EMBEDDING OBSTRUCTIONS AND 4-DIMENSIONAL THICKENINGS OF 2-COMPLEXES

VYACHESLAV S. KRUSHKAL

Abstract. The vanishing of Van Kampen's obstruction is known to be necessary and sufficient for embeddability of a simplicial n-complex into $\mathbb{R}^{2n}$ for $n \neq 2$, and it was recently shown to be incomplete for $n = 2$. We use algebraic-topological invariants of four-manifolds with boundary to introduce a sequence of higher embedding obstructions for a class of 2-complexes in $\mathbb{R}^4$.

1. Introduction

By general position any n-dimensional simplicial complex K PL embeds into $\mathbb{R}^{2n+1}$, while the image of a generic map of K into $\mathbb{R}^{2n}$ has a finite number of double points. By counting double points of an immersion one gets the cohomological obstruction to embeddability of an n-complex into $\mathbb{R}^{2n}$, introduced by Van Kampen [12]. He also constructed for each n examples which do not admit an embedding. Note that the question of embeddability is not invariant under homotopy equivalence: by a theorem of Stallings [10] every n-dimensional simplicial complex is homotopy equivalent to one which embeds into $\mathbb{R}^{2n}$. An application of Whitney trick shows that this obstruction is complete for $n > 2$, see [12], [9], [13], [4]. It follows from Kuratowski's planarity criterion for graphs [6] that this result also holds for $n = 1$. The remaining case, $n = 2$, was open until recently when the obstruction was shown in [4] to be incomplete.

This paper is centered around the question of embeddability of 2-complexes in $\mathbb{R}^4$, and is motivated by the result of [4]. We define for 2-complexes K with $H_1(K;\mathbb{Q}) = 0$ a sequence of higher embedding obstructions $\{o_m(K)\}$, using Massey products on the boundary of a 4-dimensional thickening M^4 of K. Roughly, Van Kampen's obstruction corresponds in this setting to the intersection pairing on M, modulo the choice of a thickening M. Since different thickenings may give different Massey products, $\{o_m(K)\}$ in general are *subsets* of the corresponding cohomology groups; $o_{m+1}(K)$ is defined if $o_m(K)$ contains zero. If K embeds into $\mathbb{R}^4$ then $0 \in o_m(K)$ for each m. We prove that these higher obstruction detect non-embeddability of the examples introduced in [4], by showing that $o_m(K)$ does not contain zero for some m. Our proof uses the result of Conway - Gordon and Sachs that any embedding of a complete graph on 6 vertices into S^3 contains two disjoint linking cycles ([1], [8]).

In the simplest relative case, for the disjoint union of 2-disks with a prescribed embedding of their boundaries into S^3, by a result of Turaev [11] Massey products correspond to Milnor's $\bar{\mu}$-invariants of the link in S^3, so our obstructions may be thought of as an absolute analogue of $\bar{\mu}$-invariants. As in the case of $\bar{\mu}$-invariants (for example, Whitehead double of the Hopf link is not slice, while all $\bar{\mu}$-invariants vanish), one does not expect that the entire sequence of obstructions defined here is complete, although no examples are known at this time.

The definition of Van Kampen's obstruction is recalled in section 2. In section 3 we prove its reformulation in the context of thickenings, and we introduce the sequence of higher obstructions $\{o_m(K)\}$. We review the examples of 2-complexes in [4] in section 4, and we compute the obstructions for these examples. Section 5 gives a reformulation of Van Kampen's obstruction in terms of configuration spaces, which suggests another approach to defining higher embedding obstructions.

The present study of the embedding problem for 2-complexes in $\mathbb{R}^4$ is motivated, in part, by the 4-dimensional topological surgery conjecture, via its *(A,B)-slice* reformulation [3]. More precisely, the surgery conjecture is equivalent to the relative embedding question for a certain family of 4-dimensional handlebodies – "thickenings" of 2-complexes in the sense of section 3. However, many interesting examples of these handlebodies have non-trivial first homology, and for this application the obstructions $\{o_i(K)\}$ need to be extended to the general case.

