ON CORRESPONDENCES OF A K3 SURFACE WITH ITSELF

Viacheslav V. Nikulin[1]

Abstract. Let X be a K3 surface with a polarization H of degree $H^2 = 2rs$, $r,\ s \geq 1$. The moduli space of sheaves over X with the primitive isotropic Mukai vector (r, H, s) is again a K3 surface Y. We consider the problem: When $Y \cong X$? In this case, we get a 2-dimensional algebraic cycle on the product $X \times X$, and a correspondence in the sense of Mukai (of the isometry type in periods) of X with itself.

The preprint consists of two parts.

In the Part I we treat a particular case when $H \cdot N(X) = \mathbb{Z}$ where $N(X)$ is the Picard lattice of X. In this case results are very nice, complete and very efficient. One can consider these results as ideal.

In the Part II we consider the general case: H can be non-primitive and $H \cdot N(X) = \gamma\mathbb{Z}$ where γ can be arbitrary. In this general case results are much more complicated, far from being complete and much less efficient.

There is a chance that the general results of Part II can be made so nice and efficient as in the particular case of Part I, after further study. It is the reason I puted both parts together in the preprint.

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ON CORRESPONDENCES OF A K3 SURFACE WITH ITSELF, I

Viacheslav V. Nikulin

Abstract. Let X be a K3 surface with a polarization H of degree $H^2 = 2rs$, $r,\ s \ge 1$. Assume $H \cdot N(X) = \mathbb{Z}$ for the Picard lattice $N(X)$. The moduli space of sheaves over X with the isotropic Mukai vector (r, H, s) is again a K3 surface Y.

We prove that $Y \cong X$, if there exists $h_1 \in N(X)$ with $(h_1)^2 = f(r,s)$, $H \cdot h_1 \equiv 0 \mod g(r,s)$, and h_1 satisfies some condition of primitivity. These conditions are necessary, if X is general with rk $N(X) = 2$. Existence of such kind a criterion is surprising, and it also gives some geometric interpretation of elements in $N(X)$ with negative square.

We describe all irreducible 18-dimensional components of moduli of the (X, H) with $Y \cong X$. We prove that their number is always infinite.

This generalizes results of [4, 5] for $r = s$.

0. Introduction

Let X be a K3 surface with a polarization H of degree $H^2 = 2rs$, $r,\ s \ge 1$, and a primitive Mukai vector (r, H, s). Let Y be the moduli of sheaves (coherent) over X with the Mukai vector (r, H, s). The Mukai vector $v = (r, H, s)$ is isotropic, then Y is a K3 surface with a natural *nef* element h with $h^2 = 2ab$ where $c =$ g.c.d(r, s) and $a = r/c$, $b = s/c$. The surface Y is isogenous to X in the sense of Mukai. The second Chern class of the corresponding quasi-universal sheave gives a 2-dimensional algebraic cycle $Z \subset X \times Y$, and an algebraic correspondence between X and Y. See [6, 7] and also [1] about these results.

Let $N(X)$ be the Picard lattice of X. We consider an invariant $\gamma(H) \in \mathbb{N}$ of $H \in N(X)$ defined by $H \cdot N(X) = \gamma(H)\mathbb{Z}$ (clearly, $\gamma(H) | 2rs$). In this paper, we assume that

$$\gamma(H) = 1, \text{ that is } H \cdot N(X) = \mathbb{Z}. \tag{0.1}$$

Then H is primitive and $\rho(X) = \text{rk } N(X) \ge 2$. By Mukai [6, 7], the transcendental periods $(T(X), H^{2,0}(X))$ and $(T(Y), H^{2,0}(Y))$ are isomorphic in this case. We can expect that surfaces X and Y also can be isomorphic, and we then get a cycle $Z \subset X \times X$, and a correspondence of X with itself. Thus, an interesting for us question is

Question 1. *When is Y isomorphic to X?*

We want to answer this question in terms of Picard lattices $N(X)$ and $N(Y)$ of X and Y. Then our question can be reformulated as follows:

Question 2. *Assume that N is a hyperbolic lattice, $\widetilde{H}\in N$ an element with square $2rs$ and $\widetilde{H}\cdot N=\mathbb{Z}$. What are conditions on N and $\widetilde{H}$ such that for any K3 surface X with Picard lattice $N(X)$ and s polarization $H\in N(X)$ the corresponding K3 surface Y is isomorphic to X, if the pairs $(N(X),H)$ and $(N,\widetilde{H})$ are isomorphic as abstract lattices with fixed elements?*

In other words, what are conditions on $(N(X),H)$ as an abstract lattice with a vector H which are sufficient for Y to be isomorphic to X and are necessary, if X is a general K3 surface with its Picard lattice $N(X)$?

We answered this question in [4] for $r=s=2$ (then $H^2=8$, and the condition (0.1) always satisfies, if H is primitive); in [5] for $r=s$ under the condition (0.1) (then (0.1) always satisfies, if H is primitive and $r=s$ is even). The main surprising result of [4, 5] was that $Y\cong X$, if the Picard lattice $N(X)$ has an element h_1 with some prescribed square $(h_1)^2$ and some minor additional conditions, and these conditions are necessary for a general K3 surface X with $\rho(X)=\mathrm{rk}\ N(X)=2$. Here we prove similar results in general, for arbitrary r and s, under the condition (0.1).

Theorem 1. *Let X be a K3 surface with a polarization H such that $H^2=2rs$, $r,s\ge 1$, the Mukai vector (r,H,s) is primitive and Y the moduli of sheaves on X with the Mukai vector (r,H,s).*

Then $Y\cong X$, if at least for one of signs $\pm$ there exists $h_1\in N(X)$ such that the elements H, h_1 are contained in a 2-dimensional sublattice $N\subset N(X)$ with $H\cdot N=\mathbb{Z}$, and h_1 belongs to one of a-series or b-series described below:

a-series:

$$h_1^2=\pm 2bc,\ H\cdot h_1\equiv 0 \mod bc,\ H\cdot h_1\not\equiv 0 \mod bcl_1,\ h_1/l_2\notin N(X)$$

where l_1 and l_2 are any primes such that $l_1^2|a$ and $l_2^2|b$;

b-series:

$$h_1^2=\pm 2ac,\ H\cdot h_1\equiv 0 \mod ac,\ H\cdot h_1\not\equiv 0 \mod acl_1,\ h_1/l_2\notin N(X).$$

where l_1 and l_2 are any primes such that $l_1^2|b$ and $l_2^2|a$.

The conditions above are necessary for $\gamma(H)=1$ and $Y\cong X$, if $\rho(X)\le 2$ and X is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods $(T(X),H^{2,0}(X))$ of X is ± 1.

Like in [4, 5] for $r=s$, we also describe all (irreducible) divisorial conditions on 19-dimensional moduli of polarized K3 surfaces (X,H) which imply $\gamma(H)=1$ and $Y\cong X$. It is sufficient to assume that $\rho(X)=2$. According to [10, 11], it is equivalent to description up to isomorphisms of all possible pairs $H\in N(X)$ where $\mathrm{rk}\ N(X)=2$. There are two defining invariants of such pairs: $d=-\det N(X)\in\mathbb{N}$ and $\overline{\mu}=\{\mu,-\mu\}\subset(\mathbb{Z}/2rs)^*$ where $d\equiv\mu^2 \mod 4rs$. See the definition of μ in (3.1.2) and (3.1.3)).

We introduce sets:
The a-series:

$$\begin{aligned}\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm} = \{d\in\mathbb{N}\mid d\equiv \mu^2 \mod 4abc^2,\ \exists\ (p,q)\in\mathbb{Z}\times\mathbb{Z}:\ p^2-dq^2=\pm 4ac,\\ p-\mu q\equiv 0 \mod 2ac,\ \text{g.c.d}(a,p,q)=1,\\ p-\mu q\not\equiv 0 \mod 2acl \text{ for any prime } l \text{ such that } l^2|b\ \}.\end{aligned}$$

The b-series:

$$\begin{aligned}\mathcal{D}(r,s;b)^{\overline{\mu}}_{\pm} = \{d\in\mathbb{N}\mid d\equiv \mu^2 \mod 4abc^2,\ \exists\ (p,q)\in\mathbb{Z}\times\mathbb{Z}:\ p^2-dq^2=\pm 4bc,\\ p-\mu q\equiv 0 \mod 2bc,\ \text{g.c.d}(b,p,q)=1,\\ p-\mu q\not\equiv 0 \mod 2bcl \text{ for any prime } l \text{ such that } l^2|a\ \}.\end{aligned}$$

We have

Theorem 2. *With the above notations all possible irreducible divisorial conditions on moduli of polarized K3 surfaces (X,H) with a polarization H with $H^2=2rs$, which imply $\gamma(H)=1$ and $Y\cong X$ are labelled by the set*

$$\mathcal{D}iv(r,s)=\mathcal{D}iv(r,s;a)\bigcup \mathcal{D}iv(r,s;b),$$

where

$$\begin{aligned}\mathcal{D}iv(r,s;a)=\{(d,\overline{\mu})\mid \overline{\mu}=\{\mu,-\mu\}\subset(\mathbb{Z}/2rs)^*,\ d\in\mathcal{D}(r,s;a)^{\overline{\mu}}\},\\ \mathcal{D}iv(r,s;b)=\{(d,\overline{\mu})\mid \overline{\mu}=\{\mu,-\mu\}\subset(\mathbb{Z}/2rs)^*,\ d\in\mathcal{D}(r,s;b)^{\overline{\mu}}\}\end{aligned}$$

and

$$\mathcal{D}(r,s;a)^{\overline{\mu}}=\mathcal{D}(r,s;a)^{\overline{\mu}}_{+}\bigcup\mathcal{D}(r,s;a)^{\overline{\mu}}_{-},\quad \mathcal{D}(r,s;b)^{\overline{\mu}}=\mathcal{D}(r,s;b)^{\overline{\mu}}_{+}\bigcup\mathcal{D}(r,s;b)^{\overline{\mu}}_{-}.$$

The sets

$$\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm} \text{ and } \mathcal{D}(r,s;b)^{\overline{\mu}}_{\pm}$$

are infinite if they are not empty.

The set $\mathcal{D}iv(r,s)$ is always non-empty and then infinite: i. e. at least one of sets $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm}$ or $\mathcal{D}(r,s;b)^{\overline{\mu}}_{\pm}$ is not empty for one of $\overline{\mu}$ and $\pm$.

See more exact and strong statements in Lemmas 4.3, 4.4 and Theorems 4.1, 4.5 and 4.6.

These two theorems give the main results of the paper. We remark that Theorem 1 gives many (if not all) known cases when $Y\cong X$ (e.g. see [2], [17—19] and [20]). Theorem 1 is also interesting because it gives a very clear geometric interpretation of elements of $N(X)$ with negative square (when the sign is $-$). For K3 surfaces it is well-known only for elements $\delta\in N(X)$ with $\delta^2=-2$: then δ or $-\delta$ is effective.

In Section 5 we formulate similar results for the surface Y. They are very similar to the results for X. We only formulate these results since proofs are very similar to the corresponding proofs for X, but sometimes require additional non-trivial considerations. Thus, it is surprising that the results for Y are so similar to the results for X. Using results for X and Y together, one can construct non-trivial compositions of correspondences of the K3 surfaces $X \cong Y$ with itself.

In ideas we always follow [4] where a particular case $r = s = 2$ had been treated in great details. But technically the results of this paper are much more complicated. One needs to analyze many details and calculations in general and find appropriate formulations. As in [4, 5], the fundamental tools to get the results above is the Global Torelli Theorem for K3 surfaces proved by I.I.Piatetsky-Shapiro and I.R.Shafarevich in [14], and results of Mukai [6, 7]. By results of [6, 7], we can calculate periods of Y using periods of X; by the Global Torelli Theorem for K3 surfaces [14], we can find out if Y is isomorphic to X.

Recently the general case of a primitive isotropic Mukai vector (r, H, s) was considered in [13] where many results of this paper were generalized to the general case when $\gamma(H)$ is possibly not equal to one and H is possibly not primitive in $N(X)$. Results of [13] are much more complicated and much less efficient. It is non-trivial to deduce results of this paper directly from results of [13].

Thus, in this paper we treat a very nice and important particular case when the results are very simple, efficient and complete. Using methods of this paper and [13], one can extend these results to more general types of primitive isotropic Mukai vectors (r, H, s). We hope to consider that in further publications.

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1. Some notations and results about lattices and K3 surfaces

1.1. Some notations about lattices. We use notations and terminology from [11] about lattices, their discriminant groups and forms. A *lattice* L is a non-degenerate integral symmetric bilinear form. That is L is a free $\mathbb{Z}$-module equipped with a symmetric pairing $x \cdot y \in \mathbb{Z}$ for $x,\ y \in L$, and this pairing should be non-degenerate. We denote $x^2 = x \cdot x$. The *signature* of L is the signature of the corresponding real form $L\otimes\mathbb{R}$. The lattice L is called *even* if x^2 is even for any $x \in L$. Otherwise, L is called *odd*. The *determinant* of L is defined to be $\det L = \det(e_i \cdot e_j)$ where $\{e_i\}$ is some basis of L. The lattice L is *unimodular* if $\det L = \pm 1$. The *dual lattice* of L is $L^* = Hom(L,\ \mathbb{Z}) \subset L\otimes\mathbb{Q}$. The *discriminant group* of L is $A_L = L^*/L$. It has the order $|\det L|$. The group A_L is equipped with the *discriminant bilinear form* $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ and the *discriminant quadratic form* $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ if L is even. To get this forms, one should extend the form of L to the form on the dual lattice L^* with values in $\mathbb{Q}$.

For $x \in L$, we shall consider the invariant $\gamma(x) \geq 0$ where

$$x \cdot L = \gamma(x)\mathbb{Z}. \tag{1.1.1}$$

Clearly, $\gamma(x)|x^2$ if $x \neq 0$.

We denote by $L(k)$ the lattice obtained from a lattice L by multiplication of the form of L by $k \in \mathbb{Q}$. The orthogonal sum of lattices L_1 and L_2 is denoted by $L_1 \oplus L_2$. For a symmetric integral matrix A, we denote by $\langle A\rangle$ a lattice which is given by the matrix A in some bases. We denote

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.1.2}$$

Any even unimodular lattice of the signature $(1,1)$ is isomorphic to U.

An embedding $L_1 \subset L_2$ of lattices is called *primitive* if L_2/L_1 has no torsion. We denote by $O(L)$, $O(b_L)$ and $O(q_L)$ the automorphism groups of the corresponding forms. Any $\delta \in L$ with $\delta^2 = -2$ defines a reflection $s_\delta \in O(L)$ which is given by the formula

$$x \to x + (x\cdot\delta)\delta,$$

$x \in L$. All such reflections generate the *2-reflection group* $W^{(-2)}(L) \subset O(L)$.

1.2. Some notations about K3 surfaces. Here we remind some basic notions and results about K3 surfaces, e. g. see [14—16]. A K3 surface X is a non-singular projective algebraic surface over $\mathbb{C}$ such that its canonical class K_X is zero and the irregularity $q_X = 0$. We denote by $N(X)$ the *Picard lattice* of X which is a hyperbolic lattice with the intersection pairing $x\cdot y$ for $x,\ y \in N(S)$. Since the canonical class $K_X = 0$, the space $H^{2,0}(X)$ of 2-dimensional holomorphic differential forms on X has dimension one over $\mathbb{C}$, and

$$N(X) = \{x \in H^2(X,\mathbb{Z}) \mid x\cdot H^{2,0}(X) = 0\} \tag{1.2.1}$$

where $H^2(X,\mathbb{Z})$ with the intersection pairing is a 22-dimensional even unimodular lattice of signature $(3,19)$. The orthogonal lattice $T(X)$ to $N(X)$ in $H^2(X,\mathbb{Z})$ is called the *transcendental lattice of* X. We have $H^{2,0}(X) \subset T(X)\otimes\mathbb{C}$. The pair $(T(X), H^{2,0}(X))$ is called the *transcendental periods of* X. The *Picard number* of X is $\rho(X) = \mathrm{rk}\ N(X)$. A non-zero element $x \in N(X)\otimes\mathbb{R}$ is called *nef* if $x \neq 0$ and $x\cdot C \geq 0$ for any effective curve $C \subset X$. It is known that an element $x \in N(X)$ is ample if $x^2 > 0$, x is nef, and the orthogonal complement $x^\perp$ to x in $N(X)$ has no elements with square -2. For any element $x \in N(X)$ with $x^2 \geq 0$, there exists an element $w \in W^{(-2)}(N(X))$ such that $\pm w(x)$ is nef; it then is ample, if $x^2 > 0$ and $x^\perp$ had no elements with square -2 in $N(X)$.

We denote by $V^+(X)$ the light cone of X, which is the half-cone of

$$V(X) = \{x \in N(X)\otimes\mathbb{R} \mid x^2 > 0\ \} \tag{1.2.2}$$

containing a polarization of X. In particular, all nef elements x of X belong to $\overline{V^+(X)}$: one has $x\cdot V^+(X) > 0$ for them.

The reflection group $W^{(-2)}(N(X))$ acts in $V^+(X)$ discretely, and its fundamental chamber is the closure $\overline{\mathcal{K}(X)}$ of the Kähler cone $\mathcal{K}(X)$ of X. It is the same as the set of all *nef* elements of X. Its faces are orthogonal to the set $\mathrm{Exc}(X)$ of all exceptional curves r on X which are non-singular rational curves r on X with $r^2 = -2$. Thus, we have

$$\overline{\mathcal{K}(X)} = \{0 \neq x \in \overline{V^+(X)} \mid x \cdot \mathrm{Exc}(X) \geq 0\}. \tag{1.2.3}$$

2. General results on the Mukai correspondence between K3 surfaces with primitive polarizations of degrees $2rs$ and $2ab$ which gives isomorphic K3's

2.1. The correspondence. Let X be a smooth complex projective K3 surface with a polarization H of degree $2rs$, $r,\, s \geq 1$. Let Y be the moduli space of coherent sheaves $\mathcal{E}$ on X with the primitive isotropic Mukai vector $v = (r, H, s)$. Then $\mathrm{rk}\, \mathcal{E} = r$, $\chi(\mathcal{E}) = r + s$ and $c_1(\mathcal{E}) = H$. Let

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \tag{2.1.1}$$

be the full cohomology lattice of X equipped with the Mukai product

$$(u, v) = -(u_0 \cdot v_2 + u_2 \cdot v_0) + u_1 \cdot v_1 \tag{2.1.2}$$

for $u_0, v_0 \in H^0(X, \mathbb{Z})$, $u_1, v_1 \in H^2(X, \mathbb{Z})$, $u_2, v_2 \in H^4(X, \mathbb{Z})$. We naturally identify $H^0(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ with $\mathbb{Z}$. Then the Mukai product is

$$(u, v) = -(u_0 v_2 + u_2 v_0) + u_1 \cdot v_1. \tag{2.1.3}$$

Since $H^2 = 2rs$, the element

$$v = (r, H, s) = (r, H, \chi - r) \in H^*(X, \mathbb{Z}) \tag{2.1.4}$$

is isotropic, i.e. $v^2 = 0$. In this case (for a primitive v), Mukai showed [6, 7] that Y is a K3 surface, and one has the natural identification

$$H^2(Y, \mathbb{Z}) \cong (v^\perp / \mathbb{Z}v) \tag{2.1.5}$$

which also gives the isomorphism of the Hodge structures of X and Y. The Y has the canonical polarization $h = (-a, 0, b) \mod \mathbb{Z}v$ where

$$c = \text{g.c.d}(r, s),\ a = r/c\,,\ b = s/c\,. \tag{2.1.6}$$

Then $\text{g.c.d.}(a, b) = 1$.

In particular, (2.1.5) gives an embedding

$$T(X) \subset T(Y) \tag{2.1.7}$$

of the lattices of transcendental cycles of the index

$$[T(Y) : T(X)] = n(v) = \min |v \cdot x| \tag{2.1.8}$$

where $x \in H^0(X, \mathbb{Z}) \oplus N(X) \oplus H^4(X, \mathbb{Z})$ and $v \cdot x \neq 0$.

In this paper, we are interested in the case when $Y \cong X$. By (2.1.8), it may happen, if $n(v) = 1$ only.

We can introduce the invariant $\gamma = \gamma(H) \in \mathbb{N}$ which is defined by

$$H \cdot N(X) = \gamma\mathbb{Z}. \tag{2.1.9}$$

In this paper we consider the important case when $\gamma = \gamma(H) = 1$. Thus, $N(X) \cdot H = \mathbb{Z}$. Equivalently, there exists $x \in N(X)$ such that $x \cdot H = 1$. From $\gamma = \gamma(H) = 1$, it follows that $H \in N(X)$ is primitive, the Mukai vector (r, H, s) is also primitive, and the invariant $n(v) = 1$. Thus, we may have a hope to have cases when $Y \cong X$. Since the Picard lattice $N(X)$ is even, it also follows that the Picard number $\rho(X) = \mathrm{rk}\ N(X) \geq 2$ in this case. Thus:

$$\begin{aligned}&\gamma = \gamma(H) = 1 \implies \\ &\rho(X) \geq 2,\ H \text{ is primitive},\ (T(Y), H^{2,0}(Y)) \cong (T(X), H^{2,0}(X)).\end{aligned} \tag{2.1.10}$$

The same case $\gamma = 1$ was considered in [4] for $r = s = c = 2$, and in [5] for $r = s = c$. Results of [4, 5] concerning an arbitrary Picard lattice $N(X)$ can be generalized to the general case of $\gamma = 1$.

2.2. General results for an arbitrary Picard lattice $N(X)$. First we remind *the characteristic map* defined in [4]. It is defined for any non-degenerate even lattice S and any its element $P \in S$ with $P^2 = 2m \neq 0$ and $\gamma(P) = 1$, i. e. $P \cdot S = 1$. Then P is primitive in S.

We denote by

$$K(P) = P^{\perp}_{S} \tag{2.2.1}$$

the orthogonal complement to P in S. Put $P^* = P/2m$. Then any element $x \in S$ can be written as

$$x = nP^* + k^* \tag{2.2.2}$$

where $n \in \mathbb{Z}$ and $k^* \in K(P)^*$, because

$$\mathbb{Z}P \oplus K(P) \subset S \subset S^* \subset \mathbb{Z}P^* \oplus K(P)^*.$$

Since $\gamma(P) = 1$, the map $nP^* + [P] \to k^* + K(P)$ gives an isomorphism of the groups $\mathbb{Z}/2m \cong [P^*]/[P] \cong [u^*(P) + K(P)]/K(P)$ where $u^*(P) + K(P)$ has order $2m$ in $A_{K(P)} = K(P)^*/K(P)$. It follows,

$$S = [\mathbb{Z}P, K(P), P^* + u^*(P)]. \tag{2.2.3}$$

The element $u^*(P)$ is defined canonically mod $K(P)$. Since $P^*+u^*(P)$ belongs to the even lattice S, it follows

$$(P^*+u^*(P))^2=\frac{1}{2m}+u^*(P)^2\equiv 0 \mod 2. \tag{2.2.4}$$

Let $\overline{P^*}=P^* \mod [P]\in [P^*]/[P]\cong \mathbb{Z}/2m$ and $\overline{k^*}=k^* \mod K(P)\in A_{K(P)}$. Then

$$S/[P,K(P)]=(\mathbb{Z}/2m)(\overline{P^*}+\overline{u^*(P)})\subset (\mathbb{Z}/2m)\overline{P^*}+K(P)^*/K(P). \tag{2.2.5}$$

Also $S^*\subset \mathbb{Z}P^*+K(P)^*$ since $P+K(P)\subset S$, and for $n\in\mathbb{Z}$, $k^*\in K(P)^*$ we have $x=nP^*+k^*\in S^*$, if and only if

$$(nP^*+k^*)\cdot(P^*+u^*(P))=\frac{n}{2m}+k^*\cdot u^*(P)\in\mathbb{Z}.$$

It follows,

$$\begin{aligned}S^*=\{nP^*+k^*\mid n\in\mathbb{Z},\ k^*\in K(P)^*,\ n\equiv -2m\ (k^*\cdot u^*(P)) \mod 2m\}\subset\\ \subset \mathbb{Z}P^*+K(P)^*,\end{aligned} \tag{2.2.6}$$

and

$$S^*/[P,K(P)]=\{-2m(\overline{k^*}\cdot\overline{u^*(P)})\,\overline{P^*}+\overline{k^*}\}\mid \overline{k^*}\in A_{K(P)}\}\subset (\mathbb{Z}/2m)\overline{P^*}+A_{K(P)}. \tag{2.2.7}$$

We introduce *the characteristic map of $P\in S$:*

$$\kappa(P):K(P)^*\to A_{K(P)}/(\mathbb{Z}/2m)(u^*(P)+K(P))\to A_S \tag{2.2.8}$$

where for $k^*\in K(P)^*$ we have

$$\kappa(P)(k^*)=-2m(k^*\cdot u^*(P))P^*+k^*+S\in A_S. \tag{2.2.9}$$

It is epimorphic and its kernel is $(\mathbb{Z}/2m)(u^*(P)+K(P))$. It gives the canonical isomorphism

$$\overline{\kappa(P)}:A_{K(P)}/(\mathbb{Z}/2m)(u^*(P)+K(P))\cong A_S. \tag{2.2.10}$$

For the corresponding discriminant forms we have

$$\kappa(k^*)^2 \mod 2=(k^*)^2+2m(k^*\cdot u^*(P))^2 \mod 2. \tag{2.2.11}$$

Below we shall consider the characteristic map for elements $P\in N(X)$ with $\gamma(P)=1$. It will be denoted as $\kappa(P)$.

We define $m(a,b) \mod 2ab$ by

$$m(a,b)\equiv -1 \mod 2a,\ \text{ and }\ m(a,b)\equiv 1 \mod 2b. \tag{2.2.12}$$

The $m(a,b) \mod 2ab$ was introduced by Mukai [8].

The same calculations as in [4] and also [5] (the key calculations were done in [4]) prove the fundamental for further results Theorem 2.2.1 below. Its complete proof in the general case (for any γ and possibly non-primitive H) is contained in [13]. The Global Torelli Theorem for K3 surfaces [14] and Mukai's result (2.1.5) are fundamental for the proof.

