THE KER-COKER-SEQUENCE AND ITS GENERALIZATION IN SOME CLASSES OF ADDITIVE CATEGORIES

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Abstract. We study the question of the validity of the Snake Lemma in a P-semi-abelian category. We also obtain a generalization of the Snake Lemma in a quasi-abelian category.

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Key words and phrases: strict morphism, P-semi-abelian category, quasi-abelian category, Ker-Coker-sequence.

Introduction

In the recent years, homological algebra in additive categories which are not abelian has been actively developing in connection with the study of homological aspects of functional analysis, topological algebra, some algebraic problems. Various classes of additive categories with kernels and cokernels have been considered (see, for example, [1, 4, 10, 11, 12, 13, 14, 16, 17]).

In [3, 5], the Ker-Coker-sequence

$$\operatorname{Ker}\alpha \xrightarrow{\varepsilon} \operatorname{Ker}\beta \xrightarrow{\zeta} \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \xrightarrow{\tau} \operatorname{Coker}\beta \xrightarrow{\theta} \operatorname{Coker}\gamma \qquad (I)$$

corresponding to the commutative diagram

$$\begin{array}{ccccccccc} & & A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\psi_0} & C_0 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & C_1 & & \end{array}$$

satisfying the conditions $\psi_0 = \operatorname{coker}\varphi_0$, $\varphi_1 = \ker\psi_1$, was studied in an arbitrary quasi-abelian category. It was found out how an assumption about the properties of one of the morphisms α, β, or γ influences the exactness of sequence (I) and the properties of the morphisms that constitute the sequence.

In this article, we consider two generalizations of [5].

In Section 1, we consider the question about the Ker-Coker-sequence in a P-semi-abelian category, i.e., in a category semi-abelian in the sense of Palamodov (in [1, 3, 5, 6], such categories were called pre-abelian). The difference with the case of a quasi-abelian category is that in a P-semi-abelian category, for the validity of many of the arguments of [5], we have to impose the condition of stability under

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push-outs (pull-backs) on the kernels (cokernels) of some morphisms in the diagram. For example, the construction of the connecting morphism δ for the Ker-Coker-sequence is possible if $\operatorname{coker}\psi_0$ is a stable cokernel or $\ker\varphi_1$ is a stable kernel in the above sense.

In Section 2, we consider the diagram

$$\begin{array}{ccccc} A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\psi_0} & C_0 \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\ A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & C_1, \end{array} \qquad (II)$$

in which the rows are semi-exact, in a quasi-abelian category. To this diagram, there correspond two semi-exact "halves" of the Ker-Coker-sequence

$$\operatorname{Ker}\alpha \xrightarrow{\varepsilon} \operatorname{Ker}\beta \xrightarrow{\zeta} \operatorname{Ker}\gamma \qquad (III)$$

and

$$\operatorname{Coker}\alpha \xrightarrow{\tau} \operatorname{Coker}\beta \xrightarrow{\theta} \operatorname{Coker}\gamma. \qquad (IV)$$

In [9] Nomura proved that, in a Puppe-exact category, sequence (III) is exact at $\operatorname{Ker}\beta$ if φ_1 and the canonical morphism of the rows in (II) $\chi : H(B_0) \to H(B_1)$ are monomorphisms and (IV) is exact at $\operatorname{Coker}\beta$ if ψ_0 and $\chi : H(B_0) \to H(B_1)$ are epimorphisms.

In this article, we prove that these assertions holds in a quasi-abelian category if we require that φ_1 be a kernel and φ_0 be strict (respectively, that ψ_0 be a cokernel and ψ_1 be strict).

1. Ker-Coker-**Sequence in a P-Semi-Abelian Category**

We consider additive categories satisfying the following axiom.

Axiom 1. Each morphism has kernel and cokernel.

We denote by $\ker\alpha$ ($\operatorname{coker}\alpha$) an arbitrary kernel (cokernel) of α and by $\operatorname{Ker}\alpha$ ($\operatorname{Coker}\alpha$) the corresponding object; the equality $a = \ker b$ ($a = \operatorname{coker} b$) means that a is a kernel of b (a is a cokernel of b).

In a category meeting Axiom 1, every morphism α admits a canonical decomposition $\alpha = (\operatorname{im}\alpha)\overline{\alpha}(\operatorname{coim}\alpha)$, where $\operatorname{im}\alpha = \ker\operatorname{coker}\alpha$, $\operatorname{coim}\alpha = \operatorname{coker}\ker\alpha$. A morphism α is called *strict* if $\overline{\alpha}$ is an isomorphism.

We use the following notations:

O_c is the class of all strict morphisms;

M is the class of all monomorphisms;

M_c is the class of all strict monomorphisms (= kernels);

P is the class of all epimorphisms;

P_c is the class of all strict epimorphisms (= cokernels).

