their H^k still is purely of weight k.

The notation $(-i)$ recalls that we have taken Thom-Gysin isomorphisms i times. If $N \subset M$ are smooth and proper with $\operatorname{codim}(N,M) = 1$, one has the isomorphism $H^i_N(M) \xrightarrow{\sim} H^{i-2}(N)$. This does not preserve weights, but decreases them by 2. We change the Hodge structure on $H^{i-2}(N)$ by taking $2\pi i\, H^{i-2}(N,\mathbb{Q})$ instead of $H^{i-2}(N,\mathbb{Q})$, and increasing the weights by two and the Hodge filtration by one. For this we use the notation $H^{i-2}(N)(-1)$, having pure weight i. The Thom-Gysin isomorphism in this way preserves Hodge structure.

Hence every term in $E_1^{-r,q+r}$ is purely of weight $q+r$.

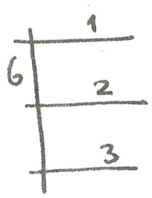
Remark that with notations from 3. and taking $\tilde{Y}_0 = D_0 \equiv E_0$, one has $\tilde{C}^{(r)} \subset \tilde{D}^{(r+1)}$ for $r \geq 0$. This induces a morphism between the spectral sequences (1) and (2) which is injective on E_1-level.

(5) Taking the cokernel of this morphism one obtains a spectral sequence (3):

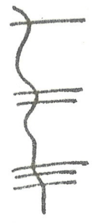
$$\begin{cases} E_1^{-r,q+r} = \bigoplus_{k\geq 0} H^{q-r-2k}(\tilde{D}^{(2k+r+1)})(-r-k) & \text{if } r>0\,; \\ E_1^{-r,q+r} = \bigoplus_{k\geq 1-r} H^{q-r-2k}(\tilde{D}^{(2k+r+1)})(-r-k) \oplus H^{q+r}(\tilde{D}^{(1-r)} - \tilde{C}^{(-r)}) & \text{if } r \leq 0\,. \end{cases}$$

Again $E_2^{-r,q+r} = Gr^W_{q+r} H^q(F)$.

<u>Example</u>: Define $Y \subset \mathbb{P}^2 \times S$ by $x^2 - y^3 = t$.



$= g^{-1}(0) \xleftarrow{\pi} \tilde{g}^{-1}(0) =$

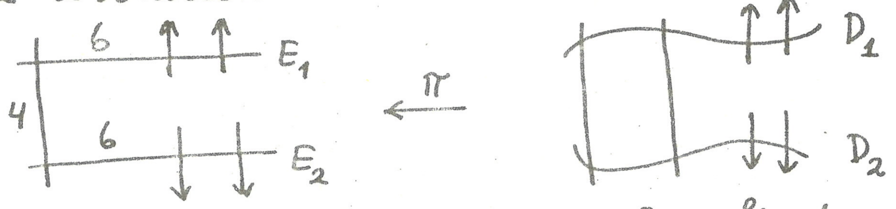


E_1 D_1

All are rational curves except D_1 which is elliptic.

$H^1(F) = H^1(D_1)$ is purely of weight 1.

<u>Example</u>: for $P(x,y) = (x^2-y^4)(x^4-y^2)$ one finds the resolution



with D_1 and D_2 of genus 2. One finds

$Gr_0^W H^1(F) = \mathbb{Q}$ (from the cycle in D!) ;

$Gr_1^W H^1(F) = H^1(D_1) \oplus H^1(D_2)$;

$Gr_2^W H^1(F) = \mathbb{Q}(-1)^3$.

(6) Denote T the limit of the monodromy acting on an equivariant way on the sequence $(*)$. It acts trivially on $H^*(Y_0)$. It is related to the mixed Hodge structures as follows. Write $T = T_s T_u = T_u T_s$ with T_u unipotent and T_s of finite order.

