Symplectic reflection algebras

Pavel Etingof and Victor Ginzburg

Introduction Given an action of a finite group Γ on a smooth affine algebraic variety X, one can study the algebraic structure of the orbifold X/Γ through its coordinate ring $\mathbb{C}[X]^\Gamma$, the ring of Γ-invariant regular functions on X. It is known, however, that if the Γ-action on X is not free the (possibly singular) space X/Γ does not necessarily reflect the features of the Γ-action quite adequately. For instance, one is lead to introduce various "stringy" topological invariants, cf. [Ba], which do not coincide with the ordinary topological invariants of X/Γ, but are rather related to certain "resolutions" of X/Γ. A standard approach to this problem, see e.g. [Co], is to replace the commutative coordinate ring $\mathbb{C}[X]^\Gamma$ by the smash-product algebra $\mathbb{C}[X]\#\Gamma$. The two algebras are *Morita equivalent*, hence lead to equivalent theories, provided the Γ-action on X is free, but are not equivalent in general. Although the algebra $\mathbb{C}[X]\#\Gamma$ is *non-commutative*, it is believed that the "right" geometry of the Γ-action on X can be read-off from the "non-commutative algebraic geometry" of $\mathbb{C}[X]\#\Gamma$. This is the approach that we adopt in the present paper.

Specifically, we will study deformations of some interesting orbifolds X/Γ through non-commutative deformations of the corresponding algebra $\mathbb{C}[X]\#\Gamma$. We will see that a systematic use of this ideology leads, via Morita equivalence, to various new results in conventional "commutative" algebraic geometry which have been unaccessible by purely "commutative" methods.

Another fascinating feature of our approach is that genuine deformations of X/Γ coming from "commutative" algebraic varieties stand on an equal footing with purely "quantum" deformations that are not associated with any algebraic varieties whatsoever. "Commutative" and "quantum" deformations only differ by the values of deformation parameters. Thus, "quantum" deformations may be viewed as quantizations of "commutative" ones, and Kontsevich's quantization plays a role here. On the other hand, the algebras arising in such "quantum" deformations are closely related to the algebras of invariant polynomial differential operators on semisimple Lie algebras and Quiver varieties. Thus, on the "quantum" side, we enter the realm of representation theory, which was our original motivation.

Symplectic reflection algebras. Let V be a finite dimensional vector space over $\mathbb{C}$, and TV its tensor algebra. Let $\Gamma \subset \mathrm{GL}(V)$ be a finite group, and $(TV)\#\Gamma$ the smash-product of TV with $\mathbb{C}\Gamma$, the group algebra of Γ. Given a skew-symmetric $\mathbb{C}$-bilinear pairing $\kappa : V \times V \to \mathbb{C}\Gamma$, put

$$\mathsf{H}_\kappa \;:=\; (TV\#\Gamma)/I\langle x\otimes y - y\otimes x - \kappa(x,y) \in T^2V \,\oplus\, \mathbb{C}\Gamma\rangle_{\,x,y\in V}\;, \tag{1}$$

where $I\langle \dots \rangle$ stands for the two-sided ideal in $(TV)\#\Gamma$ generated by the indicated set. Thus, H_κ is an associative algebra.

In the special case $\kappa = 0$, the situation simplifies drastically since all the generators $x \in V$ commute with each other. The resulting algebra is therefore isomorphic to the smash-product algebra: $\mathsf{H}_{\kappa=0} = (SV)\,\#\,\Gamma$. Note that this algebra has a natural grading obtained by placing $\mathbb{C}\Gamma$ in grade degree zero, and V in grade degree 1.

In general, if $\kappa \neq 0$, the defining relations, see (1), become inhomogeneous, hence assigning $\mathbb{C}\Gamma$ grade degree zero, and V grade degree 1, one only gets an increasing *filtration* $F_\bullet(\mathsf{H}_\kappa)$ on the corresponding algebra H_κ, such that, for any $i, j \geq 0$, we have: $F_i(\mathsf{H}_\kappa) \cdot F_j(\mathsf{H}_\kappa) \subset F_{i+j}(\mathsf{H}_\kappa)$. Let $\mathtt{gr}(\mathsf{H}_\kappa) = \bigoplus_i F_i(\mathsf{H}_\kappa)/F_{i-1}(\mathsf{H}_\kappa)$ denote the associated graded algebra. It is immediate from (1) that, for any $x, y \in V \subset \mathtt{gr}(\mathsf{H}_\kappa)$, in $\mathtt{gr}(\mathsf{H}_\kappa)$ we have: $xy - yx = 0$. Therefore the tautological imbedding: $V \hookrightarrow \mathtt{gr}(\mathsf{H}_\kappa)$ extends to a well-defined and surjective graded algebra homomorphism: $\mathsf{H}_{\kappa=0} = (SV)\#\Gamma \longrightarrow \mathtt{gr}(\mathsf{H}_\kappa)$. We say that *Poincaré-Birkhoff-Witt (PBW-)property* holds for H_κ if this morphism is an isomorphism.

From now on we assume, in addition, that V is a symplectic vector space with symplectic 2-form ω, and that $\Gamma \subset Sp(V)$. The triple (V, ω, Γ) is said to be **indecomposable** if there is no ω-orthogonal direct sum decomposition: $V = V_1 \oplus V_2$, where V_i are Γ-stable proper symplectic vector subspaces in V.

Definition 2 *An element $s \in \Gamma$ is called a symplectic reflection if* $\mathtt{rk}(\mathtt{Id} - s) = 2$.

Let S denote the set of symplectic reflections in Γ. The group Γ acts on S by conjugation. For each $s \in S$, write ω_s for the (degenerate) skew-symmetric form on V which coincides with ω on $\mathtt{Im}(\mathtt{Id} - s)$, and has $\mathtt{Ker}(\mathtt{Id} - s)$ as the radical (recall that there is an ω-orthogonal direct sum decomposition $V = \mathtt{Im}(\mathtt{Id} - s) \oplus \mathtt{Ker}(\mathtt{Id} - s)$).

Theorem 3 *Assume that (V, ω, Γ) is an indecomposable triple. Then, PBW-property holds for H_κ if and only if there exists a constant $t \in \mathbb{C}$, and an $\mathtt{Ad}\,\Gamma$-invariant function $c : S \to \mathbb{C}$, $s \mapsto c_s$, such that the pairing κ has the form:*

$$\kappa(x, y) = t \cdot \omega(x, y) \cdot 1 + \sum\nolimits_{s \in S} c_s \cdot \omega_s(x, y) \cdot s \quad , \quad \forall x, y \in V \, .$$

Remarks (i) Our proof of the Theorem shows that the algebra H_κ associated to a pairing κ of the type described by the formula above always satisfies the PBW-property, even if the triple (V, ω, Γ) is decomposable.