Theorem 2.2.1. *Assume that X is a K3 surface with a polarization H with $H^2 = 2rs$, r, $s \geq 1$, and $\gamma(H) = 1$ for $H \in N(X)$. Let Y be the moduli space of coherent sheaves on X with the Mukai vector $v = (r, H, s)$ (it is clearly primitive).*

Then the surface Y is isomorphic to X, if the following conditions (i) and (ii) hold:

(i) there exists $\widetilde{h} \in N(X)$ of the degree $\widetilde{h}^2 = 2ab$, $\gamma(\widetilde{h}) = 1$, and such that there exists an embedding

$$f : K(H) \to K(\widetilde{h})$$

of negative definite lattices such that

$$\begin{aligned} &f^*(K(\widetilde{h})) = [K(H), 2abcu^*(H)],\\ &f^*(u^*(\widetilde{h})) + f^*(K(\widetilde{h})) = m(a,b)c\, u^*(H) + f^*(K(\widetilde{h})). \end{aligned} \tag{2.2.13}$$

(ii) For the $\widetilde{h}$ and f in (i), there exists a choice of $\pm$ such that

$$\kappa(\widetilde{h})(k^*) = \pm\kappa(H)(f^*(k^*)) \tag{2.2.14}$$

for any $k^* \in K(\widetilde{h})^*$.

The conditions (i) and (ii) are necessary for a K3 surface X with $\rho(X) \leq 19$ which is general for its Picard lattice $N(X)$ in the following sense: the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is ± 1. If $\rho(X) = 20$, then always $Y \cong X$.

For instance, let $c = 1$. Then one can take $\widetilde{h} = H$ and $f = id$ in Theorem 2.2.1, if $m(a,b) \equiv 1 \mod 2ab$. This is valid, only if either $a = 1$ or $b = 1$. Thus, $Y \cong X$, if $\gamma(H) = 1$, $c = 1$ and one of a, b is equal to 1. According to Mukai [8], always $Y \cong X$, if $c = 1$ and one of a, b is equal to 1. One does not need the additional condition $\gamma(H) = 1$.

3. The case of Picard number 2

3.1. Main results for $\rho(X) = 2$. Here we apply results of Sect. 2 to X and Y with Picard number 2. A general K3 surface X with $\gamma(H) = 1$ has $\rho(X) = 2$.

First, we consider an arbitrary K3 surface X with Picard number 2 and a primitive polarization H of degree $H^2 = 2rs$ where r, $s \geq 1$. Like above, $c = \text{g.c.d.}(r, s)$ and $r = ca$, $s = cb$. Additionally, we assume that $\gamma(H) = 1$ for $H \in N(X)$. Let

$$K(H) = H^{\perp}_{N(X)} = \mathbb{Z}\delta \tag{3.1.1}$$

and $\delta^2 = -t$ where $t > 0$ is even. The $\delta \in N(X)$ is defined uniquely up to $\pm\delta$. Since $\gamma(H) = 1$, we have

$$N(X) = [\mathbb{Z}H,\ \mathbb{Z}\delta,\ \mu H^* + \frac{\delta}{2rs}] \tag{3.1.2}$$

where $H^* = H/2rs$ and $g.c.d(\mu, 2rs) = 1$. The element

$$\pm\mu \mod 2rs \in (\mathbb{Z}/2rs)^* \tag{3.1.3}$$

is *the invariant of the pair* $(N(X), H)$ up to isomorphisms of lattices with a primitive vector H of $H^2 = 2rs$ and $\gamma(H) = 1$. If δ changes to $-\delta$, then $\mu \mod 2rs$ changes to $-\mu \mod 2rs$. We have

$$(\mu H^* + \frac{\delta}{2rs})^2 = \frac{1}{2rs}(\mu^2 - \frac{t}{2rs}) \equiv 0 \mod 2. \tag{3.1.4}$$

Then $t = 2rsd$, for some $d \in \mathbb{N}$ and $\mu^2 \equiv d \mod 4rs$. Thus, $d \mod 4rs \in (\mathbb{Z}/4rs)^{*\,2}$. We have $-d = \det(N(X))$. Any element $z \in N(X)$ can be written as $z = (xH + y\delta)/2rs$ where $x \equiv \mu y \mod 2rs$. In these considerations, one can replace H by any primitive element of $N(X)$ with square $2rs$ and $\gamma(H) = 1$. Thus, we have:

Proposition 3.1.1. *Let X be a K3 surface with Picard number $\rho = 2$ equipped with a primitive polarization H of degree $H^2 = 2rs$, $r, s \geq 1$, and $\gamma(H) = 1$.*

The pair $(N(X), H)$ has the invariants $d \in \mathbb{N}$ and $\pm\mu \mod 2rs \in (\mathbb{Z}/2rs)^*$ such that $\mu^2 \equiv d \mod 4rs$. We have: $\det N(X) = -d$, *and $K(H) = H^\perp_{N(X)} = \mathbb{Z}\delta$ where $\delta^2 = -2rsd$. Moreover,*

$$N(X) = [H, \delta, \frac{(\mu H + \delta)}{2rs}], \tag{3.1.5}$$

$$N(X) = \{z = \frac{(xH + y\delta)}{2rs} \mid x, y \in \mathbb{Z} \text{ and } x \equiv \mu y \mod 2rs\}. \tag{3.1.6}$$

We have $z^2 = (x^2 - dy^2)/2rs$.

For any primitive element $H' \in N(X)$ with $(H')^2 = H^2 = 2rs$ and the same invariant $\pm\mu$, there exists an automorphism $\phi \in O(N(X))$ such that $\phi(H) = H'$.

From Proposition 3.1.1 and g.c.d$(d, 2rs) =$ g.c.d$(d, 2ab) = 1$ where $\det N(X) = -d$, we get

Proposition 3.1.2. *Under conditions and notations of Propositions 3.1.1, all elements $h' = (xH + y\delta)/(2rs) \in N(X)$ with $(h')^2 = 2ab$ are in one to one correspondence with integral solutions (x, y) of the equation*

$$x^2 - dy^2 = 4a^2b^2c^2 \tag{3.1.7}$$

such that $x \equiv \mu y \mod 2abc^2$.

For $k \in \mathbb{N}$ and $k > 1$ the element h' is divisible by k in $N(X)$ and then it is not primitive, if and only if $k \,|\, (x, y)$ and $x \equiv \mu y \mod 2abc^2k$. Then $k^2 | ab$. We call such a pair (x, y) as μ-divisible by k. If (x, y) is not μ-divisible by $k > 1$, we call such a pair (x, y) as μ-primitive. It is enough to consider only prime k. Thus, h' is primitive, if and only if the corresponding pair (x, y) is μ-primitive. If h' is primitive, then $\gamma(h') = 1$.

The crucial statement is

Theorem 3.1.3. *Let X be a K3 surface with $\rho(X) = 2$, and H its polarization of the degree $H^2 = 2rs$ with $\gamma(H) = 1$. Let Y be the moduli space of sheaves on X with the isotropic Mukai vector $v = (r, H, s)$ and the canonical nef element $h = (-a, 0, b) \mod \mathbb{Z}v$.*

As in Proposition 3.1.1, we consider the invariants $\pm\mu \mod 2rs \in (\mathbb{Z}/2rs)^*$ and $d \in \mathbb{N}$ of $(N(X), H)$. Thus, we have

$$\gamma(H) = 1,\ \det N(X) = -d \text{ where } \mu^2 \equiv d \mod 4rs. \tag{3.1.8}$$

With notations of Propositions 3.1.1, then all elements $\widetilde{h} = (xH + y\delta)/2rs \in N(X)$ with square $\widetilde{h}^2 = 2ab$ satisfying Theorem 2.2.1 are in one to one correspondence with integral solutions (x, y) of the equation

$$x^2 - dy^2 = 4a^2b^2c^2 \tag{3.1.9}$$

which satisfy the conditions (i), (ii), (iii), (iv) below:

(i) $x \equiv \mu y \mod 2rs$;

(ii) $x \equiv \pm 2abc \mod d$;

(iii) (x, y) belongs to one of a-series or b-series of solutions defined below:

a-series: $b|(x, y)$ and $x - \mu y \equiv 0 \mod 2rsa$ where $\mu^2 \equiv d \mod 4rsa$,

b-series: $a|(x, y)$ and $x - \mu y \equiv 0 \mod 2rsb$ where $\mu^2 \equiv d \mod 4rsb$;

(iv) the pair (x, y) is μ-primitive: there does not exist a prime l such that $l|(x, y)$ and $x - \mu y \equiv 0 \mod 2rs\,l$ (we always have $l^2|ab$ for such l), i. e.

$$g.c.d\left(x, y, \frac{x - \mu y}{2rs}\right) = 1. \tag{3.1.10}$$

In particular, by Theorem 2.2.1, for a general X with $\rho(X) = 2$, we have $Y \cong X$, if and only if the equation $x^2 - dy^2 = 4a^2b^2c^2$ has an integral solution (x, y) satisfying conditions (i)—(iv) above. Moreover, a nef primitive element $\widetilde{h} = (xH + y\delta)/2rs$ with $\widetilde{h}^2 = 2ab$ defines the pair $(X, \widetilde{h})$ which is isomorphic to the (Y, h), if and only if (x, y) satisfies the conditions (ii) and (iii) (it satisfies conditions (i) and (iv) since it corresponds to a primitive element of $N(X)$).

Proof. We denote

$$H^* = \frac{H}{2rs},\ \delta^* = \frac{\delta}{2rsd} \tag{3.1.11}$$

where $K(H) = \mathbb{Z}\delta = H^\perp$ in $N(X)$. Since $\mu H^* + \delta/(2rs) \in N(X)$, then $H^* + \mu^{-1}\delta/(2rs) \in N(X)$ where μ^{-1} is defined $\mod 2rs$. It follows that

$$u^*(H) = \frac{\mu^{-1}}{2rs}\delta + K(H) = \mu^{-1}d\delta^* + \mathbb{Z}\delta. \tag{3.1.12}$$

Let

$$\widetilde{h} = \frac{xH + y\delta}{2rs} \in N(X) \tag{3.1.13}$$

satisfies conditions of Theorem 2.2.1. Then $\widetilde{h}^2 = 2ab$. This is equivalent to $x,\ y \in \mathbb{Z}$, $x \equiv \mu y \mod 2rs$ and $x^2 - dy^2 = 4a^2b^2c^2$. We get (i). The $\widetilde{h}$ is primitive, if and only if (x, y) is μ-primitive. We get (iv).

Consider $K(\widetilde{h}) = \widetilde{h}^{\perp}$ in $N(X)$. If $k = a_1H + b_1\delta \in K(\widetilde{h})$, then $(k, \widetilde{h}) = a_1x - b_1yd = 0$ and $(a_1, b_1) = \lambda(yd, x)$, $\lambda \in \mathbb{Q}$. We have

$$(\lambda(ydH + x\delta))^2 = \lambda^2(2rsy^2d^2 - 2rsdx^2) = 2rs\lambda^2d(y^2d - x^2) = -2rs\lambda^2d(4c^2a^2b^2).$$

For $\lambda = 1/2rs$ we get

$$g = (ydH + x\delta)/2rs. \tag{3.1.14}$$

The element $g \in N(X)$ since yd, x are integers and $yd - \mu x \equiv \mu(\mu y - x) \equiv 0 \mod 2rs$. We have $g^2 = -2abd$. If $\widetilde{h}$ is primitive and $\gamma(\widetilde{h}) = 1$, then $K(\widetilde{h}) = \mathbb{Z}g$. Then $N(X) = [\widetilde{h}, g, (\nu\widetilde{h} + g)/2ab]$ where ν is a similar invariant for $\widetilde{h}$ as μ for H. We denote

$$\widetilde{h}^* = \frac{\widetilde{h}}{2ab} \quad \text{and} \quad g^* = \frac{g}{2abd}. \tag{3.1.15}$$

Then

$$u^*(\widetilde{h}) = \frac{\nu^{-1}g}{2ab} = \nu^{-1}dg^* + K(h). \tag{3.1.16}$$

There exists a unique (up to ± 1) embedding $f : K(H) = \mathbb{Z}\delta \to K(\widetilde{h}) = \mathbb{Z}g$ of one-dimensional lattices. It is given by $f(\delta) = \pm cg$. Its dual is defined by $f^*(g^*) = \pm c\delta^*$. We have $f^*(K(h)) = \mathbb{Z}f^*(g) = \mathbb{Z}(\delta/c) = \mathbb{Z}2abcu^*(H)$ because of (3.1.12). This gives the first part of (2.2.13).

Moreover,

$$f^*(u^*(\widetilde{h})) + f^*(K(h)) = \nu^{-1}df^*(g^*) + f^*(K(h)) =$$

$$\pm\nu^{-1}cd\delta^* + f^*(K(h)) = \pm\nu^{-1}cu^*(H) + [K(H), 2abcu^*(H)].$$

Thus, the second part of (2.2.13) is $\pm\nu^{-1} \equiv m(a, b)\mu^{-1} \mod 2ab$. Equivalently,

$$\nu \equiv \pm m(a, b)\mu \mod 2ab. \tag{3.1.17}$$

Thus, for ν given by (3.1.17) one has

$$\nu\widetilde{h}^* + g^* = \left(\left((\nu x + dy)H + (\nu y + x)\delta\right)/2rs\right)/2ab \in N(X). \tag{3.1.18}$$

This is equivalent to

$$\begin{cases} \nu x + dy \equiv 0 \mod 2ab \\ \nu y + x \equiv 0 \mod 2ab \\ \nu x + dy \equiv \mu(\nu y + x) \mod 2rs2ab \end{cases}. \tag{3.1.19}$$

For example, assume that $\nu \equiv m(a,b)\mu \mod 2ab$. Then $\nu \equiv -\mu \mod 2a$ and $\nu \equiv \mu \mod 2b$.

From (3.1.19), we then get $\mod 2a$ that $-\mu x + y\mu^2 \equiv 0 \mod 2a$, $-\mu y + x \equiv 0 \mod 2a$ and $2\mu x \equiv y(d+\mu^2) \mod 2rs2a$. First two congruences are valid since $x \equiv \mu y \mod 2rs$. The last one gives

$$\mu\frac{x-\mu y}{2rs} \equiv \frac{d-\mu^2}{4rs}y \mod a\,. \tag{3.1.20}$$

We have $d \equiv \mu^2 \mod 4rs$. Since $\mu \mod 2rs$ is invertible and $a|r$, there exists the unique its lifting $\mu \mod 2rsa$ such that $d \equiv \mu^2 \mod 4rsa$. Then (3.1.20) is equivalent to $x - \mu y \equiv 0 \mod 2rsa$.

Now let us consider (3.1.19) $\mod 2b$. First two congruences in (3.1.19) give $\mu x + yd \equiv 0 \mod 2b$, $\mu y + x \equiv 0 \mod 2b$. Since $x \equiv \mu y \mod 2b$, they are equivalent to $x \equiv y \equiv 0 \mod b$. Third congruence in (3.1.19) gives $yd \equiv \mu^2 y \mod 2rs2b$. It is valid since $b|y$ and $\mu^2 \equiv d \mod 4rs$. Thus, (x,y) belongs to the a-series in (iii). If $\nu \equiv -m(a,b)\mu \mod 2ab$, then similarly (x,y) belongs to the b-series in (iii). Thus, condition (2.2.13) is equivalent to (iii).

By (2.2.9),

$$\begin{aligned}&\kappa(H)(\delta^*) = -2rs(\delta^*\cdot u^*(H))H^* + \delta^* + K(H) =\\ &-2rs(\delta/(2rsd)\cdot\mu^{-1}\delta/(2rs))H^* + \delta^* + K(H) = \mu^{-1}H^* + \delta^* + K(H).\end{aligned}$$

Thus, we have

$$\kappa(H)(\delta^*) = \mu^{-1}H^* + \delta^* + K(H). \tag{3.1.21}$$

Similarly,

$$\kappa(\widetilde{h})(g^*) = \nu^{-1}g^* + K(\widetilde{h}) = \pm m(a,b)\mu^{-1}g^* + K(\widetilde{h}). \tag{3.1.22}$$

Here we use (3.1.17).

The condition (2.2.14) gives then

$$\kappa(\widetilde{h})(g^*) = \pm c\kappa(H)(\delta^*). \tag{3.1.23}$$

From (3.1.21) and (3.1.22) we then get

$$\begin{aligned}&\kappa(\widetilde{h})(g^*) = \pm m(a,b)\mu^{-1}\frac{\widetilde{h}}{2ab} + \frac{g}{2abd} + N(X) =\\ &\pm m(a,b)\mu^{-1}\frac{xH+y\delta}{2rs2ab} + \frac{ydH+x\delta}{2rs2abd} + N(X) =\\ &\frac{\pm m(a,b)\mu^{-1}x+y}{2ab}H^* + \frac{\pm m(a,b)\mu^{-1}dy+x}{2ab}\delta^* + N(X)\,.\end{aligned} \tag{3.1.24}$$

We have $A_{N(X)} = N(X)^*/N(X) \cong (\mathbb{Z}/d\mathbb{Z})$. The $\kappa(H)$ gives an epimorphism of $\mathbb{Z}\delta^*$ onto $A_{N(X)}$ with the kernel which is $d\delta^*$. It follows that $\kappa(\widetilde{h})(g^*) = \pm c\kappa(H)(\delta^*)$ is equivalent to

$$\frac{\pm m(a,b)\mu^{-1}dy + x}{2ab} \equiv \pm c \mod d. \tag{3.1.25}$$

Thus, $\pm m(a,b)\mu^{-1}yd + x \equiv \pm 2abc \mod 2abd$. Since g.c.d$(2abc, d) = 1$, it is equivalent to $x \equiv \pm 2abc \mod d$ and $\pm m(a,b)\mu^{-1}yd + x \equiv 0 \mod 2ab$. The last congruence is valid because of $d \equiv \mu^2 \mod 2ab$, (3.1.17) and (3.1.19).

This finishes the proof.

Now let us consider the a-series of solutions of (3.1.9) satisfying the main congruence (ii). Since $b|(x,y)$ and g.c.d$(b,d) = 1$, we have $(x,y) = (bx_1, by_1)$ where (x_1, y_1) satisfy

$$x_1^2 - dy_1^2 = 4a^2c^2 \quad \text{and} \quad x_1 \equiv \pm 2ac \mod d. \tag{3.1.26}$$

Let us find all solutions of (3.1.26). We apply the main trick of [4] (see also [5]). Considering $\pm(x_1,\ y_1)$, we can assume that $x_1 \equiv 2ac \mod d$. Then $x_1 = 2ac - kd$ where $k \in \mathbb{Z}$. We have $4a^2c^2 - 4ackd + k^2d^2 - dy_1^2 = 4a^2c^2$. Thus,

$$d = \frac{y_1^2 + 4ack}{k^2}. \tag{3.1.27}$$

Let l be prime. Like in [4], it is easy to see that if $l^{2t+1}|k$ and $l^{2t+2} \not| k$, then $l|4ac$. It follows that $k = -\alpha q^2$ where $\alpha|2ac$ and α is square-free, $q \in \mathbb{Z}$. It follows that (3.1.27) is equivalent to

$$\alpha|2ac \text{ is square-free,}\ \ p^2 - dq^2 = \frac{4ac}{\alpha}. \tag{3.1.28}$$

Then $k = -\alpha q^2$, $y = \alpha pq$. Here α can be negative. Thus, (integral) solutions α, (p,q) of (3.1.28) give all solutions

$$(x_1, y_1) = \pm(2ac + \alpha dq^2,\ \alpha pq) \tag{3.1.29}$$

of (3.1.26). We call them *associated solutions.* Thus, all solutions (x_1, y_1) of (3.1.26) are associated solutions (3.1.29) to all solutions α, (p,q) of (3.1.28). If one additionally assumes that $q \geq 0$, then (x,y) and α, (p,q) are in one to one correspondence, by our construction.

Now let us consider associated solutions (3.1.29) which satisfy the additional condition $bx_1 \equiv \mu by_1 \mod 2rs$. (It is the condition (i) of Theorem 3.1.3.) Equivalently, $x_1 \equiv \mu y_1 \mod 2ac^2$. By (3.1.29), this is equivalent

$$2ac + \alpha dq^2 \equiv \mu\alpha pq \mod 2ac^2. \tag{3.1.30}$$

We also have

$$4ac + \alpha dq^2 = \alpha p^2. \tag{3.1.31}$$

Taking (3.1.31) - (3.1.30), we get that (3.1.30) is equivalent to

$$2ac \equiv \alpha p(p-\mu q) \mod 2ac^2. \tag{3.1.32}$$

Taking 2(3.1.30) - (3.1.31), we get that (3.1.30) is equivalent to

$$\alpha p^2 + \alpha dq^2 - 2\alpha\mu pq \equiv 0 \mod 4ac^2. \tag{3.1.33}$$

Since $d \equiv \mu^2 \mod 4abc^2$, this is equivalent to

$$\alpha(p-\mu q)^2 \equiv 0 \mod 4ac^2. \tag{3.1.34}$$

Since α is square-free, it follows that

$$2c\,|p-\mu q. \tag{3.1.35}$$

From (3.1.32), we then get

$$2a \equiv \alpha p\left(\frac{p-\mu q}{c}\right) \mod 2ac \tag{3.1.36}$$

where $(p-\mu q)/c$ is an integer. Since $\alpha|2ac$, it follows that $\alpha|2a$. Thus, we get

$$\alpha|2a,\ \ c|(p-\mu q)\ \text{ and }\ \frac{2a}{\alpha} \equiv p\left(\frac{p-\mu q}{c}\right) \mod \frac{2ac}{\alpha}. \tag{3.1.37}$$

Thus, we have proved (see (3.1.34)) that the condition $x_1 \equiv \mu y_1 \mod 2ac^2$ is equivalent to

$$\alpha|2a\ \text{ and }\ (p-\mu q)^2 \equiv 0 \mod \frac{4ac^2}{\alpha}. \tag{3.1.38}$$

Let us prove that actually $\alpha|a$. Otherwise, the equation $p^2 - dq^2 = 4ac/\alpha$ has no solutions. Really, assume that $2a/\alpha$ is odd. By (3.1.38) we get $(p-\mu q)^2 \equiv 0 \mod 2c^2$. It follows $p-\mu q \equiv 0 \mod 2c$ and $p+\mu q \equiv 0 \mod 2$. Then $p^2-\mu^2q^2 \equiv 0 \mod 4c$. Since $\mu^2 \equiv d \mod 4ac^2$, we then get $p^2-dq^2 \equiv 0 \mod 4c$. On the other hand, if $2a/\alpha$ is odd, then $p^2-dq^2 = 4ac/\alpha \equiv 2c \mod 4c$. We get a contradiction.

Now we add the condition (iv) and the last condition of the a-series $x-\mu y \equiv 0 \mod 2rsa$ of Theorem 3.1.3 (here we assume that $\mu^2 \equiv d \mod 4rsa$). For $(x,y) = (bx_1, by_1)$ we then get

$$x_1 - \mu y_1 \equiv 0 \mod 2a^2c^2 \text{ and g.c.d}(bx_1,\, by_1,\, \frac{x_1-\mu y_1}{2ac^2}) = 1. \tag{3.1.39}$$

By (3.1.29), we then have $\alpha|(x_1, y_1, (x_1-\mu y_1)/2ac^2)$. It follows the fundamental result:

$$\alpha = \pm 1. \tag{3.1.40}$$

The condition $x_1 - \mu y_1 \equiv 0 \mod 2a^2c^2$ gives

$$2ac + \alpha dq^2 \equiv \mu\alpha pq \mod 2a^2c^2. \tag{3.1.41}$$

As always

$$-4ac + \alpha p^2 - \alpha dq^2 = 0. \tag{3.1.42}$$

Taking 2(3.1.41)+(3.1.42), we get that (3.1.41) is equivalent to

$$\alpha dq^2 + \alpha p^2 \equiv 2\mu\alpha pq \mod 4a^2c^2. \tag{3.1.43}$$

Since $\alpha = \pm 1$, this is equivalent to

$$dq^2 + p^2 \equiv 2\mu pq \mod 4a^2c^2. \tag{3.1.44}$$

Since $\mu^2 \equiv d \mod 4a^2c^2$, the (3.1.44) is equivalent to

$$(p - \mu q)^2 \equiv 0 \mod 4a^2c^2 \tag{3.1.45}$$

which is equivalent to

$$p - \mu q \equiv 0 \mod 2ac. \tag{3.1.46}$$

Thus, the congruence $x_1 - \mu y_1 \equiv 0 \mod 2a^2c^2$ in (3.1.39) is equivalent to (3.1.46). Now let us analyze the condition of primitivity

$$\text{g.c.d}(bx_1,\, by_1,\, \frac{x_1 - \mu y_1}{2ac^2}) = 1 \tag{3.1.47}$$

in (3.1.39). The condition (3.1.47) is equivalent to the fact that there does not exist a prime l such that

$$l|(bx_1,\, by_1,\, (x_1 - \mu y_1)/2ac^2) \text{ where always } l^2|ab. \tag{3.1.48}$$

Assume $l^2|a$. Since $a|(x_1 - \mu y_1)/2ac^2$, (3.1.48) is equivalent to $l|(x_1,\, y_1)$. We have $(x_1,\, y_1) = \pm(2ac+\alpha dq^2,\, \alpha pq) = \pm(-2ac+\alpha p^2,\, \alpha pq)$. It follows $l|(p,q)$. Thus, the $l^2|a$ in (3.1.48) does not exist, if and only if $l|(p,q)$ is impossible for $l^2|a$.

