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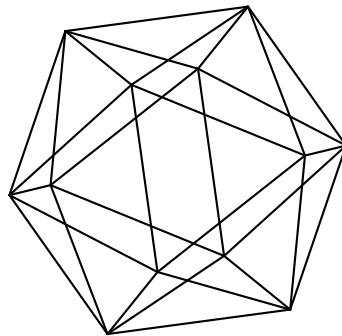
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ASYMPTOTICALLY CYLINDRICAL STEADY KÄHLER-RICCI SOLITONS

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ABSTRACT. Let D be a compact Kähler manifold with trivial canonical bundle and Γ be a finite cyclical group of order m acting on $\mathbb{C} \times D$ by biholomorphisms, where the action on the first factor is generated by rotation of angle $2\pi/m$. Furthermore, suppose that Ω_D is a trivialisation of the canonical bundle such that Γ preserves the holomorphic form $dz \wedge \Omega_D$ on $\mathbb{C} \times D$, with z denoting the coordinate on $\mathbb{C}$.

The main result of this article is the construction of new examples of gradient steady Kähler-Ricci solitons on certain crepant resolutions of the orbifolds $(\mathbb{C} \times D)/\Gamma$. These new solitons converge exponentially to a Ricci-flat cylinder $\mathbb{R} \times (\mathbb{S}^1 \times D)/\Gamma$.

1. INTRODUCTION

A *steady Ricci soliton* is a Riemannian manifold (M, g) together with a vector field X such that

$$(1) \qquad \operatorname{Ric}(g) = \frac{1}{2}\mathcal{L}_X g,$$

where $\operatorname{Ric}(g)$ denotes the Ricci tensor of g and $\mathcal{L}_X$ is the Lie derivative in direction of X. The soliton (M, g, X) is called *gradient* if X is the gradient field of some function on M.

If (M, g) is Kähler and the vector field X real holomorphic, equation (1) is equivalent to

$$(2) \qquad \operatorname{Ric}(\omega) = \frac{1}{2}\mathcal{L}_X \omega,$$

where ω is the Kähler form of g and $\operatorname{Ric}(\omega)$ the corresponding Ricci form. A Kähler manifold (M, g) which admits a real holomorphic vector field X satisfying (2) is called a *steady Kähler-Ricci soliton*.

Steady solitons may be viewed as natural generalisations of Einstein manifolds, which correspond to the case $X \equiv 0$. Non-Einstein steady solitons, however, must be non-compact ([Ive93]).

To each steady Ricci soliton (M, g, X) one can associate a self-similar Ricci-flow by rescaling and pulling back g along the flow of X. Thus, steady solitons may be possible candidates for singularity models for Ricci-flow. They are also important in the context of so-called Type II singularities, i.e. when a Ricci-flow exists up to the finite time $T > 0$, and the curvature blows up of rate at least $(T-t)^{-1}$. For recent progress

in the study of singularities as well as steady Ricci solitons, we refer the reader to [BCD+21], [CDM20], [Bam20], [CFSZ20], [DZ20], and the references therein.

This article focuses on the case of steady Kähler-Ricci solitons, and our main result is the existence of a new class of such solitons. In contrast to general Ricci-solitons, it suffices to solve a single equation of top-dimensional differential forms in order to construct a gradient Kähler-Ricci soliton. If M is a complex manifold of (complex) dimension n, together with a nowhere-vanishing holomorphic $(n, 0)$-form Ω, and a Kähler metric g whose Kähler form ω satisfies

$$(3) \qquad\qquad \omega^n = e^{-f} i^{n^2} \Omega \wedge \overline{\Omega}$$

for some function $f : M \to \mathbb{R}$, then $(M, g, \nabla^g f)$ defines a gradient steady Kähler-Ricci soliton. In fact, if M is simply-connected, then one can always associate such a form Ω to a gradient steady Kähler-Ricci soliton, compare [Bry08][Theorem 1].

However, given M and a nowhere-vanishing holomorphic n-form Ω on M it is not known if M admits a steady soliton, i.e. there is no general existence theory for steady Kähler-Ricci solitons as is the case for compact Ricci-flat Kähler manifolds due to Yau [Yau78].

All previously known examples of steady Kähler-Ricci solitons may be divided into two classes. The first group consists of solitons constructed by reducing (2) to an ODE, for instance by Hamilton [Ham88], Cao [Cao96], Dancer and Wang [DW11], Yang [Yan12] and the author [Sch20]. Most notably, we mention Hamilton's cigar on $\mathbb{C}$ ([Ham88]) and Cao's soliton on $\mathbb{C}^n$ for $n \geq 2$ ([Cao96]). The cigar is asymptotic to the cylinder $dt^2 + d\theta^2$ on the product $\mathbb{R} \times \mathbb{S}^1 \cong \mathbb{C}^*$, whereas Cao's soliton has a more complicated asymptotic behavior. (It is a so-called cigar-paraboloid whose precise asymptotics are explained in [CD20b][Section 3].)

The second group of examples are constructed by PDE methods ([BM17], [CD20b]). Here, the underlying complex manifolds are equivariant, crepant resolutions of certain orbifolds $\mathbb{C}^n/G$ ([BM17]) and of more general Calabi-Yau cones ([CD20b]). In both cases, the solitons have an asymptotic behavior similar to Cao's soliton.

In this article, we build on ideas developed in [CD20b] and find new examples of steady Kähler-Ricci solitons which are asymptotic to a product $\mathbb{C} \times D$ of Hamilton's cigar and a compact Ricci-flat Kähler manifold D. (Note that this product is also a steady Kähler-Ricci soliton.) These new examples exist on resolutions $\pi : M \to (\mathbb{C} \times D)/\Gamma$ of certain orbifolds $(\mathbb{C} \times D)/\Gamma$. Before introducing the precise conditions on D, Γ and M, consider the following example.

Example 1.1. Let $D = \mathbb{T}$ be the (real) 2-torus and let $\Gamma = \{\pm \mathrm{Id}\}$. Then $(\mathbb{C} \times \mathbb{T})/\Gamma$ has precisely four singular points, each isomorphic to a neighborhood of the origin in $\mathbb{C}^2/\{\pm \mathrm{Id}\}$. Thus, we may blow-up each

of these singular points to obtain a resolution $\pi : M \to (\mathbb{C} \times \mathbb{T})/\Gamma$. (Note that previously, certain Calabi-Yau metrics, so-called ALG gravitational instantons, were constructed on this resolution, see [BM11].)

This resolution $\pi : M \to (\mathbb{C} \times \mathbb{T})/\Gamma$ satisfies three essential properties. First, the resolution is crepant, i.e. the holomorphic (2,0)-form Ω on $(\mathbb{C}^* \times \mathbb{T})/\Gamma$, which lifts to the canonical form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$, extends to a nowhere-vanishing form on the entire resolution M.

Second, the $\mathbb{C}^*$-action on $(\mathbb{C}^* \times \mathbb{T})/\Gamma$ given by

$$\lambda * (z, w) = (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action on M, since the resolution is toric. In particular, the infinitesimal generator $z_1 \frac{\partial}{\partial z_1}$ on $(\mathbb{C}^* \times \mathbb{T})/\Gamma$ extends to a holomorphic vector field Z on M.

And third, M admits a natural complex compactification $\overline{M}$ obtained by adding the divisor $\overline{\mathbb{T}} := \mathbb{T}/\{\pm \mathrm{Id}\}$ 'at infinity', i.e. we compactify $\mathbb{C}$ by the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and let $\overline{M} = M \cup (\{\infty\} \times \overline{\mathbb{T}})$. Given a Kähler class $\kappa_{\overline{M}} \in H^2(\overline{M}, \mathbb{R})$ on $\overline{M}$, it is possible to construct a new Kähler form on M in the class $\kappa_{\overline{M}}|_M \in H^2(M, \mathbb{R})$ that is asymptotic to the cylinder

$$(4) \qquad |z_1|^{-2} \frac{i}{2} dz_1 \wedge d\bar{z}_1 + \frac{i}{2} dz_2 \wedge d\bar{z}_2.$$

(This construction follows by adapting ideas from the case of asymptotically cylindrical Calabi-Yau manifolds [HHN15].)

Thus, one may ask if there exists a steady Kähler-Ricci soliton on M which is asymptotic to the cylinder (4), whose Kähler form is contained in the class $\kappa_{\overline{M}}|_M$ and whose soliton vector field equals the real part of Z. This is indeed a non-trivial question, because M is *not* a product, but a resolution of the orbifold $(\mathbb{C} \times \mathbb{T})/\Gamma$.

Our main result (Theorem 1.2), however, implies that M *does* admits such solitons. In fact, Theorem 1.2 proves the existence of steady Kähler-Ricci solitons for a more general setup:

Theorem 1.2. *Let D^{n-1} be a compact Kähler manifold with nowhere-vanishing holomorphic $(n-1,0)$-form Ω_D. Suppose $\gamma : D \to D$ is a complex automorphism of order $m > 1$ such that*

$$\gamma^* \Omega_D = e^{-\frac{2\pi i}{m}} \Omega_D,$$

and consider the orbifold $(\mathbb{C} \times D)/\langle \gamma \rangle$, where γ acts on the product via

$$\gamma(z, w) = \left(e^{\frac{2\pi i}{m}} z, \gamma(w) \right).$$

Let $\pi : M \to (\mathbb{C} \times D)/\langle \gamma \rangle$ be a crepant resolution such that the $\mathbb{C}^*$-action on $(\mathbb{C} \times D)/\langle \gamma \rangle$ given by

$$\lambda * (z, w) = (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action of $\mathbb{C}^*$ on M.

Let $\overline{M} = M \cup \overline{D}$ be the complex compactification of M by adding the orbifold divisor $\overline{D} := D/\langle\gamma\rangle$ at infinity. Then for every orbifold Kähler class $\kappa_{\overline{M}}$ on $\overline{M}$, there exists a steady Kähler-Ricci soliton on M whose Kähler form is contained in the class $\kappa_{\overline{M}}|_M \in H^2(M,\mathbb{R})$.

As in Example 1.1, M admits a nowhere-vanishing holomorphic $(n,0)$-form because the resolution is crepant, and the infinitesimal generator of the $\mathbb{C}^*$-action on M provides a candidate for the soliton vector field. Also, the Kähler class is determined by the compactification $\overline{M}$ and the resulting Kähler-Ricci soliton is asymptotic to the cylinder $dt^2 + d\theta^2 + g_D$ on the product $(\mathbb{C}^* \times D)/\langle\gamma\rangle \cong \mathbb{R} \times (\mathbb{S}^1 \times D)/\langle\gamma\rangle$ for some Ricci-flat Kähler metric g_D on D.

The new examples of steady Kähler-Ricci solitons provided by Theorem 1.2 are geometrically different from all previously found examples in complex dimension $n \geq 2$. For instance, their volume grows linearly since they are asymptotically cylindrical, while the examples modelled on Cao's soliton in complex dimension n have volume growth equal to n, compare [Cao96], [BM17] and [CD20b].

Interestingly, our examples also seem to be the only (non-Einstein) steady Kähler-Ricci solitons whose asymptotic model is *Ricci-flat*. This contrasts with the fact that Cao's soliton has *positive* Ricci curvature ([Cao96][Lemma 2.2]).

The strategy for proving Theorem 1.2 is analogue to the proof of [CD20b][Theorem A]. We adapt Conlon and Deruelle's ideas to our setting and reduce (2) to a complex Monge-Ampère equation, whose solution exists by the following result, which is similar to [CD20b][Theorem 7.1]

Theorem 1.3. *Let (M, g, J) be an asymptotically cylindrical Kähler manifold of complex dimension n with Kähler form ω. Suppose that M admits a real holomorphic vector field X such that*

$$X = 2\Phi_* \frac{\partial}{\partial t}$$

outside some compact domain, where Φ denotes the diffeomorphism onto the cylindrical end of (M, g) and t is the radial parameter on this end. Moreover, assume that JX is Killing for g.

If $1 < \varepsilon < 2$ and $F \in C_\varepsilon^\infty(M)$ is JX-invariant, then there exists a unique, JX-invariant $\varphi \in C_\varepsilon^\infty(M)$ such that $\omega + i\partial\bar{\partial}\varphi > 0$ and

$$\left(\omega + i\partial\bar{\partial}\varphi\right)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n.$$

Note that in this theorem, we do allow more general manifolds than those appearing in Theorem 1.2. This is because the proof of Theorem 1.3 essentially only requires that we have a Kähler manifold (M, g, J), asymptotic to a cylinder (in the sense of Definition 2.1 below) and satisfying two further assumptions: Firstly, we need the radial vector field on the cylinder to be extended to a real holomorphic vector field

on (M, J) and secondly, JX must be an infinitesimal isometry of g. We will see in Proposition 3.5 below that this ensures $X = \nabla^g f$ for some function f with understood asymptotical behavior.

The spaces $C_\varepsilon^\infty(M)$ in Theorem 1.3 contain all smooth functions on M whose covariant derivatives (with respect to g) decay at least like $e^{-\varepsilon t}$ with t denoting the cylindrical parameter of (M, g) (compare Definitions 2.1 and 2.3). These function spaces are well-adapted to the cylindrical geometry and have previously been used in the construction of asymptotically cylindrical Calabi-Yau manifolds [HHN15].

Following the proof of [CD20b][Theorem 7.1], we also implement a continuity method to conclude Theorem 1.3. To this end, we need to show two things. First, that the linearisation of the Monge-Ampère operator is an isomorphism, which can be deduced from standard results on asymptotically translation invariant differential operators. Second, and most importantly, we have to derive a priori-estimates along the continuity path, where the C^0-estimate is the key part of the proof. To obtain this estimate, we adapt the C^0-estimate of Conlon and Deruelle ([CD20b][Section 7.1]) to our cylindrical setup. These authors first assume that the right-hand side F is *compactly supported* to obtain the C^0-estimate ([CD20b][Theorem 7.1]) and in a second step, they explain how to solve the Monge-Ampère equation for *decaying* F ([CD20b][Theorem 9.2]). We, however, present a modification of their argument, which allows us to achieve the C^0-estimate *directly* for F decaying exponentially in Theorem 1.3.

This article is structured as follows. In Section 2, we recall the notion of asymptotically cylindrical manifolds and the theory of linear asymptotically translation-invariant operators on such manifolds. This is later applied to the linearisation of the Monge-Ampère operator.

The basics of steady Kähler-Ricci solitons are covered in Section 3. We recall the underlying Monge-Ampère equation and also discuss when a soliton is gradient. Most notably, we show at the end of this section that, under the assumptions of Theorem 1.3, X must be a gradient field.

In Section 4, we reduce Theorem 1.2 to Theorem 1.3. We discuss the existence of cylindrical Kähler metrics on manifolds as in Theorem 1.2 in Section 4.1 and also explain which Kähler classes do indeed admit such metrics. Theorem 1.2 is then proven in Section 4.2, before we provide further examples in Section 4.3.

The fifth and final section is entirely devoted to Theorem 1.3. We explain the continuity method and reduce the proof to an a priori-estimate.

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2. Linear analysis on acyl manifolds

In this section, we review the basic definitions and theorems about asymptotically translation-invariant operators on ACyl manifolds following the presentation in [HHN15][Section 2.1] and [Nor08][Section 2.3]. The goal is to apply the general theory to the special class of operators that arise as the linearisation of the Monge-Ampère operator in Section 5 below.

We begin by recalling the definition of ACyl manifolds. For simplicity, we restrict our attention to the case of only *one* cylindrical end, i.e. a connected cross-section.

Definition 2.1. A complete Riemannian manifold (M, g) is called *asymptotically cylindrical (ACyl) of rate* $\delta > 0$ if there is a bounded open set $U \subset M$, a connected and closed Riemannian manifold (L, g_L) as well as a diffeomorphism $\Phi : [0, \infty) \times L \to M \setminus U$ such that

$$|\nabla^k (\Phi^* g - g_{cyl})| = O(e^{-\delta t})$$

for all $k \in \mathbb{N}_0$, where $g_{cyl} := dt^2 + g_L$ is the product metric and both ∇ and $|\cdot|$ are taken with respect to this metric. Here t denotes the projection onto $[0, \infty)$ and we extend the function $t \circ \Phi^{-1}$ smoothly to all of M. This extension is called a *cylindrical coordinate function*, (L, g_L) is called the *cross-section* and Φ the *ACyl map*.

Throughout this section, (M, g) denotes an ACyl manifold of rate $\delta > 0$ as defined above. It will be convenient to suppress Φ and simply view t as smooth a function on M.

Let $E, F \to M$ be tensor bundles over M and denote the corresponding space of smooth sections of E and F by $\Gamma(E)$ and $\Gamma(F)$, respectively. Then we consider a differential operator $P : \Gamma(E) \to \Gamma(F)$ of order l and we would like to understand P on the cylindrical end $M \setminus U \cong [0, \infty) \times L$.

As in [Mar02][Section 4], we cover the compact link L by charts $V_1, \ldots, V_N$ so that both E and F are trivial over each $\mathbb{R}_+ \times V_\alpha$. Given $u \in \Gamma(E)$, we denote the components of u and Pu on $\mathbb{R}_+ \times V_\alpha$ by u_j^α and $(Pu)_i^\alpha$, respectively, where $\alpha = 1, \ldots, N$, $j = 1, \ldots, \text{rank } E$ and $i = 1, \ldots, \text{rank } F$. Moreover, there are smooth functions $P_{ij}^{\alpha\beta} : \mathbb{R}_+ \times V_\alpha \to \mathbb{C}$ such that

$$(5) \qquad (Pu)_i^\alpha = \sum_{j=1}^{\text{rank } E} \sum_{0 \leq |\beta| \leq l} P_{ij}^{\alpha\beta} D^\beta u_j^\alpha$$

where the second sum runs over all multi-indices $\beta = (\beta_0, \ldots, \beta_{\dim L})$ of order $|\beta|$ at most l and D^β is defined to be

$$D^\beta := \frac{\partial^{|\beta|}}{\partial t^{\beta_0} x_1^{\beta_1} \cdots \partial x_{\dim L}^{\beta_{\dim L}}}$$

for coordinates $(x_1, \ldots, x_{\dim L})$ of V_α.

Given a second operator $Q : \Gamma(E) \to \Gamma(F)$ also of order l, we say that P is *asymptotic* to Q if the coefficients $P_{ij}^{\alpha\beta}, Q_{ij}^{\alpha\beta}$ defined by (5) satisfy

$$\sup_{\{t\} \times V_\alpha} \left| \rho_\alpha D^\gamma \left(P_{ij}^{\alpha\beta} - Q_{ij}^{\alpha\beta} \right) \right| \to 0 \quad \text{as} \quad t \to \infty$$

for all $i = 1, \ldots, \operatorname{rank} F$, $j = 1, \ldots \operatorname{rank} E$, $\alpha = 1, \ldots N$, $|\beta| \leq l$ and all multi-indices γ, where $\rho_1, \ldots, \rho_N$ is a partition of unity subordinate to the cover $V_1, \ldots, V_N$. Note that this definition does neither depend on the choice of covering nor on the partition of unity.

With this notion of asymptotic operators, we may introduce the following definitions, compare [Mar02][Section 4.2.2].

Definition 2.2. Let $P, P_\infty : \Gamma(E) \to \Gamma(F)$ be two differential operators of order l between sections of tensor bundles $E, F \to M$.

(i) P_∞ is called *translation-invariant* if the functions $(P_\infty)_{ij}^{\alpha\beta}$ defined in (5) are invariant under translation in the $\mathbb{R}_+$-factor, for all $i = 1, \ldots, \operatorname{rank} F$, $j = 1, \ldots, \operatorname{rank} E$, $\alpha = 1, \ldots, N$ and all multi-indices β of order at most l.

(ii) P is called *asymptotically translation-invariant* if P is asymptotic to some translation-invariant operator P_∞.

Important examples of asymptotically translation-invariant operators include the Laplacian Δ_g and the operator d^* associated to the ACyl metric g. These are asymptotic to the corresponding operators associated to the cylinder g_{cyl}.

Such operators may in general not be Fredholm between the usual Hölder spaces because M is noncompact. However, this changes if we introduce weight functions.

Definition 2.3. Let (M, g) be an ACyl manifold with cylindrical coordinate t and suppose $E \to M$ is a tensor bundle. The metric on E induced by g is also denoted by g, with corresponding connection ∇.

(i) For $\alpha \in (0, 1)$, the Hölder semi-norm $[\,\cdot\,]_{C^{0,\alpha}}$ is defined for any continuous tensor field v over M by

$$[v]_{C^{0,\alpha}} := \sup_{\substack{x \neq y \in M \\ d_g(x,y) < \frac{i(g)}{2}}} \frac{|v_x - v_y|_g}{d_g(x,y)^\alpha},$$

where $v_x - v_y$ is defined by parallel transport along the minimal geodesic from x to y and $i(g) > 0$ denotes the injectivity radius of g.

(ii) For $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $\varepsilon \in \mathbb{R}$, we define $C_\varepsilon^{k,\alpha}(E)$ to be the space of k-times continuously differentiable sections u of E such that the norm

$$||u||_{C_\varepsilon^{k,\alpha}} := \sum_{j=0}^{k} \sup_M \left| e^{\varepsilon t}\nabla^j u \right|_g + [e^{\varepsilon t}\nabla^k u]_{C^{0,\alpha}}$$

is finite.

(iii) $C_\varepsilon^\infty(E)$ is defined to be the intersection of $C_\varepsilon^{k,\alpha}(E)$ over all $k \in \mathbb{N}_0$.

(iv) If u is a function on M, the corresponding spaces are denoted by $C_\varepsilon^{k,\alpha}(M)$.

In other words, elements in $C_\varepsilon^\infty(E)$, as well as their covariant derivatives, are bounded from above by $e^{-\varepsilon t}$. It is not difficult to see that the definition is independent of the extension of the cylindrical coordinate t. Moreover, there are continuous inclusions

$$C_\varepsilon^{k+1}(E) \subseteq C_\varepsilon^{k,\alpha}(E) \quad \text{and} \quad C_{\varepsilon_1}^{k,\alpha}(E) \subseteq C_{\varepsilon_0}^{k,\alpha}(E),$$

if $\varepsilon_0 \le \varepsilon_1$.

This notion of weighted Hölder spaces is well-adapted to the study of asymptotically translation-invariant operators. If the operator is moreover elliptic, we have the following weighted Schauder estimates.

Theorem 2.4. *Let (M, g) be ACyl and let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic, asymptotically translation-invariant operator of order l. Suppose $h \in C_\varepsilon^{k,\alpha}(E)$ and that u is a $k+l$-times continuously differentiable solution to $Pu = h$. If $u \in C_\varepsilon^0(E)$, then $u \in C_\varepsilon^{k+l,\alpha}(E)$ and*

$$||u||_{C_\varepsilon^{k+l,\alpha}} \le C \left(||h||_{C_\varepsilon^{k,\alpha}} + ||u||_{C_\varepsilon^0} \right)$$

for some constant $C > 0$ independent of u.

Proof. This is [MP84][Theorem 3.16]. $\qquad\qquad\qquad\qquad\qquad\square$

Every translation-invariant operator $P : \Gamma(E) \to \Gamma(F)$ of order l induces a bounded map $P : C_\varepsilon^{k+l,\alpha}(E) \to C_\varepsilon^{k,\alpha}(F)$. If P is moreover elliptic, it depends on the weight $\varepsilon \in \mathbb{R}$ whether or not the induced map $P : C_\varepsilon^{k+l,\alpha}(E) \to C_\varepsilon^{k,\alpha}(F)$ is Fredholm. This naturally leads to the definition of so called critical weights.