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2. Van Kampen's obstruction

In this section we briefly review the definition of Van Kampen's obstruction, see [4] for more details. In 1933 Van Kampen [12] introduced an obstruction $o(K) \in H^{2n}_{\mathbb{Z}/2}(K^*;\mathbb{Z})$ to piecewise-linear embeddability of an n-dimensional simplicial complex K into $\mathbb{R}^{2n}$. The cohomology in question is $\mathbb{Z}/2$-equivariant cohomology where $\mathbb{Z}/2$ acts on the deleted product $K^* = K \times K \smallsetminus \Delta$ of a complex K by exchanging the factors of K^* and acts on the coefficients by multiplication with $(-1)^n$. The diagonal Δ consists of all products $\sigma \times \tau$ such that simplices σ and τ have at least one vertex in common. Note that for n even (in particular, in the case of main interest in this paper, $n = 2$) the action of $\mathbb{Z}/2$ on the coefficients is trivial, and $o(K)$ is an element of the ordinary cohomology group $H^{2n}(K^*/(\mathbb{Z}/2);\mathbb{Z})$.

Let f be any PL immersion of K into $\mathbb{R}^{2n}$. The obstruction is defined on the cochain level by counting algebraic intersection numbers of the images of disjoint top-dimensional simplices of K: $o_f(\sigma^n \times \tau^n) = f(\sigma) \cdot f(\tau)$. Here $\sigma \times \tau$ is viewed as an oriented generator of the $2n$-th chain group of $K \times K \smallsetminus \Delta$. The cohomology class $o(K)$ of o_f is independent of the chosen immersion f. Clearly $o(K)$ is trivial if K embeds into $\mathbb{R}^{2n}$. Shapiro [9] and Wu [13] made this definition precise and proved, using the Whitney trick, the converse in dimension n greater than 2.

Theorem 2.1 ([12], [9], [13], [6]). *For $n \neq 2$ an n-dimensional simplicial complex K embeds into $\mathbb{R}^{2n}$ if and only if $o(K)$ vanishes.*

See [4] for a modern exposition of the proof for $n > 2$. For $n = 1$ this theorem follows from Kuratowski's planarity criterion. The obstruction in the remaining case, for $n = 2$, was shown to be incomplete in [4]. We recall the construction of examples in [4] in section 4.

3. Obstructions via 4-dimensional thickenings of 2-complexes

In this section we give a rational reformulation of Van Kampen's obstruction $o(K)$ in terms of thickenings of K, and we introduce the sequence of higher embedding obstructions $\{o_m(K)\}$ for 2-complexes whose rational first homology vanishes. Throughout this section all coefficients are $\mathbb{Q}$, unless stated otherwise, and K denotes a simplicial 2-complex.

Definition 3.1. A *thickening* of K is a 4-manifold M with boundary, obtained by replacing each i-simplex of K with a 4-dimensional i-handle, $i = 0, 1, 2$. The attaching map of each 2-handle is required to be isotopic, within the union of 0- and 1-handles, to the attaching map of the corresponding 2-dimensional simplex.

In general, K has different thickenings depending on the choice of attaching maps of the 2-handles. For example, the boundary-connected sum $S^2 \times D^2 \natural S^2 \times D^2$ and $S^2 \times S^2 \smallsetminus$ 4-cell are both thickenings of $S^2 \vee S^2$.

The intersection pairing on M defines an element $\iota \in Hom(H_2(M) \otimes H_2(M), \mathbb{Q})$. Let $\bar{\iota} \in H^4(K \times K \smallsetminus \Delta; \mathbb{Q})$ denote its image under the homorphism

$$Hom(H_2(M) \otimes H_2(M), \mathbb{Q}) \cong Hom(H_2(K) \otimes H_2(K), \mathbb{Q}) \cong$$

$$\cong H^4(K \times K) \longrightarrow H^4(K \times K \smallsetminus \Delta)$$

where the last map is the restriction homomorphism.

Theorem 1. *The image of (rational) Van Kampen's obstruction $o(K)$ under the homomorphism*

$$H^4_{\mathbb{Z}/2}(K \times K \smallsetminus \Delta; \mathbb{Q}) \longrightarrow H^4(K \times K \smallsetminus \Delta; \mathbb{Q})$$

coincides with $-\bar{\iota}$.