Assume $l^2|b$. Then $l|(bx_1,\, by_1,\, (x_1 - \mu y_1)/2ac^2)$ is equivalent to $x_1 - \mu y_1 \equiv 0 \mod 2ac^2l$. This is equivalent to $x_1 - \mu y_1 \equiv 0 \mod 2a^2c^2l$ because we know that $x_1 - \mu y_1 \equiv 0 \mod 2a^2c^2$ and $\text{g.c.d}(l, a) = 1$. From (3.1.29), we get

$$2ac + \alpha dq^2 \equiv \mu\alpha pq \mod 2a^2c^2l. \tag{3.1.49}$$

By $-4ac + \alpha p^2 - \alpha dq^2 = 0$ and $\alpha = \pm 1$, we then get like above that (3.1.49) is equivalent to

$$dq^2 + p^2 - 2\mu pq \equiv 0 \mod 4a^2c^2l. \tag{3.1.50}$$

Since $\mu^2 \equiv d \mod 4abc^2a$ and $l|b$, the (3.1.50) is equivalent to

$$(p - \mu q)^2 \equiv 0 \mod 4a^2c^2\,l. \tag{3.1.51}$$

The (3.1.51) is equivalent to

$$p - \mu q \equiv 0 \mod 2ac\,l. \tag{3.1.52}$$

Thus, we finally get that (3.1.39) is equivalent to

$$\begin{cases} p \equiv \mu q \mod 2ac, \\ \text{g.c.d}(a, p, q) = 1, \\ p \not\equiv \mu q \mod 2acl \text{ for any prime } l|b. \end{cases} \tag{3.1.53}$$

We get the final result:

Theorem 3.1.4. *With conditions of Theorem 3.1.3, for a general K3-surface X with $\rho(X) = 2$ and $\gamma(H) = 1$, we have $Y \cong X$, if and only if for one of $\alpha = \pm 1$ there exists an integral solution (p, q) for one of a-series or b-series:*

a-series:

$$\begin{cases} p^2 - dq^2 = 4ac/\alpha \\ p \equiv \mu q \mod 2ac \\ g.c.d(a, p, q) = 1 \\ p \not\equiv \mu q \mod 2acl \textit{ for any prime } l|b \end{cases}. \tag{3.1.54}$$

A solution (p, q) of (3.1.54) *gives a solution (x, y) of Theorem 3.1.3 as associated solution*

$$(x, y) = \pm(2abc + \alpha bdq^2,\ \alpha bpq). \tag{3.1.55}$$

For b-series, one should just change a and b places:

b-series:

$$\begin{cases} p^2 - dq^2 = 4bc/\alpha \\ p \equiv \mu q \mod 2bc \\ g.c.d(b, p, q) = 1 \\ p \not\equiv \mu q \mod 2bcl \textit{ for any prime } l|a \end{cases}. \tag{3.1.56}$$

A solution (p, q) of (3.1.56) *gives a solution (x, y) of Theorem 3.1.3 as associated solution*

$$(x, y) = \pm(2abc + \alpha adq^2,\ \alpha apq). \tag{3.1.57}$$

All $\widetilde{h}$ (or solutions (x, y)) of Theorem 3.1.3 are given by (3.1.55) *and* (3.1.57) *as associated solutions.*

Now let us interpret solutions (p, q) of Theorem 3.1.4 as elements of the Picard lattice $N(X)$. Further ± 1 denotes α. If there is an ambiguity with other $\pm$, we introduce α (e.g. see (3.1.63) below).

Let us consider a solution (p,q) of the a-series (3.1.54). The congruence $p \equiv \mu q \mod 2ac$ is equivalent to

$$tp \equiv t\mu q \mod 2abc^2\,, \tag{3.1.58}$$

where

$$t = bc. \tag{3.1.59}$$

The congruence (3.1.58) is equivalent to

$$h_1 = \frac{t(pH + q\delta)}{2abc^2} \in N(X) \text{ and } h_1 \cdot H \equiv 0 \mod t. \tag{3.1.60}$$

We have $p^2 - dq^2 = \pm 4ac$ is equivalent to $(h_1)^2 = t^2(p^2 - dq^2)/(2abc^2) = \pm 2bc$. Moreover, $H \cdot h_1 \equiv 0 \mod bc$.

The condition $\text{g.c.d}(a,p,q) = 1$ can be interpreted as l does not divide $p = H \cdot h_1/t$, if $l^2|a$. Really, if $l|a$ and $l|p$, then $l|q$ since $p^2 - dq^2 = \pm 4ac$. Thus, we get $l|(H \cdot h_1)/bc$ is impossible for $l^2|a$.

Assume that $p \equiv \mu q \mod 2acl$ for a prime $l|b$. This is equivalent to $pt/l \equiv \mu qt/l \mod 2abc^2$. Equivalently, $h_1/l = \left((pt/l)H + (qt/l)\delta\right)/(2abc^2) \in N(X)$. Thus, such a prime l does not exist, if and only if h_1 is not divisible by $l|b$ in $N(X)$. It is enough to consider $l^2|b$.

Changing a and b places, we get similar statements for the b-series.

Thus, we finally get

Theorem 3.1.5. *With notations and conditions of Theorem 3.1.3, for a general K3-surface X with $\rho(X) = 2$ and $\gamma(H) = 1$ for $H \in N(X)$, we have $Y \cong X$, if and only if at least for one of signs $\pm$ (i. e. $\alpha = \pm 1$) there exists $h_1 \in N(X)$ such that h_1 belongs to one of a-series or b-series:*

a-series:

$$h_1^2 = \pm 2bc,\ H \cdot h_1 \equiv 0 \mod bc,\ H \cdot h_1 \not\equiv 0 \mod bcl_1,\ h_1/l_2 \notin N(X) \tag{3.1.61}$$

where l_1 and l_2 are any primes such that $l_1^2|a$ and $l_2^2|b$;

b-series:

$$h_1^2 = \pm 2ac,\ H \cdot h_1 \equiv 0 \mod ac,\ H \cdot h_1 \not\equiv 0 \mod acl_1,\ h_1/l_2 \notin N(X) \tag{3.1.62}$$

where l_1 and l_2 are any primes such that $l_1^2|b$ and $l_2^2|a$.

Important Remark. By Theorem 3.1.3 and the formulae (3.1.55), (3.1.57) for the associated solution, we get that

$$\pm\widetilde{h} = \begin{cases} -\frac{H}{c} + \frac{\alpha(H\cdot h_1)h_1}{bc^2}, & \text{if } h_1 \text{ is from } a\text{-series,} \\ -\frac{H}{c} + \frac{\alpha(H\cdot h_1)h_1}{ac^2}, & \text{if } h_1 \text{ is from } b\text{-series} \end{cases} \tag{3.1.63}$$

belongs to $N(X)$ and

$$(Y,h) \cong \left(X, \pm w(\widetilde{h})\right) \text{ for some } w \in W^{(-2)}(N(X)). \tag{3.1.64}$$

It describes moduli (Y,h) of sheaves in terms of X where $h = (-a,0,b) \mod \mathbb{Z}v$.

Applying additionally Theorem 2.2.1, we get the following simple sufficient condition when $Y \cong X$ which is valid for X with any $\rho(X)$. This is one of the main results of the paper.

Theorem 3.1.6. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, r, $s \geq 1$, the Mukai vector (r, H, s) is primitive and Y the moduli of sheaves on X with the Mukai vector $v = (r, H, s)$.*

Then $Y \cong X$, if at least for one of signs $\pm$ (i. e. $\alpha = \pm 1$) there exists $h_1 \in N(X)$ such that elements H, h_1 are contained in a 2-dimensional sublattice $N \subset N(X)$ with $H \cdot N = \mathbb{Z}$ and h_1 belongs to one of a-series or b-series described below:

a-series:

$$h_1^2 = \pm 2bc,\ H \cdot h_1 \equiv 0 \mod bc,\ H \cdot h_1 \not\equiv 0 \mod bcl_1,\ h_1/l_2 \notin N(X) \quad (3.1.65)$$

where l_1 and l_2 are any primes such that $l_1^2|a$ and $l_2^2|b$;

b-series:

$$h_1^2 = \pm 2ac,\ H \cdot h_1 \equiv 0 \mod ac,\ H \cdot h_1 \not\equiv 0 \mod acl_1,\ h_1/l_2 \notin N(X) \quad (3.1.66)$$

where l_1 and l_2 are any primes such that $l_1^2|b$ and $l_2^2|a$.

We also have the formulae (3.1.63) *and* (3.1.64) *for (Y, h) where $h = (-a, 0, b) \mod \mathbb{Z}v$.*

The conditions above are necessary for $\gamma(H) = 1$ and $Y \cong X$, if $\rho(X) \leq 2$ and X is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ of X is ± 1.

4. Divisorial conditions on moduli of (X, H) which imply $\gamma(H) = 1$ and $Y \cong X$

Further we use the following notations. We fix:
$r, s \in \mathbb{N}$, $r, s \geq 1$, $r = ca$, $s = cb$ where $c = \text{g.c.d.}(r, s)$ and then $\text{g.c.d.}(a, b) = 1$;
$\overline{\mu} = \{\mu, -\mu\} \subset (\mathbb{Z}/2rs)^*$.

For any choice of the sign $\pm$ (equivalently, $\alpha = \pm 1$), we denote

$$\begin{aligned}\mathcal{D}(r, s; a)^{\overline{\mu}}_{\pm} = \{d \in \mathbb{N} \mid d \equiv \mu^2 \mod 4abc^2,\ \exists\ (p, q) \in \mathbb{Z} \times \mathbb{Z} :\ p^2 - dq^2 = \pm 4ac,\\ p - \mu q \equiv 0 \mod 2ac,\ \text{g.c.d}(a, p, q) = 1,\\ p - \mu q \not\equiv 0 \mod 2acl \text{ for any prime } l \text{ such that } l^2|b\}. \quad (4.1)\end{aligned}$$

The sets $\mathcal{D}(r, s; a)^{\overline{\mu}}_{\pm}$ are called the a-series.

For any choice of the sign $\pm$ we denote

$$\begin{aligned}\mathcal{D}(r, s; b)^{\overline{\mu}}_{\pm} = \{d \in \mathbb{N} \mid d \equiv \mu^2 \mod 4abc^2,\ \exists\ (p, q) \in \mathbb{Z} \times \mathbb{Z} :\ p^2 - dq^2 = \pm 4bc,\\ p - \mu q \equiv 0 \mod 2bc,\ \text{g.c.d}(b, p, q) = 1,\\ p - \mu q \not\equiv 0 \mod 2bcl \text{ for any prime } l \text{ such that } l^2|a\}. \quad (4.2)\end{aligned}$$

The sets $\mathcal{D}(r, s; b)^{\overline{\mu}}_{\pm}$ are called the b-series.

Let X be a K3 surface with a polarization H of the degree $H^2 = 2rs$ and a primitive Mukai vector (r, H, s). The moduli space Y of sheaves on X with the Mukai vector $v = (r, H, s)$ has the canonical *nef* element $h = (-a, 0, b) \mod \mathbb{Z}v$ with $h^2 = 2ab$.

If $\gamma(H) = 1$, then $\rho(X) \geq 2$. Since the dimension of the moduli space of (X, H) is equal to $20 - \rho(X)$, it follows that describing general (X, H) with $\rho(X) = 2$, $\gamma(H) = 1$ and $Y \cong X$, we at the same time describe all possible divisorial conditions on moduli of (X, H) which imply $\gamma(H) = 1$ and $Y \cong X$. They are described by the invariants of the pairs $(N(X), H)$ where rk $N(X) = 2$. Here we use results of [10,11] (see also [3]) about irreducibility of moduli of K3 surfaces with a fixed Picard lattice.

Using Theorem 3.1.4, we get

Theorem 4.1. *With the above notations all possible divisorial conditions on moduli of polarized K3 surfaces (X, H) with a polarization H of the degree $H^2 = 2rs$, which imply $\gamma(H) = 1$ and $Y \cong X$ are labelled by the set*

$$\mathcal{D}iv(r, s) = \mathcal{D}iv(r, s; a) \bigcup \mathcal{D}iv(r, s; b), \tag{4.3}$$

where

$$\begin{aligned} \mathcal{D}iv(r, s; a) &= \{(d, \overline{\mu}) \mid \overline{\mu} = \{\mu, -\mu\} \subset (\mathbb{Z}/2rs)^*,\ d \in \mathcal{D}(r, s; a)^{\overline{\mu}}\}, \\ \mathcal{D}iv(r, s; b) &= \{(d, \overline{\mu}) \mid \overline{\mu} = \{\mu, -\mu\} \subset (\mathbb{Z}/2rs)^*,\ d \in \mathcal{D}(r, s; b)^{\overline{\mu}}\} \end{aligned} \tag{4.4}$$

and

$$\mathcal{D}(r, s; a)^{\overline{\mu}} = \mathcal{D}(r, s; a)^{\overline{\mu}}_{+} \bigcup \mathcal{D}(r, s; a)^{\overline{\mu}}_{-}, \quad \mathcal{D}(r, s; b)^{\overline{\mu}} = \mathcal{D}(r, s; b)^{\overline{\mu}}_{+} \bigcup \mathcal{D}(r, s; b)^{\overline{\mu}}_{-}. \tag{4.5}$$

The sets

$$\mathcal{D}(r, s; a)^{\overline{\mu}}_{\pm} \text{ and } \mathcal{D}(r, s; b)^{\overline{\mu}}_{\pm} \tag{4.6}$$

are infinite if they are not empty.

The set $\mathcal{D}(r, s; a)^{\overline{\mu}}_{\pm}$ is not empty, if and only if the congruence

$$\mu q t + a c t^2 \equiv \pm 1 \mod bcq^2 \tag{4.7}$$

has a solution $q \in \mathbb{N}$, $t \in \mathbb{Z}$. E.g. it is true for $q = 1$, if $b|c^2$. The set $\mathcal{D}(r, s; b)^{\overline{\mu}}_{\pm}$ is not empty, if and only if the congruence

$$\mu q t + b c t^2 \equiv \pm 1 \mod acq^2 \tag{4.8}$$

has a solution for $q \in \mathbb{N}$, $t \in \mathbb{Z}$. E. g. it is true for $q = 1$, if $a|c^2$. See other general results of this type in Theorems 4.5 and 4.6 below.

The set $\mathcal{D}iv(r,s)$ is always non-empty and then infinite: i. e. at least one of sets $\mathcal{D}(r,s;a)_{\pm}^{\overline{\mu}}$ or $\mathcal{D}(r,s;b)_{\pm}^{\overline{\mu}}$ is not empty for one of $\overline{\mu}$ and $\pm$. Its subset $\mathcal{D}iv(r,s;a) \subset \mathcal{D}iv(r,s)$ is non-empty and then infinite, if either abc is odd or ac is even; its subset $\mathcal{D}iv(r,s;b) \subset \mathcal{D}iv(r,s)$ is non-empty and then infinite, if either abc is odd or bc is even.

Below we prove statements of Theorem 4.1 which don't follow from Theorem 3.1.4 straightforward. These considerations are important as itself, and they give more exact and strong statements as we have formulated in Theorem 4.1.

Further we consider the a-series. The same results are valid for the b-series changing a and b places. For $q \in \mathbb{N}$ such that g.c.d$(a,q)=1$ we introduce

$$\begin{aligned}\mathcal{D}(r,s;a)_{\pm}^{\overline{\mu}}(q) = \{d \in \mathbb{N} \mid d \equiv \mu^2 \mod 4abc^2,\ \exists\, p \in \mathbb{Z}:\ p^2 - dq^2 = \pm 4ac,\\ p-\mu q \equiv 0 \mod 2ac,\ \text{g.c.d}(a,p,q)=1,\\ p-\mu q \not\equiv 0 \mod 2acl \text{ for any prime } l \text{ such that } l^2|b\}.\end{aligned} \tag{4.9}$$

By definition, the set $\mathcal{D}(r,s;a)_{\pm}^{\overline{\mu}}(q) = \emptyset$, if g.c.d$(a,q)>1$.

Obviously,

$$\mathcal{D}(r,s;a)_{\pm}^{\overline{\mu}} = \bigcup_{q\in\mathbb{N}} \mathcal{D}(r,s;a)_{\pm}^{\overline{\mu}}(q). \tag{4.10}$$

We have

Lemma 4.2. *For any $\pm$ and any $q \in \mathbb{N}$ the set $\mathcal{D}(r,s;a)_{\pm}^{\overline{\mu}}(q)$ is either empty or infinite. It is infinite, if and only if the system of congruences*

$$\begin{cases} p \equiv \mu q \mod 2ac,\\ p^2 \equiv \pm 4ac + \mu^2 q^2 \mod 4abc^2q^2,\\ p \not\equiv \mu q \mod 2acl \end{cases} \tag{4.11}$$

has a solution. Here l is any prime such that $l^2|b$, the condition $p \not\equiv \mu q \mod 2acl$ must satisfy for all these primes. The set $\mathcal{D}(r,s;a)_{\alpha}^{\overline{\mu}}(q)$ consists of all $d=(p^2 \mp 4ac)/q^2$ where p satisfies the above system of congruences and $p^2 \mp 4ac > 0$.

Proof. It at once follows from definitions using that $d=(p^2\mp 4ac)/q^2>0$. Clearly, if $p=p_0$ satisfies the system (4.11), then $p=p_0+t(4abc^2q^2)$, $t\in\mathbb{Z}$, satisfies (4.11), and $p^2 \mp 4ac > 0$ for $|t| >> 0$. It gives an infinite subset of $\mathcal{D}(r,s;a)_{\pm}^{\overline{\mu}}(q)$. It finishes the proof.

We denote

$$\alpha = \pm 1. \tag{4.12}$$

Let us rewrite (4.11). We have $p=\mu q + 2act$, $t\in\mathbb{Z}$ where g.c.d$(t,b)=1$. Then $(\mu q+2act)^2 \equiv 4\alpha ac + \mu^2q^2 \mod 4abc^2q^2$. It follows $\mu qt + act^2 \equiv \pm 1 \mod bcq^2$.

Then g.c.d$(t,b)=1$ and g.c.d$(q,a)=1$. It follows that (4.11) is equivalent to the congruence

$$\mu q t + act^2 \equiv \pm 1 \mod bcq^2. \tag{4.13}$$

A solution $t\in\mathbb{Z}$ of (4.13) gives the infinite series of elements from $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm}(q)$ which is

$$\frac{\left(\mu q + 2ac(t+bcq^2m)\right)^2 \mp 4ac}{q^2} > 0, \quad m\in\mathbb{Z}. \tag{4.14}$$

Thus, additionally to Lemma 4.2 we get

Lemma 4.3. *For $q\in\mathbb{N}$ the set $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm}(q)$ is non-empty, if and only if the congruence* (4.13) *has a solution $t\in\mathbb{Z}$. Each such solution t generates an infinite series of elements from $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm}(q)$ given by* (4.14)*. Together they give all elements of $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm}(q)$.*

In particular, the set $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm}$ is non-empty (and then it is infinite), if and only if the congruence (4.13) *has a solution (q,t) for $q\in\mathbb{N}$ and $t\in\mathbb{Z}$.*

For $q=1$, the congruence (4.13) gives

$$\mu t + act^2 \equiv \alpha \mod bc. \tag{4.15}$$

We have $\mu t\equiv\alpha \mod c$ and $t=\mu^{-1}\alpha+kc$, $k\in\mathbb{Z}$. Simple calculations show that (4.15) is equivalent to the quadratic equation

$$k^2ac^2 + k(\mu+2\alpha ac\mu^{-1}) + a\mu^{-2} \equiv 0 \mod b. \tag{4.16}$$

For $A\neq 0$, the quadratic equation

$$Ax^2+Bx+C\equiv 0 \mod b \tag{4.17}$$

is equivalent to

$$(2Ax+B)^2 \equiv B^2-4AC \mod 4Ab. \tag{4.18}$$

It follows that (4.15) is equivalent to

$$(2ac^2k+\mu+2\alpha ac\mu^{-1})^2 \equiv (\mu^2+4\alpha ac) \mod 4abc^2. \tag{4.19}$$

In particular, (4.16) (equivalently (4.19)) has a solution, if $c^2\equiv 0 \mod b$. Then $A=ac^2\equiv 0 \mod b$ and $B=\mu+2\alpha ac\mu^{-1}$ is invertible $\mod b$. Thus, we get

Lemma 4.4. *The set $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\pm}(1)$ is non-empty and then infinite, if and only if the quadratic equation* (4.16) *(equivalently,* (4.19)*) has a solution k.*

In particular, this is valid, if $b|c^2$.

Using Lemma 4.4, we also get the following general result:

Theorem 4.5. *The set $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\alpha}(1)$ is non-empty for $\mu\equiv\theta^{-1}ac-\theta\alpha \mod 2abc^2$, if $\theta\in(\mathbb{Z}/2abc^2)^*$ and g.c.d$(2b,\theta^{-1}ac-\theta\alpha)=1$.*

It follows that $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\alpha}(1)$ is non-empty at least for one of α and $\overline{\mu}$, if ac is even. In particular, the set $\mathcal{D}iv(r,s;a)$ is not empty, if ac is even.

Proof. By conditions on θ, we have that $\mu\equiv\theta^{-1}ac-\theta\alpha \mod 2abc^2\in(\mathbb{Z}/2abc^2)^*$. Then $\mu^2+4\alpha ac\equiv\nu^2 \mod 4abc^2$ where $\nu\equiv\theta^{-1}ac+\theta\alpha \mod 2abc^2$. Thus, (4.19) satisfies, if

$$2ac^2k+\mu+2\alpha\, ac\mu^{-1}\equiv-\nu \mod 2abc^2$$

has a solution k. Thus, we should have

$$2ac^2k+(\theta^{-1}ac-\theta\alpha)+\frac{2\alpha ac}{\theta^{-1}ac-\theta\alpha}\equiv-(\theta^{-1}ac+\theta\alpha) \mod 2abc^2.$$

This gives

$$(\theta^{-1}ac-\theta\alpha)2ac^2k\equiv-2\theta^{-2}a^2c^2 \mod 2abc^2.$$

Equivalently,

$$(\theta^{-1}ac-\theta\alpha)k\equiv-\theta^{-2}a \mod b.$$

It has a solution k since g.c.d$(\theta^{-1}ac-\theta\alpha,b)=1$. By Lemma 4.4, the set $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\alpha}(1)$ is not then empty.

Assume that ac is even. We can find α such that for any prime $p_i|2b$ there exists $x_i \mod p_i$ such that $x_i\not\equiv0 \mod p_i$ and $-\alpha x_i^2+ac\not\equiv0 \mod p_i$. Really, if $p_i=2$, we can take $x_i\equiv1 \mod 2$. If $p_i=3$, we can take $x_i\equiv\pm1 \mod 3$. If $ac\equiv0 \mod 3$, we can take any $\alpha=\pm1$. If $ac\equiv1 \mod 3$, we take $\alpha=-1$. If $ac\equiv-1 \mod 3$, we take $\alpha=1$. If $p_i\ge5$, the equation $-\alpha x^2+ac\equiv0 \mod p_i$ has not more then two solutions $\mod p_i$, and there always exists $x_i \mod p_i$ with the above conditions.

By standard results, there exists $x \mod 2b$ such that $x\equiv x_i \mod p_i$. It then follows that $x\in(\mathbb{Z}/2b)^*$ and $(-\alpha x^2+ac,2b)=1$. By standard results and considerations, there exists $\theta\in(\mathbb{Z}/2abc^2)^*$ such that $\theta\equiv x \mod 2b$. Then $(-\alpha\theta^2+ac,2b)=1$ and also $(-\alpha\theta+ac\theta^{-1},2b)=1$. For the found α and θ, the set $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\alpha}(1)$ is not empty, if $\mu\equiv-\alpha\theta+ac\theta^{-1} \mod 2abc^2$.

This finishes the proof.

Considering Theorem 4.5 also for the b-series, we get that the set $\mathcal{D}iv(r,s)=\mathcal{D}iv(r,s;a)\bigcup\mathcal{D}iv(r,s;b)$ is not empty, if at least one of a, b or c is even.

Thus, to prove that the set $\mathcal{D}iv(r,s)$ is not empty, we need to consider only the case when abc is odd. Then we have

Theorem 4.6. *Assume that abc is odd. Then the set $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\alpha}(1)$ is non-empty for $\mu \mod 2abc^2\in(\mathbb{Z}/2abc^2)^*$, if $\mu\equiv\theta^{-1}ac-\theta\alpha \mod abc^2$ where $\theta\in(\mathbb{Z}/abc^2)^*$ and g.c.d$(b,\theta^{-1}ac-\theta\alpha)=1$.*

It follows that $\mathcal{D}(r,s;a)^{\overline{\mu}}_{\alpha}(1)$ is not empty at least for one of α and $\overline{\mu}$. In particular, the set $\mathcal{D}iv(r,s;a)$ is not empty.

Proof. The congruence (4.19) is the identity $\mod 4$. Then, since abc is odd, (4.19) is equivalent to

$$(2ac^2k+\mu+2\alpha ac\mu^{-1})^2 \equiv (\mu^2+4\alpha ac) \mod abc^2. \tag{4.20}$$

Thus, it is enough to consider $\mu \mod abc^2$. Then the same considerations as for Theorem 4.5 prove the statement.

We can summarize the main results:

Theorem 4.7. *For any r, s the set $\mathcal{D}iv(r,s)$ of divisorial conditions on moduli is non-empty and then infinite.*

Its subset $\mathcal{D}iv(r,s;a)$ is non-empty and then infinite, if either abc is odd or ac is even; its subset $\mathcal{D}iv(r,s;b)$ is non-empty and then infinite, if either abc is odd or bc is even.

5. The results from the viewpoint of Y.

It is interesting and important to consider the same problems from the viewpoint of Y. In this case, we fix a, $b\in\mathbb{N}$ with g.c.d$(a,b)=1$, and we fix $c\in\mathbb{N}$. Further we denote $r=ac$ and $s=bc$.

We consider a K3 surface Y with a *nef* element $h\in N(Y)$ such that $h^2=2ab$. We ask: *When (Y,h) is the moduli space $\mathcal{M}$ of sheaves on Y with the Mukai vector $v=(r,H,s)$ and the canonical nef element $\widetilde{h}=(-a,0,b) \mod \mathbb{Z}v$ where $H\in N(Y)$ is nef, $H^2=2rs$ and $\gamma(H)=1$? More exactly, when there exists an isomorphism $\xi:\mathcal{M}\to Y$ such that $\xi(\widetilde{h})=h$?*

This question is very similar to the previous ones for X, and, in principle, the corresponding results can be deduced from the results for X above. But it is simpler to reconsider the previous calculations from the viewpoint of Y. They are very similar but sometimes very non-trivial. It is surprising that the results for Y below are so similar to the results for X.

Theorem 5.1 below is a reformulation of Theorem 2.2.1. We use notations from Sect. 2 as in Theorem 2.2.1.

Theorem 5.1. *Let Y be a K3 surface with a nef element h of the degree $h^2=2ab$.*

Then (Y,h) is (isomorphic to) the moduli space $(\mathcal{M},\widetilde{h})$ of coherent sheaves on Y with the Mukai vector $v=(r,H,s)$, where $H^2=2rs$, $\gamma(H)=1$, and the canonical nef element $\widetilde{h}=(-a,0,b) \mod \mathbb{Z}v$, if $\gamma(h)=1$ and the conditions (i) and (ii) below hold:

(i) there exists $\widetilde{H}\in N(Y)$ with $\widetilde{H}^2=2rs$, $\gamma(\widetilde{H})=1$, and such that there exists an embedding

$$f: K(\widetilde{H})\to K(h)$$

of negative definite lattices such that

$$\begin{aligned} &f^*(K(h))=[K(\widetilde{H}),2abcu^*(\widetilde{H})],\\ &f^*(u^*(h))+f^*(K(h))=m(a,b)c\,u^*(\widetilde{H})+f^*(K(h)). \end{aligned}\tag{5.1}$$

(ii) For the h and f in (i), there exists a choice of $\pm$ such that

$$\kappa(h)(k^*) = \pm\kappa(\widetilde{H})(f^*(k^*)) \tag{5.2}$$

for any $k^* \in K(h)^*$.

The condition $\gamma(h) = 1$ is necessary for any Y.

