p- ADIC HYPERBOLIC SURFACES

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ABSTRACT. Let X be a hypersurface of degree d in the projective space $\mathbb{P}^n(\mathbb{C}_p)$. We prove that if X is a pertubation of the Fermat hypersurface, and if d is sufficiently large with respect to n and to the number of non-zero monomials in the equation defining X, then every holomorphic map from $\mathbb{C}_p$ into X has the image contained in a proper algebraic subset of X. As a consequence, we give explicit examples of p-adic hyperbolic surfaces of degree ≥ 24 in $\mathbb{P}^3(\mathbb{C}_p)$ and of curves of degree ≥ 24 with hyperbolic complements in $\mathbf{P}^2(\mathbb{C}_p)$, as well as examples of hyperbolic surfaces of degree ≥ 50 in $\mathbb{P}^3(\mathbb{C}_p)$ with hyperbolic complements. For the proof, the main tool is the height of p-adic holomorphic functions defined in author's previous papers.

§1. INTRODUCTION

A holomorphic curve in a projective variety X is said to be degenerate if it is contained in a proper algebraic subset of X. In 1979 ([GG]) M. Green and Ph. Griffiths conjectured that every holomorphic curve in a complex projective variety of general type is degenerate. Up to now this conjecture seems still far completly proved, but some progress are made. M. Green ([G]) proved the degeneracy of holomorphic curves in the Fermat variety of large degree. In [N] A. M. Nadel gives a class of projective hypersurfaces for which the conjecture is valid. Using the results on degeneracy of holomorphic curves Nadel constructed some explicit examples of hyperbolic hypersurfaces in $\mathbb{P}^3$. To receive the mentioned results, M. Green used the Nevanlinna theory for holomorphic curves, and A. Nadel's

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techniques are based on Siu's theory of meromorphic connections. We refer the reader to the survey [Z2] for related topics.

For the p-adic case, the degeneracy of holomorphic curves in the Fermat variety of large degree is established in [HM]. In this note we are going to show that if X is a pertubation of the Fermat variety in $\mathbb{P}^n(\mathbb{C}_p)$ of degree large enough with respect to n and to the number of non-zero coefficients in the defining equation, then every holomorphic curve in X is degenerate. The proof provides sufficiently precise information of the position of the curve in X, that is useful in applications. As a consequence, we give some explicit examples of p-adic hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C}_p)$, and curves in $\mathbb{P}^2(\mathbb{C}_p)$ with hyperbolic complements. Recall that a variety X is said to be p-adic hyperbolic if every holomorphic map from $\mathbb{C}_p$ into X is constant. The examples to be given here are different to ones in [HM], given by using the p-adic Nevanlinna-Cartan theorem. While the degree of surfaces in [HM], as well as in all known explicit examples of complex hyperbolic surfaces, is divided by some integer > 1, for the examples in this note, the degree d is arbitrary, required only ≥ 24 for hyperbolic surfaces and cuves with hyperbolic complements. As in [HM], the main tool of this note is the height function defined in [H1]-[H3], [HM]. This function plays a role similar to one of the Nevanlinna characteristic function in Green's arguments. Moreover, the height of a p-adic holomorphic function $f(z)$ gives information on distribution of zeros of f, and describes the growth of $|f(z)|$. Then, in many cases we can use the height in the study of p-adic holomorphic functions as the degree in the study of complex polynomials. The proof of Lemma 3.2 is such an example.

The paper is planed as follows. In §2 we recall some facts on heights of p-adic holomorphic functions and of p-adic holomorphic curves. Section 3 is devoted to the proof of degeneracy of holomorphic curves in pertubations of the Fermat variety. These results are used in the last section to give explicit examples of p-adic hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C}_p)$, curves with hyperbolic complements in $\mathbb{P}^2(\mathbb{C}_p)$.

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§2. Height of p-adic holomorphic functions

We recall some facts on heights of p-adic holomorphic functions for later use in this note. More details can be found in [H1]-[H3], [HM].

Let p be a prime number, Q_p the field of p-adic numbers, and C_p the p-adic completion of the algebraic closure of Q_p. The absolute value in Q_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on C_p which extends ord_p.