Proof. Suppose a thickening M is induced by an immersion $f\colon K \looparrowright \mathbb{R}^4$, so that f extends to an immersion $M \looparrowright \mathbb{R}^4$. By subdividing the complex K, if necessary, one may assume that $f(\sigma) \cap f(\tau) \neq \emptyset$ for all 2-simplices $\sigma \neq \tau$ with $\sigma \times \tau \in \Delta$, and $f|_\sigma$ is an embedding for each σ. Let $\bar{o}_f\colon C_4(K \times K) \longrightarrow \mathbb{Q}$ denote the extension by zero on the diagonal of Van Kampen's cochain o_f. It suffices to prove that $[\bar{o}_f]$ and $-\bar{\iota}$ define identical elements in $Hom(H_2(K) \otimes H_2(K), \mathbb{Q})$. Let a, b be two classes in $H_2(K)$ and let $\alpha = \Sigma \alpha_i \sigma_i$, $\beta = \Sigma \beta_i \sigma_i$ be their cycle representatives, where $\{\sigma_i\}$ is the set of 2-simplices of K. In order to compute $a \cdot b$ in M, we perturb α and β to $\tilde{\alpha}$ and $\tilde{\beta}$ which intersect each other transversely (in a finite number of double points). Since f is an immersion into $\mathbb{R}^4$, the intersection number of two cycles $f(\tilde{\alpha})$ and $f(\tilde{\beta})$ is trivial. On the other hand, $f(\tilde{\alpha}) \cdot f(\tilde{\beta})$ is the sum of two terms: one is the intersection number of $\tilde{\alpha}$ and $\tilde{\beta}$ in M, the other is obtained by considering the intersections of

$f(\tilde{\alpha})$ and $f(\tilde{\beta})$, which are singular points of f. This last term is equal to $\bar{o}_f(\alpha \times \beta)$, and this proves

$$[\bar{o}_f] = -\bar{\iota}\colon\ H_2(K) \otimes H_2(K) \longrightarrow \mathbb{Q}.$$

The restriction of $[\bar{o}_f]$ to $H^4(K \times K \smallsetminus \Delta; \mathbb{Q})$ coincides with $o(K)$, thus the result is proved for thickenings induced by immersions.

In general not every thickening of K may be immersed into $\mathbb{R}^4$. Let M^4 be an arbitrary thickening of K and let $f\colon\ K \looparrowright \mathbb{R}^4$ be any immersion. Again one may assume that $f(\sigma) \cap f(\tau) = \emptyset$ if $\sigma \times \tau \in \Delta$, $\sigma \neq \tau$, and $f|_\sigma$ is an embedding for each simplex σ. The immersion f extends to an embedding of 0- and 1-handles of M. There is an integer obstruction to extending it over each 2-handle, due to a possible difference of the framing of the 2-handle and of the normal bundle of the 2-simplex in $\mathbb{R}^4$. However, each 2-handle may be mapped into $\mathbb{R}^4$ as a bundle over the corresponding 2-simplex, pinched over several points.

The proof, given above in the case of an immersion, carries through, if one extends o_f to $\bar{o}_f$ by setting $\bar{o}_f(\sigma \times \sigma)$ to be equal to the difference in framings, discussed above, and setting $\bar{o}_f(\sigma \times \tau) = 0$ for all $\sigma \times \tau \in \Delta$, $\sigma \neq \tau$. □

Remark. In general the intersection pairing varies within the homotopy type of a 4-manifold M. In the example above the intersection pairing on $S^2 \times D^2 \natural S^2 \times D^2$ is trivial, while the pairing on $S^2 \times S^2 \smallsetminus$ 4-cell is non-degenerate. However, theorem 1 shows that the pull-back of the intersection pairing on thickenings to a cohomology class on $K \times K \smallsetminus \Delta$ is an invariant of K, which coincides with (negative) Van Kampen's obstruction.

As a corollary to the proof of theorem 1, we have the following result.