(ii) An analogue of Theorem 3 holds in a more general setup, where V is any (not necessarily symplectic) vector space, and $\Gamma \subset \mathrm{GL}(V)$ is a finite subgroup containing no elements $s \in \Gamma$ such that $\mathtt{rk}(\mathtt{Id} - s) = 1$. The proof of Theorem 3 shows that, in this generality, PBW-property holds for H_κ if and only if $\kappa(x, y) = \omega(x, y) \cdot 1 + \sum_{s \in S} \omega_s(x, y) \cdot s$, $\forall x, y \in V$, where ω is a (possibly *degenerate*) skew-symmetric Γ-invariant form on V, symplectic reflections are defined as in Definition 2 (involving no reference to any symplectic structure), and $\{\omega_s\}_{s \in S}$ stands for an arbitrary Γ-equivariant collection of skew-symmetric forms such that: $\mathtt{Ker}(\mathtt{Id} - s) \subset \mathtt{Radical}(\omega_s)$, $\forall s \in S$.

Theorems 5–7 below also hold in this generality. $\lozenge$

Let C denote the vector space of $\mathtt{Ad}\,\Gamma$-invariant functions on S. We have

$$\dim \mathsf{C} = \textit{ number of } \mathtt{Ad}\,\Gamma\textit{-conjugacy classes in } S \, . \tag{4}$$

From now on, we assume that the pairing κ in (1) has the form described in Theorem 3, for a certain $t \in \mathbb{C}$ and $c \in \mathsf{C}$. The corresponding algebra $\mathsf{H}_{t,c} := \mathsf{H}_\kappa$ will be referred to as a *symplectic reflection algebra*.

Let $\mathbf{e} = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g \in \mathbb{C}\Gamma$, be the 'symmetriser' idempotent, viewed as an element of $\mathsf{H}_{t,c}$. The algebra $\mathbf{e}\mathsf{H}_{t,c}\mathbf{e} \subset \mathsf{H}_{t,c}$ will be called the *spherical subalgebra* in $\mathsf{H}_{t,c}$. For $t = 0$ and $c = 0$, we have: $\mathbf{e}\mathsf{H}_{0,0}\mathbf{e} = \mathbf{e}\big(SV\#\Gamma\big)\mathbf{e} \simeq (SV)^\Gamma$, the algebra of Γ-invariants. Furthermore, for any $(t, c) \in \mathbb{C} \oplus \mathsf{C}$, Theorem 3 yields: $\mathtt{gr}\big(\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}\big) \simeq (SV)^\Gamma$.

The space $\mathsf{H}_{t,c}\mathbf{e}$ has an obvious *left* $\mathsf{H}_{t,c}$-module structure, and *right* $\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}$-module structure. Given an algebra A, and a left, resp. right, A-module M, we set: $M^\vee = \mathtt{Hom}_{A\text{-left}}(M, A)$, resp. ${}^\vee M = \mathtt{Hom}_{A\text{-right}}(M, A)$.

Theorem 5 (i) $\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}$ *is a finitely generated Gorenstein algebra without zero divisors.*
(ii) $\mathsf{H}_{t,c}\mathbf{e}$ *is a finitely generated, Cohen-Macaulay right* $\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}$*-module.*
(iii) *We have:* ${}^\vee(\mathsf{H}_{t,c}\mathbf{e}) = \mathbf{e}\mathsf{H}_{t,c}$, *and* $(\mathbf{e}\mathsf{H}_{t,c})^\vee = \mathsf{H}_{t,c}\mathbf{e}$. *In particular,* $\mathsf{H}_{t,c}\mathbf{e}$ *is reflexive.*
(vi) *The left* $\mathsf{H}_{t,c}$*-action induces an algebra isomorphism:* $\mathsf{H}_{t,c} \xrightarrow{\sim} \mathtt{End}_{\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}}(\mathsf{H}_{t,c}\mathbf{e})$.

For any $t \neq 0$, let $\mathsf{A}_t = TV/I\langle x\cdot y - y\cdot x - t\cdot\omega(x,y)\rangle_{x,y\in V}$ be the *Weyl algebra* of the symplectic vector space $(V, t\cdot\omega)$; the algebras A_t are isomorphic to each other, for all $t \neq 0$. We observe that, for $c = 0$ (and any $t \neq 0$), we have: $\mathsf{H}_{t,0} = \mathsf{A}_t\#\Gamma$. We show that, for any fixed $t \neq 0$, the family $\{\mathsf{H}_{t,c}\}_{c\in\mathsf{C}}$ gives a universal deformation of the algebra $\mathsf{A}_t\#\Gamma$. Furthermore, the spherical subalgebra $\mathbf{e}\mathsf{H}_{t,0}\mathbf{e}$ is isomorphic to A_t^Γ, the subalgebra of Γ-invariants in the Weyl algebra, and (for any fixed $t \neq 0$) the family $\{\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}\}_{c\in\mathsf{C}}$ gives a universal deformation of the algebra A_t^Γ.

Theorem 6 *For any* $c \in \mathsf{C}$, *the algebra* $\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}$ *is commutative if and only if* $t = 0$.