Let $f(z)$ be a p-adic holomorphic function on $\mathbb{C}_p$ represented by a convergent series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since we have

$$\lim_{n\to\infty} \{v(a_n) + nv(z)\} = \infty$$

for every $z \in \mathbb{C}_p$, it follows that for every $t \in \mathbb{R}$ there exists an n for which $v(a_n) + nt$ is minimal.

Definition 2.1. The *height* of $f(z)$ is defined by

$$h(f,t) = \min_{0\leq n<\infty} \{v(a_n) + nt\}.$$

Now let us give a geometric interpretation of height. For each n we draw the graph Γ_n which depicts $v(a_n z^n)$ as a function of $v(z)$. This graph is a straight line with slope n. Then $h(f,t)$ is the boundary of the intersection of all of the half-planes lying under the lines Γ_n. Then in any finite segment $[r,s], 0 < r, s < +\infty$, there are only finitely many Γ_n which appear in $h(f,t)$. Thus, $h(f,t)$ is a polygonal line. The point t at which $h(f,t)$ has vertices are called *the critical points* of $f(z)$. A finite segment $[r,s]$ contains only a finitely many critical points. It is clear that if t is a critical point, then $v(a_n) + nt$ attains its minimum at least at two values of n.

If $v(z) = t$ is not a critical point, then $f(z) \neq 0$ and $|f(z)| = p^{-h(f,t)}$. The function $f(z)$ has zeros when $v(z) = t_i$, where $t_o > t_1 > \ldots$ is the sequence of critical points; and the number of zeros (counting multiplicity) for which $v(z) = t_i$ is equal to the difference $n_{i+1} - n_i$ between the slope of $h(f,t)$ at $t_i - 0$ and its slope at $t_i + 0$. It is easy to see that n_i and n_{i+1}, respectively, are the smallest and the largest values of n at which $v(n) + nt$ attains minimum.

Lemma 2.2. *Let $f(z)$ be a non-constant holomorphic function on $\mathbb{C}_p$. Then we have*

$$h(f',t) - h(f,t) \geq -t + O(1),$$

where $O(1)$ is bounded when $t \to -\infty$

Lemma 2.3. *For a non-constant holomorphic function $f(z)$ in $\mathbb{C}_p$, $h(f,t) \longrightarrow -\infty$ as $t \to -\infty$*

Lemma 2.4. *For holomorphic functions $f(z)$, $g(z)$ in $\mathbb{C}_p$ we have:*

i) $h(f+g,t) \geq \min\{h(f,t), h(g,t)\}$.

ii) $h(fg,t) = h(f,t) + h(g,t)$.

The proof of Lemmas 2.2- 2.4 follows immediately from Definition 2.1, and the geometric interpretation of height.

Now let f be a p-adic holomorphic curve in the projective space $\mathbb{P}^n(\mathbb{C}_p)$, i.e., a holomorphic map from $\mathbb{C}_p$ to $\mathbb{P}^n(\mathbb{C}_p)$. We identify f with its representation by a collection of holomorphic functions on $\mathbb{C}_p$:

$$f = (f_1, f_2, \ldots, f_{n+1}),$$

where the functions f_i have no common zeros.

Definition 2.5. The *height* of the holomorphic curve f is defined by:

$$h(f,t) = \min_{1 \le i \le n+1} h(f_i, t).$$

We need the following Lemma.

Lemma 2.6. *Let $(g_1, \ldots, g_{n+1})$ be a representation of the same projective map as $(f_1, \ldots, f_{n+1})$, where g_i are holomorphic functions. Then for t sufficiently small we have*

$$h(f,t) \ge \min_{1 \le i \le n+1} h(g_i, t) + 0(1).$$

Proof. By the hypothesis there is a meromorphic function $\lambda(z)$ such that for every $i = 1, \ldots, n+1$ we have

$$g_i(z) = \lambda(z) f_i(z).$$

Since $g_i(z)$ are holomorphic functions, and $f_i(z)$ have no common zeros, λ is a holomorphic function. Then by Lemma 2.3 $h(\lambda, t) < 0$ for t sufficiently small, or $\lambda(z)$ is constant. Lemma 2.6 is proved.

From Lemma 2.6 we can see that the height of a holomorphic curve is well defined modulo a bounded value.