Lemma 3.2. *Let K be a 2-complex such that Van Kampen's obstruction $o(K)$ vanishes. Then there is a 4-dimensional thickening M of K with the trivial intersection pairing $\iota = 0 \in Hom(H_2(M) \otimes H_2(M); \mathbb{Q})$.*

Proof. Any cochain representative of the obstruction $o(K)$ is given by o_f for some immersion f, see [12] or [4]. Since $o(K)$ vanishes, there exists an immersion $f\colon\ K \looparrowright \mathbb{R}^4$, giving rise to the trivial Van Kampen's cochain $o_f = 0$. Let M denote the thickening, induced by f. It follows from the proof of theorem 1 that if one extends o_f by zero on the diagonal to a cochain $\bar{o}_f$ on $K \times K$, then $\bar{\iota} = [\bar{o}_f] = 0 \in Hom(H_2(M) \otimes H_2(M); \mathbb{Q})$. This completes the proof of lemma 3.2. □

Before introducing higher embedding obstructions, we review the definition of Massey products. See [5] for proofs and additional properties.

Definition 3.3. Let X be a space, and let $\alpha_1, \dots, \alpha_m$ be elements in $H^1(X)$. Suppose there is a collection of 1-cochains $S = \{c_{ij} \in C^1(X) | 1 \leq i \leq j \leq m, (i,j) \neq (1,m)\}$ satisfying

$$[c_{ii}] = \alpha_i \text{ for each i} = 1, \dots, \text{m},$$

$$\delta c_{ik} = \sum_{j=i}^{k-1} c_{ij} \cup c_{j+1,k} \text{ for i < k.}$$

A computation shows that the cochain $\sum_{j=1}^{m-1} c_{1j} \cup c_{j+1,m}$ is a cocycle, and its cohomology class in $H^2(X)$ is called the *Massey product* of $\alpha_1, \ldots, \alpha_m$ defined by the system S. The set of Massey products corresponding to all such defining systems is denoted by $<\alpha_1, \ldots, \alpha_m>$.

Massey product of two elements is just a cup product. Note that given some classes $\alpha_1, \ldots, \alpha_m$, $<\alpha_1, \ldots, \alpha_m>$ is not necessarily defined. However, if all Massey products of less than m elements vanish, then for any $\alpha_1, \ldots, \alpha_m$, $<\alpha_1, \ldots, \alpha_m>$ is a well-defined element.

The following lemma justifies our definition of higher embedding obstructions.

Lemma 3.4. *Let M be a 4-manifold with boundary and with $H_1(M; \mathbb{Q}) = 0$, and suppose M admits an embedding into $\mathbb{R}^4$. Then all Massey products on $H^1(\partial M; \mathbb{Z})$ vanish.*

Proof. Let N denote the complement $\mathbb{R}^4 \smallsetminus M$. By Alexander duality, $H_2(N)$ and $H^2(N)$ are trivial. The map $i^*\colon H^1(N) \longrightarrow H^1(\partial M)$ in the cohomology sequence of the pair $(N, \partial M)$ is an isomorphism, since by assumption and by Poincare duality $H^1(N, \partial M) \cong H_3(N)$ and $H^2(N, \partial M) \cong H_2(N)$ are trivial. Since Massey products are functorial, for any $\alpha_1, \ldots, \alpha_m \in H^1(\partial M)$ one has

$$<\alpha_1, \ldots, \alpha_m> = i^* < (i^*)^{-1}\alpha_1, \ldots, (i^*)^{-1}\alpha_m> \in H^2(\partial M).$$

However, this is the image of an element in $H^2(N) = 0$, and the result follows. $\square$

Let K be a 2-complex with $H_1(K; \mathbb{Q}) = 0$, and assume $o(K)$ vanishes. Let M be a thickening of K with trivial intersection pairing (its existence is given by lemma 3.2.) Note that the map $H_2(\partial M) \longrightarrow H_2(M)$ is an isomorphism, since by assumption on K, $H_2(M, \partial M) \cong H^1(M) = 0$, and the map $H_2(M) \longrightarrow H_2(M, \partial M)$ is trivial by assumption on the intersection pairing.

