QUASI-ASSOCIATIVE ALGEBRAS

A.S. DZHUMADIL'DAEV

ABSTRACT. We consider the class $\mathfrak{Ass}^{(q)}$ of algebras that appear from associative algebras under the taking of q-commutator $a \circ_q b = a \circ b + q\, b \circ a,$ where $q \in K, q^2 \neq 0, 1$, and $p = char\, K \neq 2, 3.$ We establish that this class forms a variety and satisfies the identity $ass^{(q)} = 0,$ where $ass^{(q)} = (q-1)^2(t_1, t_3, t_2) + q\,[t_3, [t_1, t_2]].$ If $q^2 - 4q + 1 \neq 0$, the variety of q-associative algebras is generated by this identity. If $q^2 - 4q + 1 = 0$, then this identity is not enough to generate $\mathfrak{Ass}^{(q)}$ and it should be supplemented by the Lie-Admissibility identity $[[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2] = 0.$ In the exceptional case the variety generated by the identity $ass^{(q)} = 0$ is equivalent to the variety of alternative algebras.

It is well known that every associative algebra under the taking of commutator becomes a Lie algebra, and under the taking of anticommutator becomes a Jordan algebra. In this paper we study the class of algebras that appear from associative algebras under the taking of q-commutator, with $q^2 \neq 0, 1.$

Let $A = (A, \circ)$ be an algebra with multiplication $\circ$. Throughout, all vector spaces are considered over a field K of characteristic $p \neq 2, 3$. Denote a q-commutator of $\circ$ by $\circ_q$:

$$a \circ_q b = a \circ b + q\, b \circ a.$$

For example, $[a, b] = a \circ_{-1} b$ is a Lie commutator, and $\{a, b\} = a \circ_1 b$ is a Jordan commutator. Let

$$ass(a, b, c) = (a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$$

denote an associator. Given $q \in K$, define $A^{(q)} = (A, \circ_q)$ to be the algebra with the underlying vector space A and the multiplication $\circ_q$.

For a non-commutative non-associative polynomial $g = g(t_1, t_2, \dots)$, denote by $F_q g$ the polynomial that appears from g under the multiplication rule $t_i \cdot_q t_j = t_i t_j + q\, t_j t_i$. For example,

$$F_q(t_1(t_2t_3)) = t_1(t_2t_3) + q\, t_1(t_3t_2) + q\,(t_2t_3)t_1 + q^2\,(t_3t_2)t_1, \qquad (1)$$

$$F_q((t_1t_2)t_3) = (t_1t_2)t_3 + q\, t_3(t_1t_2) + q\,(t_2t_1)t_3 + q^2\, t_3(t_2t_1) \qquad (2)$$

and

$$F_q\, ass(t_1,t_2,t_3) = ass(t_1,t_2,t_3) + \tag{3}$$
$$q(ass(t_1,t_3,t_2) - ass(t_3,t_1,t_2) - ass(t_2,t_3,t_1) +$$
$$ass(t_2,t_1,t_3) + [[t_1,t_3],t_2]) - q^2\, ass(t_3,t_2,t_1).$$

Similarly, for an algebra $A=(A,\circ)$, we also denote the algebra $(A,\circ_q)$ by F_qA.

For $q\in K, q^2\neq 0,1$, q*-associativity* polynomial $ass^{(q)}$ is defined as

$$ass^{(q)}(t_1,t_2,t_3) = (q-1)^2(t_1,t_3,t_2) + q\,[t_3,[t_1,t_2]].$$

Then

$$ass^{(0)} = ass.$$

The *Lie-Admissibility* polynomial lia is defined as

$$lia(t_1,t_2,t_3) = [[t_1,t_2],t_3] + [[t_2,t_3],t_1] + [[t_3,t_1],t_2].$$

Recall that here $(t_i,t_j,t_s) = t_i(t_jt_s) - (t_it_j)t_s$ is the associator and $[t_i,t_j] = t_it_j - t_jt_i$ is the commutator. An algebra A satisfying the identity $lia=0$ is called *Lie-Admissible*.