§3. DEGENERACY OF HOLOMORPHIC CURVES

Let

$$M_j = z_1^{\alpha_{j,1}} \dots z_{n+1}^{\alpha_{j,n+1}}, \qquad 1 \le j \le s,$$

be distinct monomials of degree d with non-negative exponents. Let X be a hypersurface of degree d of $\mathbb{P}^n(\mathbb{C}_p)$ defined by

$$X: \quad c_1 M_1 + \dots c_s M_s = 0$$

where $c_j \in \mathbb{C}_p^*$ are non-zero constants. We call X *a pertubation of the Fermat hypersurface of degree d* if $s \ge n+1$ and

$$M_j = z_j^d, \qquad j = 1, \dots, n+1.$$

We prove the following

Theorem 3.1. *Let X be a pertubation of the Fermat hypersurface of degree d in $\mathbb{P}^n(\mathbb{C}_p)$ and let f be a holomorphic curve in X. Assume that*

$$d \ge \frac{(n+1)(s-1)(s-2)}{2}.$$

Then the image of f lies in a proper algebraic subset of X.

If there is $f_i \equiv 0$, then f is degenerate, and we can assume that any $f_i \not\equiv 0$. The proof uses some Lemmas.

Lemma 3.2. *Let $f = (f_1, \dots, f_{n+1})$ be a holomorphic curve and let M be a monomial as above. Then for every $k \ge 0$ we have the following representation*

$$\frac{(M \circ f)^{(k)}}{M \circ f} = \frac{Q_k}{f_1^k \dots f_{n+1}^k},$$

where Q_k is a holomorphic function and

$$h(Q_k, t) \geq k \sum_{i=1}^{n+1} h(f_i, t) - kt + 0(1).$$

Proof. We prove the Lemma by induction on k. The case $k = 0$ is trivial. Assume for k we have the representation as in the Lemma. For simplicity we set

$$\varphi = f_1 \dots f_{n+1}. \tag{1}$$

Then we have

$$h(\varphi, t) = \sum_{i=1}^{n+1} h(f_i, t).$$

The induction hypothesis gives us

$$(M \circ f)^{(k)} = \frac{Q_k . M \circ f}{\varphi^k}.$$

Then we have

$$\frac{(M \circ f)^{(k+1)}}{M \circ f} = \frac{Q_{k+1}}{\varphi^{k+1}},$$

where

$$Q_{k+1} = \varphi . Q_k' + \varphi . Q_k . \frac{(M \circ f)'}{M \circ f} - kQ_k . \varphi'.$$

Note that the functions $\dfrac{(M \circ f)'}{(M \circ f)}$ has only simple poles at the zeros of $f_1, \dots, f_{n+1}$. Therefore, the function $\varphi . \dfrac{(M \circ f)'}{(M \circ f)}$ is holomorphic. Hence, Q_{k+1} is a holomorphic function.

On the other hand, by Lemmas 2.3 and 2.4,

$$\begin{aligned} h(Q_{k+1}, t) \geq \min\{ & h(\varphi, t) + h(Q_k', t), \\ & h(\varphi, t) + h(Q_k, t) + h((M \circ f)', t) - h(M \circ f, t), \\ & v(k) + h(Q_k, t) + h(\varphi', t)\} \end{aligned}$$

Then by Lemma 2.2 we obtain

$$
\begin{aligned}
h(Q_{k+1},t) \geq &\min\{h(\varphi,t)+h(Q_k,t)-t+0(1), h(\varphi,t)+h(Q_k,t)-t+0(1), \\
&v(k)+h(Q_k,t)+h(\varphi,t)-t+0(1)\} \qquad (2)\\
&= h(\varphi,t)+h(Q_k,t)-t+0(1)
\end{aligned}
$$

The Lemma is proved by (1), (2) and the induction hypothesis.

Notice that, the representation in Lemma 3.2 does not depend on the degree d, that is important in applications.

Lemma 3.3. *Let X be a pertubation of the Fermat hypersurface of degree d in $\mathbb{P}^n(\mathbb{C}_p)$ and let f is a holomorphic curve in X. Assume that*

$$d \geq \frac{(n+1)(s-1)(s-2)}{2}.$$

If $\{M_j \circ f,\ j=1,\dots,s-1\}$ are linearly independent, then f is a constant map.