The conditions (i) and (ii) are necessary for a K3 surface Y with $\rho(Y) \le 19$ which is general for its Picard lattice $N(Y)$ in the following sense: the automorphism group of the transcendental periods $(T(Y), H^{2,0}(Y))$ is ± 1.

If $\rho(Y) = 20$ the theorem is always valid, if $\gamma(h) = 1$.

Now assume that rk $N(Y) = 2$ and $h \in N(Y)$ has the degree $h^2 = 2ab$, $\gamma(h) = 1$. Applying Proposition 3.1.1 to $c = 1$, we get

Proposition 5.2. *Let Y be a K3 surface with rk $N(Y) = 2$ equipped with a primitive nef element h of the degree $h^2 = 2ab$, and $\gamma(h) = 1$.*

The pair $(N(Y), h)$ has the invariants $d \in \mathbb{N}$ and $\pm\nu \mod 2ab \in (\mathbb{Z}/2ab)^*$ such that $\nu^2 \equiv d \mod 4ab$. We have: $\det N(Y) = -d$, and $K(h) = h^\perp_{N(Y)} = \mathbb{Z}\delta_1$ where $\delta_1^2 = -2abd$. Moreover,

$$N(Y) = [h, \delta_1, \frac{(\nu h + \delta_1)}{2ab}], \tag{5.3}$$

$$N(Y) = \{z = \frac{(xh + y\delta_1)}{2ab} \mid x, y \in \mathbb{Z} \text{ and } x \equiv \nu y \mod 2ab\}. \tag{5.4}$$

We have $z^2 = (x^2 - dy^2)/2ab$.

For any primitive element $h' \in N(Y)$ of the degree $(h')^2 = h^2 = 2ab$ and the same invariant $\pm\nu$, there exists an automorphism $\phi \in O(N(Y))$ such that $\phi(h) = h'$.

From Proposition 3.1.1 applied to $(N(Y), \widetilde{H})$ where $\widetilde{H}^2 = 2abc^2$ and $\gamma(\widetilde{H}) = 1$, and Proposition 5.2 we get that g.c.d$(d, 2rs) =$ g.c.d$(d, 2ab) = 1$, and we get similar statement as Proposition 3.1.2.

Proposition 5.3. *Under conditions and notations of Propositions 5.2, all elements $\widetilde{H} = (xh + y\delta_1)/(2ab) \in N(Y)$ with $\widetilde{H}^2 = 2abc^2$ are in one to one correspondence with integral solutions (x, y) of the equation $x^2 - dy^2 = 4a^2b^2c^2$ such that $x \equiv \nu y \mod 2ab$.*

For $k \in \mathbb{N}$ and $k > 1$ the element $\widetilde{H}$ is divisible by k in $N(Y)$ and then it is not primitive, if and only if $k \mid (x, y)$ and $x \equiv \nu y \mod 2abk$. Then $k^2 | abc^2$. We call such a pair (x, y) as ν-divisible by k. If (x, y) is not ν-divisible by $k > 1$, we call such a pair (x, y) as ν-primitive. It is enough to consider only prime k. Thus, $\widetilde{H}$ is primitive, if and only if the corresponding pair (x, y) is ν-primitive. If g.c.d$(d, c) = 1$ and $\widetilde{H}$ is primitive, then $\gamma(\widetilde{H}) = 1$.

The crucial statement is the statement which is similar to Theorem 3.1.3:

Theorem 5.4. *Let Y be a K3 surface with $\rho(Y)=2$, and $h\in N(Y)$ a nef element of the degree $h^2=2ab$ with $\gamma(h)=1$. As in Proposition 5.2, we consider invariants d and $\nu\in(\mathbb{Z}/2ab)^*$ of the pair $(N(Y),h)$. Thus, $d\equiv\nu^2 \mod 4ab$. Assume that g.c.d$(c,d)=1$.*

With notations of Proposition 5.2, all elements $\widetilde{H}=(xh+y\delta_1)/(2ab)\in N(Y)$ with $\widetilde{H}^2=2abc^2$ satisfying Theorem 5.1 are in one to one correspondence with integral solutions (x,y) of the equation

$$x^2-dy^2=4a^2b^2c^2 \tag{5.5}$$

which satisfy the conditions (i), (ii), (iii), (iv) below:

(i) $x\equiv\nu y \mod 2ab$;

(ii) $x\equiv\pm 2abc \mod d$;

(iii) (x,y) belongs to one of a-series or b-series of solutions defined below:

a-series: $b|(x,y)$ and $x-\nu y\equiv 0 \mod 2a^2b$ where $\nu^2\equiv d \mod 4a^2b$,

b-series: $a|(x,y)$ and $x-\nu y\equiv 0 \mod 2ab^2$ where $\nu^2\equiv d \mod 4ab^2$;

(iv) the pair (x,y) is ν-primitive: there does not exist a prime l such that $l|(x,y)$ and $x-\nu y\equiv 0 \mod 2ab\,l$ (we always have $l^2|abc^2$ for such l), i. e. g.c.d$(x,y,(x-\nu y)/(2ab))=1$.

In particular, by Theorem 5.1, for a general Y with $\rho(Y)=2$, the pair (Y,h) is the moduli space $(\mathcal{M},\widetilde{h})$ of coherent sheaves over Y with the Mukai vector $v=(r,H,s)$, where $H^2=2rs$, $\gamma(H)=1$, and the canonical nef element $\widetilde{h}=(-a,0,b) \mod \mathbb{Z}v$, if g.c.d$(d,c)=1$ and the equation $x^2-dy^2=4a^2b^2c^2$ has an integral solution (x,y) satisfying conditions (i)—(iv) above. Moreover, a nef primitive element $\widetilde{H}=(xh+y\delta_1)/(2ab)\in N(Y)$ with $\widetilde{H}^2=2abc^2$ defines the Mukai vector $(r,\widetilde{H},s)$ with the moduli space $(\mathcal{M},\widetilde{h})$ of coherent sheaves on Y which is isomorphic to (Y,h), if and only if (x,y) satisfies the conditions (ii) and (iii) (it satisfies the conditions (i) and (ii) since it corresponds to a primitive element of $N(Y)$).

We can deduce from Theorem 5.4 a similar result as Theorem 3.1.4

Theorem 5.5. *With conditions and notations of Theorem 5.4, for a general Y with $\rho(Y)=2$ and $\gamma(h)=1$, we have (Y,h) is the moduli space $(\mathcal{M},\widetilde{h})$ of sheaves over Y with the Mukai vector $v=(r,\widetilde{H},s)$ where $\widetilde{H}^2=2rs$, $\gamma(\widetilde{H})=1$, and the canonical nef element $\widetilde{h}=(-a,0,b) \mod \mathbb{Z}v$, if and only if g.c.d$(c,d)=1$, for one of $\alpha=\pm1$ there exists an integral solution (p,q) for one of the a-series or the b-series:*

The a-series:

$$\begin{cases} p^2-dq^2=4ac/\alpha \\ p\equiv\nu q \mod 2a \\ g.c.d(l_1,p,q)=1 \text{ for any prime } l_1|ac \text{ and } l_1\not| b \\ p\not\equiv\nu q \mod 2al_2 \text{ for any prime } l_2|b \end{cases}. \tag{5.6}$$

A solution (p,q) of (5.6) *gives a solution (x,y) of Theorem 5.4 as the associated solution*

$$(x,y)=\pm(2abc+\alpha bdq^2,\ \alpha bpq). \tag{5.7}$$

The b-series:

$$\begin{cases} p^2-dq^2=4bc/\alpha \\ p\equiv \nu q \mod 2b \\ g.c.d(l_1,p,q)=1 \text{ for any prime } l_1|bc \text{ and } l_1 \not| a \\ p\not\equiv \nu q \mod 2bl_2 \text{ for any prime } l_2|a \end{cases}. \tag{5.8}$$

A solution (p,q) of (5.8) *gives a solution (x,y) of Theorem 5.4 as the associated solution*

$$(x,y)=\pm(2abc+\alpha adq^2,\ \alpha apq). \tag{5.9}$$

All $\widetilde{H}$ (or solutions (x,y)) of Theorem 5.4 are given by (5.7) *and* (5.9) *as associated solutions.*

Like in Theorems 3.1.5 and 3.1.6 we can interpret (p,q) of Theorem 5.5 as elements $h_1=b(ph+q\delta_1)/(2ab)\in N(Y)$ for the a-series and $h_1=a(ph+q\delta_1)/(2ab)\in N(Y)$ for the b-series. We formulate only the statement which is analogous to Theorem 3.1.6. It contains the statement which is similar to Theorem 3.1.5.

Theorem 5.6. *Let Y be a K3 surface with a nef element h such that $h^2=2ab$ and $\gamma(h)=1$. Then (Y,h) is the moduli space $(\mathcal{M},\widetilde{h})$ of sheaves over Y with the Mukai vector $v=(r,\widetilde{H},s)$ where $\widetilde{H}^2=2rs$, $\gamma(\widetilde{H})=1$, and the canonical nef element $\widetilde{h}=(-a,0,b) \mod \mathbb{Z}v$, if at least for one of signs $\pm$ (i. e. $\alpha=\pm1$) there exists $h_1\in N(Y)$ such that elements h, h_1 are contained in a 2-dimensional sublattice $N\subset N(Y)$ with $h\cdot N=\mathbb{Z}$, $g.c.d(\det(N),c)=1$, and h_1 belongs to one of the a-series or the b-series:*

The a-series:

$$h_1^2=\pm 2bc,\ h\cdot h_1\equiv 0 \mod b,\ h\cdot h_1\not\equiv 0 \mod bl_1,\ h_1/l_2\notin N(Y) \tag{5.10}$$

where l_1 and l_2 are any primes such that $l_1|ac$ and $l_1\not| b$, and $l_2|b$.

The b-series:

$$h_1^2=\pm 2ac,\ h\cdot h_1\equiv 0 \mod a,\ h\cdot h_1\not\equiv 0 \mod al_1,\ h_1/l_2\notin N(Y) \tag{5.11}$$

where l_1 and l_2 are any primes such that $l_1|bc$ and $l_1\not| a$, and $l_2|a$.

The conditions above are necessary, if $\rho(Y)\le 2$ and Y is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods $(T(Y),H^{2,0}(Y))$ of Y is ± 1.

Important Remark. By Theorem 5.4 and the formulae (5.7) and (5.9) we get

$$\pm w(\widetilde{H})=\begin{cases} -ch+\frac{\alpha(h\cdot h_1)h_1}{b}, & \text{if } h_1 \text{ is from } a\text{-series},\\ -ch+\frac{\alpha(h\cdot h_1)h_1}{a}, & \text{if } h_1 \text{ is from } b\text{-series}\end{cases} \tag{5.12}$$

where $w\in W^{(-2)}(N(Y))$.

Using results of Sects 3 and 5 both, one can construct many compositions of the correspondences from Sects 3 and 5 of the K3 surfaces $X \cong Y$ with itself.

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ON CORRESPONDENCES OF A K3 SURFACE WITH ITSELF, II

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Abstract. Let X be a K3 surface with a polarization H of degree $H^2 = 2rs$, $r,\ s \geq 1$, and with a primitive Mukai vector (r, H, s). The moduli space of sheaves over X with the isotropic Mukai vector (r, H, s) is again a K3 surface Y.

We prove that $Y \cong X$, if the Picard lattice $N(X)$ has an element h_1 with $(h_1)^2 = f(r, s)$, and the pair (H, h_1) satisfies a finite number of congruence conditions modulo $N_i(r, s)$. All these conditions are exactly written, and they are necessary, if X is general with rk $N(X) = 2$.

Existence of such kind a criterion is very surprising, and it also gives some geometric interpretation of elements in $N(X)$ with negative square. We also describe all irreducible divisorial conditions on moduli of (X, H) which imply $Y \cong X$, and we prove that their number is infinite if non-empty.

Thus, this paper treats in general problems considered in [MN1], [MN2] and [N4], when the additional condition $H \cdot N(X) = \mathbb{Z}$ had been imposed. In general, the results are much more complicated and less efficient.

0. Introduction

Let X be a K3 surface with a polarization H of degree $H^2 = 2rs$, $r,\ s \geq 1$. Assume that the Mukai vector $v = (r, H, s)$ is primitive.

Let Y be the moduli of coherent sheaves over X with the isotropic Mukai vector $v = (r, H, s)$ (see [Mu1]—[Mu4] and also [A]). The Y is a K3 surface which is equipped with a natural primitive *nef* element h with $h^2 = 2ab$ where we denote $c = \text{g.c.d}(r, s)$ and $a = r/c$, $b = s/c$. The surface Y is isogenous to X in the sense of Mukai. The second Chern class of the corresponding quasi-universal sheave gives then a 2-dimensional algebraic cycle $Z \subset X \times Y$ and an algebraic correspondence between X and Y.

Let H be divisible by $d \in \mathbb{N}$ where $\widetilde{H} = H/d$ is primitive in the Picard lattice $N(X)$ of X. Primitivity of $v = (r, H, s)$ means that $\text{g.c.d}(r, d, s) = 1$. We have $d^2 | rs$. Let $\gamma = \gamma(\widetilde{H})$ is defined by $\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}$, i.e. $H \cdot N(X) = \gamma d$. Clearly, $\gamma | (2rs/d^2) = \widetilde{H}^2$.

We denote
$$n(v) = \text{g.c.d}(r, s, d\gamma).$$

By Mukai, [Mu2], [Mu3], $n(v) = [T(Y) : T(X)]$ where $T(X)$ and $T(Y)$ are transcendental lattices of X and Y. We assume that

$$n(v) = \text{g.c.d}(r, s, d\gamma) = \text{g.c.d}(c, d\gamma) = 1. \tag{0.1}$$

By Mukai [Mu2], [Mu3], the transcendental periods $(T(X), H^{2,0}(X))$ and $(T(Y), H^{2,0}(Y))$ are isomorphic in this case. We can expect that sometimes the surfaces X and Y are also isomorphic, and we then get a cycle $Z \subset X\times X$, and a correspondence of X with itself. Thus, an interesting for us question is

Question 1. *When is Y isomorphic to X?*

We want to answer this question in terms of Picard lattices $N(X)$ and $N(Y)$ of X and Y. Then our question can be reformulated as follows:

Question 2. *Assume that N is a hyperbolic lattice, $H_1 \in N$ an element with square $2rs$. What are conditions on N and H_1 such that for any K3 surface X with Picard lattice $N(X)$ and s polarization $H \in N(X)$ the corresponding K3 surface Y is isomorphic to X, if the the pairs $(N(X), H)$ and (N, H_1) are isomorphic as abstract lattices with fixed elements?*

In other words, what are conditions on $(N(X), H)$ as an abstract lattice with an element H which are sufficient for Y to be isomorphic to X, and they are necessary, if X is a general K3 surface with the Picard lattice $N(X)$?

We answered this question in [MN1], [MN2] and [N4] under the condition $\gamma = 1$: in [MN1] for $r = s = 2$ and $d = 1$; in [MN2] for $r = s$ and $d = 1$; in [N4] for arbitrary r and s and $d = 1$.

The main surprising result of [MN1], [MN2] and [N4] was that $Y \cong X$, if the Picard lattice $N(X)$ has an element h_1 with some prescribed square $(h_1)^2$ and some minor additional conditions. Moreover, these conditions are necessary to have $Y \cong X$ for a general K3 surface X with $\rho(X) = 2$. Here we prove similar results in general.

We assume (0.1). Then $d^2|ab$ and $\gamma|2ab/d^2$. Moreover, $\text{g.c.d}(a, b) = 1$. We have $d = d_a d_b$ where $d_a = \text{g.c.d}(d, a)$ and $d_b = \text{g.c.d}(d, b)$. We define

$$a_1 = \frac{a}{d_a^2},\ b_1 = \frac{b}{d_b^2}$$

and

$$a_2 = \frac{a_1}{\gamma_a},\ b_2 = \frac{b_1}{\gamma_b},\ e_2 = \frac{2}{\gamma_2}$$

where $\gamma_a = \text{g.c.d}(a_1, \gamma)$, $\gamma_b = \text{g.c.d}(b_1, \gamma)$, $\gamma_2 = \gamma/(\gamma_a\gamma_b)$.

We prove (see Theorem 3.1.16)

Theorem 0.1. *Assume that*

$$g.c.d(c, d\gamma) = 1.$$

Then $Y \cong X$, if at least for one square-free $\alpha \,|\, 2a_1b_1$ there exists $\widetilde{h}_1 \in N(X)$ which belongs to the a-series or to the b-series described below (moreover, one has (3.1.153) *and* (3.1.154)*).*

The $\widetilde{h}_1$ belongs to the a-series, if there exists a primitive sublattice $N \subset N(X)$ of the rank two containing $\widetilde{H}$ and $\widetilde{h}_1$ such that $\widetilde{H} \cdot N = \gamma\mathbb{Z}$ and

$$(\widetilde{h}_1)^2 = \frac{2\gamma^2 b_1^2 c}{\alpha},\ \ \widetilde{H}\cdot\widetilde{h}_1 \equiv 0 \mod b_1c,\ \ f(\widetilde{H})\cdot\widetilde{h}_1 \equiv 0 \mod b_1c\det N,$$

and

$$p = \frac{\widetilde{H}\cdot\widetilde{h}_1}{b_1c},\quad q = \frac{f(\widetilde{H})\cdot\widetilde{h}_1}{b_1c\det N}$$

satisfy conditions (G) (see (3.1.113)—(3.1.117)*) and (A) (see* (3.1.118)—(3.1.126)*).*

The $\widetilde{h}_1$ belongs to the b-series, if there exists a primitive sublattice $N \subset N(X)$ of the rank two containing $\widetilde{H}$ and $\widetilde{h}_1$ such that $\widetilde{H} \cdot N = \gamma\mathbb{Z}$ and

$$(\widetilde{h}_1)^2 = \frac{2\gamma^2 a_1^2 c}{\alpha},\ \ \widetilde{H}\cdot\widetilde{h}_1 \equiv 0 \mod a_1c,\ \ f(\widetilde{H})\cdot\widetilde{h}_1 \equiv 0 \mod a_1c\det N,$$

and

$$p = \frac{\widetilde{H}\cdot\widetilde{h}_1}{a_1c},\quad q = \frac{f(\widetilde{H})\cdot\widetilde{h}_1}{a_1c\det N}$$

satisfy the conditions (G) and (B) (see (3.1.127)—(3.1.135)*).*

These conditions are necessary to have $Y \cong X$, if $\rho(X) \le 2$ and X is a general X K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is ± 1.

It seems many (if not all) known examples when $Y \cong X$ follow from this Theorem. E.g. see [C], [T1]—[T3] and [V].

In Theorem 0.1 $\mathbb{Z}f(\widetilde{H})$ is the orthogonal complement to $\widetilde{H}$ in N. In conditions (G), (A) and (B) we use the invariant $\mu \in \left(\mathbb{Z}/2a_1b_1c^2/\gamma\right)^*$ of the pair $\widetilde{H} \in N$ (use (3.1.2) for $S = N$) and $\delta = -\det N/\gamma$.

Like in [MN1], [MN2] and [N4] we also describe irreducible divisorial conditions on moduli of polarised K3 surfaces (X, H) which imply $\widetilde{H}\cdot N(X) = \gamma\mathbb{Z}$ and $Y \cong X$. We show that they are labelled by pairs $(\pm\mu, \delta)$ where $\pm\mu \in \left(\mathbb{Z}/(2a_1b_1c^2/\gamma)\right)^*$, $\delta \in \mathbb{N}$ where $\delta \equiv \mu^2\gamma \mod 2a_1b_1c^2/\gamma$, moreover at least for one $\alpha|2a_1b_1$ the equation

$$p^2 - \gamma\delta q^2 = \frac{4a_1b_1c}{\alpha}$$

has an integral solution (p,q) where (p,q) satisfy either conditions (G) and (A) (the a-series) or (G) and (B) (the b-series). See Sect. 4. As a result, we prove that the number of the divisorial conditions is either empty or infinite.

We remark that similar results in [N4] for the particular case $d=\gamma=1$ are much more exact and simple. They don't follow from the general results of this paper. At least, now we don't know how to get results of [N4] from main results above of this paper.

As in [MN1], [MN2] and [N4], the fundamental tools to get the results above is the Global Torelli Theorem for K3 surfaces proved by Piatetsky-Shapiro and Shafarevich in [PS], and results of Mukai [Mu2], [Mu3]. By results of [Mu2], [Mu3], we can calculate periods of Y using periods of X; by the Global Torelli Theorem for K3 surfaces [PS], we can find out if Y is isomorphic to X.

These paper treats in general problems considered in [MN1], [MN2] and [N4] where the additional condition $\gamma=d=1$ had been imposed. It makes results of this paper much more complicated and less efficient. For instance, in these paper we don't consider the question of non-emptiness of the divisorial conditions on moduli. It is much more complicated in the general setting of this paper. We hope to consider this problem later.

1. Preliminary notations and results about lattices and K3 surfaces

1.1. Some notations about lattices. We use notations and terminology from [N2] about lattices, their discriminant groups and forms. A *lattice* L is a non-degenerate integral symmetric bilinear form. I. e. L is a free $\mathbb{Z}$-module equipped with a symmetric pairing $x\cdot y\in\mathbb{Z}$ for $x,\ y\in L$, and this pairing should be non-degenerate. We denote $x^2=x\cdot x$. The *signature* of L is the signature of the corresponding real form $L\otimes\mathbb{R}$. The lattice L is called *even* if x^2 is even for any $x\in L$. Otherwise, L is called *odd*. The *determinant* of L is defined to be $\det L=\det(e_i\cdot e_j)$ where $\{e_i\}$ is some basis of L. The lattice L is *unimodular* if $\det L=\pm1$. The *dual lattice* of L is $L^*=Hom(L,\ \mathbb{Z})\subset L\otimes\mathbb{Q}$. The *discriminant group* of L is $A_L=L^*/L$. It has the order $|\det L|$. The group A_L is equipped with the *discriminant bilinear form* $b_L:A_L\times A_L\to\mathbb{Q}/\mathbb{Z}$ and the *discriminant quadratic form* $q_L:A_L\to\mathbb{Q}/2\mathbb{Z}$ if L is even. To get this forms, one should extend the form of L to the form on the dual lattice L^* with values in $\mathbb{Q}$.

For $x\in L$, we shall consider the invariant $\gamma(x)\geq 0$ where

$$x\cdot L=\gamma(x)\mathbb{Z}. \tag{1.1.1}$$

Clearly, $\gamma(x)|x^2$ if $x\neq 0$.

We denote by $L(k)$ the lattice obtained from a lattice L by multiplication of the form of L by $k\in\mathbb{Q}$. The orthogonal sum of lattices L_1 and L_2 is denoted by $L_1\oplus L_2$. For a symmetric integral matrix A, we denote by $\langle A\rangle$ a lattice which is given by the matrix A in some bases. We denote

$$U=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{1.1.2}$$

Any even unimodular lattice of the signature $(1,1)$ is isomorphic to U.

An embedding $L_1 \subset L_2$ of lattices is called *primitive* if L_2/L_1 has no torsion. We denote by $O(L)$, $O(b_L)$ and $O(q_L)$ the automorphism groups of the corresponding forms. Any $\delta \in L$ with $\delta^2 = -2$ defines a reflection $s_\delta \in O(L)$ which is given by the formula

$$x \to x + (x \cdot \delta)\delta,$$

$x \in L$. All such reflections generate the *2-reflection group* $W^{(-2)}(L) \subset O(L)$.

1.2. Some notations about K3 surfaces. Here we remind some basic notions and results about K3 surfaces, e. g. see [PS], [S-D], [Sh]. A K3 surface X is a non-singular projective algebraic surface over $\mathbb{C}$ such that its canonical class K_X is zero and the irregularity $q_X = 0$. We denote by $N(X)$ the *Picard lattice* of X which is a hyperbolic lattice with the intersection pairing $x \cdot y$ for $x,\ y \in N(S)$. Since the canonical class $K_X = 0$, the space $H^{2,0}(X)$ of 2-dimensional holomorphic differential forms on X has dimension one over $\mathbb{C}$, and

$$N(X) = \{x \in H^2(X,\mathbb{Z}) \mid x \cdot H^{2,0}(X) = 0\} \tag{1.2.1}$$

where $H^2(X,\mathbb{Z})$ with the intersection pairing is a 22-dimensional even unimodular lattice of signature $(3,19)$. The orthogonal lattice $T(X)$ to $N(X)$ in $H^2(X,\mathbb{Z})$ is called the *transcendental lattice of* X. We have $H^{2,0}(X) \subset T(X) \otimes \mathbb{C}$. The pair $(T(X), H^{2,0}(X))$ is called the *transcendental periods of* X. The *Picard number* of X is $\rho(X) = \mathrm{rk}\ N(X)$. A non-zero element $x \in N(X) \otimes \mathbb{R}$ is called *nef* if $x \neq 0$ and $x \cdot C \geq 0$ for any effective curve $C \subset X$. It is known that an element $x \in N(X)$ is ample if $x^2 > 0$, x is *nef*, and the orthogonal complement $x^\perp$ to x in $N(X)$ has no elements with square -2. For any element $x \in N(X)$ with $x^2 \geq 0$, there exists a reflection $w \in W^{(-2)}(N(X))$ such that the element $\pm w(x)$ is nef; it then is ample, if $x^2 > 0$ and $x^\perp$ had no elements with square -2 in $N(X)$.

We denote by $V^+(X)$ the light cone of X, which is the half-cone of

$$V(X) = \{x \in N(X) \otimes \mathbb{R} \mid x^2 > 0\ \} \tag{1.2.2}$$

containing a polarization of X. In particular, all *nef* elements x of X belong to $\overline{V^+(X)}$: one has $x \cdot V^+(X) > 0$ for them.

The reflection group $W^{(-2)}(N(X))$ acts in $V^+(X)$ discretely, and its fundamental chamber is the closure $\overline{\mathcal{K}(X)}$ of the Kähler cone $\mathcal{K}(X)$ of X. It is the same as the set of all *nef* elements of X. Its faces are orthogonal to the set $\mathrm{Exc}(X)$ of all exceptional curves r on X which are non-singular rational curves r on X with $r^2 = -2$. Thus, we have

$$\overline{\mathcal{K}(X)} = \{0 \neq x \in \overline{V^+(X)} \mid x \cdot \mathrm{Exc}(X) \geq 0\,\}. \tag{1.2.3}$$

2. Condition of $Y \cong X$ for a general K3 surface X with a given Picard lattice

2.1. The correspondence. Let X be a smooth complex projective K3 surface with a polarization (or more generally a *nef* element) H of degree $2rs$, $r,\ s \geq 1$.