We now give the definition of higher embedding obstructions. Let a_1, a_2, a_3 be classes in $H_2(K)$, and let $\alpha_1, \alpha_2, \alpha_3 \in H^1(\partial M)$ denote their images under the isomorphism

$$H_2(K) \cong H_2(M) \cong H_2(\partial M) \cong H^1(\partial M).$$

The triple cup product $(\alpha_1 \cup \alpha_2 \cup \alpha_3)[\partial M]$ defines a homomorphism $H_2(K) \otimes H_2(K) \otimes H_2(K) \longrightarrow \mathbb{Q}$, and an element $o_3(K, M) \in H^6(K \times K \times K)$. The cohomology class $o_3(K, M)$ depends in general on the choice of a thickening M, thus we define the third obstruction $o_3(K)$ to be the subset $\{o_3(K, M)\} \subset H^6(K^3; \mathbb{Q})$ where M is to vary over all thickenings of K with trivial intersection pairing.

Definition 3.5. Assume by induction that $o_{m-1}(K)$ is defined and contains zero for some $m > 3$, so there is a thickening M of K such that all Massey products on $H^1(\partial M)$ of at most $(m-2)$ elements vanish. Let $a_1, \dots, a_m$, be classes in $H_2(K)$, and let $\alpha_1, \dots, \alpha_m$ denote the corresponding elements in $H^1(\partial M)$. The class $o_m(K, M) \in H^{2i}(K^i; \mathbb{Q})$ is defined by the homomorphism

$$H_{2m}(K^i) \cong \otimes_1^m H_2(K) \cong \otimes_1^m H^1(\partial M) \longrightarrow \mathbb{Q}$$

which sends $a_1 \otimes \cdots \otimes a_m$ to $(<\alpha_1, \dots, \alpha_{m-1}> \cup\, \alpha_m)[\partial M]$.

Here since $o_{m-1}(K, M)$ vanishes, all Massey products on $H^1(\partial M)$ of less than $(m-1)$ elements vanish, and $<\alpha_1, \dots, \alpha_{m-1}> \in H^2(\partial M)$ is a well-defined element.

Definition 3.6. The obstruction $o_m(K)$ is defined to be the subset

$$\{o_m(K, M)\} \subset H^{2m}(K^m),$$

where M is to vary over all thickenings such that $o_{m-1}(K, M) = 0$.

Lemma 3.4 implies the following corollary.

Corollary 3.7. *If K admits an embedding into $\mathbb{R}^4$ then $o_m(K)$ is defined and contains zero for each m.*

Some computations are given in section 4. We show that $o_m(K)$ does not contain zero for some m for examples in [4], thus giving another proof that they do not embed into $\mathbb{R}^4$.

The relative embedding problem

Let K be a 2-complex with $H_1(K; \mathbb{Q}) = 0$, and let L be a 1-dimensional subcomplex of K with a prescribed embedding $\phi\colon L \hookrightarrow S^3$. Consider the relative embedding problem: does there exist an embedding $K \hookrightarrow B^4$ which extends ϕ? Denote $B^4 \cup_\phi$(thickening of K) by M, where thickening is taken in the sense of definition 3.1. Note that various embeddings ϕ correspond to different thickenings of $K\cup$(cone on L). Let K_L^m denote the subset in K^m consisting of all m-tuples $(x_1, \dots, x_m)$ such that $x_i \in L$ for some i. Assume that $\pi_0(L) \longrightarrow \pi_0(K)$ is injective to have $H_1(M) = 0$. Analogously to the absolute case, Massey products on ∂M define an element, depending on M, in the relative cohomology group $H^{2m}(K^m, K_L^m)$. Let $o_m(K, L, \phi)$ denote the set of these elements in $H^{2m}(K^m, K_L^m)$, where M is to vary over all thickenings for which the $(m-1)$-st obstruction is zero. If there is an embedding of K into B^4, extending ϕ, clearly there is a 4-dimensional thickening M which embeds into S^4, so $0 \in o_m(K, L, \phi)$ for each m.

Consider the simplest relative case, when $(K, L) = (D^2 \amalg \dots \amalg D^2, S^1 \amalg \dots \amalg S^1)$. By the result of Turaev [11], the first non-trivial obstruction coincides in this case with the first non-trivial Milnor's $\bar{\mu}$-invariants of the link $\phi(L)$ in S^3. In this this sense the obstructions $o_m(K)$ may be thought of as an absolute analogue of $\bar{\mu}$-invariants. However, since 2-complexes in general have a more complicated topology, $\{o_m(K)\}$ have a larger indeterminacy.