We write $\alpha \,|\, \beta$ if $\alpha = \ker\beta$ and $\beta = \operatorname{coker}\alpha$.

Lemma 1. [1, 2, 7, 12] *The following assertions hold in an additive category meeting Axiom 1:*

(1) $\ker\alpha \in M_c$ *and* $\operatorname{coker}\alpha \in P_c$ *for every* α;

(2) $\alpha \in M_c \Longleftrightarrow \alpha = \operatorname{im}\alpha$, $\alpha \in P_c \Longleftrightarrow \alpha = \operatorname{coim}\alpha$;

(3) *a morphism* α *is strict if and only if it is representable in the form* $\alpha = \alpha_1\alpha_0$ *with* $\alpha_0 \in P_c$, $\alpha_1 \in M_c$; *in every such representation,* $\alpha_0 = \operatorname{coim}\alpha$ *and* $\alpha_1 = \operatorname{im}\alpha$;

(4) *if a commutative square*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ A & \xrightarrow{\beta} & B \end{array} \qquad (1)$$

is a pull-back then $f \in M \Longrightarrow g \in M$, $f \in M_c \Longrightarrow g \in M_c$, *if the square is push-out then* $g \in P \Longrightarrow f \in P$, $g \in P_c \Longrightarrow f \in P_c$.

An additive category meeting Axiom 1 is abelian if and only if $\overline{\alpha}$ is an isomorphism for every α.

An additive category is called *P-semi-abelian*, *semi-abelian in the sense of Palamodov* [10, 11], or *pre-abelian* [1] if it meets Axiom 1 and the following

Axiom 2. For every morphism α, $\overline{\alpha}$ is a monomorphism and an epimorphism.

Lemma 2. [5] *The following hold in a P-semi-abelian category*:

(1) $gf \in M_c \Longrightarrow f \in M_c$, $gf \in P_c \Longrightarrow g \in P_c$;

(2) *if* $f, g \in M_c$ *and* fg *is defined then* $fg \in M_c$; *if* $f, g \in P_c$ *and* fg *is defined then* $fg \in P_c$;

(3) *if* $fg \in O_c$, $f \in M$ *then* $g \in O_c$; *if* $fg \in O_c$, $g \in P$ *then* $f \in O_c$.

An additive category satisfying Axiom 1 is called *quasi-abelian* [16] (*semiabelian* in the sense of Raĭkov [12], or *almost abelian* [13]) if it meets the following

Axiom 3. If square (1) is a pull-back then $f \in P_c \Longrightarrow g \in P_c$. If (1) is a push-out then $g \in M_c \Longrightarrow f \in M_c$.

As is well-known [7, 12, 13, 16], every quasi-abelian category is P-semi-abelian. As has been recently discovered by Rump [15], there exist semi-abelian categories that are not quasi-abelian.

In [7][Theorem 1], Kuz′minov and Cherevikin established the following fact:

Lemma 3. *An additive category* $\mathcal{A}$ *with kernels and cokernels is P-semi-abelian if and only if the following two conditions are fulfilled:*

(P1) *if* (1) *is a pull-back then* $f \in P_c \Longrightarrow g \in P$;

(P2) *if* (1) *is a push-out then* $g \in M_c \Longrightarrow f \in M$.

If, for a morphism $f \in P_c$ in a pull-back (1) in an additive category with kernels and cokernels, $g \in P_c$ (for a morphism $g \in M_c$ in a push-out (1), $f \in M_c$) then f is called a *stable cokernel* (g is called a *stable kernel*).

A sequence $\dots \xrightarrow{a} B \xrightarrow{b} \dots$ in a P-semi-abelian category is said to be *exact at the term* B if $\operatorname{im} a = \ker b$ (or, equivalently, $\operatorname{coker} a = \operatorname{coim} b$).

For a commutative square (1), denote by $\hat{g} : \operatorname{Ker}\alpha \to \operatorname{Ker}\beta$ the morphism defined by the condition $g(\ker\alpha) = (\ker\beta)\hat{g}$ and by $\hat{f} : \operatorname{Coker}\alpha \to \operatorname{Coker}\beta$, the morphism defined by the condition $\hat{f}(\operatorname{coker}\alpha) = (\operatorname{coker}\beta)f$.

Lemma 4. [2] *For an arbitrary pull-back* (1) *in an additive category meeting Axiom* 1, $\hat{g}$ *is an isomorphism.*

The dual assertion also holds.

Throughout the rest of the section, the ambient category $\mathcal{A}$ is assumed P-semi-abelian.