Let H is divisible by $d \in \mathbb{N}$ and $\widetilde{H} = H/d$ is primitive in $N(X)$. Then $\widetilde{H}^2 = 2rs/d^2$ and $d^2|rs$. We denote

$$c = \text{g.c.d}(r, s),\ \ a = r/c,\ \ b = s/c. \tag{2.1.1}$$

We assume that the Mukai vector (r, H, s) is primitive, i. e.

$$\text{g.c.d}(r, s, d) = \text{g.c.d}(c, d) = 1. \tag{2.1.2}$$

Let Y be the moduli space of (coherent) sheaves $\mathcal{E}$ on X with the primitive isotropic Mukai vector $v = (r, H, s)$. Then $\text{rk}\ \mathcal{E} = r$, $\chi(\mathcal{E}) = r + s$ and $c_1(\mathcal{E}) = H$. Let

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \tag{2.1.3}$$

be the full cohomology lattice of X equipped with the Mukai product

$$(u, v) = -(u_0 \cdot v_2 + u_2 \cdot v_0) + u_1 \cdot v_1 \tag{2.1.4}$$

for $u_0, v_0 \in H^0(X, \mathbb{Z})$, $u_1, v_1 \in H^2(X, \mathbb{Z})$, $u_2, v_2 \in H^4(X, \mathbb{Z})$. We naturally identify $H^0(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ with $\mathbb{Z}$. Then the Mukai product is

$$(u, v) = -(u_0 v_2 + u_2 v_0) + u_1 \cdot v_1. \tag{2.1.5}$$

The element

$$v = (r, H, s) = (r, H, \chi - r) \in H^*(X, \mathbb{Z}) \tag{2.1.6}$$

is isotropic, i.e. $v^2 = 0$, since $H^2 = 2rs$. In this case (for a primitive v), Mukai showed [Mu2], [Mu3] that Y is a K3 surface, and one has the natural identification

$$H^2(Y,\ \mathbb{Z}) \cong (v^\perp/\mathbb{Z}v) \tag{2.1.7}$$

which also gives the isomorphism of the Hodge structures of X and Y. The Y has the canonical *nef* element $h = (-a, 0, b) \mod \mathbb{Z}v$ with $h^2 = 2ab$.

In particular, (2.1.5) gives the embedding

$$T(X) \subset T(Y) \tag{2.1.8}$$

of the transcendental periods of the index

$$[T(Y) : T(X)] = n(v) = \min |v \cdot x| \tag{2.1.9}$$

where $x \in H^0(X,\mathbb{Z}) \oplus N(X) \oplus H^4(X,\mathbb{Z})$ and $v \cdot x \neq 0$ (see [Mu2], [Mu3]). In this paper, we are interested in the case when $Y \cong X$. By (2.1.10), it may happen, if $n(v) = 1$ only.

We can introduce the invariant $\gamma = \gamma(\widetilde{H}) \in \mathbb{N}$ which is defined by

$$\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}, \tag{2.1.10}$$

equivalently, $H \cdot N(X) = \gamma d\mathbb{Z}$. Clearly, $\gamma | 2rs/d^2 = \widetilde{H}^2$, and

$$n(v) = \text{g.c.d}(r, s, \gamma d) = \text{g.c.d}(c, \gamma d). \tag{2.1.11}$$

Thus, $n(v) = 1$, and it is possible to have $Y \cong X$, only if

$$\text{g.c.d}(r, s, \gamma d) = \text{g.c.d}(c, \gamma) = \text{g.c.d}(c, d) = 1. \tag{2.1.12}$$

This is exactly the case when, according to Mukai, the transcendental periods

$$(T(X), H^{2,0}(X)) \cong (T(Y), H^{2,0}(Y)) \tag{2.1.13}$$

are isomorphic.

2.2. The characteristic map of a primitive element of a lattice. Let S be an even lattice and $P \in S$ its primitive element with $P^2 = 2m \neq 0$ and $\gamma(P) = \gamma | 2m$ (in S), i. e. $P \cdot S = \gamma\mathbb{Z}$.

We want to calculate the discriminant quadratic form of S. Consider

$$K(P) = P^{\perp}_S \tag{2.2.1}$$

the orthogonal complement to P in S. Put $P^* = P/2m$. Then any element $x \in S$ can be written as

$$x = n\gamma P^* + k^* \tag{2.2.2}$$

where $n \in \mathbb{Z}$ and $k^* \in K(P)^*$, because

$$\mathbb{Z}P \oplus K(P) \subset S \subset S^* \subset \mathbb{Z}P^* \oplus K(P)^*$$

and $P \cdot S = \gamma\mathbb{Z}$. Since $\gamma(P) = \gamma$, the map $n\gamma P^* + [P] \to k^* + K(P)$ gives an isomorphism of the groups $\mathbb{Z}/\frac{2m}{\gamma} \cong [\gamma P^*]/[P] \cong [u^*(P) + K(P)]/K(P)$ where $u^*(P) + K(P)$ has order $2m/\gamma$ in $A_{K(P)} = K(P)^*/K(P)$. Clarify in [N2]. It follows,

$$S = [\mathbb{Z}P, K(P), \gamma P^* + u^*(P)]. \tag{2.2.3}$$

The element $u^*(P)$ is defined canonically mod $K(P)$ by the condition that $\gamma P^* + u^* \in S$. The element

$$u^*(P) + K(P) \in K(P)^*/K(P) \tag{2.2.4}$$

is called *the canonical element of* P. Since $\gamma P^* + u^*(P)$ belongs to the even lattice S, it follows

$$(\gamma P^* + u^*(P))^2 = \frac{\gamma^2}{2m} + u^*(P)^2 \equiv 0 \mod 2. \tag{2.2.5}$$

For $n \in \mathbb{Z}$ and $k^* \in K(P)^*$, we have $x = nP^* + k^* \in S^*$, if and only if

$$(nP^* + k^*) \cdot (\gamma P^* + u^*(P)) = \frac{n\gamma}{2m} + k^* \cdot u^*(P) \in \mathbb{Z}.$$

It follows,

$$S^* = \{nP^* + k^* \mid n \in \mathbb{Z},\ k^* \in K(P)^*,\ n \equiv -\frac{2m}{\gamma}\ (k^* \cdot u^*(P)) \mod \frac{2m}{\gamma}\} \subset \\ \subset \mathbb{Z}P^* + K(P)^*. \tag{2.2.6}$$

It gives the calculation of the discriminant group $A_S = S^*/S$ where S is given by (2.2.3) and S^* is given by (2.2.6).

We define *the canonical submodule* $\widetilde{K}(P)^* \subset \mathbb{Z} \oplus K(P)^*$ by the condition

$$\widetilde{K}(P)^* = \{n \in \mathbb{Z},\, k^* \in K(P)^* \mid n \equiv -\frac{2m}{\gamma}\ (k^* \cdot u^*(P)) \mod \frac{2m}{\gamma}\}. \tag{2.2.7}$$

Now we define the *characteristic map*

$$\kappa(P) : \widetilde{K}(P)^* \to A_S, \tag{2.2.8}$$

by the condition

$$\kappa(P)(n, k^*) = nP^* + k^* \mod S. \tag{2.2.9}$$

Obviously, the characteristic map is epimorphic. Its kernel is

$$\widetilde{K}(P)^*_0 = [(2m\mathbb{Z},\, K), \mathbb{Z}(\gamma,\, u^*(P))] \cong S. \tag{2.2.10}$$

Thus, we correspond to a primitive $P \in S$ with $P^2 = 2m$ and $\gamma(P) = \gamma$ the canonical triplet

$$(K(P), u^*(P) + K(P), \kappa(P)). \tag{2.2.11}$$

This triplet is important because of the trivial but very important for us

Lemma 2.2.1. *Let $P_1 \in S$ and $P_2 \in S$ are two primitive elements of an even lattice S with $P_1^2 \neq 0$ and $P_2^2 \neq 0$.*

There exists an automorphism $f \in O(S)$ such that $f(P_1) = P_2$ and f gives ± 1 on the discriminant group $A_S = S^*/S$, if and only if $P_1^2 = P_2^2$, $\gamma(P_1) = \gamma(P_2)$, and there exists an isomorphism of lattices $\phi : K(P_1) \to K(P_2)$ such that $\phi^*(u^*(P_2) + K(P_2)) = u^*(P_1) + K(P_1)$ and $\widetilde{\phi}^* = (id, \phi^*) : \widetilde{K}(P_2)^* \to \widetilde{K}(P_1)^*$ is $\pm$ commuting with the characteristic maps $\kappa(P_1)$ and $\kappa(P_2)$, i. e.

$$\kappa(P_1)\widetilde{\phi^*} = \pm\kappa(P_2).$$

Proof. Trivial.

We also mention that

$$\det(S) = \gamma^2 \det K(P)/2m. \tag{2.2.12}$$

because $[S : \mathbb{Z}P \oplus K(P)] = 2m/\gamma$

2.3. Relation between periods of X and Y. Here we consider the case and notations of Sect. 2.1. Thus, for the primitive isotropic Mukai vector $v=(r,H,s)$, r, $s\ge 1$, $H^2=2rs$, we assume that for $d\in\mathbb{N}$, the element $\widetilde{H}=H/d$ is primitive in $N(X)$, $\gamma(\widetilde{H})=\gamma$ (in $N(X)$). We remind that $c=\text{g.c.d}(r,s)$, $a=r/c$, $b=s/c$, and $n(v)=\text{g.c.d}(r,s,d\gamma)=(c,d\gamma)=1$. It follows, $d^2|ab$ and $\gamma d^2|2ab$. Thus, our data are defined by

$$a,\ b,\ c,\ d,\ \gamma\ \in\mathbb{N},\ \text{such that}\ \ (a,b)=(d,c)=(d,\gamma)=1,\ d^2|ab,\ \gamma d^2|2ab, \quad (2.3.1)$$

and by a primitive polarization

$$\widetilde{H}\in N(X)\ \ \text{such that}\ \ \widetilde{H}^2=2abc^2/d^2,\ \gamma(\widetilde{H})=\gamma. \quad (2.3.2)$$

Then, the Mukai vector $v=(r,H,s)=(ac,d\widetilde{H},bc)$.

Let us denote by e_1 the canonical generator of $H^0(X,\mathbb{Z})$ and by e_2 the canonical generator of $H^4(X,\mathbb{Z})$. They generate the sublattice U in $H^*(X,\mathbb{Z})$ with the Gram matrix U. Consider Mukai vector $v=(re_1+se_2+H)$. We have

$$N(Y)=v^{\perp}_{U\oplus N(X)}/\mathbb{Z}v. \quad (2.3.3)$$

Let us calculate $N(Y)$. Let $K(H)=K(\widetilde{H})=(H)^{\perp}_{N(X)}$. We denote $H^*=H/(2rs)\in(\mathbb{Z}H)^*=\mathbb{Z}H^*$, and $\widetilde{H}^*=\widetilde{H}/(2rs/d^2)=dH^*\in(\mathbb{Z}\widetilde{H})^*=\mathbb{Z}\widetilde{H}^*$. Then we have an embedding of lattices of finite index

$$\mathbb{Z}\widetilde{H}\oplus K(H)\subset N(X)\subset N(X)^*\subset\mathbb{Z}\widetilde{H}^*\oplus K(H)^* \quad (2.3.4)$$

We have the orthogonal decomposition up to finite index

$$U\oplus\mathbb{Z}\widetilde{H}\oplus K(H)\subset U\oplus N(X)\subset U\oplus\mathbb{Z}\widetilde{H}^*\oplus K(H)^*. \quad (2.3.5)$$

Let $f=x_1e_1+x_2e_2+yH^*+z^*\in v^{\perp}_{U\oplus N(X)}$, $z^*\in K(H)^*$. Then $-sx_1-rx_2+y=0$ since $f\in v^{\perp}$ and hence $(f,v)=0$. Thus, $y=(sx_1+rx_2)$ and

$$f=x_1e_1+x_2e_2+(sx_1+rx_2)H^*+z^*. \quad (2.3.6)$$

Here $f\in U\oplus N(X)$ if and only if

$$x_1,x_2\in\mathbb{Z},\quad sx_1+rx_2\equiv 0\mod d,\quad \frac{sx_1+rx_2}{d}\widetilde{H}^*+z^*\in N(X). \quad (2.3.7)$$

Equivalently, by Sect. 2.2, we have

$$sx_1+rx_2\equiv 0\mod d\gamma,\quad z^*=\frac{sx_1+rx_2}{d\gamma}u^*(H)\mod K(H). \quad (2.3.8)$$

We denote
$$h' = (-a, b) \oplus 0 \in U \oplus N(X).$$
Clearly, $h' \in v^\perp$ and $h = h' \mod \mathbb{Z}v \in N(Y)$. Thus, the orthogonal complement contains
$$[\mathbb{Z}v, \mathbb{Z}h', K(H)] \tag{2.3.9}$$
where $h' = -ae_1 + be_2$, and (2.3.9) is a sublattice of finite index in $(v^\perp)_{U\oplus N(X)}$. The generators v, h' and generators of $K(H)$ are free, and we can rewrite f above using these generators with rational coefficients. We have
$$e_1 = \frac{v - ch' - H}{2r},\ e_2 = \frac{v + ch' - H}{2s}. \tag{2.3.10}$$
It follows,
$$f = \frac{sx_1 + rx_2}{2rs}v + \frac{c(-sx_1 + rx_2)}{2rs}h' + z^* \tag{2.3.11}$$
where x_1, x_2, z^* satisfy (2.3.8). Considering $\mod \mathbb{Z}v$, we finally get
$$N(Y) = \frac{c(-sx_1 + rx_2)}{2rs}h + \frac{sx_1 + rx_2}{d\gamma}u^*(\widetilde{H}) + K(H),\ \text{where } sx_1 + rx_2 \equiv 0 \mod d\gamma. \tag{2.3.12}$$

Let us calculate the lattice $K(h) = h^\perp_{N(Y)}$. It is equal to $(sx_1+rx_2)/(d\gamma)u^*(\widetilde{H})+K(H)$ where $sx_1+rx_2 \equiv 0 \mod d\gamma$ and $-sx_1+rx_2 = 0$. It follows, $x_1 = ax$, $x_2 = bx$, $x \in \mathbb{Z}$, and $sx_1+rx_2 = 2abcx \equiv 0 \mod d\gamma$, which is always true. Thus, $K(h) = [2abc/(d\gamma)u^*(\widetilde{H}) + K(H)]$. By Sect. 2.2, $u^*(\widetilde{H}) + K(H)$ has the order $2abc^2/(d^2\gamma)$. We have $(2abc/(d\gamma), 2abc^2/(d^2\gamma) = 2abc/(d^2\gamma)(d, c)$ where $2abc/(d^2\gamma) \in \mathbb{N}$ and $(d, c) = 1$. It follows, $K(h) = [K(H), 2abc/(d^2\gamma)u^*(\widetilde{H})]$, and the index $[K(h) : K(H)] = c$.

Let us show that $d|h$ in $N(Y)$ and h/d is primitive in $N(Y)$. By (2.3.12), the primitive submodule in $N(Y)$, generated by h, is $c(-sx_1 + rx_2)/(2rs)h$ where $(sx_1 + rx_2)/(d\gamma)u^*(\widetilde{H}) \in K(h)$ and $sx_1 + rx_2 \equiv 0 \mod d\gamma$. From calculation of $K(h)$ above, we then get $(sx_1 + rx_2)/(d\gamma) \equiv 0 \mod 2abc/(d^2\gamma)$. It follows, $bx_1 + ax_2 \equiv 0 \mod 2ab/d$. Let $d = d_ad_b$ where $d_a|a$ and $d_b|b$. Then $(a/d_a)|x_1$ and $(b/d_b)|x_2$. It follows, $x_1 = (a/d_a)\widetilde{x}_1$ and $x_2 = (b/d_b)\widetilde{x}_2$ where $\widetilde{x}_1,\ \widetilde{x}_2 \in \mathbb{Z}$ and $d_b\widetilde{x}_1 + d_a\widetilde{x}_2 \equiv 0 \mod 2$. It follows that the module $c(-sx_1 + rx_2)/(2rs)h = (-d_b\widetilde{x}_1 + d_a\widetilde{x}_2)/(2d_ad_b)$ where $d_b\widetilde{x}_1 + d_a\widetilde{x}_2 \equiv 0 \mod 2$. It follows that this module is $\mathbb{Z}(h/d)$. It proves the statement. We denote
$$\widetilde{h} = \frac{h}{d}\ . \tag{2.3.13}$$
We had proved that $\widetilde{h}$ is primitive in $N(Y)$ and $h = d\widetilde{h}$. We have $\widetilde{h}^2 = 2ab/d^2$.

Let us show that $\gamma(\widetilde{h}) = \gamma(H) = \gamma$. We remind that $\gamma(\widetilde{h})\mathbb{Z} = \widetilde{h} \cdot N(Y)$. By Sect. 2.2 and (2.3.12), $\widetilde{h}^2/\gamma(\widetilde{h})$ is equal to the index $[[(sx_1 + rx_2)/(d\gamma)u^*(\widetilde{H}) +$

$K(H)] : K(h)]$ where $sx_1 + rx_2 \equiv 0 \mod d\gamma$. Such elements x_1, x_2 give $cd\gamma\mathbb{Z}$. Thus, $[(sx_1 + rx_2)/(d\gamma)u^*(\widetilde{H}) + K(H)] = [cu^*(\widetilde{H}) + K(H)]$. We had proved that $K(h) = [2abc/(d^2\gamma)u^*(\widetilde{H}) + K(H)]$. Thus, the index is equal to $2ab/d^2\gamma$. Then we get $\widetilde{h}^2/\gamma(\widetilde{h}) = 2ab/(d^2\gamma(\widetilde{h})) = 2ab/(d^2\gamma)$. It follows $\gamma(\widetilde{h}) = \gamma$. We had also proved that $u^*(\widetilde{h}) = mcu^*(H) + K(h)$ where m is defined $\mod 2ab/(d^2\gamma)$. We shall calculate m below.

Now we can rewrite (2.3.12) in the form (2.2.2). We denote $\widetilde{h}^* = \widetilde{h}/\widetilde{h}^2 = \widetilde{h}/(2ab/d^2)$. We have

$$N(Y) = \frac{-bx_1 + ax_2}{d}\widetilde{h}^* + \frac{(bx_1 + ax_2)}{d\gamma}cu^*(\widetilde{H}) + K(H) \tag{2.3.14}$$

where $bx_1 + ax_2 \equiv 0 \mod d\gamma$. We have proved that the elements $(-bx_1 + ax_2)/d$ give $\mathbb{Z}\gamma$.

Let us write $d = d_a d_b$ where $d_a|a$ and $d_b|b$. Since $\gamma|2ab/d^2$ and g.c.d$(a, b) = 1$, we can write $\gamma = \gamma_2\gamma_a\gamma_b$ where $\gamma_a =$ g.c.d$(\gamma, a/d_a^2)$, $\gamma_b =$ g.c.d$(\gamma, b/d_b^2)$ and $\gamma_2 = \gamma/(\gamma_a\gamma_b)$. Clearly, $\gamma_2|2$.

We define $m = m(a, b, d, \gamma) \mod 2ab/(d^2\gamma)$ by the conditions

$$m \equiv -1 \mod 2a/(d_a^2\gamma_a\gamma_2) \ \text{ and } \ m \equiv 1 \mod 2b/(d_b^2\gamma_b\gamma_2). \tag{2.3.15}$$

The congruences (2.3.15) define m uniquely $\mod 2ab/(d^2\gamma)$. Really, assume that $x \equiv 0 \mod 2b/(d_b^2\gamma_b\gamma_2)$ and $x \equiv 0 \mod 2a/(d_a^2\gamma_a\gamma_2)$. Then $x/(2/\gamma_2) \equiv 0 \mod b/(d_b^2\gamma_b)$ and $x/(2/\gamma_2) \equiv 0 \mod a/(d_a^2\gamma_a)$. It follows that $x/(2/\gamma_2) \equiv 0 \mod ab/(d^2\gamma_a\gamma_b)$. Thus, $x \equiv 0 \mod 2ab/(d^2\gamma_a\gamma_b\gamma_2)$ where $2ab/(d^2\gamma_a\gamma_b\gamma_2) = 2ab/(d^2\gamma)$. Thus $x \equiv 0 \mod 2ab/(d^2\gamma)$. Clearly, there exists a unique $m(a, b)$ $\mod 2ab$ defined by the condition

$$m(a, b) \equiv -1 \mod 2a \text{ and } m(a, b) \equiv 1 \mod 2b. \tag{2.3.16}$$

Then

$$m(a, b, d, \gamma) \equiv m(a, b) \mod \frac{2ab}{d^2\gamma}, \tag{2.3.17}$$

and one can take (2.3.16) and (2.3.17) as definition of $m(a, b, d, \gamma)$.

Let us prove that $u^*(\widetilde{h}) + K(h) = m(a, b, d, \gamma)cu^*(\widetilde{H}) + K(h)$. We had proved that $u^*(\widetilde{h}) + K(h) = mcu^*(\widetilde{H}) + K(h)$ where m is defined $\mod 2ab/(d^2\gamma)$. To find $m \mod 2ab/(d^2\gamma)$, one should put $(-bx_1 + ax_2)/d = \gamma$ in (2.3.14) or

$$-bx_1 + ax_2 = d\gamma. \tag{2.3.18}$$

From (2.3.18) we have $d_a\gamma_a|x_1$ and $d_b\gamma_b|x_2$. From (2.3.14) and (2.3.18), we get

$$m \equiv \frac{ax_2 + bx_1}{d\gamma} \equiv 1 + \frac{2bx_1}{d\gamma} \mod 2ab/(d^2\gamma), \tag{2.3.19}$$

and

$$m \equiv -1 + \frac{2ax_2}{d\gamma} \mod 2ab/(d^2\gamma). \tag{2.3.20}$$

Since $d_a\gamma_a|x_1$, we get from (2.3.19) that $m \equiv 1 \mod 2b/(d_b^2\gamma_b\gamma_2)$, and from (2.3.20) that $m \equiv -1 \mod 2a/(d_a^2\gamma_a\gamma_2)$. Thus, $m = m(a,b,d,\gamma) \mod 2ab/(d^2\gamma)$. It proves the statement.

Since $h^2 = 2ab$ and $H^2 = 2abc^2$, we can formally put $h = H/c$. By our construction, $K(H) \subset K(h)$ is a sublattice. Thus, we can consider $N(X)$ and $N(Y)$ as extensions of finite index of a common sublattice $\mathbb{Z}\widetilde{H} + K(H)$.

Finally, we get the very important for us

Proposition 2.3.1. *The Picard lattice of X is*

$$N(X) = [\widetilde{H}, K(H), \gamma\widetilde{H}^* + u^*(\widetilde{H})] \tag{2.3.21}$$

where $\widetilde{H} = H/d$ is primitive in $N(X)$ with $H^2 = 2abc^2$ and $\widetilde{H}^* = d^2\widetilde{H}/(2abc^2)$, the lattice $K(H) = H^\perp_{N(X)}$, and $u^*(\widetilde{H}) + K(H)$ has the order $2abc^2/(d^2\gamma)$ in $K(H)^*/K(H)$.

The Picard lattice of Y is

$$N(Y) = [\widetilde{h} = h/d, K(h), \gamma\widetilde{h}^* + u^*(\widetilde{h})], \tag{2.3.22}$$

where the element $h = (-a, 0, b) \mod \mathbb{Z}v$, $h^2 = 2ab$ and $\widetilde{h} = h/d$ is primitive in $N(Y)$, the element $\widetilde{h}^* = d^2\widetilde{h}/(2ab)$, and $u^*(\widetilde{h}) + K(h)$ has the order $2ab/(d^2\gamma)$ in $K(h)^*/K(h)$.

They are related as follows:

$$K(h) = h^\perp_{N(Y)} = [K(H), \frac{2ab}{d^2\gamma}\, cu^*(\widetilde{H})], \tag{2.3.23}$$

and

$$u^*(\widetilde{h}) + K(h) = m(a,b)cu^*(\widetilde{H}) + K(h) \tag{2.3.24}$$

where $m(a,b) \mod 2ab$ is defined by $m(a,b) \equiv -1 \mod 2a$ and $m(a,b) \equiv 1 \mod 2b$. To define $u^*(\widetilde{h}) + K(h)$ above, it is enough to consider $m(a,b) \mod 2ab/(d^2\gamma)$.

We can formally put $h = H/c$, equivalently $\widetilde{h} = \widetilde{H}/c$. Then $N(X)$ and $N(Y)$ become the extensions of a common sublattice:

$$N(X) \supset [\widetilde{H} = c\,\widetilde{h},\ K(H)] \subset N(Y). \tag{2.3.25}$$

Since $n(v) = 1$, the transcendental lattices $T(X) = T(Y)$ are canonically identified in $v^\perp$. It follows that we have the canonical identifications

$$N(X)^*/N(X) = T(X)^*/T(X) = T(Y)^*/T(Y) = N(Y)^*/N(Y). \tag{2.3.26}$$

Here we use that discriminant groups of orthogonal complements in a unimodular lattice are canonically isomorphic. For example, here the identification $N(X)^*/N(X) = T(X)^*/T(X)$ is given by $n^* + N(X) \to t^* + T(X)$, if $n^* + t^* \in H^2(X, \mathbb{Z})$.

Let us calculate the identification

$$N(X)^*/N(X) = N(Y)^*/N(Y). \tag{2.3.27}$$

Obviously (from the description above), it is given by the canonical maps

$$N(X)^* \leftarrow (U \oplus N(X))^* \supset (U \oplus N(X))^{*\perp}_{\ v} \to (v^\perp)^*_0 \leftarrow (v^\perp/\mathbb{Z}v)^*. \tag{2.3.28}$$

Here $(U \oplus N(X))^{*\perp}_{\ v} = \{x \in (U \oplus N(X))^* | x \cdot v = 0\}$, $(v^\perp) = \{x \in U \oplus N(X) | x \cdot v = 0\}$ and $(v^\perp)^*_0 = \{x \in (v^\perp)^* | x \cdot v = 0\}$.