Example: Kleinian singularity. Let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup acting on $V = \mathbb{C}^2$. The action on $V \smallsetminus \{0\}$ is free, and any element of $\Gamma \smallsetminus \{1\}$ is a symplectic reflection. This case has been studied by Crawley-Boevey and Holland [CBH]. $\Diamond$

Quasi-classical case: $\kappa = (\mathbf{0}, \mathbf{c})$. Assume next that $t = 0$ and $c \neq 0$. The algebra $\mathsf{H}_{0,c}$, $c \neq 0$, has some very interesting special features making it quite different from other members of the family $\{\mathsf{H}_\kappa\}_{\kappa\in\mathbb{C}\oplus\mathsf{C}}$. First of all, the algebra $\mathsf{H}_{0,c}$ has a large center, $\mathsf{Z}_{0,c} = \mathsf{Z}(\mathsf{H}_{0,c})$, as opposed to the algebras H_κ for generic κ. In more detail, the standard filtration on $\mathsf{H}_{0,c}$ induces a filtration on $\mathsf{Z}(\mathsf{H}_{0,c})$, and we have:

$$\mathtt{gr}\,\mathsf{Z}(\mathsf{H}_{0,c}) \;=\; \mathsf{Z}(\mathsf{H}_{0,0}) \;=\; \mathsf{Z}(SV\#\Gamma) \;=\; (SV)^\Gamma .$$

Further, for any $c \in \mathsf{C}$, we use the construction of Hayashi [Ha], to make both the spherical subalgebra $\mathbf{e}\mathsf{H}_{0,c}\mathbf{e}$ and the center $\mathsf{Z}_{0,c}$ Poisson algebras, which reduce to the standard Poisson structure on $(SV)^\Gamma$, if $c = 0$. Unlike the case $c = 0$, the center of $\mathsf{H}_{0,c}$ is harder to describe explicitly. To this end, we generalize the classical Satake isomorphism for an affine Hecke algebra (as proposed by Lusztig), and show that the map: $z \mapsto z\cdot\mathbf{e}$ establishes a Poisson algebra isomorphism: $\mathsf{Z}_{0,c} \xrightarrow{\sim} \mathbf{e}\mathsf{H}_{0,c}\mathbf{e}$, which we call the *Satake isomorphism.*

Let $\mathcal{R}$ denote the coherent sheaf on $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ corresponding to $\mathsf{H}_{0,c}\mathbf{e}$, viewed as a (finitely generated) $\mathsf{Z}_{0,c}$-module, that is: $\Gamma(\mathtt{Spec}\,\mathsf{Z}_{0,c}, \mathcal{R}) = \mathsf{H}_{0,c}\mathbf{e}$. We will view points of $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ as algebra homomorphisms $\chi : \mathsf{Z}_{0,c} \longrightarrow \mathbb{C}$. Then, for any point $\chi \in \mathtt{Spec}\,\mathsf{Z}_{0,c}$, the geometric fiber of the sheaf $\mathcal{R}$ at χ equals: $\mathsf{H}_{0,c}\mathbf{e}\bigotimes_{\mathsf{Z}_{0,c}}\chi$. The left action of $\mathsf{H}_{0,c}$ on $\mathsf{H}_{0,c}\mathbf{e}$ gives an $\mathsf{H}_{0,c}$-module structure on each geometric fiber of $\mathcal{R}$.

In the theorem below, we fix $U \subset \mathtt{Spec}\,\mathsf{Z}_{0,c}$, a Zariski-open affine subset, and use the subscript 'U' to denote 'restriction to U'. In particular we write $\mathsf{Z}_U := \mathbb{C}[U] \supset \mathsf{Z}_{0,c}$, and $\mathsf{H}_U := \mathsf{Z}_U \bigotimes_{\mathsf{Z}_{0,c}} \mathsf{H}_{0,c}$, and also $\mathcal{R}_U := \mathcal{R}\big|_U$.

Theorem 7 *Let U be an affine Zariski-open subset contained in the smooth locus of* $\mathtt{Spec}\,\mathsf{Z}_{0,c}$. *Then we have:*

(i) *The sheaf $\mathcal{R}_U$ is locally free, and the algebra H_U is isomorphic to $\mathtt{End}_{\mathsf{Z}_U}(\mathcal{R}_U)$.*

(ii) *The algebra H_U is Morita equivalent to Z_U, i.e. the categories of finitely generated H_U-modules and Z_U-modules are equivalent.*

(iii) *Any simple H_U-module is isomorphic to one of the form: $\mathsf{H}_U\mathbf{e}\bigotimes_{\mathsf{Z}_U}\chi$, for a certain character $\chi : \mathsf{Z}_U \to \mathbb{C}$. In particular, the action of the center defines a bijection between the isomorphism classes of simple H_U-modules and points of U.*

(vi) *Any simple H_U-module has dimension $|\Gamma|$ and is isomorphic, as a Γ-module, to the regular representation of Γ.*

If $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ is itself smooth then claims (i) -(vi) *hold for $U = \mathtt{Spec}\,\mathsf{Z}_{0,c}$, and $\mathsf{H}_U = \mathsf{H}_{0,c}$.*

Following [Alv], on the group algebra $\mathbb{C}\Gamma$, define an increasing filtration $F_\bullet(\mathbb{C}\Gamma)$ by letting $F_k(\mathbb{C}\Gamma)$, $k \geq 0$, be the $\mathbb{C}$-linear span of the elements $g \in \Gamma$ such that $\mathtt{rk}(\mathtt{Id}-g) \leq k$. This filtration is obviously compatible with the algebra structure on $\mathbb{C}\Gamma$. Let $F_\bullet(\mathsf{Z}\Gamma)$ denote the induced filtration on $\mathsf{Z}\Gamma$, the center of $\mathbb{C}\Gamma$. Write $\mathtt{gr}^F_\bullet(\mathsf{Z}\Gamma)$ for the corresponding associated graded algebra.

Further, write $HH^\bullet(\mathcal{A})$ for the Hochschild cohomology of an associative algebra $\mathcal{A}$, and $H^\bullet(-)$ for the ordinary singular cohomology of a topological space (with complex coefficients). Part (i) of the theorem below is a generalization of [AFLS] and [Alv], while part (ii), may be viewed as a 'symplectic generalization' of **McKay correspondence**, cf. [Re].

Theorem 8 *For any $c \in \mathsf{C}$ we have:*

(i) *For all $t \in \mathbb{C}^*$, except possibly a countable set, there is a natural graded algebra isomorphism: $HH^\bullet(\mathsf{H}_{t,c}) \simeq \mathtt{gr}^F_\bullet(\mathsf{Z}\Gamma)$.*

(ii) *If the variety $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ is smooth, then the Poisson structure on $\mathsf{Z}_{0,c}$ makes $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ a symplectic manifold; Furthermore, there is a graded algebra isomorphism: $H^\bullet(\mathtt{Spec}\,\mathsf{Z}_{0,c}) \simeq \mathtt{gr}^F_\bullet(\mathsf{Z}\Gamma)$.*

Corollary 9 *If $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ is smooth, then $\dim H^{\mathsf{odd}}(\mathtt{Spec}\,\mathsf{Z}_{0,c}) = 0$, and for all $i \geq 0$, we have: $\dim H^{2i}(\mathtt{Spec}\,\mathsf{Z}_{0,c}) = \mathbf{n}(i)$, where*

$$\mathbf{n}(i) := \textit{ number of conjugacy classes of } g \in \Gamma \textit{ such that } \mathtt{rk}(\mathtt{Id}-g) = 2i\,. \qquad \square$$