Suppose that square (1) is a pull-back, $\beta = \beta_1\beta_0$, $\beta_0 \in P$, $\beta_1 \in M_c$. Consider the pull-back

$$\begin{array}{ccc} E & \xrightarrow{\alpha_1} & D \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ F & \xrightarrow{\beta_1} & B. \end{array} \tag{2}$$

Since square (2) is a pull-back, there exists a morphism $\alpha_0 : C \to E$ such that $\alpha_1\alpha_0 = \alpha$ and $h\alpha_0 = \beta_0 g$.

Lemma 5. *If $\beta \in O_c$ then $\alpha_0 \in P$. If $f \in M_c$ and g is an isomorphism then h is an isomorphism and $\alpha_0 \in P$.*

Proof. The square

$$\begin{array}{ccc} C & \xrightarrow{\alpha_0} & E \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ A & \xrightarrow{\beta_0} & F \end{array}$$

is a pull-back [2]. By Lemma 3, the condition $\beta \in O_c$ (i.e., $\beta_0 \in P_c$) implies that $\alpha_0 \in P$. Suppose that $f \in M_c$ and g is an isomorphism. By Lemma 1, $h \in M_c$. Since $\beta_0 g \in P$ and $\beta_0 g = h\alpha_0$, we have $h \in P$. Consequently, h is an isomorphism and $\alpha_0 = h^{-1}\beta_0 g$ is an isomorphism. The lemma is proved.

Lemma 6. *Suppose that square (1) is a pull-back. If $\beta \in O_c$, then $\hat{f} \in M$.*

Proof coincides almost literally with the proof of Lemma 6 in [5].

Lemma 7. *Suppose that the square*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & D \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ A & \xrightarrow{\beta} & B \end{array} \tag{3}$$

is commutative, $f \in M$, and a morphism $h : \operatorname{Coker}\beta \to \operatorname{Coker} f$ is defined by the condition $\operatorname{coker} f = h(\operatorname{coker}\beta)$. Then

(1) *if $\beta \in O_c$ then $\hat{f} \in M$;*

(2) *if $f \in M_c$ and $\operatorname{coker}\beta$ is a stable cokernel then $\hat{f} = \ker h$.*

Proof. (1) Since $f \in M$, square (3) is a pull-back. Therefore, item 1 follows from Lemma 6.

(2) Suppose that $f \in M_c$. By Lemma 5 we may assume that $\beta \in M_c$.

Let $x : X \to \operatorname{Coker}\beta$ be a morphism with $hu = 0$. Consider the pull-back

$$\begin{array}{ccc} Y & \xrightarrow{s} & X \\ {\scriptstyle v}\big\downarrow & & {\scriptstyle x}\big\downarrow \\ B & \xrightarrow{\operatorname{coker}\beta} & \operatorname{Coker}\beta. \end{array} \tag{4}$$

Since $(\operatorname{coker} f)v = hxs = 0$ and $f = \ker\operatorname{coker} f$, there is a morphism $w : Y \to D$ such that $v = fw$. Since (4) is a pull-back, $v(\ker s) = \ker(\operatorname{coker}\beta) = \beta$. Therefore,

$$\hat{f}(\operatorname{coker}\alpha)w(\ker s) = (\operatorname{coker}\beta)fw(\ker s) = (\operatorname{coker}\beta)v(\ker s) = (\operatorname{coker}\beta)\beta = 0.$$

Since $\operatorname{coker}\beta$ is a stable cokernel, $s \in P_c$ and hence $s = \operatorname{coker}(\ker s)$. Consequently, there exists a morphism $c : X \to \operatorname{Coker}\alpha$ such that $(\operatorname{coker}\alpha)w = cs$. We infer

$$\hat{f}cs = \hat{f}(\operatorname{coker}\alpha)w = (\operatorname{coker}\beta)fw = (\operatorname{coker}\beta)v = xs.$$

Since $s \in P$, it follows that $x = \hat{f}c$. This condition determines c uniquely because $\hat{f} \in M$ (Lemma 6). Thus, $\hat{f} = \ker h$.

The lemma is proved.

Suppose that in a commutative diagram

$$\begin{array}{ccccccc} & & A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\psi_0} & C_0 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & C_1 & & \end{array} \tag{5}$$

$\psi_0 = \operatorname{coker}\varphi_0$, $\varphi_1 = \ker\psi_1$. As in the case of an abelian category, diagram (5) gives rise to two semi-exact parts of a Ker-Coker-sequence:

$$\operatorname{Ker}\alpha \xrightarrow{\varepsilon} \operatorname{Ker}\beta \xrightarrow{\zeta} \operatorname{Ker}\gamma \tag{6}$$

and

$$\operatorname{Coker}\alpha \xrightarrow{\tau} \operatorname{Coker}\beta \xrightarrow{\theta} \operatorname{Coker}\gamma. \tag{7}$$

Here $\varepsilon = \widehat{\varphi}_0$, $\zeta = \hat{\psi}_0$, $\tau = \widehat{\varphi}_1$, $\theta = \hat{\psi}_1$.