4. Examples

First we recall the construction of examples in [4]. Let C denote the 2-skeleton of the 6-simplex with vertices $v_1, \ldots, v_7$, with one 2-cell, with vertices $v_1v_2v_3$, removed. Take another copy C' of C, with vertices $v_1', \ldots, v_7'$, and denote by $\overline{C}$ the union of C and C', identified along their last vertices, $v_7 = v_7'$. This 2-complex is easily seen to admit an embedding into $\mathbb{R}^4$ (see [12]). Let γ (resp. γ') denote the loop $v_7v_1v_2v_3v_1v_7$ (resp. $v_7v_1'v_2'v_3'v_1'v_7$) in $\overline{C}$.

Denote by F the free group on two generators, and fix a positive integer m. Let α be an element in F^m, the m-th term of the lower central series of F. We identify F with $\pi_1(\gamma \vee \gamma')$, and we associate to each word in F its "standard" representative curve in the wedge of two circles $\gamma \vee \gamma'$. Finally, we construct the 2-complex K_α by attaching a 2-cell to $\overline{C}$ along α.

Theorem 4.1 ([4])**.** *Let α be a non-trivial element in F^m for some $m \geq 2$. Then Van Kampen's obstruction $o(K_\alpha)$ vanishes, but K_α does not admit an embedding into $\mathbb{R}^4$.*

We now present a computation of the obstructions $\{o_i(K_\alpha)\}$. The class m of the commutator α is reflected in non-vanishing of the obstruction $o_{m+1}(K_\alpha)$.

Theorem 2. *Let α be an element in F^m for some $m \geq 2$, and assume $\alpha \notin F^{m+1}$. Then $o_{m+1}(K_\alpha)$ is defined and does not contain zero. In particular, K_α does not embed into $\mathbb{R}^4$.*

Proof. First we construct a thickening M of K_α with trivial intersection pairing and such that $o_i(K, M) = 0$ for all $i \leq m$. The complex K_α is obtained from $\overline{C} = C \vee C'$ by attaching a 2-cell along the commutator $\alpha \in F^m$. Van Kampen constructed in [12] an immersion of the 2-skeleton of the 6-simplex with vertices $v_1, \ldots, v_7$ into $\mathbb{R}^4$ such that the 2-cells with vertices $v_1v_2v_3$ and $v_4v_5v_6$ intersect in one point, and all other simplices are disjoint and embedded. Consider the corresponding embedding of $\overline{C}$, and let $\overline{M}$ denote its thickening in $\mathbb{R}^4$. Clearly the intersection pairing on $\overline{M}$ is trivial, and all Massey products on $H^1(\partial\overline{M})$ vanish. Recall that C and C' have the vertex v_7 in common. Consider the handle decomposition of $\overline{M}$, given by thickenings of simplices of K_α in $\mathbb{R}^4$. The union of the handles in $\overline{M}$ corresponding to all simplices in $\overline{C}$, containing v_7, is a 4-ball B. The remaining 2-handles are attached to B along a link $\overline{L}$ in $S^3 = \partial B$. Each attaching curve is isotopic, within the union of 0- and 1-handles, to the boundary curve of the corresponding 2-simplex. There is no 2-cell attached to $v_1v_2v_3$, however we introduce in S^3 a circle, isotopic to it. Because of the choice of the embedding of K into $\mathbb{R}^4$, $\overline{L}$ is a slice link, and the curves isotopic to $v_1v_2v_3$ and $v_4v_5v_6$ (respectively $v_1'v_2'v_3'$ and $v_4'v_5'v_6'$) have linking number one. The remaining 2-cell of K_α is attached along the commutator of $v_1v_2v_3$ and $v_1'v_2'v_3'$. Choosing appropriately the corresponding curve l in S^3, we get the link $L = \overline{L} \cup l$ such that all Milnor's $\bar{\mu}$-invariants of L of length less than $m+1$ vanish, and a $\bar{\mu}$-invariant of length $m+1$ of the 3-component link $(v_4v_5v_6, v_4'v_5'v_6', l)$ is non-trivial. It is a result of Turaev [11] that the first non-vanishing $\bar{\mu}$-invariant is equal to the corresponding Massey product on ∂M, where $M = \overline{M} \cup_l$ 2-handle. (Note that the framings of the components of $\overline{L}$ are zero, since the intersection pairing on $\overline{M}$ vanishes, and we choose the framing of l also to be zero.) This proves that $o_i(K, M) = 0$ for all $i \leq m$, and $o_{m+1}(K_\alpha, M) \neq 0$.