Recall the result of Batyrev [Ba, Thm. 4.8] saying that, for any *crepant* (hence, symplectic, see e.g. [Ve]) resolution: $\widehat{V/\Gamma} \twoheadrightarrow V/\Gamma$, one has: $\dim H^{2i}(\widehat{V/\Gamma}) = \mathbf{n}(i)$. It has been conjectured by E. Vasserot [Va] (and also by the second author) that, for a resolution $\widehat{V/\Gamma}$ as above, there is a graded algebra isomorphism: $H^\bullet(\widehat{V/\Gamma}) \simeq \mathtt{gr}^F_\bullet(\mathsf{Z}\Gamma)$. This conjecture is related to Theorem 8 as follows. According to [Ka2], if the symplectic orbifold V/Γ has a smooth symplectic resolution of singularities, $\widehat{V/\Gamma}$, then this resolution can be deformed to a smooth affine symplectic manifold. We expect that the latter coincides with the variety $\mathtt{Spec}\,\mathsf{Z}_{0,c}$, for an appropriate value of the parameter $c \in \mathsf{C}$, in particular, $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ is smooth for generic $c \in \mathsf{C}$. Our expectation is based on the conjecture saying that *any* Poisson deformation of V/Γ, in particular the one constructed by Kaledin, is given by a certain variety $\mathtt{Spec}\,\mathsf{Z}_{0,c}$. If our expectations are correct, then any smooth symplectic resolution, $\widehat{V/\Gamma}$, has the same cohomology algebra as $\mathtt{Spec}\,\mathsf{Z}_{0,c}$. Thus, Theorem 8 would imply the conjecture in [Va]. In many concrete examples the space $\widehat{V/\Gamma}$ has a natural hyper-Kähler structure such that $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ is obtained from $\widehat{V/\Gamma}$ by 'rotating' the complex structure. Hence the two spaces are even diffeomorphic, and Theorem 8 applies. This is the case, for instance, for $\Gamma = S_n$ acting diagonally on $V = \mathbb{C}^n \oplus \mathbb{C}^n$. In that case, $\widehat{V/\Gamma} = \mathsf{Hilb}^n(\mathbb{C}^2)$ is the Hilbert scheme of n points on the plane, and the corresponding result on the algebra structure of $H^\bullet\big(\mathsf{Hilb}^n(\mathbb{C}^2)\big)$ has been recently proved in [LSo] and [Va] (independently). Their approach is totally different from ours.

Further, write $K(X)$ for the algebraic K-theory of an algebraic variety X, that is, for the Grothendieck group formed by algebraic vector bundles on X. Now let $X = \mathtt{Spec}\,\mathsf{Z}_{0,c}$, and assume that it is smooth so that the sheaf $\mathcal{R}$ is locally free, by Theorem 7(i). We view $\mathcal{R}$ as an algebraic vector bundle on $\mathtt{Spec}\,\mathsf{Z}_{0,c}$. Given an irreducible Γ-representation E, we let $\mathcal{R}^E := E \otimes_\Gamma \mathcal{R} = \mathtt{Hom}_\Gamma(E^*, \mathcal{R})$ denote the E-isotypic component of $\mathcal{R}$, which is again an algebraic vector bundle on $\mathtt{Spec}\,\mathsf{Z}_{0,c}$.

Proposition 10 *If the variety* $\mathtt{Spec}\,\mathsf{Z}_{0,c}$ *is smooth, then* $K(\mathtt{Spec}\,\mathsf{Z}_{0,c})$ *is a free abelian group with the basis* $\{\mathcal{R}^E\}_{E\in\mathsf{Irreps}(\Gamma)}$. *Moreover, the Chern character map gives an isomorphism:* $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathtt{Spec}\,\mathsf{Z}_{0,c}) \xrightarrow{\sim} H^{\mathsf{ev}}(\mathtt{Spec}\,\mathsf{Z}_{0,c})$.

Wreath-product case. An infinite series of interesting examples of groups generated by symplectic reflections[1] is provided by a so-called *wreath-product* construction. Fix a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$ and an integer $n > 1$. We write $L = \mathbb{C}^2$ for the tautological 2-dimensional Γ-module, form the vector space $V = L^{\oplus n}$ with the symplectic structure ω_V induced from the standard one on $L = \mathbb{C}^2$, and let S_n, the Symmetric group, act on V by permutations of the direct summands. The group $\mathbf{\Gamma}_n := S_n \ltimes (\Gamma \times \Gamma \times \ldots \times \Gamma) \subset Sp(V)$ acts naturally on $V = L^{\oplus n}$ and is called the wreath-product of S_n with Γ.

It is known that $\mathsf{C}(\mathbf{\Gamma}_n)$, the space of class functions on the set of symplectic reflections in $\mathbf{\Gamma}_n$ can be identified with $\mathsf{Z}\Gamma$, the center of the group algebra of Γ. Thus there is a symplectic reflection algebra, $\mathsf{H}_{0,c}(\mathbf{\Gamma}_n)$, attached to the triple $(V, \omega_V, \mathbf{\Gamma}_n)$, and any parameter $c \in \mathsf{Z}\Gamma \simeq \mathsf{C}(\mathbf{\Gamma}_n)$.

On the other hand, let $\mathbf{V} = \mathbb{C}\Gamma^{\oplus n}$ be the direct sum of n copies of the regular representation of Γ. Following Nakajima [Na], to any $c \in \mathsf{Z}\Gamma$ one associates an affine algebraic variety of *quiver data*:

$$\mathbb{M}^c_\Gamma(\mathbf{V}) \;:=\; \big\{(\nabla, I, J) \in \big(\mathtt{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}\otimes_{\mathbb{C}} L) \,\bigoplus\, \mathbf{V} \,\bigoplus\, \mathbf{V}^*\big)^\Gamma \;\Big|\; [\nabla,\nabla] + I\otimes J = c|_{\mathbf{V}}\big\}\,, \tag{11}$$

where $I \otimes J \in \mathtt{End}_\Gamma\mathbf{V}$, and $[\nabla,\nabla] \in (\mathtt{End}_\Gamma\mathbf{V}) \bigotimes \bigwedge^2 L \simeq \mathtt{End}_\Gamma\mathbf{V}$. The natural action on $\big(\mathtt{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}\otimes_{\mathbb{C}} L) \bigoplus \mathbf{V} \bigoplus \mathbf{V}^*\big)^\Gamma$ of the group $G_\Gamma(\mathbf{V})$ of Γ-equivariant automorphisms of $\mathbf{V}$ preserves the subvariety $\mathbb{M}^c_\Gamma(\mathbf{V})$. The quotient: $\mathfrak{M}_{\Gamma,n,c} = \mathbb{M}^c_\Gamma(\mathbf{V})/G_\Gamma(\mathbf{V})$ is a particular case of the Nakajima *quiver variety*. In general, the quotient is understood in the sense of the Geometric Invariant Theory, but in our case the GIT quotient coincides with the naive one because of the following well-known result, see also [Na]

[1]see [Coh] for a classification of finite groups generated by symplectic reflections.