Suppose now that ψ_0 is a stable cokernel. Then there is a connecting morphism $\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ that unites (6) and (7) in the Ker-Coker-sequence

$$\operatorname{Ker}\alpha \xrightarrow{\varepsilon} \operatorname{Ker}\beta \xrightarrow{\zeta} \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \xrightarrow{\tau} \operatorname{Coker}\beta \xrightarrow{\theta} \operatorname{Coker}\gamma. \tag{8}$$

The morphism δ is constructed as follows. Let

$$\begin{array}{ccc} X & \xrightarrow{s} & \operatorname{Ker}\gamma \\ {\scriptstyle u}\downarrow & & {\scriptstyle \ker\gamma}\downarrow \\ B_0 & \xrightarrow{\psi_0} & C_0 \end{array} \tag{9}$$

be a pull-back and let

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_1} & B_1 \\ {\scriptstyle \operatorname{coker}\alpha}\downarrow & & {\scriptstyle v}\downarrow \\ \operatorname{Coker}\alpha & \xrightarrow{t} & Y \end{array} \tag{10}$$

be a push-out. Since $\psi_0\varphi_0 = 0$, there exists a unique morphism $w : A \to \operatorname{Ker}\psi_0$ such that $\varphi_0 = (\ker\psi_0)w$. Since $\ker\psi_0 = \operatorname{im}\varphi_0$, it follows that $w = \overline{\varphi}_0(\operatorname{coim}\varphi_0) \in P$. Since $\psi_1\beta u = \gamma(\ker\gamma)s = 0$ and $\varphi_1 = \ker\psi_1$, there is a unique morphism $m : X \to A_1$ with $\varphi_1 m = \beta u$. Next, by Lemma 4, $\ker s = \ker\psi_0$, which implies the chain of equalities

$$\varphi_1 m(\ker s)w = \beta u(\ker s)w = \beta(\ker\psi_0)w = \beta\varphi_0 = \varphi_1\alpha,$$

from which $\alpha = m(\ker s)w$. Hence, $(\operatorname{coker}\alpha)m(\ker s)w = (\operatorname{coker}\alpha)\alpha = 0$, whence $(\operatorname{coker}\alpha)m(\ker s) = 0$ because $w \in P$. Since ψ_0 is a stable cokernel, $s = \operatorname{coker}\ker s$. Therefore, there exists a unique morphism $\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ satisfying the relation $(\operatorname{coker}\alpha)m = \delta s$. For the morphism δ, we have $t\delta s = t(\operatorname{coker}\alpha)m = v\varphi_1 m = v\beta u$. Since $t \in M$ (Lemma 3), the condition

$$t\delta s = v\beta u \tag{11}$$

defines δ uniquely once squares (9) and (10) have been chosen and makes it possible to prove the naturality of δ.

The dual argument shows that, in the case where φ_1 is a stable kernel, a connecting morphism $\tilde{\delta}$ satisfying (11) is also defined. Since $t \in M$, $s \in P$, when ψ_0 is a stable cokernel and φ_1 is a stable kernel, the two so-constructed connecting morphisms δ coincide.

Suppose that the morphism β in (5) is represented as $\beta = \beta_1\beta_0$, $\beta_0 : B_0 \to B$, $\beta_1 : B \to B_1$. Consider the pull-back

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\beta_1}\downarrow \\ A_1 & \xrightarrow{\varphi_1} & B_1. \end{array}$$

Suppose that $\psi = \operatorname{coker}\varphi$, $\psi : B \to C$. By Lemma 1 $\varphi \in M_c$ and hence $\varphi = \ker\psi$. There are morphisms $\alpha_0 : A_0 \to A$, $\gamma_0 : C_0 \to C$, $\gamma_1 : C \to C_1$ for which $\alpha_1\alpha_0 = \alpha$ and the diagram

$$\begin{array}{ccccccccc} & & A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\psi_0} & C_0 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha_0}\downarrow & & {\scriptstyle\beta_0}\downarrow & & {\scriptstyle\gamma_0}\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\beta_1}\downarrow & & {\scriptstyle\gamma_1}\downarrow & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & C_1 & & \end{array} \tag{12}$$

commutes. It is easy to verify that $\gamma_1\gamma_0 = \gamma$.