Proof. For simplicity we set

$$g_j(z) = c_j M_j \circ f(z)/c_s M_s \circ f, \quad j=1,\dots,s-1.$$

Then the meromorphic functions $\{g_1,\dots,g_{s-1}\}$ satisfy the following relation:

$$g_1+\cdots+g_{s-1} \equiv -1.$$

We are going to show that $\{g_1,\dots,g_{s-1}\}$ are linearly dependent. For this purpose we apply the Wronskian techniques of Nevanlinna, Bloch, Cartan ([C],see also [L], Ch. VII).

Define the following logarithmic Wronskian:

$$L_s(g) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \frac{g_1'}{g_1} & \frac{g_2'}{g_2} & \dots & \frac{g_{s-1}'}{g_{s-1}} \\ \dots & \dots & \dots & \dots \\ \frac{g_1^{(s-2)}}{g_1} & \frac{g_2^{(s-2)}}{g_2} & \dots & \frac{g_{s-1}^{(s-2)}}{g_{s-1}} \end{vmatrix}$$

We further define the logarithmic Wronskians $L_i = L_i(g_1, \dots, g_{s-1})$:

$$L_1(g) = L_1(g_1, \dots, g_{s-1}) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & \frac{g_2'}{g_2} & \dots & \frac{g_{s-1}'}{g_{s-1}} \\ \dots & \dots & \dots & \dots \\ 0 & \frac{g_2^{(s-2)}}{g_2} & \dots & \frac{g_{s-1}^{(s-2)}}{g_{s-1}} \end{vmatrix}$$

and similarly for all i $(i = 1, \dots, s-1)$. where the column $\{1, 0, \dots, 0\}$ is the i-th column.

If $\{g_1, \dots, g_{s-1}\}$ are linearly independent, then the projective maps

$$(M_1 \circ f, \dots, M_s \circ f) \text{ and } L = (L_1, L_2, \dots, L_s)$$

are equal (see [L]).

Now we can apply Lemma 3.2 to the determinants. Typically, the first term in the expansion of $L_1(g)$ can be written in the form

$$\frac{Q_1 \dots Q_{s-2}}{\varphi \dots \varphi^{s-2}} = \frac{R}{\varphi^{(s-1)(s-2)/2}}.$$

The denominator $\varphi^{(s-1)(s-2)/2}$ is a common denominator of all the terms in all the expansions of all the determinants $L_i(g)$. Hence, we have an equality of projective maps:

$$(M_1 \circ f, \dots, M_s \circ f) = (L_1 \dots, L_s) = (R_1, \dots, R_s),$$

where, by Lemma 3.2, the R_j are holomorphic functions and satisfy the following condition

$$\begin{aligned}
h(R_j, t) &= \sum_{k=1}^{s-2} h(Q_k, t) \\
&\geq (h(\varphi, t) - t) \sum_{k=1}^{s-2} k + 0(1) \\
&= \frac{(s-1)(s-2)}{2} h(\varphi, t) - \frac{(s-1)(s-2)}{2} t + 0(1) \\
&\geq \frac{(n+1)(s-1)(s-2)}{2} h(f, t) - \frac{(s-1)(s-2)}{2} t + 0(1)
\end{aligned}$$

Since $M_1 \circ f, \dots, M_s \circ f$ have no common zeros, by Lemma 2.6 we have

$$\begin{aligned}
\min_{1 \leq j \leq s} h(M_j \circ f, t) &\geq \min_j h(R_j, t) \\
&\geq \frac{(n+1)(s-1)(s-2)}{2} h(f, t) - \frac{(s-1)(s-2)}{2} t + 0(1).
\end{aligned}$$

Because X is a pertubation of the Fermat hypersurface of degree d we have

$$\min_{1 \leq j \leq n+1} h(M_j \circ f.t) = d \min_{1 \leq j \leq n+1} h(f_j, t) = dh(f, t). \tag{3}$$

For other monomials we have

$$h(M_j \circ f, t) = \sum_{k=0}^{n} \alpha_{jk} h(f_k, t) \geq dh(f, t).$$

Thus we obtain

$$dh(f, t) \geq \frac{(n+1)(s-1)(s-2)}{2} h(f, t) - \frac{(s-1)(s-2)}{2} t + 0(1) \tag{4}$$

When $d = (n+1)(s-1)(s-2)/2$ we have a contradiction as $t \to -\infty$, and when $d > \dfrac{(n+1)(s-1)(s-2)}{2}$ the inequality (4) gives us

$$h(f,t) \geq -Nt + 0(1),$$

where N is a positive number, so by Lemma 2.4, f is a constant map. The Lemma is proved.

