# COMPACTIFYING INFINITE GROUP ACTIONS

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ABSTRACT. Conditions are given under which discrete co-compact group actions on $S^n \times \mathbf{R}^k$ extend to actions on $S^{n+k}$.

## 1. INTRODUCTION

In a previous paper [6] we studied free, properly discontinuous co-compact actions of certain infinite discrete groups $\Gamma$ on $S^n \times \mathbf{R}^k$. The goal was to find restrictions on the finite subgroups of $\Gamma$ by showing that the action $(S^n \times \mathbf{R}^k, \Gamma)$ restricted to any finite subgroup $G \subset \Gamma$ could be completed to an action of $G$ on the sphere $S^{n+k}$, free on the complement of a standardly embedded $G$-invariant subsphere $S^{k-1}$.

In this paper we consider the problem of completing the $\Gamma$ action. The examples we obtain give many new actions of discrete groups on spheres $S^{n+k}$ with limit sets contained in a $\Gamma$-invariant subsphere $S^{k-1}$. There is an extensive literature on this subject (see [9]) arising from the classical theory of Kleinian groups.

To state our main criterion for compactifying $\Gamma$ actions on $S^n \times \mathbf{R}^k$ we will use the definitions of *Lipschitz homotopy equivalence* from Section 1 (introduced in [7, §11]) and of an action which is *eventually small at infinity* given in Section 3. This material owes a lot to the foundational work of M. Gromov (see for example [4]). Recall that a torsion-free group $\Gamma_0$ has a classifying space $B\Gamma_0$ with contractible universal covering space $E\Gamma_0$ on which $\Gamma_0$ acts freely and properly discontinuously. We always assume that $B\Gamma_0$ is compact and give $E\Gamma_0$ the metric induced from a metric on $B\Gamma_0$, so that $\Gamma_0$ acts by isometries on $E\Gamma_0$.

**Definition 1.1.** A group $\Gamma$ is said to be *eventually* $(\alpha, k)$*-euclidean* if $vcd(\Gamma) < \infty$ and it has a torsion-free normal subgroup $\Gamma_0$ of finite index with $B\Gamma_0$ compact, such that

(i) $\Gamma$ acts by isometries on $E\Gamma_0$ extending the $\Gamma_0$ action, properly discontinuously, co-compactly and with finite isotropy,
(ii) $E\Gamma_0$ is Lipschitz homotopy equivalent to $\mathbf{R}^k$,
(iii) $E\Gamma_0$ has a $\Gamma$-equivariant compactification $(\overline{E\Gamma}_0, \Gamma) = (D^k, \Gamma)$ where the action is eventually small at infinity, and
(iv) the action of $\Gamma$ restricted to the boundary of $D^k$ is given by a homomorphism $\alpha\colon \Gamma \to Homeo(S^{k-1})$.

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**Theorem A.** *Let $\Gamma$ be a group which is eventually $(\alpha, k)$-euclidean. If $\Gamma$ acts freely, properly discontinuously and co-compactly on $S^n \times \mathbf{R}^k$ then there exists a compactification $(S^{n+k}, \Gamma)$ such that*

(i) *there is a $\Gamma$-invariant linear subsphere $S^{k-1}$ in $S^n \times \mathbf{R}^k$,*
(ii) *the action on $S^{n+k} - S^{k-1} = S^n \times \mathbf{R}^k$ is topologically conjugate to the given action, and*
(iii) *the $\Gamma$ action on $S^{k-1}$ is given by $\alpha$.*

These conditions hold if $\Gamma$ is a group of isometries of a complete Riemannian manifold with non-positive curvature. We therefore obtain examples of the form $\Gamma = \mathbf{Z}^k \rtimes D$ or $\Gamma = \Delta \rtimes D$, where $D$ is a finite group acting freely on a sphere and $\Delta$ is the fundamental group of a hyperbolic manifold. More examples arise from the existence results of [6,8.3].