(6.1) Denote λ the automorphism of $\tilde{X}$ induced by multiplication with $\exp 2\pi i/e$ on $\tilde{S}$. Then λ preserves every D_i and $D_i/\langle\lambda\rangle \cong E_i$. The action of T_s on $(*)$ is induced by the action of λ^* on the spectral sequences (1), (2), (3) and T_s preserves the mixed Hodge structures.

(6.2) Let $N = \log T_u = (T_u - I) - \frac{1}{2}(T_u - I)^2 + \dots$ Then N is nilpotent, $N W_i \subset W_{i-2}$ and $N F^p \subset F^{p-1}$. Hence N is a morphism of mixed Hodge structures of type $(-1,-1)$. Moreover for all $r \geq 0$ one has

$$N^r : Gr^W_{n+r} H^n(Y_s) \xrightarrow{\sim} Gr^W_{n-r} H^n(Y_s) .$$

(6.3) By the invariant cycle theorem one has

$$H^n(Y_s)^{T_s} \cap \operatorname{Ker}(N) \cong H^n(Y_0) .$$

These facts permit one to deduce the Jordan type of T from the pair $(H^n(Y_s), T_s)$.

Remark that one can obtain the T_s-invariants of $H^*(Y_s)$ and $H^*(F)$ replacing $\tilde{D}^{(r)}$ by $\tilde{E}^{(r)}$ in the E_1-terms of the spectral sequences (2) and (3).

7 The mixed Hodge structure and the intersection form for F are related as follows. Define the Hodgenumbers $c^{pq} = \dim_{\mathbb{C}} Gr_F^p \, Gr_{p+q}^W H^n(F)$ and the unipotent Hodgenumbers $c_1^{pq} = \dim_{\mathbb{C}} Gr_F^p \, Gr_{p+q}^W H^n(F)^{T_s}$.
Let $c_{\neq 1}^{pq} = c^{pq} - c_1^{pq}$.

Suppose that after diagonalization of the intersection form on $H_n(F)$ we get μ_+ positive numbers, μ_- negative numbers and μ_0 zeroes on the diagonal.
Then:

$$\mu_0 = \sum_{p+q \leq n+1} c_1^{pq} - \sum_{p+q \geq n+3} c_1^{pq} \;;$$

$$\mu_+ = \sum_{\substack{p+q=n+2 \\ q \text{ even}}} c_1^{pq} + 2\sum_{\substack{p+q \geq n+3 \\ q \text{ even}}} c_1^{pq} + \sum_{q \text{ even}} c_{\neq 1}^{pq} \;;$$

$$\mu_- = \sum_{\substack{p+q=n+2 \\ q \text{ odd}}} c_1^{pq} + 2\sum_{\substack{p+q \geq n+3 \\ q \text{ odd}}} c_1^{pq} + \sum_{q \text{ odd}} c_{\neq 1}^{pq} \,.$$

Let us consider the case $n=2$. Then $c_1^{pq} = 0$ if $p=0$ or $q=0$. Moreover $c_1^{1,1} = c_1^{2,2}$, as we show now.

$$c_1^{2,2} = \cancel{\dim \operatorname{Coker} \big(H^2(\tilde{E}^{(2)})(-1) \to \bigoplus_{i \geq 1} H^4(E_i) \big)}$$

$$= \dim \operatorname{Ker} \big(H^0(\tilde{E}^{(3)})(-2) \to H^2(\tilde{E}^{(2)})(-1) \big) \;;$$

$$c_1^{1,1} = \dim H \Big(H^0(\tilde{E}^{(2)})(-1) \to \bigoplus_{i \geq 1} H^2(E_i) \oplus H^0(\tilde{E}^{(3)})(-1) \to \bigoplus_{1 \leq i < j} H^2(E_i \cap E_j) \Big) .$$