Definition 2.5. Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator asymptotic to a translation-invariant operator $P_\infty : \Gamma(E) \to \Gamma(F)$. $\varepsilon \in \mathbb{R}$ is called a *critical weight* if there exists a non-trivial solution $v = e^{i\lambda t}u : \mathbb{R} \times L \to \mathbb{C}$ to

$$(6) \qquad\qquad\qquad\qquad P_\infty(v) = 0$$

for some $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda = \varepsilon$ and for some smooth section $u = u(t, x)$ of E over $\mathbb{R} \times L$ that is a polynomial in t.

Note that the set of critical weights is discrete in $\mathbb{R}$. In the case of functions, i.e. if E is the trivial line bundle, u in the above definition is simply a polynomial in t with smooth functions on L as coefficients. This is crucial because it allows us to explicitly compute critical weights in examples.

The fundamental result in the theory of asymptotically translation-invariant operators is the following

Theorem 2.6. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic, translation-invariant operator of order l. If ε is not a critical weight, then the map $P : C_{\varepsilon}^{k+l,\alpha}(E) \to C_{\varepsilon}^{k,\alpha}(F)$ is Fredholm.*

This result was originally formulated for weighted Sobolev spaces ([LMO85][Theorem 6.2]). However, as explained in [HHN15][Section 2.1], the same proof applies in the Hölder setting as well.

Knowing that the induced map $P : C_{\varepsilon}^{k+l,\alpha}(E) \to C_{\varepsilon}^{k,\alpha}(F)$ is Fredholm for all non-critical weights ε, we would now like to have a better understanding of its kernel and image.

Proposition 2.7. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic, translation-invariant operator of order l. If an interval $[\varepsilon_1, \varepsilon_2]$ contains no critical weights, then the kernels of P in $C_{\varepsilon_1}^{k,\alpha}(M)$ and $C_{\varepsilon_2}^{k,\alpha}(M)$ are equal.*

This is proven in [LMO85][Lemma 7.1]. To give a precise characterization of the image of P, we need to introduce the formal adjoint $P^* : \Gamma(F) \to \Gamma(E)$. It is uniquely defined by the condition that

(7) $$\langle Pu, v \rangle_{L^2} = \langle u, P^*v \rangle_{L^2}$$

holds for all smooth, compactly supported sections u, v. Here, the L^2-inner product is defined with respect to the ACyl metric g. Observe that the identity (7) extends to sections u, v in certain Hölder spaces.

Lemma 2.8. *Let $P : \Gamma(E) \to \Gamma(F)$ be an asymptotically translation-invariant operator of order l with formal adjoint $P^* : \Gamma(F) \to \Gamma(E)$. Suppose that $u \in C_{\varepsilon_1}^{l,\alpha}(E)$ and $v \in C_{\varepsilon_2}^{l,\alpha}(F)$ with $\varepsilon_1 + \varepsilon_2 > 0$. Then*

$$\langle Pu, v \rangle_{L^2} = \langle u, P^*v \rangle_{L^2}.$$

The proof is straight forward, and written out in ([Nor08][Lemma 2.3.15]), for example.

Proposition 2.9. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic, asymptotically translation-invariant operator of order l with formal adjoint $P^* : \Gamma(F) \to \Gamma(E)$. If ε is not a critical weight, then the image of $P : C_{\varepsilon}^{k+l,\alpha}(E) \to C_{\varepsilon}^{k,\alpha}(F)$ is precisely the L^2-orthogonal complement to the kernel of $P^* : C_{-\varepsilon}^{k+l,\alpha}(F) \to C_{-\varepsilon}^{k,\alpha}(E)$ in $C_{\varepsilon}^{k,\alpha}(F)$.*

Proof. This can be deduced from Theorem 2.6 and Proposition 2.7, compare [Nor08][Proposition 2.3.16] for details. □

We seek to apply this general theory to a certain subclass of asymptotically translation-invariant operators, which naturally arise as the linearisation of the Monge-Ampère operator in Section 5 below.

Definition 2.10. Let f be a smooth function on an ACyl manifold (M, g). Then the following operator

$$\Delta_f u := \Delta_g u + g(\nabla^g f, \nabla^g u)$$

is called the *drift Laplacian with potential function f*. If additionally $f - 2t \in C^\infty_{\delta_0}(M)$ for some $\delta_0 > 0$, we refer to Δ_f as an *ACyl drift Laplace operator*.

Any such operator Δ_f is self-adjoint with respect to the L^2-inner product induced by the measure $e^f \, dV_g$, i.e.

$$\int_M (\Delta_f u) v \, e^f \, dV_g = \int_M u (\Delta_f v) \, e^f \, dV_g$$

for all smooth, compactly supported functions u, v. If Δ_f is moreover an ACyl drift Laplacian, then it is asymptotic to the translation-invariant operator

$$(8) \qquad \Delta_{2t} u = \Delta_{g_{cyl}} u + g_{cyl}\left(2\frac{\partial}{\partial t}, \nabla^{g_{cyl}} u\right) = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + \Delta_{g_L} u$$

where $g_{cyl} = dt^2 + g_L$ is the product metric. From the general theory, we deduce the next

Theorem 2.11. *Let (M, g) be an ACyl manifold and suppose that Δ_f is an ACyl drift Laplacian with potential function f. Then for any $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$ and $0 < \varepsilon < 2$ the operator*

$$\Delta_f : C^{k+2,\alpha}_\varepsilon(M) \to C^{k,\alpha}_\varepsilon(M)$$

is an isomorphism.

Proof. Since $\varepsilon > 0$, the injectivity of Δ_f follows immediately from the standard maximum principle, so we only need to show surjectivity. Before using Proposition 2.9, we need to prove the following

Claim. There are no critical weights for Δ_f in the interval $(0, 2)$.

Since Δ_f is asymptotic to the operator given in (8), the definition of critical weights requires us to show that there are no solutions v to the equation

$$(9) \qquad\qquad \frac{\partial^2 v}{\partial t^2} + 2\frac{\partial v}{\partial t} + \Delta_{g_L} v = 0$$

of the form

$$(10) \qquad v = e^{i\lambda t} \sum_{j=0}^{d} a_j t^j$$

with $\mathrm{Im}\,\lambda = \varepsilon \in (0,2)$ and functions a_j on L. To see this, we plug (10) into (9) and by considering the coefficient of t^d, we observe that (9) can only be satisfied if

$$(11) \qquad \Delta_{g_L} a_d + (-\lambda^2 + 2i\lambda)a_d = 0.$$

This implies that $-\lambda^2 + 2i\lambda$ must be real and non-negative because Δ_{g_L} is a negative and self-adjoint operator on the closed manifold (L, g_L). Writing $\lambda = \gamma + i\varepsilon$, this translates into

$$(12) \qquad 2\gamma(1-\varepsilon) = 0, \quad \text{and} \quad -\gamma^2 + \varepsilon(\varepsilon - 2) \geq 0.$$

If $\varepsilon = 1$, the second equation in (12) gives a contradiction, and so $\gamma = 0$. Then, however, the second equation implies $\varepsilon \geq 2$ since $\varepsilon > 0$. Thus, there cannot be a solution to (9) of the form (10) with $\varepsilon \in (0,2)$, proving the claim.

Hence, according to Proposition 2.9, it suffices to show that the formal adjoint Δ_f^* of Δ_f is injective when viewed as a map $\Delta_f^* : C_{-\varepsilon}^{k+2,\alpha}(M) \to C_{-\varepsilon}^{k,\alpha}(M)$ with $0 < \varepsilon < 2$. A simple computation shows that Δ_f^* is given by

$$\Delta_f^* u = \Delta_g u - g(\nabla^g f, \nabla^g u) - u\,\Delta_g f.$$

Assuming that $u \in C_{-\varepsilon}^{k+2,\alpha}(M)$ satisfies $\Delta_f^* u = 0$, we compute

$$\begin{aligned}
\Delta_f(e^{-f}u) &= u\,\Delta_f e^{-f} + e^{-f}\Delta_f u + 2g(\nabla^g e^{-f}, \nabla^g u) \\
&= -ue^{-f}\Delta_f f + e^{-f}\Delta_g u - e^{-f}g(\nabla^g f, \nabla^g u) \\
&= e^{-f}\Delta_f^* u \\
&= 0.
\end{aligned}$$

Since $\varepsilon < 2$, the function $e^{-f}u$ tends to zero as $t \to \infty$, and so the maximum principle implies that $e^{-f}u$ vanishes identically. Thus, the kernel of Δ_f^* is trivial and the theorem follows. $\qquad\square$

We end this section by proving a (global) Poincaré-inequality for a certain drift Laplace operator, which is needed later on to obtain L^2-estimates for the Monge-Ampère operator as in [CD20b].

Proposition 2.12. *Let (M, g) be an ACyl manifold. If f is a C^2-function on M satisfying $f - 2t \in C_{\delta_0}^2(M)$ for some $\delta_0 > 0$ then there exists a constant $\lambda > 0$ such that*

$$\lambda \int_M u^2 \frac{e^f}{(f+c)^2}\,\mathrm{d}V_g \leq \int_M |\nabla^g u|_g^2 \frac{e^f}{(f+c)^2}\,\mathrm{d}V_g$$

holds for all smooth, compactly supported functions u on M, where $c > 0$ is chosen so that $f + c > 0$.

Proof. First of all note that we can assume that $f + c > 0$ for some $c > 0$ because f is proper since $f - 2t \in C^2_{\delta_0}(M)$. By [CD20b][Lemma 5.1], it is sufficient to find a positive C^2-function v on M and a positive constant λ_0 such that $\Delta_{f-2\log(f+c)} v \leq -\lambda_0 v$ outside some compact subset $K \subset M$.

We claim that this condition holds for the function $v := e^{-\frac{f}{2}}$. Indeed, we first calculate

$$\Delta_{f-2\log(f+c)} e^{-\frac{f}{2}} = -e^{-\frac{f}{2}} \left(\frac{1}{2}\Delta_g f + \left(\frac{1}{4} - \frac{1}{f+c} \right) g(\nabla^g f, \nabla^g f) \right),$$

and, since $f - 2t \in C^2_{\delta_0}(M)$, we then observe that $(f+c)^{-1} \to 0$ in the limit $t \to \infty$, as well as

$$\Delta_g f \to \Delta_{g_{cyl}} t = 0, \text{ and } |\nabla^g f|^2_g \to |\nabla^{g_{cyl}} t|^2_{g_{cyl}} = 1 \text{ if } t \to \infty.$$

The claim now follows by taking for instance $\lambda_0 = 1/8$ and $K := \{x \in M \mid t(x) \leq m\}$ for $m \gg 1$ large enough. $\square$

3. Preliminaries on Kähler-Ricci solitons

In this section, we recall some basic definitions and facts about steady Kähler-Ricci solitons. In particular, we review when a solitons is gradient. The main result in this direction is Proposition 3.5, which states a criterion for the radial vector field on an ACyl Kähler manifold to be a gradient field.

Definition 3.1. A triple (M, g, X) consisting of a complete Kähler manifold (M, g) and a complete real holomorphic vector field X on M is a *steady Kähler-Ricci soliton* if the corresponding Kähler form ω satisfies

(13) $$\mathrm{Ric}(\omega) = \frac{1}{2}\mathcal{L}_X \omega,$$

where $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω and $\mathcal{L}_X$ is the Lie derivative in direction of X. The vector field X is called the *soliton vector field*.

We say that a steady Kähler-Ricci soliton (M, g, X) is *gradient* if $X = \nabla^g f$ for some smooth real-valued function f on M. In this case, f is called the *soliton potential* and equation (13) becomes

(14) $$\mathrm{Ric}(\omega) = i\partial\bar{\partial} f.$$

For us, the most important example is Hamilton's cigar soliton [Ham88].

Example 3.2 (Cigar soliton)**.** Let $M = \mathbb{C}$ and consider the following metric

$$g_{cig} = \frac{1}{1 + x^2 + y^2} \left(dx^2 + dy^2 \right)$$

which is Kähler with Kähler form

$$\omega_{cig} = \frac{1}{1 + |z|^2} \frac{i}{2} dz \wedge d\bar{z},$$

where $z = x + iy$ is the standard coordinate for $\mathbb{C}$. A straight forward computation then shows that $(\mathbb{C}, g_{cig})$ defines a Kähler-Ricci soliton with real holomorphic vector field

$$X = 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} = 4 \operatorname{Re}\left(z \frac{\partial}{\partial z} \right).$$

In fact, $(\mathbb{C}, g_{cig})$ is also an ACyl manifold in the sense of Definition 2.1, with ACyl map $\Phi : \mathbb{R} \times S^1 \to \mathbb{C}^*$ given by

$$\Phi(t, e^{2\pi i\theta}) := e^{t+2\pi i\theta}.$$

Under this change of coordinates, we have

$$\Phi_* \frac{\partial}{\partial t} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{and} \quad \Phi^* g_{cig} = \frac{1}{1 + e^{2t}}(dt^2 + d\theta^2),$$

from which it is easy to see that g_{cig} is indeed asymptotic to $g_{cyl} = dt^2 + d\theta^2$.

In higher dimension, further examples of ACyl Kähler-Ricci solitons can be obtained by taking the product of the cigar soliton with a compact, Ricci-flat Kähler manifold. Such examples, and their finite quotients, are the asymptotic model for the solitons constructed in Theorem 4.6.

Under certain conditions, the soliton equation (13) can be reduced to solving a Monge-Ampère equation, as shown in [CD20b][Proposition 4.5], for example. We adapt their arguments to obtain the next

Lemma 3.3. *Let (M, g, J) be a Kähler manifold of dimension n with Kähler form ω. Let X be a real holomorphic vector field such that $X = \nabla^g f$, for some smooth function $f : M \to \mathbb{R}$, and suppose M admits a nowhere-vanishing, holomorphic $(n, 0)$-form Ω. If there is a smooth function $\varphi : M \to \mathbb{R}$ satisfying*

$$(15) \qquad \left(\omega + i\partial\bar{\partial}\varphi \right)^n = e^{-f - \frac{X}{2}(\varphi)} i^{n^2} \Omega \wedge \overline{\Omega},$$

then $\omega + i\partial\bar{\partial}\varphi$ defines a steady Kähler-Ricci soliton with vector field X. Moreover, if φ is JX-invariant, the resulting soliton is gradient.

Proof. We closely follow the computation provided in the proof of [CD20b][Proposition 4.5]. Suppose $\omega_\varphi := \omega + i\partial\bar{\partial}\varphi$ satisfies (15) and compute:

$$\begin{aligned}
\operatorname{Ric}(\omega_\varphi) &= -i\partial\bar{\partial} \log \frac{\omega_\varphi^n}{i^{n^2}\Omega \wedge \overline{\Omega}} \\
&= i\partial\bar{\partial}f + \frac{X}{2}(\varphi) \\
&= \frac{1}{2}\mathcal{L}_X \omega + \frac{1}{2}\mathcal{L}_X \left(i\partial\bar{\partial}\varphi \right) = \frac{1}{2}\mathcal{L}_X \omega_\varphi,
\end{aligned}$$

where we used in the last line that X is real holomorphic, and

$$\frac{1}{2}\mathcal{L}_X\omega = \frac{1}{2}d\iota_X\omega = \frac{1}{2}dJ\iota_{JX}\omega = -\frac{1}{2}dJdf = i\partial\bar{\partial}f$$

since $X = \nabla^g f$. So, if g_φ is the metric corresponding to ω_φ, the triple (M, g_φ, X) defines a steady Kähler-Ricci soliton.

For the second claim, assume φ to be JX-invariant. It is not difficult to see that $\iota_{JX}\left(2i\partial\bar{\partial}\varphi\right) = -dX(\varphi)$, compare the proof of [CD20b][Lemma 7.3] for instance. Then we conclude

$$\iota_{JX}\left(\omega + i\partial\bar{\partial}\varphi\right) = -d\left(f + \frac{X}{2}(\varphi)\right)$$

i.e. $X = \nabla^{g_\varphi}\left(f + \frac{X}{2}(\varphi)\right)$ as claimed. $\qquad\square$

We conclude this section by addressing the question when a given Kähler-Ricci soliton is gradient. It is not difficult to see that if it is gradient, then JX is a Killing vector field for the metric. Under certain conditions, the converse is true as well.

Lemma 3.4. *Let (M, g, X) be a steady Kähler-Ricci soliton and suppose that JX is Killing for g, where J denotes the complex structure of (M, g). If $H^1(M, \mathbb{R}) = 0$, then the soliton (M, g, X) is gradient.*

Proof. This is a special case of [CD20a][Corollary A.7]. $\qquad\square$

In the special case of ACyl manifolds, we can replace the condition $H^1(M, \mathbb{R}) = 0$ in Lemma 3.4 by an asymptotic condition on the vector field X. In fact, there is the following statement for more general ACyl Kähler manifolds.

Proposition 3.5. *Let (M, g) be an ACyl Kähler manifold of rate $\delta > 0$ with complex structure J and ACyl map Φ. Suppose X is a real holomorphic vector field on M such that*

$$(16) \qquad\qquad X = 2\Phi_*\frac{\partial}{\partial t}$$

outside some compact domain. If JX is Killing for g, then there exists a smooth function $f : M \to \mathbb{R}$ with $f - 2t \in C_\delta^\infty(M)$ such that $X = \nabla^g f$.

Proof. The idea is to adapt a proof of Frankel for compact manifolds ([Fra59]) to the ACyl setting. This is possible because there is a version of Hodge splitting on such manifolds, see for example [Nor08][Section 2.3.3].

Let ω be the Kähler form of (M, g, J). First, since JX is Killing and X is real holomorphic, we have $\mathcal{L}_{JX}g = \mathcal{L}_{JX}J = 0$ and so $\mathcal{L}_{JX}\omega = 0$. In particular, the 1 form $\iota_{JX}\omega$ is closed. We would like to show that it is in fact exact, for which we need to understand its asymptotic behaviour.

Let $\Phi^{-1} \circ t$ be a cylindrical coordinate function for (M, g), whose smooth extension to all of M is denoted by τ. Then we claim that

$$\text{(17)} \qquad \iota_{JX}\omega + d\tau \in C_\delta^\infty(\Lambda^1(M)).$$

Indeed, outside of a sufficiently large domain so that (16) is satisfied, we can estimate

$$|d\tau + \iota_{JX}\omega|_g = |\iota_{\Phi_*\partial_t}(\Phi_* g_{cyl}) - \iota_X g|_g \leq |X|_g \cdot |\Phi_* g_{cyl} - g|_g = O(e^{-\delta t})$$

because (M, g) is ACyl of rate $\delta > 0$ and the norm of X is uniformly bounded on M. Here we used that on the product $\mathbb{R} \times L$, the tensors dt and g_{cyl} are related by $\iota_{\partial_t} g_{cyl} = dt$. A similar estimate holds for the first covariant derivative

$$|\nabla^g (\iota_{\Phi_*\partial_t}(\Phi_* g_{cyl}) - \iota_X g)|_g \leq |\nabla^g X|_g \cdot |\Phi_* g_{cyl} - g|_g + |X|_g \cdot |\nabla^g g_{cyl}|_g$$
$$= O(e^{-\delta t})$$

since $|\nabla^g X|_g = O(1)$ and $|\nabla^g g_{cyl}|_g$ decays exponentially of rate δ. Similarly, we can proceed by induction to obtain bounds on higher derivatives, which implies (17).

By the ACyl version of Hodge splitting ([Nor08][Theorem 2.3.27]), there are 1-forms $h, \alpha, \beta \in C_\varepsilon^\infty(\Lambda^1 M)$ such that

$$\text{(18)} \qquad \iota_{JX}\omega + d\tau = h + \alpha + \beta,$$

where h is Δ_g-harmonic, α exact and β co-exact. Here, $0 < \varepsilon < \min\{\delta, \lambda\}$, with λ denoting the smallest (positive) critical weight of the Laplace operator Δ_g acting on 1-forms. Moreover, we can write

$$\alpha = d\tilde{f} \quad \text{and} \quad \beta = d^*\gamma$$

for some $\tilde{f} \in C_\varepsilon^\infty(M)$ and $\gamma \in C_0^\infty(\Lambda^2 M)$. (Note that the growth of γ follows from [Nor08][Theorem 2.3.27] since translation-invariant forms on the cylinder $\mathbb{R} \times L$ are bounded with respect to g_{cyl}, and that we can indeed assume $\tilde{f}$ decays at infinity because the only translation-invariant harmonic functions are constants.)

We have to show that both h and β vanish identically. We begin by observing that h is closed. Since h is Δ_g-harmonic and in $C_\varepsilon^\infty(\Lambda^1 M)$, we may, according to Lemma 2.8, integrate by parts to obtain

$$\text{(19)} \qquad 0 = \langle h, \Delta_g h \rangle_{L^2} = \langle dh, dh \rangle_{L^2} + \langle d^*h, d^*h \rangle_{L^2},$$

i.e. $dh = 0$ and $d^*h = 0$. Hence, we deduce immediately from the decomposition (18) that β is also closed. Integrating by parts then yields

$$\langle \beta, \beta \rangle_{L^2} = \langle \beta, d^*\gamma \rangle_{L^2} = \langle d\beta, \gamma \rangle_{L^2} = 0,$$

so $\beta \equiv 0$ as desired.

Next, we follow the proof of [Fra59][Lemma 2] to show that $h \equiv 0$. By assumption, JX is Killing for g and so

$$\Delta_g \left(\mathcal{L}_{JX} h\right) = \mathcal{L}_{JX} \left(\Delta_g h\right) = 0,$$

but also $\mathcal{L}_{JX} h = d(\iota_{JX} h)$, i.e. $\mathcal{L}_{JX} h$ is a harmonic and exact 1-form in $C_\varepsilon^\infty(\Lambda^1 M)$. Using the orthogonality of Hodge's decomposition, we conclude $\mathcal{L}_{JX} h = 0$

Moreover, the 1-form $Jh(\cdot) := h(J \cdot)$ is also harmonic since the Laplace operator on a Kähler manifold preserves the bi-degree decomposition of the cotangent bundle. Using the same computation as in (19), we conclude that Jh is closed, from which we further deduce that

$$d \left(\iota_{JX}(Jh)\right) = \mathcal{L}_{JX}(Jh) = \mathcal{L}_{JX}(J)h + J\mathcal{L}_{JX}h = 0$$

because JX is real holomorphic, i.e. $\mathcal{L}_{JX} J = 0$. In particular, the function $\iota_{JX}(Jh) = -h(X)$ is constant on M, and thus it can only be identically zero as $h(X)$ tends to zero at infinity. This, together with integration by parts, in turn gives

$$\begin{aligned}
\langle h, h \rangle_{L^2} &= \langle \iota_{JX}\omega + d\tau, h \rangle_{L^2} \\
&= \langle \iota_{JX}\omega, h \rangle_{L^2} + \langle \tau, d^* h \rangle_{L^2} \\
&= -\int_M h(X) \, \mathrm{dV}_g \\
&= 0,
\end{aligned}$$

where we used in the penultimate line that $\iota_{JX}\omega$ is the negative g-dual of X and $d^* h = 0$. We conclude $h \equiv 0$, and consequently

$$\iota_{JX}\omega = d\tilde{f} - d\tau$$

with $\tilde{f} \in C_\varepsilon^\infty(M)$. It remains to improve the decay rate of $\tilde{f}$, i.e. we need to show $\tilde{f} \in C_\delta^\infty(M)$ instead of merely $\tilde{f} \in C_\varepsilon^\infty(M)$. It clearly suffices to prove $\tilde{f} \in C_\delta^0(M)$ because we already know from (17) that $df \in C_\delta^\infty(\Lambda^1 M)$.