## 2. Lipschitz Homotopy Equivalence

We work in the category of proper metric spaces and proper maps. Recall that a metric space is *proper* if all closed metric balls are compact. A proper map $f: X \to Y$ between metric spaces is *proper* if the inverse image of any bounded set is bounded. A map $f: X \to Y$ is *eventually Lipschitz* if there are constants $K > 0$, $L \geq 0$ such that $d(f(x), f(x')) \leq K d(x, x') + L$ for all $x, x' \in X$. If $L = 0$ the map is called Lipschitz.

**Definition 2.1.** Let $f_0, f_1: X \to Y$ be proper, Lipschitz maps between proper metric spaces. They are called *Lipschitz homotopy equivalent* (written $f_0 \simeq_{Lip} f_1$ if there exists a proper Lipschitz map $H: X \times \mathbf{R} \to Y \times \mathbf{R}$ of the form $H(x,t) = (h_t(x), t)$ and a continuous function $\phi: X \to [0, \infty)$ such that

(i) $h_t(x) = f_1(x)$ if $t \geq \phi(x)$, and
(ii) $h_t(x) = f_0(x)$ if $t \leq 0$.

We remark that Lipschitz homotopy equivalence is a reflexive relation, but it is not clear whether it is symmetric or transitive.

**Example 2.2.** Let $X = Y = O(K)$ with $f_0(tx) = 2tx$, $f_1(tx) = tx$. Then $f_1 \simeq_{Lip} f_2$ using the map $\phi(tx) = 2t$.

**Remark 2.3.** The inclusion map of the subspace $N = \{(x, \phi(x)) \,|\, x \in X\}$ is not necessarily a Lipschitz map into $X \times \mathbf{R}$, but $N$ is homeomorphic to $X$.

**Definition 2.4.** Two proper metric spaces $X$,$Y$ are *Lipschitz homotopy equivalent* (written $X \simeq_{Lip} Y$) if there exist proper Lipschitz maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \simeq_{Lip} id_X$ and $f \circ g \simeq_{Lip} id_Y$.

The analogous definition using eventually Lipschitz maps will be called *eventually Lipschitz homotopy equivalent*. For example, two metric spaces $X$ and $Y$ which are quasi-isometric are eventually Lipschitz homotopy equivalent. A special case to keep in mind is any subgroup $\Gamma_0 \subset \Gamma$ of finite index, where $\Gamma$ is a finitely generated discrete group with the word metric. The inclusion map is a quasi-isometry, hence $\Gamma_0$ and $\Gamma$ are eventually Lipschitz homotopy equivalent.

**Example 2.5.** The subspace of $\mathbf{R}^3$ given by the union of the half cylinder

$$\{(x, y, z) \in \mathbf{R}^3 \,|\, x^2 + y^2 = 1, z \geq 0\}$$

together with its circular base $\{x^2 + y^2 \leq 1, z = 0\}$ is homeomorphic to $\mathbf{R}^2$, but not Lipschitz homotopy equivalent to $\mathbf{R}^2$.

**Example 2.6.** Let $M$ be a complete simply-connected Riemannian $k$-manifold of non-positive curvature. Then $M \simeq_{Lip} \mathbf{R}^k$ using the exponential map $\exp_x\colon T_x M \to M$ and its inverse, the logarithm map. The logarithm map is Lipschitz but the exponential map must be modified by composing with a radial contraction to make it Lipschitz.

**Theorem 2.7.** *Suppose that $f_0$ and $f_1$ are proper Lipschitz maps from $X \to Y$ and $f_0 \simeq_{Lip} f_1$. Then the induced functors $(f_0)_*$ and $(f_1)_*$ from $\mathcal{C}_X(R) \to \mathcal{C}_Y(R)$ give the same maps on $K$-theory and $L$-theory.*

*Proof.* We include a sketch of the proof (following the argument in [7, 11.3]). Consider the following subspaces of $X \times \mathbf{R}$: let $M = \{(x, 0) \,|\, x \in X\}$, $N = \{(x, \phi(x)) \,|\, x \in X\}$ and $W = \{(x, t) \,|\, 0 \leq t \leq \phi(x)\}$. We have the inclusion maps $\iota_0\colon M \to W$ and $\iota_1\colon N \to W$. By excision, both $\iota_0$ and $\iota_1$ induce isomorphisms on bounded $K$ or $L$ theory with appropriate decorations.