Lemma 8. *If $\beta_0 \in P$, $\beta_1 \in M_c$,* $\operatorname{coker}\alpha$ *is a stable cokernel, the morphism* $\widehat{\alpha}_1 : \operatorname{Coker}\alpha_0 \to \operatorname{Coker}\alpha$ *is defined by the condition* $(\operatorname{coker}\alpha)\alpha_1 = \widehat{\alpha}_1(\operatorname{coker}\alpha_0)$*, and the morphism* $\tau : \operatorname{Coker}\alpha \to \operatorname{Coker}\beta$ *is defined by the condition* $\tau(\operatorname{coker}\alpha) = (\operatorname{coker}\beta)\varphi_1$ *then* $\widehat{\alpha}_1 = \ker\tau$.

Proof. The stability of $\operatorname{coker}\alpha$ justifies application of Lemma 7 to the commutative square

$$\begin{array}{ccc} \operatorname{Im}\alpha_0 & \xrightarrow{\operatorname{im}\alpha_0} & A \\ {\scriptstyle\mathrm{id}}\downarrow & & {\scriptstyle\alpha_1}\downarrow \\ \operatorname{Im}\alpha_0 & \xrightarrow{\alpha_1(\operatorname{im}\alpha_0)} & B_1, \end{array}$$

which yields $\widehat{\alpha}_1 = \ker h$, where the morphism $h : \operatorname{Coker}\alpha \to \operatorname{Coker}\alpha_1$ is defined by the condition $h(\operatorname{coker}\alpha) = \operatorname{coker}\alpha_1$. Now follow the proof of Lemma 10 in [5]. The lemma is proved.

Consider the commutative diagram

$$\begin{array}{ccccccccc} & & A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\psi_0} & C_0 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha_0}\big\downarrow & & {\scriptstyle\beta_0}\big\downarrow & & {\scriptstyle\gamma_0}\big\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C, & & \end{array} \qquad (13)$$

in which $\psi_0 = \operatorname{coker}\varphi_0$, $\varphi = \ker\psi$. Let $\bar{\bar{\delta}} : \operatorname{Ker}\gamma_0 \to \operatorname{Coker}\alpha_0$ be the connecting morphism for (13) (which is defined if φ is a stable kernel or ψ_0 is a stable cokernel).

Lemma 9. *If the connecting morphism* $\bar{\bar{\delta}} : \operatorname{Ker}\gamma_0 \to \operatorname{Coker}\alpha_0$ *is defined for* (13) *and* $\beta_0 \in P_c$ *then* $\bar{\bar{\delta}} \in P$.

Proof. Consider the pull-back

$$\begin{array}{ccc} X & \xrightarrow{s} & \operatorname{Ker}\gamma_0 \\ {\scriptstyle u}\big\downarrow & & {\scriptstyle\ker\gamma_0}\big\downarrow \\ B_0 & \xrightarrow{\psi_0} & C_0 \end{array}$$

and the push-out

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\operatorname{coker}\alpha_0}\big\downarrow & & {\scriptstyle v}\big\downarrow \\ \operatorname{Coker}\alpha_0 & \xrightarrow{t} & Y. \end{array}$$

Since $\psi\beta_0 u = \gamma_0(\ker\gamma_0)s = 0$, there exists a morphism $h : X \to A$ with $\varphi h = \beta_0 u$. Reasoning as in the proof of Lemma 11 in [5], we conclude that the square

$$\begin{array}{ccc} X & \xrightarrow{u} & B_0 \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle\beta_0}\big\downarrow \\ A & \xrightarrow{\varphi} & B \end{array}$$

is a pull-back.

By Lemma 3 $h \in P$; therefore, $(\operatorname{coker}\alpha_0)h \in P$. Since $t\bar{\bar{\delta}}s = v\beta_0 u = v\varphi h = t(\operatorname{coker}\alpha_0)h$ and $t \in M$ by Lemma 3, we have $\bar{\bar{\delta}}s = (\operatorname{coker}\alpha_0)h \in P$, which yields $\bar{\bar{\delta}} \in P$. The lemma is proved.

Now we are in a condition to prove the following version of Theorem 1 of [5] for P-semi-abelian categories.

Theorem 1. *The following hold*:

(1) *if in* (5) ψ_0 *and* $\operatorname{coker}\alpha$ *are stable cokernels and* $\beta \in O_c$ *then sequence* (8) *is exact at* $\operatorname{Coker}\alpha$;

(2) *if in* (5) φ_1 *and* $\ker\gamma$ *are stable kernels and* $\beta \in O_c$ *then sequence* (8) *is exact at* $\operatorname{Ker}\gamma$.