It remains to show that $o_{m+1}(K_\alpha)$ does not contain zero. Let M be any thickening with trivial intersection pairing and with $o_m(K, M) = 0$. As above, the union of the handles in M corresponding to all simplices, containing v_7, is a 4-ball B. By a theorem of Conway - Gordon [1] and Sachs [8] any embedding of the complete graph on 6 vertices in S^3 contains two disjoint linking cycles. Consider the complete graph on vertices $v_1, \ldots, v_6$ in $\overline{C}$. According to the definition of M, the attaching curves of the 2-handles in $S^3 = \partial B$ are isotopic to the attaching maps of simplices of K_α. As above, we introduce in S^3 a curve isotopic to $v_1 v_2 v_3$. Now we have in S^3 a perturbed version of the complete graph on 6 vertices. Since, according to definition 3.1, these perturbations take place in the union of 0- and 1-handles, at least two of the curves must have a non-trivial linking number. Since the intersection pairing on M vanishes, these two circles are the ones isotopic to $v_1 v_2 v_3$ and to $v_4 v_5 v_6$. Similarly we have in S^3 another, disjoint, copy of a perturbed graph on $v'_1, \ldots, v'_6$, and two linking circles isotopic to $v'_1 v'_2 v'_3$ and to $v'_4 v'_5 v'_6$. Recall that there are no 2-handles attached to $v_1 v_2 v_3$ or $v'_1 v'_2 v'_3$, however there is a 2-handle whose attaching curve l is a commutator of these circles. It is easily seen that the link $(v_4 v_5 v_6, v'_4 v'_5 v'_6, l)$ has a non-trivial $\bar{\mu}$-invariant of length $m+1$. As above, this is translated into non-vanishing of $o_{m+1}(K, M)$. $\square$

Remark. The idea of the proof of the fact that any embedding of the complete graph on 6 vertices in S^3 contains two linking cycles ([1], [8]) is conceptually similar to the proof of Van Kampen that the 2-skeleton of the 6-simplex does not embed into $\mathbb{R}^4$ [12]. (In both cases one shows that a certain number is invariant mod 2 for different maps - in one case, the total linking number, in other case, the total number of singular points of an immersion.) In this sense our proof of theorem 2 is similar to the proof of theorem 4.1 in [4].

5. A note on configuration spaces

In this section we consider an approach to the embedding problem, suggested by obstruction theory and configuration spaces. We give a reformulation of Van Kampen's obstruction in this context, which suggests another approach to defining higher embedding obstructions. Given a space X, $C^m(X)$ will denote its m-fold symmetric product:

$$C^m(X) = \{(x_1, \ldots, x_m) \in X^m \,|\, x_i \neq x_j \text{ if } \mathrm{i} \neq \mathrm{j}\}.$$

In the simplicial category, the symmetric product of a complex K is defined to be

$$C^m(K) = \{\sigma_1 \times \ldots \times \sigma_m \subset K^m \,|\, \text{simplices } \sigma_\mathrm{i},\ \sigma_\mathrm{j} \text{ have no vertices in common, } \mathrm{i} \neq \mathrm{j}\}.$$

The 2-fold symmetric product is sometimes called deleted product and is also denoted by X^*. The symmetric groups are denoted by S_m; S_m acts freely on $C^m(K)$, and on its i-skeleton $(C^m(K))^i$ for each i, by exchanging the coordinates.