These are equal, for $c_1^{0,0} = 0$ implies exactness of

$$0 \to \bigoplus_{i \geq 1} H^0(E_i) \to \bigoplus_{j > i \geq 1} H^0(E_i \cap E_j) \to \bigoplus_{j > k > i \geq 1} H^0(E_i \cap E_j \cap E_k) \to 0$$

and because of Grauert's criterion for a set of curves on a surface to be able to be blown down to a point, the map $H^2(E_0) \to \bigoplus_{i\geq 1} H^2(E_0 \cap E_i)$ is surjective. Consider now the ~~exact~~ sequence arising from $(*)$ by taking $Gr_F^1 Gr_2^W$. [1]

Consequently one gets for $n=2$:

$$\mu_0 = c_1^{11} + 2c_1^{1,2} \; ; \; \mu_+ = c_1^{1,1} + \sum_{q\ \text{even}} c_{\neq 1}^{pq} = c_1^{1,1} + 2c_{\neq 1}^{00} + 2c_{\neq 1}^{02} + 2c_{\neq 1}^{12}.$$

Hence we have

<u>Theorem</u>: if $n=2$ then $\mu_0 + \mu_+$ is even.

For $\mu_+ - \mu_- = \mathrm{sign}(F)$ one obtains in this case:

$$\mathrm{sign}\, F = \sum_{i=1}^{m} \mathrm{sign}(D_i) + A - B$$

where $A = \# \tilde{D}^{(3)}$ and $B = \#$ components of $\tilde{C}^{(1)} = \#$ components of $\coprod_{i\geq 1} E_0 \cap E_i$.

[1] Denote by $H^2(Y_s)_1$ resp. $H^2(F)_1$ the T_s-invariants. Then by Grauert's criterion $Gr_2^W H^3(Y_0) = 0$. Moreover $Gr_2^W H^2(Y_s)_1 \cong Gr_2^W H^2(Y_0) \oplus N\, Gr_4^W H^2(Y_s)_1$ hence $(*)$ gives that $N: Gr_4^W H^2(F)_1 \xrightarrow{\sim} Gr_2^W H^2(F_1)$. Hence $c_1^{1,1} = c_1^{2,2}$.

Title: Nearly Flat Triangulations of Riemannian Manifolds

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In 1934-36 S.S. Cairns proved that smooth manifolds admit smooth triangulations and introduced the concept of combinatorial approximation of manifolds by means of uniformly fine subdivisions. The uniqueness of the PL structure on the complex defined by a smooth triangulation was proved in 1940 by J.H.C. Whitehead.

In recent years some attention has been drawn to the question of including in the combinatorial approximation theory of manifolds such structural concepts as Riemannian metrics and, in the case of compact manifolds, Hodge's theory of harmonic forms and other invariants related to the generalized potential theory of a Riemannian manifold.

In 1972-73 Jozef Dodziuk obtained the following approximation theorem for the Hodge-de Rham operator on compact manifolds.

Theorem 1 Let M be a compact Riemannian manifold. For any smooth triangulation of M, one can introduce an inner product structure on the real cochain group of the finite complex, such that, if one merely restricts the concept of allowable directed systems of subdivisions ("sufficiently regular subdivisions"), the corresponding combinatorial Laplace operators on the various cochain groups approach, in their spectral decomposition, the Hodge operator on M.

The above theorem is insufficient to include the convergence of some ~~some~~ more delicate potential theoretic invariants, like the kernel of the heat diffusion equation for forms or the zeta function for forms of each degree. It seems that part of the difficulty in attempting to achieve these stronger convergence properties may be overcome, if we can require that the duality operator on each approximating finite cochain group (associated with the quadratic form) be support preserving, or in other words, that the positive definite quadratic form on each cochain group, in terms of the natural basis consisting of each cell with unit coefficient, be represented by a positive, diagonal matrix.

Let M be a complete Riemannian manifold, $\delta : M \to \mathbb{R}^+$ a positive valued function satisfying a Lipschitz condition $|\delta(x) - \delta(y)| \leq \frac{1}{2}\rho(x,y)$ (ρ = geodesic distance). Let K be a simplicial (finite) complex and K^* its dual complex, whose underlying spaces are combinatorially equivalent to M. We say that a pair of triangulations $T : |K| \to X$, $T^* : |K^*| \to X$ is a nearly flat, orthogonal pair of triangulations of X of mesh δ, if they satisfy the following conditions

1) for each pair (σ, σ^*) ~~of the~~ mutually dual cells of K and K^*, the union of their images in M has diameter ~~not~~ greater than the minimum value of δ on $T(\sigma) \cup T^*(\sigma^*)$, although they do not necessarily intersect.