For a non-commutative, non-associative polynomial $f=f(t_1,\ldots,t_k)$ in k variables, we say that $f=0$ is an *identity* on $(A,\circ)$ if $f(a_1,\ldots,a_k)=0$ for any substitutions $t_1=a_1,\ldots,t_k=a_k\in A$. Here the multiplications are done in terms of $\circ$. Given polynomials $f_1,f_2,\ldots,f_r$, we denote the variety generated by the identities $f_1=0, f_2=0,\ldots,f_r=0$ by $Var(f_1,f_2,\ldots,f_r)$. Let $\mathfrak{Ass}$ be the category of associative algebras, and $\mathfrak{Ass}^{(q)}$ the category of algebras of the form $(A,\circ_q)$, where $(A,\circ)$ is an associative algebra. It is well known that $\mathfrak{Ass} = Var(ass)$ is a variety, and $\mathfrak{Ass}^{(-1)}$ is a variety generated by the anti-commutativity and Jacobi polynomials (Lie algebras). In that time $\mathfrak{Ass}^{(1)}$ (class of special Jordan algebras) is not a variety,

$$\mathfrak{Ass}^{(1)} \subset Var(acom, jor), \quad \mathfrak{Ass}^{(1)} \neq Var(acom, jor),$$

where

$$acom = t_1t_2 - t_2t_1, \quad jor = (t_1^2, t_2, t_1).$$

For details we refer, for example, to [4]. Let

$$lalt = (t_1,t_2,t_3) - (t_1,t_3,t_1),$$
$$ralt = (t_1,t_2,t_3) - (t_2,t_1,t_3)$$

be the left and right-alternative polynomials, and let $\mathfrak{Alt} = Var(lalt, ralt)$ be the variety of alternative algebras.

In this paper we study the category $\mathfrak{Ass}^{(q)}$ for $q^2\neq 0,1$. Recall that every q-algebra $(A,\circ_q)$ of an associative algebra $(A,\circ)$ is said to be *quasi-associative.* We call such a q-algebra q*-associative.* Recall also that a flexible algebra with the Jordan identity $(t_1^2, t_2, t_1) = 0$ is called non-commutative Jordan. Connections between quasi-associative algebras, associative algebras and noncommutative Jordan algebras were studied in [1],[2],[3].

Let $\mathfrak{L}^\circ$ and $\mathfrak{L}^\star$ be categories of algebras. The objects of $\mathfrak{L}^\circ$ are algebras $(A, \circ)$ with multiplication $\circ$, and the objects of $\mathfrak{L}^\star$ are algebras $(A, \star)$ with multiplication $\star$. The morphisms between objects are usual homomorphisms between the corresponding algebras. Recall that a map between categories

$$F : \mathfrak{L}^\circ \to \mathfrak{L}^\star, \quad F(A, \circ) = (A, \star)$$

is called a morphism of categories if each morphism $\alpha \in Mor((A, \circ), (B, \circ))$ of the category $\mathfrak{L}^\circ$ goes to a morphism in the category $\mathfrak{L}^\star$. In our case this means that if $\alpha : (A, \circ) \to (B, \circ)$ is a homomorphism of algebras, then

$$\alpha : (A, \star) \to (B, \star)$$

is a homomorphism of algebras too. In this sense the map

$$F_q : \mathfrak{L}^\circ \to \mathfrak{L}^{\circ_q}$$

is a morphism of categories.