Projection $(x, t) \mapsto (x, 0)$ gives maps $p_1\colon N \to M$ and $p_W\colon W \to M$, with $p_W \circ \iota_1 = p_1$ and $p_W \circ \iota_0 = id_M$. By construction, $H \circ \iota_1 = f_1 \circ p_1$ and $H \circ \iota_0 = f_0$ Therefore

$$\begin{aligned} H_* \circ (\iota_1)_* &= (f_1)_* \circ (p_1)_* \\ &= (f_1)_* \circ (p_W)_* \circ (\iota_1)_* \end{aligned}$$

But $(\iota_1)_*$ is an isomorphism, so $H_* = (f_1)_* \circ (p_W)_*$. Now

$$(f_1)_* = (f_1)_* \circ (p_W)_* \circ (\iota_0)_* = H_* \circ (\iota_0)_* = (f_0)_*$$

□

**Corollary 2.8.** *Let $\Gamma$ be eventually $(\alpha, k)$-euclidean. Then the bounded TOP structure set of $S^n \times E\Gamma_0$, bounded with respect to the second factor projection $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$, contains only the base point if $n + k \geq 5$.*

*Proof.* We compare the bounded surgery exact sequences [3] for $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$ and $\ell \circ p\colon S^n \times E\Gamma_0 \to \mathbf{R}^k$, by composing with a Lipschitz homotopy equivalence $\ell\colon E\Gamma_0 \to \mathbf{R}^k$ at the control space level. This gives a well-defined map of surgery exact sequences, inducing an isomorphism on the normal invariant and $L$-group terms. Therefore the bounded structure set of $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$ has a bijection to the bounded structure set of $\ell \circ p\colon S^n \times E\Gamma_0 \to \mathbf{R}^k$. By assumption, there is a homeomorphism $E\Gamma_0 = int(D^k)$ and we compose with a radial identifcation $int(D^k) \approx \mathbf{R}^k$ to get a homeomorphism $h\colon E\Gamma_0 \to \mathbf{R}^k$. The map $1 \times h\colon S^n \times E\Gamma_0 \to S^n \times \mathbf{R}^k$ gives a bijection of bounded structure sets, and we note that the bounded structure set of $p_2\colon S^n \times \mathbf{R}^k \to \mathbf{R}^k$ contains only the base point. □

## 3. Control at Infinity

Let $X$ be a topological space.

**Definition 3.1.** A topological action $(X, \Gamma)$ is *continuously controlled* at a $\Gamma$-invariant subset $A \subset X$ provided that: for all compact subsets $K \subset X - A$, and for each neighbourhood $U$ of $x \in A$, there exists a neighbourhood $V \subset U$ of $x$ such that whenever $\gamma \cdot K \cap V \neq \emptyset$, for some $\gamma \in \Gamma$, it follows that $\gamma \cdot K \subset U$.

Our main application is to the compactifications of classifying spaces for discrete groups. For our purposes, a compactification of $E\Gamma_0$ is a compact, contractible topological space $\overline{E\Gamma}_0$ containing $E\Gamma_0$ as a dense open subset. The *frontier* of $\overline{E\Gamma}_0$ is $\overline{E\Gamma}_0 - E\Gamma_0$, denoted by $\partial\overline{E\Gamma}_0$.

**Definition 3.2.** Let $\Gamma_0 \subset \Gamma$ be a torsion-free subgroup and suppose that $\Gamma$ acts by isometries on $E\Gamma_0$. If $(E\Gamma_0, \Gamma)$ has a $\Gamma$-equivariant compactification $(\overline{E\Gamma}_0, \Gamma)$, then we say that the action is *eventually small at infinity* if $(\overline{E\Gamma}_0, \Gamma)$ is continuously controlled at the frontier $\partial\overline{E\Gamma}_0$.

There is another control condition at infinity using the metric on $E\Gamma_0$.