Lemma 12 *If $c \in \mathsf{Z}\Gamma$ is generic enough, then the induced $G_\Gamma(\mathbf{V})$-action on $\mathbb{M}_\Gamma^c(\mathbf{V})$ is free. Furthermore, $\mathfrak{M}_{\Gamma,n,c}$ is a smooth symplectic affine algebraic variety of dimension $2n$.*

Write $\mathsf{Z}_{0,c}(\mathbf{\Gamma}_n)$ for the center of the algebra $\mathsf{H}_{0,c}(\mathbf{\Gamma}_n)$. One of our main results is

Theorem 13 *If $c \in \mathsf{Z}\Gamma$ is generic enough, then $\mathsf{Spec}\,\mathsf{Z}_{0,c}(\mathbf{\Gamma}_n)$ is isomorphic to $\mathfrak{M}_{\Gamma,n,c}$, as a Poisson algebraic variety.*

Corollary 14 *If $c \in \mathsf{Z}\Gamma$ is generic enough, then we have:*

(i) $\mathsf{Spec}\,\mathsf{Z}_{0,c}(\mathbf{\Gamma}_n)$ *is a smooth symplectic variety.*

(ii) *Simple $\mathsf{H}_{0,c}(\mathbf{\Gamma}_n)$-modules all have dimension $n! \cdot |\Gamma|^n$, and are parametrized by points of the quiver variety $\mathfrak{M}_{\Gamma,n,c}$.*

(iii) *There is a canonical algebraic vector bundle $\mathcal{R}$ on $\mathfrak{M}_{\Gamma,n,c}$ whose fibers carry the regular representation of the group $\mathbf{\Gamma}_n$. The classes: $\{\mathcal{R}^E\}_{E\in\mathsf{Irrep}(\mathbf{\Gamma}_n)}$ form a basis of $K(\mathfrak{M}_{\Gamma,n,c})$.*

(vi) *There is an algebra isomorphism: $H^\bullet(\mathfrak{M}_{\Gamma,n,c}) \simeq \mathtt{gr}^F_\bullet \mathsf{Z}\mathbf{\Gamma}_n$.* □

Rational Cherednik algebra. As we have mentioned earlier, all indecomposable triples (V, ω, Γ) fall into two large groups: **(1)** those where V is an irreducible Γ-module, and **(2)** those where $V = \mathfrak{h} \oplus \mathfrak{h}^*$ is a direct sum of two Γ-stable irreducible Lagrangian subspaces. In the second case, the action on $\mathfrak{h}$ by any symplectic reflection $g \in \Gamma$ has a pointwise fixed codimension 1 hyperplane in $\mathfrak{h}$, hence is a *complex reflection* in $\mathfrak{h}$. Thus, in case (2), saying that Γ is generated by symplectic reflections amounts to saying that $\Gamma = W$ is a finite complex reflection group in a vector space $\mathfrak{h}$, acting on $V = T^*\mathfrak{h} \simeq \mathfrak{h} + \mathfrak{h}^*$ by induced symplectic automorphisms. Any (complex) reflection $s \in W$ induces the corresponding symplectic reflection of $\mathfrak{h} + \mathfrak{h}^*$.

We are mostly interested in the special case where W is the Weyl group of a finite reduced root system $R \subset \mathfrak{h}^*$. The set S of reflections in W forms either one or two conjugacy classes, depending on whether all roots $\alpha \in R$ are of the same length or not. Thus, giving a W-invariant function $c : S \to \mathbb{C}$ amounts to giving a map: $\alpha \mapsto c_\alpha \in \mathbb{C}$, where c_α depends only on the length of $\alpha \in R$.

Let $\mathsf{H}_{t,c}$ be the symplectic reflection algebra associated to a complex number $t \in \mathbb{C}$, a map c as above, and the vector space $V = \mathfrak{h} + \mathfrak{h}^*$. Explicitly, write $\alpha^\vee \in \mathfrak{h}$ for the coroot corresponding to a root $\alpha \in R$, and $s_\alpha \in W$ for the reflection relative to α. Then, formula (1) shows that the algebra $\mathsf{H}_{t,c}$ is generated by the spaces $\mathfrak{h}, \mathfrak{h}^*$, and the group W, subject to the following defining relations:

$$\begin{aligned}
& w \cdot x \cdot w^{-1} = w(x) \;\;,\;\; w \cdot y \cdot w^{-1} = w(y)\,, && \forall y \in \mathfrak{h}\,,\, x \in \mathfrak{h}^*\,,\, w \in W \\
& [x_1, x_2] = 0 = [y_1, y_2]\,, && \forall y_1, y_2 \in \mathfrak{h},\; x_1\,,\, x_2 \in \mathfrak{h}^* \\
& [y, x] = t \cdot \langle y, x\rangle - \tfrac{1}{2} \cdot \textstyle\sum_{\alpha\in R} c_\alpha \cdot \langle y, \alpha\rangle\langle \alpha^\vee, x\rangle \cdot s_\alpha\,, && \forall y \in \mathfrak{h}\,,\, x \in \mathfrak{h}^*\,,
\end{aligned} \tag{15}$$

(the factor $\frac{1}{2}$ on the RHS of the last equation accounts to the fact that roots α and $-\alpha$ give rise to the same reflection).

We will refer to the algebra $\mathsf{H}_{t,c}$ above as a *rational Cherednik algebra* because it is a certain 'rational' degeneration of the double affine Hecke algebra studied by Cherednik [Ch]. In the case of type $\mathbf{A_1}$, the algebra H_κ was discussed in detail in [ChM].