We say that F is an *equivalence* of categories if there exists a morphism

$$G : \mathfrak{L}^\star \to \mathfrak{L}^\circ$$

such that the compositions

$$GF : \mathfrak{L}^\circ \to \mathfrak{L}^\circ, \qquad FG : \mathfrak{L}^\star \to \mathfrak{L}^\star$$

are the identity morphisms:

$$GF(A, \circ) = (A, \circ), \qquad FG(A, \star) = (A, \star).$$

Theorem 1. *The class $\mathfrak{Ass}^{(q)}$ of q-associative algebras forms a variety if $q^2 \neq 0, 1$. Namely,*

$$\mathfrak{Ass}^{(q)} = Var(ass^{(q)}) \text{ if } q^2 - 4q + 1 \neq 0,$$

$$\mathfrak{Ass}^{(q)} = Var(ass^{(q)}, lia) \text{ if } q^2 - 4q + 1 = 0.$$

Moreover, the following categories (varieties) are equivalent:

$$Var(ass^{(q)}) \sim \mathfrak{Ass}, \quad \text{if } q^2 - 4q + 1 \neq 0,$$

$$Var(ass^{(q)}, lia) \sim \mathfrak{Ass}, \quad \text{if } q^2 - 4q + 1 = 0,$$

$$Var(ass^{(q)}) \sim \mathfrak{Alt}, \quad \text{if } q^2 - 4q + 1 = 0.$$

The equivalence morphisms are given by the map F_{-q}:

$$F_{-q} : Var(ass^{(q)} \to Var(ass), \quad \text{if } q^2 - 4q + 1 \neq 0,$$

$$F_{-q} : Var(ass^{(q)}) \to Var(ralt, lalt), \quad if \;\; q^2 - 4q + 1 = 0,$$

$$F_{-q} : Var(ass^{(q)}, lia) \to Var(ass), \quad if \;\; q^2 - 4q + 1 = 0.$$

The inverse morphisms are given by the map $(1 - q^2)^{-2} F_q$.

Notice that in the exceptional case $q^2 - 4q + 1 = 0$, we have

$$ass^{(q)} = q\, ass_0^{(q)},$$

where

$$ass_0^{(q)} = 2(t_1, t_3, t_2) + [t_3, [t_1, t_2]].$$

Corollary 2. *The following identities are consequences of the identity* $ass^{(q)} = 0$ *:*

$$elast(t_1, t_2, t_3) = (t_1, t_2, t_3) + (t_3, t_2, t_1),$$

$$assq_1(t_1, t_2, t_3, t_4) =$$
$$(t_1t_2)(t_3t_4)+(t_1t_3)(t_4t_2)+(t_1t_4)(t_2t_3)-(t_1(t_3t_4))t_2-(t_1(t_4t_2))t_3-(t_1(t_2t_3))t_4,$$

$$assq_2(t_1, t_2, t_3, t_4) =$$
$$(t_1t_2)(t_3t_4)+(t_2t_3)(t_1t_4)+(t_3t_1)(t_2t_4)-t_3((t_1t_2)t_4)-t_1((t_2t_3)t_4)-t_2((t_3t_1)t_4),$$

$$assq_3(t_1, t_2, t_3, t_4) =$$
$$t_1([t_2, t_4]t_3) - (t_3t_1)[t_2, t_4] + [t_3, (t_1t_2)t_4 - (t_1t_4)t_2].$$

In particular, every q *-associative algebra is non-commutative Jordan.*

One can show that the identities $elast = 0, assq_1 = 0, assq_2 = 0, assq_3 = 0$ are independent.

Lemma 3. *If* $g = g(t_1, \ldots, t_l)$ *is a homogeneous non-commutative, non-associative polynomial of degree* k, *then*

$$F_{-q}F_q g = F_q F_{-q} g = (1 - q^2)^{k-1} g.$$

Proof. There exist $\frac{1}{n}\binom{2(n-1)}{n-1}$ bracketing types on n letters. For example, there exist two bracketing types on three letters: $t_1(t_2t_3)$ and $(t_1t_2)t_3$.

It is enough to prove that

$$F_q F_{-q} \sigma = (1 - q^2)^{k-1} \sigma$$

for any bracketing type σ on k letters.

We use the induction on k.