**Definition 3.3.** Let $\Gamma$ act properly discontinuously and co-compactly on $E\Gamma_0$, and suppose that $(E\Gamma_0, \Gamma)$ has a $\Gamma$-equivariant compactification $(\overline{E\Gamma}_0, \Gamma)$. The group $\Gamma$ is *small at infinity* provided that: for all metric balls $B(k) \subset E\Gamma_0$ and for each $x \in \overline{E\Gamma}_0 - E\Gamma_0$ and neighbourhood $U$ of $x$ in $\overline{E\Gamma}_0$, there exists a neighbourhood $V \subset U$ of $x$ such that whenever $B(k) \cap V \neq \emptyset$, it follows that $B(k) \subset U$.

Note that since $E\Gamma_0$ is a proper metric space and $\Gamma$ acts by isometries, a group $\Gamma$ which is small at infinity has the given action $(\overline{E\Gamma}_0, \Gamma)$ eventually small at infinity. In our situation, these two conditions are actually equivalent.

**Proposition 3.4.** *Suppose that the action $(\overline{E\Gamma}_0, \Gamma)$ is eventually small at infinity. Then the group $\Gamma$ is small at infinity.*

*Proof.* Since the action $(E\Gamma_0, \Gamma_0)$ is co-compact, we can choose a point $x_0 \in E\Gamma_0$ and $k$ so large that any metric ball $B(k)$ contains a point of the form $\gamma \cdot x_0$ for some $\gamma \in \Gamma$. Given $x \in \overline{E\Gamma}_0 - E\Gamma_0$, and a neighbourhood $U$ of $x$ in $\overline{E\Gamma}_0$, there exists a neigbourhood $V \subset U$ such that

$$\gamma \cdot B(x_0, 2k) \cap V \neq \emptyset \qquad \text{implies} \qquad \gamma \cdot B(x_0, 2k) \subset U \ . \tag{3.5}$$

This is just the condition that the $\Gamma$ action is eventually small at infinity.

Now suppose that $B(k) \cap V \neq \emptyset$ for some metric ball $B(k)$. Then $\gamma \cdot x_0 \in B(k)$ for some $\gamma \in \Gamma$ and it follows that

$$B(k) \subset B(\gamma \cdot x_0, 2k) = \gamma \cdot B(x_0, 2k)$$

where the last equality follows since $\Gamma$ acts by isometries on $E\Gamma_0$. But now we conclude from (3.5) that $B(k) \subset U$. $\square$

## 4. Almost Equivariant Projections

We suppose now that $\Gamma$ is eventually $(\alpha, k)$-euclidean. If $\Gamma$ acts freely, properly discontinuously and co-compactly on $S^n \times \mathbf{R}^k$ then we have a compact manifold $M = S^n \times \mathbf{R}^k/\Gamma_0$ and a classifying map $M \to B\Gamma_0$. Up to homotopy this is a spherical fibration with fibre $S^n$, so we can replace it by a block fibration $\overline{M} \to B\Gamma_0$ with $\overline{M}$ still compact and homotopy equivalent to $M$. The universal cover $\widehat{M}$ of $\overline{M}$ is a block fibration $q\colon \widehat{M} \to E\Gamma_0$ over $E\Gamma_0$, which is contractible, so it is block and hence boundedly homotopy equivalent to the trivial block fibration $S^n \times E\Gamma_0 \to E\Gamma_0$. Here we are using the blocked structures with respect to a $\Gamma_0$-equivariant triangulation of $E\Gamma_0$, so the simplices have a bounded diameter.

Now if $\widetilde{M}$ denotes the universal covering of $M$, we obtain a bounded homotopy equivalence $f\colon \widetilde{M} \to S^n \times E\Gamma_0$, bounded with respect to the second factor projection $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$.

**Definition 4.1.** Let $X$ be a $\Gamma$ space and $Z$ be a metric space on which $\Gamma$ acts by isometries. Given a $\Gamma$ equivariant map $p\colon X \to Z$, we say that $p$ is *almost equivariant* if there exists a constant $k > 0$ such that $d(\gamma \cdot p(x), p(\gamma \cdot x)) < k$ for all $x \in X$. If the bound $k$ is independent of $\gamma \in \Gamma$, we say that $p$ is *uniformly almost equivariant*