Let $f^* = n\widetilde{H}^* + k^* \in N(X)^*$ where $n \in \mathbb{Z}$ and $k^* \in K(H)^*$. The element $\widetilde{f}^* = x_1e_1 + x_2e_2 + n\widetilde{H}^* + k^*$ is its lift to $(U \oplus N(X))^*$ where $x_1,\ x_2 \in \mathbb{Z}$. We have $\widetilde{f}^* \in (U \oplus N(X))^{*\perp}_{\ v}$, if $-sx_1 - rx_2 + dn = 0$. It follows that $n = c(bx_1 + ax_2)/d$ where $bx_1 + ax_2 \equiv 0 \mod d$, and $\widetilde{f}^* = x_1e_1 + x_2e_2 + (c(bx_1 + ax_2)/d)\,\widetilde{H}^* + k^*$. It follows that

$$f^* = \frac{c(bx_1 + ax_2)}{d}\widetilde{H}^* + k^*, \quad bx_1 + ax_2 \equiv 0 \mod d. \tag{2.3.29}$$

Like in (2.3.11), (2.3.12) and (2.3.14), we finally get that the corresponding to $f^* + N(X)$ element in $N(Y)^*/N(Y)$ is $\epsilon(f^*) + N(Y)$ where

$$\epsilon(f^*) = \frac{-bx_1 + ax_2}{d}\widetilde{h}^* + k^* \in N(Y)^*. \tag{2.3.30}$$

Thus, the identification (2.3.27) is given by $f^* = c\,((bx_1 + ax_2)/d)\,\widetilde{H}^* + k^* + N(X) \to \epsilon(f^*) = (-bx_1 + ax_2)\widetilde{h}^* + k^* + N(Y)$ where $bx_1 + ax_1 \equiv 0 \mod d$. We have $x_1 = d_a\widetilde{x}_1$ and $x_2 = d_b\widetilde{x}_2$ where $\widetilde{x}_1,\ \widetilde{x}_2 \in \mathbb{Z}$. Then

$$f^* = c\,((b/d_b)\widetilde{x}_1 + (a/d_a)\widetilde{x}_2)\,\widetilde{H}^* + k^* \to \epsilon(f^*) = (-(b/d_b)\widetilde{x}_1 + (a/d_a)\widetilde{x}_2)\,\widetilde{h}^* + k^*.$$

Let us denote $n_1 = (b/d_b)\widetilde{x}_1 + (a/d_a)\widetilde{x}_2$ and $n_2 = -(b/d_b)\widetilde{x}_1 + (a/d_a)\widetilde{x}_2$. We have $n_1 \equiv m(a,b)n_2 \mod 2ab/d^2$. Really, $n_1 - m(a,b)n_2 \equiv (2a/d_a)\widetilde{x}_2 \equiv 0 \mod 2a/d_a^2$ because $m(a,b) \equiv -1 \mod 2a/d_a^2$. We have $n_1 - m(a,b)n_2 \equiv (2b/d_b)\widetilde{x}_1 \equiv 0 \mod 2b/d_b^2$ because $m(a,b) \equiv 1 \mod 2b/d_b^2$. It follows the statement. Here we consider $m(a,b) \mod 2ab/d^2$.

The expression $(b/d_b)\widetilde{x}_1 + (a/d_a)\widetilde{x}_2$ gives all integers $n \in \mathbb{Z}$. Thus, we have proved that if $f^* = cn\widetilde{H}^* + k^* \in N(X)^*$ where $n \in \mathbb{Z}$, $k^* \in K(H)^*$, then $\epsilon(f^*) = m(a,b)\widetilde{h}^* + k^* \in N(Y)$. Thus, we have proved

Proposition 2.3.2. *In notations of Proposition 2.3.1, the canonical identification $\epsilon : N(X)^*/N(X) \cong N(Y)^*/N(Y)$ of the discriminant groups given by periods:*

$$\epsilon : N(X)^*/N(X) = T(X)^*/T(X) = T(Y)^*/T(Y) = N(Y)^*/N(Y) \qquad (2.3.31)$$

is given by

$$\epsilon : cn\widetilde{H}^* + k^* + N(X) \mapsto m(a,b)n\widetilde{h}^* + k^* \qquad (2.3.32)$$

where $n \in \mathbb{Z}$, $k^* \in K(H)^*$. Here one should consider $m(a,b) \mod 2ab/d^2$.

Equivalently, the characteristic maps $\kappa(\widetilde{H}) : \widetilde{K}(H) \to N(X)^*/N(X)$ and $\kappa(\widetilde{h}) : \widetilde{K}(h) \to N(Y)^*/N(Y)$ are related as follows:

$$\epsilon(\kappa(\widetilde{H})((cn, k^*) + \widetilde{K}(H)_0) = \kappa(\widetilde{h})(m(a,b)n, k^*) + \widetilde{K}(h)_0 \qquad (2.3.33)$$

(here one should consider $m(a,b) \mod 2ab/d^2$). Here we set $\widetilde{K}(H) = \widetilde{K}(\widetilde{H})$ and $\widetilde{K}(h) = \widetilde{K}(\widetilde{h})$ (see (2.2.7)).

This finishes the calculation of the periods of $N(Y)$ in terms of the periods of $N(X)$.

Applying Lemma 2.2.1 and Propositions 2.3.1 and 2.3.2, by Global Torelli Theorem for K3 surfaces [PS], we get

Theorem 2.3.3. *Assume that X is a K3 surface with a polarization H with $H^2 = 2rs$, r, $s \geq 1$, and a primitive Mukai vector $v = (r, H, s)$ with the invariants*

$$(a, b, c, d, \gamma) \qquad (2.3.34)$$

introduced above, i.e. $\widetilde{H} = H/d$ is primitive. Let Y be the moduli space of coherent sheaves on X with the Mukai vector $v = (r, H, s)$. We denote by $(K(\widetilde{H}), u^*(\widetilde{H}), \kappa(\widetilde{H}))$ the invariants (2.2.11) of $\widetilde{H} \in N(X)$.

The transcendental periods $(T(X), H^{2,0}(X))$ and $(T(Y), H^{2,0}(Y))$ are isomorphic, if and only if

$$n(v) = g.c.d(c,\, d\gamma) = 1 \qquad (2.3.35)$$

(this is Mukai's result). We denote by $(K(H) = K(\widetilde{H}), u^*(\widetilde{H}), \kappa(\widetilde{H}))$ the invariants (2.2.11) of $\widetilde{H} \in N(X)$.

Assume that (2.3.35) is valid. Then $Y \cong X$, if the following conditions (a), (b) and (c) are valid:

(a) there exists a primitive $\widetilde{h} \in N(X)$ with $\widetilde{h}^2 = 2ab/d^2$ and $\gamma(\widetilde{h}) = \gamma$.

(b) There exists an embedding $\phi : K(H) \subset K(\widetilde{h})$ of lattices such that $\phi^*(K(\widetilde{h})) = [K(H), 2abc/(d^2\gamma)u^*(\widetilde{H})]$ and $\phi^*(u^*(\widetilde{h})) + \phi^*(K(\widetilde{h})) = m(a,b)cu^*(\widetilde{H}) + \phi^*(K(\widetilde{h}))$. Here $m(a,b) \mod 2ab/(d^2\gamma)$ is considered.

(c) There exists a choice of $\pm$ such that $\kappa(\widetilde{h})(m(a,b)n, z^*) = \pm\kappa(H)(cn, \phi^*(z^*))$, if $(cn, \phi^*(z^*)) \in \widetilde{K}(H)^* = \widetilde{K}(\widetilde{H})^*$. Here $m(a,b) \mod 2ab/d^2$ is considered.

The conditions (a), (b) and (c) are necessary for a K3 surface X with $\rho(X) \leq 19$ which is general for its Picard lattice $N(X)$ in the following sense: the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is ± 1. If $\rho(X) = 20$, then always $Y \cong X$, if (2.3.35) holds.

Example 2.3.4. Let us assume that $\rho(X) = 1$. Thus, $N(X) = \mathbb{Z}\widetilde{H}$. Assume conditions (a), (b) and (c) of Theorem 2.3.3 satisfy. Then $\widetilde{h} = \pm\widetilde{H}$. It follows $c = 1$. The lattices $K(H)$ and $K(\widetilde{h})$ are zero, and then (b) is valid. The discriminant group $N(X) = \mathbb{Z}\widetilde{H}^*/\mathbb{Z}\widetilde{H} \equiv \mathbb{Z}/(2ab/d^2)\mathbb{Z}$. The condition (c) is valid, if and only if $m(a,b) \equiv \pm 1 \mod 2ab/d^2$. This is true, if and only if $(a/d_a^2) = 1$ or $(b/d_b^2) = 1$.

Thus, $X \cong Y$, if $c = 1$ and either $a/d_a^2 = 1$ or $b/d_b^2 = 1$. These conditions are necessary to have $Y \cong X$, if X is a general K3 surface with $\rho(X) = 1$. We recover the result of Mukai from [Mu3].

Example 2.3.5. Following Example 2.3.5, let us consider the case when we can satisfy conditions of Theorem 2.3.3 taking $\widetilde{h} = \pm H$ and $\phi = \pm id$. Again we get $c = 1$. (b) satisfies, if and only if $m(a,b) \equiv \pm 1 \mod 2ab/(d^2\gamma)$. This is equivalent to either $2a/(d_a^2\gamma_a\gamma_2) \le 2$ or $2b/(d_b^2\gamma_b\gamma_2) \le 2$. (c) satisfies, if and only if $m(a,b) \equiv \pm 1 \mod 2ab/d^2$. Thus either $a/d_a^2 = 1$ or $b/d_b^2 = 1$. Thus, always $Y \cong X$, if $c = 1$ and either $a/d_a^2 = 1$ or $b/d_b^2 = 1$. This is just a specialization of the $\rho = 1$ case above. This result is also due to Mukai [Mu3]

In Sect. 3 we consider $\rho = 2$. We shall analyse when we can satisfy conditions of Theorem 2.3.3 in this case. By specialisation of these cases, we shall get results about K3 surfaces with any Picard number $\rho \ge 2$.

3. Conditions of $Y \cong X$ for a general K3 surface X with $\rho = 2$

3.1. Main results for $\rho(X) = 2$. Here we apply results of Sect. 2 to X and Y with Picard number 2. Thus, we assume that $\rho(X) = \mathrm{rk}\ N(X) = 2$.

We start with some preliminary considerations on a primitive element $P \in S$ of an even hyperbolic lattice S or $\mathrm{rk}\ S = 2$. We assume that $P^2 = 2n$, $n \in \mathbb{N}$, and $\gamma(P) = \gamma | 2n$.

Let

$$K(P) = P_S^\perp = \mathbb{Z}f(P) \tag{3.1.1}$$

and $f(P)^2 = -t$ where $t > 0$ is even. Then $\pm f(P) \in S$ is defined uniquely by P. Below we set $f = f(P)$.

By elementary considerations, we have

$$S = [\mathbb{Z}P,\, \mathbb{Z}f,\, \frac{\gamma(\mu P + f)}{2n}] \tag{3.1.2}$$

where

$$\text{g.c.d}(\mu, \frac{2n}{\gamma}) = 1. \tag{3.1.3}$$

The element

$$\pm\mu \mod \frac{2n}{\gamma} \in (\mathbb{Z}/\frac{2n}{\gamma})^* \tag{3.1.4}$$

is *the invariant of the pair* $P \in S$ up to isomorphisms of lattices with a primitive vector P of $P^2 = 2n$ and $\gamma(P) = \gamma$. If f changes to $-f$, then $\mu \mod 2n/\gamma$ changes to $-\mu \mod 2n/\gamma$.

We have $(\gamma(\mu P+f)/(2n))^2 = \gamma^2(\mu^2 - t/2n)/2n \equiv 0 \mod 2$. It follows $2n\mu^2 - t \equiv 0 \mod 8n^2/\gamma^2$. It follows that for some $\delta \in \mathbb{N}$ we have

$$f^2 = -\frac{2n\delta}{\gamma}, \quad n\delta \equiv 0 \mod \gamma, \quad \text{and} \quad \delta \equiv \mu^2\gamma \mod \frac{4n}{\gamma}. \tag{3.1.5}$$

We have

$$\det S = -\gamma\delta. \tag{3.1.6}$$

Any element $z \in S$ can be written as $z = \gamma(xP + yf)/2n$ where $x \equiv \mu y \mod 2n/\gamma$. We have

$$z^2 = \frac{\gamma x^2 - \delta y^2}{(2n/\gamma)} \tag{3.1.7}$$

It is convenient to put

$$\widetilde{n} = \frac{2n}{\gamma}\ . \tag{3.1.8}$$

Thus, the considered above case of a primitive $P \in S$ where S is an even hyperbolic lattice of rk $S = 2$ is described by the invariants

$$\widetilde{n},\ \gamma,\ \delta,\ \pm\mu \in (\mathbb{Z}/\widetilde{n})^*, \tag{3.1.9}$$

where $\widetilde{n}$, γ, $\delta \in \mathbb{N}$. The invariants (3.1.9) must satisfy

$$\widetilde{n}\gamma \equiv \widetilde{n}\delta \equiv 0 \mod 2,\ \delta \equiv \mu^2\gamma \mod 2\widetilde{n}. \tag{3.1.10}$$

Then $P^2 = \widetilde{n}\gamma$, $f^2 = -\widetilde{n}\delta$, $P \perp f$, and

$$S = \{\frac{xP + yf}{\widetilde{n}} \mid x,\, y \in \mathbb{Z},\ x \equiv \mu y \mod \widetilde{n}\}. \tag{3.1.11}$$

We have

$$z^2 = \frac{\gamma x^2 - \delta y^2}{\widetilde{n}}\ . \tag{3.1.12}$$

Moreover,

$$\det S = -\delta\gamma. \tag{3.1.13}$$

We denote

$$P^* = \frac{P}{\widetilde{n}\gamma},\ f^* = \frac{f}{\widetilde{n}\delta}. \tag{3.1.14}$$

Then

$$S^* = \{vP^* + wf^* \mid \mu v - w \equiv 0 \mod \widetilde{n}\} \tag{3.1.15}$$

and

$$S=\{vP^*+wf^*\mid v\equiv 0 \mod \gamma,\ w\equiv 0 \mod \delta,\ \frac{v}{\gamma}\equiv\frac{\mu w}{\delta} \mod \widetilde{n}\ \}. \quad (3.1.16)$$

Here $v,\, w\in\mathbb{Z}$. From (3.1.15), $w=\mu v+\widetilde{n}t$, and

$$S^*=v(P^*+\mu f^*)+t\widetilde{n}f^*,\quad v,\,t\in\mathbb{Z}, \quad (3.1.17)$$

and $v(P^*+\mu f^*)+t\widetilde{n}f^*\in S$, if and only if

$$v\equiv 0 \mod \gamma,\ \mu v+\widetilde{n}t\equiv 0 \mod \delta,\ \ \delta v\equiv\gamma\mu(\mu v+\widetilde{n}t) \mod \delta\gamma\widetilde{n}. \quad (3.1.18)$$

We have

$$u^*(P)=\frac{\mu^{-1}f}{\widetilde{n}}+\mathbb{Z}f=\mu^{-1}\delta f^*+\mathbb{Z}f. \quad (3.1.19)$$

We remind notations we have used in Sect. 2:

$$a_1=\frac{a}{d_a^2},\ b_1=\frac{b}{d_b^2} \quad (3.1.20)$$

where $d_a=\text{g.c.d}(d,a)$ and $d_b=\text{g.c.d}(d,b)$. We put

$$a_2=\frac{a_1}{\gamma_a},\ b_2=\frac{b_1}{\gamma_b},\ e_2=\frac{2}{\gamma_2}. \quad (3.1.21)$$

where $\gamma_a=\text{g.c.d}(a_1,\gamma)$, $\gamma_b=\text{g.c.d}(b_1,\gamma)$, $\gamma_2=\gamma/(\gamma_a\gamma_b)$. Then $\gamma_2|2$.

Applying calculations above to $S=N(X)$ and primitive $P=\widetilde{H}$ with $n=2a_1b_1c^2$, $\widetilde{n}=2a_1b_1c^2/\gamma=e_2a_2b_2c^2$, and $\gamma(\widetilde{H})=\gamma$, we get

Proposition 3.1.1. *Let X be a K3 surface with Picard number $\rho=2$ equipped with a primitive polarization (or vector) $\widetilde{H}\in N(X)$ of degree $\widetilde{H}^2=2a_1b_1c^2$ and $\gamma(\widetilde{H})=\gamma|2a_1b_1$.*

Let $K(\widetilde{H})=(\widetilde{H})^\perp=\mathbb{Z}f(\widetilde{H})$. We have $f(\widetilde{H})^2=-2a_1b_1c^2\delta/\gamma$ where $\det N(X)=-\delta\gamma$.

For some $\mu\in(\mathbb{Z}/(2a_1b_1c^2/\gamma))^*$ (the $\pm\mu$ is the invariant of the pair $\widetilde{H}\in N(X)$) where $\delta\equiv\mu^2\gamma \mod 4a_1b_1c^2/\gamma$, one has

$$N(X)=[\widetilde{H},f(\widetilde{H}),\frac{(\mu\widetilde{H}+f(\widetilde{H}))}{2a_1b_1c^2/\gamma}], \quad (3.1.22)$$

$$N(X)=\{z=\frac{x\widetilde{H}+yf(\widetilde{H})}{2a_1b_1c^2/\gamma}\mid x,y\in\mathbb{Z}\text{ and }x\equiv\mu y \mod \frac{2a_1b_1c^2}{\gamma}\}. \quad (3.1.23)$$

We have

$$z^2=\frac{\gamma x^2-\delta y^2}{2a_1b_1c^2/\gamma}. \quad (3.1.24)$$

For any primitive element $P\in N(X)$ with $P^2=2a_1b_1c^2$, $\gamma(P)=\gamma$ and the same invariant $\pm\mu$, there exists an automorphism $\phi\in O(N(X))$ such that $\phi(\widetilde{H})=P$.

Applying calculations above to $S=N(Y)$ and primitive $P=\widetilde{h}\in N(Y)$ with $n=2a_1b_1$, $\widetilde{n}=2a_1b_1/\gamma=e_2a_2b_2$ and $\gamma(\widetilde{h})=\gamma$, we get

Proposition 3.1.2. *Let Y be a K3 surface with Picard number $\rho = 2$ equipped with a primitive polarization (or vector) $\widetilde{h} \in N(Y)$ of degree $\widetilde{h}^2 = 2a_1b_1$ and $\gamma(\widetilde{h}) = \gamma | 2a_1b_1$.*

Let $K(\widetilde{h}) = (\widetilde{h})^\perp = \mathbb{Z}f(\widetilde{h})$. We have $f(\widetilde{h})^2 = -2a_1b_1\delta/\gamma$ where $\det N(Y) = -\delta\gamma$.

For some $\nu \in (\mathbb{Z}/(2a_1b_1/\gamma))^*$ (the $\pm\nu$ is the invariant of the pair $\widetilde{h} \in N(Y)$) where $\delta \equiv \nu^2\gamma \mod 4a_1b_1/\gamma$, one has

$$N(Y) = [\widetilde{h}, f(\widetilde{h}), \frac{(\nu\widetilde{h} + f(\widetilde{h}))}{2a_1b_1/\gamma}], \tag{3.1.25}$$

$$N(Y) = \{z = \frac{x\widetilde{h} + yf(\widetilde{h})}{2a_1b_1/\gamma} \mid x, y \in \mathbb{Z} \text{ and } x \equiv \nu y \mod \frac{2a_1b_1}{\gamma}\}. \tag{3.1.26}$$

We have

$$z^2 = \frac{\gamma x^2 - \delta y^2}{2a_1b_1/\gamma}. \tag{3.1.27}$$

For any primitive element $P \in N(Y)$ with $P^2 = 2a_1b_1$, $\gamma(P) = \gamma$ and the same invariant $\pm\nu$, there exists an automorphism $\phi \in O(N(Y))$ such that $\phi(\widetilde{h}) = P$.

The crucial statement is

Theorem 3.1.3. *Let X be a K3 surface, $\rho(X) = 2$ and H a polarization of X of degree $H^2 = 2rs$, r, $s \geq 1$, and Mukai vector (r, H, s) is primitive. Let Y be the moduli space of sheaves on X with the isotropic Mukai vector $v = (r, H, s)$ and the canonical nef element $h = (-a, 0, b) \mod \mathbb{Z}v$. We assume that*

$$g.c.d(c, d\gamma) = 1.$$

With notations of Propositions 3.1.1, all elements

$$\widetilde{h} = \frac{x\widetilde{H} + yf(\widetilde{H})}{2a_1b_1c^2/\gamma} \in N(X)$$

with square $\widetilde{h}^2 = 2a_1b_1$ satisfying Theorem 2.2.3 are in one to one correspondence with integral solutions (x, y) of the equation

$$\gamma x^2 - \delta y^2 = 4a_1^2b_1^2c^2/\gamma \tag{3.1.28}$$

which satisfy conditions (i) — (v) below:

(i)

$$x \equiv \mu y \mod 2a_1b_1c^2/\gamma\,, \tag{3.1.29}$$

$$\mu\gamma x \equiv \delta y \mod 2a_1b_1c^2; \tag{3.1.30}$$

(ii) (x, y) belongs to one of a-series (the sign $+$) or b-series (the sign $-$) of solutions defined below:

$$\pm m(a,b)\mu x+(\delta y/\gamma)\equiv 0 \mod 2a_1b_1/\gamma\,, \tag{3.1.31}$$

$$x\pm m(a,b)\mu y\equiv 0 \mod 2a_1b_1/\gamma\,, \tag{3.1.32}$$

$$\pm m(a,b)\mu x+(\delta y/\gamma)\equiv \mu(x\pm m(a,b)\mu y) \mod (2a_1b_1c^2/\gamma)(2a_1b_1/\gamma)\,; \tag{3.1.33}$$

(iii) there exists a choice of $\beta=\pm1$ such that

$$\begin{cases} \left(\frac{m(a,b)\gamma x\pm\mu\gamma y}{2a_1b_1}-\beta c\right)\equiv 0 \mod \gamma \\ \left(\frac{\delta m(a,b)y\pm\mu\gamma x}{2a_1b_1}-\beta\mu c\right)\equiv 0 \mod \delta \\ \delta\left(\frac{m(a,b)\gamma x\pm\mu\gamma y}{2a_1b_1}-\beta c\right)\equiv \mu\gamma\left(\frac{\delta m(a,b)y\pm\mu\gamma x}{2a_1b_1}-\beta\mu c\right) \mod 2a_1b_1c^2\delta \end{cases} \tag{3.1.34}$$

and

$$\begin{cases} c\delta y\equiv 0 \mod \gamma \\ cx-(\pm\beta)\frac{2a_1b_1c^2}{\gamma}\equiv 0 \mod \delta \\ \delta y\equiv \mu\left(\gamma x-(\pm\beta)2a_1b_1c\right) \mod 2a_1b_1c\delta \end{cases} \tag{3.1.35}$$

where $+$ is taken for a-series, and $-$ is taken for b-series.

(iv) the pair (x, y) is μ-primitive:

$$g.c.d\left(x,\,y,\,\frac{x-\mu y}{2a_1b_1c^2/\gamma}\right)=1; \tag{3.1.36}$$

(v) $\gamma(\widetilde{h})=\gamma$, equivalently

$$g.c.d\left(\gamma x,\,\delta y,\,\frac{\mu\gamma x-\delta y}{2a_1b_1c^2/\gamma}\right)=\gamma. \tag{3.1.37}$$

In particular (by Theorem 2.3.3), for a general X with $\rho(X)=2$ we have $Y\cong X$, if and only if the equation $\gamma x^2-\delta y^2=4a_1^2b_1^2c^2/\gamma$ has an integral solution (x,y) satisfying conditions (i)—(v) above. Moreover, a nef primitive element $P=(x\widetilde{H}+f(\widetilde{H}))/(2a_1b_1c^2/\gamma)$ with $P^2=2a_1b_1$ and $\gamma(P)=\gamma$ defines a pair (X,P) which is isomorphic to the $(Y,\widetilde{h})$, if and only if (x,y) satisfies the conditions (ii) and (iii) (it satisfies conditions (i), (iv) and (v) since it corresponds to a primitive element of $N(X)$ with $\gamma(P)=\gamma$).

Proof. We denote

$$H^*=\frac{\widetilde{H}}{2a_1b_1c^2},\ \ f(\widetilde{H})^*=\frac{\gamma f(\widetilde{H})}{2a_1b_1c^2\delta} \tag{3.1.38}$$

where $K(H)=\mathbb{Z}f(\widetilde{H})=H^\perp$ in $N(X)$. By (3.1.19), we have

$$u^*(\widetilde{H})=\frac{\mu^{-1}\gamma f(\widetilde{H})}{2a_1b_1c^2}+\mathbb{Z}f(\widetilde{H})=\mu^{-1}\delta f(\widetilde{H})^*+\mathbb{Z}f(\widetilde{H}). \tag{3.1.39}$$

Let

$$\widetilde{h}=\frac{x\widetilde{H}+yf(\widetilde{H})}{2a_1b_1c^2/\gamma}\in N(X) \tag{3.1.40}$$

satisfies conditions of Theorem 2.2.1. Then $x\,,y\in\mathbb{Z}$ and $x\equiv \mu y \mod 2a_1b_1c^2/\gamma$. We get (3.1.29) in (i). Moreover, $\widetilde{h}$ is primitive which is equivalent to (iv). We also have $\gamma(\widetilde{h})=\gamma$. It is equivalent to

$$\text{g.c.d}\left(\widetilde{H}\cdot\widetilde{h},\ f(\widetilde{H})\cdot\widetilde{h},(\mu\widetilde{H}+f(\widetilde{H}))/(2a_1b_1c^2/\gamma)\cdot\widetilde{h}\right)=\gamma.$$

It follows (3.1.30) in (i), and (v). We have $\widetilde{h}^2=2a_1b_1$. This is equivalent to $\gamma x^2-\delta y^2=4a_1^2b_1^2c^2/\gamma$.

Consider $K(\widetilde{h})=\widetilde{h}^\perp$ in $N(X)$. Let us denote

$$f(\widetilde{h})=\frac{(\delta y/\gamma)\widetilde{H}+xf(\widetilde{H})}{2a_1b_1c^2/\gamma}. \tag{3.1.41}$$

The element $f(\widetilde{h})\in N(X)$ because of (i). We have $f(\widetilde{h})\perp\widetilde{h}$ and $f(\widetilde{h})^2=-2a_1b_1\delta/\gamma$. Since $\gamma(\widetilde{h})=\gamma$, it follows that $K(\widetilde{h})=\mathbb{Z}f(\widetilde{h})$ and $f(\widetilde{h})$ is primitive. We have

$$N(X)=\left[\widetilde{h},f(\widetilde{h}),\frac{\nu\widetilde{h}+f(\widetilde{h})}{2a_1b_1/\gamma}\right] \tag{3.1.42}$$

where $\nu\in(\mathbb{Z}/(2a_1b_1/\gamma))^*$ (according to Proposition 3.1.2). We denote

$$\widetilde{h}^*=\frac{\widetilde{h}}{2a_1b_1},\quad f(\widetilde{h})^*=\frac{\gamma f(\widetilde{h})}{2a_1b_1\delta}. \tag{3.1.43}$$

By (3.1.19), we have

$$u^*(\widetilde{h})=\frac{\nu^{-1}f(\widetilde{h})}{2a_1b_1/\gamma}+\mathbb{Z}f(\widetilde{h})=\nu^{-1}\delta f(\widetilde{h})^*+\mathbb{Z}f(\widetilde{h})\,. \tag{3.1.44}$$

There exists a unique (up to ±1) embedding

$$\phi:K(\widetilde{H})=\mathbb{Z}f(\widetilde{H})\to K(\widetilde{h})=\mathbb{Z}f(\widetilde{h}),\quad \phi(f(\widetilde{H}))=\pm cf(\widetilde{h}). \tag{3.1.45}$$

of one-dimensional lattices. Its dual is defined by $\phi^*(f(\widetilde{h})^*)=\pm cf(\widetilde{H})^*$.