In case $k=2$, we have only one bracketing type: t_1t_2. In this case our statement is true:

$$F_q(t_{i_1}t_{i_2}) = t_{i_1}t_{i_2} + q\,t_{i_2}t_{i_1},$$

and

$$\begin{gathered} F_{-q}F_q(t_{i_1}t_{i_2}) \\ = t_{i_1}t_{i_2} - q\,t_{i_2}t_{i_1} + q\,t_{i_2}t_{i_1} - q^2 t_{i_1}t_{i_2} \\ = (1-q^2)t_{i_1}t_{i_2} = (1-q^2)(t_{i_1}t_{i_2}). \end{gathered}$$

Suppose that our statement is true for $k-1$. Let σ be one of the bracketing types on k letters. It can be presented in the form

$$\sigma(t_{i_1},\dots,t_{i_k}) = \sigma'(t_{i_1},\dots,t_{i_{k'}})\sigma''(t_{i_{k'+1}},\dots,t_{i_k})$$

for some bracketing types σ' and σ'' on k' and $k''=k-k'$ letters, $1\le k'\le k$.

Then

$$F_q\sigma(t_{i_1},\dots,t_{i_k})$$

$$\begin{gathered} = F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k}) \\ +q\,F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k})F_q\sigma'(t_{i_1},\dots,t_{i_{k'}}) \end{gathered}$$

and

$$F_{-q}F_q\sigma(t_{i_1},\dots,t_{i_k})$$

$$\begin{gathered} = F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k}) \\ -q\,F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k})F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}}) \\ +q\,F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k})F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}}) \\ -q^2F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k}) \end{gathered}$$

$$\begin{gathered} = F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k}) \\ -q^2F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k}) \end{gathered}$$

By the induction hypothesis,

$$\begin{gathered} F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}}) \\ = (1-q^2)^{k'-1}\sigma'(t_{i_1},\dots,t_{i_{k'}}), \end{gathered}$$

$$\begin{gathered} F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k}) \\ = (1-q^2)^{k-k'-1}\sigma''(t_{i_{k'+1}},\dots,t_{i_k}). \end{gathered}$$

Therefore,

$$F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k})$$

$$= (1-q^2)^{k-2}\sigma(t_{i_1},\dots,t_{i_k}),$$

$$-q^2F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k})$$
$$= -q^2(1-q^2)^{k-2}\sigma(t_{i_1},\dots,t_{i_k})$$

and

$$F_{-q}F_q\sigma(t_{i_1},\dots,t_{i_k})$$
$$= F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k})$$
$$-q^2F_{-q}F_q\sigma'(t_{i_1},\dots,t_{i_{k'}})F_{-q}F_q\sigma''(t_{i_{k'+1}},\dots,t_{i_k})$$
$$= (1-q^2)^{k-1}\sigma(t_{i_1},\dots,t_{i_k}).$$

The lemma is proved.

Lemma 4. *If $q^2-4q+1\neq 0$, then the identity $lia=0$ follows from the identity $ass^{(q)}=0$.*

If $q^2-4q+1=0$ and $p\neq 2,3$, then the identities $ass^{(q)}=0$ and $lia=0$ are independent.

Proof. The following relation holds:

$$ass^{(q)}(t_1,t_2,t_3)+ass^{(q)}(t_2,t_3,t_1)+ass^{(q)}(t_3,t_1,t_2)-$$
$$ass^{(q)}(t_1,t_3,t_2)-ass^{(q)}(t_2,t_1,t_3)-ass^{(q)}(t_3,t_2,t_1)=$$
$$(q^2-4q+1)\,lia(t_1,t_2,t_3).$$

Therefore,

$$ass^{(q)}=0,\ q^2-4q+1\neq 0,\ \Rightarrow lia=0.$$

Let $\mathbf{Ca}$ be the 8-dimensional Cayley-Dickson algebra corresponding to the Cayley octonians (the algebra $\mathbf{C}(-4/5,-1,-1)$ in the notation of [4], Chapter 2, §2 , p. 48, Exercise 3). Recall that $\mathbf{Ca}$ is an alternative algebra. Take a basis $e_0,e_1,\dots,e_7$ for $\mathbf{Ca}$ with the multiplication table of [4], p. 48, Exercise 3.

