Topological invariants of singular varieties
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We know that any singular algebraic variety over the complex numbers has a resolution of singularities. A fundamental difficulty in algebraic geometry is that these resolutions are far from unique. A good approach to the difficulty is to consider only resolutions which are "minimal" in the sense of minimal model theory. This approach works perfectly in dimensions at most 2, where every singular algebraic variety has a unique minimal resolution.

In higher dimensions, there are various complications, of which we focus on one: even when minimal resolutions exist, they are not unique. There is an operation called a "flop" which can change one minimal resolution into another. It seems that there is no reason to prefer one minimal resolution to any other. So it becomes natural to ask which invariants of smooth varieties (topological invariants, for example) are unchanged under flops. We can hope that such an invariant extends to singular varieties in such a way as to agree with the given invariant for any minimal resolution.

A typical example is that the cohomology groups (say, with integer coefficients) are unchanged under a flop, whereas the cohomology ring may change. Goresky and MacPherson used one version of this observation to argue that intersection homology for singular varieties cannot have a "natural" ring structure [4]. Recently Batyrev and Kontsevich have gone further by showing that Betti numbers and Hodge numbers are unchanged under a flop of the most general sense used in minimal model theory [1], [3]. Their work strongly suggests that there should be a generalization of intersection homology groups, "stringy cohomology groups," which are invariant under minimal resolutions, not just under small resolutions.

Goresky and MacPherson also emphasized the question of which characteristic numbers (meaning linear combinations of Chern numbers), which are a priori invariants of smooth compact varieties, can be defined for singular varieties, in a suitably precise sense [4]. In particular, they invented intersection homology in order to define the signature of singular varieties, as Sullivan had suggested should be possible.

I found that not every characteristic number can be defined for singular varieties in the sense considered by Goresky and MacPherson. In fact, such a characteristic number must be invariant under flops, and it turns out that the only characteristic numbers which are invariant under a certain class of flops are the linear combinations of coefficients of the complex elliptic genus, which goes back to Witten and was described in detail by Krichever [6] and Höhn [5]. This gives a purely algebro-geometric description of the elliptic genus [7], which even works over the ring $\mathbf{Z}[1/2]$, not just over $\mathbf{Q}$.

A remaining problem is to show that the elliptic genus is invariant under flops in the most general sense used in minimal model theory. There is some progress on this question by Borisov and Libgober [2].

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