We have

$$\phi^*(K(\widetilde{h})) = \mathbb{Z}\phi^*(f(\widetilde{h})) = \mathbb{Z}f(\widetilde{H})/c =$$

$$[K(\widetilde{H}),\ (2a_1b_1c/\gamma)u^*(\widetilde{H})] = [K(\widetilde{H}),\ (2abc/(d^2\gamma))u^*(\widetilde{H})]$$

because of (3.1.39). This gives the first part of (b) in Theorem 2.3.3.

We have

$$\phi^*(u^*(\widetilde{h})) + \phi^*(K(\widetilde{h})) = \nu^{-1}\delta\phi^*(f(\widetilde{h})^*) + \mathbb{Z}\phi^*(f(\widetilde{h})) = \pm\nu^{-1}\delta cf(\widetilde{H})^* + \mathbb{Z}f(\widetilde{H})/c =$$

$$\pm\nu^{-1}\delta cf(\widetilde{H})^* + \mathbb{Z}(2a_1b_1c\delta/\gamma)f(\widetilde{H})^*.$$

On the other hand, by (3.1.39)

$$m(a,b)cu^*(\widetilde{H}) + \mathbb{Z}(2a_1b_1c\delta/\gamma)f(\widetilde{H})^* = m(a,b)\mu^{-1}\delta cf(\widetilde{H})^* + \mathbb{Z}(2a_1b_1c\delta/\gamma)f(\widetilde{H})^*.$$

Thus, by (3.1.44), second part of (b) in Theorem 2.3.3 is $\pm\nu^{-1} \equiv m(a,b)\mu^{-1} \mod 2a_1b_1/\gamma$. Equivalently,

$$\nu \equiv \pm m(a,b)\mu \mod \frac{2a_1b_1}{\gamma}. \tag{3.1.46}$$

Thus, for ν given by (3.1.46) one has (this is the definition of ν)

$$\frac{\nu\widetilde{h} + f(\widetilde{h})}{2a_1b_1/\gamma} = \left(\frac{\nu x + (\delta y/\gamma)}{2a_1b_1/\gamma}\widetilde{H} + \frac{x+\nu y}{2a_1b_1/\gamma}f(\widetilde{H})\right)/(2a_1b_1c^2/\gamma) \in N(X). \tag{3.1.47}$$

This is equivalent for $\nu \equiv m(a,b)\mu \mod 2a_1b_1/\gamma$ to

$$\left\{\begin{array}{l} \pm m(a,b)\mu x + (\delta y/\gamma) \equiv 0 \mod 2a_1b_1/\gamma \\ x \pm m(a,b)\mu y \equiv 0 \mod 2a_1b_1/\gamma \\ \pm m(a,b)\mu x + (\delta y/\gamma) \equiv \mu(x \pm m(a,b)\mu y) \mod (2a_1b_1c^2/\gamma)(2a_1b_1/\gamma) \end{array}\right. . \tag{3.1.48}$$

This gives (ii).

Let us consider the condition (c) of Theorem 2.3.3. For a choice of $\beta = \pm 1$, one has

$$m(a,b)n\widetilde{h}^* + z^* \equiv \beta(cn\widetilde{H}^* + \phi^*(z^*)) \mod N(X), \quad n \in \mathbb{Z},\ z \in K(\widetilde{h})^* \tag{3.1.49}$$

if

$$cn\widetilde{H}^* + \phi^*(z^*) \in N(X)^*. \tag{3.1.50}$$

Let $z^* = kf(\widetilde{h})^*$, $k \in \mathbb{Z}$, then $\phi^*(kf(\widetilde{h})^*) = \pm kcf(\widetilde{H})^*$ and $cn\widetilde{H}^* + \phi^*(z^*) = cn\widetilde{H}^* \pm kcf(\widetilde{H})^*$. By (3.1.15), $cn\widetilde{H}^* \pm kcf(\widetilde{H})^* \in N(X)^*$, if and only if $\mu cn \mp kc \equiv 0$

mod $2a_1b_1c^2/\gamma$. This is equivalent to $k \equiv \pm\mu n \mod 2a_1b_1c/\gamma$. Like in (3.1.17), we get $k = \pm\mu n + (2a_1b_1c/\gamma)t$ where $n,\ t \in \mathbb{Z}$. We get

$$cn\widetilde{H}^* \pm kcf(\widetilde{H})^* = cn(\widetilde{H}^* + \mu f(\widetilde{H})^*) \pm t(2a_1b_1c^2/\gamma)f(\widetilde{H})^*. \tag{3.1.51}$$

Thus, it is enough to check

$$m(a,b)n\widetilde{h}^* + kf(\widetilde{h})^* \equiv \beta(cn\widetilde{H}^* \pm kcf(\widetilde{H})^*) \mod N(X), \tag{3.1.52}$$

where $(n,k) = (1, \pm\mu)$ or $(n,k) = (0, 2a_1b_1c/\gamma)$. Thus, one should check for one of $\beta = \pm 1$ that

$$m(a,b)\widetilde{h}^* \pm \mu f(\widetilde{h})^* \equiv \beta(c\widetilde{H}^* + c\mu f(\widetilde{H})^*) \mod N(X) \tag{3.1.53}$$

and

$$\frac{2a_1b_1cf(\widetilde{h})^*}{\gamma} \equiv \pm\beta\frac{2a_1b_1c^2f(\widetilde{H})^*}{\gamma} \mod N(X). \tag{3.1.54}$$

We have

$$m(a,b)n\widetilde{h}^* + kf(\widetilde{h})^* = m(a,b)n\frac{\widetilde{h}}{2a_1b_1} + k\frac{\gamma f(\widetilde{h})}{2a_1b_1\delta} =$$

$$\frac{1}{2a_1b_1\delta}\left(\delta m(a,b)n\widetilde{h} + \gamma kf(\widetilde{h})\right) =$$

$$\frac{1}{2a_1b_1\delta}\left(\delta m(a,b)n\frac{x\widetilde{H} + yf(\widetilde{H})}{2a_1b_1c^2/\gamma} + \gamma k\frac{(\delta y/\gamma)\widetilde{H} + xf(\widetilde{H})}{2a_1b_1c^2/\gamma}\right) =$$

$$\frac{1}{2a_1b_1\delta}\left(\delta m(a,b)n(\gamma x\widetilde{H}^* + \delta yf(\widetilde{H})^*) + \gamma k(\delta y\widetilde{H}^* + \delta xf(\widetilde{H})^*\right) =$$

$$\frac{m(a,b)n\gamma x + \gamma ky}{2a_1b_1}\widetilde{H}^* + \frac{\delta m(a,b)ny + \gamma kx}{2a_1b_1}f(\widetilde{H})^*. \tag{3.1.55}$$

Thus, (3.1.53) is

$$\left(\frac{m(a,b)\gamma x \pm \mu\gamma y}{2a_1b_1} - \beta c\right)\widetilde{H}^* + \left(\frac{\delta m(a,b)y \pm \mu\gamma x}{2a_1b_1} - \beta\mu c\right)f(\widetilde{H})^* \in N(X) \tag{3.1.56}$$

and (3.1.54) is

$$cy\widetilde{H}^* + \left(cx - (\pm\beta)\frac{2a_1b_1c^2}{\gamma}\right)f(\widetilde{H})^* \in N(X). \tag{3.1.57}$$

Here one should take $+$, if (x,y) belongs to a-series, and one should take $-$, if (x,y) belongs to b-series.

By (3.1.16), we can reformulate (3.1.56) as (3.1.34) in (iii), and we can reformulate (3.1.57) as (3.1.35) in (iii).

This finishes the proof.

Now we analyse conditions of Theorem 3.1.3. The most important are congruences $\mod \delta$ since δ is not bounded by a constant depending on (r, s).

Let us consider the congruence $cx - (\pm\beta)(2a_1b_1c^2/\gamma) \equiv 0 \mod \delta$ in (3.1.35). We know that $\delta \equiv \gamma\mu^2 \mod (2a_1b_1/\gamma)c^2$ where $\mu \mod (2a_1b_1/\gamma)c^2$ is invertible, $\gamma|2a_1b_1$ and g.c.d$(\gamma, c) = 1$. It follows that g.c.d$(c, \delta) = 1$. Thus, the considered congruence is equivalent to

$$x \equiv \pm\frac{2a_1b_1c}{\gamma} \mod \delta \tag{3.1.58}$$

since $\beta = \pm 1$. Later we shall see that all congruences $\mod \delta$ which follow from conditions of Theorem 3.1.3 are consequences of (3.1.58). Thus, (3.1.58) is the most important condition of Theorem 3.1.3.

Let us consider all integral (x, y) which satisfy (3.1.28) and (3.1.58), i. e.

$$\gamma x^2 - \delta y^2 = \frac{4a_1^2b_1^2c^2}{\gamma} \text{ and } x \equiv \pm\frac{2a_1b_1c}{\gamma} \mod \delta. \tag{3.1.59}$$

We apply the main trick used in [MN1], [MN2] and [N4]. Considering $\pm(x, y)$, we can assume that $x \equiv 2a_1b_1c/\gamma \mod \delta$, i. e. $x = 2a_1b_1c/\gamma - k\delta$ where $k \in \mathbb{Z}$. We have $\gamma^2x^2 = 4a_1^2b_1^2c^2 - 4a_1b_1ck\delta\gamma + \gamma^2k^2\delta^2 = \gamma\delta y^2 + 4a_1^2b_1^2c^2$. It follows

$$\delta = \frac{y^2 + 4a_1b_1ck}{\gamma k^2}. \tag{3.1.60}$$

Consider a prime l such that $l \not| 2a_1b_1c$. Assume that $l^{2t+1}|k$, but $l^{2t+2} \not| k$. We have $k|y^2$. Then $l^{2t+1}|y^2$. Then $l^{2t+2}|y^2$. Since $k^2|y^2 + 4a_1b_1ck$, it follows that $l^{2t+2}|y^2 + 4a_1b_1ck$ We then get $l^{2t+2}|k$. We get a contradiction. It follows that $k = -\alpha q^2$ where $q \in \mathbb{Z}$ (we can additionally assume that $q \geq 0$), $\alpha|2a_1b_1c$ and α is square-free. Remark that α can be negative.

From (3.1.60), we get

$$\delta = \frac{y^2 - 4a_1b_1c\alpha q^2}{\gamma\alpha^2q^4}. \tag{3.1.61}$$

It follows $\alpha q|y$ and $y = \alpha pq$ where $p \in \mathbb{Z}$. From (3.1.61),

$$\delta = \frac{p^2 - 4a_1b_1c/\alpha}{\gamma q^2}. \tag{3.1.62}$$

Equivalently,

$$p^2 - \gamma\delta q^2 = \frac{4a_1b_1c}{\alpha}. \tag{3.1.63}$$

If integral p, q satisfy (3.1.63), we have

$$(x,\,y)=\pm\left(\frac{2a_1b_1c}{\gamma}+\alpha\delta q^2,\,\alpha pq\right). \tag{3.1.64}$$

satisfy (3.1.59). We call solutions (3.1.64) of (3.1.59) as associated solutions. Thus, we get the very important for us

Theorem 3.1.4. *All integral solutions (x,y) of*

$$\gamma x^2-\delta y^2=\frac{4a_1^2b_1^2c^2}{\gamma}\quad \textit{and } x\equiv\pm\frac{2a_1b_1c}{\gamma}\mod \delta \tag{3.1.65}$$

are associated solutions

$$(x,\,y)=\pm\left(\frac{2a_1b_1c}{\gamma}+\alpha\delta q^2,\,\alpha pq\right) \tag{3.1.66}$$

to integral solutions α, (p,q) of

$$\alpha|2a_1b_1c\ \textit{where } \alpha \textit{ is square-free, and }\ p^2-\gamma\delta q^2=\frac{4a_1b_1c}{\alpha}. \tag{3.1.67}$$

Any solution (x,y) of (3.1.65) *can be written in the form* (3.1.66) *where α, (p,q) is solution of* (3.1.67)*. Any solution of* (3.1.67) *gives a solution* (3.1.66) *of* (3.1.65)*.*

Solutions of (3.1.65) *and* (3.1.67) *are in one-to-one correspondence if we additionally assume that $q\ge 0$.*

Now we can write (x,y) of Theorem 3.1.3 in the form (3.1.66) as associated solutions to (3.1.67). Putting that (x,y) to relations (i)—(v) of Theorem 3.1.3, we get some relations on α and (p,q). They are a finite number of congruences mod N_i where N_i depend only on (r,s) (or (a,b,c)). All N_i are bounded by functions depending only on (r,s). These congruences have a lot of relations between them and with (3.1.67). All together they give many very strong restrictions on α and (p,q) (except (3.1.67)). Unfortunately, it is hard to analyse all these relations and restrictions completely. In this paper, we were able to analyse only the most important these relations and restrictions. We present them below.

We fix

$$\mu\in\left(\mathbb{Z}/(2a_1b_1c^2/\gamma)\right)^* \tag{3.1.68}$$

and consider $\delta\in\mathbb{N}$ such that

$$\delta\equiv\mu^2\gamma\mod \frac{4a_1b_1c^2}{\gamma} \tag{3.1.69}$$

The relation (3.1.29) in (i) of Theorem 3.1.3 is equivalent to

$$2a_1b_1c+\alpha\gamma\delta q^2\equiv\gamma\mu\alpha pq\mod 2a_1b_1c^2. \tag{3.1.70}$$

By (3.1.67), we have

$$-4a_1b_1c + \alpha p^2 - \alpha\gamma\delta q^2 = 0. \tag{3.1.71}$$

Taking sum, we get

$$2a_1b_1c \equiv \alpha p(p - \mu\gamma q) \mod 2a_1b_1c^2. \tag{3.1.72}$$

The relation (3.1.72) is equivalent to (3.1.29). Taking 2(3.1.70)+(3.1.71), we get

$$\alpha p^2 + \alpha\gamma\delta q^2 \equiv 2\gamma\mu\alpha pq \mod 4a_1b_1c^2 \tag{3.1.73}$$

which is also equivalent to (3.1.29). From (3.1.69), we get

$$\alpha p^2 + \alpha\mu^2\gamma^2 q^2 \equiv 2\gamma\mu\alpha pq \mod \frac{4a_1b_1c^2}{\gamma} \tag{3.1.74}$$

and

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \mod \frac{4a_1b_1c^2}{\gamma}. \tag{3.1.75}$$

It is the corollary $\mod 4a_1b_1c^2/\gamma$ of the congruence (3.1.73). Since $\gamma|2a_1b_1$, it follows $\alpha(p-\mu\gamma q)^2 \equiv 0 \mod 2c^2$. Since α is square-free, it follows

$$c|(p - \mu\gamma q). \tag{3.1.76}$$

From (3.1.72), we get

$$2a_1b_1 \equiv \alpha p\frac{p - \mu\gamma q}{c} \mod 2a_1b_1c \tag{3.1.77}$$

where $(p - \mu\gamma q)/c$ is an integer. It follows, $\alpha|2a_1b_1$. Thus, we get

Proposition 3.1.5. *The condition* (3.1.29) *of Theorem 3.1.3 is equivalent to*

$$\alpha p^2 + \alpha\gamma\delta q^2 \equiv 2\gamma\mu\alpha pq \mod 4a_1b_1c^2. \tag{3.1.78}$$

From (3.1.78) *we get*

$$\alpha|2a_1b_1,\ c|(p - \mu\gamma q),\ \textit{and}\ \alpha(p - \mu\gamma q)^2 \equiv 0 \mod \frac{4a_1b_1c^2}{\gamma}. \tag{3.1.79}$$

The (3.1.78) *is also equivalent to*

$$2a_1b_1c \equiv \alpha p(p - \mu\gamma q) \mod 2a_1b_1c^2. \tag{3.1.80}$$

Considering the condition (3.1.30) of Theorem 3.1.3, we similarly get

Proposition 3.1.6. *The condition* (3.1.30) *of Theorem 3.1.3 is equivalent to*

$$\mu\alpha p^2 + \mu\alpha\gamma\delta q^2 \equiv 2\alpha\delta pq \mod 4a_1b_1c^2. \tag{3.1.81}$$

Congruences (3.1.78) *and* (3.1.81) *are equivalent* $\mod (4a_1b_1c^2/\gamma)$*, and* (3.1.81) *is equivalent to* (3.1.78) *together with*

$$2\alpha pq(\delta - \gamma\mu^2) \equiv 0 \mod 4a_1b_1c^2 \tag{3.1.82}$$

where $\delta - \gamma\mu^2 \equiv 0 \mod 4a_1b_1c^2/\gamma$*.*

Proof. To get (3.1.82), consider μ (3.1.78) - (3.1.81).

Now consider (3.1.31) of Theorem 3.1.3. Let us consider the a-series (the sign $+$). Since $m(a,b) \equiv -1 \mod 2a_1/(\gamma_2\gamma_a)$, we get $-\mu x + (\delta y/\gamma) \equiv 0 \mod 2a_1/(\gamma_2\gamma_a)$. It is satisfied because of (3.1.30). Since $m(a,b) \equiv 1 \mod 2b_1/(\gamma_2\gamma_b)$, we get $\mu x + (\delta y/\gamma) \equiv 0 \mod 2b_1/(\gamma_2\gamma_b)$. By (3.1.30), we get $\mu x - (\delta y/\gamma) \equiv 0 \mod 2b_1/(\gamma_2\gamma_b)$. Thus, we get $2\mu x \equiv 0 \mod 2b_1/(\gamma_2\gamma_b)$. If $\gamma_2 = 1$, this is equivalent to $\mu x \equiv 0 \mod b_1/\gamma_b$ and $x \equiv 0 \mod b_1/\gamma_b$. If $\gamma_2 = 2$, then b_1/γ_b is odd, and we get $2\mu x \equiv 0 \mod (b_1/\gamma_b)$ which is equivalent to $x \equiv 0 \mod b_1/\gamma_b$. Thus, at any case we get $(b_1/\gamma_b)|x$, equivalently, $x \equiv 0 \mod (b_1/\gamma_b)$. By (3.1.66), this is equivalent to $\alpha\delta q^2 \equiv 0 \mod (b_1/\gamma_b)$. We have $\delta \equiv \mu^2\gamma \mod (b_1/\gamma_b)$ and $\mu \mod (b_1/\gamma_b)$ is invertible. Thus, we get $\alpha\gamma q^2 \equiv 0 \mod (b_1/\gamma_b)$ which is equivalent to $\alpha(\gamma_b q)^2 \equiv 0 \mod b_1$.

Let us consider (3.1.32) of Theorem 3.1.3. We consider the a-series. We get $x - \mu y \equiv 0 \mod 2a_1/(\gamma_2\gamma_a)$. It satisfies because of (3.1.29). We get $x + \mu y \equiv 0 \mod 2b_1/(\gamma_2\gamma_b)$. Using (3.1.29), we similarly get that this is equivalent to $(b_1/\gamma_b)|x$.

Thus, we finally get

Proposition 3.1.7. *The conditions* (3.1.31) *and* (3.1.32) *in (ii) of Theorem 3.1.3 are equivalent to*

$$x \equiv 0 \mod \frac{b_1}{\gamma_b} \tag{3.1.83}$$

or to

$$\alpha(\gamma_b q)^2 \equiv 0 \mod b_1 \tag{3.1.84}$$

for a-series (the sign $+$), and it is equivalent to

$$x \equiv 0 \mod \frac{a_1}{\gamma_a} \tag{3.1.85}$$

or to

$$\alpha(\gamma_a q)^2 \equiv 0 \mod a_1 \tag{3.1.86}$$

for b-series (the sign $-$).

Consider (3.1.33) in Theorem 3.1.3. We consider the a-series. We get $\bmod 2a_1/(\gamma_2\gamma_a)$ that

$$2\mu\gamma x \equiv (2\delta + (\mu^2\gamma - \delta))y \mod (2a_1b_1c^2)\,(2a_1/(\gamma_2\gamma_a))\,.$$

We get $\bmod 2b_1/(\gamma_2\gamma_b)$ that

$$(\delta - \mu^2\gamma)y \equiv 0 \mod (2a_1b_1c^2)(2b_1/\gamma_2\gamma_b).$$

Using (3.1.66) (and (3.1.71) to make the relations homogeneous), we finally get

Proposition 3.1.8. *The condition* (3.1.33) *of Theorem 3.1.3 is equivalent to*

$$2\mu\gamma x \equiv (\delta + \mu^2\gamma)y \mod (2a_1b_1c^2)\,(2a_1/(\gamma_2\gamma_a)) \tag{3.1.87}$$

and

$$(\delta - \mu^2\gamma)y \equiv 0 \mod (2a_1b_1c^2)\,(2b_1/(\gamma_2\gamma_b)) \tag{3.1.88}$$

or

$$\mu\alpha p^2 - \alpha(\delta + \mu^2\gamma)pq + \mu\alpha\gamma\delta q^2 \equiv 0 \mod (2a_1b_1c^2)\,(2a_1/(\gamma_2\gamma_a)) \tag{3.1.89}$$

and

$$\alpha(\delta - \mu^2\gamma)pq \equiv 0 \mod (2a_1b_1c^2)\,(2b_1/(\gamma_2\gamma_b)) \tag{3.1.90}$$

for the a-series (the sign $+$), and it is equivalent to

$$2\mu\gamma x \equiv (\delta + \mu^2\gamma)y \mod (2a_1b_1c^2)\,(2b_1/(\gamma_2\gamma_b)) \tag{3.1.91}$$

and

$$(\delta - \mu^2\gamma)y \equiv 0 \mod (2a_1b_1c^2)\,(2a_1/(\gamma_2\gamma_a)) \tag{3.1.92}$$

or

$$\mu\alpha p^2 - \alpha(\delta + \mu^2\gamma)pq + \mu\alpha\gamma\delta q^2 \equiv 0 \mod (2a_1b_1c^2)\,(2b_1/(\gamma_2\gamma_b)) \tag{3.1.93}$$

and

$$\alpha(\delta - \mu^2\gamma)pq \equiv 0 \mod (2a_1b_1c^2)\,(2a_1/(\gamma_2\gamma_a)) \tag{3.1.94}$$

for the b-series (the sign $-$).

Consider the condition (3.1.35) of Theorem 3.1.3. Consider the a-series (the sign $+$). Second condition in (3.1.35) gives $x \equiv \beta(2a_1b_1c/\gamma) \mod \delta$. By (3.1.66), we get $(x,y) = \beta(2a_1b_1c/\gamma + \alpha\delta q^2,\ \alpha pq)$. Then third condition in (3.1.35) is equivalent to

$$\beta\delta\alpha pq \equiv \mu(\beta 2a_1b_1c + \beta\alpha\gamma\delta q^2 - \beta 2a_1b_1c) \mod 2a_1b_1c\delta.$$

This is equivalent to $\alpha q(p - \mu\gamma q) \equiv 0 \mod 2a_1b_1c$. Since $\gamma | 2a_1b_1c$, it follows the first condition in (3.1.35) which is $c\delta\beta\alpha pq \equiv 0 \mod \gamma$.

For b-series we get the same. Thus, we have

Proposition 3.1.9. *The condition* (3.1.35) *of Theorem 3.1.3 is equivalent to*

$$\alpha q(p-\mu\gamma q)\equiv 0 \mod 2a_1b_1c. \tag{3.1.95}$$

Consider (3.1.34) of Theorem 3.1.3. We consider the a-series. Then

$$(x,y)=\beta(\frac{2a_1b_1c}{\gamma}+\alpha\delta q^2,\ \alpha pq) \tag{3.1.96}$$

The first relation of (3.1.34) gives

$$m(a,b)\gamma x+\mu\gamma y\equiv 2a_1b_1\beta c \mod 2a_1b_1\gamma.$$

Using (3.1.96), we get

$$m(a,b)\alpha\gamma\delta q^2+\mu\alpha\gamma pq\equiv 2a_1b_1c(1-m(a,b)) \mod 2a_1b_1\gamma.$$

This is equivalent to

$$-\alpha\delta q^2+\mu\alpha pq\equiv \frac{4a_1b_1c}{\gamma} \mod 2a_1$$

and

$$\alpha\delta q^2+\mu\alpha pq\equiv 0 \mod 2b_1.$$

Using (3.1.71), we can rewrite the first relation in the homogeneous form

$$\alpha p(p-\mu\gamma q)\equiv 0 \mod 2a_1\gamma.$$

The second relation of (3.1.34) gives

$$\delta m(a,b)y+\mu\gamma x-2a_1b_1\beta\mu c\equiv 0 \mod 2a_1b_1\delta.$$

By (3.1.96) it is equivalent to

$$m(a,b)\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0 \mod 2a_1b_1.$$

This is equivalent to

$$-\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0 \mod 2a_1$$

and

$$\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0 \mod 2b_1.$$

The third relation in (3.1.34) is

$$\begin{aligned}&\delta\left(m(a,b)\gamma x+\mu\gamma y-2\beta a_1b_1c\right)\equiv\\&\mu\gamma\left(\delta m(a,b)y+\mu\gamma x-2\mu\beta a_1b_1c\right) \mod (2a_1b_1c^2\delta)(2a_1b_1).\end{aligned}$$

Using (3.1.96), one can calculate that it is equivalent to

$$\mu\alpha\gamma pq\,(m(a,b)-1)+\alpha\gamma q^2\left(\gamma\mu^2-m(a,b)\delta\right)$$
$$\equiv 2a_1b_1c\,(m(a,b)-1)\mod (2a_1b_1c^2)(2a_1b_1).$$

This is equivalent to

$$4a_1b_1c\equiv\alpha\gamma q\left(2\mu p-(\gamma\mu^2+\delta)q\right)\mod (2a_1b_1c^2)(2a_1)$$

and

$$\alpha\gamma q^2(\delta-\gamma\mu^2)\equiv 0\mod (2a_1b_1c^2)(2b_1).$$

Using (3.1.71), we can rewrite the first relation in the homogeneous form

$$\alpha(p-\mu\gamma q)^2\equiv 0\mod (2a_1b_1c^2)(2a_1).$$

Thus, we get

Proposition 3.1.10. *The condition* (3.1.34) *of Theorem 3.1.3 is equivalent to the system of congruences:*
For the a-series (the sign +)

$$\alpha p(p-\mu\gamma q)\equiv 0\mod 2a_1\gamma,\tag{3.1.97}$$
$$\alpha\delta q^2+\alpha\mu pq\equiv 0\mod 2b_1,\tag{3.1.98}$$
$$-\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0\mod 2a_1,\tag{3.1.99}$$
$$\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0\mod 2b_1,\tag{3.1.100}$$
$$\alpha(p-\mu\gamma q)^2\equiv 0\mod (2a_1b_1c^2)(2a_1),\tag{3.1.101}$$
$$\alpha\gamma q^2(\delta-\gamma\mu^2)\equiv 0\mod (2a_1b_1c^2)(2b_1).\tag{3.1.102}$$

For the b-series (the sign −):

$$\alpha p(p-\mu\gamma q)\equiv 0\mod 2b_1\gamma,\tag{3.1.103}$$
$$\alpha\delta q^2+\alpha\mu pq\equiv 0\mod 2a_1,\tag{3.1.104}$$
$$-\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0\mod 2b_1,\tag{3.1.105}$$
$$\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0\mod 2a_1,\tag{3.1.106}$$
$$\alpha(p-\mu\gamma q)^2\equiv 0\mod (2a_1b_1c^2)(2b_1),\tag{3.1.107}$$
$$\alpha\gamma q^2(\delta-\gamma\mu^2)\equiv 0\mod (2a_1b_1c^2)(2a_1).\tag{3.1.108}$$

Consider the condition (iv) of Theorem 3.1.3. It means that the corresponding element $\widetilde{h}$ is primitive. Since $\widetilde{h}^2=2a_1b_1$ and the lattice $N(X)$ is even, it is not valid only if $\widetilde{h}/l\in N(X)$ for some prime l such that $l^2|a_1b_1$. Thus, (3.1.36) is not valid if and only if

$$x\equiv y\equiv\frac{x-\mu y}{2a_1b_1c^2/\gamma}\equiv 0\mod l$$

for some prime l such that $l^2|a_1b_1$. Using (3.1.66) and (3.1.71), we get

Proposition 3.1.11. *The condition (iv) of Theorem 3.1.3 is equivalent to the non-existence of a prime l such that $l^2|a_1b_1$ and*

$$x \equiv y \equiv \frac{x-\mu y}{2a_1b_1c^2/\gamma} \equiv 0 \mod l. \tag{3.1.109}$$

Equivalently, the system of congruences

$$\begin{cases} \alpha p^2 + \alpha\gamma\delta q^2 \equiv 0 \mod 2\gamma l \\ \alpha pq \equiv 0 \mod l \\ \alpha p^2 + \alpha\gamma\delta q^2 \equiv 2\alpha\gamma\mu pq \mod 4a_1b_1c^2 l \end{cases} \tag{3.1.110}$$

is not satisfied for any prime l such that $l^2|a_1b_1$.