It is not hard to see that the three-linear maps lia and $ass^{(q)}$ on $\mathbf{Ca}$ are skew-symmetric. Moreover,

$$(q-1)^2(q^2-4q+1)\,lia(a,b,c)-6\,ass^{(q)}(a,b,c)=0,$$

for any $a,b,c\in\mathbf{Ca}$. It is easy to check that

$$lia(e_1,e_2,e_4)=12\,e_7,$$

$$ass^{(q)}(e_1,e_2,e_4)=2(q-1)^2(q^2-4q+1)\,e_7.$$

Thus, the octonian algebra is not Lie-Admissible and satisfies the identity $ass^{(q)}=0$ if $q^2-4q+1=0$ and $p\neq 2,3$. This means that the identities $ass^{(q)}=0$ and $lia=0$ are independent if $q^2-4q+1=0$.

Lemma 5. *If* $q^2-4q+1=0$ *, then*

$$2\,F_{-q}\,ralt(t_1,t_2,t_3)=$$
$$(1-q)(ass^{(q)}(t_1,t_2,t_3)+ass^{(q)}(t_1,t_3,t_2))$$
$$+(1-3\,q)(ass^{(q)}(t_2,t_1,t_3)+ass^{(q)}(t_3,t_1,t_2)),$$

$$2\,F_{-q}\,lalt(t_1,t_2,t_3)=$$
$$(1-3\,q)\,ass^{(q)}(t_1,t_2,t_3)+2q\,ass^{(q)}(t_1,t_3,t_2)+(-1+3\,q)\,ass^{(q)}(t_2,t_1,t_3)$$
$$+2\,(1-2\,q)\,ass^{(q)}(t_2,t_3,t_1)+2\,(1-3\,q)\,ass^{(q)}(t_3,t_1,t_2)$$

and

$$ass^{(q)}(t_1t_2,t_3)=$$
$$-2\,lalt(t_1,t_2,t_3)+6q\,lalt(t_1,t_2,t_3)+4q\,lalt(t_1,t_3,t_2)+2\,ralt(t_1,t_2,t_3)$$
$$-4q\,ralt(t_1,t_2,t_3)+2\,ralt(t_2,t_1,t_3)-8q\,ralt(t_2,t_1,t_3)-2q\,ralt(t_3,t_1,t_2).$$

Proof. In the forthcoming calculations we use (1), (2) and (3). We have:

$$ass^{(q)}(t_1,t_2,t_3)+ass^{(q)}(t_1,t_3,t_2)=$$
$$t_1(t_2t_3)-2q\,t_1(t_2t_3)+q^2\,t_1(t_2t_3)+t_1(t_3t_2)$$
$$-2q\,t_1(t_3t_2)+q^2\,t_1(t_3t_2)+q\,t_2(t_1t_3)-q\,t_2(t_3t_1)$$
$$+q\,t_3(t_1t_2)-q\,t_3(t_2t_1)-(t_1t_2)t_3+q\,(t_1t_2)t_3$$
$$-q^2\,(t_1t_2)t_3-(t_1t_3)t_2+q\,(t_1t_3)t_2-q^2\,(t_1t_3)t_2$$
$$+q\,(t_2t_1)t_3+q\,(t_3t_1)t_2,$$

$$ass^{(q)}(t_2,t_1,t_3)+ass^{(q)}(t_3,t_1,t_2)=$$
$$-q\,t_2(t_1t_3)+t_2(t_3t_1)-q\,t_2(t_3t_1)+q^2\,t_2(t_3t_1)$$
$$-q\,t_3(t_1t_2)+t_3(t_2t_1)-q\,t_3(t_2t_1)+q^2\,t_3(t_2t_1)$$
$$+q\,(t_1t_2)t_3+q\,(t_1t_3)t_2-q\,(t_2t_1)t_3-(t_2t_3)t_1$$
$$+2q\,(t_2t_3)t_1-q^2\,(t_2t_3)t_1-q\,(t_3t_1)t_2-(t_3t_2)t_1$$
$$+2q\,(t_3t_2)t_1-q^2\,(t_3t_2)t_1$$