Olshanetsky-Perelomov [OM], have associated a Calogero-Moser type integrable system to any Weyl group W. We show that the space $\mathtt{Spec}\,\mathsf{Z}(\mathsf{H}_{0,c})$ for the rational Cherednik algebra attached to W is a natural completion of the phase space of that integrable system. The affine algebraic variety $\mathtt{Spec}\,\mathsf{Z}(\mathsf{H}_{0,c})$ is always an irreducible normal Gorenstein variety with a Poisson structure. We propose to call it the *Calogero-Moser space associated to* the Weyl group W, with parameter c. For Weyl groups of type: $\mathbf{A_n}, \mathbf{B_n}, \mathbf{C_n}$ and sufficiently general parameters $c \in \mathsf{C}$, the variety $\mathtt{Spec}\,\mathsf{Z}(\mathsf{H}_{0,c})$ is a *smooth* symplectic manifold, by Theorem 13.

Relation to the double-affine Hecke algebra. The Cherednik algebra $\mathsf{H}_{t,c}$ may be thought of as the following two-step degeneration of the double affine Hecke algebra $\mathbf{H}$ defined in [Ch]:

$$\mathbf{H} \;=\; \mathbf{H}^{\text{elliptic}} \;\rightsquigarrow\; \mathbf{H}^{\text{trigonometric}} \;\rightsquigarrow\; \mathbf{H}^{\text{rational}} \;=\; \mathsf{H}_{t,c}\,.$$

In more detail, let G be the Lie group corresponding to the Lie algebra $\mathfrak{g}$, and T a maximal torus of G. The algebra $\mathbf{H}$ contains the group algebras of two (dual) lattices: $X = \mathtt{Hom}(\mathbb{C}^*, T)$ and $Y = \mathtt{Hom}(T, \mathbb{C}^*)$. Degeneration from $\mathbf{H}$ to $\mathbf{H}^{\text{trigonometric}}$ amounts, effectively, to replacing the group algebra of the lattice $X = \mathtt{Hom}(\mathbb{C}^*, T)$ by the Symmetric algebra of the vector space $\mathtt{Hom}_{\mathbb{C}}(\mathbb{C}, \mathtt{Lie}\,T) = \mathfrak{h}$, while the group algebra of the second lattice Y remains unaffected. Degeneration from $\mathbf{H}^{\text{trigonometric}}$ to $\mathbf{H}^{\text{rational}} = \mathsf{H}_{t,c}$ is obtained similarly, by further replacing the group algebra of the lattice $Y = \mathtt{Hom}(T, \mathbb{C}^*)$ by the Symmetric algebra of the vector space $\mathtt{Hom}_{\mathbb{C}}(\mathtt{Lie}\,T, \mathbb{C}) = \mathfrak{h}^*$.

Harish-Chandra homomorphism. It turns out that the spherical subalgebra, $\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}$, of the symplectic reflection algebra associated to the root system of type $\mathbf{A_{n-1}}$ can be described as a 'quantum Hamiltonian reduction' of the algebra $\mathcal{D}(\mathfrak{gl}_n)$ of polynomial differential operators on the Lie algebra $\mathfrak{gl}_n$. A similar description conjecturally exists for the spherical subalgebra of the symplectic reflection algebra associated to any wreath-product $\mathbf{\Gamma}_n$, in which case $\mathfrak{gl}_n$ gets replace by a (vector) space of representations of a Dynkin quiver ($\mathfrak{gl}_n$ is the space of n-dimensional representations of the quiver with one vertex and one edge-loop).

In more detail. Fix an arbitrary complex reductive Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Let $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ be the algebra of $\mathtt{ad}\,\mathfrak{g}$-invariant polynomial differential operators on $\mathfrak{g}$, and $\mathcal{D}(\mathfrak{h})^W$ the algebra of W-invariant polynomial differential operators on $\mathfrak{h}$. Write $\Delta_{\mathfrak{g}}$, resp. $\Delta_{\mathfrak{h}}$, for the second order Laplacian on $\mathfrak{g}$, resp. on $\mathfrak{h}$, associated to a nondegenerate invariant bilinear form on $\mathfrak{g}$. In 1964, Harish-Chandra [HC] defined an algebra homomorphism $\Phi : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{D}(\mathfrak{h})^W$, that reduces to the restriction map: $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$, $f \mapsto f\big|_{\mathfrak{h}}$, on zero order differential operators, and such that $\Phi(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}}$.

We construct a 1-parameter deformation of the Harish-Chandra homomorphism Φ in the special case $\mathfrak{g} = \mathfrak{gl}_n$, and show that this deformed homomorphism may (and should) be viewed as an isomorphism between the quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{gl}_n)$ and the spherical subalgebra $\mathbf{e}\mathsf{H}_{t,c}\mathbf{e}$.

Now let $\mathfrak{h} = \mathbb{C}^n$ be the Cartan subalgebra of diagonal matrices in $\mathfrak{g} = \mathfrak{gl}_n$. Write $x_1, \ldots, x_n$ for coordinates in $\mathbb{C}^n$, and $\mathfrak{h}^{\text{reg}} \subset \mathbb{C}^n$ for the open subset of points with pairwise distinct coordinates. We apply the formalism of *radial parts* of invariant differential operators to construct a family of algebra homomorphisms $\Phi_k : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{D}(\mathfrak{h}^{\text{reg}})^W$

depending on a parameter $k \in \mathbb{C}$, which specializes to the Harish-Chandra homomorphism Φ at $k = 0$. For each $k \in \mathbb{C}$, this homomorphism Φ_k reduces to the restriction map: $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$, $f \mapsto f\big|_{\mathfrak{h}}$, on zero order differential operators. Further, we have: $\Phi_k(\Delta_{\mathfrak{g}}) = \mathsf{L}_k$, where

$$\mathsf{L}_k = \sum\nolimits_j \frac{\partial^2}{\partial x_j^2} - \sum\nolimits_{i \neq j} \frac{k(k+1)}{(x_i - x_j)^2}\,. \tag{16}$$

is the Calogero-Moser operator with rational potential, corresponding to the parameter k.