Working on the cylindrical end, we write $\tilde{f}(t, x) := \tilde{f} \circ \Phi(t, x)$ for $t \in \mathbb{R}_+$ and $x \in L$, and express $\tilde{f}$ as an integral of the radial derivative as follows:

$$\tilde{f}(t, x) = -\int_t^\infty \partial_s \tilde{f}(s, x) ds.$$

This, together with $d\tilde{f}(X) = O(e^{-\delta t})$, implies $\tilde{f} = O(e^{-\delta t})$ as required. Proposition 3.5 now follows by setting $f := -\tilde{f} + \tau$.

$\square$

4. The existence theorem

The goal of this section is to show the main result of this article (Theorem 1.2). We begin by introducing a more general setup and discussing the existence of ACyl Kähler metrics on the considered manifolds. Step

by step, we then add further assumptions and point out their importance for Theorem 1.2. This discussion will also be accompanied by a simple, but illustrative example.

Throughout this section, let $D = D^{n-1}$ be a compact Kähler manifold of complex dimension $n-1$ and assume that $\gamma : D \to D$ is a biholomorphism of order $m > 1$. Consider the orbifold $M_{orb} := (\mathbb{C} \times D) / \Gamma$, where we set $\Gamma := \langle \gamma \rangle \cong \mathbb{Z}_m$ and let γ act on the product via

$$(20) \qquad \gamma(z, w) := \left(e^{\frac{2\pi i}{m}} z, \gamma(w) \right).$$

The singular part M_{orb}^{sing} of M_{orb} is clearly contained in the slice $(\{0\} \times D) / \Gamma$ and corresponds to the fixed points of γ on D.

Let $\pi : M \to M_{orb}$ be a resolution of M_{orb}, with exceptional set $E = \pi^{-1}(M_{orb}^{sing})$. Then we use π to identify $M \setminus E \cong M_{orb} \setminus M_{orb}^{sing}$ and, in particular, we view $(\mathbb{C}^* \times D) / \Gamma$ as an (open) complex submanifold of M.

It is instructive to keep the following example in mind.

Example 4.1 (A first example). Let $D = \mathbb{T}$ be the (real) 2-torus and define $\gamma = -\operatorname{Id}$. Then consider the orbifold $(\mathbb{C} \times D) / \langle \gamma \rangle$ with four isolated singular points contained inside the slice $\{0\} \times D / \langle \gamma \rangle$ and locally isomorphic to a neighborhood of the origin in $\mathbb{C}^2 / \mathbb{Z}_2$. Blowing-up each of these rational double points then yields a resolution $\pi : M \to (\mathbb{C} \times D) / \langle \gamma \rangle$.

We point out that this complex manifold M does admit Kähler metrics, and in fact, certain Calabi-Yau metrics (so-called ALG gravitational instantons) were constructed on M in [BM11][Theorem 2.3].

Before finding ACyl Kähler metrics on a resolution $\pi : M \to M_{orb}$, we have to fix an asymptotic model g_{cyl} on $(\mathbb{C}^* \times D) / \Gamma$. For this, we choose a γ-invariant Kähler metric g_D on D and define the cylindrical parameter $t : \mathbb{C}^* \times D \to \mathbb{R}$ to be

$$(21) \qquad t(z, w) := \log |z|.$$

If $g_{\mathbb{C}}$ denotes the standard flat metric on $\mathbb{C}$, then the product metric

$$(22) \qquad g_{cyl} := e^{-2t} g_{\mathbb{C}} + g_D$$

is Γ-invariant and can thus be viewed as a metric on the quotient $(\mathbb{C}^* \times D) / \Gamma$. Note that if we let

$$(23) \qquad \begin{aligned} \Phi : \mathbb{R} \times \mathbb{S}^1 \times D &\to \mathbb{C}^* \times D, \\ (t, e^{2\pi i \theta}, w) &\mapsto (e^{t + 2\pi i \theta}, w) \end{aligned}$$

then $\Phi^*(g_{cyl}) = dt^2 + g_{\mathbb{S}^1} + g_D$, where $g_{\mathbb{S}^1}$ denotes the metric on $\mathbb{S}^1$ of length 1. So g_{cyl} is indeed a Γ-invariant cylinder with cross-section $(\mathbb{S}^1 \times D, g_{\mathbb{S}^1} + g_D)$. The corresponding Kähler form ω_{cyl} on $\mathbb{C}^* \times D$ is

given by

$$\text{(24)} \qquad \omega_{cyl} = |z|^{-2}\frac{i}{2}dz \wedge d\bar{z} + \omega_D = i\partial\bar{\partial}t^2 + \omega_D,$$

where ω_D is the Kähler form associated to g_D.

We would like to understand how to construct ACyl Kähler metris on M that are asymptotic to g_{cyl} as in (22) for some choice of Kähler metric g_D on D. Moreover, we wish to know which de Rham cohomology classes contain the corresponding Kähler forms.

To simplify notation, we introduce the following notion of Kähler class.

Definition 4.2. Let $\pi : M \to M_{orb}$ be as above. A class $\kappa \in H^2(M,\mathbb{R})$ is said to be *Kähler* if there exists a Kähler form $\omega \in \kappa$.

A Kähler class is called *ACyl* if it contains a Kähler form whose metric g is ACyl and satisfies

$$\text{(25)} \qquad |\left(\nabla^{g_{cyl}}\right)^k (g - g_{cyl})|_{g_{cyl}} = O(e^{-\delta t}) \quad \text{as} \quad t \to \infty,$$

for some $\delta > 0$ and all $k \in \mathbb{N}_0$, where g_{cyl} is given by (22) for some γ-invariant Kähler metric g_D on D.

We point out that this notion of ACyl Kähler classes is quite restrictive since we only allow ACyl metrics with ACyl diffeomorphism Φ defined by (23). In particular, the ACyl Kähler metric g and its asymptotic cylinder are Kähler with respect to the *same* complex structure since $M \setminus E$ is biholomorphic to $(\mathbb{C}^* \times D)/\Gamma$.

One way to describe ACyl classes is by introducing a complex compactification $\overline{M}$ of M, whose construction we now describe.

Recall that $\mathbb{C}$ can naturally be compactified to the Riemann sphere $\mathbb{CP}^1$ by adding one point 'at infinity'. We denote this point by ∞, i.e. $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Consequently, the orbifold M_{orb} is naturally compactified by $\left(\mathbb{CP}^1 \times D\right)/\Gamma$ and, since $(\mathbb{C}^* \times D)/\Gamma$ and M are biholomorphic outside of the exceptional set E, we also obtain a compactification $\overline{M}$ of M.

In other words, $\overline{M}$ is constructed from M by gluing in the orbifold divisor $\overline{D} := (\{\infty\} \times D)/\Gamma$ at 'infinity'. We emphasize this by writing $\overline{M} = M \cup \overline{D}$. Then the following theorem provides equivalent characterisations of ACyl Kähler classes.

Theorem 4.3. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4, and suppose that $\overline{M} = M \cup \overline{D}$ is the compacification obtained by adding an orbifold divisor $\overline{D}$ at infinity. For a given $\kappa \in H^2(M,\mathbb{R})$, the following are equivalent:*

(i) κ is an ACyl Kähler class.

(ii) $\kappa = \kappa_{\overline{M}}|_M$ for some orbifold Kähler class $\kappa_{\overline{M}}$ on $\overline{M}$.

Moreover, if the $\mathbb{C}^*$-action $(\mathbb{C} \times D)/\langle\gamma\rangle$ given by

$$\text{(26)} \qquad \lambda * (z,w) = (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action of $\mathbb{C}^*$ on M, then (i) is equivalent to the following:

> *(iii) There exists some Kähler form $\omega_0 \in \kappa$ such that the 1-form $\iota_{J\frac{\partial}{\partial t}}\omega_0$ is exact on $(\mathbb{C}^* \times D)/\langle \gamma \rangle$, where J denotes the complex structure on M and t is defined in (21).*

The equivalence of (i) and (ii) originates in work on ACyl Calabi-Yau manifolds [HHN15], however, it is impractical to verify in concrete examples. This is why we introduce criterion (iii). In fact, this condition allows us to prove:

Corollary 4.4. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4 and assume that the $\mathbb{C}^*$-action given by (26) extends π-equivariantly to a holomorphic action on M.*

If every closed, γ-invariant 1-form on D is exact, then each Kähler class is ACyl.

The proof of this corollary also partly justifies extending the $\mathbb{C}^*$-action (26) to the resolution.

Proof. Let ω_0 a Kähler form on M. Since $\mathbb{S}^1$ is compact and connected, we can average ω_0 over this group to obtain a new closed 2-form $\hat{\omega}_0$ such that $[\hat{\omega}_0] = [\omega_0] \in H^2(M, \mathbb{R})$. In fact, $\hat{\omega}_0$ is a positive $(1,1)$-form because $\mathbb{S}^1$ acts by biholomorphisms and the averaging does not affect the positivity.

As the $\mathbb{C}^*$-action (26) extends to M, the radial vector field $\partial/\partial t$ also extends to a real holomorphic vector field Y on M. In particular,

$$(27) \qquad Y = \frac{\partial}{\partial t} \quad \text{on} \quad (\mathbb{C}^* \times D)/\Gamma \subset M$$

and JY is a generator of the corresponding $\mathbb{S}^1$-action, so that

$$\mathcal{L}_{JY}(\hat{\omega}_0) = 0.$$

Hence, the 1-form $\iota_{JY}(\hat{\omega}_0)$ is closed and to apply (iii) of Theorem 4.3, we need to show that its restriction to $M \setminus E \cong (\mathbb{C}^* \times D)/\langle \gamma \rangle$ is exact.

Observe that it is sufficient for the lift of $\iota_{JY}(\hat{\omega}_0)$ to $\mathbb{C}^* \times D$ to be exact. This lift, in turn, is clearly exact if its restriction to a slice $\{0\} \times \mathbb{S}^1 \times D \subset \mathbb{R} \times S^1 \times D \cong \mathbb{C}^* \times D$ is exact. Since we have $\iota_{JY}(\hat{\omega}_0)(JY) = 0$, this restriction, however, is of the form $p_D^*\alpha$ for some 1-form α on D, where $p_D : \mathbb{S}^1 \times D \to D$ denotes the projection. Using that $\iota_{JY}(\hat{\omega}_0)$ is also closed and γ-invariant, we conclude that α must be closed and γ-invariant as well, and hence exact by assumption. $\square$

The proof of Theorem 4.3 is postponed to Section 4.1.

Remark 4.5. Let us examine the usefulness of this corollary by considering Example 4.1. Recall that in this case, the resolution $\pi : M \to (\mathbb{C}^* \times \mathbb{T})/\langle \gamma \rangle$ is obtained by blowing-up the four fixed points of $\gamma = -\operatorname{Id}$ on $\mathbb{C} \times \mathbb{T}$. For showing that the $\mathbb{C}^*$-action given by (26)

extends to the blow-up M, it suffices to do so locally near each singularity because these are isolated points. This, however, is clearly true because the blow-up

$$(28) \qquad\qquad \mathcal{O}_{\mathbb{CP}^1}(-2) \to \mathbb{C}^2/\{\pm \operatorname{Id}_{\mathbb{C}^2}\}$$

is a toric resolution (with respect to the standard action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$).

Verifying the condition in Corollary 4.4 is also straight forward. Indeed, denoting the holomorphic coordinate of the universal cover $\mathbb{C}$ of $\mathbb{T}^1$ by $w = u + iv$, we see that the translation-invariant 1-forms du and dv are clearly *not* fixed by the action of $-\operatorname{Id}$ on $\mathbb{C}$. Thus, *every* Kähler class of the blow-up M admits an ACyl Kähler metric.

Having understood when a resolution $\pi : M \to M_{orb}$ admits ACyl Kähler metrics, we may continue adding further assumptions in order to find steady Kähler-Ricci solitons on M. Namely, assume that D^{n-1} admits a nowhere-vanishing holomorphic $(n-1,0)$-form Ω_D such that

$$\gamma^* \Omega_D = e^{-\frac{2\pi i}{m}} \Omega_D.$$

This, together with (20), implies that the holomorphic $(n,0)$-form $\Omega := dz \wedge \Omega_D$ is γ-invariant and descends to M_{orb}. Thus, we may require the resolution $\pi : M \to M_{orb}$ to be *crepant*, i.e. we assume that Ω extends to a nowhere-vanishing form on M.

As in Theorem 4.3, we additionally assume the extension of the $\mathbb{C}^*$-action (26) from M_{orb} to M. This guarantees that the infinitessimal generator Y of the corresponding $\mathbb{R}_+$-action is a real holomorphic vector field and thus, multiples of Y are candidates for the soliton field of the desired solitons.

With these conditions, we recall the main result of this article.

Theorem 4.6. *Let D^{n-1} be a compact Kähler manifold with nowhere-vanishing holomorphic $(n-1,0)$-form Ω_D. Suppose $\gamma : D \to D$ is a complex automorphism of order $m > 1$ such that*

$$(29) \qquad\qquad \gamma^* \Omega_D = e^{-\frac{2\pi i}{m}} \Omega_D,$$

and consider the orbifold $(\mathbb{C} \times D)/\langle \gamma \rangle$, where γ acts on the product via

$$(30) \qquad\qquad \gamma(z,w) = \left(e^{\frac{2\pi i}{m}} z, \gamma(w) \right).$$

Let $\pi : M \to (\mathbb{C} \times D)/\langle \gamma \rangle$ be a crepant resolution such that the $\mathbb{C}^*$-action on $(\mathbb{C} \times D)/\langle \gamma \rangle$ given by

$$\lambda * (z,w) = (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action of $\mathbb{C}^*$ on M.

Then every ACyl Kähler class admits a gradient steady Kähler-Ricci soliton. Moreover, the soliton metric is ACyl of rate ε for each $0 <$

ε < 2 and with asymptotic cylinder given by

$$g_{cyl} = e^{-2t}g_{\mathbb{C}} + g_{RF},$$

where g_{RF} is a Ricci-flat Kähler metric on D.

Looking back at our Example 4.1, we see that the resolution $\pi : M \to (\mathbb{C} \times \mathbb{T})/\{\pm \operatorname{Id}\}$ satisfies all requirements because the blow-up (28) of each singularity is indeed crepant, and $\gamma = -\operatorname{Id}$ acts on the holomorphic 1-form on $\mathbb{T}^1$ by multiplication with -1. Hence, Theorem 4.6, together with Remark 4.5, imply the existence of a steady Kähler-Ricci soliton in *each* Kähler class on M.

Following ideas of Conlon and Deruelle developed in [CD20b][Section 4.2], the strategy for proving Theorem 4.6 is reducing it to a complex Monge-Ampère equation. As explain before Theorem 4.6, the assumptions ensure the existence of a nowhere-vanishing holomorphic $(n,0)$-form as well as suitable real holomorphic vector fields, so that Lemma 3.3 may indeed be used to set up a Monge-Ampère equation for finding a steady Kähler-Ricci soliton. The technical argument for solving the resulting equation is then provided by Theorem 4.7 below, whose proof we postpone to Section 5.

Theorem 4.7. *Let (M, g, J) be an ACyl Kähler manifold of complex dimension n with Kähler form ω. Suppose that M admits a real holomorphic vector field X such that*

$$X = 2\Phi_* \frac{\partial}{\partial t}$$

outside some compact domain, where Φ is the ACyl map and t the cylindrical coordinate function. Moreover, assume that JX is Killing for g.

If $1 < \varepsilon < 2$ and $F \in C_\varepsilon^\infty(M)$ is JX-invariant, then there exists a unique, JX-invariant $\varphi \in C_\varepsilon^\infty(M)$ such that $\omega + i\partial\bar{\partial}\varphi > 0$ and

$$(31) \qquad \left(\omega + i\partial\bar{\partial}\varphi\right)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n$$

The remainder of this section is structured as follows. In Section 4.1, we focus on proving Theorem 4.3. In fact, we provide a detailed construction of the ACyl metrics, and thus obtain more precise statements than those in Theorem 4.3.

Having derived the necessary tools, we then present the proof of Theorem 4.6 by reducing it to Theorem 4.7. Further examples to which Theorem 4.6 may be applied are then discussed in Section 4.3.

4.1. **Constructing ACyl Kähler metrics.** The goal is to prove Theorem 4.3, and we use the notation introduced at the beginning of Section 4.

Let $\pi : M \to M_{orb} := (\mathbb{C} \times D)/\Gamma$ be a resolution, where D denotes some compact Kähler manifold, and the action of $\Gamma = \langle\gamma\rangle \cong \mathbb{Z}_m$ is given

by (20). Also, recall that the cylindrical parameter $t : \mathbb{C}^* \times D \to \mathbb{R}$ is defined as $t(z, w) = \log |z|$.

We begin by focusing on the equivalence of Conditions (i) and (iii) in Theorem 4.3 as this is most relevant to our purpose. That (iii) implies (i) is settled in the next proposition.

Proposition 4.8. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4 and let the function t be defined by (21). Suppose that g_0 is a Kähler metric on M, whose Kähler form ω_0 satisfies*

$$(32) \qquad \iota_{J\frac{\partial}{\partial t}} \omega_0 = df \quad on \quad \{t \geq 0\} \subset (\mathbb{C}^* \times D)/\Gamma,$$

for some smooth $f : \{t \geq 0\} \to \mathbb{R}$, where J denotes the complex structure on M. Then there exists an ACyl Kähler metric g on M, with Kähler form ω, such that $[\omega] = [\omega_0] \in H^2(M, \mathbb{R})$.

Moreover, if g is lifted to $\mathbb{C}^* \times D$, it is explicitly given by

$$(33) \qquad g = g_{cyl} = e^{-2t} g_{\mathbb{C}} + g_D \quad on \quad \{t \geq t_0\} \subset \mathbb{C}^* \times D$$

for some $t_0 > 1$, where $g_{\mathbb{C}}$ denotes the Euclidean metric on $\mathbb{C}$ and g_D is the restriction of g_0 to the slice $\{1\} \times D \subset \mathbb{C}^* \times D$.

Interestingly, the ACyl metrics obtained by the previous proposition are of *optimal rate*, i.e. they are *equal* to its asymptotic model g_{cyl} outside some compact domain. This is an even stronger statement than claimed in Theorem 4.3.

Proof. Analogously to [HHN15][Section 4.2], the idea is to glue the Kähler form ω_0 to a certain cylindrical Kähler form ω_{cyl} outside of some compact domain. Doing so, however, requires that the difference of these two (1,1)-forms is $\partial\bar{\partial}$-exact.

Thus, before we can perform any gluing, we need to have a description of ω_0 in terms of a Kähler potential, at least on the set $\{t \geq 0\}$. We begin by explaining the construction of such a potential function.

Suppose that ω_0 is a Kähler form satisfying

$$(34) \qquad \iota_{J\frac{\partial}{\partial t}} \omega_0 = df \quad on \quad \{t \geq 0\} \subset (\mathbb{C}^* \times D)/\Gamma,$$

for some smooth function f. Working on $\mathbb{C}^* \times D$, we lift ω_0 and f to Γ-invariant forms denoted by the same letters. We view $\mathbb{C}^* \times D$ as a (trivial) fibre bundle over D, and introduce two holomorphic maps

$$j : D \to \mathbb{C}^* \times D \quad and \quad p : \mathbb{C}^* \times D \to D,$$

where j is the inclusion of the slice $\{1\} \times D \subset \mathbb{C}^* \times D$, and p the projection onto D. Then we *define* a Kähler form ω_D on D by setting

$$\omega_D := j^* \omega_0.$$

Using the cylindrical parameter t as defined in (21), we identify $\mathbb{C}^* \cong \mathbb{R} \times \mathbb{S}^1$ and define a new function φ by

$$\varphi(t, y) := 2 \int_0^t f(s, y) ds \quad \text{for } t \in \mathbb{R}_{\geq 0} \text{ and } y \in \mathbb{S}^1 \times D.$$

Then we claim that

(35) $$\omega_0 = i\partial\bar{\partial}\varphi + p^*\omega_D \quad \text{on} \quad \{t \geq 0\} \cong \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \times D.$$

In other words, we have to show that the $(1,1)$-form $\alpha := \omega_0 - i\partial\bar{\partial}\varphi$ is a basic form for the fibre bundle $p : \mathbb{C}^* \times D \to D$. This means that

(36) $$\mathcal{L}_V \alpha = 0 \quad \text{and} \quad \iota_V \alpha = 0$$

for all vector fields V on $\mathbb{C}^* \times D$ which are tangent to the fibres of the projection p. However, since α is d-closed, it suffices to show the second condition in (36), and thus we only have to prove that

(37) $$\iota_{\frac{\partial}{\partial t}} \alpha = 0 \quad \text{and } \iota_{J\frac{\partial}{\partial t}} \alpha = 0$$

since any vector field tangent to fibres of p can be written in terms of $\partial/\partial t$ and $J\partial/\partial t$.

Let us begin by considering the first equation in (37). Keeping in mind that $(J\partial/\partial t)(f) = 0$ by (34), we split $df = d_t f + d_D f$, where d_t and d_D are the differentials in direction of the $\mathbb{R}$- and D-factor, respectively. Using the definition of φ and the fact that $\partial\bar{\partial}t = 0$, we observe

$$2i\partial\bar{\partial}\varphi = dJd\varphi = 2df \wedge Jdt + d_t Jd_D\varphi + d_D Jd_D\varphi,$$

so that we conclude from (34)

$$\iota_{\frac{\partial}{\partial t}} \left(i\partial\bar{\partial}\varphi \right) = \frac{\partial}{\partial t} f Jdt + \frac{1}{2} Jd_D \frac{\partial}{\partial t}\varphi = Jdf = \iota_{\frac{\partial}{\partial t}} \omega_0,$$

as claimed. The second equation in (37) follows similarly:

$$\iota_{J\frac{\partial}{\partial t}} \left(i\partial\bar{\partial}\varphi \right) = -df \cdot (Jdt) \left(J\frac{\partial}{\partial t} \right) = df = \iota_{J\frac{\partial}{\partial t}} \omega_0.$$

This finishes the proof of (35).

Let us define the cylindrical Kähler form ω_{cyl} on $\mathbb{C}^* \times D$ to be

$$\omega_{cyl} := i\partial\bar{\partial}t^2 + p^*\omega_D.$$

The goal is to construct a new Kähler form ω, cohomologous to ω_0, such that

(38) $$\omega = \begin{cases} \omega_{cyl} & \text{on } \{t \geq t_2\}, \\ \omega_0 & \text{on } \{t \leq t_1\} \end{cases}$$

for some positive numbers $t_1 < t_2$. The following gluing procedure is an adaptation of the one contained on [HHN15][p. 247]. For this

construction, we first fix $t_0 > 1$ and choose a cut-off function $\chi = \chi(t)$ satisfying

$$\chi(t) = \begin{cases} 1 & \text{if } t \geq t_0, \\ 0 & \text{if } t \leq 1, \end{cases}$$

and then we define a Γ-invariant $(1,1)$-form ω on $\{t \geq 0\}$ by

$$\omega := i\partial\bar{\partial}\left(\chi(t) \cdot t^2 + (1 - \chi(t)) \cdot \varphi\right) + \rho(t)dt \wedge d^c t + p^*\omega_D,$$

where $\rho(t)dt \wedge d^c t$ is an exact bump-form supported inside a neighborhood of $[1, t_0]$, say $[1/2, t_0 + 1/2]$. Clearly, $\omega - \omega_0$ is exact and ω agrees with ω_0 inside the region $\{t \leq 1/2\}$, so that ω extends to a $(1,1)$-form on M.