Proof. (1) Suppose that $\beta = \beta_1\beta_0$, $\beta_0 \in P_c$, $\beta_1 \in M_c$. In diagram (12) corresponding to this factorization of β, we have $\gamma_1 \in M$ by Lemma 6 and hence $\ker\gamma = \ker\gamma_0$. Since ψ_0 is a stable cokernel, the connecting morphism $\bar{\bar{\delta}}$ for (13) is defined. By the naturality of the conecting morphism, $\delta = \widehat{\alpha}_1\bar{\bar{\delta}}$. By Lemma 9, $\bar{\bar{\delta}} \in P$, and by Lemma 8, $\widehat{\alpha}_1 = \ker\tau$. Consequently, $\operatorname{im}\delta = \ker\tau$, i.e., sequence (9) is exact at $\operatorname{Coker}\alpha$.

Item (2) is obtained from (1) by duality.

The theorem is proved

The proof of the following lemma repeats the first part of the proof of Theorem 2 in [5].

Lemma 10. *Suppose that in* (5) $\varphi_0 = \ker\psi_0$ ($\psi_1 = \operatorname{coker}\psi_1$). *Then* $\varepsilon = \ker\zeta$ ($\theta = \operatorname{coker}\tau$).

Theorem 2. *The following hold*:

(1) *if in* (5) $\varphi_0 \in O_c$ *and* $\ker\alpha$ *is a stable kernel then sequence* (6) *is exact at* $\operatorname{Ker}\beta$ *and* $\varepsilon \in O_c$; *if in* (5) $\alpha \in O_c$ *then sequence* (6) *is exact;*

(2) *if in* (5) $\psi_1 \in O_c$ *and* $\operatorname{coker}\gamma$ *is a stable cokernel then sequence* (7) *is exact at* $\operatorname{Coker}\beta$ *and* $\theta \in O_c$; *if in* (5) $\alpha \in O_c$ *then sequence* (7) *is exact.*

Proof. Represent φ_0 as a composition $\varphi_0 = \varphi_0''\varphi_0'$, $\varphi_0' \in P$, $\varphi_0'' \in M_c$, $\varphi_0' : A_0 \to A_0'$, $\varphi_0'' : A_0' \to B_0$. Since $\varphi_0'' = \ker\psi_0$, there exists a morphism $\alpha' : A_0' \to A_1$ such that $\varphi_1\alpha' = \beta\varphi_0''$. Then $\alpha'\varphi_0' = \alpha$. Since $\varphi_0' \in P$, the square

$$\begin{array}{ccc} A_0 & \xrightarrow{\alpha} & A_1 \\ {\scriptstyle\varphi_0'}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} \\ A_0' & \xrightarrow{\alpha'} & A_1 \end{array}$$

is a push-out. The morphism φ_0' induces the morphism $\widehat{\varphi}_0{}' : \operatorname{Ker}\alpha \to \operatorname{Ker}\alpha'$ and φ_0'' induces the morphism $\widehat{\varphi}_0{}'' : \operatorname{Ker}\alpha' \to \operatorname{Ker}\beta$; by Lemma 10 $\widehat{\varphi}_0{}'' = \ker\zeta$.

Let $\varphi_0 \in O_c$. Then $\varphi_0' \in P_c$. By the assertion dual to item (2) of Lemma 7, we conclude that $\widehat{\varphi}_0{}' \in P_c$. Hence, $\varepsilon \in O_c$ and $\operatorname{im}\varepsilon = \ker\zeta$.

Assume now that $\alpha \in O_c$. The assertion dual to item (1) of Lemma 7 yields the relation $\widehat{\varphi}_0' \in P$. We again have $\widehat{\varphi}_0{}'' = \ker\zeta$.

Item (1) is proved, and item (2) is obtained from it by duality. The theorem is proved.

Theorem 3. *If in* (5) $\alpha \in O_c$ *and* $\ker\gamma$ *and* φ_1 *are stable kernels then sequence* (8) *is exact at* $\operatorname{Ker}\beta$ *and* $\operatorname{Ker}\gamma$. *If* $\gamma \in O_c$ *and* $\operatorname{coker}\alpha$ *and* ψ_0 *are stable cokernels then sequence* (8) *is exact at* $\operatorname{Coker}\beta$ *and* $\operatorname{Coker}\alpha$.