2) Each image of a k-cell $\sigma \in K$, $T(\sigma)$ is a compact subset in a k-dimensional submanifold S_σ containing geodesick balls of radius $\geq \delta(x)$ around each $x \in T(\sigma)$, and similarly for each $\sigma^* \in K^*$; furthermore the Riemannian norm of the second fundamental form at each point of S_σ and $S^*_{\sigma^*}$ is uniformly bounded from above by $c_1 \delta$, where c_1 is an absolute constant depending only on n.

3) The extended k-submanifold S_σ and the $(n-k)$-submanifold $S^*_{\sigma^*}$ associated to the dual cell intersect orthogonally at a ~~some~~ unique point p_σ; for $k=0$, $p_\sigma = T(\sigma) \in T^*(\sigma^*)$, while for $k=n$, $p_\sigma = T^*(\sigma^*) \in T(\sigma)$. The geodesic diameter of each $S_\sigma \cup S^*_{\sigma^*}$ is $\leq 6\,\delta(p_\sigma)$.

4). There is a positive absolute constant c_2 depending only on n, such that the volume of each image k-cell $T(\sigma)$ or each image k-cell $T^*(\sigma^*)$ is $\geq c_2 \delta^k$.

Theorem 2 For each positive function $\delta : M \to \mathbb{R}^+$, satisfying the Lipschitz condition $|\delta(x) - \delta(y)| \leq \frac{1}{2}\rho(x,y)$ and sufficiently small at each x in terms of the curvature and injectivity radius at x one can find a pair of nearly flat, orthogonal dual triangulations T, T^* of M of mesh δ, as specified above.

The construction of T and T^* is uniquely determined by a suitable choice of the restriction of T to a 0-skeleton, subject to complicated regularity conditions; the 1-dimensional skeleton of $T(K)$, and indeed the extended cell of each 1-simplex $\sigma \in K$, denoted earlier by S_σ, is a geodesic segment.

Let u be a k-cochain on T and let $u_\sigma = u(\sigma)$ be its coefficient at each k-simplex $\sigma \in K_k$: Then the norm of u is defined to be

$$\|u\|^2 = (u,u) = \sum_{\sigma \in K_k} \frac{\mu_{n-k}(\sigma^*)}{\mu_k(\sigma)} |u(\sigma)|^2$$

where $\mu_{n-k}(\sigma^*)$ and $\mu_k(\sigma)$ denote respectively the $(n-k)$-dimensional volume of $T^*(\sigma^*)$ and the k-dimensional volume of $T(\sigma)$ determined by the Riemannian metric. This quadratic norm defines a duality operator $* = *_K$ on K, a boundary operator δ_K and a combinatorial Laplace operator Δ_K induced by T, T^*.

The above definitions are justified by the following result

Theorem 3. Let ω be a differentiable k-form on a compact manifold M of n dimensions and assume that its L_2-norm is, without loss of generality, equal to unity. Let $\|\omega\|_1$ denote the uniform upper bound of the sum of the pointwise norm of ω and of its covariant derivative, and let $T^*\omega$ be the cochain induced on K by integrating ω along each $T(\sigma)$ for each k-simplex $\sigma \in K$. Then the norm of the difference $T^*\Delta\omega - \Delta_K T^*\omega$, if (T, T^*) are nearly flat, orthogonal triangulations of mesh δ, is small of order $\delta \cdot \|\omega\|_1$. Furthermore, if we denote by G_K the combinatorial Green's operator on k-cochains of K associated with Δ_K, the norm of $G_K T^*\omega - T^* G\omega$ becomes small of order ~~[illegible]~~ δ in a similar way.