Moreover, we notice that $\omega = \omega_{cyl}$ if $t \geq t_0 + 1/2$, and thus, the only thing left to show is the positivity of ω on the region $\{1/2 \leq t \leq t_0 + 1/2\}$. For $t \in [1/2, t_0 + 1/2] \setminus [1, t_0]$, this is clear because we have

$$\omega = \begin{cases} \omega_{cyl} + \rho dt \wedge d^c t & \text{on } \{t \geq t_0\}, \\ \omega_0 + \rho dt \wedge d^c t & \text{on } \{t \leq 1\} \end{cases}$$

and $\rho \geq 0$, so we only need to focus on the case $t \in [1, t_0]$.

To show that $\omega > 0$ on this region, it suffices to check that ω is positive in the direction of the D-factor since we can then compensate for potentially negative terms by choosing ρ sufficiently large inside $[1, t_0]$. Hence, consider $0 \neq v \in T_{\mathbb{C}}D$ and observe

$$\begin{aligned} \omega(v, \overline{v}) &= (1 - \chi(t)) \cdot \left(i\partial\bar{\partial}\varphi\right)(v, \overline{v}) + p^*\omega_D(v, \overline{v}) \\ &= (1 - \chi(t)) \cdot \omega_0(v, \overline{v}) + \chi(t) \cdot p^*\omega_D(v, \overline{v}) \\ &> 0, \end{aligned}$$

where we used in the first line that χ only depends on t, and the second equation follows from (35). As explain before, ω is positive on $\{1 \leq t \leq t_0\}$ once we choose $\rho \gg 1$ on $[1, t_0]$, and so we constructed a Kähler form ω on M in the same cohomology class as ω_0, which also satisfies (38). The corresponding ACyl metric g then fulfills (33), since both g and g_{cyl} are Kähler with respect to the same complex structure. $\qquad\square$

For the converse to Proposition 4.8, i.e. that (i) of Theorem 4.3 implies (iii), we additionally assume that the $\mathbb{C}^*$-action on M_{orb} given by (26) extends π-equivariantly to a holomorphic action on the resolution $\pi : M \to M_{orb}$. Hence, the infinitesimal generators of this action extend to real holomorphic vector fields on all of M. Let Y denote the generator of the induced $\mathbb{R}_+$-action (corresponding to translation in the cylindrial parameter t), i.e.

$$(39) \qquad\qquad Y = \frac{\partial}{\partial t} \quad \text{on} \quad (\mathbb{C}^* \times D)/\Gamma \subset M.$$

Note that if J is the complex structure on M, then JY is generating the $\mathbb{S}^1$-action on M.

Next, we show that Condition (iii) in Theorem 4.3 is in fact necessary for a Kähler class to be ACyl.

Proposition 4.9. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4 and assume that the $\mathbb{C}^*$-action given by (26) extends π-equivariantly to a holomorphic action on M.*

Then every ACyl Kähler class contains an ACyl Kähler form $\hat{\omega}$ such that

$$\iota_{JY}\hat{\omega} = df,$$

where JY is the infinitessimal generator of the $\mathbb{S}^1$-action.

Proof. Let g be an ACyl Kähler metric, with Kähler form ω, such that (25) holds. First, average ω over the $\mathbb{S}^1$-action to obtain a Kähler form $\hat{\omega}$ in the same cohomology class. Then observe that the averaging does not change the asymptotic behavior since both g_{cyl} and t are $\mathbb{S}^1$-invariant, so that the corresponding metric $\hat{g}$ is ACyl and satisfies (25). In particular, the function $t = \log|z|$ is also the cylindrical parameter for $\hat{g}$.

Then we notice that JY, for Y given by (39), is a Killing field for $\hat{g}$ because $\mathcal{L}_{JY}\hat{\omega} = 0$. Thus, Proposition 3.5 implies that Y is the gradient field of some function on M, or equivalently that $\iota_{JY}\hat{\omega}$ is exact. $\square$

It only remains to show the equivalence of (i) and (ii) in Theorem 4.3, i.e. that each ACyl Kähler class is the restriction of some orbifold Kähler class on the complex compactification $\overline{M}$.

This goes back to a construction of Haskins, Hein and Nordström [HHN15]. In fact, their ideas can be used to prove the following

Proposition 4.10. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4, and suppose that $\overline{M} = M \cup \overline{D}$ is the compacification obtained by adding the orbifold divisor $\overline{D} = D/\Gamma$ at infinity.*

For a given $\kappa \in H^2(M, \mathbb{R})$, the following are equivalent:

(i) κ is an ACyl Kähler class.
(ii) $\kappa = \kappa_{\overline{M}}|_M$ for some orbifold Kähler class $\kappa_{\overline{M}}$ on $\overline{M}$.

Proof. That (i) implies (ii) is a direct consequence of [HHN15][Theorem 3.2], which can be applied here since g and g_{cyl} are Kähler with respect to the same complex structure.

The construction required for the converse implication can be found on [HHN15][p. 247], so we only briefly sketch the idea.

If $\omega_{\overline{M}}$ is a Kähler form on $\overline{M}$, then we define ω_D to be the restriction of $\omega_{\overline{M}}$ to the orbifold divisor $\overline{D} = \{\infty\} \times D/\Gamma$. Note that ω_D lifts to a smooth Γ-invariant form on D, and so we can define the asymptotic

model ω_{cyl} on $\mathbb{C}^* \times D$ to be

$$\omega_{cyl} := i\partial\bar{\partial}t^2 + \omega_D.$$

The new ACyl Kähler form asymptotic to ω_{cyl} is then constructed as

$$\omega := \omega_{\overline{M}} + i\partial\bar{\partial}\left(\chi \cdot t^2\right) + \rho dt \wedge d^c t,$$

for some cut-off function χ and a bump-function ρ. The cut-off χ is equal to 1 in a neighborhood of $\overline{D}$ and 0 if $t \leq 0$, and ρ is chosen sufficiently large to ensure positivity. $\square$

This concludes the proof of Theorem 4.3, and so we focus on proving Theorem 4.6 next.

4.2. **Proof of Theorem 4.6.** Let $D^{n-1}, \Omega_D, \Gamma = \langle\gamma\rangle$ and $\pi : M \to (\mathbb{C} \times D)/\langle\gamma\rangle$ be defined as in Theorem 4.6. In particular, the discussion of the previous subsection applies and we use the same notation as introduced at the beginning of Section 4. We also assume that the $\mathbb{C}^*$-action on M_{orb} defined by

$$\lambda * (z, w) := (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action on M. As a consequence, the infinitesimal generators of this action extend to real holomorphic vector fields on M. Let X be two-times the generator of the induced $\mathbb{R}$-action (corresponding to translation in the cylindrical parameter t), i.e.

$$X = 2\frac{\partial}{\partial t} \quad \text{on} \quad (\mathbb{C}^* \times D)/\Gamma \subset M.$$

Then JX is two-times the generator of the $\mathbb{S}^1$-action, where J is the complex structure on M.

Moreover, we point out that the action of γ given by (30) preserves the holomorphic $(n, 0)$ form Ω on $\mathbb{C}^* \times D$ defined as

$$\Omega := dz \wedge \Omega_D$$

since γ satisfies (29). In particular, Ω descends to M_{orb} and, because the resolution $\pi : M \to M_{orb}$ is crepant, Ω then extends to a holomorphic $(n, 0)$-form on M, which we also denote by Ω.

Let $\kappa \in H^2(M, \mathbb{R})$ be an ACyl Kähler class, i.e. there exists an ACyl metric g satisfying (25) and with Kähler form $\omega \in \kappa$. We need to find a different ACyl metric g_0 with Kähler form ω_0 also contained in the given class κ, such that $X = \nabla^{g_0} f$ and

$$(40) \qquad \left(\omega_0 + i\partial\bar{\partial}\varphi\right)^n = \alpha e^{-f - \frac{X}{2}(\varphi)} i^{n^2} \Omega \wedge \overline{\Omega},$$

for some JX-invariant functions $f, \varphi : M \to \mathbb{R}$ and some constant $\alpha \in \mathbb{R}$. According to Lemma 3.3, $\omega_0 + i\partial\bar{\partial}\varphi$ is then a gradient steady Kähler-Ricci soliton, as required. To achieve this, we begin by modifying ω near infinity to improve the convergence rate and to ensure that it is asymptotic to a *Ricci-flat* cylinder.

First, we improve the asymptotic behavior of ω by applying Proposition 4.8, so that there exists an ACyl Kähler form $\omega_1 \in [\omega]$ which, if lifted to $\mathbb{C}^* \times D$, is of the form

$$\omega_1 = i\partial\bar{\partial}t^2 + \omega_D \quad \text{on} \quad \{t \geq t_0\}$$

for some $t_0 > 0$. Here, ω_D denotes the restriction of ω to the slice $\{1\} \times D$.

In a second step, we modify ω_0 so that it becomes Ricci-flat if restricted to $\{t\} \times D$ for $t \gg t_0$. Recall that by Yau's Theorem [Yau78], there exists $u_D : D \to \mathbb{R}$ such that $\omega_{RF} := \omega_D + i\partial\bar{\partial}u_D > 0$ and

$$(41) \qquad (\omega_{RF})^{n-1} = ci^{(n-1)^2}\Omega_D \wedge \overline{\Omega}_D.$$

Moreover, the uniqueness of solutions to (41) implies that u_D is γ-invariant, because γ preserves both ω_D and $\Omega_D \wedge \overline{\Omega}_D$.

Choosing a cut-off function χ with

$$\chi(t) = \begin{cases} 1 & \text{if } t \geq t_0 + 2 \\ 0 & \text{if } t \leq t_0 + 1, \end{cases}$$

we then define a Γ-invariant $(1,1)$-form by

$$\omega_0 := \omega_1 + i\partial\bar{\partial}\left(\chi \cdot u_D\right) + \rho dt \wedge d^c t,$$

where ρ is a bump-function supported in a small neighborhood of $[t_0 + 1, t_0 + 2]$. By the same reasoning as in the proof of Proposition 4.8, ω_0 is positive if ρ is sufficiently large and thus, ω_0 defines a Kähler metric on M in the class $\kappa = [\omega]$. Note that by construction we have

$$(42) \qquad \omega_0 = i\partial\bar{\partial}t^2 + \omega_{RF}$$

on the region $\{t \geq t_0 + 3\}$.

The next step is to further modify ω_0 so that it satisfies the requirements of Theorem 4.7. Note that after averaging ω_0 over the compact and connected group $\mathbb{S}^1$ we can assume that ω_0 is invariant under the $\mathbb{S}^1$-action because averaging neither affects the cohomology class, nor the positivity of ω_0. Hence, JX is a Killing field for the corresponding Kähler metric g_0 and by Proposition 3.5, there exists a function f such that

$$X = \nabla^{g_0}f \quad \text{and} \quad f - 2t \in C_\delta^\infty(M),$$

for each $\delta > 0$. In fact, we conclude from (42) that

$$(43) \qquad f = 2t \quad \text{on} \quad \{t \geq t_0 + 3\}.$$

In particular, we notice that (M, g_0) satisfies the assumptions of Theorem 4.7.

Let us define a JX-invariant function $F : M \to \mathbb{R}$ by

$$F := \log \frac{\alpha i^{n^2} \Omega \wedge \overline{\Omega}}{\omega_0^n} - f$$

for some constant α to be fixed later. For an appropriate choice of α, we claim that F has compact support. To see this, first observe from (41) and (42) that the cylindrical volume form of ω_{cyl} can be computed as

$$\omega_{cyl}^n = \frac{c\,n}{2}|z|^{-2}i^{n^2}dz \wedge \Omega_D \wedge d\bar{z} \wedge \overline{\Omega}_D,$$

so we set $\alpha := cn/2$, and obtain

$$\begin{aligned} F &= \log \frac{\alpha i^{n^2}\Omega \wedge \overline{\Omega}}{\omega_{cyl}^n} + \log \frac{\omega_{cyl}^n}{\omega_0^n} - f \\ &= 2t - f \\ &= 0, \end{aligned}$$

if $t \geq t_0 + 3$. Thus, F is compactly supported.

If we fix *some* $0 < \varepsilon < 2$, Theorem 4.7 yields a JX-invariant $\varphi \in C_\varepsilon^\infty(M)$ such that

$$(44) \qquad \left(\omega_0 + i\partial\bar{\partial}\varphi\right)^n = e^{F - \frac{X}{2}(\varphi)}\omega_0^n = \frac{cn}{2}e^{-f - \frac{X}{2}(\varphi)}i^{n^2}\Omega \wedge \overline{\Omega},$$

which is precisely (40), so that $\omega_0 + i\partial\bar{\partial}\varphi$ defines a gradient steady Kähler-Ricci soliton. The underlying Kähler metric is clearly ACyl of rate ε.

However, since $F \in C_\varepsilon^\infty(M)$ for *all* $0 < \varepsilon < 2$ and since solutions to (44) contained in $C_\varepsilon^\infty(M)$ are *unique*, we may conclude that indeed $\varphi \in C_\varepsilon^\infty(M)$ for all $0 < \varepsilon < 2$, finishing the proof.

4.3. **Examples.** We begin by providing further examples in complex dimension two. The manifolds M_k considered below are defined as in [BM11][Section 2.2], and their construction is similar to Example 4.1.

Example 4.11. For $k = 2, 3, 4, 6$ we consider the maps $\gamma_k : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\gamma_k(z_1, z_2) := \left(e^{\frac{2\pi i}{k}}z_1, e^{-\frac{2\pi i}{k}}z_2\right)$$

If we let $\mathbb{T}$ be the (real) 2-torus, then γ_k descends to $\mathbb{C} \times \mathbb{T}$, provided the lattice in $\mathbb{C}$ is chosen appropriately: For $k = 2, 4$, let $\mathbb{T}$ be obtained from the square-lattice, and for $k = 3, 6$ use the hexagonal one instead.

In any case, we may define orbifolds $M_{orb}^k := (\mathbb{C} \times \mathbb{T})/\langle\gamma_k\rangle$ with isolated singular points which are locally modelled on a neighborhood of the origin in $\mathbb{C}^2/\mathbb{Z}_j$, with $\mathbb{Z}_j$-action induced by the map

$$(z_1, z_2) \mapsto (e^{\frac{2\pi i}{j}}z_1, e^{-\frac{2\pi i}{j}}z_2)$$

for $j \in \{2, 3, 4, 6\}$. More precisely,

- If $k = 2$, M_{orb}^2 has four singularities, all isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$.
- If $k = 4$, the corresponding orbifold M_{orb}^4 has one $\mathbb{C}^2/\mathbb{Z}_2$ and two $\mathbb{C}^2/\mathbb{Z}_4$ singularities.

- If $k = 3$, there are three singular points in M_{orb}^3 and all are isomorphic to $\mathbb{C}^2/\mathbb{Z}_3$.
- If $k = 6$, M_{orb}^6 also has three singularities: one $\mathbb{C}^2/\mathbb{Z}_2$, one $\mathbb{C}^2/\mathbb{Z}_3$ and one $\mathbb{C}^2/\mathbb{Z}_6$ singularity.

In each case, condition (29) is fulfilled and the blow-up of all singularities results in a complex manifold denoted by M_k. The corresponding resolution is indeed crepant since all singularities are isolated points and because blowing-up the origin in the local models $\mathbb{C}^2/\mathbb{Z}_j$ yields in fact a crepant resolution. Similar to the reasoning in Example 4.1 and Remark 4.5, one can show that M_k satisfies the requirements of both Theorem 4.6 and Corollary 4.4.

Thus, there is a steady Kähler-Ricci soliton in *each* Kähler class of M_k. Interestingly, these manifolds also admit ALG gravitational instantons by [BM11][Theorem 2.3], for instance.

For finding examples of complex dimension 3, we may take D to be a product $\mathbb{T} \times \mathbb{T}$, but then we consider a different resolution, as the next example shows.

Example 4.12. Let $\mathbb{T}$ be constructed from the hexagonal lattice in $\mathbb{C}$. By setting $D := \mathbb{T} \times \mathbb{T}$ we define $\gamma : \mathbb{C} \times D \to \mathbb{C} \times D$ by

$$\gamma(z_1, z_2, z_3) = e^{\frac{2\pi i}{3}}(z_1, z_2, z_3)$$

and note that $\gamma^*(dz_2 \wedge dz_3) = e^{-\frac{2\pi i}{3}} dz_2 \wedge dz_3$, i.e. (29) is satisfied. Each of the $3^2 = 9$ singularities of $(\mathbb{C} \times D)/\mathbb{Z}_3$ is modelled on $\mathbb{C}^3/\mathbb{Z}_3$, and so we may consider the blow-up M of all singular points.

As before, this resolution is crepant and the $\mathbb{C}^*$-action on the first factor extends, because the same is true for the resolution

$$\mathcal{O}_{\mathbb{CP}^2}(-3) \to \mathbb{C}^3/\mathbb{Z}_3.$$

Moreover, the only closed, γ-invariant 1-forms on D are clearly exact, so that again *each* Kähler class admits a steady Kähler-Ricci soliton.

We conclude this section by discussing another class of examples with D a K3-surface and γ an antisymplectic involution. Explicit examples of such K3-surfaces can for instance be obtain form the Kummer's construction.

Example 4.13. Let D be a K3-surface together with a trivialisation Ω_D of the canonical bundle. Suppose that γ_D is a holomorphic involution on D such that

$$\gamma_D^*\Omega_D = -\Omega_D.$$

Also assume that the fixed point set $\text{Fix}(\gamma_D)$ is non-empty. This implies that $\text{Fix}(\gamma_D)$ is the disjoint union of smooth, complex curves. (In fact, there is a classification for all possibilities of $\text{Fix}(\gamma_D)$, compare [Nik83].)

At any $p \in \mathrm{Fix}(\gamma_D)$, we may linearise γ_D so that its action in a suitable chart is given by

$$
(45) \qquad
\begin{aligned}
\mathbb{C}^2 &\to \mathbb{C}^2 \\
(z_1, z_2) &\to (-z_1, z_2)
\end{aligned}
$$

In particular, the singular set of the orbifold $D/\langle \gamma_D \rangle$ locally corresponds to $\{z_1 = 0\}$ inside $\mathbb{C}^2/\mathbb{Z}_2$, with $\mathbb{Z}_2$-action defined by (45).

As in Theorem 4.6, we let $\gamma : \mathbb{C} \times D \to \mathbb{C} \times D$ be

$$
\gamma(z_0, z) := (-z_0, \gamma_D(z)).
$$

Then the singularities of $M_{orb} = (\mathbb{C} \times D)/\langle \gamma \rangle$ are locally isomorphic to $\mathbb{C}^3/\mathbb{Z}_2 \cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, where $\mathbb{Z}_2$ acts by -1 in the first two factors, and trivially in the third one. This orbifold, however, admits a *unique* crepant resolution

$$
(46) \qquad \mathcal{O}_{\mathbb{CP}^1}(-2) \times \mathbb{C} \to \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C},
$$

so that the local resolutions may be patched together to yield a crepant resolution $M \to M_{orb}$. Moreover, the $\mathbb{C}^*$-action by multiplication in the first factor extends to M, because this is clearly true for the local model (46).

Since $H^1(D, \mathbb{R}) = 0$, we deduce that each Kähler class on M admits a steady Kähler-Ricci soliton, thanks to Theorem 4.6 and Corollary 4.4.

5. The monge-ampère equation

In this section, we present the proof of Theorem 4.7. We consider a more general setting as in Theorem 4.6 in order to clarify which assumptions are used for the a priori estimates below. The following list of properties is assumed throughout this section:

Assumption 5.1. Let (M, g) be an ACyl manifold of (real) dimension $2n$ in the sense of Definition 2.1.

A.1 Suppose there exists a complex structure J on M, so that (M, g, J) is Kähler and denote the Kähler form by ω.

A.2 There exists a real holomorphic vector field X on M such that

$$
X = 2\Phi_* \frac{\partial}{\partial t},
$$

where Φ denotes the ACyl map and t the cylindrical coordinate function of (M, g).

A.3 JX is a Killing field of g. In particular, $\mathcal{L}_{JX}\omega = 0$ and according to Proposition 3.5, there exists a smooth $\tilde{f} : M \to \mathbb{R}$ such that $X = \nabla^g \tilde{f}$ and

$$
\tilde{f} - 2t \in C_\delta^\infty(M),
$$

where $\delta > 0$ is the convergence rate of (M, g) to its asymptotic model. We normalise the proper function $\tilde{f}$ by choosing a $c > 0$ such that $f := \tilde{f} + c \geq 1$ so that we still have $X = \nabla^g f$.

The reader may recall that the ACyl metric constructed in Section 4.2 satisfies all of these requirements.

We define new function spaces $C^\infty_{\varepsilon, JX}(M)$ consisting of all elements in $C^\infty_\varepsilon(M)$ which are JX-invariant, i.e.

$$C^\infty_{\varepsilon, JX}(M) := \{u \in C^\infty_\varepsilon(M) \,|\, JX(u) = 0\}.$$

Using this notation, the main result of this section is the next

Theorem 5.2. *Let (M, g) be an ACyl manifold of real dimension $2n$ satisfying the assumptions A.1, A.2 and A.3. Given $F \in C^\infty_{\varepsilon, JX}(M)$ for some $1 < \varepsilon < 2$, there exists a unique $\varphi \in C^\infty_{\varepsilon, JX}(M)$ such that $\omega + i\partial\bar{\partial}\varphi$ is Kähler and satisfies*

$$(47) \qquad \left(\omega + i\partial\bar{\partial}\varphi\right)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n.$$

This theorem is analogue to [CD20b][Theorem 7.1], and we also follow the same strategy as in [CD20b][Section 7] to prove it, i.e. we set up a continuity method.

For given $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$ and $F \in C^\infty_{\varepsilon, JX}(M)$ with $1 < \varepsilon < 2$, we define the Monge-Ampère operator on the set $\mathcal{U}$ containing all $\varphi \in C^{k+2,\alpha}_{\varepsilon, JX}(M)$ with $\omega + i\partial\bar{\partial}\varphi > 0$ as follows:

$$(48) \qquad \begin{aligned} \mathcal{M} : \mathcal{U} \times [0, 1] &\to C^{k,\alpha}_{\varepsilon, JX}(M) \\ (\varphi, s) &\mapsto \log \frac{(\omega + i\partial\bar{\partial}\varphi)^n}{\omega^n} + \frac{X}{2}(\varphi) - sF \end{aligned}$$

It is worth mentioning that the function $\mathcal{M}(\varphi, s)$ is indeed JX-invariant since F is assumed to be invariant under JX, and also $\mathcal{L}_{JX}\omega = 0$ by A.3. Before applying the implicit function theorem, we need to compute the linearization of $\mathcal{M}$, i.e. the derivative at the point (φ, s) in direction of $(u, 0)$:

$$(49) \qquad D\mathcal{M}_{(\varphi, s)}(u, 0) = \frac{1}{2}\Delta_{g_\varphi}(u) + \frac{X}{2}(u).$$

Here Δ_{g_φ} denotes the Riemannian Laplace operator of the metric g_φ associated to the Kähler form $\omega + i\partial\bar{\partial}\varphi$.

As in [CD20b], the first step is to show that the linearized operator is an isomorphism $C^{k+2,\alpha}_{\varepsilon, JX}(M) \to C^{k,\alpha}_{\varepsilon, JX}(M)$, which is covered in the next

Proposition 5.3. *Let (M, g) be an ACyl manifold of real dimension $2n$ satisfying the assumptions A.1, A.2 and A.3. Given $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$ and $0 < \varepsilon < 2$, the operator*

$$\Delta_g + X : C^{k+2,\alpha}_{\varepsilon, JX}(M) \to C^{k,\alpha}_{\varepsilon, JX}(M)$$

is an isomorphism.

Here, our arguments differ from those in [CD20b][Theorem 6.6], because the metrics we consider have a different asymptotic behavior. Instead, we reduce the proof to Theorem 2.11.