We now describe the image of the map Φ_k. Let $\mathcal{C}_k$ denote the centraliser of L_k in the algebra $\mathcal{D}(\mathfrak{h}^{\mathsf{reg}})^W$ of W-invariant differential operators on $\mathfrak{h}^{\mathsf{reg}}$. If $k = 0$, then it is classically known that $\mathcal{C}_0 = S\mathfrak{h}^W$, is the algebra of W-invariant differential operators with constant coefficients. In general, it is known (see e.g. [O]) that for each k, there exists an algebra isomorphism $\sigma_k : S\mathfrak{h}^W \xrightarrow{\sim} \mathcal{C}_k$, such that, for any homogeneous element $u \in S\mathfrak{h}^W$ of degree $\ell \geq 0$, the differential operator $\sigma_k(u)$ has order ℓ, and its principal symbol equals u; moreover, $\sigma_k(\Delta_{\mathfrak{h}}) = \mathsf{L}_k$. It turns out that the image of the homomorphism Φ_k equals $\mathcal{B}_k$, the associative subalgebra in $\mathcal{D}(\mathfrak{h}^{\mathsf{reg}})^W$ generated by $\mathcal{C}_k$ and by the polynomial subalgebra $\mathbb{C}[\mathfrak{h}]^W \subset \mathcal{D}(\mathfrak{h})^W$, of zero order W-invariant operators on $\mathfrak{h}$.

Thus, we summarize:

Theorem 17 *There exists a (flat) family of surjective algebra homomorphisms* $\Phi_k : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \twoheadrightarrow \mathcal{B}_k$, *all of which reduce to the restriction map:* $f \mapsto f\big|_{\mathfrak{h}}$, *on zero order differential operators. This family specializes, at* $k = 0$, *to the Harish-Chandra homomorphism* Φ*; furthermore,* $\Phi_k(\Delta_{\mathfrak{g}}) = \mathsf{L}_k$, *for any* $k \in \mathbb{C}$.

A drawback of Theorem 17 is that elements of the algebra $\mathcal{B}_k \subset \mathcal{D}(\mathfrak{h}^{\mathsf{reg}})^W$ may have quite complicated singularites at the divisor $\mathfrak{h} \smallsetminus \mathfrak{h}^{\mathsf{reg}}$. Therefore, it is tricky to give an explicit description of $\mathcal{B}_k$, as a subalgebra inside $\mathcal{D}(\mathfrak{h}^{\mathsf{reg}})^W$. Remarkably, the algebra $\mathcal{B}_k$ admits an imbedding into H_κ, the rational Cherednik algebra of type $\mathbf{A_{n-1}}$. The image of $\mathcal{B}_k$ has a very simple description that we now explain.

If $R = \mathbf{A_{n-1}}$, is the root system of $\mathfrak{g} = \mathfrak{gl}_n$ then all roots are of the same length, and $\mathsf{C} = \mathbb{C}$. We treat the parameter $\kappa = (t, c)$ as a point on the Riemann sphere $\mathbb{P}^1 = (\mathbb{C} \oplus \mathsf{C})/\mathbb{C}^*$ with homogeneous coordinates $(c : t)$, i.e., as a complex number $\kappa = c/t \in \mathbb{C} \cup \{\infty\}$; rescaling: $(t, c) \mapsto (r \cdot t, r \cdot c)$, $r \in \mathbb{C}^*$, makes no essential effect on the algebra $\mathsf{H}_{t,c})$. Thus, in the $\mathbf{A_{n-1}}$-case we make no distinction between the parameter: $\kappa = c/t$, and the complex parameter 'k' entering the deformed Harish-Chandra homomorphism Φ_k. Note that the limit $\kappa \to \infty$ corresponds to $t \to 0$.

For any Weyl group, Cherednik defined a *faithful* representation of the double affine Hecke algebra, given by the so-called Demazure-Lusztig-Dunkl operators. A version of Cherednik's construction yields a faithful representation of H_κ in the vector space $\mathbb{C}[\mathfrak{h}^{\mathsf{reg}}]$. It is easy to see that the restriction of this representation to the spherical subalgebra $\mathsf{eH}_\kappa\mathsf{e} \subset \mathsf{H}_\kappa$ keeps the subspace $\mathbb{C}[\mathfrak{h}^{\mathsf{reg}}]^W \subset \mathbb{C}[\mathfrak{h}^{\mathsf{reg}}]$ stable, and the resulting $\mathsf{eH}_\kappa\mathsf{e}$-action on $\mathbb{C}[\mathfrak{h}^{\mathsf{reg}}]^W$ is given by W-invariant polynomial differential operators on $\mathfrak{h}^{\mathsf{reg}}$. This gives an algebra homomorphism

$$\Theta_\kappa^{\mathsf{spher}} : \mathsf{eH}_\kappa\mathsf{e} \longrightarrow \mathcal{D}(\mathfrak{h}^{\mathsf{reg}})^W. \tag{18}$$

Thus, for $\mathfrak{g} = \mathfrak{gl}_n$ and $W = S_n$, we get the following diagram:

$$\mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\ \Phi_\kappa\ } \mathcal{D}(\mathfrak{h}^{\text{reg}})^W \xleftarrow{\ \Theta_\kappa^{\text{spher}}\ } \mathsf{eH}_\kappa\mathbf{e} \tag{19}$$

The remarkable fact is, that the homomorphisms Φ_κ and $\Theta_\kappa^{\text{spher}}$ above have the same image, the subalgebra $\mathcal{B}_\kappa \subset \mathcal{D}(\mathfrak{h}^{\text{reg}})^{S_n}$. Further, the map $\Theta_\kappa^{\text{spher}}$ in (19) is a bijection onto its image. Inverting this map, we thus obtain an algebra homomorphism:

$$\Phi_\kappa^{\text{spher}} = (\Theta_\kappa^{\text{spher}})^{-1} \circ \Phi_\kappa \ : \quad \mathcal{D}(\mathfrak{gl}_n)^{\mathfrak{gl}_n} \longrightarrow \mathsf{eH}_\kappa\mathbf{e}\,. \tag{20}$$

This homomorphism provides a link between invariant differential operators on $\mathfrak{gl}_n$ and the Cherednik algebra H_κ. Furthermore, replacing the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ by the group $G = GL_n$, one can define a 'trigonometric' version of the deformed Harish-Chandra homomorphism (20):

$$\Phi_\kappa^{\text{trigonometric}} \ : \ \mathcal{D}(G)^G \longrightarrow \mathbf{e} \cdot \mathbf{H}^{\text{trigonometric}} \cdot \mathbf{e}\,. \tag{21}$$

One might expect that replacing further $\mathcal{D}(G)$ by the so-called *Heisenberg double* of the corresponding quantum group one gets an elliptic analogue of the sperical Harish-Chandra homomorphism with image $\mathbf{eHe}$.