and

$$F_{-q}\,ralt(t_1,t_2,t_3)=$$
$$t_1(t_2t_3)-q\,t_1(t_2t_3)+t_1(t_3t_2)-q\,t_1(t_3t_2)$$
$$+q\,t_2(t_1t_3)-q^2\,t_2(t_3t_1)+q\,t_3(t_1t_2)-q^2\,t_3(t_2t_1)$$
$$-(t_1t_2)t_3-(t_1t_3)t_2+q\,(t_2t_1)t_3-q\,(t_2t_3)t_1$$
$$+q^2\,(t_2t_3)t_1+q\,(t_3t_1)t_2-q\,(t_3t_2)t_1+q^2\,(t_3t_2)t_1.$$

Thus,

$$(1-q)(ass^{(q)}(t_1,t_2,t_3)+ass^{(q)}(t_1,t_3,t_2))+(1-3q)(ass^{(q)}(t_2,t_1,t_3)+ass^{(q)}(t_3,t_1,t_2))$$

$$-2\, F_{-q}\, ralt(t_1,t_2,t_3) =$$

$$2(1-4q+q^2)(ass(t_2,t_3,t_1)+ass(t_3,t_2,t_1)).$$

Similarly,

$$(1-3q)ass^{(q)}(t_1,t_2,t_3)+2q\, ass^{(q)}(t_1,t_3,t_2)+(-1+3q)ass^{(q)}(t_2,t_1,t_3)$$

$$+2(1-2q)ass^{(q)}(t_2,t_3,t_1)+2(1-3q)ass^{(q)}(t_3,t_1,t_2)-2\, F_{-q}\, lalt(t_1,t_2,t_3) =$$

$$2(1-4q+q^2)(ass(t_3,t_1,t_2)+ass(t_3,t_2,t_1))$$

and

$$-2\, lalt(t_1,t_2,t_3)+6q\, lalt(t_1,t_2,t_3)+4q\, lalt(t_1,t_3,t_2)+2\, ralt(t_1,t_2,t_3)$$

$$-4q\, ralt(t_1,t_2,t_3)+2\, ralt(t_2,t_1,t_3)-8q\, ralt(t_2,t_1,t_3)-2q\, ralt(t_3,t_1,t_2)$$

$$-F_q\, ass(t_1,t_2,t_3) =$$

$$(1-4q+q^2)(2\, ass(t_2,t_3,t_1)+[[t_1,t_2],t_3]).$$

The lemma is proved.

Calculations in the next lemmas are similar to those above and thus omitted.

Lemma 6.

$$F_q\, ass^{(q)}(t_1,t_2,t_3) =$$

$$(q-1)^2\,(q\, ass(t_1,t_2,t_3)+ass(t_1,t_3,t_2))-$$

$$q(q-1)^2\,(ass(t_2,t_1,t_3)+q\, ass(t_2,t_3,t_1))+$$

$$q(q-1)^2\,(ass(t_3,t_1,t_2)-ass(t_3,t_2,t_1)).$$

Lemma 7.