Proof. First, we observe by Assumption A.3, that $\Delta_g + X$ is an ACyl drift operator in the sense of Definition 2.10. Thus, according to Theorem 2.11, the map

$$\Delta_g + X : C_\varepsilon^{k+2,\alpha}(M) \to C_\varepsilon^{k,\alpha}(M)$$

is an isomorphism for $k \in \mathbb{N}_0, \alpha \in (0,1)$ and $0 < \varepsilon < 2$. Consequently, it only remains to show that $u \in C_\varepsilon^{k+2,\alpha}(M)$ is JX-invariant, provided $(\Delta_g + X)(u)$ is invariant under JX. To see this, we use that X is real holomorphic and obtain

$$[X, JX] = J[X, X] = 0,$$

so that $JX(X(u)) = X(JX(u))$. Moreover, we have $JX(\Delta_g u) = \Delta_g(JX(u))$ which follows directly from the relation

$$\frac{1}{2}\Delta_g u\,\omega^n = n\,i\partial\bar{\partial}u \wedge \omega^{n-1}$$

by applying $\mathcal{L}_{JX}\omega = 0$. Hence, we conclude that if $(\Delta_g + X)(u)$ is JX-invariant for some $u \in C_\varepsilon^{k+2,\alpha}(M)$, then

$$0 = JX((\Delta_g + X)(u)) = (\Delta_g + X)(JX(u)).$$

As $|X|_g$ is bounded, $JX(u)$ tends to 0 as $t \to \infty$, and the maximum principle yields $JX(u) = 0$, as desired. $\qquad\square$

Remark 5.4 (on the decay rate ε)**.** The reader may notice that Proposition 5.3 holds for all $0 < \varepsilon < 2$, whereas Theorem 5.2 only includes the case $F \in C_{\varepsilon,JX}^\infty$ with $1 < \varepsilon < 2$. This is because Conlon and Deruelle's approach to the uniform C^0-estimate requires the convergence of certain weighted functionals, compare Definition 5.8 below.

However, it seems plausible to use Theorem 5.2 together with ideas contained in [CD20b][Section 9] to cover the case $0 < \varepsilon \le 1$ as well, but we do not pursue this further in this article.

We also obtain the following regularity statement for the Monge-Ampère operator.

Proposition 5.5 (Regularity)**.** *Let* $(M,g), F \in C_{\varepsilon,JX}^\infty(M)$ *and* $1 < \varepsilon < 2$ *be as in Theorem 5.2. Suppose that* $\varphi \in C_{\varepsilon',JX}^{3,\alpha}(M)$ *for some* $0 < \varepsilon' \le \varepsilon$ *satisfies*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n.$$

Then $\varphi \in C_{\varepsilon,JX}^\infty(M)$.

Note that this statement only gives *qualitative* information about the function φ, i.e. it does *not* provide uniform estimates for the $C_\varepsilon^\infty(M)$-norm of φ. The crucial part of the continuity method, however, is precisely to obtain uniform a priori bounds on $\|\varphi\|_{C_\varepsilon^{k,\alpha}}$. This is achieved in the next

Theorem 5.6 (A priori estimates). *Let (M, g), $F \in C_{\varepsilon, JX}^\infty(M)$ and $1 < \varepsilon < 2$ be as in Theorem 5.2. Suppose that $(\varphi_s)_{0 \le s \le 1}$ is a family in $C_{\varepsilon, JX}^\infty(M)$ such that $\omega + i\partial\bar\partial\varphi_s$ is Kähler for each $s \in [0, 1]$ and satisfies*

$$(50) \qquad \left(\omega + i\partial\bar\partial\varphi_s\right)^n = e^{s \cdot F - \frac{X}{2}(\varphi_s)}\omega^n.$$

Then, for given $k \in \mathbb{N}_0, \alpha \in (0, 1)$, there exists a constant $C > 0$ such that

$$\sup_{s \in [0,1]} \|\varphi_s\|_{C_\varepsilon^{k,\alpha}} \le C,$$

where C only depends on k, α, F and the geometry of (M, g).

The strategy for proving Proposition 5.5 and Theorem 5.6 is to follow, up to some minor adjustments, the arguments provided by Conlon and Deruelle ([CD20b][Section 7]). In particular, we use their idea to achieve the uniform C^0-bound, but we present a variation of their arguments which allows us to immediately assume $F \in C_{\varepsilon, JX}^\infty(M)$ with $1 < \varepsilon < 2$, instead of first considering functions F with compact support as in [CD20b][Theorem 7.1].

We postpone the proofs of both Proposition 5.5 and Theorem 5.6 to subsequent sections and for now assume these results to conclude Theorem 5.2.

Proof of Theorem 5.2. First, we point out that we only need to show the existence statement since the uniqueness part is a direct consequence of the maximum principle, see [BM17][Proposition 1.2].

For the proof of existence, assume we are given $F \in C_{\varepsilon, JX}^\infty(M)$, and consider the set

$$S := \{s \in [0, 1] \,|\, \text{there exists a } \varphi_s \in C_{\varepsilon, JX}^\infty(M) \text{ satisfying (50)}\}.$$

Clearly, $0 \in S$ and so it is sufficient to show that S is both open and closed.

The openness is a consequence of Proposition 5.3. To see this, let $\mathcal{U}$ be the set of all $\psi \in C_{\varepsilon, JX}^{3;\alpha}(M)$ such that $\omega + i\partial\bar\partial\psi > 0$ and consider the Monge-Ampère operator $\mathcal{M}$ defined by

$$\mathcal{M} : \mathcal{U} \times [0, 1] \to C_{\varepsilon, JX}^{1;\alpha}(M)$$

$$(\psi, s) \mapsto \log \frac{(\omega + i\partial\bar\partial\psi)^n}{\omega^n} + \frac{X}{2}(\psi) - sF$$

Suppose we are given $s_0 \in S$, i.e. $\varphi_{s_0} \in C_{\varepsilon, JX}^\infty(M)$ solving (50). Since φ_{s_0} is JX-invariant and $\varphi_{s_0} \in C_\varepsilon^\infty(M)$, the Riemannian metric $g_{\varphi_{s_0}}$

corresponding to $\omega + i\partial\bar{\partial}\varphi_{s_0}$ is ACyl, with the same ACyl map as g, and satisfies Assumptions A.1, A.2 and A.3. Hence, the linearization of $\mathcal{M}$ at the point (φ_s, s), which is given by (49), is injective if restricted to the subspace $C^{3,\alpha}_{\varepsilon, JX}(M)$ and also surjective according to Proposition 5.3. Thus, the implicit function theorem implies the existence of a $\delta_0 > 0$ such that for all $s \in (s_0 - \delta_0, s_0 + \delta_0)$ there exists a $\varphi_s \in C^{3,\alpha}_{\varepsilon}(M)$ solving (50). But then $\varphi_s \in C^{\infty}_{\varepsilon, JX}(M)$ by Proposition 5.5, and consequently $(s_0 - \delta_0, s_0 + \delta_0) \cap [0, 1] \subset S$.

That S is closed follows from Theorem 5.6. Indeed, consider a sequence $(s_k)_{k \in \mathbb{N}}$ in S which converges to some $s_\infty \in [0, 1]$, and denote the corresponding sequence in $C^{\infty}_{\varepsilon, JX}(M)$ of solutions to (50) by (φ_{s_k}). According to Theorem 5.6, this sequence (φ_{s_k}) is uniformly bounded in $C^{3,\alpha}_{\varepsilon}(M)$. Choosing $\varepsilon' \in (0, \varepsilon)$ and $\beta \in (0, \alpha)$, the inclusion $C^{3,\alpha}_{\varepsilon}(M) \subset C^{3,\beta}_{\varepsilon'}(M)$ is compact (by [Mar02][Theorem 4.3] for instance), so that we can extract a subsequence of (φ_{s_k}) converging in $C^{3,\beta}_{\varepsilon'}(M)$ to some limit $\varphi_{s_\infty} \in C^{3,\beta}_{\varepsilon'}(M)$. Note that we must have $JX(\varphi_{s_\infty}) = 0$ and that φ_{s_∞} satisfies

$$(\omega + i\partial\bar{\partial}\varphi_{s_\infty})^n = e^{s_\infty F - \frac{X}{2}(\varphi_{s_\infty})}\omega^n,$$

as we can take the point-wise limit $k \to \infty$ in (50). From this, we immediately see that $\omega + i\partial\bar{\partial}\varphi_{s_\infty}$ is a Kähler form, and applying Proposition 5.5 then implies $\varphi_{s_\infty} \in C^{\infty}_{\varepsilon, JX}(M)$, i.e. $s_\infty \in S$. This concludes the proof. $\square$

The rest of this section is devoted to proving Proposition 5.5 and Theorem 5.6. We begin in Section 5.1 by deriving the C^0-estimate which is the key part of the proof. Then we move on to higher-order estimates in Section 5.2 to finish the proof of Theorem 5.6. Afterwards, we conclude by verifying Proposition 5.5.

5.1. **The C^0-estimate.** Throughout this section, let (M, g) satisfy Assumptions A.1, A.2 and A.3. The goal is to obtain uniform estimates for solutions $(\varphi_s)_{0 \le s \le 1}$ to (50), among which the C^0-bound is the most difficult one to achieve.

The proof of the C^0-estimate is split into three parts: First, we obtain an upper bound on $\sup_M \varphi_s$, then an L^2-bound with a certain weight and finally, we can conclude a lower bound on $\inf_M \varphi_s$. The last two steps closely follow the ideas developed in [CD20b][Section 7.1]. Before beginning with the preparations, let us fix some notation.

Notation. We denote the metric associated with $\omega + i\partial\bar{\partial}\varphi_s$ by g_{φ_s}, and $\nabla^{g_{\varphi_s}}$, $\Delta_{g_{\varphi_s}}$, etc. denote the Levi-Civita connection, the Laplace operator, etc. of g_{φ_s}. We point out that $\Delta_{g_{\varphi_s}}$ is the *Riemannian* Laplace operator, i.e. it satisfies

(51) $$\frac{1}{2}\Delta_{g_{\varphi_s}} u \, \omega^n_{\varphi_s} = n \, i\partial\bar{\partial}u \wedge \omega^{n-1}_{\varphi_s}$$

for each C^2-function u.

5.1.1. *An upper bound on φ_s.* We begin by estimating φ_s from above:

Proposition 5.7 (Weighted upper bound on φ_s)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} \sup_M e^{\varepsilon t} \varphi_s \leq C,$$

where C only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

We present a proof based on the use of a barrier function, so our argument differs from the one given in [CD20b][Proposition 7.9].

Proof. Since $F \in C^\infty_\varepsilon(M)$ with $0 < \varepsilon < 2$ and because of Assumption A.3, Theorem 2.11 implies the existence of a function $u_F \in C^\infty_\varepsilon(M)$ such that

$$\frac{1}{2}\Delta_g u_F + \frac{X}{2}(u_F) = F.$$

Using $\log(1 + \tau) \leq \tau$, we deduce from (50) that φ_s satisfies

(52) $$\frac{1}{2}\Delta_g(\varphi_s) + \frac{X}{2}(\varphi_s) \geq sF,$$

which then leads to

$$(\Delta_g + X)(\varphi_s - su_F) \geq 2s(F - F) = 0.$$

Choosing a sequence $(t_k)_{k \in \mathbb{N}}$ with $t_k \to \infty$ and applying Hopf's maximum principle to a sequence of domains of the form $\{t \leq t_k\} \subset M$ then yields

$$\sup_M (\varphi_s - su_F) \leq \lim_{t \to \infty}(\varphi_s - su_F) = 0,$$

i.e. $\varphi_s \leq su_F$ holds on all of M. In particular, we observe that

$$e^{\varepsilon t}\varphi_s \leq su_F e^{\varepsilon t} \leq ||u_F||_{C^0_\varepsilon} =: C,$$

which proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

For obtaining a lower bound on φ_s, we need to work considerably harder. The important idea in [CD20b] is to first obtain a weighted L^2-bound.

5.1.2. *A weighted L^2-bound.* As in [CD20b][Subsection 7.1.1.], we consider two functionals which were used by Tian and Zhu [TZ00] to study shrinking Kähler-Ricci solitons on compact Fano manifolds.

Definition 5.8. Consider $1 < \varepsilon < 2$ and let $(\psi_\tau)_{0 \leq \tau \leq 1}$ be a C^1-path in $C^\infty_{\varepsilon, JX}(M)$ from $\psi_0 = 0$ to $\psi_1 = \psi$ and assume for each $\tau \in [0, 1]$ that $\omega_{\psi_\tau} := \omega + i \partial \bar{\partial} \psi_\tau > 0$. Define:

$$I_{\omega, X}(\psi) := \int_M \psi \left(e^f \omega^n - e^{f + \frac{X}{2}(\psi)} \omega_\psi^n \right),$$

$$J_{\omega, X}(\psi) := \int_0^1 \int_M \dot{\psi}_\tau \left(e^f \omega^n - e^{f + \frac{X}{2}(\psi_\tau)} \omega_{\psi_\tau}^n \right) \wedge d\tau,$$

where $\dot{\psi}_\tau = \frac{\partial}{\partial \tau} \psi_\tau$.

Since M is non-compact, we need to show that $I_{\omega, X}$ and $J_{\omega, X}$ are well-defined, i.e. that the resulting integrals are finite. Given $\psi \in C^\infty_{\varepsilon, JX}(M)$ with $1 < \varepsilon < 2$, we deduce from (A.3) that $\psi \, e^f = O(e^{(2-\varepsilon)t})$, so it suffices to show

$$(53) \qquad\qquad |\omega^n - e^{\frac{X}{2}(\psi)} \omega_\psi^n|_g = O(e^{-\varepsilon t})$$

since $\varepsilon > 1$. To see that this is true, we expand ω_ψ^n and obtain

$$\omega^n - e^{\frac{X}{2}(\psi)} \omega_\psi^n = \left(1 - e^{\frac{X}{2}(\psi)} \right) \omega^n - e^{\frac{X}{2}(\psi)} \sum_{k=1}^n \binom{n}{k} \left(i \partial \bar{\partial} \psi \right)^k \wedge \omega^{n-k}$$

from which (53) follows because $\frac{X}{2}(\psi) = O(e^{-\varepsilon t})$ and $|i \partial \bar{\partial} \psi|_g = O(e^{-\varepsilon t})$ by definition of $C^\infty_\varepsilon(M)$. Thus $I_{\omega, X}(\psi)$ is finite, and the same argument also proves that $J_{\omega, X}$ is well-defined. The crucial starting point is the next

Theorem 5.9. Let $(\psi_\tau)_{0 \leq \tau \leq 1}$ be a C^1-path as in Definition 5.8. Then the first variation of the difference $I_{\omega, X} - J_{\omega, X}$ is given by

$$\frac{d}{d\tau}(I_{\omega, X} - J_{\omega, X})(\psi_\tau) = -\int_M \psi_\tau \left(\frac{1}{2} \Delta_{g_{\psi_\tau}}(\dot{\psi}_\tau) + \frac{X}{2}(\dot{\psi}_\tau) \right) e^{f + \frac{X}{2}(\psi_\tau)} \omega_{\psi_\tau}^n,$$

where g_{ψ_τ} is the metric with Kähler form $\omega_{\psi_\tau} = \omega + i \partial \bar{\partial} \psi_\tau$. Moreover, $J_{\omega, X}$ does not depend on the choice of path $(\psi_\tau)_{0 \leq \tau \leq 1}$, but only on the end points $\psi_0 = 0$ and $\psi_1 = \psi$.

Proof. This is [CD20b][Theorem 7.5], whose proof in turn relies on [TZ00]. The reader may observe that this proof is a completely formal calculation, which applies word-by-word to our case if Stokes theorem holds. This, however, is only used once on [CD20b][p. 50]. Given our asymptotics, it is clear from Lemma 2.8 that we as well can integrate by parts because the integrands decay exponentially in the parameter t. $\qquad \square$

Before we can continue with the weighted L^2 bounds, we need another lemma as preparation.

Lemma 5.10 (A first bound on $\inf_M X(\varphi_s)$). *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\text{(54)} \qquad \inf_{s \in [0,1]} \inf_M \left(f + \frac{X}{2}(\varphi_s) \right) \geq 1,$$

where C only depends on the geometry of (M, g).

Proof. Since both f and φ_s are JX-invariant, the argument [CD20b][(7.6)] applies and we obtain that

$$\text{(55)} \qquad X = \nabla^{g_{\varphi_s}} \left(f + \frac{X}{2}(\varphi_s) \right).$$

Also observe that $\frac{X}{2}(\varphi_s) \to 0$ as $t \to \infty$ because X is bounded with respect to the norm g_{φ_s}. Thus, we conclude from (A.3) that $f + \frac{X}{2}(\varphi_s)$ converges to the function $2t + c$ with $c > 0$ and consequently, $f + \frac{X}{2}(\varphi_s)$ attains a global minimum at some point $p \in M$. By (55), we see that X must vanish at p, so we conclude that

$$\inf_M \left(f + \frac{X}{2}(\varphi_s) \right) = \min_{\{X=0\}} \left(f + \frac{X}{2}(\varphi_s) \right) = \min_{\{X=0\}} f$$

holds for all $s \in [0, 1]$. In particular, (54) follows since we normalised f such that $f \geq 1$ on M. $\qquad\square$

Proposition 5.11 (A priori bound on weighted energy). *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\text{(56)} \qquad \sup_{0 \leq s \leq 1} \int_M |\varphi_s|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \leq C,$$

where C only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and on the geometry of (M, g).

Proof. We follow [CD20b][Proposition 7.7]. The idea is to consider two different paths in $C^\infty_{\varepsilon, JX}(M)$ with $1 < \varepsilon < 2$ and to use Theorem 5.9 for obtaining the required bound.

We begin by considering a linear path from 0 to φ_s. Given $s \in [0, 1]$, define this path $(\psi_\tau)_{0 \leq \tau \leq 1}$ by $\psi_\tau := \tau \varphi_s$. Since $\omega + i\partial\bar{\partial}\psi_\tau > 0$, Theorem 5.9 implies that

(57)

$$(I_{\omega,X} - J_{\omega,X})(\varphi_s) = -\int_0^1 \int_M \frac{\tau \varphi_s}{2} \left(\Delta_{g_{\tau\varphi_s}} + X \right)(\varphi_s) e^{f + \tau \frac{X}{2}(\varphi_s)} \omega^n_{\tau\varphi_s} \wedge d\tau$$

Recalling that $X = \nabla^{g_{\tau\varphi_s}}(f + \frac{X}{2}(\varphi_s))$, we integrate by parts and obtain

$$(58) \quad (I_{\omega,X} - J_{\omega,X})(\varphi_s) = n \int_0^1 \int_M \tau e^{f + \tau \frac{X}{2}(\varphi_s)} i\partial\varphi_s \wedge \bar\partial\varphi_s \wedge \omega_{\tau\varphi_s}^{n-1} \wedge d\tau$$

$$= n \int_0^1 \int_M \tau e^{f + \tau \frac{X}{2}(\varphi_s)} i\partial\varphi_s \wedge \bar\partial\varphi_s \wedge ((1-\tau)\omega + \tau\omega_{\varphi_s})^{n-1} \wedge d\tau$$

$$\geq n \int_0^1 \int_M \tau(1-\tau)^{n-1} e^{f + \tau \frac{X}{2}(\varphi_s)} i\partial\varphi_s \wedge \bar\partial\varphi_s \wedge \omega^{n-1} \wedge d\tau$$

$$\geq n \int_0^1 \int_M \tau(1-\tau)^{n-1} e^{(1-\tau)f} i\partial\varphi_s \wedge \bar\partial\varphi_s \wedge \omega^{n-1} \wedge d\tau$$

$$= n \int_M \left(\int_0^1 \tau(1-\tau)^{n-1} e^{(1-\tau)f} d\tau \right) \wedge i\partial\varphi_s \wedge \bar\partial\varphi_s \wedge \omega^{n-1},$$

where the penultimate line holds since $\frac{X}{2}(\varphi_s) \geq -f$ by Lemma 5.10. Thanks to [CD20b][Claim 7.8], there exists a constant $C > 0$ such that

$$(59) \qquad n \int_0^1 \tau(1-\tau)^{n-1} e^{(1-\tau)f} d\tau \geq C \frac{e^f}{f^2},$$

which, in combination with (58), then leads to

$$(60) \qquad (I_{\omega,X} - J_{\omega,X})(\varphi_s) \geq C \int_M \frac{e^f}{f^2} i\partial\varphi_s \wedge \bar\partial\varphi_s \wedge \omega^{n-1}.$$

To estimate $(I_{\omega,X} - J_{\omega,X})(\varphi_s)$ from below, we recall from Theorem 5.9 that $J_{\omega,X}$ is independent of the choice of path from 0 to φ_s. Thus, we can compute $(I_{\omega,X} - J_{\omega,X})(\varphi_s)$ by defining a new path $(\psi_\tau)_{0 \leq \tau \leq 1}$ as $\psi_\tau := \varphi_{\tau s}$. We point out that $\psi_0 = \varphi_0 \equiv 0$ follows from the maximum principle applied to the Monge-Ampère equation (50). For calculating $\dot\psi_\tau$, differentiate (50) with respect to s and obtain

$$n\, i\partial\bar\partial\dot\varphi_s \wedge \omega^{n-1} = \left(F - \frac{X}{2}(\dot\varphi_s) \right) \omega_{\varphi_s}^n.$$

Combining with (51) and using $\dot\psi_\tau = s\dot\varphi_{\tau s}$, we arrive at

$$\frac{1}{2}\Delta_{\psi_\tau}\dot\psi_\tau + \frac{X}{2}(\dot\psi_\tau) = sF,$$

to which we further apply Theorem (5.9) and continue:

$$(I_{\omega,X} - J_{\omega,X})(\varphi_s) = -\int_0^1 \int_M \psi_\tau \cdot sF e^{f + \frac{X}{2}(\psi_\tau)} \omega_{\psi_\tau}^n \wedge d\tau$$

$$= -\int_0^1 \int_M \psi_\tau \cdot sF e^{f + \tau F} \omega^n \wedge d\tau$$

$$\leq C \int_0^1 \int_M |F| f^2 \cdot |\psi_\tau| \frac{e^f}{f^2} \omega^n \wedge d\tau$$

$$\leq C \int_0^1 \left(\int_M |\psi_\tau|^2 \frac{e^f}{f^2} \omega^n \right)^{\frac{1}{2}} d\tau.$$

Here, we applied (50) in the second line and Cauchy-Schwarz in the last one. Also note that $C > 0$ denotes a uniform constant, different from line to line, which only depends on $\sup_M |F|$ and $\int |F|^2 f^2 \omega^n < \infty$. From (58) and (61), we thus conclude

$$\int_M |\nabla^g \varphi_s|_g^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \leq C \int_0^1 \left(\int_M |\varphi_{\tau s}|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \right)^{\frac{1}{2}} d\tau$$

$$= \frac{C}{s} \int_0^s \left(\int_M |\varphi_\tau|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \right)^{\frac{1}{2}} d\tau.$$

Together with Proposition 2.12, we finally arrive at

$$(61) \qquad \lambda \int_M |\varphi_s|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \leq \frac{C}{s} \int_0^s \left(\int_M |\varphi_\tau|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \right)^{\frac{1}{2}} d\tau.$$

As observed by Conlon and Deruelle [CD20b][Proposition 7.7], this is a Grönwall-type differential inequality for the function $U : (0,1] \to \mathbb{R}_+$ defined by

$$U(s) := \int_0^s \left(\int_M |\varphi_\tau|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \right)^{\frac{1}{2}} d\tau.$$

Indeed, it is immediate that (61) becomes

$$\frac{\dot{U}(s)}{\sqrt{U(s)}} \leq \frac{C}{\sqrt{s}},$$

so that we integrate to obtain $\sqrt{U(s)} \leq C\sqrt{s}$ with $s \in (0,1]$. Hence,

$$\left(\int_M |\varphi_s|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \right)^{\frac{1}{2}} d\tau = \dot{U}(s) \leq C,$$

where $C = C(M, g, F)$ is independent of $s \in [0,1]$, as claimed. $\qquad \square$

5.1.3. *A lower bound on* φ_s. For proving a uniform bound on $\sup_M |\varphi_s|$, it remains to bound $\inf_M \varphi_s$ from below. This is the main result of this subsection:

Proposition 5.12 (Lower bound on $\inf_M \varphi_s$)**.** *Let* $1 < \varepsilon < 2$ *and suppose* $(\varphi_s)_{0 \leq s \leq 1}$ *is a family in* $C^\infty_{\varepsilon, JX}(M)$ *solving (50). Then there exists a constant* $C > 0$ *such that*

$$\inf_{s \in [0,1]} \inf_M \varphi_s \geq -C,$$

where C *only depends* $F \in C^\infty_{\varepsilon, JX}(M)$ *and on the geometry of* (M, g)*.*

If we assumed that F was compactly supported, the same argument as in [CD20b][Proposition 7.10] would go through verbatim and provide the required bound on $\inf_M \varphi_s$, since we already obtained uniform bounds on $\sup_M \varphi_s$ (Proposition 5.7) and on the weighted L^2-norm (Proposition 5.11).