**Lemma 4.2.** *The map $pf\colon \widetilde{M} \to E\Gamma_0$ is uniformly almost $\Gamma_0$-equivariant.*

*Proof.* The map $f$ is the composite of a $\Gamma_0$-equivariant bounded homotopy equivalence $\widetilde{M} \to \widehat{M}$ (covering the homotopy equivalence $M \to \overline{M}$) and a bounded homotopy equivalence $j\colon \widehat{M} \to S^n \times E\Gamma_0$. Since $p \circ j$ is bounded homotopy equivalent to $q$, the distance $d(pj(\gamma \cdot x), \gamma \cdot pj(x))$ differs by a bounded amount independent of $\gamma$ from $d(q(\gamma \cdot x), \gamma \cdot q(x))$. But $q$ is $\Gamma_0$-equivariant so this last distance is zero. $\square$

Since $\Gamma$ is eventually $(\alpha, k)$-euclidean, the bounded structure set of $S^n \times E\Gamma_0 \to E\Gamma_0$ contains just one element (represented by the identity map, see Corollary 2.8). Therefore there is a bounded homotopy from $f$ to a homeomorphism $h\colon \widetilde{M} \to S^n \times E\Gamma_0$. On $\widetilde{M}$ we have the given free $\Gamma$ action, so we can consider the conjugate $\Gamma$ action by $h$ on $S^n \times E\Gamma_0$.

**Lemma 4.3.** *The second factor projection map $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$ is uniformly almost equivariant with respect to the conjugated $\Gamma$ action restricted to $\Gamma_0$.*

*Proof.* We consider the quantity

$$d(phzh^{-1}x, zpx) \leq d(phzh^{-1}x, pf(zh^{-1}x)) + d(pf(zh^{-1}x), zpx)$$

where $z \in \Gamma_0$ and $x \in S^n \times E\Gamma_0$. The first term is bounded since $f$ is boundedly homotopic to $h$. The second term is a bounded distance from $d(zpf(h^{-1}x), zpx)$ since $pf$ is uniformly almost $\Gamma_0$-equivariant by (4.2). Since $\Gamma_0$ acts by isometries on $E\Gamma_0$, this last term is equal to $d(pf(h^{-1}x), px)$ which is bounded since $f$ is boundedly homotopic to $h$. $\square$

The main result of this section is that this almost equivariance property holds for the $\Gamma$ action as well.

**Theorem 4.4.** *The second factor projection map $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$ is uniformly almost equivariant with respect to the conjugated $\Gamma$ action.*

*Proof.* Choose a compact fundamental domain $U$ for the $\Gamma_0$ action on $S^n \times E\Gamma_0$, so that the sets $\{z \cdot U \,|\, z \in \Gamma_0\}$ cover $S^n \times E\Gamma_0$. Let $\{g_i \,|\, 1 \leq i \leq m\}$ be a set of coset representatives for $\Gamma/G_0$, and let

$$U_1 = \bigcup_{1 \leq i \leq m} g_i \cdot U \ .$$

Since $U$ is compact, the set

$$\overline{U}_1 = \bigcup_{1 \leq i \leq m} g_i \cdot p(U_1) \subset E\Gamma_0$$

has finite diameter $d$.

For $x \in S^n \times E\Gamma_0$, write $x = zu$ where $z \in \Gamma_0$ and $u \in U$, and let $g = g_i$ for some $i$, $1 \leq i \leq m$. Then

$$\begin{aligned} d(pgx, gpx) &= d(pgzu, gpzu) \\ &\leq d(pgzu, gzg^{-1}pgu) + d(gzg^{-1}pgu, gzpu) + d(gzpu, gpzu) \end{aligned}$$

The first term is

$$d(pgzu, gzg^{-1}pgu) = d(pgzg^{-1}gu, gzg^{-1}pgu) < k$$

since $gzg^{-1} \in \Gamma_0$ by (4.3). This is the place where we need *uniform* almost equivariance for the map $p$ with respect to the $\Gamma_0$ action. The second term is

$$d(gzg^{-1}pgu, gzpu) = d(gzg^{-1}pgu, gzg^{-1}gpu) = d(pgu, gpu) < d$$

since $\Gamma_0$ acts by isometries on $E\Gamma_0$, and both $p(gu)$ and $gp(u)$ are in $\overline{U}_1$ which has diameter $d$. Finally, the third term

$$d(gzpu, gpzu) = d(zpu, pzu) < k$$

since $\Gamma$ acts by isometries and the projection $p$ is $\Gamma_0$-almost equivariant. From these estimates we get

$$d(p\gamma x, \gamma px) < 3k + d$$

for all $\gamma \in \Gamma$ and all $x \in S^n \times E\Gamma_0$ so the projection $p$ is uniformly almost equivariant. □