Proof. Represent β as $\beta = \beta_1\beta_0$, $\beta_0 \in P_c$, $\beta_1 \in M$, and consider the corresponding diagram (12). By Lemma 1 $\alpha_1 \in M$, and by Lemma 6 $\gamma_1 \in M$. Consequently, $\ker\alpha = \ker\alpha_0$, $\ker\beta = \ker\beta_0$, $\ker\gamma = \ker\gamma_0$. The Ker-Coker-sequences corresponding to (5) and (13) are connected with each other by the following commutative diagram:

$$\begin{array}{ccccccccccc} \operatorname{Ker}\alpha_0 & \xrightarrow{\bar\varepsilon} & \operatorname{Ker}\beta_0 & \xrightarrow{\bar\zeta} & \operatorname{Ker}\gamma_0 & \xrightarrow{\bar\delta} & \operatorname{Coker}\alpha_0 & \to & 0 & \to & 0 \\ \| & & \| & & \| & & {\scriptstyle\widehat{\alpha}_1}\downarrow & & \downarrow & & \downarrow \\ \operatorname{Ker}\alpha & \xrightarrow{\varepsilon} & \operatorname{Ker}\beta & \xrightarrow{\zeta} & \operatorname{Ker}\gamma & \xrightarrow{\delta} & \operatorname{Coker}\alpha & \xrightarrow{\tau} & \operatorname{Coker}\beta & \xrightarrow{\theta} & \operatorname{Coker}\gamma. \end{array} \tag{15}$$

By Lemma 7, the strictness of α implies that $\widehat{\alpha}_1 \in M$. Therefore, $\ker\bar\delta = \ker\delta$. By item (2) of Theorem 1, the upper row of (15) is exact at $\operatorname{Ker}\gamma_0$. Therefore, the lower row is exact at $\operatorname{Ker}\gamma$.

Exactness at $\operatorname{Ker}\beta$ follows from Theorem 2.

The second claim of the theorem is obtained from the first by duality.

The theorem is proved.

2. A Generalization of the Ker-Coker-Sequence in a Quasi-Abelian Category

Throughout the section we work in a quasi-abelian category $\mathcal{A}$.

Consider a commutative diagram

$$\begin{array}{ccccc} A_0 & \xrightarrow{\varphi_0} & B_0 & \xrightarrow{\psi_0} & C_0 \\ \alpha\big\downarrow & & \beta\big\downarrow & & \gamma\big\downarrow \\ A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & C_1, \end{array} \tag{16}$$

where $\psi_0\varphi_0 = 0$ and $\psi_1\varphi_1 = 0$.

As above, there are two semi-exact sequences

$$\operatorname{Ker}\alpha \xrightarrow{\varepsilon} \operatorname{Ker}\beta \xrightarrow{\zeta} \operatorname{Ker}\gamma \tag{17}$$

and

$$\operatorname{Coker}\alpha \xrightarrow{\tau} \operatorname{Coker}\beta \xrightarrow{\theta} \operatorname{Coker}\gamma. \tag{18}$$

By the (co)homology $H(B)$ of a sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ such that $\psi\varphi = 0$ we mean the cokernel of the natural morphism $r : \operatorname{Im}\varphi \to \operatorname{Ker}\psi$ or, equivalently, the kernel of the natural morphism $q : \operatorname{Coker}\varphi \to \operatorname{Coim}\psi$ (see [6]).

For diagram (16), we have a commutative diagram of natural morphisms:

$$\begin{array}{ccc} \operatorname{Im}\varphi_0 & \xrightarrow{r_0} & \operatorname{Ker}\psi_0 \\ s\big\downarrow & & t\big\downarrow \\ A_1 = \operatorname{Im}\varphi_1 & \xrightarrow{r_1} & \operatorname{Ker}\psi_1 \end{array}$$

Here $t : \operatorname{Ker}\psi_0 \to \operatorname{Ker}\psi_1$ is the morphism of the kernels of the rows of the square

$$\begin{array}{ccc} B_0 & \xrightarrow{\psi_0} & C_0 \\ \beta\big\downarrow & & \gamma\big\downarrow \\ B_1 & \xrightarrow{\psi_1} & C_1, \end{array}$$

$s : \operatorname{Im}\varphi_0 \to \operatorname{Im}\varphi_1$ is the morphism of the kernels of the rows of the square

$$\begin{array}{ccc} B_0 & \xrightarrow{\operatorname{coker}\varphi_0} & \operatorname{Coker}\varphi_0 \\ \beta\big\downarrow & & \hat\beta\big\downarrow \\ B_1 & \xrightarrow{\operatorname{coker}\varphi_1} & \operatorname{Coker}\varphi_1, \end{array} \tag{19}$$

where $\hat\beta$ is the morphism of the cokernels of the rows of the square

$$\begin{array}{ccc} A_0 & \xrightarrow{\varphi_0} & B_0 \\ \alpha\big\downarrow & & \beta\big\downarrow \\ A_1 & \xrightarrow{\varphi_1} & B_1, \end{array}$$

and the morphisms r_0 and r_1 arise from the semi-exactness of the rows in (16). Thus, we have a natural norphism in homology $\chi : H(B_0) \to H(B_1)$, the morphism of the cokernels of the rows of (19).

We prove the following assertion which is a quasi-abelian version of Corollary B2 in Nomura's article [9].