In our situation, however, we do *not* assume that F has compact support, but merely $F \in C^\infty_{\varepsilon, JX}(M)$ with $1 < \varepsilon < 2$. Thus, we proceed as follows.

First, we construct a compact domain $K \subset M$ so that we obtain a suitable barrier function on its complement $M \setminus K$, which will be useful for arguments relying on the maximum principle. In a second step, the argument in [CD20b][Proposition 7.10] gives a lower bound on $\inf_K \varphi_s$. And finally, we will see that the maximum principle yields a lower bound on $\inf_{M \setminus K} \varphi_s$.

In other words, our strategy is to prove the following lemma, as well as the next two propositions:

Lemma 5.13 (Construction of K)**.** *Let* $1 < \varepsilon < 2$ *and suppose* $(\varphi_s)_{0 \leq s \leq 1}$ *is a family in* $C^\infty_{\varepsilon, JX}(M)$ *solving (50). Then there exists a constant* $0 < \varepsilon_0 < 1$ *and a compact domain* $K \subset M$ *such that for all* $s \in [0,1]$, *we have*

(62)
$$\left(\Delta_{g_{\varphi_s}} + X \right) \left(e^{-\varepsilon_0 \left(f + \frac{X}{2}(\varphi_s) \right)} \right) \leq -\frac{\varepsilon_0}{2} e^{-\varepsilon_0 \left(f + \frac{X}{2}(\varphi_s) \right)} < 0 \quad on \quad M \setminus K,$$

where both ε_0 *and* K *only depend on* $F \in C^\infty_{\varepsilon, JX}(M)$ *and the geometry of* (M, g)*.*

Proposition 5.14 (Lower bound on a compact set)**.** *Let* $1 < \varepsilon < 2$ *and suppose* $(\varphi_s)_{0 \leq s \leq 1}$ *is a family in* $C^\infty_{\varepsilon, JX}(M)$ *solving (50). For the compact domain* $K \subset M$ *given by Lemma 5.13, there exists a constant* $C > 0$ *such that*

$$\inf_{s \in [0,1]} \inf_K \varphi_s \geq -C,$$

where C *only depends on* K, $F \in C^\infty_{\varepsilon, JX}(M)$ *and the geometry of* (M, g)*.*

Proposition 5.15 (Lower bound outside of a compact set). *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). For the compact domain $K \subset M$ constructed in Lemma 5.13, there exists a constant $C > 0$ such that*

$$\inf_{s \in [0,1]} \inf_{M \setminus K} \varphi_s \geq -C,$$

where C only depends on K, $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Clearly, Proposition 5.15, together with Proposition 5.14, yield a uniform lower bound on $\inf_M \varphi_s$, as claimed in Proposition 5.12.

Since Lemma 5.13 requires some preparation, let us for the moment assume that we are given the compact set $K \subset M$ from Lemma 5.13 and see how this implies the lower bound on $\inf_K \varphi_s$, i.e. Proposition 5.14.

Proof of Proposition 5.14. We follow the proof of [CD20b][Proposition 7.10], which in turn relies on Błocki's local argument [Bło05].

Let $K \subset M$ be the compact domain constructed in Lemma 5.13. For each $p \in K$, let V be a chart around p so that ω can be written as $\omega = i\partial\bar\partial G$. According to the proof of [Bło05][Theorem 4], there are constants $a, r > 0$ only depending on the local geometry of (M, g) around p such that $G < 0$ on $B_g(p, 2r)$, G is minimal at p and $G \geq G(p) + a$ on $B_g(p, 2r) \setminus B_g(p, r)$, where $B_g(p, 2r) \subset V$ is the geodesic ball of radius $2r$ around p. Since K is compact, we can cover K by a *finite* number of such balls $B_g(p, 2r)$.

For a given $s \in [0, 1]$, we consider φ_s solving (50) and point out that there exists a $p_s \in K$ such that $\varphi_s(p_s) = \inf_K \varphi_s$. Then $p_s \in B_g(p, 2r)$ for one of the balls constructed above. Define a plurisubharmonic function $u : B_g(p, 2r) \to \mathbb{R}_{\leq 0}$ by

$$u = \begin{cases} \varphi_s + G & \text{if } \sup_M \varphi_s \leq 0, \\ \varphi_s - \sup_M \varphi_s + G & \text{otherwise,} \end{cases}$$

so that [Bło05][Proposition 3] implies the following estimate

$$(63) \quad \sup_{B_g(p,2r)} |u| \leq a + \left(c_n \cdot 2r \cdot a^{-1}\right)^{2n} \int_{B_g(p,2r)} |u| \, \mathrm{d}V_g \cdot \left(\sup_{B_g(p,2r)} \frac{\omega^n_{\varphi_s}}{\omega^n}\right)^2$$

where $\omega_{\varphi_s} = \omega + i\partial\bar\partial\varphi_s$ and $c_n > 0$ is a constant only depending on the dimension n of M.

We now explain how to estimate the terms appearing on the right hand side of (63). We begin by using (50) together with Lemma 5.10 to obtain

$$(64) \quad \sup_{B_g(p,2r)} \frac{\omega^n_{\varphi_s}}{\omega^n} = \sup_{B_g(p,2r)} e^{s \cdot F - \frac{X}{2}(\varphi_s)} \leq \sup_{N_{2r}(K)} e^{|F|+f} =: C_1.$$

Here $N_{2r}(K)$ denotes the tabular neighborhood of radius $2r$ around K. Note that since K is compact, the constant C_1 is indeed finite.

Next, we focus on the integral appearing in (63) and first consider the case $\sup_M \varphi_s \leq 0$. We continue:

$$\fint_{B_g(p,2r)} |u| \, \mathrm{d}V_g$$

$$\leq \fint_{B_g(p,2r)} |\varphi_s| \, \mathrm{d}V_g + \sup_M \varphi_s - G(p)$$

$$\leq \max\left\{1, \mathrm{Vol}(B_g(p,2r))\right\} \left(\left(\fint_{B_g(p,2r)} |\varphi_s|^2 \, \mathrm{d}V_g\right)^{\frac{1}{2}} + C - G(p)\right)$$

$$\leq \max\left\{1, \mathrm{Vol}(N_{2r}(K))\right\} \left(\sup_M \frac{e^{-f}}{f^2} \cdot \left(\int_M |\varphi_s|^2 \frac{e^f}{f^2} \, \mathrm{d}V_g\right)^{\frac{1}{2}} + C - G(p)\right)$$

$$\leq \max\left\{1, \mathrm{Vol}(N_{2r}(K))\right\} \left(\sup_M \frac{e^{-f}}{f^2} \cdot C + C - G(p)\right) =: C_2,$$

where we used Cauchy-Schwarz and Proposition 5.7 in the second line and Proposition 5.11 in the last one. Combining this estimate with (63) and (64) then leads to

$$-\inf_K \varphi_s = -\varphi_s(p_s) = -u(p_s) - \sup_M \varphi_s + G(p_s)$$

(65)
$$\leq \sup_{B_g(p,2r)} |u|$$

$$\leq a + \left(c_n \cdot 2r \cdot a^{-1}\right)^{2n} \cdot C_2 \cdot C_1^2.$$

Note that a priori, the constants in the last line of (65) depend on the ball $B_g(p,2r)$ containing the point in which φ_s attains its minimum inside K. However, since K is covered by only *finitely* many of such balls $B_g(p,2r)$, (65) does indeed prove the required uniform lower bound on $\inf_K \varphi_s$. Observing that the above estimates hold in the case $\sup_M \varphi_s \leq 0$ as well then finishes the proof. $\square$

Thus, it only remains to show Lemma 5.13 and Proposition 5.15. We begin with the following crucial observation.

Lemma 5.16 (Uniform bound on $X^2(\varphi_s)$). *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} \sup_M |X(X(\varphi_s))| \leq C,$$

where C only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof. The idea is to obtain a differential equality to which the maximum principle applies, so that the desired estimate follows.

First, we differentiate (50) in the direction of $\frac{X}{2}$, i.e. apply $\mathcal{L}_{\frac{X}{2}}$, which leads to

$$(66) \quad ni\partial\bar\partial\left(f + \frac{X}{2}(\varphi_s)\right)\wedge\omega_{\varphi_s}^{n-1} = \left(\frac{X}{2}(sF) - \frac{X^2}{4}(\varphi_s) + \frac{1}{2}\Delta_g f\right)\omega_{\varphi_s}^n$$

where we abbreviated $X(X(\cdot)) = X^2(\cdot)$. Here, we also used two formulas, $\mathcal{L}_{\frac{X}{2}}\omega = i\partial\bar\partial f$ and $\mathcal{L}_{\frac{X}{2}}\omega_{\varphi_s} = i\partial\bar\partial f + \frac{X}{2}(\varphi_s)$, whose computations can be found in the proof Lemma 3.3. Next, recall that for any real $(1,1)$-form α, we have

$$(67) \qquad n(n-1)\alpha^2\wedge\omega_{\varphi_s}^{n-2} = \left((\mathrm{tr}_{\omega_{\varphi_s}}(\alpha))^2 - |\alpha|_{g_{\varphi_s}}^2\right)\omega_{\varphi_s}^n,$$

where $\mathrm{tr}_{\omega_{\varphi_s}}(\alpha)$ is is defined by the formula $\mathrm{tr}_{\omega_{\varphi_s}}(\alpha)\,\omega_{\varphi_s}^n = n\alpha\wedge\omega_{\varphi_s}^{n-1}$. Differentiating (66) once more in direction of $\frac{X}{2}$ and using (67) with $\alpha = i\partial\bar\partial f + \frac{X}{2}(\varphi_s)$, then yields the following equation:

$$(68) \quad (\Delta_{g_{\varphi_s}} + X)\left(X(f) + \frac{X^2}{2}(\varphi_s)\right) = H_1 + \left|\partial\bar\partial f + \frac{X}{2}(\varphi_s)\right|_{g_{\varphi_s}}^2$$

with $H_1 := X^2(sF) + X(\Delta_g f) + X(f)$. We continue to estimate the right-hand side of (68) from below :

$$\left|\partial\bar\partial f + \frac{X}{2}(\varphi_s)\right|_{g_{\varphi_s}}^2 \geq \frac{1}{n}\left(\Delta_{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right)\right)^2$$

$$= \frac{1}{n}\left(X(f) + \frac{X^2}{2}(\varphi_s) - H_2\right)^2,$$

where $H_2 := X(f) + X(sF) + \Delta_g f$ and we made use of (66) in the second line. Combining the previous inequality with (68), we then arrive at

$$(\Delta_{g_{\varphi_s}} + X)\left(X(f) + \frac{X^2}{2}(\varphi_s)\right) \geq H_1 + \frac{1}{n}\left(X(f) + \frac{X^2}{2}(\varphi_s) - H_2\right)^2.$$

Note that by Assumption A.3, the function $X(f) + \frac{X^2}{2}(\varphi_s)$ tends to 1 as $t\to\infty$, and so either $X(f) + \frac{X^2}{2}(\varphi_s) \leq 1$, or $X(f) + \frac{X^2}{2}(\varphi_s)$ attains its maximum at some point. In the first case, we are done so we assume that $X(f) + \frac{X^2}{2}(\varphi_s)$ is maximal at $p_{\max}\in M$. Then we observe that the previous inequality gives at $p_{\max}$

$$X(f) + \frac{X^2}{2}(\varphi_s) \leq \sqrt{n\sup_M|H_1|} + \sup_M|H_2| < \infty,$$

i.e. $X(f) + \frac{X^2}{2}(\varphi_s)$ is uniformly bounded from above. This, in turn, implies a uniform upper bound on $X^2(\varphi_s)$ since $X(f)$ is bounded.

For the lower bound on $X^2(\varphi_s)$ we recall from (55) that

$$X = \nabla^{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right),$$

so that we can estimate as follows:

$$X\left(\frac{X}{2}(\varphi_s)\right) = -X(f) + |X|^2_{g_{\varphi_s}} \geq -\sup_M |X(f)|,$$

which is finite. This completes the proof.

$\square$

With the previous lemma, we can finish the proof of Lemma 5.13.

Proof of Lemma 5.13. Let us define the barrier function $v := e^{\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}$ for some $0 < \varepsilon_0 < 1$ to be chosen later on. Since $X = \nabla^{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right)$, we compute

$$(\Delta_{g_{\varphi_s}} + X)\left(v^{-1}\right) = \varepsilon_0 v^{-1}\left(|X|^2_{g_{\varphi_s}} \cdot (\varepsilon_0 - 1) - \Delta_{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right)\right)$$

$$= \varepsilon_0 v^{-1}\left(\varepsilon_0 |X|^2_{g_{\varphi_s}}(\varphi_s) - X(f) - X(sF) - \Delta_g f\right)$$

$$\leq \varepsilon_0 v^{-1}\left(\varepsilon_0 C - X(f) - X(sF) - \Delta_g f\right),$$

where we used (66) in the second line and the uniform bound $C > 0$ from Lemma 5.16 in the last one. Since $X(f) \to 1$, and $X(F), \Delta_g f \to 0$ as $t \to \infty$, there exists a compact domain $K \subset M$ such that

$$X(f) \geq \frac{3}{4} \quad \text{and} \quad |\Delta_g f| + |X(F)| \leq \frac{1}{8} \quad \text{on} \quad M \setminus K.$$

Moreover, we can assume that K is of the form

(69) $$K = \{x \in M \,|\, t(x) \leq t_0\}$$

for some $t_0 > 0$. Choosing $\varepsilon_0 > 0$ sufficiently small so that

$$\varepsilon_0 C \leq \frac{1}{8},$$

we thus obtain

$$(\Delta_{g_{\varphi_s}} + X)\left(v^{-1}\right) \leq \frac{\varepsilon_0}{2} v^{-1} \quad \text{on} \quad M \setminus K,$$

as claimed.

$\square$

Before obtaining Proposition 5.15, we require yet another

Lemma 5.17 (Bounding $X(\varphi_s)$ on a compact set). *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). For the compact domain $K \subset M$ given by Lemma 5.13, there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} \sup_K X(\varphi_s) \leq C \quad \text{and} \quad \sup_{s \in [0,1]} X(\varphi_s) \leq Ct + C \text{ on } M \setminus K,$$

where C only depends on K, $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof. For the first part of the statement, we essentially follow the argument in [CD20b][Proposition 7.11].

Consider the flow $(\Phi_\tau)_{\tau \in \mathbb{R}}$ of the complete vector field $\frac{X}{2}$. In particular, the map Φ_τ corresponds to translation by τ in the radial parameter t on the cylindrical end $[0, \infty) \times L$. Then we let $\psi_x(\tau) := \varphi_s(\Phi_\tau(x))$ for $(x, \tau) \in M \times \mathbb{R}$ and observe that for each fixed $x \in M$, the limit $\lim_{\tau \to \pm\infty} \psi_x(\tau)$ always exists because φ tends to zero as $t \to \infty$. Keeping this in mind, we consider $\eta_-(\tau) := e^{-\tau}$ and integrate by parts as follows

$$
(70) \quad
\begin{aligned}
\int_0^\infty \eta_-''(\tau)\psi_x(\tau)d\tau &= -\int_0^\infty \eta_-'(\tau)\psi_x'(\tau)d\tau + \psi_x(0) \\
&= \int_0^\infty \eta_- \psi_x''(\tau)d\tau + \psi_x'(0) + \psi_x(0).
\end{aligned}
$$

By choosing $x \in K$, rearranging (70) and using $\frac{X}{2}(\varphi_s)(x) = \psi_x'(0)$, we consequently estimate

$$
\begin{aligned}
\frac{X}{2}(\varphi_s)(x) &\leq -\inf_K \varphi_s + \sup_M \varphi_s \int_0^\infty e^{-\tau}d\tau - \inf_M \frac{X^2}{4}(\varphi_s) \int_0^\infty e^{-\tau}d\tau \\
&\leq C_1,
\end{aligned}
$$

where $C_1 > 0$ only depends on F and the geometry of (M, g), thanks to Propositions 5.14 and 5.7 as well as Lemma 5.16. This shows the first part of this lemma.

For the second part, recall from (69), that we can identify $M \setminus K \cong (t_0, \infty) \times L$ for some $t_0 > 0$. To emphasize this splitting, we write $x = (t, y)$ for points $x \in M \setminus K$. Under this identification, $X = 2\partial/\partial t$ and so we can write

$$
(71) \quad
\begin{aligned}
X(\varphi_s)(t, y) &= \int_{t_0}^t \frac{X^2}{2}(\varphi_s)(\sigma, y)d\sigma + X(\varphi_s)(t_0, y) \\
&\leq C_2(t - t_0) + C_1,
\end{aligned}
$$

since $(0, y) \in K$ and $X^2(\varphi_s) \leq C_2$ for some uniform constant $C_2 > 0$ given by Lemma 5.16. As the right-hand side of (71) is independent of $s \in [0, 1]$, the lemma follows. $\qquad\square$

Now we can deduce Proposition 5.15.

Proof of Proposition 5.15. As in [CD20b][Proposition 7.20], we use a barrier function to show the claim. Let $0 < \varepsilon_0 < 1$ and $K \subset M$ be given by Lemma 5.13, i.e. on $M \setminus K$, we have

$$
(72) \quad \left(\Delta_{g_{\varphi_s}} + X\right)\left(e^{-\varepsilon_0\left(f + \frac{X}{2}(\varphi_s)\right)}\right) \leq -\frac{\varepsilon_0}{2}e^{-\varepsilon_0\left(f + \frac{X}{2}(\varphi_s)\right)} < 0.
$$

The reader may observe from the proof that (72) holds as long as $0 < \varepsilon_0 \ll 1$ is sufficient small. In particular, we are free to choose $\varepsilon_0 > 0$ as small was we require and (72) is still valid.

Similar to (52), which was used for proving the upper bound, the Monge-Ampère equation (50) implies

$$(\Delta_{g_{\varphi_s}} + X)(\varphi_s) \leq |F|,$$

and so for some $A > 0$ to be specified later on, we obtain

$$(73) \quad \left(\Delta_{g_{\varphi_s}} + X\right)\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right) \leq |F| - A\frac{\varepsilon_0}{2}e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}.$$

The idea is to choose $A \gg 1$ sufficiently large so that the right term in (73) becomes negative.

Note that by Lemma 5.17 and Assumption A.3 there exists a constant $C > 0$ only depending on F and the geometry of (M, g) such that

$$(74) \qquad \varepsilon_0\left(f + \frac{X}{2}(\varphi_s)\right) \leq \varepsilon_0 Ct + C \leq \varepsilon t + C,$$

where the second inequality holds if we fix some $\varepsilon_0 > 0$ with

$$\varepsilon_0 < \frac{C}{\varepsilon}.$$

Applying (74) to the right-hand side of (73) then yields

$$|F| - A\frac{\varepsilon_0}{2}e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} \leq e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\left(e^{\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)-\varepsilon t}||F||_{C^0_\varepsilon} - A\frac{\varepsilon_0}{2}\right)$$

$$\leq e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\left(e^C||F||_{C^0_\varepsilon} - A\frac{\varepsilon_0}{2}\right).$$

Thus, choosing $A > 0$ sufficiently large so that

$$A > \frac{2}{\varepsilon_0}e^C||F||_{C^0_\varepsilon},$$

and plugging this back into (73), we arrive at

$$\left(\Delta_{g_{\varphi_s}} + X\right)\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right) \leq 0 \quad \text{on} \quad M \setminus K.$$

Hence, Hopf's maximum principle states that

$$\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} \geq \min\left\{0, \min_{\partial K}\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right)\right\}$$

$$(75) \qquad\qquad\qquad\qquad \geq \min\left\{0, \min_K \varphi_s\right\}$$

$$\geq -C$$

holds on $M \setminus K$ because $\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}$ goes to 0 as $t \to \infty$ and $\min_K \varphi_s$ is, according to Lemma 5.17, uniformly bounded from below by some constant $-C < 0$. To conclude the Proposition, we observe that

$$f + \frac{X}{2}(\varphi_s) \geq 1$$

by Lemma 5.10 and consequently,

$$\varphi_s \geq -C - Ae^{-\varepsilon_0} \quad \text{on} \quad M \setminus K,$$

as claimed.

$\square$

Having finally finished the proof of Proposition 5.12, we can now strengthen the estimates in Lemma 5.17, i.e. achieve a uniform bound on the radial derivative of φ_s.

Corollary 5.18 (Uniform bound on $X(\varphi_s)$). *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \le s \le 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} \sup_M |X(\varphi_s)| \le C,$$

where C only depends on $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M,g).

Proof. We apply the same idea as in the proof of Lemma 5.17. Namely, by (70), we can estimate for each $x \in M$:

$$\frac{X}{2}(\varphi_s)(x) \le -\inf_M \varphi_s + \sup_M \varphi_s - \inf_M \frac{X^2}{4}(\varphi_s),$$

so that the uniform upper bound follows from Propositions 5.7, 5.12 and Lemma 5.16. The lower bound is similar. Using $\eta_+ = e^\tau$ instead of $\eta_- = e^{-\tau}$ leads to

$$\int_{-\infty}^0 \eta''_+(\tau)\psi_x(\tau)d\tau = \int_{-\infty}^0 \eta_+(\tau)\psi''_x(\tau)d\tau - \psi'_x(0) + \psi_x(0),$$

and estimating as before then yields

$$\frac{X}{2}(\varphi_s)(x) \ge \inf_M \varphi_s - \sup_M \varphi_s + \inf_M \frac{X^2}{4}(\varphi_s),$$

finishing the proof. $\square$

This new bound on $X(\varphi_s)$ enables us to conclude a weighted lower bound on φ_s, at least for *some* $\varepsilon_0 < \varepsilon$.

Proposition 5.19 (A first weighted lower bound on φ_s). *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \le s \le 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). Then there exist two constants $0 < \varepsilon_0 < 1$ and $C > 0$ such that*

$$\inf_{s \in [0,1]} \inf_M e^{\varepsilon_0 t} \varphi_s \ge -C,$$

where both ε_0 and C only depend on $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M,g).

Proof. Using Corollary 5.18, the proof of Proposition 5.12 can be refined by following the argument in [CD20b][Proposition 7.20].