Calogero-Moser space. It is a well-known principle that the 'classical' limit of the algebra $\mathcal{D}(\mathfrak{g})$ is the commutative algebra of regular functions on $T^*\mathfrak{g}$, the cotangent bundle on $\mathfrak{g}$. Since $T^*\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^* \simeq \mathfrak{g} \oplus \mathfrak{g}$, the 'classical' limit of $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ is the algebra $\mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]^{\mathfrak{g}}$ of $\mathtt{ad}\,\mathfrak{g}$-invariant polynomials on $\mathfrak{g} \oplus \mathfrak{g}$. Now, let $\mathfrak{g} = \mathfrak{gl}_n$. There are reasons to think of the limit: $\kappa = (t, c) \longrightarrow \infty = (0, 1) \in \mathbb{P}^1$, as a 'classical' limit. Accordingly, we show that the family of Harish-Chandra homomorphisms $\Phi_\kappa^{\text{spher}} : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \twoheadrightarrow \mathsf{eH}_\kappa\mathbf{e}$, see (20), has a 'classical' limit, a surjective algebra homomorphism $\Phi_\infty^{\text{spher}} : \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]^{\mathfrak{g}} \twoheadrightarrow \mathsf{eH}_\infty\mathbf{e}$. It turns out that this homomorphism is intimately related to the so-called *Calogero-Moser space* $\mathcal{M}_n$, a symplectic algebraic manifold introduced by Kazhdan-Kostant-Sternberg [KKS], and studied further by G. Wilson [Wi]. This space, which is a particular case of the quiver variety $\mathcal{M}_{\Gamma,n,c}$, for $\Gamma = \{1\}$, is defined as a categorical quotient: $\mathcal{M}_n = M//\mathrm{Ad}\ \mathrm{GL}_n$, where

$$M := \{(X, Y) \in \mathfrak{g} \times \mathfrak{g} \ \big|\ [X, Y] + \mathtt{Id} = \text{ rank 1 matrix}\}\,, \tag{22}$$

is an affine subvariety in $\mathfrak{g} \oplus \mathfrak{g}$, a special case of the quiver data variety $\mathbb{M}_\Gamma^c(\mathbf{V})$. Thus, by definition, the coordinate ring of $\mathcal{M}_n$ equals $\mathbb{C}[\mathcal{M}_n] = \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]^{\mathfrak{g}}/(\mathcal{I}_M)^{\mathfrak{g}}$, where $\mathcal{I}_M \subset \mathbb{C}[\mathfrak{g}\oplus\mathfrak{g}]$ denotes the defining ideal of M, and the superscript '$\mathfrak{g}$' stands for '$\mathtt{ad}\,\mathfrak{g}$-invariants'. A 'commutative' counterpart of homomorphism (20) reads:

Theorem 23 *The kernel of the homomorphism $\Phi_\infty^{\text{spher}}$ equals $(\mathcal{I}_M)^{\mathfrak{g}}$; hence one has the following chain of Poisson algebra isomorphisms:*

$$\mathbb{C}[\mathcal{M}_n] = \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]^{\mathfrak{g}}/\mathtt{Ker}\big(\Phi_\infty^{\text{spher}}\big) \xrightarrow[\sim]{\Phi_\infty^{\text{spher}}} \mathsf{eH}_\infty\mathsf{e} \xrightarrow[\sim]{\mathsf{Satake}^{-1}} \mathsf{Z}(\mathsf{H}_\infty)\,.$$

This result provides, in the special case $\Gamma = \{1\}$, a transparent conceptual interpretation of the isomorphism of Theorem 13. Thus, for $\mathbf{\Gamma}_n = S_n$ and $\kappa = \infty$, Corollary 14 yields

Theorem 24 *If $W = S_n$ is the Symmetric group and $\kappa = \infty = (0, c)$, $c \neq 0$, then*

(i) *The algebra $\mathsf{H}_\infty = \mathsf{H}_{0,c}(S_n)$ is Morita equivalent to the algebra: $\mathsf{Z}_\infty \simeq \mathbb{C}[\mathcal{M}_n]$.*

(ii) *Any simple H_∞-module is isomorphic to one of the form: $\mathsf{H}_\infty \mathbf{e} \otimes_{\mathsf{Z}_\infty} \chi$, for a certain character $\chi : \mathsf{Z}_\infty \to \mathbb{C}$. Thus, the isomorphism classes of simple H_∞-modules are parametrized by points of the Calogero-Moser space $\mathcal{M}_n$.*

(iii) *Any simple H_∞-module has dimension $n!$ and is isomorphic, as an S_n-module, to the regular representation of S_n.*

Berest and Wilson have introduced an (infinite-dimensional) group $\mathbb{G}$ of the automorphisms of $\mathbb{C}\langle x, y\rangle$, the free associative algebra on two generators, that keep the element $xy - yx \in \mathbb{C}\langle x, y\rangle$ fixed. We show that the group $\mathbb{G}$ acts naturally on $\mathcal{D}(\mathfrak{gl}_n)$ and on H_κ by algebra automorphisms, preserving subalgebras $\mathcal{D}(\mathfrak{gl}_n)^{\mathfrak{gl}_n}$ and $\mathbf{e}\mathsf{H}_\kappa\mathbf{e}$, respectively. Moreover, the map $\Phi_\kappa^{\mathtt{spher}} : \mathcal{D}(\mathfrak{gl}_n)^{\mathfrak{gl}_n} \longrightarrow \mathbf{e}\mathsf{H}_\kappa\mathbf{e}$ commutes with the $\mathbb{G}$-action. Further, in [BW], Berest-Wilson defined a transitive action of $\mathbb{G}$ on the Calogero-Moser space $\mathcal{M}_n$. It follows from our results that the $\mathbb{G}$-action on $\mathcal{M}_n$ can be canonically lifted to a $\mathbb{G}$-action on the vector bundle $\mathcal{R}$, see Theorem 7. It seems to us quite important to obtain a better geometric understanding of this vector bundle.

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P.E.: Department of Mathematics, Rm 2-165, MIT, 77 Mass. Ave, Cambridge, MA 02139
`etingof@math.mit.edu`
V.G.: Department of Mathematics, University of Chicago, Chicago, IL 60637, USA;
`ginzburg@math.uchicago.edu`