$$(q-1)(q^2-4q+1)\, F_{-q}\, ass(t_1,t_2,t_3) =$$

$$(3\,q-1)\, ass^{(q)}(t_1,t_3,t_2)-q\,(q-1)\, ass^{(q)}(t_1,t_2,t_3)+q(q-1)\, ass^{(q)}(t_2,t_1,t_3)+$$

$$q(1-q)\, ass^{(q)}(t_2,t_3,t_1)+3\,q^2\, ass^{(q)}(t_3,t_1,t_2)-q^3\, ass^{(q)}(t_3,t_1,t_2)-$$

$$q\, ass^{(q)}(t_3,t_2,t_1)+q^2\, ass^{(q)}(t_3,t_2,t_1).$$

Lemma 8. *If* $q^2-4q+1=0$ *, then*

$$2\,q\, F_{-q}\, ass(t_1,t_2,t_3) =$$

$$(1-3\,q)\, ass^{(q)}(t_3,t_1,t_2)+(1-q)\, ass^{(q)}(t_1,t_3,t_2)-2\,q^2\, lia(t_1,t_2,t_3).$$

Proof of Theorem 1. By Lemma 4, $lia = 0$ is a consequence of the identity $ass^{(q)} = 0$ if $q^2 - 4q + 1 \neq 0$. In other words,

$$\mathfrak{Ass}^{(q)} = \mathfrak{Ass}^{(q,\, lia)} \text{ if } \quad q^2 - 4q + 1 \neq 0.$$

By Lemma 6, if $(A, \circ)$ an associative algebra, then the algebra $(A, \circ_q)$ satisfies the identity $ass^{(q)} = 0$.

Let $q^2 - 4q + 1 \neq 0$. By Lemma 7, if an algebra $(A, \circ)$ satisfies the identity $ass^{(q)} = 0$, then the algebra $(A, \circ_{-q})$ satisfies the identity $ass = 0$.

Now, consider the case of $q^2 - 4q + 1 = 0$. By Lemma 8, if $(A, \circ)$ satisfies the identities $ass^{(q)} = 0$ and $lia = 0$, then $(A, \circ_{-q})$ satisfies the identity $ass = 0$. By Lemma 5, if $(A, \circ)$ satisfies the identity $ass^{(q)} = 0$, then $(A, \circ_{-q})$ is alternative and, vice versa, if $(A, \circ)$ is alternative then $(A, \circ_q)$ satisfies the identity $ass^{(q)} = 0$.

So, if $(A, \circ)$ is an associative algebra, then $(A, \circ_q)$ satisfies the identities $ass^{(q)} = 0$ and $lia = 0$. Vice versa, if $(A, \star)$ satisfies the identities $ass^{(q)} = 0, lia = 0$ then $(A, \star_{-q})$ is associative. Thus, by Lemma 3 the functor

$$F_q : \mathfrak{Ass} \to \mathfrak{Ass}^{(q)}, \qquad (A, \circ) \to (A, \circ_q)$$

is well defined and has the inverse

$$(1 - q^2)^{-2} F_{-q} : \mathfrak{Ass}^{(q)} \to \mathfrak{Ass},$$

if $q^2 - 4q + 1 \neq 0$. Similarly, the functors

$$F_q : \mathfrak{Ass} \to \mathfrak{Ass}^{(q,lia)}, \qquad (A, \circ) \to (A, \circ_q),$$

$$(1 - q^2)^{-2} F_{-q} : \mathfrak{Ass}^{(q,lia)} \to \mathfrak{Ass}, \quad (1 - q^2)^{-2} F_{-q} : \mathfrak{Ass}^{(q)} \to \mathfrak{Alt},$$

are also well defined and

$$(1 - q^2)^{-2} F_{-q} F_q = id$$

if $q^2 - 4q + 1 = 0$.

References

[1] K. McCrimmon, *A note on quasi-associtive algebras,* Proc. AMS, **17**(1966), No.6, 1455-1459.

[2] K. McCrimmon, *Noncommutative Jordan rings,* Trans. AMS, **158**(1971), No.1, 1-33.

[3] K. McCrimmon, *Noncomutative Jordan division algebras,* Trans. AMS, **163**(1972), 215-224.

[4] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I, Shirshov, *Rings which are nearly associative* [Russian], Nauka, Moscow, 1976.

Kazakh-British University, Tole bi 59, Almaty, 050000, Kazakhstan

E-mail address: askar@math.kz