We repeat the proof until arriving at (75), so that we have on $M \setminus K$:

$$(76) \quad \varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} \ge \min\left\{0, \min_{\partial K}\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right)\right\}.$$

By Corollary 5.18 and Assumption A.3, there is a uniform constant $C > 0$ such that

$$(77) \qquad C^{-1}e^{-2\varepsilon_0 t} \leq e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} \leq Ce^{-2\varepsilon_0 t}$$

holds on M. In particular, since $\inf_K \varphi_s$ is uniformly bounded from below by Proposition 5.14, we can choose $A \gg 1$ even larger, so that

$$\min_{\partial K}\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right) \geq \inf_K \varphi_s + A\inf_K C^{-1}e^{-2\varepsilon_0 t} \geq 0.$$

Consequently, we arranged that (76) becomes

$$\varphi_s \geq -Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} \geq -ACe^{-2\varepsilon_0 t} \quad \text{on} \quad M \setminus K,$$

because of (77). This is precisely what we wanted to prove. $\qquad\square$

Before improving the weighted bound from ε_0 to ε, we require uniform bounds on all derivatives of φ_s, which is the content of the subsequent section.

5.2. **Higher order estimates.** In the previous section, we obtained uniform bounds on φ_s and its radial derivative up to second order. Using these results, we begin by deriving bounds on the C^2- and C^3-norms of φ_s, which then leads to estimates for all derivatives. We purse essentially the same strategy as in [CD20b][Section 7], but occasionally we present different computations.

5.2.1. *The C^2-estimate.* The C^2-estimate for φ_s is equivalent to bounding the associated metric g_{φ_s} uniformly in terms of g.

Proposition 5.20 (Uniform bound on the metric)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). If g_{φ_s} denotes the metric associated to the Kähler form $\omega + i\partial\bar\partial\varphi_s$, then there exists a constant $C > 0$ such that*

$$(78) \qquad C^{-1}g \leq g_{\varphi_s} \leq Cg,$$

where C only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Before proceeding with the proof, we immediately obtain a uniform bound on the volume form by looking at (50) and applying Corollary 5.18.

Corollary 5.21 (Uniform bound on volume form)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$C^{-1}\omega^n \leq (\omega + i\partial\bar\partial\varphi_s)^n \leq C\omega^n,$$

where $C > 0$ only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof of Proposition 5.20. We argue as in [CD20b][Proposition 7.14], but present different calculations. The bound (78) amounts to bounding both $\operatorname{tr}_\omega \omega_{\varphi_s}$ and $\operatorname{tr}_{\omega_{\varphi_s}} \omega$ uniformly from above. However, there is the well-known formula

$$\operatorname{tr}_{\omega_{\varphi_s}} \omega \leq n \cdot \frac{\omega^n}{\omega_{\varphi_s}^n} \left(\operatorname{tr}_\omega \omega_{\varphi_s} \right)^{n-1}$$

compare for example ([BEG13][Lemma 4.1.1]). Thus, it suffices to estimate $\operatorname{tr}_\omega \omega_{\varphi_s}$ since the volume form $\omega_{\varphi_s}^n$ is uniformly bounded by Corollary 5.21.

In this proof, $C > 0$ denotes a uniform constant, which may increase from line to line but only depends on the geometry of (M, g) and the C^∞-norm of F.

Recall that a standard computation yields the following inequality

$$(79) \qquad \frac{1}{2} \Delta_{g_{\varphi_s}} \log \operatorname{tr}_\omega \omega_{\varphi_s} \geq -\frac{\operatorname{tr}_\omega \operatorname{Ric}(\omega_{\varphi_s})}{\operatorname{tr}_\omega \omega_{\varphi_s}} - C \operatorname{tr}_{\omega_{\varphi_s}} \omega,$$

where $\operatorname{Ric}(\omega_{\varphi_s})$ is the Ricci form of ω_{φ_s} and $C > 0$ a constant such that the holomorphic bisectional curvature of g is bounded from below by $-C$. For a proof of this inequality, we refer the reader to [BEG13][Proposition 4.1.2]. Also observe that in our case the bisectional curvature of g is bounded since g is asymptotically cylindrical. Starting from (50), we compute the Ricci form of ω_{φ_s}:

$$(80) \quad \operatorname{Ric}(\omega_{\varphi_s}) = -i\partial\bar\partial \log \omega_{\varphi_s}^n = \operatorname{Ric}(\omega) - i\partial\bar\partial sF + i\partial\bar\partial \left(\frac{X}{2}(\varphi_s) \right).$$

As both $\|F\|_{C^2}$ and the curvature of g are uniformly bounded, we continue to estimate

$$(81) \qquad -\operatorname{tr}_\omega \operatorname{Ric}(\omega_{\varphi_s}) \geq -C - \operatorname{tr}_\omega i\partial\bar\partial \left(\frac{X}{2}(\varphi_s) \right).$$

Also recall from [BEG13][Lemma 4.1.1] that

$$(82) \qquad \operatorname{tr}_\omega \omega_{\varphi_s} \geq n \cdot \left(\frac{\omega_{\varphi_s}^n}{\omega^n} \right)^{\frac{1}{n}} \geq C^{-1} > 0,$$

where the lower bound again follows from Corollary 5.21. Combining (82) and (81) with (79), we consequently arrive at

$$(83) \qquad \frac{1}{2} \Delta_{g_{\varphi_s}} \log \operatorname{tr}_\omega \omega_{\varphi_s} \geq -\frac{\operatorname{tr}_\omega i\partial\bar\partial \left(\frac{X}{2}(\varphi_s) \right)}{\operatorname{tr}_\omega \omega_{\varphi_s}} - C - C \operatorname{tr}_{\omega_{\varphi_s}} \omega.$$

Next, we calculate the radial derivative of $\operatorname{tr}_\omega \omega_{\varphi_s}$ by considering its defining equation:

$$\operatorname{tr}_\omega \omega_{\varphi_s} \cdot \omega^n = n \cdot \omega_{\varphi_s} \wedge \omega^{n-1}.$$

Taking the Lie derivative in direction $\frac{X}{2}$ on both sides of this equation and then dividing by ω^n leads to

$$\frac{X}{2}(\operatorname{tr}_\omega \omega_{\varphi_s}) + \operatorname{tr}_\omega \omega_{\varphi_s} \cdot \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega)$$

(84)
$$= \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega_{\varphi_s}) + n(n-1) \cdot \frac{\omega_{\varphi_s} \wedge \mathcal{L}_{\frac{X}{2}}(\omega) \wedge \omega^{n-2}}{\omega^n}$$

$$= \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega_{\varphi_s}) + \operatorname{tr}_\omega \omega_{\varphi_s} \cdot \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega) - \langle \omega_{\varphi_s}, \mathcal{L}_{\frac{X}{2}}(\omega) \rangle_g,$$

or equivalently,

(85)
$$\frac{X}{2}(\operatorname{tr}_\omega \omega_{\varphi_s}) = \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega_{\varphi_s}) - \langle \omega_{\varphi_s}, \mathcal{L}_{\frac{X}{2}}(\omega) \rangle_g$$

Here, the last equation in (84) is a straight forward computation, which can be found in [Szé14][Lemma 4.6], and $\langle,\rangle_g$ denotes the metric on 2-forms induced by g. We recall that $\mathcal{L}_{\frac{X}{2}}(\omega) = i\partial\bar{\partial}f$ since $X = \nabla^g f$ and also that the norm $|i\partial\bar{\partial}f|_g \leq C$ is uniformly bounded by some $C > 0$ because of A.3. Applying these observations to the previous equation, we obtain

$$\frac{X}{2}(\log \operatorname{tr}_\omega \omega_{\varphi_s}) = \frac{\operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right)}{\operatorname{tr}_\omega \omega_{\varphi_s}} + \frac{\operatorname{tr}_\omega i\partial\bar{\partial}f}{\operatorname{tr}_\omega \omega_{\varphi_s}} - \frac{\langle \omega_{\varphi_s}, \mathcal{L}_{\frac{X}{2}}(\omega) \rangle_g}{\operatorname{tr}_\omega \omega_{\varphi_s}}$$

$$\geq \frac{\operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right)}{\operatorname{tr}_\omega \omega_{\varphi_s}} - \frac{C}{\operatorname{tr}_\omega \omega_{\varphi_s}} - \frac{C \cdot |\omega_{\varphi_s}|_g}{\operatorname{tr}_\omega \omega_{\varphi_s}}$$

$$\geq \frac{\operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right)}{\operatorname{tr}_\omega \omega_{\varphi_s}} - C,$$

where we used the bound on $|i\partial\bar{\partial}f|_g$ in the second line and (82) in the last one. Altogether, we finally arrive at

(86)
$$\frac{1}{2}\left(X + \Delta_{g_{\varphi_s}}\right) \log \operatorname{tr}_\omega \omega_{\varphi_s} \geq -C - C \operatorname{tr}_{\omega_{\varphi_s}} \omega.$$

From there, it is standard to conclude an upper bound on $\operatorname{tr}_\omega \omega_{\varphi_s}$. We begin by considering the following inequality

$$\frac{1}{2}\left(X + \Delta_{g_{\varphi_s}}\right) \varphi_s \leq C + n - \operatorname{tr}_{\omega_{\varphi_s}} \omega$$

where we used the upper bound on $X(\varphi_s)$ from Proposition 5.18 and the definition of ω_{φ_s}. In combination with (86), we then obtain

(87)
$$\frac{1}{2}\left(X + \Delta_{g_{\varphi_s}}\right) \left(\log \operatorname{tr}_\omega \omega_{\varphi_s} - (C+1)\varphi_s\right) \geq -C + \operatorname{tr}_{\omega_{\varphi_s}} \omega.$$

Applying the maximum principle to this equation, yields the desired estimate for $\operatorname{tr}_\omega \omega_{\varphi_s}$ as follows. Note that we can assume $\log \operatorname{tr}_\omega \omega_{\varphi_s} - (C+1)\varphi_s > n$ at least somewhere on M, because otherwise we are done by the uniform upper bound on φ_s (Proposition 5.7). Thus, there exists $p_{\max} \in M$ such that $\log \operatorname{tr}_\omega \omega_{\varphi_s} - (C+1)\varphi_s$ is maximal at $p_{\max}$.

Then at this point, we obtain from (87) that $\mathrm{tr}_{\omega_{\varphi_s}} \omega \leq C$, so that at p_{max}:

$$\mathrm{tr}_\omega \omega_{\varphi_s} \cdot e^{-(C+1)\varphi_s} \leq n e^{-(C+1)\varphi_s} \cdot \frac{\omega_{\varphi_s}^n}{\omega^n} (\mathrm{tr}_{\omega_{\varphi_s}} \omega)^{n-1} \leq C.$$

Hence, $\log \mathrm{tr}_\omega \omega_{\varphi_s} - (C+1)\varphi_s$ is uniformly bounded from above, and so is $\mathrm{tr}_\omega \omega_{\varphi_s}$, finishing the proof. $\square$

5.2.2. *The C^3-estimate.*

Proposition 5.22 (Uniform C^3-estimate). *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). If g_{φ_s} denotes the metric associated to the Kähler form $\omega + i\partial\bar\partial\varphi_s$, then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} \sup_M |\nabla^g g_{\varphi_s}|_g \leq C,$$

where the constant C only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof. We define

$$S := |\nabla^g g_{\varphi_s}|_g^2,$$

and then the computation in [CD20b][Proposition 7.16] goes through verbatim. In particular, if $\mathrm{Rm}(g)$ denotes the curvature tensor of g, there exists a constant $C > 0$, which only depends on the constant in Proposition 5.20 as well as on bounds for covariant derivatives of both F and $\mathrm{Rm}(g)$, such that

$$(88) \qquad \frac{1}{2}\left(\Delta_{g_{\varphi_s}} - X\right) S \geq -C(S+1).$$

Moreover, recall that the standard Schwarz-Lemma calculation in holomorphic coordinates yields

(89)
$$\frac{1}{2}\Delta_{g_{\varphi_s}} \mathrm{tr}_\omega \omega_{\varphi_s} = -\mathrm{tr}_\omega \mathrm{Ric}(\omega_{\varphi_s}) + g_{\varphi_s}^{\bar l k} R_{k\bar l}{}^{j\bar i} g_{i\bar j}^{\varphi_s} + g^{\bar j i} g_{\varphi_s}^{\bar q p} g_{\varphi_s}^{\bar l k} \nabla_i^g g_{p\bar l}^{\varphi_s} \nabla_{\bar j}^g g_{k\bar q}^{\varphi_s},$$

where $g_{i\bar j}^{\varphi_s}$ denotes the components of g_{φ_s} in coordinates, with inverse $g_{\varphi_s}^{\bar j i}$, and $R_{k\bar l i \bar j}$ is the local expression of $\mathrm{Rm}(g)$. For the computation, we refer the reader to [BEG13][(3.67)], for example. Starting from (89), and keeping Proposition 5.20 as well as (81) in mind, we estimate

$$\frac{1}{2}\Delta_{g_{\varphi_s}} \mathrm{tr}_\omega \omega_{\varphi_s} \geq -\mathrm{tr}_\omega i\partial\bar\partial\left(\frac{X}{2}(\varphi_s)\right) - C + C^{-1}S.$$

Proposition 5.20 applied to (85) also leads to

$$\frac{X}{2}(\mathrm{tr}_\omega \omega_{\varphi_s}) \geq \mathrm{tr}_\omega i\partial\bar\partial\left(\frac{X}{2}(\varphi_s)\right) - C,$$

and hence,

$$(90) \qquad \frac{1}{2} \left(\Delta_{g_{\varphi_s}} + X \right) \operatorname{tr}_\omega \omega_{\varphi_s} \geq -C + C^{-1} S$$

for some constant $C > 0$ only depending on $|| \operatorname{Rm}(g) ||_{C^0(M)}$, $|| \partial \bar\partial f ||_{C^0(M)}$, $||F||_{C^2(M)}$ and the constant in Proposition 5.20. If we choose a sufficiently large constant $C_1 > 0$ and then add (88) to C_1-times (90), we can arrange that

$$(91) \qquad \frac{1}{2} (\Delta_{g_{\varphi_s}} - X) \left(S + C_1 \operatorname{tr}_\omega \omega_{\varphi_s} \right) \geq -C + S.$$

Again, there are two cases to consider. If $S + C_1 \operatorname{tr}_\omega \omega_{\varphi_s} \leq \lim_{t\to\infty} S + C_1 \operatorname{tr}_\omega \omega_{\varphi_s} = n$, there is nothing to show, so we can assume $S + C_1 \operatorname{tr}_\omega \omega_{\varphi_s} > n$. Thus, there exists a point $p_{\max} \in M$, where $S + C_1 \operatorname{tr}_\omega \omega_{\varphi_s}$ is maximal. Applying the maximum principle to (91), we have at $p_{\max}$:

$$(92) \qquad S + C_1 \operatorname{tr}_\omega \omega_{\varphi_s} \leq C + C_1 \sup_M \operatorname{tr}_\omega \omega_{\varphi_s} \leq C.$$

This implies a uniform upper bound on S, as claimed. $\qquad \square$

Since the C^1-norm of g_{φ_s} is uniformly bounded, we obtain a uniform $C^{0,\alpha}$-bound on g_{φ_s}, as in [CD20b][Corollary 7.17].

Corollary 5.23. *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). If g_{φ_s} denotes the Riemannian metric corresponding to ω_{φ_s}, and $g_{\varphi_s}^{-1}$ the induced metric on 1-forms, then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} \left(||g_{\varphi_s}||_{C^{0,\alpha}} + ||g_{\varphi_s}^{-1}||_{C^{0,\alpha}} \right) \leq C,$$

where C only depends on α, $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof. Recall that the natural embedding

$$C^1(TM \otimes TM) \subseteq C^{0,\alpha}(TM \otimes TM)$$

is continuous, so that the operator norm of this inclusion only depends on (M, g) and α. Hence, the $C^{0,\alpha}$-norm of g_{φ_s} is uniformly bounded from above by $||g_{\varphi_s}||_{C^1}$, which in turn is uniformly bounded according to Proposition 5.20 and 5.22.

Similarly, we find a uniform $C > 0$, only depending on (M, g) and α, such that

$$(93) \qquad ||g_{\varphi_s}^{-1}||_{C^{0,\alpha}} \leq C \left(||g_{\varphi_s}^{-1}||_{C^0} + ||\nabla^g g_{\varphi_s}^{-1}||_{C^0} \right).$$

Moreover, there is the following point-wise estimate

$$(94) \qquad |\nabla^g g_{\varphi_s}^{-1}|_g \leq |g_{\varphi_s}|_g^2 \cdot |\nabla^g g_{\varphi_s}|_g.$$

Indeed, this inequality follows immediately by using holomorphic normal coordinates and differentiating the relation

$$g_{\varphi_s}^{\bar j i} \cdot g_{k \bar j}^{\varphi_s} = \delta_k^i,$$

where $g_{\varphi_s}^{\bar{j}i}$ and $g_{i\bar{j}}^{\varphi_s}$ are the components of $g_{\varphi_s}^{-1}$ and g_{φ_s}, respectively.

Thus, we conclude the required uniform bound on $||g_{\varphi_s}^{-1}||_{C^{0,\alpha}}$ from (93) and (94), together with Proposition 5.20 and 5.22. $\square$

The standard Schauder theory for ACyl metrics then implies uniform $C^{2,\alpha}$-bounds for φ_s.

Proposition 5.24 (Uniform $C^{2,\alpha}$-bound on φ_s). *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C_{\varepsilon, JX}^{\infty}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} ||\varphi_s||_{C^{2,\alpha}} \leq C,$$

where C only depends on α, $F \in C_{\varepsilon, JX}^{\infty}(M)$ and the geometry of (M, g).

Proof. Recall that Δ_g is an asymptotically translation-invariant operator of order 2, and so the Schauder estimates (Theorem 2.4) apply and yield a constant $C > 0$, only depending on (M, g), such that

$$||\varphi_s||_{C^{2,\alpha}} \leq C \left(||\varphi_s||_{C^0} + ||\Delta_g \varphi_s||_{C^{0,\alpha}} \right).$$

As $||\Delta_g \varphi_s||_{C^{0,\alpha}}$ can be bounded from above in terms of $||g_{\varphi_s}||_{C^{0,\alpha}}$, the claim then follows immediately from Corollary 5.23 and the uniform bound on $\sup_M |\varphi_s|$ given by Propositions 5.7 and 5.12. $\square$

5.2.3. *Local $C^{k,\alpha}$-estimates.* Uniform higher order estimates can be obtained similarly to the compact case considered by Yau [Yau78]. The idea is to use *local* Schauder estimates to conclude higher regularity in a uniform way. Since our manifold (M, g) is non-compact, we require the use of special coordinates in which the metric g, and all its derivatives, are uniformly bounded. This is provided by the following

Theorem 5.25. *Let (M, g) be an n-dimensional ACyl Kähler manifold and $i(g) > 0$ the corresponding injectivity radius. For each $q \in \mathbb{N}_0$, suppose that $C_q > 0$ is a constant such that the curvature tensor $\mathrm{Rm}(g)$ satisfies*

$$\sup_M |\nabla^q \mathrm{Rm}(g)|_g \leq C_q.$$

Then there are two constants $r_2 > r_1 > 0$, depending only on $n, i(g), C_q$, such that for each $x \in M$, there exists a chart $\phi : U \subset \mathbb{C}^n \to M$ satisfying the following properties:

  (i) *$B_{\mathbb{C}^n}(0, r_1) \subset U \subset B_{\mathbb{C}^n}(0, r_2)$ and $\phi(0) = x$, where $B_{\mathbb{C}^n}(0, r_i)$ denotes the Euclidean ball of radius r_i around the origin.*
  (ii) *There exists a constant $C > 0$, depending only on r_1, r_2 such that the Euclidean metric $g_{\mathbb{C}^n}$ satisfies*

$$C^{-1} g_{\mathbb{C}^n} \leq \phi^* g \leq C g_{\mathbb{C}^n} \quad on \ \ U.$$

(iii) For each $l \in \mathbb{N}_0$, there exist constants $A_l > 0$, depending only on l, r_1, r_2, such that

$$\sup_U \left| \frac{\partial^{|\mu|+|\nu|} g_{i\bar{j}}}{\partial z^\mu \partial \bar{z}^\nu} \right| \le A_l \quad \textit{for all} \ \ |\mu| + |\nu| \le l,$$

where $g_{i\bar{j}}$ are the components of g in the holomorphic coordinates $(z_1, \ldots, z_n)$ induced by ϕ, and μ, ν are multi-indices with $|\mu| = \mu_1 + \cdots + \mu_n$.

This theorem follows because the asymptotic cylinder is given explicitly. Similar results have previously been used to solve complex Monge-Ampère equations on non-compact Kähler manifolds, see for instance [CY80] and [TY90]. More generally, Theorem 5.25 is also valid for every non-compact Kähler manifold of positive injectivity radius and bounded geometry, compare [WY20][Theorem 9].

Using the coordinates given by Theorem 5.25, we can apply the local Schauder theory and conclude estimates on the $C_{\text{loc}}^{k,\alpha}$-norm of φ_s. This argument is by induction on k starting at $k = 3$.

Proposition 5.26 (Local $C^{3,\alpha}$-bound on φ_s)**.** *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$ and suppose $(\varphi_s)_{0 \le s \le 1}$ is a family in $C_{\varepsilon, JX}^\infty(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} ||\varphi_s||_{C_{\text{loc}}^{3,\alpha}} \le C,$$

where C only depends on α, $F \in C_{\varepsilon, JX}^\infty(M)$ and the geometry of (M, g).

Proof. As in [CD20b][Proposition 7.19], we follow the argument given in the compact case [Yau78].

We consider $x \in M$ and work in the holomorphic chart $\phi : U \to M$ as in Theorem 5.25. To simplify notation, we suppress ϕ and simply view U as a subset of M. The conditions (ii) and (iii) ensure that the Euclidean Hölder norm $|| \cdot ||_{C^{k,\alpha}(B_x)}$ on the ball $B_x := B(0, r_1) \subset M$ is uniformly equivalent to $|| \cdot ||_{C^{k,\alpha}(B_x,g)}$, the Hölder norm on B_x induced by the ACyl metric g. In other words, there exists a constant $C_1 > 0$, only depending on k, α and the constants in Theorem 5.25, such that

$$(95) \qquad C_1^{-1} || \cdot ||_{C^{k,\alpha}(B_x)} \le || \cdot ||_{C^{k,\alpha}(B_x,g)} \le C_1 || \cdot ||_{C^{k,\alpha}(B_x)}.$$

In particular, the interior Schauder estimates ([GT01][Theorem 6.2, 6.17]) on B_x are valid for the norms $|| \cdot ||_{C^{k,\alpha}(B_x,g)}$. The goal is to apply these estimates to the equation

$$(96) \qquad \frac{1}{2} \Delta_{g_{\varphi_s}} (\partial_j \varphi_s) = \partial_j \left(sF - \frac{X}{2}(\varphi_s) \right) + (\text{tr}_\omega - \text{tr}_{\omega_{\varphi_s}}) \mathcal{L}_{\partial_j}(\omega),$$

where ∂_j denotes the coordinate field $\partial / \partial z_j$ induced by the chart ϕ and $j = 1, \ldots, n$. Observe that (96) is obtained by applying the Lie derivative $\mathcal{L}_{\partial_j}$ to the Monge-Ampère equation (50) and dividing by $\omega_{\varphi_s}^n$.