## 5. The Proof of Theorem A

We assume that the group $\Gamma$ is $(\alpha, k)$-euclidean and acts freely, properly discontinuously, and co-compactly on $S^n \times \mathbf{R}^k$. The results of the previous section say that $(S^n \times \mathbf{R}^k, \Gamma)$ is $\Gamma$ equivariantly homeomorphic to an action $(S^n \times E\Gamma_0, \Gamma)$ which has the additional property that the second factor projection $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$ is uniformly almost equivariant.

**Proposition 5.1.** *Suppose that $\Gamma$ is small at infinity. Then for each $x_0 \in \partial\overline{E\Gamma}_0$, and each neighbourhood $U$ of $x_0$, there exists a neighbourhood $V \subset U$ of $x_0$ such that $p(x) \in V$, for $x \in S^n \times E\Gamma_0$, implies that $\gamma^{-1} \cdot p(\gamma \cdot x) \in U$ for all $\gamma \in \Gamma$.*

*Proof.* Let $k$ denote a uniform bound for the $\Gamma$-almost equivariance of the map $p: S^n \times E\Gamma_0 \to E\Gamma_0$, with respect to the action on $S^n \times E\Gamma_0$ described above. For $U$ and $x_0$ given, according to Definition 3.3 there exists a neigbourhood $V \subset U$ of $x_0$ such that $B(k) \cap V \neq \emptyset$ implies that $B(k) \subset U$. Now if $p(x) \in V$, for some $x \in S^n \times E\Gamma_0$, then we apply this to the $k$-ball centered at $p(x)$, and obtain $B(p(x), k) \subset U$. But

$$d(\gamma^{-1} \cdot p(\gamma \cdot x), p(x)) = d(\gamma^{-1} \cdot p(\gamma \cdot x), p(\gamma^{-1}\gamma \cdot x)) < k$$

so that $\gamma^{-1} \cdot p(\gamma \cdot x) \in B(p(x), k)$ implies $\gamma^{-1} \cdot p(\gamma \cdot x) \in U$. □

**Corollary 5.2.** *The action $(S^n \times \mathbf{R}^k, \Gamma)$ extends to a topological $\Gamma$-action on $S^{n+k} = S^n \times \mathbf{R}^k \cup S^{k-1}$ where the action on $S^{k-1}$ is given by the action $\alpha$ on $\partial\overline{E\Gamma}_0 = \partial D^k = S^{k-1}$.*

*Proof.* The preceeding result shows that for each $\gamma \in \Gamma$, the given action of $\gamma$ on $S^n \times \mathbf{R}^k$ together with the action given by $\alpha(\gamma)$ on $S^{k-1}$ fit together to give a homeomorphism of $S^{n+k}$. □

## 6. Future Developments

Some of the definitions given earlier suggest questions for further study. Probably the formulations below are too naive.

**Question 6.1.** Let $(X, d)$ be a metric space homeomorphic to $\mathbf{R}^k$, and suppose that $(X, d)$ is also Lipschitz homotopy equivalent to $\mathbf{R}^k$. Does a finite group acting by isometries on $X$ necessarily have a fixed point ?

We remark that there exist smooth fixed-point free actions of finite cyclic groups on $\mathbf{R}^k$ (see [8]), but these actions do not preserve the standard metric on $\mathbf{R}^k$.

**Question 6.2.** Suppose that $(\overline{E\Gamma}_1, \Gamma_1)$ and $(\overline{E\Gamma}_2, \Gamma_2)$ are actions which are eventually small at infinity. Does there exist an equivariant compactification of $E\Gamma_1 \times E\Gamma_2$ which is eventually small at infinity ?