To complete the proof of Theorem 3.1, it suffices to notice that, by Lemma 3.3 the image of f is contained in the proper algebraic subset of X defined by the equation:

$$a_1 z_1^d + a_2 z_2^d + \cdots + a_{n+1} z_{n+1}^d + a_{n+1} M_{n+2} + \cdots + a_{s-1} M_{s-1} = 0,$$

where not all a_j are zeros. Theorem 3.1. is proved.

§4. Hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C}_p)$

In this section we apply Theorem 3.1 to give explicit examples of p-adic surfaces in $\mathbb{P}^3(\mathbb{C}_p)$, as well as examples of curves in $\mathbb{P}^2(\mathbb{C}_p)$ with hyperbolic complements.

Without loss of generality we may assume that in the defining equation of X, the first coefficients $c_i = 1,\ i = 1, \ldots, n+1$.

Theorem 4.1. *Let X be a surface in $\mathbb{P}^3(\mathbb{C}_p)$ defined by the equation*

$$(5) \qquad X:\ z_1^d + z_2^d + z_3^d + z_4^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0,$$

where $c \neq 0$, $\sum_{i=1}^{4} \alpha_i = d$, and if there is an exponent $\alpha_i = 0$, the others must be at least two. Then X is hyperbolic if $d \geq 24$.

Proof. First of all let us recall a result from [HM] (Theorem 4.3).

Lemma 4.2. *Let X be the Fermat hypersurface of degree d in $\mathbb{P}^n(\mathbb{C}_p)$, and let $f = (f_1, \dots, f_{n+1})$ be a holomorphic curve in X. If $d \geq n^2 - 1$, then either f is a constant curve, or there is a decomposition of the set of indices $\{1, \dots, n+1\} = \cup I_\xi$ such that every I_ξ contains at least two elements, and if $i, j \in I_\xi$, f_i is equal to f_j multiple a constant.*

Now let X be a hypersurface satisfying the hypothesis of Theorem 4.1, and let $f = (f_1, f_2, f_3, f_4) : \mathbb{C}_p \longrightarrow X$ be a holomorphic curve in X. We consider all possible cases.

1) Suppose that for some i, $f_i \equiv 0$, for example, $f_4 \equiv 0$.

i) $\alpha_4 > 0$. Then $f_1^d + f_2^d + f_3^d \equiv 0$, and f is a constant map by Lemma 4.2.

ii) $\alpha_4 = 0$. We have

$$f_1^d + f_2^d + f_3^d + c f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} \equiv 0.$$

From the proof of Theorem 3.1 it follows that $\{f_1^d, f_2^d, f_3^d\}$ are lineraly dependent:

$$c_1 f_1^d + c_2 f_2^d + c_3 f_3^d \equiv 0,$$

where not all $c_i = 0$. Then either f is a constant map. or we can assume, for examples, that $f_1 = a_1 f_2$ and obtain:

$$(a_1^d + 1) f_2^d + f_3^d + c a_1^{\alpha_1} f_2^{\alpha_1 + \alpha_2} f_3^{\alpha_3} \equiv 0.$$

By the hypothesis, $\alpha_1 + \alpha_2 \neq 0, d$, and in any case we see that $f_2/f_3 =$ const, so f is a constant map.

2) Hence, we can assume that any $f_i \not\equiv 0$. From the proof of Theorem 3.1 it follows that $\{f_1^d, \dots, f_4^d\}$ are linearly dependent. Suppose that

$$a_1 f_1^d + \cdots + a_4 f_4^d \equiv 0,$$

where not all a_i are zeros. Consider the following possible cases:

i) $a_i \neq 0,\ i = 1, \ldots, 4$. By Lemma 4.2, f is a onstant map, or we can assume that $f_1 = c_1 f_2,\ f_3 = c_2 f_4$. Then we can substitute this relation to (5) and show that f is a constant map by the same arguments as in 1-ii).

ii) Only one coefficient, say, $a_4 = 0$. Then (f_1, f_2, f_3) is a constant map by Lemma 4.2, and it is easy to show that f is constant.

iii) Two coefficients, say, $a_1 = a_2 = 0$. Then we have $f_3 = c_3 f_4$. Substitute this relation into (5) we obtain

$$f_1^d + f_2^d + \varepsilon_1 f_3^d + \varepsilon_2 f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3+\alpha_4} \equiv 0, \tag{6}$$

where $\varepsilon_2 \neq 0$. If $\varepsilon_1 \neq 0$, then we return to the case 1-ii).