Recall that in holomorphic coordinates, we have

$$\Delta_{g_{\varphi_s}} = g_{\varphi_s}^{\bar{j}i} \partial_i \partial_{\bar{j}},$$

so that applying Schauder requires to bound the coefficients of $\Delta_{g_{\varphi_s}}$ uniformly in $C^{0,\alpha}(B_x)$, i.e. we have to find a constant $D > 0$, only depending on α, F and the geometry of (M, g), such that

$$(97) \qquad\qquad ||g_{\varphi_s}^{\bar{j}i}||_{C^{0,\alpha}(B_x)} \leq D \quad \text{and} \quad g_{\varphi_s}^{-1} \geq D\, g_{\mathbb{C}^n}.$$

The first inequality is clear by (95) together with Corollary 5.23 and the second bound follows immediately from Proposition 5.20 and condition (ii) in Theorem 5.25. Thus, interior Schauder estimates provide a constant $C_2 > 0$, only depending on n, α and D, such that

$$(98) \qquad \begin{aligned} ||\partial_j \varphi_s||_{C^{2,\alpha}(B_x)} &\leq C_2 \left(||\Delta_{g_{\varphi_s}} \partial_j \varphi_s||_{C^{0,\alpha}(B_x)} + ||\partial_j \varphi_s||_{C^0(B_x)} \right) \\ &\leq C_2 \left(||\Delta_{g_{\varphi_s}} \partial_j \varphi_s||_{C^{0,\alpha}(B_x)} + C_1 ||\varphi_s||_{C^{2,\alpha}(M,g)} \right), \end{aligned}$$

where we used (95) for the second inequality. We continue to estimate the first term on the right-hand side of (98) as follows

$$(99) \qquad \begin{aligned} &||\Delta_{g_{\varphi_s}} \partial_j \varphi_s||_{C^{0,\alpha}(B_x)} \\ &\leq ||\varphi_s||_{C^{2,\alpha}(B_x)} + ||F||_{C^{1,\alpha}(B_x)} + ||(\mathrm{tr}_\omega - \mathrm{tr}_{\omega_{\varphi_s}})\mathcal{L}_{\partial_j}(\omega)||_{C^{0,\alpha}(B_x)} \\ &\leq C_1 \left(||\varphi_s||_{C^{2,\alpha}(M,g)} + ||F||_{C^{1,\alpha}(M,g)} + A \cdot ||g^{-1} - g_{\varphi_s}^{-1}||_{C^{0,\alpha}(M,g)} \right), \end{aligned}$$

for some constant $A > 0$ determined by condition (iii) of Theorem 5.25. Here, the first inequality is a consequence of (96) and the second one is obtained from (95). In combination with (99), inequality (98) then becomes

$$(100) \qquad\qquad ||\partial_j \varphi_s||_{C^{2,\alpha}(B_x)} \leq C_3$$

for some constant $C_3 > 0$ which only depends on C_1, C_2, A, F and the uniform bounds on $||\varphi_s||_{C^{2,\alpha}}$ and $||g_{\varphi_s}^{-1}||_{C^{0,\alpha}}$ given by Proposition 5.24 and Corollary 5.23, respectively.

To conclude the proof, we point out that the same arguments for (100) also yield

$$||\partial_{\bar{j}} \varphi_s||_{C^{2,\alpha}(B_x)} \leq C_3,$$

and hence

$$\begin{aligned} ||\varphi_s||_{C^{3,\alpha}(B_x)} &\leq \sum_{j=1}^n ||\partial_j \varphi_s||_{C^{2,\alpha}(B_x)} + ||\partial_{\bar{j}} \varphi_s||_{C^{2,\alpha}(B_x)} + ||\varphi_s||_{C^0(B_x)}, \\ &\leq 2nC_3 + C_1 ||\varphi_s||_{C^{2,\alpha}(M,g)} \\ &\leq C_4 \end{aligned}$$

with $C_4 > 0$ only depending on n, C_1, C_3 and the uniform bound on $||\varphi_s||_{C^{2,\alpha}}$. In particular, the constant C_4 is independent of both $x \in M$ and $s \in [0, 1]$, so that the proposition then follows.

$\square$

The standard bootstrapping argument then leads to uniform $C^{k,\alpha}$-estimates.

Proposition 5.27 (Local $C^{k,\alpha}$-bounds on φ_s). *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$, $k \in \mathbb{N}_{\geq 1}$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^{\infty}_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$(101) \qquad \sup_{s \in [0,1]} ||\varphi_s||_{C^{k+2,\alpha}_{\mathrm{loc}}} \leq C,$$

where C only depends on k, α, $F \in C^{\infty}_{\varepsilon, JX}(M)$ and the geometry of (M,g).

Proof. As in [CD20b][Proposition 7.19], the proof is by induction on $k \geq 1$, with the $k = 1$ case being settled by Proposition 5.26. Thus, we consider $k \geq 2$ and can assume that the statement holds for $k - 1$, i.e. that there is a $C_{k-1} > 0$, only depending on k, α, F and the geometry of (M,g), such that

$$(102) \qquad ||\varphi_s||_{C^{k+1,\alpha}_{\mathrm{loc}}} \leq C_{k-1}.$$

Using the same notation as in the previous proof, we work near a given $x \in M$ in the chart $\phi : U \to M$ given by Theorem 5.25. Because of (95), it suffices to show (101) for the Euclidean ball $B_x := B(0, r_1)$ and the Euclidean Hölder norm $|| \cdot ||_{C^{k,\alpha}(B_x)}$.

This time, we aim at applying interior Schauder estimates (of higher order) to equation (96), for which we require a constant D_{k-1}, depending only on k, α, F and the geometry of (M,g), such that

$$(103) \qquad ||g_{\varphi_s}^{\bar{j}i}||_{C^{k-1,\alpha}(B_x)} \leq D_{k-1} \quad \text{and} \quad g_{\varphi_s}^{-1} \geq D_{k-1} \, g_{\mathbb{C}}.$$

The second inequality is again clear by Proposition 5.20 and condition (ii) in Theorem 5.25, and for the first, recall that

$$g_{i\bar{j}}^{\varphi_s} = g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi_s.$$

Together with condition (iii) in Theorem 5.25, we obtain

$$\begin{aligned} ||g_{i\bar{j}}^{\varphi_s}||_{C^{k-1,\alpha}(B_x)} &\leq ||\partial\bar{\partial}\varphi_s||_{C^{k-1,\alpha}}(B_x) + A_{k-1} \\ &\leq ||\varphi_s||_{C^{k+1,\alpha}(B_x)} + A_{k-1} \\ &\leq C_{k-1} + A_{k-1}, \end{aligned}$$

where we used the induction hypothesis (102) in the last line. Consequently, the entries of the inverse matrix can be bounded as well since there exists a $C_0 > 0$, depending only on the uniform bound on $||g_{\varphi_s}^{-1}||_{C^0(M)}$ from Proposition 5.20, such that

$$||g_{\varphi_s}^{\bar{j}i}||_{C^{k-1,\alpha}(B_x)} \leq C_0 ||g_{i\bar{j}}^{\varphi_s}||_{C^{k-1,\alpha}(B_x)}.$$

Note that this follows by differentiating the identity

$$g_{\varphi_s}^{\bar{j}i} g_{l\bar{j}}^{\varphi_s} = \delta_l^i$$

and using the fact that for functions u with $\inf u > 0$, one has

$$||u||_{C^{0,\alpha}} \leq (\inf u)^{-1} \left(1 + ||u||_{C^{0,\alpha}} (\inf u)^{-1} \right).$$

Thus, (103) holds if $D_{k-1} := C_0(C_k + A_{k-1})$. Then the interior Schauder estimates [GT01][Theorem 6.17] provide a constant $E_{k-1} > 0$, depending only on n, k, α and D_{k-1}, such that

$$||\partial_j \varphi_s||_{C^{k+1,\alpha}(B_x)}$$
$$\leq E_{k-1} \left(||\Delta_{g_{\varphi_s}}(\partial_j \varphi_s)||_{C^{k-1,\alpha}(B_x)} + ||\partial_j \varphi_s||_{C^0(B_x)} \right)$$
$$\leq E_{k-1} \left(||\varphi_s||_{C^{k+1,\alpha}(B_x)} + ||F||_{C^{k,\alpha}(B_x)} + ||(\mathrm{tr}_\omega - \mathrm{tr}_{\omega_{\varphi_s}})\mathcal{L}_{\partial_j}(\omega)||_{C^{k-1,\alpha}(B_x)} \right)$$
$$\leq E_{k-1} \left(||\varphi_s||_{C^{k+1,\alpha}(B_x)} + ||F||_{C^{k,\alpha}(B_x)} + A_{k-1}||g^{-1} - g_{\varphi_s}^{-1}||_{C^{k-1,\alpha}(B_x)} \right)$$

where (96) implies the second inequality, and for the third one, we used the bounds in (iii) of Theorem 5.25. Hence, we conclude from this, together with the induction hypothesis (102) and (103), that

$$||\partial_j \varphi_s||_{C^{k+1,\alpha}(B_x)} \leq C_k$$

for some $C_k > 0$ only depending on E_{k-1}, C_{k-1}, F and the constants in Theorem 5.25. As in the previous proof, we finally arrive at

$$||\varphi_s||_{C^{k+2,\alpha}(B_x)} \leq \sum_{j=1}^n ||\partial_j \varphi_s||_{C^{k+1,\alpha}(B_x)} + ||\partial_{\bar{j}} \varphi_s||_{C^{k+1,\alpha}(B_x)} + ||\varphi_s||_{C^0(B_x)}$$
$$\leq 3nC_k,$$

as required. $\square$

5.2.4. *Weighted $C^{k,\alpha}$-estimates.* Recall from Propositions 5.7 and 5.19 that $|\varphi_s|$ is uniformly bounded from above by $e^{-\varepsilon_0 t}$ for some $0 < \varepsilon_0 \ll 1$. First, we will see that the $C^{k,\alpha}_{\varepsilon_0}$-norms of φ_s are also uniformly bounded and, in a second step, we explain how to improve the decay from ε_0 to ε.

Proposition 5.28 (Weighted $C^{k,\alpha}$-bounds on φ_s). *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$, $k \in \mathbb{N}_0$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). For the constant $0 < \varepsilon_0 < 1$ given by Proposition 5.19, exists a $C > 0$ such that*

$$\sup_{s \in [0,1]} ||e^{\varepsilon_0 t} \varphi_s||_{C^{k,\alpha}} \leq C,$$

where C only depends on ε_0, k, α, $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M, g).

Proof. We follow the argument given in [CD20b][Proposition 7.22]. For $\tau \in [0,1]$, consider the function

$$(104) \qquad H(\tau) := \log \frac{\left(\omega + i\partial\bar{\partial}(\tau \cdot \varphi_s)\right)^n}{\omega^n},$$

so that

$$H'(\tau) = \frac{1}{2}\Delta_{g_{\tau\varphi_s}}(\varphi_s),$$

where $g_{\tau\varphi_s}$ denotes the metric with Kähler form $\omega + i\partial\bar{\partial}(\tau\varphi_s)$. By using (50) and $H(0) = 0$, we can write

$$(105) \qquad sF - \frac{X}{2}(\varphi_s) = H(1) = \int_0^1 H'(\tau)d\tau = \frac{1}{2}\int_0^1 \Delta_{g_{\tau\varphi_s}}(\varphi_s)d\tau.$$

The goal is to apply local Schauder estimates to this differential equation. For any $x \in M$, let $\phi : U \to M$ be the holomorphic chart with $\phi(0) = x$ given by Theorem 5.25. Then (105) becomes

$$sF = \left(\int_0^1 g_{\tau\varphi_s}^{\bar{j}i}d\tau\right)\partial_i\partial_{\bar{j}}\varphi_s + \frac{X}{2}(\varphi_s) =: a^{\bar{j}i}\partial_i\partial_{\bar{j}}\varphi_s + b_j\partial_j\varphi_s,$$

where we use Einstein's sum convention, and the fact that X is real-holomorphic as well as $JX(\varphi_s) = 0$.

Let $k \geq 0$ be an integer and $\alpha \in (0,1)$. Recall that by conditions (ii), (iii) of Theorem 5.25 and Proposition 5.27, there exists a constant $C_1 > 0$ such that

$$a^{\bar{j}i} \geq C_1^{-1}\delta_{i\bar{j}}, \quad \text{and} \quad ||a^{\bar{j}i}||_{C^{k,\alpha}(B_x,g)} \leq C_1,$$

where B_x is the holomorphic ball of radius r_1 around x and $||\cdot||_{C^{k,\alpha}(B_x,g)}$ the Hölder norm on B_x induced by the restriction of g. Moreover, we can arrange that $||b_j||_{C^{k,\alpha}(B_x,g)} \leq C_1$ since X and all its covariant derivatives (w.r.t. g) are uniformly bounded. Recall from (95) that the norms on B_x induced by g are uniformly equivalent to the Euclidean Hölder norms, so that interior Schauder estimates ([GT01][Theorem 6.17]) can be applied. Hence, there exists a constant $C_2 > 0$, depending only on n, k, α and C_1, such that

$$(106) \qquad \begin{aligned} ||\varphi_s||_{C^{k+2,\alpha}(B_x,g)} &\leq C_2\left(||\varphi_s||_{C^0(B_x)} + ||F||_{C^{k,\alpha}(B_x,g)}\right) \\ &\leq C_2\left(C_3 + C_3||F||_{C_{\varepsilon_0}^{k,\alpha}(M,g)}\right)e^{-\varepsilon_0 t(x)}, \end{aligned}$$

for some $C_3 > 0$ only depending on the radius r_1 of the ball B_x and the bounds from Propositions 5.7 and 5.19. Note that in the last inequality, we also used that the function t is uniformly equivalent to the distance function of (M,g) to some fixed point.

As the constants in (106) are independent of the considered point $x \in M$, we conclude the desired estimate for $||e^{\varepsilon_0 t}\varphi_s||_{C^{k,\alpha}}$ as follows. Let $0 \leq l \leq k+1$ and notice that (106) implies

$$|(\nabla^g)^l\varphi_s|_g(x) \leq ||\varphi_s||_{C^{k,\alpha}(B_x,g)} \leq C_2C_3\left(1 + ||F||_{C_{\varepsilon_0}^{k,\alpha}(M,g)}\right)e^{-\varepsilon_0 t(x)}$$

$$=: Ce^{-\varepsilon_0 t(x)}$$

holds for all $x \in M$, or equivalently,

$$||e^{\varepsilon_0 t}\varphi_s||_{C^{k+1}(M,g)} \leq C.$$

This finishes the proof because the inclusion $C^{k+1}_{\varepsilon_0}(M) \subset C^{k,\alpha}_{\varepsilon_0}(M)$ is continuous.

$\square$

It remains to improve the uniform decay rate of φ_s from $e^{-\varepsilon_0 t}$ to $e^{-\varepsilon t}$, which is achieved in the next

Proposition 5.29 (Improved weighted $C^{k,\alpha}$-bounds on φ_s). *Let* $1 < \varepsilon < 2$, $\alpha \in (0,1)$, $k \in \mathbb{N}_0$ *and suppose* $(\varphi_s)_{0 \leq s \leq 1}$ *is a family in* $C^\infty_{\varepsilon,JX}(M)$ *solving* (50). *Then there exists a constant* $C > 0$ *such that*

$$\sup_{s \in [0,1]} ||e^{\varepsilon t} \varphi_s||_{C^{k,\alpha}} \leq C,$$

where C *only depends on* k, α, $F \in C^\infty_{\varepsilon,JX}(M)$ *and the geometry of* (M,g).

Proof. This improvement of the rate based on [CD20b][p. 63]. We begin by noting that $H(\tau)$ define by (104) satisfies

$$H''(\tau) = -|\partial\bar\partial\varphi_s|^2_{g_{\tau\varphi_s}},$$

so that we can write

$$(107) \quad sF + \int_0^1 \int_0^\tau |\partial\bar\partial\varphi_s|^2_{g_{\tau\varphi_s}} d\sigma\, d\tau = sF + H'(0) - H(1) + H(0)$$
$$= \frac{1}{2}(\Delta_g + X)(\varphi_s),$$

where we used (50) for the second inequality. From (107) and Proposition 5.28, we conclude that there exists a uniform constant $C > 0$ such that

$$(108) \quad (\Delta_g + X)(\varphi_s) \leq Ce^{-\varepsilon_1 t} \quad \text{with} \quad \varepsilon_1 := \min\{2\varepsilon_0, \varepsilon\} > \varepsilon_0.$$

Starting from this equation, we can obtain a uniform lower bound on $e^{\varepsilon_1 t}\varphi_s$ by using the maximum principle and arguing as in Proposition 5.7. Let $v \in C^\infty_{\varepsilon_1}(M)$ be the unique solution to

$$(\Delta_g + X)(v) = Ce^{-\varepsilon_1 t},$$

so that we have

$$(\Delta_g + X)(\varphi_s - v) \leq 0 \quad \text{on} \quad M.$$

Thus, the maximum principle implies

$$(109) \quad \varphi_s - v \geq \lim_{t \to \infty}(\varphi_s - v) = 0 \quad \text{on} \quad M$$

which is a uniform weighted lower bound on φ_s since $v \in C^\infty_{\varepsilon_1}(M)$ only depends on ε_1, C and (M,g). Combining (109) with the upper bound in Proposition 5.7, the term $||e^{\varepsilon_1 t}\varphi_s||_{C^0}$ is uniformly bound from above.

The next step is to prove that for each $k \in \mathbb{N}_0$, $\alpha \in (0,1)$, there exists a uniform constant $C > 0$ such that

$$(110) \quad ||e^{\varepsilon_1 t}\varphi_s||_{C^{k,\alpha}} \leq C.$$

Indeed, the same argument as in Proposition 5.28 goes through verbatim, starting this time from the uniform bound on $||e^{\varepsilon_1 t}\varphi_s||_{C^0}$ instead of merely $||e^{\varepsilon_0 t}\varphi_s||_{C^0}$. Hence, we improved the uniform decay from ε_0 to ε_1.

If $\varepsilon_1 = \varepsilon$, we are done, so we assume $\varepsilon_1 = 2\varepsilon_0 < \varepsilon$. Notice that (110) and (107) can then be used to further improve the uniform decay of $(\Delta_g + X)\varphi_s$ in (108) to

$$\varepsilon_2 := \min\{2\varepsilon_1, \varepsilon\} > \varepsilon_1 = 2\varepsilon_0$$

so that repeating the entire argument then gives a uniform bound on $||e^{\varepsilon_2 t}\varphi_s||_{C^{k,\alpha}}$.

After iterating this process a bounded number of times, we finally conclude the required uniform estimate on $||e^{\varepsilon t}\varphi_s||_{C^{k,\alpha}}$. $\qquad\square$

Since the previous Proposition is precisely the content of Theorem 5.6, the only statement left to show is the regularity result in Proposition 5.5.

Proof of Proposition 5.5. Let $F \in C^{\infty}_{\varepsilon, JX}(M)$ for some $1 < \varepsilon < 2$ and suppose that $\varphi \in C^{3,\alpha}_{\varepsilon', JX}(M)$ solves

$$(111) \qquad\qquad (\omega + i\partial\bar\partial\varphi)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n$$

with $0 < \varepsilon' \leq \varepsilon$. We have essentially seen all required arguments in Propositions 5.27, 5.28 and 5.29, but the difference is that we now only require *qualitative* information on the solution φ, i.e. all the constants below a priori *do* depend on φ.

First, we improve the regularity and claim that $\varphi \in C^{k,\alpha}_{\text{loc}}(M)$ for each integer $k \geq 3$ and $\alpha \in (0,1)$. As in Proposition 5.27, we work around some $x \in M$ in the holomorphic chart $\phi : B_x = B(0, r_1) \to M$ given by Theorem 5.25. Differentiating (111) in direction of $\partial_j = \partial/\partial z_j$, we obtain

$$(112) \qquad \frac{1}{2}\Delta_{g_\varphi}(\partial_j\varphi) = \partial_j\left(F - \frac{X}{2}(\varphi)\right) + \left(\text{tr}_\omega - \text{tr}_{\omega_\varphi}\right)\mathcal{L}_{\partial_j}(\omega)$$

Then we notice that the coefficients $g_\varphi^{\bar j i}$ of Δ_{g_φ}, as well as the right-hand side of (112), are in $C^{1,\alpha}(B_x)$, so that the local regularity for elliptic equations ([GT01][Theorem 6.17]) implies $\partial_j\varphi \in C^{3,\alpha}(B_x)$ for all $j = 1, \ldots, n$. Similarly, one can show that each $\partial_{\bar j}\varphi$ is also in $C^{3,\alpha}(B_x)$, implying $\varphi \in C^{4,\alpha}(B_x)$. Hence, the standard bootstrapping gives $\varphi \in C^{k,\alpha}(B_x)$ for any given $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Indeed, using $\varphi \in C^{4,\alpha}(B_x)$, we observe that the coefficients $g_\varphi^{\bar j i}$ and the right-hand side of (112) are in $C^{2,\alpha}(B_x)$, so that $\partial_j\varphi, \partial_{\bar j}\varphi \in C^{4,\alpha}(B_x)$. This implies φ is $C^{5,\alpha}(B_x)$, and so forth, until we finally arrive that $\varphi \in C^{k,\alpha}(B_x)$, as claimed.

In the second step, we show that $\varphi \in C^{k,\alpha}_{\varepsilon'}(M)$, i.e. that higher order derivatives of φ decay as $e^{-\varepsilon' t}$. In the same notation as in Proposition

5.28, consider the following equation on B_x:

$$F = \left(\int_0^1 g_{\tau\varphi}^{\bar{j}i} d\tau \right) \partial_i \partial_{\bar{j}} \varphi + \frac{X}{2}(\varphi) =: a^{\bar{j}i} \partial_i \partial_{\bar{j}} \varphi + b_j \partial_j \varphi.$$

Then, by local Schauder estimates, there exists a constant $C > 0$, depending on k, α, $||g_\varphi||_{C^{k,\alpha}(M)}$ and $||X||_{C^{k,\alpha}(M)}$, such that

$$||\varphi||_{C^{k+2,\alpha}(B_x)} \leq C \left(||\varphi||_{C^0(B_x)} + ||F||_{C^{k,\alpha}(B_x)} \right).$$

Since $\varphi = O(e^{-\varepsilon' t})$, $F \in C_\varepsilon^\infty(M)$ and because the Euclidean Hölder norms on B_x are uniformly equivalent to the ones induced by the restriction of g, we conclude from this equation that $\varphi \in C_{\varepsilon'}^{k,\alpha}(M)$ by the same argument used in Proposition 5.28.

Finally, it remains to show $\varphi \in C_\varepsilon^{k,\alpha}(M)$, i.e. to improve the decay rate from ε' to ε. Similarly to Proposition 5.29, consider the equation

$$(113) \qquad F + \int_0^1 \int_0^\tau |\partial\bar{\partial}\varphi|_{g_{\tau\varphi}}^2 d\sigma \, d\tau = \frac{1}{2} \left(\Delta_g + X \right)(\varphi),$$

and deduce that

$$\left(\Delta_g + X \right)(\varphi) \in C_{\varepsilon_1}^\infty(M) \quad \text{for} \quad \varepsilon_1 = \min\{2\varepsilon', \varepsilon\}.$$

Applying Theorem 2.11, we find a unique $v \in C_{\varepsilon_1}^\infty(M)$ such that

$$(\Delta_g + X)(v) = (\Delta_g + X)(\varphi),$$

but then the maximum principle implies $\varphi = v \in C_{\varepsilon_1}^\infty(M)$. If $\varepsilon_1 < \varepsilon$, iterate this process starting from (113) a bounded number of times, and conclude that $\varphi \in C_\varepsilon^\infty(M)$, finishing the proof. $\qquad\square$

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