Consider the condition (v) of Theorem 3.1.3. This is equivalent to $\gamma(\widetilde{h}) = \gamma$ where $\widetilde{h}\cdot N(X) = \gamma(\widetilde{h})\mathbb{Z}$. All other conditions of Theorem 3.1.3 give that $\gamma|\widetilde{h}\cdot N(X)$, and $\gamma|\gamma(\widetilde{h})$. Since $\widetilde{h}^2 = 2a_1b_1$, it follows that $(\gamma(\widetilde{h})/\gamma)\,|\,2a_1b_1/\gamma$. Equivalently, (v) is not satisfied, if and only if for some prime $l|2a_1b_1/\gamma$ one has

$$x \equiv \frac{\delta y}{\gamma} \equiv \frac{\mu\gamma x - \delta y}{2a_1b_1c^2} \equiv 0 \mod l.$$

Using (3.1.66) and (3.1.71), we get

Proposition 3.1.12. *The condition (v) of Theorem 3.1.3 is equivalent to the non-existence of a prime $l\,|\,(2a_1b_1/\gamma)$ such that*

$$x \equiv \frac{\delta y}{\gamma} \equiv \frac{\mu\gamma x - \delta y}{2a_1b_1c^2} \equiv 0 \mod l. \tag{3.1.111}$$

Equivalently, the system of congruences

$$\begin{cases} \alpha p^2 + \alpha\gamma\delta q^2 \equiv 0 \mod 2\gamma l \\ \delta\alpha pq \equiv 0 \mod \gamma l \\ \mu\alpha p^2 + \mu\alpha\gamma\delta q^2 \equiv 2\alpha\delta pq \mod 4a_1b_1c^2 l \end{cases} \tag{3.1.112}$$

is not satisfied for any prime $l\,|\,(2a_1b_1/\gamma)$.

Now we collect analysed conditions of Theorem 3.1.3 all together. We divide them in *general conditions: (G)* which are valid for both a and b-series, *conditions (A)* which are valid for the a-series, and *conditions (B)* which are valid for the b-series.

(G): General conditions

$$\alpha p^2+\alpha\gamma\delta q^2\equiv 2\gamma\mu\alpha pq \mod 4a_1b_1c^2, \tag{3.1.113}$$

$$\mu\alpha p^2+\mu\alpha\gamma\delta q^2\equiv 2\alpha\delta pq \mod 4a_1b_1c^2, \tag{3.1.114}$$

$$\alpha q(p-\mu\gamma q)\equiv 0 \mod 2a_1b_1c, \tag{3.1.115}$$

$$\begin{cases} \alpha p^2+\alpha\gamma\delta q^2\equiv 0 \mod 2\gamma l\\ \alpha pq\equiv 0 \mod l\\ \alpha p^2+\alpha\gamma\delta q^2\equiv 2\alpha\gamma\mu pq \mod 4a_1b_1c^2l \end{cases} \tag{3.1.116}$$

is not satisfied for any prime l such that $l^2|a_1b_1$,

$$\begin{cases} \alpha p^2+\alpha\gamma\delta q^2\equiv 0 \mod 2\gamma l\\ \delta\alpha pq\equiv 0 \mod \gamma l\\ \mu\alpha p^2+\mu\alpha\gamma\delta q^2\equiv 2\alpha\delta pq \mod 4a_1b_1c^2l \end{cases} \tag{3.1.117}$$

is not satisfied for any prime $l\,|\,(2a_1b_1/\gamma)$.

(A): Conditions of the a-series

$$\alpha(\gamma_a q)^2\equiv 0 \mod b_1, \tag{3.1.118}$$

$$\mu\alpha p^2-\alpha(\delta+\mu^2\gamma)pq+\mu\alpha\gamma\delta q^2\equiv 0 \mod (2a_1b_1c^2)\,(2a_1/(\gamma_2\gamma_a)) \tag{3.1.119}$$

$$\alpha(\delta-\mu^2\gamma)pq\equiv 0 \mod (2a_1b_1c^2)\,(2b_1/(\gamma_2\gamma_b))\,, \tag{3.1.120}$$

$$\alpha p(p-\mu\gamma q)\equiv 0 \mod 2a_1\gamma, \tag{3.1.121}$$

$$\alpha\delta q^2+\alpha\mu pq\equiv 0 \mod 2b_1, \tag{3.1.122}$$

$$-\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0 \mod 2a_1, \tag{3.1.123}$$

$$\alpha pq+\alpha\mu\gamma\delta q^2\equiv 0 \mod 2b_1, \tag{3.1.124}$$

$$\alpha(p-\mu\gamma q)^2\equiv 0 \mod (2a_1b_1c^2)(2a_1), \tag{3.1.125}$$

$$\alpha\gamma q^2(\delta-\gamma\mu^2)\equiv 0 \mod (2a_1b_1c^2)(2b_1). \tag{3.1.126}$$

(B): Conditions of the b-series

$$\alpha(\gamma_a q)^2\equiv 0 \mod a_1, \tag{3.1.127}$$

$$\mu\alpha p^2-\alpha(\delta+\mu^2\gamma)pq+\mu\alpha\gamma\delta q^2\equiv 0 \mod (2a_1b_1c^2)\,(2b_1/(\gamma_2\gamma_b)) \tag{3.1.128}$$

$$\alpha(\delta-\mu^2\gamma)pq\equiv 0 \mod (2a_1b_1c^2)\,(2a_1/(\gamma_2\gamma_a))\,, \tag{3.1.129}$$

$$\alpha p(p-\mu\gamma q)\equiv 0 \mod 2b_1\gamma, \tag{3.1.130}$$

$$\alpha\delta q^2+\alpha\mu pq\equiv 0 \mod 2a_1, \tag{3.1.131}$$

$$-\alpha pq + \alpha\mu\gamma\delta q^2 \equiv 0 \mod 2b_1, \tag{3.1.132}$$

$$\alpha pq + \alpha\mu\gamma\delta q^2 \equiv 0 \mod 2a_1, \tag{3.1.133}$$

$$\alpha(p-\mu\gamma q)^2 \equiv 0 \mod (2a_1b_1c^2)(2b_1), \tag{3.1.134}$$

$$\alpha\gamma q^2(\delta-\gamma\mu^2) \equiv 0 \mod (2a_1b_1c^2)(2a_1). \tag{3.1.135}$$

Now we can formulate the main results which follows from Theorem 3.1.3 and the calculations above.

Theorem 3.1.13. *Let X be a K3 surface with $\rho(X)=2$, and H a polarization of X of degree $H^2=2rs$, $r,\ s\geq 1$. Assume that the Mukai vector (r,H,s) is primitive. Let Y be the moduli space of sheaves on X with the isotropic Mukai vector $v=(r,H,s)$ and the canonical nef element $h=(-a,0,b) \mod \mathbb{Z}v$. Let $\widetilde{H}=H/d$ be the corresponding primitive polarization with $\widetilde{H}\cdot N(X)=\gamma\mathbb{Z}$. We denote by μ the invariant of the pair $(N(X),\widetilde{H})$ and use notations of Proposition 3.1.1.*

We have $Y\cong X$, if

$$g.c.d(c,d\gamma)=1$$

and there exists a square-free $\alpha\,|\,2a_1b_1$ such that the equation

$$p^2-\gamma\delta q^2=\frac{4a_1b_1c}{\alpha} \tag{3.1.136}$$

has an integral solution (p,q) which satisfies the general condition (G) and one of conditions (A) ((p,q) belongs to the a-series) or (B) ((p,q) belongs to the b-series) above.

These conditions are necessary to have $Y\cong X$, if X is a general K3 surface with $\rho(X)=2$, i. e. the automorphism group of the transcendental periods $(T(X),H^{2,0}(X))$ is ± 1.

Solutions (3.1.136) *give all solution (x,y) of Theorem 3.1.3 as associated solutions*

$$(x,y)=\pm(\frac{2a_1b_1c}{\gamma}+\alpha\delta q^2,\ \alpha pq). \tag{3.1.137}$$

Now we want to interpret solutions (p,q) of Theorem 3.1.13 as elements of the Picard lattice $N(X)$.

Let (p,q) be a solution of (3.1.136). Then $p^2-\gamma\delta q^2=4a_1b_1c/\alpha$ and

$$\gamma p^2-\delta(\gamma q)^2=\frac{4a_1b_1c\gamma}{\alpha}. \tag{3.1.138}$$

Assume that (p,q) satisfies the condition (A) (i.e. it belongs to the a-series). By (3.1.125), $\alpha(p-\mu\gamma q)^2\equiv 0 \mod 4a_1^2c^2$. Since α is square-free, it follows that

$$p-\mu\gamma q\equiv 0 \mod 2a_1c. \tag{3.1.139}$$

Let us put
$$t = b_1c. \tag{3.1.140}$$
Then
$$tp - t\mu q \equiv 0 \mod 2a_1b_1c^2. \tag{3.1.141}$$
It follows, $tp - t\mu q \equiv 0 \mod 2a_1b_1c^2/\gamma$ and
$$\widetilde{h}_1 = t\frac{p\widetilde{H} + \gamma q f(\widetilde{H})}{2a_1b_1c^2/\gamma} \in N(X). \tag{3.1.142}$$
We have
$$\widetilde{h}_1^2 = \frac{t^2(\gamma p^2 - \delta(\gamma q)^2)}{2a_1b_1c^2/\gamma} = \frac{t^2 4a_1b_1c\gamma}{2\alpha a_1b_1c^2/\gamma} = \frac{2t^2\gamma^2}{\alpha c} = \frac{2\gamma^2 b_1^2 c}{\alpha}. \tag{3.1.143}$$
We have
$$\widetilde{H}\cdot\widetilde{h}_1 = b_1cp \equiv 0 \mod b_1c \text{ and } p = \frac{\widetilde{H}\cdot\widetilde{h}_1}{b_1c}. \tag{3.1.144}$$
Also
$$-f(\widetilde{H})\cdot\widetilde{h}_1 = b_1c\gamma\delta q \equiv 0 \mod b_1c\gamma\delta \text{ and } q = -\frac{f(\widetilde{H})\cdot\widetilde{h}_1}{b_1c\gamma\delta}. \tag{3.1.145}$$
Here
$$-\gamma\delta = \det N(X) \tag{3.1.146}$$
and we can rewrite (3.1.145) as follows:
$$f(\widetilde{H})\cdot\widetilde{h}_1 = b_1c\det N(X)q \equiv 0 \mod b_1c\det N(X) \text{ and } q = \frac{f(\widetilde{H})\cdot\widetilde{h}_1}{b_1c\det N(X)}. \tag{3.1.147}$$
Another calculation of p and q is as follows. We have
$$\widetilde{h}_1 = \frac{u\widetilde{H} + vf(\widetilde{H})}{2a_1b_1c^2/\gamma} \tag{3.1.148}$$
where
$$u \equiv 0 \mod b_1c,\quad p = \frac{u}{b_1c},$$
and
$$v \equiv 0 \mod b_1c\gamma,\quad q = \frac{v}{b_1c\gamma}. \tag{3.1.149}$$
We remind that here $\mathbb{Z}f(\widetilde{H})$ is the orthogonal complement to $\widetilde{H}$ in $N(X)$. Both these calculations of p and q are equivalent.

Changing the letters a and b places we get the same results for the b-series.

We get

Theorem 3.1.14. *Let X be a K3 surface with $\rho(X)=2$, and H a polarization of X of degree $H^2=2rs$, $r,\ s\ge 1$. Assume that the Mukai vector (r,H,s) is primitive. Let Y be the moduli space of sheaves on X with the isotropic Mukai vector $v=(r,H,s)$ and the canonical nef element $h=(-a,0,b) \mod \mathbb{Z}v$. Let $\widetilde{H}=H/d$ be the corresponding primitive polarization with $\widetilde{H}\cdot N(X)=\gamma\mathbb{Z}$. We denote by μ the invariant of the pair $(N(X),\widetilde{H})$ and use notations of Proposition 3.1.1.*

We have $Y\cong X$, if

$$g.c.d(c,d\gamma)=1,$$

and at least for one square-free $\alpha\,|\,2a_1b_1$ there exists $\widetilde{h}_1\in N(X)$ which belongs to the a-series or to the b-series described below.

The $\widetilde{h}_1$ belongs to the a-series, if

$$(\widetilde{h}_1)^2=\frac{2\gamma^2b_1^2c}{\alpha},\ \widetilde{H}\cdot\widetilde{h}_1\equiv 0 \mod b_1c,\ f(\widetilde{H})\cdot\widetilde{h}_1\equiv 0 \mod b_1c\det N(X), \tag{3.1.150}$$

and

$$p=\frac{\widetilde{H}\cdot\widetilde{h}_1}{b_1c},\quad q=\frac{f(\widetilde{H})\cdot\widetilde{h}_1}{b_1c\det N(X)}$$

satisfy conditions (G) and (A) above (see (3.1.148) *and* (3.1.149) *about an alternative definition of p and q).*

The $\widetilde{h}_1$ belongs to the b-series, if

$$(\widetilde{h}_1)^2=\frac{2\gamma^2a_1^2c}{\alpha},\ \widetilde{H}\cdot\widetilde{h}_1\equiv 0 \mod a_1c,\ f(\widetilde{H})\cdot\widetilde{h}_1\equiv 0 \mod a_1c\det N(X), \tag{3.1.151}$$

and

$$p=\frac{\widetilde{H}\cdot\widetilde{h}_1}{a_1c},\quad q=\frac{f(\widetilde{H})\cdot\widetilde{h}_1}{a_1c\det N(X)} \tag{3.1.152}$$

satisfy the conditions (G) and (B) above.

These conditions are necessary to have $Y\cong X$, if X is a general K3 surface with $\rho(X)=2$, i. e. the automorphism group of the transcendental periods $(T(X),H^{2,0}(X))$ is ± 1.

Important Remark 3.1.15. Applying Theorem 3.1.3 and the formula (3.1.137) for the associated solution, we get that

$$\pm h_1=\begin{cases}\frac{-H}{c}+\frac{\alpha(\widetilde{H}\cdot\widetilde{h}_1)\widetilde{h}_1}{b_1^2c^2\gamma}, & \text{if } \widetilde{h}_1 \text{ is from } a\text{-series},\\ \frac{-H}{c}+\frac{\alpha(\widetilde{H}\cdot\widetilde{h}_1)\widetilde{h}_1}{a_1^2c^2\gamma}, & \text{if } \widetilde{h}_1 \text{ is from } b\text{-series}\end{cases} \tag{3.1.153}$$

belongs to $N(X)$ and

$$(Y,\widetilde{h})\cong(X,\pm w(h_1)) \text{ for some } w\in W^{(-2)}(N(X)). \tag{3.1.154}$$

Applying additionally Theorem 2.3.3, we get the following sufficient condition to have $Y \cong X$ which is valid for X with any $\rho(X)$. This is one of the main results of the paper.

In Theorem 3.1.16 below, for $\widetilde{H} \in N$ we apply the same definitions and notations: $f(\widetilde{H})$, δ, μ, as for $\widetilde{H} \in N = N(X)$ of Proposition 3.1.1.

Theorem 3.1.16. *Let X be a K3 surface and H a polarization of X of degree $H^2 = 2rs$, r, $s \geq 1$. Assume that the Mukai vector (r, H, s) is primitive. Let Y be the moduli space of sheaves on X with the isotropic Mukai vector $v = (r, H, s)$ and the canonical nef element $h = (-a, 0, b) \mod \mathbb{Z}v$. Let $\widetilde{H} = H/d$ be the corresponding primitive polarization with $\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}$. We assume that*

$$g.c.d(c, d\gamma) = 1.$$

Then $Y \cong X$, if at least for one square-free $\alpha \,|\, 2a_1b_1$ there exists $\widetilde{h}_1 \in N(X)$ which belongs to the a-series or to the b-series described below (moreover, one has (3.1.153) *and* (3.1.154)*).*

The $\widetilde{h}_1$ belongs to the a-series, if there exists a primitive sublattice $N \subset N(X)$ of the rank two containing $\widetilde{H}$ and $\widetilde{h}_1$ such that $\widetilde{H} \cdot N = \gamma\mathbb{Z}$ and

$$(\widetilde{h}_1)^2 = \frac{2\gamma^2 b_1^2 c}{\alpha},\ \widetilde{H} \cdot \widetilde{h}_1 \equiv 0 \mod b_1 c,\ f(\widetilde{H}) \cdot \widetilde{h}_1 \equiv 0 \mod b_1 c \det N, \tag{3.1.155}$$

and

$$p = \frac{\widetilde{H} \cdot \widetilde{h}_1}{b_1 c},\quad q = \frac{f(\widetilde{H}) \cdot \widetilde{h}_1}{b_1 c \det N} \tag{3.1.156}$$

satisfy conditions (G) and (A) above.

The $\widetilde{h}_1$ belongs to the b-series, if there exists a primitive sublattice $N \subset N(X)$ of the rank two containing $\widetilde{H}$ and $\widetilde{h}_1$ such that $\widetilde{H} \cdot N = \gamma\mathbb{Z}$ and

$$(\widetilde{h}_1)^2 = \frac{2\gamma^2 a_1^2 c}{\alpha},\ \widetilde{H} \cdot \widetilde{h}_1 \equiv 0 \mod a_1 c,\ f(\widetilde{H}) \cdot \widetilde{h}_1 \equiv 0 \mod a_1 c \det N, \tag{3.1.157}$$

and

$$p = \frac{\widetilde{H} \cdot \widetilde{h}_1}{a_1 c},\quad q = \frac{f(\widetilde{H}) \cdot \widetilde{h}_1}{a_1 c \det N} \tag{3.1.158}$$

satisfy the conditions (G) and (B) above.

These conditions are necessary to have $Y \cong X$, if $\rho(X) \leq 2$ and X is a general X K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods $(T(X), H^{2,0}(X))$ is ± 1.

These paper treats in general problems considered in [MN1], [MN2] and [N4] where the additional condition $\gamma = d = 1$ had been imposed. It makes results of this paper much more complicated and less efficient.

4. Divisorial conditions on moduli of (X, H) which imply $Y \cong X$ and $\gamma(\widetilde{H}) = \gamma$

Using notations above, for $\overline{\mu} = \{\mu, -\mu\} \subset (\mathbb{Z}/(2a_1b_1c^2/\gamma))^*$, $\alpha\,|2a_1b_1$ and g.c.d$(c, d\gamma) = 1$ we denote by

$$\mathcal{D}(r, s, d, \gamma; A)^{\overline{\mu}}_{\alpha} \tag{4.1}$$

the set of all $\delta \in \mathbb{N}$ such that $\delta \equiv \gamma\mu^2 \mod 4a_1b_1c^2/\gamma$ and the equation $p^2 - \gamma\delta q^2 = 4a_1b_1c/\alpha$ has an integral solution (p, q) satisfying (G) and (A) (the a-series). Similarly, we define

$$\mathcal{D}(r, s, d, \gamma; B)^{\overline{\mu}}_{\alpha} \tag{4.2}$$

(the b-series).

We denote

$$\mathcal{D}(r, s, d, \gamma)^{\overline{\mu}} = \left(\bigcup_{\alpha|2a_1b_1} \mathcal{D}(r, s, d, \gamma; A)^{\overline{\mu}}_{\alpha}\right) \bigcup \left(\bigcup_{\alpha|2a_1b_1} \mathcal{D}(r, s, d, \gamma; B)^{\overline{\mu}}_{\alpha}\right). \tag{4.3}$$

By Theorem 3.1.13, the set $\mathcal{D}(r, s, d, \gamma)^{\overline{\mu}}$ describes all possible pairs $H \in N(X)$ of general polarized K3 surfaces (X, H) with rk $N(X) = 2$, the primitive polarization $\widetilde{H} = H/d \in N(X)$, the invariant $\gamma(\widetilde{H}) = \gamma$ (i. e. $\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}$) and the invariant $\pm\mu$ for $\widetilde{H} \in N(X)$), such that $Y \cong X$. By general results of [N1] and [N2] the pair $\widetilde{H} \in N(X)$ defines the irreducible 18-dimensional moduli of such pairs $(X, \widetilde{H})$, i. e. a (irreducible) divisorial condition on 19-dimensional moduli of polarized K3 surfaces (X, H) which implies that $\gamma(\widetilde{H}) = \gamma$ and $Y \cong X$. Thus, we can interpret our results as follows.

Theorem 4.1. *The set*

$$\mathcal{D}(r, s, d, \gamma) = \{(\overline{\mu}, \delta) \mid \{\mu, -\mu\} \subset (\mathbb{Z}/(2a_1b_1/\gamma))^*\,,\ \delta \in \mathcal{D}(r, s, d, \gamma)^{\overline{\mu}}\} \tag{4.4}$$

describes all (irreducible) divisorial conditions on moduli of polarized K3 surfaces (X, H) with $H^2 = 2rs$ and the primitive polarization $\widetilde{H} = H/d$ (here $d^2|rs$), which imply that $\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}$ (i.e. $\gamma(\widetilde{H}) = \gamma$)) and $Y \cong X$.

We have (see (4.3)*)*

$$\mathcal{D}(r, s, d, \gamma)^{\overline{\mu}} = \left(\bigcup_{\alpha|2a_1b_1} \mathcal{D}(r, s, d, \gamma; A)^{\overline{\mu}}_{\alpha}\right) \bigcup \left(\bigcup_{\alpha|2a_1b_1} \mathcal{D}(r, s, d, \gamma; B)^{\overline{\mu}}_{\alpha}\right). \tag{4.5}$$

where each set $\mathcal{D}(r, s, d, \gamma; A)^{\overline{\mu}}_{\alpha}$ and $\mathcal{D}(r, s, d, \gamma; B)^{\overline{\mu}}_{\alpha}$ is infinite, if it is not empty.

Proof. We need to prove the last statement only. Assume that $\mathcal{D}(r,s,d,\gamma;A)_{\alpha}^{\overline{\mu}}$ is not empty. Thus, there exist integral (p_0,q_0) such that

$$\begin{cases} \frac{p_0^2-4a_1b_1c/\alpha}{\gamma q_0^2} \equiv \gamma\mu^2 \mod \frac{4a_1b_1c^2}{\gamma} \\ p_0^2-4a_1b_1c/\alpha > 0 \\ (p_0,q_0) \text{ satisfies } (G) \textit{ and } (A) \end{cases}. \tag{4.6}$$

Then

$$\delta_0 = \frac{p_0^2-4a_1b_1c/\alpha}{\gamma q_0^2} \in \mathcal{D}(r,s,d,\gamma;A)_{\alpha}^{\overline{\mu}}. \tag{4.7}$$

The (4.6) is equivalent to

$$\begin{cases} p_0^2-4a_1b_1c/\alpha \equiv \gamma^2\mu^2 q_0^2 \mod 4a_1b_1c^2q_0^2 \\ p_0^2-4a_1b_1c/\alpha > 0 \\ (p_0,q_0) \text{ satisfies } (G) \textit{ and } (A) \end{cases}. \tag{4.8}$$

Clearly, (p,q_0) where

$$p \equiv p_0 \mod 4a_1b_1c^2q_0^2a_1b_1, \text{ and } p^2-4a_1b_1c/\alpha > 0 \tag{4.9}$$

also satisfies (4.8) and defines

$$\delta = \frac{p^2-4a_1b_1c/\alpha}{\gamma q_0^2} \in \mathcal{D}(r,s,d,\gamma;A)_{\alpha}^{\overline{\mu}}. \tag{4.10}$$

Obviously, their number is infinite. This proves the statement.

The key question is:

Problem 4.2. *When $\mathcal{D}(r,s,d,\gamma;A)_{\alpha}^{\overline{\mu}}$ and $\mathcal{D}(r,s,d,\gamma;B)_{\alpha}^{\overline{\mu}}$ are non-empty?*

We hope to consider this question in further publications on the subject. About $d=1$ and $\gamma=1$ see [N4] and also [MN1], [MN2].

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