An action of a discrete group is uniformly continuous if the usual $\epsilon$-$\delta$ continuity condition for each group element $\gamma \in \Gamma$ allows a $\delta$ depending only on $\epsilon$ (and not on the group element $\gamma$).

**Question 6.3.** *Suppose that $\Gamma_0 \subset \Gamma$ is a torsion-free normal subgroup of finite index and the action on $(\overline{E\Gamma}_0, \Gamma)$ is eventually small at infinity. Is the $\Gamma$-action on $\overline{E\Gamma}_0$ uniformly continuous ?*

We would certainly like to relax some of our assumptions on the discrete group $\Gamma$ in order to include more of the interesting classes of groups already appearing in the literature (see [1], [4]). As above, we let $\Gamma_0 \subset \Gamma$ be a torsion-free normal subgroup of finite index. Suppose that $(\overline{E\Gamma}_0, \Gamma)$ is an equivariant compactification of $E\Gamma_0$ with $\Gamma$-action eventually small at infinity. We will assume that there exists an embedding of $\overline{E\Gamma}_0$ in $S^{n+k}$ as a neighbourhood retract. Furthermore, we want to assume that $E\Gamma_0$ is compactified by a $Z$-set. In other words, $\overline{E\Gamma}_0$ comes equipped with a homotopy

$$h_t\colon \overline{E\Gamma}_0 \times [0,1] \to \overline{E\Gamma}_0$$

such that $h_0 = id$, and the image $h_t(x) \notin \partial\overline{E\Gamma}_0$ for $t > 0$. Using the retract and this homotopy, we can define a proper map $q\colon S^{n+k} - \partial\overline{E\Gamma}_0 \to E\Gamma_0$ by the formula $h_t(r(x))$ where

$$t = \frac{d(x, \overline{E\Gamma}_0)}{1 + d(x, \overline{E\Gamma}_0)}$$

and $r$ is the retract. We can now ask if the bounded structure set of

$$q\colon S^{n+k} - \partial\overline{E\Gamma}_0 \to E\Gamma_0$$

contains just one element for $n+k \geq 5$. This is true for the bounded structure set of $S^{n+k} - K$ bounded over the open cone $O(K)$, provided that $K$ is a finite simplicial complex (see our paper [5,3.2]). Extending our techniques to handle complements of $\partial\overline{E\Gamma}_0$ instead of a finite complex $K$ looks like an interesting project, with other possible applications.

If $\Gamma$ acts freely, properly discontinuously, and co-compactly on $S^{n+k} - \partial\overline{E\Gamma}_0$ then all the additional information we need to extend the action to $S^{n+k}$ is the analogue of Corollary 2.8. The bounded homotopy type of the complement of the frontier (if $n \geq 2$) is again just given by the second factor projection $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$. This follows by the same arguments given at the beginning of Section 4, provided that $S^{n+k} - \partial\overline{E\Gamma}_0$ is homotopy equivalent to $S^n$. Since $\overline{E\Gamma}_0$ is contractible, this will follow for example if $(\overline{E\Gamma}_0, \partial\overline{E\Gamma}_0)$ is a $k$-dimensional Poincaré pair. In this situation, we need an affirmative answer from surgery theory to the following question.

**Question 6.4.** Suppose that $E\Gamma_0$ is a topological manifold. Does the bounded structure set of $S^n \times E\Gamma_0 \to E\Gamma_0$ contain only the base point ? Equivalently, is every bounded homotopy equivalence $W \to S^n \times E\Gamma_0$, bounded with respect to the second factor projection $p\colon S^n \times E\Gamma_0 \to E\Gamma_0$, boundedly homotopic to a homeomorphism ?

We conclude this collection of informal questions and remarks with a bounded version of the Borel conjecture:

**Question 6.5.** Suppose that $E\Gamma_0$ is a topological manifold. Is every bounded homotopy equivalence $W \to E\Gamma_0$, bounded with respect to the identity map $E\Gamma_0 \to E\Gamma_0$, boundedly homotopic to a homeomorphism ?

We remark that if the bounded Borel conjecture is true for a torsion-free group $\Gamma_0$, then the integral $L$-theory assembly map for $\Gamma_0$ is a split monomorphism of spectra [2], [10]. This is a strong version of the Novikov conjecture.

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