Now suppose that $\varepsilon_1 = 0$. Then the image of the map (f_1, f_2, f_3) is contained in the following curve in $\mathbb{P}^2(\mathbb{C}_p)$:

$$Y:\ z_1^d + z_2^d + \varepsilon_2 z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3+\alpha_4} = 0.$$

We are going to show that under the hypothesis of Theorem 4.1, the genus of Y is at least 1, then Theorem 4.1 follows from Berkovich's theorem ([Be]).

The genus of Y is equal to the number of integer points in the triangle with the vertices $(d, 0), (0, d)$ and (α_1, α_2) (see, for example, [Ho]). It is easy to see that this triangle contains at least one integer point, unless the cases $\alpha_1 + \alpha_2 = d$ or $\alpha_1 + \alpha_2 = d - 1$. These cases are excluded by the hypothesis of Theorem 4.1. The proof is completed.

Remark 4.1. In [HM] by using the method of K. Masuda and J. Noguchi (MN]), we give the following examples of hyperbolic hypersurfaces in $\mathbb{P}^3(\mathbb{C}_p)$:

$$z_1^{4d} + \cdots + z_4^{4d} + t(z_1 z_2 z_3 z_4)^d = 0,\ d \geq 6(\deg\ X = 4d \geq 24), t \in \mathbb{C}_p^*$$

Here we have the examples with arbitrary degree ≥ 24 (not necessarily divided by 4). Notice that all known explicit examples of hyperbolic hypersurfaces in the complex case are of degree d divided by some number > 1 (2 in the case of Brody-Green's example, 3 in Nadel's example, and 3, 4 in Masuda-Noguchi's examples). Indeed, in [MN] it is given an algorithm to construct hyperbolic hypersurfaces of degree $d > 54$, here we have hyperbolic hypersurfaces with $d \geq 24$.

Remark 4.2. 1) The following examples show that if among the exponents α_i two of them are (0, 1) or (0,0), then X may not be hyperbolic. The surface

$$X: \; z_1^{25} + z_2^{25} + z_3^{25} + z_4^{25} + z_1 z_2^{24} = 0$$

contains the holomorphic curve $(-1 - z^{25}, 1, 1 + z^{25}, z)$.

2) The surface

$$X: \; z_1^{25} + z_2^{25} + z_3^{25} + z_4^{25} - 2z_1^{10} z_2^{15} = 0$$

contains the holomorphic curve $f = (z, z, 1, -1)$

Now we use Theorem 4.1 to give explicit examples of curves in $\mathbb{P}^2(\mathbb{C}_p)$ with hyperbolic complements.

Theorem 4.3. *Let X be a curve in $\mathbb{P}^2(\mathbb{C}_p)$ defined by the following equation:*

$$X: \; z_1^d + z_2^d + z_3^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0,$$

where $d \geq 24$, $\alpha_i \geq 2$, $\sum \alpha_i = d$. Then the complement of X is p-adic hyperbolic in $\mathbb{P}^2(\mathbb{C}_p)$

Proof. Let $f = (f_1, f_2, f_3) : \; \mathbb{C}_p \longrightarrow \mathbb{P}^2$ be a holomorphic curve with the image contained in the complement of X. Then the function

$$f_1^d + f_2^d + f_3^d + c f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} \neq 0$$

for $z \in \mathbb{C}_p$, and then is identically equal to a non-zero constant a. Hence, the image of the folowing holomorphic curve

$$(f_1, f_2, f_3, 1): \mathbb{C}_p \longrightarrow \mathbb{P}^3$$

is contained in the surface Y of $\mathbb{P}^3$ defined by the equation

$$Y: z_1^d + z_2^d + z_3^d - az_4^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0.$$

By Theorem 4.1, Y is hyperbolic, and f is a constant map. Theorem 4.3 is proved.

Remark 4.3. In [MN] K. Masuda and J. Noguchi give an algorithm to construct curves of degree $d \geq 48$ in $\mathbb{P}^2(\mathbb{C})$ with hyperbolic complements. Here we have explicit examples of such curves in $\mathbb{P}^2(\mathbb{C}_p)$ of degree $d \geq 24$.

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