Theorem 4. *The following hold*:

(1) *if in* (16) $\varphi_0 \in O_c$, $\varphi_1 \in M_c$, *and the morphism* $\chi : H(B_0) \to H(B_1)$ *is a mononmorphism then sequence* (17) *is exact*;

(2) *if in* (16) $\psi_1 \in O_c$, $\psi_0 \in P_c$, *and* χ *is an epimorphism then sequence* (18) *is exact.*

Proof. (1) Take a morphism $x : X \to \operatorname{Ker}\beta$ with $\zeta x = 0$. Show that $x = (\operatorname{im}\varepsilon)\tilde{x}$ for some unique $\tilde{x}$.

We may assume without loss of generality that $x = \operatorname{im} x \in M_c$.

We have the commutative diagram

$$\begin{array}{ccccc} \operatorname{Im}\varphi_0 & \xrightarrow{r_0} & \operatorname{Ker}\psi_0 & \xrightarrow{\operatorname{coker} r_0} & H(B_0) = \operatorname{Coker} r_0 \\ {\scriptstyle s}\big\downarrow & & {\scriptstyle t}\big\downarrow & & {\scriptstyle \chi}\big\downarrow \\ A_1 = \operatorname{Im}\varphi_1 & \xrightarrow{r_1} & \operatorname{Ker}\psi_1 & \xrightarrow{\operatorname{coker} r_1} & H(B_1) = \operatorname{Coker} r_1. \end{array}$$

Since $0 = (\ker\gamma)\zeta x = \psi_0(\ker\beta)x$, there exists a morphism $z : X \to \operatorname{Ker}\psi_0$ with $(\ker\beta)x = (\ker\psi_0)z$. Furthermore, $(\ker\psi_1)tz = \beta(\ker\psi_0)z = \beta(\ker\beta)x = 0$ and $\ker\psi_1 \in M_c$; therefore, $tz = 0$. Moreover, since $\chi(\operatorname{coker} r_0)z = (\operatorname{coker} r_1)tz = 0$ and $\chi \in M$, we have $(\operatorname{coker} r_0)z = 0$. Since $r_0 \in M_c$, it follows that $r_0 = \ker\operatorname{coker} r_0$; therefore, there is a morphism $\sigma : X \to \operatorname{Im}\varphi_0$ with $z = r_0\sigma$.

We have $r_1 s\sigma = t r_0 \sigma = tz = 0$. Since $r_1 \in M_c$, we infer that $s\sigma = 0$.

Represent φ_0 as $\varphi_0 = (\operatorname{im}\varphi_0)\varphi_0'$. Then $\varphi_0' \in P_c$. Consider the pull-back

$$\begin{array}{ccc} Y & \xrightarrow{y_2} & X \\ {\scriptstyle y_1}\big\downarrow & & {\scriptstyle \sigma}\big\downarrow \\ A_0 & \xrightarrow{\varphi_0'} & \operatorname{Im}\varphi_0. \end{array}$$

Note that $s = \beta(\operatorname{im}\varphi_0)$. We have

$$\varphi_1\alpha y_1 = \beta\varphi_0 y_1 = \beta(\operatorname{im}\varphi_0)\varphi_0' y_1 = \beta(\operatorname{im}\varphi_0)\sigma y_2 = 0.$$

Since $\varphi_1 \in M$, this implies that $\alpha y_1 = 0$. Hence, there exists a morphism $y : Y \to \operatorname{Ker}\alpha$ with the property $y_1 = (\ker\alpha)y$. We infer

$$\begin{aligned} (\ker\beta)xy_2 &= (\ker\psi_0)zy_2 = (\ker\psi_0)r_0\sigma y_2 \\ &= (\operatorname{im}\varphi_0)\sigma y_2 = (\operatorname{im}\varphi_0)\varphi_0' y_1 = \varphi_0 y_1 = \varphi_0(\ker\alpha)y = (\ker\beta)\varepsilon y. \end{aligned}$$

Since $\ker\beta \in M_c$, this gives

$$xy_2 = \varepsilon y. \tag{20}$$

Let $\varepsilon = (\operatorname{im}\varepsilon)\varepsilon'$. In (20), $x \in M_c$, $y_2 \in P_c$ and hence $x = \operatorname{im}(xy_2) = (\operatorname{im}\varepsilon)(\operatorname{im}(\varepsilon' y))$. We may take $\tilde{x} = \operatorname{im}(\varepsilon' y)$. The condition $x = (\operatorname{im}\varepsilon)\tilde{x}$ defines $\tilde{x}$ uniquely since $\operatorname{im}\varepsilon$ is a monomorphism.

Item (1) of the theorem is proved. Item (2) follows by duality.

The theorem is proved.

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