A necessary condition for the existence of an embedding $K^n \hookrightarrow \mathbb{R}^{2n}$ is the existence, for each m, of an S_m-equivariant map $C^m(K) \longrightarrow C^m(\mathbb{R}^{2n})$. We will now analyze the first embedding obstruction, corresponding to $m = 2$, that is, the obstruction to existence of a $\mathbb{Z}_2$-equivariant map

$$K \times K \setminus \Delta \longrightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n} \setminus \Delta \simeq S^{2n-1}.$$

The $\mathbb{Z}_2$-equivariant homotopy equivalence above is given by the projection of $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \setminus \Delta$ onto the unit sphere in the antidiagonal $\{(x, -x)\} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. The diagonal Δ in $K \times K$ is the "simplicial" diagonal, as defined in section 2, while $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is the usual set-theoretic diagonal. Recall that the spaces above are denoted in short by K^* and $(\mathbb{R}^{2n})^*$ respectively.

Theorem 3. *The obstruction to existence of a $\mathbb{Z}_2$-equivariant map $K^* \longrightarrow (\mathbb{R}^{2n})^*$ lies in $H^{2n}_{\mathbb{Z}/2}(K^*; \mathbb{Z})$ and coincides with Van Kampen's obstruction $o(K)$.*

Proof. Since K^* is a $(2n)$-dimensional CW-complex, the only non-trivial obstruction group in this setting is $H^{2n}_{\mathbb{Z}_2}(K^*; \pi_{2n-1}(S^{2n-1})) \cong H^{2n}_{\mathbb{Z}/2}(K^*; \mathbb{Z})$.

Let $f\colon K \looparrowright \mathbb{R}^{2n}$ be any immersion. Since $f(\sigma)$ and $f(\nu)$ are disjoint for any n-simplex σ and any $(n-1)$-simplex ν, $f \times f$ restricted to the $(2n-1)$-skeleton of K^* is a $\mathbb{Z}_2$-equivariant embedding into $(\mathbb{R}^{2n})^*$. Let σ, τ be two n-dimensional simplices of K and consider $\sigma \times \tau$ as an oriented generator of $(2n)$-dimensional cellular chains on K^*. The obstruction cochain c_f assigns to $\sigma \times \tau$ the element

$$c_f(\sigma \times \tau) = [(f \times f)(\partial(\sigma \times \tau))] \in \pi_{2n-1}(S^{2n-1}).$$

The map $f \times f$ sends $\sigma \times \tau$ into $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, and one has

$$o_f(\sigma \times \tau) = f(\sigma) \cdot f(\tau) = (f \times f)(\sigma \times \tau) \cap \Delta_{\mathbb{R}^{2n}} = (f \times f)(\partial(\sigma \times \tau)) = c_f(\sigma \times \tau).$$

This shows that the homotopy-theoretic obstruction coincides with Van Kampen's obstruction even on the cochain level, when the map of $(2n-1)$-skeleton of K^* corresponds to the chosen immersion f. This completes the proof, since the cohomology class $[c_f]$ is independent of the choice of a map of $(2n-1)$-skeleton of K^*, being the first non-trivial obstruction. $\square$

Remarks. (1) Theorem 3 has been independently stated in [7].
(2) It is interesting to note that by theorems 2.1 and 3, the existence of a $\mathbb{Z}_2$ - equivariant map $K^* \longrightarrow (\mathbb{R}^{2n})^*$ is equivalent to existence of an embedding $K^n \hookrightarrow \mathbb{R}^{2n}$ for $n \neq 2$.

We conclude by suggesting the following approach to defining higher embedding obstructions. Suppose $o(K)$ vanishes, so there exists a $\mathbb{Z}/2$-equivariant map $K^* \longrightarrow (\mathbb{R}^4)^*$. One may consider the obstructions to existence of an equivariant, with respect to the free action of the symmetric group S_m, map $C^m(K) \longrightarrow C^m(\mathbb{R}^4)$, $m = 3, 4, \ldots$. Note that Fadell - Neuwirth [2] have determined homotopy types of the symmetric products of Euclidean spaces, thus (rationally) explicitely giving the coefficients of obstruction groups.

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Michigan State University, East Lansing, MI 48824-1027, USA

Current address: Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26, D-53225 Bonn, Germany

E-mail address: krushkal@math.msu.edu